Lecture 23 - Eigenvalues and Eigenvectors

November 28, 2022

Goals: Check if a given vector is an eigenvector of a matrix and compute its eigenvalue if so, and find eigenvalues and eigenvectors of 2×2 matrices.

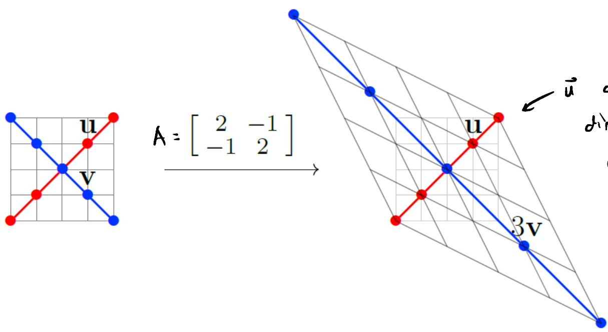
Definition: Let A be an $n \times n$ matrix. A vector $\mathbf{v} \in \mathbb{R}^n$ is called an **eigenvector** of A if it is nonzero and if there is a scalar $\lambda \in \mathbb{R}$ for which $A\mathbf{v} = \lambda\mathbf{v}$. → λ is called the eigenvalue

Example 1: Verify that $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigen~~values~~ vectors of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

$A\vec{u} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\vec{u}$ ∴ $\vec{u}$ is eigen vector w/ eigenvalue $\lambda = 1$

$A\vec{v} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3\vec{v}$ ∴ $\vec{v}$ is eigenvec w/ eigenval $\lambda = 3$



$\vec{u}$ and $\vec{v}$ are same dir, just scales in length

Example 2: For A as in Example 2, here is another we can calculate $A\mathbf{w}$ for any $\mathbf{w} \in \mathbb{R}^2$.

Suppose $\vec{u}$ & $\vec{v}$ are a basis for $\mathbb{R}^2$

$\vec{w} = a\vec{u} + b\vec{v} \rightarrow A\vec{w} = A(a\vec{u} + b\vec{v})$
$= a(A\vec{u}) + b(A\vec{v})$
$= a(1\vec{u}) + b(3\vec{v})$
$= a\vec{u} + 3b\vec{v}$

Coords of $\vec{w}$ are just linear comb of eigen vectors

The takeaway from the last example is that if an $n \times n$ matrix A exhibits a basis of eigenvectors of $\mathbb{R}^n$, then A "looks diagonal" if we represent vectors in $\mathbb{R}^n$ in terms of the aforementioned eigenvectors.

↳ $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Suppose $\vec{w} = w_1\vec{e_1} + w_2\vec{e_2}$

Then, $D\vec{w} = Dw_1\vec{e_1} + Dw_2\vec{e_2}$
$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} w_2$
$= w_1\vec{e_1} + 3w_2\vec{e_2}$

diagonal matrix scales each corresponding component of $\vec{w}$ by the diagonal entry

Example 3: Suppose $\mathbf{v}$ is an eigenvector of a matrix A with eigenvalues λ and μ. How are λ and μ related?

λ must equal μ

Proof! Assume $A\vec{v} = \lambda\vec{v}$ and $A\vec{v} = \mu\vec{v}$. Thus, $\lambda\vec{v} = \mu\vec{v}$. Since $\vec{v}$ is nonzero by definition & eigen vectors, $\lambda = \mu$.

Each eigenvector only has one eigenvalue

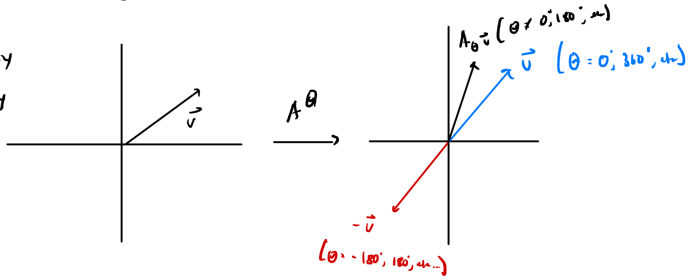
Eigenvalues can have infinitely many eigenvectors
$A\vec{v} = \lambda\vec{v}$, $A(2\vec{v}) = \lambda(2\vec{v})$, etc...

