Jack Le - PSET 9 - M51

Exercise 23.1. For the following matrices calculate the eigenvalues for the given eigenvectors.

(a) $A = \begin{bmatrix} 8 & 0 & 1 \\ 1 & 7 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ with eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 19 \\ -20 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 11 & -3 & 5 \\ -4 & 7 & 10 \\ 2 & 3 & 8 \end{bmatrix}$ with eigenvectors $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

a) For an eigenvector $\vec{v}$, $A\vec{v} = x\vec{v}$

$$A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0 \\ 0+7+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \therefore \boxed{7 \text{ is the eigenvalue for } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.}$$

$$A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8+0+0 \\ 1+7+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 0 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \therefore \boxed{8 \text{ is the eigenvalue for } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.}$$

$$A\begin{bmatrix} 4 \\ 19 \\ -20 \end{bmatrix} = \begin{bmatrix} 32+0-20 \\ 4+133-80 \\ 0+0-60 \end{bmatrix} = \begin{bmatrix} 12 \\ 57 \\ -60 \end{bmatrix} = 3\begin{bmatrix} 4 \\ 19 \\ -20 \end{bmatrix} \quad \therefore \boxed{3 \text{ is the eigenvalue for } \begin{bmatrix} 4 \\ 19 \\ -20 \end{bmatrix}.}$$

b) For an eigenvector $\vec{v}$, $B\vec{v} = x\vec{v}$

$$B\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -11+6+5 \\ 4-14+10 \\ -2-6+8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \quad \therefore \boxed{0 \text{ is the eigenvalue for } \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.}$$

$$B\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -33-6+0 \\ 12+14+0 \\ -6+6+0 \end{bmatrix} = \begin{bmatrix} -39 \\ 26 \\ 0 \end{bmatrix} = 13\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} \quad \therefore \boxed{13 \text{ is the eigenvalue for } \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}.}$$

$$B\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 55+0+10 \\ -20+0+20 \\ 10+0+16 \end{bmatrix} = \begin{bmatrix} 65 \\ 0 \\ 26 \end{bmatrix} = 13\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} \quad \therefore \boxed{13 \text{ is the eigenvalue for } \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}.}$$

Exercise 23.3. Let $A = \begin{bmatrix} 1 & -8 \\ -2 & 1 \end{bmatrix}$. Find its eigenvalues and an eigenvector for each of them.

$$p_A(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3) = 0$$

$\lambda_1 = 5, \quad \lambda_2 = -3$

<u>5-eigenspace</u>

Find solutions to $(A - 5I_2)\vec{x} = 0 \rightarrow$ Find $N(A - 5I_2)$

$$(A - 5I_2)\vec{x} = 0 \rightarrow \begin{bmatrix} -4 & -8 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{matrix} -4x - 8y = 0 \\ -2x - 4y = 0 \end{matrix} \rightarrow x = -2y \rightarrow \vec{x} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Thus, $N(A - 5I_2) = span\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$. Therefore, $\boxed{\vec{w}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ is an eigen vector with } \lambda_1 = 5.}$

<u>-3-eigenspace</u>

Find solutions to $(A + 3I_2)\vec{x} = 0 \rightarrow$ Find $N(A + 3I_2)$

$$(A + 3I_2)\vec{x} = 0 \rightarrow \begin{bmatrix} 4 & -8 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{matrix} 4x - 8y = 0 \\ -2x + 4y = 0 \end{matrix} \rightarrow \begin{matrix} 4x = 8y \\ 2x = 4y \end{matrix} \rightarrow x = 2y \rightarrow \vec{x} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus, $N(A + 3I_2) = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. Therefore, $\boxed{\vec{w}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is an eigen vector with } \lambda_2 = -3.}$

Exercise 23.5. For each eigenvalue λ of $A = \begin{bmatrix} 7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 8 & 5 \end{bmatrix}$, compute a basis for the nonzero linear subspace $N(A - \lambda I_3)$ in $\mathbf{R}^3$ (the "λ-eigenspace" of A), and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

Since A is lower-triangular, the eigenvalues are the diagonal entries : $\boxed{\lambda = 7,\ 4,\ 5}$

$\boxed{\text{7-eigenspace:}}$ Find $N(A - 7I_3)$

$$(A - 7I_3)\vec{x} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -3 & 0 \\ -2 & 8 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \rightarrow \begin{array}{l} 3x - 3y = 0 \rightarrow y = x \\ -2x + 8y - 2z = 0 \rightarrow -2x + 8x - 2z = 0 \rightarrow 6x = 2z \rightarrow z = 3x \end{array}$$

Thus, $\vec{x} = \begin{bmatrix} x \\ x \\ 3x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Thus, $\boxed{\text{the basis of } N(A - 7I_3) \text{ is } \left\{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right\}.}$

Check: $A\vec{v} = \begin{bmatrix} 7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 8 & 5 \end{bmatrix}\left(t\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right) = t\left(\begin{bmatrix} 7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 8 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right) = t\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = t\left(7\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right) = 7\left(t\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right)$ ✓

$\boxed{\text{4-eigenspace:}}$ Find $N(A - 4I_3)$

$$(A - 4I_3)\vec{x} = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ -2 & 8 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \rightarrow \begin{array}{l} 3x = 0 \rightarrow x = 0 \\ -2x + 8y + z = 0 \rightarrow z = -8y \end{array}$$

