Exercise 10.2. Let $f(x, y) = x^3 - 3x^2 - 6xy + 9x + 3y^2$.

(a) Show that this function f has exactly 2 critical points: $(1, 1)$ and $(3, 3)$.
(b) By examining the behavior of f on the lines $x = 1$ and $y = x$ that pass through $(1, 1)$, explain why $(1, 1)$ is a saddle point. (It turns out that $(3, 3)$ is a local minimum.)
(c) By examining the behavior of f on the x-axis, show that f has no global maximum and no global minimum.

a) The function f has a critical point at (a,b) when $f_x(a,b)=0$ and $f_y(a,b)=0$.

$f_y(x,y) = -6x + 6y = 0 \longrightarrow 6y = 6x \rightarrow \underline{y = x}$

$f_x(x,y) = 3x^2 - 6x - 6y + 9 = 0$

$x^2 - 2x - 2y + 3 = 0$

$x^2 - 2x - 2(x) + 3 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\underline{x = 3, 1}$

Therefore, the critical points are $(1,1)$ and $(3,3)$.

b) We first move along the line $x=1$. We pick two points on the line near $(1,1)$ to test this.

$f(1, 0.9) = (1)^3 - 3(1)^2 - 6(1)(0.9) + 9(1) + 3(0.9)^2$

$= 1 - 3 - 5.4 + 9 + 3(0.81)$

$= 7 - 5.4 + 2.43 = 4.03$

$f(1,1) = 1^3 - 3(1)^2 - 6(1)(1) + 9(1) + 3(1)^2 = 4$

$f(1, 1.1) = 1^3 - 3(1)^2 - 6(1)(1.1) + 9(1) + 3(1.1)^2 = 4.03$

We now check the line $y=x$. We pick two points near $(1,1)$ on this line.

$$f(0.9, 0.9) = (0.9)^3 - 3(0.9)^2 - 6(0.9)(0.9) + 9(0.9) + 3(0.9)^2 = 3.969$$

$$f(1,1) = 4$$

$$f(1.1, 1.1) = (1.1)^3 - 3(1.1)^2 - 6(1.1)(1.1) + 9(1.1) + 3(1.1)^2 = 3.971$$

Based on these, we see that when we move along $y=x$, $f(1,1)$ is a local maximum. When we move along $x=1$, $f(1,1)$ is a local minimum. Therefore, the point $(1,1)$ is a saddle point.

c) The x-axis is the line $y=0$. Plugging this into equation, we get

$$f(x,0) = x^3 - 3x^2 + 9x$$

Suppose this were a single variable function $g(x) = x^3 - 3x^2 + 9x$

$$g'(x) = 3x^2 - 6x + 9$$

We can use the determinant of the quadratic equation to find if this equation has any real roots.

$$b^2 - 4ac \longrightarrow (-6)^2 - 4(3)(9) = 36 - 108 = -72$$

Since $b^2 - 4ac < 0$ for $g'(x)$, there are no real roots. This means $g'(x)$ is always positive. Therefore, when we move towards $-\infty$ on the x-axis, $f(x,0)$ continues to decrease. When we move towards ∞ on the x-axis, $f(x,0)$ continues to increase. Therefore, $f(x,0)$ is unbounded both above and below. As such, $f(x,y)$ can never have a global maximum or a global minimum.

This was my original solution for part (c). However, I was told in office hours that it is not the intended solution. If you don't have time, just ignore this. If you do, can I know if the solution below is still correct?

c) The x-axis is the line $y=0$. Plugging this into equation, we get

$$f(x,0) = x^3 - 3x^2 + 9x = x(x^2 - 3x + 9)$$

We showed in (a) that f has exactly two critical points. This means the only possible global extrema are $(1,1)$ and $(3,3)$. However, in (b), we showed that $(1,1)$ was a saddle point and $(3,3)$ was a local minimum. Based on this, $f(x,y)$ has no global maximum.

We now show that $(3,3)$ is not a global minimum using the line at $f(x,0)$.

