Pre-reading due Monday

Lecture 9 - Partial Derivatives and Contour Plots

October 14, 2022

Goals: Symbolically and numerically compute first and second order partial derivatives and be able to determine their sign given the contour plot or graph of a function.

Let $f(x_1, x_2)$ be a scalar function of two variables. $f: \mathbb{R}^n \Rightarrow \mathbb{R}$

Definition: The **partial derivative of** f with respect to x_1 at the point (a, b) denoted in any of the following ways

$$\frac{\partial f}{\partial x_1}(a,b), \quad \left.\frac{\partial f}{\partial x_1}\right|_{(a,b)}, \quad f_{x_1}(a,b).$$

The partial derivative of f with respect to x_1 at (a, b) is the instantaneous rate of change of f at the point (a, b) if we only move in the x_1 direction (so x_2 is held constant at the value b). Formally,

$$\frac{\partial f}{\partial x_1}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

Similarly,

$$\frac{\partial f}{\partial x_2}(a,b) = \left.\frac{\partial f}{\partial x_2}\right|_{(a,b)} = f_{x_2}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

More generally, if we have a function of n variables $f(x_1, \ldots, x_n)$, the partial derivative of f with respect to x_i at the point $(a_1, \ldots, a_n)$ is denoted ~~as~~

$f'(x)$

$$\frac{\partial f}{\partial x_i}(a_1,\ldots,a_n), \quad \left.\frac{\partial f}{\partial x_i}\right|_{(a_1,\ldots,a_n)}, \quad f_{x_i}(a_1,\ldots,a_n).$$

$\frac{\partial f}{\partial x_1}$

We can compute this in two ways (the first one is always used in practice; the second is convenient to see why some concepts from single-variable derivatives transfer to partial derivatives).

Method 1 (Symbolic): Think of the x_j's, $j \neq i$ as constant and then apply the usual single-variable derivative rules. After, plug in the point $(a_1, \ldots, a_n)$ for $(x_1, \ldots, x_n)$.

Method 2 (Numerical): Replace each x_j with a_j in f for $j \neq i$. Then differentiate the resulting single-variable function with respect to x_i. Finally, replace x_i with a_i.

$f(x_1, a_2, a_3, \ldots, a_n)$
$\Rightarrow$
single variable
$\Rightarrow$ differentiate

Symbolic

Example 1: Compute $f_x(2, \frac{1}{8})$, where $f(x,y) = x\cos(\pi xy)$. Symbolically, what is $f_y(x,y)$?

① $f_x(x,y) = \cos(\pi xy) \quad - x\sin(\pi xy)\cdot \pi y$

$f_x(2, \frac{1}{8}) = \cos(\frac{\pi}{4}) - \frac{\pi}{4}\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8}$

Numerical

② $f(x, \frac{1}{8}) = x\cos(\frac{\pi}{8}x)$, $\frac{d}{dx} f(x, \frac{1}{8}) = \cos(\frac{\pi}{8}x) - \frac{\pi}{8}x\sin$

Plug in y first then differentiate

$f_x(2, \frac{1}{8}) = \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8}$

$f_y(x,y) = -\pi x^2 \sin(\pi xy)$

Example 2: Compute $f_y(\sqrt{17}, 2)$, where $f(x,y) = 8^y + \ln(y) + \pi x e^{e^x}\sqrt{x+1}$.

① $f_y(x,x) = 8^y \ln y + \frac{1}{y} + 0 = \ln 8 \cdot 8^y + \frac{1}{y}$

$f_y(\sqrt{17}, 2) = 64\ln 8 + \frac{1}{2}$

② $f(\sqrt{17}, y) = 8^y + \ln y + \pi\sqrt{17}e^{\sqrt{17}}\sqrt{\sqrt{17}+1}$

$\frac{d}{dy} f(\sqrt{17}, 4) = \ln 8 \cdot 8^y + \frac{1}{y} + 0 \Rightarrow f_y(\sqrt{17}, 2)$

Example 3: Compute $f_z(1,2,3)$ where $f(x,y,z) = xz + yz + z^2$.

① $f_z(x,y,z) = x + y + 2z$

$f_z(1,2,3) = 1 + 2 + 2(3) = 4$

② $f(1,2,z) = z + 2z + z^2 = 3z + z^2$

$\frac{d}{dz} f(1,2,3) = 3 + 2z \Rightarrow 4$

Example 4: Approximate $f(1, 4.1)$ where $f(x,y) = \sqrt{x+2y}$.

S.V. $f(x) \approx f(a) + f'(a)(x-a)$ (for a close to x)

$(1, 4.1)$ is close to $(1,4)$

$f(1,4.1) \approx f(1,4) + f_y(1,4)(4.1-4)$

$f(1,4) = \sqrt{1+8} = 3 \qquad \frac{\partial f}{\partial y} = \frac{1}{2}(x+2y)^{-1/2}\cdot 2$

$f_y(1,4) = 1/3$

$f(1,4.1) \approx 3 + \frac{1}{3}\cdot\frac{1}{10} = \frac{91}{30} \approx 3.03$

Interpreting Partial Derivatives on a contour plot: Visualize $f(x, y)$ as the height above (x, y) on the surface graph $z = f(x, y)$, where x is the east-west coordinate and y the north-south. Then

- $f_x(a, b)$ equals the slope experience by someone walking on the surface $z = f(x, y)$ just as they go past the point (a, b) from *west to east.*

- $f_y(a, b)$ equals the slope experienced by someone walking on the surface $z = f(x, y)$ as they walk past the point (a, b) from *south to north.*

In other words,

- $f_x(a, b)$ tells us whether the contour labels (values of f) are increasing/decreasing as we walk through (a, b) from west to east.