Thus, $\vec{x} = \begin{bmatrix} 0 \\ y \\ -8y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ -8 \end{bmatrix}$. Thus, $\boxed{\text{the basis of } N(A - 4I_3) \text{ is } \left\{\begin{bmatrix} 0 \\ 1 \\ -8 \end{bmatrix}\right\}.}$

Check: $A\vec{v} = \begin{bmatrix} 7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 8 & 5 \end{bmatrix}\left(t\begin{bmatrix} 0 \\ 1 \\ -8 \end{bmatrix}\right) = t\left(\begin{bmatrix} 7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 8 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -8 \end{bmatrix}\right) = t\begin{bmatrix} 0 \\ 4 \\ -32 \end{bmatrix} = t\left(4\begin{bmatrix} 0 \\ 1 \\ -8 \end{bmatrix}\right) = 4\left(t\begin{bmatrix} 0 \\ 1 \\ -8 \end{bmatrix}\right)$ ✓

$\boxed{\text{5-eigenspace:}}$ Find $N(A - 5I_3)$

$$(A - I_3)\vec{x} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -2 & 8 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \rightarrow \begin{array}{l} 2x = 0 \rightarrow x = 0 \\ 3x - y = 0 \rightarrow 0 - y = 0 \rightarrow y = 0 \\ -2x + 8y = 0 \rightarrow 0 = 0 \checkmark \end{array}$$

Thus, $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus, $\boxed{\text{the basis of } N(A - 5I_3) \text{ is } \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}.}$

Check: $A\vec{v} = \begin{bmatrix} 7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 8 & 5 \end{bmatrix}\left(t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = t\left(\begin{bmatrix} 7 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 8 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = t\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = t\left(5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 5\left(t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$ ✓

Exercise 24.1. Let A be an $n \times n$ matrix with an eigenvector $\mathbf{v}$ having eigenvalue λ.

(a) Let B be an $n \times n$ matrix. Assume that $\mathbf{v}$ is also an eigenvector to B, with eigenvalue μ. Show that $\mathbf{v}$ is an eigenvector of $A + B$ with eigenvalue $\lambda + \mu$.

(b) Now consider the matrix $M = A^2 + 2A + 3\mathrm{I}_n$. Show that $\mathbf{v}$ is also an eigenvector of M, with eigenvalue $\lambda^2 + 2\lambda + 3$.

a) Since $\vec{v}$ is an eigenvector of A and B, we know $A\vec{v} = \lambda\vec{v}$ and $B\vec{v} = \mu\vec{v}$.

$$(A+B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \mu\vec{v} = (\lambda+\mu)\vec{v}$$

Thus, by the distributive laws of matrices, $\vec{v}$ is an eigenvector of $A+B$ with eigenvalue $\lambda+\mu$.

b) $$M\vec{v} = (A^2+2A+3I_n)\vec{v} = A^2\vec{v} + 2A\vec{v} + 3\vec{v} = A(\lambda\vec{v}) + 2(\lambda\vec{v}) + 3\vec{v}$$
$$= \lambda(\lambda\vec{v}) + 2(\lambda\vec{v}) + 3\vec{v}$$
$$= (\lambda^2 + 2\lambda + 3)\vec{v}$$

Thus, $\vec{v}$ is an eigenvector of M with eigenvalue $\lambda^2 + 2\lambda + 3$.

Exercise 24.2. Use the eigenvalues of the corresponding symmetric matrix to classify each of the following quadratic forms as positive-definite, negative-definite, positive-semidefinite, negative-semidefinite, or indefinite:

(a) $q_A(x, y) = 3x^2 - 12xy + 12y^2$.
(b) $q_B(x, y) = -5x^2 + 4xy - 5y^2$.

a) $A = \begin{bmatrix} 3 & -6 \\ -6 & 12 \end{bmatrix}$ Eigenvalues: $\lambda^2 - tr(A)\lambda + det(A) = 0$

$\lambda^2 - 15\lambda - 0 = \lambda(\lambda - 15) = 0$

$\rightarrow \lambda = 0, 15$

Since all eigenvalues are ≥ 0, q_A is $\boxed{\text{positive-semidefinite.}}$

a) $B = \begin{bmatrix} -5 & 2 \\ 2 & -5 \end{bmatrix}$ Eigenvalues: $\lambda^2 - tr(B)\lambda + det(B) = 0$

$\lambda^2 + 10\lambda + 21 = (\lambda + 7)(\lambda + 3) = 0$

$\rightarrow \lambda = -7, -3$

Since all eigenvalues are < 0, q_B is $\boxed{\text{negative-definite.}}$

Exercise 24.4. Suppose A is an $n \times n$ symmetric matrix.

(a) Explain why each of A^2, A^3, etc. is also symmetric. (Hint: if you are stuck, look back at Example 20.3.6.)