Example 4: For an angle θ that is not an integer multiple of 0° or 180°, the rotation matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has no eigenvectors.

$A\vec{v} = \lambda\vec{v}$ means scaling $\vec{v}$ by λ. The only line rotation scales is when $\theta = 0°, 180°$, etc...



$\Rightarrow$ no eigenvectors for A when $\theta \neq 0°, 180°$

Example 5: If the null space of an $n \times n$ matrix A contains a nonzero vector $\mathbf{v}$, then is $\mathbf{v}$ an eigenvector of A?

Since $\vec{v} \in N(A)$, $A\vec{v} = 0 \rightarrow A\vec{v} = 0\vec{v}$ $\therefore$ $\vec{v}$ is an eigenvector w/ eigenvalue 0

Bc $\exists$ nonzero $\vec{v}$ in $N(A)$

Building off the previous example, we see that 0 is an eigenvalue of A precisely when A is **not** invertible. Because of this, we obtain the following proposition, which gives us a way to compute eigenvalues/eigenvectors.

<u>**Proposition 23.1.11:**</u> A scalar λ is an eigenvalue for an $n \times n$ matrix A precisely when $A - \lambda I_n$ is **not** invertible, or equivalently the null space $N(A - \lambda I_n)$ contains a nonzero vector. The eigenvectors for A with eigenvalue λ are the nonzero vectors in $N(A - \lambda I_n)$ (which is called the **λ-eigenspace** for A).

$(A - \lambda I_n)\vec{v} = \vec{0} \Rightarrow A\vec{v} = \lambda\vec{v}$

Solutions to homogenous system, nonzero vectors in null space

Example 5: Find the eigenvalues of an $n \times n$ diagonal matrix.

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$ everything in this area is 0

$$\rightarrow D - \lambda I_n = \begin{bmatrix} d_1 - \lambda & & 0 \\ & \ddots & \\ 0 & & d_n - \lambda \end{bmatrix}$$

from Lec 22, if there's a zero in diag, then it's not invertible

$\Rightarrow$ If λ is equal to any diag entry, D is not invertible. Therefore, the diagonal entries $d_1, \ldots, d_n$ are the eigenvalues

$\left(D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \lambda = 1, 3, -2 \right)$

The result of Example 5 can in fact be extended more generally to upper and lower triangular matrices.

<u>**Theorem 23.2.2:**</u> Let M be an $n \times n$ upper or lower triangular matrix. The eigenvalues of M are exactly the diagonal entries. For each eigenvalue λ, the corresponding eigenvectors are the nonzero vectors in the null space $N(M - \lambda I_n)$, i.e. the non-zero solutions $\mathbf{x}$ to the upper or lower triangular system

$$(M - \lambda I_n)\mathbf{x} = \mathbf{0}$$

of n linear equations in n-variables which can be solved via back-substitution.

upper - triangular

Example 7. Let $M = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 8 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Find the eigenvectors and the corresponding eigenvalues of M.

Find vectors $\vec{v}$ where $(A - \lambda I_n)\vec{v} = 0$,

since $(A - \lambda I_n)\vec{v} = 0 \to A\vec{v} - \lambda\vec{v} = 0 \to A\vec{v} = \lambda\vec{v}$

$\lambda = 2, 8$

① 2-eigenspace: $N(M - 2I_3) \longrightarrow$ solve $(M - 2I_3)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & 3 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \longrightarrow \begin{matrix} 3x_2 + x_3 = 0 \\ 6x_2 + 2x_3 = 0 \end{matrix} \longrightarrow x_2 = -\tfrac{1}{3}x_3$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ -\frac{1}{3}x_3 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix}$$