- $f_y(a, b)$ tells us whether the contour labels are increasing/decreasing as we walk through (a, b) from south to north.

If $f_x(a_1, b_1) > f_x(a_2, b_2) > 0$, then the slope in the x-direction at (a_1, b_1) is *steeper* than the slope in the x-direction at (a_2, b_2). $\frac{\partial f}{\partial y}$ is similar (except in the y-direction).

Example 5: Consider the following contour plot.

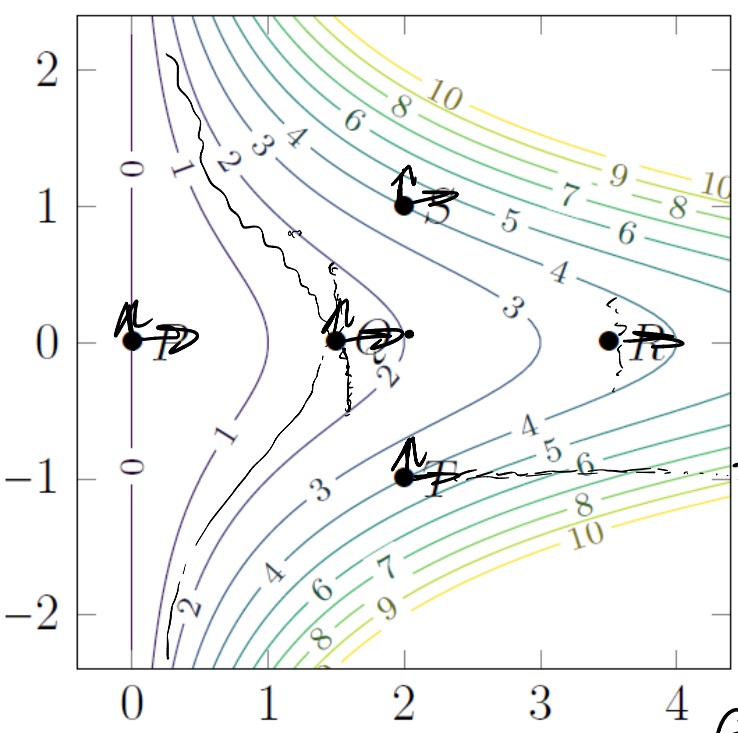
this plot is in 2-d

f_x: Determine the sign at each point

$f_x(P) > 0 \to$ positive

$f_x(Q) > 0 \to$ positive

$f_x(R), f_x(S), f_x(T) > 0$



$f_x(S), f_x(T) > f_x(P), f_x(Q), f_x(R)$

$f_y(P) = 0$ (f is not changing)

$f_y(Q), f_y(R) \approx 0$

$f_y(S) > 0, f_y(T) > 0$

Move to the right (Is f increasing? If so, positive)

Move up (Is f increasing? If so, positive)

Definition: For a function $f(x_1, \ldots, x_n)$ of n variables that is differentiable in each x_i separately, the **second partial derivatives** are defined to be

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = x_i\text{-th partial derivative of } \frac{\partial f}{\partial x_j}.$$

$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$

If $j = i$, we denote the second partial by $\frac{\partial^2 f}{\partial x_i^2}$.

Theorem 9.6.4: (Clairaut-Schwarz) Consider a function $f(x_1, \ldots, x_n)$ that is continuous, and for $1 \le i, j \le n$ suppose the partial derivatives $\partial f / \partial x_i$ and $\partial f / \partial x_j$ as well as the second partial derivatives $\partial^2 f / \partial x_i \partial x_j$ and $\partial^2 f / \partial x_j \partial x_i$ exist and are continuous. Then the order of applying the partial derivatives does not matter:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Order does not matter.

We often denote the second partial by the notation $f_{x_i x_j}$ or $f_{x_j x_i}$.

Example 6: Compute all second partials of the function $f(x, y) = x^4 - 6x^2y^2 + y^4$. What is $f_{xx} + f_{yy}$?

$f_x = 4x^3 - 12xy^2$

$f_{xx} = 12x^2 - 12y^2$

$f_{xy} = -24xy$

$f_y = -12x^2y + 4y^3$

$f_{yy} = -12x^2 + 12y^2$

$f_{yx} = -24xy$

$f_{xx} + f_{yy} = 0$

Note: Functions that satisfy $f_{xx} + f_{yy} = 0$ are called *harmonic*. The equation $f_{xx} + f_{yy} = 0$ is a *partial differential equation* known as Laplace's equation. See Math 53 for further discussion on PDEs.

(If time permits, we will do more derivative examples)

Ex 7 $\quad f(x,y) = 2xy + y^8 \ln(2^y - 3\sqrt{y})$

compute $f_{yx} = f_{xy}$

$\left\{ \begin{array}{l} f_x = 2y \\ f_{xy} = 2 \end{array} \right. \rightarrow$ other order is annoying