(b) Using the expression $A = WDW^{-1}$, show that A is invertible exactly when its eigenvalues are all nonzero. (You are being asked to show that if A is invertible then 0 is not an eigenvalue, and that if all eigenvalues are nonzero then A is invertible.)

a) By definition, a symmetric matrix satisfies $A^T = A$.

$$\text{For } A^2,\ (A^2)^T = (AA)^T = \overbrace{A^T A^T}^{A^T = A} = AA = A^2$$

$$\text{For } A^3,\ (A^3)^T = (AA^2)^T = (A^2)^T A^T = A^2 A = A^3$$

$$\text{For } A^r,\ (A^r)^T = (\underbrace{AA\cdots A}_{r \text{ times}})^T = \underbrace{A^T A^T \cdots A^T}_{r \text{ times}} = \underbrace{AA\cdots A}_{r \text{ times}} = A^r$$

Thus, the powers of a symmetric matrix are also symmetric because they <u>equal their transpose.</u>

b) If A is invertible, there exists A^{-1} such that $A^{-1} = (WDW^{-1})^{-1} = (W^{-1})^{-1} D^{-1} W^{-1} = WD^{-1}W^{-1}$.

Since diagonal matrices scales the matrix being multiplied, the inverse would be to "unscale" it. This means the inverse of a diagonal matrix replaces the diagonals with its reciprocal.

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} \longrightarrow D^{-1} = \begin{bmatrix} 1/\lambda_1 & & 0 \\ & 1/\lambda_2 & \\ 0 & & \ddots \\ & & & 1/\lambda_n \end{bmatrix}$$

However, this can only happen if all the eigenvalues λ_i are non-zero since $1/0$ is undefined. Thus, if A is invertible, 0 is not an eigenvalue.

If all the eigenvalues are nonzero, then the diagonal matrix D is

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}, \quad \lambda_i \neq 0$$

Since all $\lambda_i \neq 0$, the inverse D^{-1} exists since we do not face the issue of $1/0$ being undefined.

$$\text{Thus, } A^{-1} = WD^{-1}W^{-1} \text{ exists, with } D^{-1} = \begin{bmatrix} 1/\lambda_1 & & 0 \\ & 1/\lambda_2 & \\ 0 & & \ddots \\ & & & 1/\lambda_n \end{bmatrix}$$

We multiply this to check if it is an inverse: $AA^{-1} = (WDW^{-1})(WD^{-1}W^{-1}) = WDD^{-1}W^{-1} = WW^{-1} = I_n$

Therefore, A is invertible when its eigenvalues are all nonzero.

Exercise 24.5. Let M be an $m \times n$ matrix (where m doesn't necessarily equal n).

(a) Explain why the $n \times n$ matrix $M^\top M$ is symmetric. (See Example 26.1.10 for discussion of the interest in such matrices.)

(b) Consider the n-variable quadratic form $q(\mathbf{x}) = \mathbf{x} \cdot (M^\top M\mathbf{x})$. Show that $q(\mathbf{x}) = \|M\mathbf{x}\|^2$. Conclude that q is positive-semidefinite.

(c) Show that q is positive-definite exactly when $N(M) = \{\mathbf{0}\}$. (Hint: when is the length of $M\mathbf{x}$ equal to zero?)

a) To be symmetric, $A^T = A$. Thus, we wish to show $(M^TM)^T = M^TM$

Remember that $(AB)^T = B^TA^T$, and $A^{TT} = A$. Using this, we have

$$\underline{(M^TM)^T = M^TM^{TT} = M^TM}$$

Thus, M^TM is symmetric.

b) $q(x) = \vec{x} \cdot (M^TM\vec{x}) = \vec{x}^TM^TM\vec{x} = (M\vec{x})^T(M\vec{x}) = (M\vec{x}) \cdot (M\vec{x}) = \|M\vec{x}\|^2$

Thus, $\underline{q(\vec{x}) = \|M\vec{x}\|^2}$

This means $q(\vec{x})$ represents the length of vector $(M\vec{x})$, squared. The length of vectors is always nonnegative. It is positive when $M\vec{x} \neq \vec{0}$, and zero when $M\vec{x} = 0$. In addition, there exists $\vec{x} \neq 0$ such that $M\vec{x} = 0$ and $M\vec{x} \neq 0$, depending on the choice of M.

For example:

$$M\vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0} \qquad M\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \vec{0}$$

Thus, $\boxed{q(\vec{x}) \text{ is positive-semidefinite.}}$

c) Since $q(\vec{x}) = \|M\vec{x}\|^2$ is the length of $M\vec{x}$ squared, $q(\vec{x}) = 0$ exactly when the length of $M\vec{x}$ equals zero. In order for $q(\vec{x})$ to be considered positive-definite, this can only happen when $\vec{x} = \vec{0}$, as positive-definite means $q(\vec{x}) > 0$ for all $\vec{x} \neq 0$.

As such, the homogenous system $M\vec{x} = \vec{0}$ can only have $\vec{0}$ as its solution in order for $q(\vec{x})$ to be positive definite. Since the null space is all vectors that map to $\vec{0}$ when M is applied, this is the same as saying $q(\vec{x})$ is positive-definite exactly when $N(M) = \{\vec{0}\}$.

Exercise 24.7. Consider the symmetric matrix $A = \begin{bmatrix} -5 & -14 & 2 \\ -14 & 4 & -16 \\ 2 & -16 & 10 \end{bmatrix}$.

(a) Check that the following are eigenvectors for A, determining the eigenvalue for each (all eigenvalues are integers):

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

(These $\mathbf{v}_i$'s are readily seen to be nonzero and pairwise orthogonal, as the Spectral Theorem guarantees can always be arranged, so these $\mathbf{v}_i$'s constitute an orthogonal basis of $\mathbf{R}^3$.)