$\Rightarrow$ The 2-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix}\right\}$

eigenvectors w/ eigenvalue 2 are all vectors in span above

② 8-eigenspace: $N(M - 8I_3) \longrightarrow$ solve $(M - 8I_3)\vec{x} = \vec{0}$

$$\begin{bmatrix} -6 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \longrightarrow \begin{matrix} -6x_1 + 3x_2 + x_3 = 0 \\ 2x_3 = 0 \\ -6x_3 = 0 \end{matrix} \Rightarrow x_3 = 0 \longrightarrow \begin{matrix} 3x_2 = 6x_1 \\ x_2 = 2x_1 \end{matrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ 2x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\Rightarrow$ The 8-eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right\}$

Note that the matrix $A - \lambda I_n$ being non-invertible is equivalent to the problem of when $\det(A - \lambda I_n) = 0$. Since we only discussed the determinant for 2×2 matrices, we will focus on that case. It turns out that $\det(A - \lambda I_n)$ is a polynomial in the variable λ, and hence finding eigenvalues of A amounts to finding the roots of that polynomial.

$A - \lambda I_n = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

<u>**Theorem 23.3.1:**</u> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. Define its **trace** $\operatorname{tr}(A) = a + d$ to be the sum of its diagonal entries, and recall its determinant $\det(A) = ad - bc$. We have the following:

1. The eigenvalues of A in $\mathbb{R}$ are exactly the roots of $P_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$ (this polynomial is called the **characteristic polynomial** of A).

2. For each such root λ, the corresponding eigenvectors are the nonzero vectors in $N(A - \lambda I_2)$.

<u>**Corollary 23.3.2:**</u> A 2×2 matrix A has an eigenvalue precisely when $P_A(\lambda)$ has a real root, which is to say $\operatorname{tr}(A)^2 - 4\det(A) \geq 0$.

$b^2 - 4ac$

Example 8: Find the characteristic polynomial of A_θ. $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\operatorname{tr}(A) = 2\cos\theta$, $\det A_\theta = ad - bc = \cos^2\theta + \sin^2\theta = 1$

$P_{A_\theta}(\lambda) = \lambda^2 - 2\cos\theta\,\lambda + 1$

$\Rightarrow \operatorname{tr}(A)^2 - 4\det(A) \geq 0 \longrightarrow 4\cos^2\theta - 4 \geq 0 \longrightarrow 4(\cos^2\theta - 1) \geq 0$

this is neg for $\theta \neq 0°, 180°$, etc..

Characteristic Polynomial $A - \lambda I_n = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$P_A(\lambda) = \det(A - \lambda I_n)$
$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - a\lambda - d\lambda + ad - bc$
$= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$

If $P_A(\lambda) = 0$, then there exists a $N(A - \lambda I_n) \neq 0$, which means there is an $\vec{x}$ where $A\vec{x} = \lambda\vec{x}$.
$\therefore$ solutions of $P_A(\lambda)$ are the eigenvalues of A

Fact: $P_A(A) = 0$

Example 9: Find the eigenvalues of $A = \begin{bmatrix} 3 & 16 \\ 2 & -1 \end{bmatrix}$.

$\operatorname{tr} A = 3 + (-1) = 2$

$\det A = -3 - 32 = -35$

$P_A(\lambda) = \lambda^2 - 2\lambda - 35$

Find roots:
$P_A(\lambda) = \lambda^2 - 2\lambda - 35 = 0$
$= (\lambda + 5)(\lambda - 7) = 0$

$\Rightarrow \boxed{\lambda = -5, 7}$

Find eigenvectors

① -5-eigenspace

$$A - (-5)I_n = \begin{bmatrix} 8 & 16 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \longrightarrow \begin{matrix} 8x_1 + 16x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{matrix} \longrightarrow x_1 = -2x_2$$

$\hookrightarrow \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow \boxed{\operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\} \text{ is space}}$

② 7-eigenspace

$$A - 7I_n = \begin{bmatrix} -4 & 16 \\ 2 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \longrightarrow \begin{matrix} -4x_1 + 16x_2 = 0 \\ 2x_1 - 8x_2 = 0 \end{matrix} \longrightarrow x_1 = 4x_2$$

$\hookrightarrow \begin{bmatrix} 4x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 4 \\ 1 \end{bmatrix} \Rightarrow \boxed{\operatorname{span}\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\} \text{ is space}}$