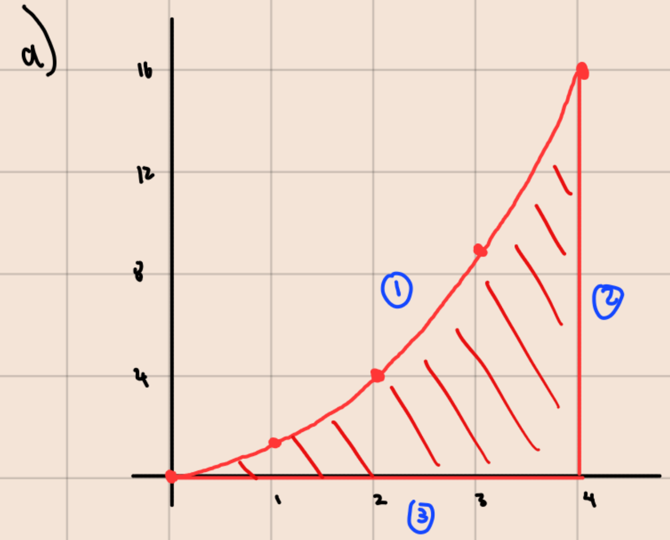
$$f(3,3) = 3^3 - 3(3)^2 - 6(3)(3) + 9(3) + 3(3)^2 = 0$$

Since there are no other possible global minima besides $(3,3)$, we can show it does not exist if there is a point $f(x,0) < f(3,3)$. One such value is $x=-1$, as $f(-1,0) = -13 < 0$. Therefore, $f(x,y)$ has no global maximum and no global minimum.

Exercise 10.5. Let $D = \{(x, y) \in \mathbf{R}^2 : 0 \le x \le 4, 0 \le y \le x^2\}$ (the region over the interval $[0, 4]$ in the x-axis and "below $y = x^2$"). Let $f(x, y) = x^3 + y^3 - 3xy$. It is a fact that f has extrema on D.

(a) Draw a picture of D and show that there are no critical points of f on the *interior* of D (the region given by strict inequalities: $0 < y < x^2$, $0 < x < 4$); in particular, f has no extrema on the interior.
(b) By analyzing f on the boundary of D (which consists of 3 parts: the parabolic arc of points (x, x^2) for $0 \le x \le 4$ and two line segments), use single-variable calculus to find the extrema on the boundary.
(c) Find the maximal and minimal values of f on D, and where they occur.

a)



$f_x = 3x^2 - 3y = 0$

$3x^2 = 3y$

$x^2 = y$

$f_y = 3y^2 - 3x = 0$

$3y^2 = 3x$

$y^2 = x$

Critical Points: $(0,0)$, $(1,1)$

However, the inequality states that to be in the interior, $y < x^2$. This makes both of the above critical points invalid.

$\boxed{\text{Therefore, } f \text{ has no critical points or extrema on the interior of } D.}$

b) Case 1 : $y = x^2$, $0 \le x \le 4$

$f(x, x^2) = x^3 + (x^2)^3 - 3x(x^2) = x^6 - 2x^3$

Let $g(x) = x^6 - 2x^3$

critical points of g

$g'(x) = 6x^5 - 6x^2 = 6x^2(x^3 - 1) = 0 \rightarrow x = 0, 1$

We now test $g(x)$ on critical points and endpoints.

Test $x = 0, 1, 4$: $g(0) = 0$, $g(1) = -1$, $g(4) = 3968$

Case 2 : $x = 4$, $0 \le y \le 16$

$f(4, y) = 4^3 + y^3 - 3(4)y = y^3 - 12y + 64$

Let $g(y) = y^3 - 12y + 64$

not in interval

$g'(y) = 3y^2 - 12 = 0 \rightarrow y^2 = 4 \rightarrow y = \cancel{-2}$ and 2

We now test $g(y)$ on critical points and endpoints.

Test $y = 0, 2, 4$: $g(0) = 64$, $g(2) = 48$, $g(4) = 80$

Case (3) : $y = 0$, $0 \le x \le 4$

$f(x, 0) = x^3 + (0)^3 - 3x(0) = x^3$

Let $g(x) = x^3$

$g'(x) = 3x^2 = 0 \rightarrow x^2 = 0 \rightarrow x = 0$

We now test $g(x)$ on critical points and endpoints.