(b) Write each standard basis vector $\mathbf{e}_i$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ by computing the projection of $\mathbf{e}_i$ onto each of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(c) Use the following strategy to compute A^{10} (you can leave expressions like λ^{10}, where λ is an eigenvalue of A, in your answer, rather than multiplying them all out): for each $i = 1, 2, 3$, the ith column of A^{10} is given by $A^{10}\mathbf{e}_i$, which we can compute by writing $\mathbf{e}_i$ in the orthogonal basis of eigenvectors from part (a). As a safety check, make sure that your answer is symmetric, since any power of a symmetric matrix is symmetric (you might like to think on your own about why that is true).

(d) Use another strategy to compute A^{10}: since each $\mathbf{v}_i$ has length 3, the eigenvectors $\mathbf{v}_i' = \mathbf{v}_i/3$ are unit vectors and hence constitute an orthonormal basis. Thus, the matrix Q with ith column $\mathbf{v}_i'$ is orthogonal and we know that $A = QDQ^{-1} = QDQ^{\top}$ where D is the diagonal 3×3 matrix with ith entry λ_i, so $A^{10} = QD^{10}Q^{-1} = QD^{10}Q^{\top}$.
Write out Q and compute this expression for A^{10}. (Hint: before multiplying matrices, you should be able to cancel out the 3's in the denominators, so all work is then with integers, not fractions.) You should get the same answer as for (c)!

a)

$$A\vec{v}_1 = \begin{bmatrix} -5 & -14 & 2 \\ -14 & 4 & -16 \\ 2 & -16 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -5+28+4 \\ -14-8-32 \\ 2+32+20 \end{bmatrix} = \begin{bmatrix} 27 \\ -54 \\ 54 \end{bmatrix} = 27\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \rightarrow \boxed{\lambda_1 = 27}$$

$$A\vec{v}_2 = \begin{bmatrix} -5 & -14 & 2 \\ -14 & 4 & -16 \\ 2 & -16 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10-28+2 \\ -28+8-16 \\ 4-32+10 \end{bmatrix} = \begin{bmatrix} -36 \\ -36 \\ -18 \end{bmatrix} = -18\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \rightarrow \boxed{\lambda_2 = -18}$$

$$A\vec{v}_3 = \begin{bmatrix} -5 & -14 & 2 \\ -14 & 4 & -16 \\ 2 & -16 & 10 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10-14+4 \\ 28+4-32 \\ -4-16+20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \rightarrow \boxed{\lambda_3 = 0}$$

b) $\vec{e}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \text{Proj}_{\vec{v}_1}(\vec{e}_1) + \text{Proj}_{\vec{v}_2}(\vec{e}_1) + \text{Proj}_{\vec{v}_3}(\vec{e}_1)$

$$\boxed{\vec{e}_1 = \frac{1}{9}\vec{v}_1 + \frac{2}{9}\vec{v}_2 - \frac{2}{9}\vec{v}_3}$$

$\vec{e}_2 = \text{Proj}_{\vec{v}_1}(\vec{e}_2) + \text{Proj}_{\vec{v}_2}(\vec{e}_2) + \text{Proj}_{\vec{v}_3}(\vec{e}_2)$

$$\boxed{\vec{e}_2 = \frac{-2}{9}\vec{v}_1 + \frac{2}{9}\vec{v}_2 + \frac{1}{9}\vec{v}_3}$$

$\vec{e}_3 = \text{Proj}_{\vec{v}_1}(\vec{e}_3) + \text{Proj}_{\vec{v}_2}(\vec{e}_3) + \text{Proj}_{\vec{v}_3}(\vec{e}_3)$

$$\boxed{\vec{e}_3 = \frac{2}{9}\vec{v}_1 + \frac{1}{9}\vec{v}_2 + \frac{2}{9}\vec{v}_3}$$

c) Each column of A^{10} can be found using $A^{10}\vec{e}_i$.

$$\text{Column 1: } A^{10}\vec{e}_1 = A^{10}\left(\frac{1}{9}\vec{v}_1 + \frac{2}{9}\vec{v}_2 - \frac{2}{9}\vec{v}_3\right) = \overbrace{\frac{1}{9}A^{10}\vec{v}_1 + \frac{2}{9}A^{10}\vec{v}_2 - \frac{2}{9}A^{10}\vec{v}_3}^{v_1, v_2, v_3 \text{ are eigenvectors}}$$

$$= \frac{1}{9}\left(27^{10}\vec{v}_1\right) + \frac{2}{9}\left((-18)^{10}\vec{v}_2\right) - \frac{2}{9}\left(0^{10}\vec{v}_3\right)$$

$$= \frac{1}{9}\begin{bmatrix} 27^{10} \\ -2\cdot 27^{10} \\ 2\cdot 27^{10} \end{bmatrix} + \frac{2}{9}\begin{bmatrix} 2\cdot 18^{10} \\ 2\cdot 18^{10} \\ 18^{10} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 27^{10} + 4\cdot 18^{10} \\ -2\cdot 27^{10} + 4\cdot 18^{10} \\ 2\cdot 27^{10} + 2\cdot 18^{10} \end{bmatrix}$$

$$\text{Column 2: } A^{10}\vec{e}_2 = A^{10}\left(\frac{-2}{9}\vec{v}_1 + \frac{2}{9}\vec{v}_2 + \frac{1}{9}\vec{v}_3\right) = \frac{-2}{9}A^{10}\vec{v}_1 + \frac{2}{9}A^{10}\vec{v}_2 + \frac{1}{9}A^{10}\vec{v}_3$$

$$= -\frac{2}{9}\left(27^{10}\vec{v}_1\right) + \frac{2}{9}\left((-18)^{10}\vec{v}_2\right) - \frac{2}{9}\left(0^{10}\vec{v}_3\right)$$

$$= -\frac{2}{9}\begin{bmatrix} 27^{10} \\ -2\cdot 27^{10} \\ 2\cdot 27^{10} \end{bmatrix} + \frac{2}{9}\begin{bmatrix} 2\cdot 18^{10} \\ 2\cdot 18^{10} \\ 18^{10} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -2\cdot 27^{10} + 4\cdot 18^{10} \\ 4\cdot 27^{10} + 4\cdot 18^{10} \\ -4\cdot 27^{10} + 2\cdot 18^{10} \end{bmatrix}$$

$$\text{Column 3: } A^{10}\vec{e}_3 = A^{10}\left(\frac{2}{9}\vec{v}_1 + \frac{1}{9}\vec{v}_2 + \frac{2}{9}\vec{v}_3\right) = \frac{2}{9}A^{10}\vec{v}_1 + \frac{1}{9}A^{10}\vec{v}_2 + \frac{2}{9}A^{10}\vec{v}_3$$

$$= \frac{2}{9}\left(27^{10}\vec{v}_1\right) + \frac{1}{9}\left((-18)^{10}\vec{v}_2\right) + \frac{2}{9}\left(0^{10}\vec{v}_3\right)$$

$$= \frac{2}{9}\begin{bmatrix} 27^{10} \\ -2\cdot 27^{10} \\ 2\cdot 27^{10} \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 2\cdot 18^{10} \\ 2\cdot 18^{10} \\ 18^{10} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2\cdot 27^{10} + 2\cdot 18^{10} \\ -4\cdot 27^{10} + 2\cdot 18^{10} \\ 4\cdot 27^{10} + 18^{10} \end{bmatrix}$$

$$\boxed{\text{Thus, } A^{10} = \frac{1}{9}\begin{bmatrix} 27^{10} + 4\cdot 18^{10} & -2\cdot 27^{10} + 4\cdot 18^{10} & 2\cdot 27^{10} + 2\cdot 18^{10} \\ -2\cdot 27^{10} + 4\cdot 18^{10} & 4\cdot 27^{10} + 4\cdot 18^{10} & -4\cdot 27^{10} + 2\cdot 18^{10} \\ 2\cdot 27^{10} + 2\cdot 18^{10} & -4\cdot 27^{10} + 2\cdot 18^{10} & 4\cdot 27^{10} + 18^{10} \end{bmatrix}}$$

d) $$\vec{v}_1' = \frac{\vec{v}_1}{3} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \quad \vec{v}_2' = \frac{\vec{v}_2}{3} = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \quad \vec{v}_3' = \frac{\vec{v}_3}{3} = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} | & | & | \\ \frac{v_1}{\|v_1\|} & \frac{v_2}{\|v_2\|} & \frac{v_3}{\|v_3\|} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} = \begin{bmatrix} 27 & & 0 \\ & -18 & \\ 0 & & 0 \end{bmatrix}$$

$$A^{10} = QDQ^T = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} (27)^{10} & & 0 \\ & (-18)^{10} & \\ 0 & & 0^{10} \end{bmatrix}\begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 27^{10} & & 0 \\ & 18^{10} & \\ 0 & & 0^{10} \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 27^{10} & 2\cdot 18^{10} & 0 \\ -2\cdot 27^{10} & 2\cdot 18^{10} & 0 \\ 2\cdot 27^{10} & 18^{10} & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} = \boxed{\frac{1}{9}\begin{bmatrix} 27^{10} + 4\cdot 18^{10} & -2\cdot 27^{10} + 4\cdot 18^{10} & 2\cdot 27^{10} + 2\cdot 18^{10} \\ -2\cdot 27^{10} + 4\cdot 18^{10} & 4\cdot 27^{10} + 4\cdot 18^{10} & -4\cdot 27^{10} + 2\cdot 18^{10} \\ 2\cdot 27^{10} + 2\cdot 18^{10} & -4\cdot 27^{10} + 2\cdot 18^{10} & 4\cdot 27^{10} + 18^{10} \end{bmatrix}}$$

Exercise 25.4. Let $f(x,y) = e^{3x-2y}$.

(a) Compute $(\nabla f)(x,y)$ and $(\mathrm{H}f)(x,y)$ symbolically (please check your work with others to catch errors).

(b) Compute the quadratic approximation $f(2+h, 3+k)$ to f at $(2,3)$ (with h, k near 0).