Test $x = 0, 4$: $g(0) = 0$, $g(4) = 64$

Therefore, using these three cases, the extrema on the boundary of f is a minimum of -1 at $f(1,1)$ and a maximum of 3968 at $f(4, 16)$.

c) We found in (a) that f has no critical points in the interior of D, so it has no extrema on the interior. Therefore, we focus on the extrema at the boundary, found in (b).

From (b), we found various possible possible points:

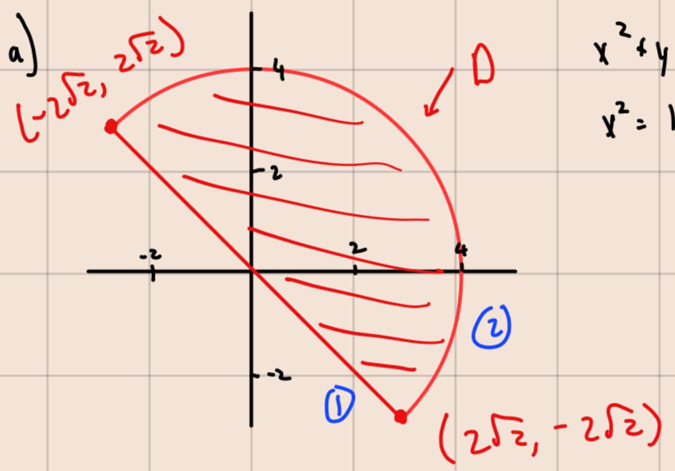
Point	$f(x,y)$
$(0,0)$	0
$(1,1)$	-1 ← min
$(4,16)$	3968 ← max
$(4,0)$	64
$(4,2)$	48
$(4,4)$	80

Therefore, the maximum value of f on D is 3968 at point $(4, 16)$ and the minimum value of f on D is -1 at point $(1,1)$

Exercise 10.6. Consider the region $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 16, x + y \geq 0\}$ (the half of the disk of radius 4 centered at the origin that lies on or above the line $x + y = 0$ through the center). Let $f(x, y) = x^2 + y^2 - 4y$. It is a fact that f has extrema on D.

(a) Draw a picture of D, labeling the coordinates of any "corners", and show that there is one critical point of f in the interior of D. Evaluate f there.
(b) By analyzing f on the boundary of D (which consists of 2 parts: the circular arc and the diameter of the circle), use single-variable calculus to find the extrema on the boundary.
(c) Find the maximal and minimal values of f on D, and where they occur.

a)



$x^2 + y^2 = 16$

$x^2 = 16 - y^2$

$x + y = 0 \rightarrow y = -x$

$y^2 = x^2 \rightarrow y^2 = 16 - y^2$

$2y^2 = 16 \rightarrow y^2 = 8$

$y = \pm\sqrt{8} = \pm 2\sqrt{2}$

$x = -y$

$$\nabla f(x, y) = \begin{bmatrix} 2x \\ 2y - 4 \end{bmatrix}$$

For (a, b) to be a critical point, $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

$f_x = 2x = 0 \qquad f_y = 2y - 4 = 0$

$x = 0 \qquad y = 2$

Since $\nabla f(x,y)$ can't be undefined, the point $(0, 2)$ is the only critical point of f. This point is also contained in D.

$$f(0, 2) = 0^2 + 2^2 - 4(2) = \boxed{-4}$$

b) Case ①: $x + y = 0 \rightarrow y = -x$, $-2\sqrt{2} \leq x \leq 2\sqrt{2}$, $-2\sqrt{2} \leq y \leq 2\sqrt{2}$

$f(x, -x) = x^2 + (-x)^2 - 4(-x) = x^2 + x^2 + 4x = 2x^2 + 4x$

Let $g(x) = 2x^2 + 4x$

$g'(x) = 4x + 4 = 0 \rightarrow 4x = -4 \rightarrow x = -1$

We now test $g(x)$ on critical points and endpoints.