(c) Use your answer in (b) to estimate $f(2.2, 2.9)$ and compare with the corresponding linear approximation (i.e., omitting the Hessian term) and the "exact" answer on a calculator. Is the quadratic approximation more accurate than the linear approximation?

a) $\boxed{\nabla f(x,y) = \begin{bmatrix} 3e^{3x-2y} \\ -2e^{3x-2y} \end{bmatrix}}$, $\boxed{(\mathrm{H}f)(x,y) = \begin{bmatrix} 9e^{3x-2y} & -6e^{3x-2y} \\ -6e^{3x-2y} & 4e^{3x-2y} \end{bmatrix}}$

b) $f(\vec{a}+\vec{h}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + \frac{1}{2} q_{\mathrm{H}f(\vec{a})}(\vec{h})$

$f(x+h, y+k) \approx f(x,y) + \nabla f(x,y) \cdot \begin{bmatrix} h \\ k \end{bmatrix} + \frac{1}{2} q_{\mathrm{H}f(x,y)}(h,k)$

$f(2,3) = 1$, $\nabla f(2,3) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $(\mathrm{H}f)(2,3) = \begin{bmatrix} 9 & -6 \\ -6 & 4 \end{bmatrix}$, $q_{\mathrm{H}f(2,3)}(x,y) = 9x^2 + 4y^2 - 12xy$

$$f(2+h, 3+k) \approx f(2,3) + \nabla f(2,3) \cdot \begin{bmatrix} h \\ k \end{bmatrix} + \frac{1}{2} q_{\mathrm{H}f(2,3)}(h,k)$$

$$= 1 + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} h \\ k \end{bmatrix} + \frac{1}{2}(9h^2 + 4k^2 - 12hk)$$

$$= 1 + 3h - 2k + \frac{9}{2}h^2 + 2k^2 - 6hk$$

$$\boxed{f(2+h, 3+k) \approx \frac{9}{2}h^2 + 2k^2 - 6hk + 3h - 2k + 1}$$

c) Quadratic: $f(2.2, 2.9) \approx \frac{9}{2}(.2)^2 + 2(-.1)^2 - 6(.2)(-.1) + 3(.2) - 2(-.1) + 1$

$\boxed{= 2.12}$

Linear: $f(2.2, 2.9) \approx f(2,3) + \nabla f\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) \cdot \begin{bmatrix} 2.2-2 \\ 2.9-3 \end{bmatrix}$

$= 1 + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} .2 \\ -.1 \end{bmatrix} = 1 + 3(.2) - 2(-.1)$

$\boxed{= 1.8}$

Actual: $f(2.2, 2.9) = e^{3(2.2) - 2(2.9)} = \boxed{2.23}$

The $\boxed{\text{quadratic approximation is more accurate}}$ than the linear approximation.

Exercise 25.8. This exercise provides practice with a technique that will relate contour plots to the multivariable second derivative test in Chapter 26.

For each quadratic form $q(x,y)$ below, compute the eigenvalues λ_1, λ_2 of the associated symmetric 2×2 matrix and find an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ of corresponding eigenvectors. (The eigenvalues are integers in both cases.)

Also sketch qualitatively correct level sets, including justification in terms of the eigenvalues: in a definite case draw ellipses aligned with the eigenlines and longer along the correct eigenline, and in an indefinite case draw hyperbolas $q(x,y) = \pm c$ aligned with the eigenlines and with asymptotes drawn "closer" to the correct eigenline.

In an indefinite case, indicate as well (with justification in terms of eigenvalue information) which hyperbolas are $q(x,y) = c$ with $c > 0$ and which are $q(x,y) = c$ with $c < 0$.

(a) $q(x,y) = -x^2 + 12xy - y^2$
(b) $q(x,y) = 12x^2 - 6xy + 4y^2$

a) $q_A(x,y) = -x^2 + 12xy - y^2 \longrightarrow A = \begin{bmatrix} -1 & 6 \\ 6 & -1 \end{bmatrix} \longrightarrow \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$

$\lambda^2 + 2\lambda - 35 = (\lambda - 5)(\lambda + 7) = 0$

$\boxed{\lambda_1 = 5, \quad \lambda_2 = -7}$

$\boxed{\underline{5\text{-eigenspace:}}}$ Find $N(A - 5I_2)$

$(A - 5I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} -6x + 6y = 0 \\ 6x - 6y = 0 \end{matrix} \rightarrow y = x \rightarrow \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$N(A - 5I_2) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\boxed{\underline{-7\text{-eigenspace:}}}$ Find $N(A + 7I_2)$

$(A + 7I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 6x + 6y = 0 \rightarrow y = -x \rightarrow \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$N(A + 7I_2) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

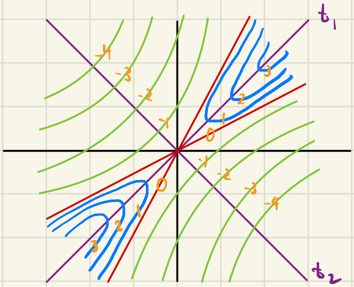
Using the diagonalization formula, $q_A(t_1, t_2) = 5t_1^2 - 7t_2^2$

Level sets:

$q_A = 5t_1^2 - 7t_2^2 = 0$ line ~

$q_A = 5t_1^2 - 7t_2^2 = -1 < 0$ opens to t_2 ~

$q_A = 5t_1^2 - 7t_2^2 = 1 > 0$ opens to t_1 ~



For general hyperbolas, $x^2 - y^2 = 1$ opens towards the x-axis, and $x^2 - y^2 = -1$ opens towards the y-axis. Thus, for $q_A = 5t_1^2 - 7t_2^2 = c$, when $c < 0$, it opens to the t_2-axis (green curves), and when $c > 0$, it opens to the t_1-axis (blue curves).

b) $q_A(x,y) = 12x^2 - 6xy + 4y^2 \longrightarrow A = \begin{bmatrix} 12 & -3 \\ -3 & 4 \end{bmatrix} \longrightarrow \lambda^2 - tr(A)\lambda + det(A) = 0$