Test $x = -2\sqrt{2}, -1, 2\sqrt{2}$

$$g(-2\sqrt{2}) = 2(-2\sqrt{2})^2 + 4(-2\sqrt{2}) = 16 - 8\sqrt{2}$$

$$g(-1) = 2(-1)^2 + 4(-1) = -2$$

$$g(2\sqrt{2}) = 2(2\sqrt{2})^2 + 4(2\sqrt{2}) = 16 + 8\sqrt{2}$$

Case 2: $x^2 + y^2 = 16 \to y^2 = 16 - x^2$

i) $y = \sqrt{16 - x^2}$, $-2\sqrt{2} \le x \le 4$, $0 \le y \le 4$

ii) $y = -\sqrt{16 - x^2}$, $2\sqrt{2} \le x \le 4$, $-2\sqrt{2} \le y \le 0$

i) $$f(x, \sqrt{16 - x^2}) = x^2 + \left(\sqrt{16 - x^2}\right)^2 - 4\left(\sqrt{16 - x^2}\right)$$

$$= x^2 + 16 - x^2 - 4\sqrt{16 - x^2}$$

$$= 16 - 4\sqrt{16 - x^2}$$

Let $g(x) = 16 - 4\sqrt{16 - x^2}$

$$g'(x) = -2(16 - x^2)^{-1/2}(-2x) = 4x / \sqrt{16 - x^2}$$

$\to$ Check $g'(x)$ undefined

$$\sqrt{16 - x^2} = 0$$

$$16 - x^2 = 0 \to x^2 = 16$$

not in D $\to$ $x = \cancel{-4}$, $x = 4$

$\to$ Check $g'(x) = 0$

$$4x / \sqrt{16 - x^2} = 0$$

$$4x = 0$$

$$x = 0$$

We now test $g(x)$ on critical points and endpoints.

Test $x = -2\sqrt{2}, 0, 4$

$$g(-2\sqrt{2}) = 16 - 4\sqrt{16 - (-2\sqrt{2})^2} = 16 - 4\sqrt{16 - 8} = 16 - 4\sqrt{8} = 16 - 8\sqrt{2}$$

$$g(0) = 16 - 4\sqrt{16 - (0)^2} = 16 - 4(4) = 0$$

$$g(4) = 16 - 4\sqrt{16 - (4)^2} = 16 - 4(0) = 16$$

ii) $f(x, -\sqrt{16-x^2}) = x^2 + (-\sqrt{16-x^2})^2 - 4(-\sqrt{16-x^2})$

$$= x^2 + 16 - x^2 + 4\sqrt{16-x^2}$$

$$= 16 + 4\sqrt{16-x^2}$$

Let $h(x) = 16 + 4\sqrt{16-x^2}$

$$h'(x) = 2(16-x^2)^{-1/2}(-2x) = -4x/\sqrt{16-x^2}$$

$\rightarrow$ Check $h'(x)$ is undefined

$$\sqrt{16-x^2} = 0$$

$$16 - x^2 = 0 \rightarrow x^2 = 16$$

not in D

$$x = \cancel{-4}, \ x = 4$$

$\rightarrow$ Check $h'(x) = 0$

$$-4x/\sqrt{16-x^2} = 0$$

$$-4x = 0$$

$$x = \cancel{0}$$

not in D bc $y = -4$

We now test $g(x)$ on critical points and endpoints.

Test $x = 2\sqrt{2}, 4$

$$g(2\sqrt{2}) = 16 + 4\sqrt{16-(2\sqrt{2})^2} = 16 + 4\sqrt{16-8} = 16 + 4\sqrt{8} = 16 + 8\sqrt{2}$$
$$g(4) = 16 + 4\sqrt{16-(4)^2} = 16 + 4(0) = 16$$

Therefore, using these three cases, the extrema on the boundary of f is a minimum of -2 at $f(-1, 1)$ and a maximum of $16 + 8\sqrt{2}$ at $(2\sqrt{2}, -2\sqrt{2})$.

c) We found in (a) that f has a critical point at $(0, 2)$ in the interior of D. We also found multiple critical points on the boundary in (b). We check all of these

Point	$f(x,y)$
$(0,2)$	-4
$(-2\sqrt{2}, 2\sqrt{2})$	$16-8\sqrt{2}$
$(-1,1)$	-2
$(2\sqrt{2}, -2\sqrt{2})$	$16+8\sqrt{2}$
$(0,4)$	0
$(4,0)$	16

Therefore, the maximum value of f on D is $16+8\sqrt{2}$ at point $(2\sqrt{2}, -2\sqrt{2})$. The minimum value of f on D is -4 at point $(0,2)$.

Exercise 10.10. Let $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. Find all critical points of f. (There are three such points; in your solution you should find that $x^3 = -y^3$ at a critical point, so $x = -y$ at such points.)

$$\nabla f(x,y) = \begin{bmatrix} 4x^3 - 4x + 4y \\ 4y^3 + 4x - 4y \end{bmatrix}$$

$f_x = 4x^3 - 4x + 4y = 0$

$x^3 - x + y = 0$

$x - y = x^3$

$f_y = 4y^3 - 4y + 4x = 0$

$y^3 - y + x = 0$

$y^3 + x^3 = 0$

$x^3 = -y^3$

$x = -y \rightarrow y = -x$

We plug this back into our equations above

$f_x = x^3 - x - x = 0$

$x^3 = 2x \rightarrow x = 0$

$x^2 = 2$

$x = -\sqrt{2}, \sqrt{2}$

$f_y = (-x)^3 - (-x) + x = 0$

$-x^3 + 2x = 0$

$x^3 = 2x \rightarrow x = 0$

$x^2 = 2$

$x = -\sqrt{2}, \sqrt{2}$

Therefore, our critical points are $(-\sqrt{2}, \sqrt{2})$, $(0,0)$, and $(\sqrt{2}, -\sqrt{2})$

Exercise 11.5. Consider the contour map shown in Figure 11.5.1 below.

(a) For every indicated point draw an approximate unit vector in the direction of the negative gradient when nonzero, or indicate when it vanishes.

(b) For each indicated point, where on the map will the gradient descent from there likely end?

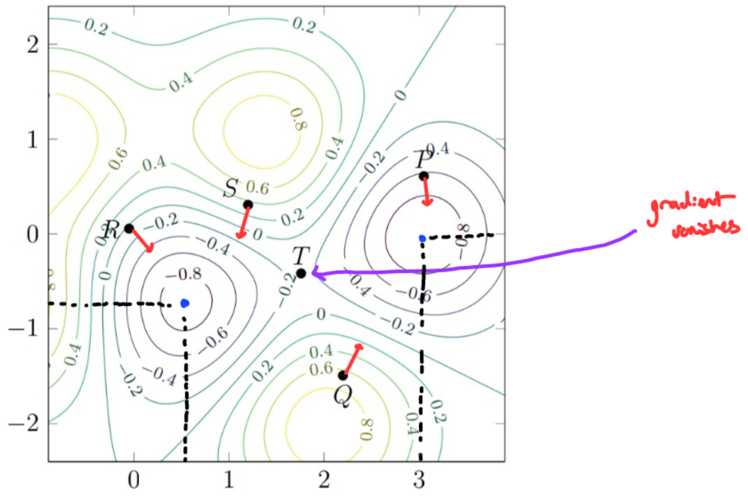


FIGURE 11.5.1. A contour map with five labeled points P, Q, R, S, T.

b) Point R: It will end around $(0.5, -0.8)$

Point S: It will end around $(0.5, -0.8)$

Point P: It will end around $(3, 0)$

Point Q: It will end around $(3, 0)$

Point T: The gradient is already vanishing.

Exercise 11.8. Consider the surface

$$S = \{x^2 + 2y^2 + 3z^2 = 5\}$$

in $\mathbf{R}^3$. Describe the tangent plane to S at $(0, 1, 1)$ in each of two ways:

(a) equation form,

(b) parametric form (multiple answers are possible for this).

a) Suppose $f(x, y, z) = x^2 + 2y^2 + 3z^2 - 5$

$$\nabla f = \begin{bmatrix} 2x \\ 4y \\ 6z \end{bmatrix} \rightarrow \nabla f(0,1,1) = \begin{bmatrix} 2(0) \\ 4(1) \\ 6(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$$

Tangent Plane: This gradient vector is the normal vector to the plane.