$\lambda^2 - 16\lambda + 39 = (\lambda - 13)(\lambda - 3)$

$\boxed{\lambda_1 = 3, \quad \lambda_2 = 13}$

$\boxed{\underline{3\text{-eigenspace:}}}$ Find $N(A - 3I_2)$

$(A - 3I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} 9x - 3y = 0 \\ -3x + y = 0 \end{matrix} \rightarrow y = 3x \rightarrow \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$N(A - 3I_2) = span\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$

$\boxed{\underline{13\text{-eigenspace:}}}$ Find $N(A - 13I_2)$

$(A - 13I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} -x - 3y = 0 \\ -3x - 9y = 0 \end{matrix} \rightarrow x = -3y \rightarrow \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3y \\ y \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$N(A - 13I_2) = span\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}}$

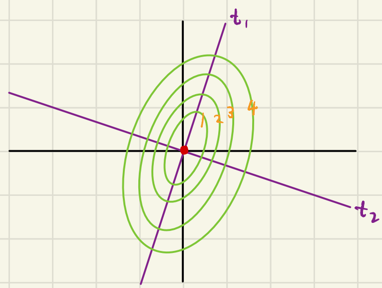
Using the diagonalization formula, $q_A(t_1, t_2) = 3t_1^2 + 13t_2^2$

Level sets:

$q_A = 3t_1^2 + 13t_2^2 = 0$ point

$q_A = 3t_1^2 + 13t_2^2 = 1 > 0$ ellipse

$q_A = 3t_1^2 + 13t_2^2 = -1 < 0$ not possible



Exercise 26.1.

(a) Show that the function $f(x, y) = -\frac{17}{2}x^2 + 4xy - y^2 + 5x + 2y + 1$ has exactly one critical point by finding it. (The coordinates of the critical point are positive integers.)

(b) Determine the eigenvectors and eigenvalues for the symmetric Hessian matrix at the critical point in (a) (as a safety check, make sure the eigenvectors you find are perpendicular to each other), and use this information to determine if f has a local maximum, local minimum or a saddle point there.

(c) Sketch a contour plot of f near the critical point by computing eigenvalues and corresponding eigenvectors for the Hessian there. (It only matters to sketch approximate ellipses or hyperbolas aligned with the appropriate perpendicular lines through the critical point, indicating the longer axis direction in the ellipse case and the eigenline to which the asymptotes are "closer" in the hyperbola case.)

a)

$$\nabla f = \begin{bmatrix} -17x + 4y + 5 \\ 4x - 2y + 2 \end{bmatrix} = \vec{0} \rightarrow \begin{array}{l} -17x + 4y + 5 = 0 \rightarrow -17x + 8x + 4 + 5 = 0 \rightarrow x = 1 \\ 4x - 2y + 2 = 0 \rightarrow y = 2x + 1 \rightarrow y = 3 \end{array}$$

$\boxed{\text{Thus, } (1,3) \text{ is the only critical point.}}$

b)

$$Hf(x,y) = \begin{bmatrix} -17 & 4 \\ 4 & -2 \end{bmatrix} \rightarrow Hf(1,3) = \begin{bmatrix} -17 & 4 \\ 4 & -2 \end{bmatrix} \rightarrow \begin{array}{l} \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0 \\ \lambda^2 + 19\lambda + 18 = (\lambda+18)(\lambda+1) = 0 \end{array}$$

$$\boxed{\lambda_1 = -1, \quad \lambda_2 = -18}$$

$\boxed{\underline{-1\text{-eigenspace:}}}$ Find $N(A + 1 I_2)$

$$(A + 1 I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} -16 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} -16x + 4y = 0 \\ 4x - y = 0 \end{array} \rightarrow y = 4x \rightarrow \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 4x \end{bmatrix} = x\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$N(A + 1 I_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}}$$

$\boxed{\underline{-18\text{-eigenspace:}}}$ Find $N(A + 18 I_2)$

$$(A + 18 I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} x + 4y = 0 \\ 4x + 16y = 0 \end{array} \rightarrow x = -4y \rightarrow \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4y \\ y \end{bmatrix} = y\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$N(A + 18 I_2) = \text{span}\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}}$$

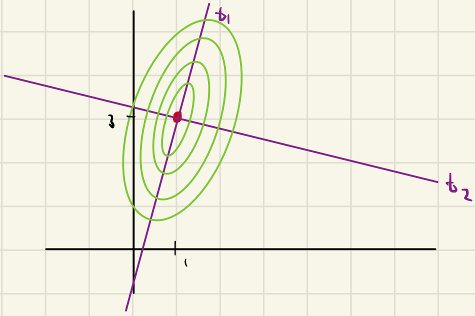
Since all the eigenvalues are negative, there is a $\boxed{\text{local max at point } (1,3).}$

c) $q_{Hf(a)}(t_1, t_2) = -t_1^2 - 18t_2^2$

$q = -t_1^2 - 18t_2^2 = 0$ point

$q = -t_1^2 - 18t_2^2 = 1 > 0$ ellipse

$q = -t_1^2 - 18t_2^2 = -1 < 0$ not possible



Exercise 26.3. The function $f(x, y) = 8x^2 + 6xy + 4x^3 + 3xy^2$ appears in Example I.2.3, where its critical points are found: $(0, 0)$, $(0, -2)$, $(-3/2, -1)$, $(1/6, -1)$.