Let $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector on the plane. The gradient vector will be orthogonal to the displacement between $\vec{w}$ and $(0, 1, 1)$.

$$\vec{n} \cdot \left(\vec{w} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = 0 \longrightarrow \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix} \cdot \left(\begin{bmatrix} x-0 \\ y-1 \\ z-1 \end{bmatrix}\right) = 0$$

$$0(x-0) + 4(y-1) + 6(z-1) = 0$$

$$4y - 4 + 6z - 6 = 0$$

$$\boxed{4y + 6z = 10}$$

b) Suppose P is our point $(0, 1, 1)$.

We find two more points using the equation. Since x is not in the equation above, any value of x with a valid value of y and z will be on the plane.

1. Let $x = 1$, $y = 1 \to 4(1) + 6z = 10 \to 6z = 6$, $z = 1$

   Therefore, point $Q = (1, 1, 1)$

2. Let $x = 2$, $y = 4 \to 4(4) + 6z = 10 \longrightarrow 6z = -6$, $z = -1$

   Therefore, point $R = (2, 4, -1)$

We now find displacement vectors $\vec{e}$ and $\vec{e}\,'$ using the points

$$\vec{e} = \vec{Q} - \vec{P} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{e}\,' = \vec{R} - \vec{P} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

Using this, the parametric form of the plane is

$$\boxed{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}}$$

Exercise 11.9. Find all points P on the curve $C = \{(x, y) \in \mathbf{R}^2 : 3yx^2 + 3xy^2 + y^3 + 2x^3 = 27\}$ for which the tangent of C at P is parallel to the y-axis.

Let $f(x,y) = 3yx^2 + 3xy^2 + y^3 + 2x^3 - 27$

$$\nabla f = \begin{bmatrix} 6yx + 3y^2 + 6x^2 \\ 3x^2 + 6xy + 3y^2 \end{bmatrix}$$

This gradient is the normal of the tangent line. For the tangent line to be parallel to the y-axis, the normal must be a scalar multiple of the normal of the y-axis.

The normal of the y-axis is $\begin{bmatrix} a \\ 0 \end{bmatrix}$, for any $a \in \mathbb{R}$. Therefore, the normal of the tangent line will be a scalar multiple if $f_y = 0$.

$$f_y = 3x^2 + 6xy + 3y^2 = 0$$

$$3(x^2 + 2xy + y^2) = 0$$

$$3(x+y)^2 = 0$$

$$x + y = 0 \to y = -x$$

We plug this back into our equation for C to find points P.

$$3yx^2 + 3xy^2 + y^3 + 2x^3 = 27$$

$$x^3 + 3yx^2 + 3xy^2 + y^3 + x^3 = 27$$

$$(x+y)^3 + x^3 = 27$$

$$(x + (-x))^3 + x^3 = 27$$

$$x^3 = 27 \to x = 3 \to y = -3$$

Therefore, the only point is $\boxed{(3, -3)}$

Exercise 11.10. Figure 11.5.2 shows a yellow ellipsoid $2x^2 + 3y^2 + z^2 = 9$ and blue "2-sheeted" hyperboloid $-3x^2 + 6y^2 + z^2 = 7$ meeting along a curve C consisting of 2 parts (one per "sheet"); zoom in to see the 2 parts of the blue surface. The point $P = (1, -1, 2)$ is on both surfaces (i.e., it satisfies the equation defining each), so it lies on the curve C along which they meet.

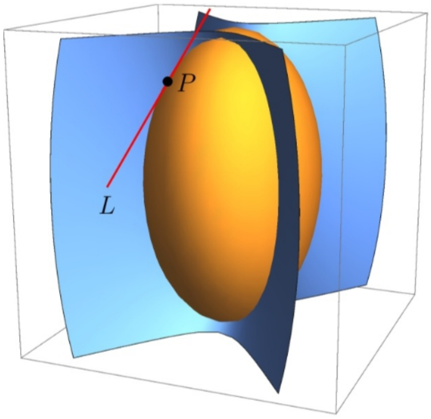


FIGURE 11.5.2. Two surfaces $2x^2 + 3y^2 + z^2 = 9$ and $-3x^2 + 6y^2 + z^2 = 7$ meet along a curve C which contains a point P. The line L is the tangent line to C at P.

It is a plausible-sounding fact (which you do *not* need to justify) that the tangent line L to C at P is the overlap of the tangent planes to the surfaces at P. (That fact is not specific to this point and these surfaces.)

(a) Compute a normal vector to the tangent plane at P for each surface.
(b) Find a nonzero vector orthogonal to both normal vectors you found in (a), and use it to give a parametric form for L.

a) Let $f(x,y,z) = 2x^2 + 3y^2 + z^2$ $\qquad$ $g(x,y,z) = -3x^2 + 6y^2 + z^2$

$$\nabla f = \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix} \qquad \nabla g = \begin{bmatrix} -6x \\ 12y \\ 2z \end{bmatrix}$$

$$\nabla f(1,-1,2) = \begin{bmatrix} 4(1) \\ 6(-1) \\ 2(2) \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 4 \end{bmatrix} \qquad \nabla g(1,-1,2) = \begin{bmatrix} -6(1) \\ 12(-1) \\ 2(2) \end{bmatrix} = \begin{bmatrix} -6 \\ -12 \\ 4 \end{bmatrix}$$

The normal vector for the yellow surface at P is $(4, -6, 4)$
The normal vector for the blue surface at P is $(-6, -12, 4)$

b) Suppose $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We wish to find $\vec{v} \cdot \begin{bmatrix} 4 \\ -6 \\ 4 \end{bmatrix} = 0$ and $\vec{v} \cdot \begin{bmatrix} -6 \\ -12 \\ 4 \end{bmatrix} = 0$.

$$4x - 6y + 4z = 0$$

$$-6x - 12y + 4z = 0 \rightarrow 6x + 12y - 4z = 0$$

$$10x = -6y \rightarrow x = -\frac{3}{5}y$$

$$4\left(-\frac{3}{5}y\right) - 6y + 4z = 0$$

$$-\frac{12}{5}y - 6y + 4z = 0$$

$$4z = \frac{42}{5}y \rightarrow z = \frac{21}{10}y$$

We plug this back in to get $\begin{bmatrix} -3/5y \\ y \\ 21/10y \end{bmatrix} = y \begin{bmatrix} -3/5 \\ 1 \\ 21/10 \end{bmatrix}$.

We set $y=10$ to get $\begin{bmatrix} -6 \\ 10 \\ 21 \end{bmatrix}$ as our desired vector.

Therefore, the parametric form of L is $\boxed{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -6 \\ 10 \\ 21 \end{bmatrix}}$.

Exercise 11.13. Give an estimate for $\sqrt{(4.2)^2 + (2.9)^2}$ by considering the linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ at a suitable point. (There is no need to use a calculator for this.)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

We choose $(a,b) = (4,3)$ since it is close to our point.

$$\nabla f(x,y) = \begin{bmatrix} \frac{1}{2}(x^2+y^2)^{-1/2}(2x) \\ \frac{1}{2}(x^2+y^2)^{-1/2}(2y) \end{bmatrix} = \begin{bmatrix} x/\sqrt{x^2+y^2} \\ y/\sqrt{x^2+y^2} \end{bmatrix}$$

$$f(4,3) = \sqrt{4^2+3^2} = 5$$

$$f_x(4,3) = 4/\sqrt{4^2+3^2} = 4/5$$

$$f_y(4,3) = 3/\sqrt{4^2+3^2} = 3/5$$

$$f(4.2, 2.9) \approx f(4,3) + f_x(4,3)(4.2-4) + f_y(4,3)(2.9-3)$$

$$\approx 5 + 4/5\,(0.2) + 3/5\,(-0.1)$$

$$\approx 5 + 4/5\,(1/5) + 6/10\,(-1/10)$$

$$\approx 5 + 4/25 - 6/100$$

$$\approx 500/100 + 16/100 - 6/100$$

$$\approx 510/100$$

$$\boxed{f(4.2, 2.9) \approx 5.1}$$