(a) Compute the Hessian matrix at each of these points and for each determine if it is a local maximum, local minimum, or a saddle point.

(b) Sketch the contour plot of f near each of the points $(-3/2, -1)$ and $(1/6, -1)$ by computing eigenvalues and corresponding eigenvectors for the Hessian there. (It only matters to sketch approximate ellipses or hyperbolas aligned with the appropriate perpendicular lines through the critical point, indicating the longer axis direction in the ellipse case and the eigenline to which the asymptotes are "closer" in the hyperbola case.)

a) $\nabla f(x,y) = \begin{bmatrix} 16x + 6y + 12x^2 + 3y^2 \\ 6x + 6xy \end{bmatrix}$, $(Hf)(x,y) = \begin{bmatrix} 16 + 24x & 6 + 6y \\ 6 + 6y & 6x \end{bmatrix}$

$$\boxed{(Hf)(0,0) = \begin{bmatrix} 16 & 6 \\ 6 & 0 \end{bmatrix}}$$

$\lambda^2 - 16\lambda - 36 = (\lambda - 18)(\lambda + 2) = 0$

$\boxed{\lambda = 18, -2}$

Opposite signs, therefore there is a saddle point at $(0,0)$.

$$\boxed{(Hf)(0,-2) = \begin{bmatrix} 16 & -6 \\ -6 & 0 \end{bmatrix}}$$

$\lambda^2 - 16\lambda - 36 = (\lambda - 18)(\lambda + 2) = 0$

$\boxed{\lambda = 18, -2}$

Opposite signs, therefore there is a saddle point at $(0,-2)$.

$$\boxed{(Hf)(-3/2, -1) = \begin{bmatrix} -20 & 0 \\ 0 & -9 \end{bmatrix}}$$

$\lambda^2 + 29\lambda + 180 = (\lambda + 20)(\lambda + 9) = 0$

$\boxed{\lambda = -20, -9}$

Both negative, therefore there is a local max at $(-3/2, -1)$

$$\boxed{(Hf)(1/6, -1) = \begin{bmatrix} 20 & 0 \\ 0 & 1 \end{bmatrix}}$$

$\lambda^2 - 21\lambda + 20 = (\lambda - 20)(\lambda - 1) = 0$

$\boxed{\lambda = 1, 20}$

Both positive, therefore there is a local min at $(1/6, -1)$

b) For $(-3/2, -1)$, we have $\lambda_1 = -20$, $\lambda_2 = -9$. We find the eigenvectors.

$\boxed{\text{-20- eigenspace:}}$ Find $N(A + 20 I_2)$

$$(A + 20 I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x \text{ is free} \\ 11y = 0 \end{matrix} \rightarrow \begin{matrix} x \text{ is free} \\ y = 0 \end{matrix} \rightarrow \vec{x} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$N(A + 20 I_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, $\boxed{\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\boxed{\text{-9 - eigenspace:}}$ Find $N(A + 9 I_2)$

$$(A + 9 I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} -11 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} -11x = 0 \\ y \text{ is free} \end{matrix} \rightarrow \begin{matrix} x = 0 \\ y \text{ is free} \end{matrix} \rightarrow \vec{x} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$N(A + 9 I_2) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, $\boxed{\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

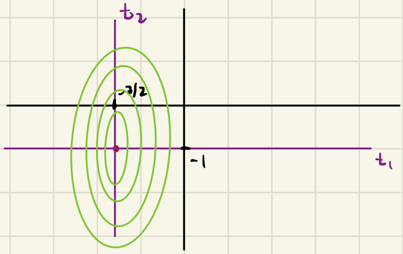
Using the diagonalization formula, $q_A(t_1, t_2) = -20t_1^2 - 9t_2^2$

Level sets:

$q_A = -20t_1^2 - 9t_2^2 = 0$ point

$q_A = -20t_1^2 - 9t_2^2 = 1 > 0$ ellipse

$q_A = -20t_1^2 - 9t_2^2 = -1 < 0$ not possible



For $(1/6, -1)$, we have $\lambda_1 = 1$, $\lambda_2 = 20$. We find the eigenvectors.

$\boxed{\underline{1\text{-eigenspace}}:}$ Find $N(A - 1 I_2)$

$$(A - 1 I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 19 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} 19x = 0 \\ y \text{ is free} \end{matrix} \rightarrow \begin{matrix} x = 0 \\ y \text{ is free} \end{matrix} \rightarrow \vec{x} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$N(A - 1 I_2) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$\boxed{\underline{20\text{-eigenspace}}:}$ Find $N(A - 20 I_2)$

$$(A - 20 I_2)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x \text{ is free} \\ -19y = 0 \end{matrix} \rightarrow \begin{matrix} x \text{ is free} \\ y = 0 \end{matrix} \rightarrow \vec{x} = \begin{bmatrix} x \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$N(A - 20 I_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \quad \boxed{\vec{w}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Using the diagonalization formula, $q_A(t_1, t_2) = t_1^2 + 20t_2^2$

Level sets:

$q_A = t_1^2 + 20t_2^2 = 0$ point

$q_A = t_1^2 + 20t_2^2 = 1 > 0$ ellipse

$q_A = t_1^2 + 20t_2^2 = -1 < 0$ not possible