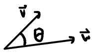
Vector Basics

1) $\|\vec v\| = \sqrt{v_1^2 + \dots + v_n^2}$
2) $\|\vec v - \vec w\| \to$ distance btwn $\vec v$ & $\vec w$
3) $\cos\theta = \frac{\vec v \cdot \vec w}{\|\vec v\|\|\vec w\|}$ 
4) Correlation Coefficient: $r = \frac{\vec x \cdot \vec y}{\|x\|\|y\|}$

Planes

Eqn form: $ax + by + cz = d$

Parametric: $\vec P + t\vec e + t\vec e'$

1) Eqn ⟷ Normal
   ① coefficients is normal, $\vec n = \begin{bmatrix}a\\b\\c\end{bmatrix}$
   ② plug in point to find d
2) Eqn ⟶ Parametric
   ① use eqn to find 3 points
   ② pick one point to be $\vec P$
   ③ find displacement to others
   $\vec e = \vec Q - \vec P$, $\vec e' = \vec R - \vec P$
3) Parametric → Normal
   ① solve for $\vec n$, $ax + by + cz = d$
   $\vec n \cdot \vec e = 0$, $\vec n \cdot \vec e' = 0$
   ③ plug in point to find d

Linear Regression

1) let $X = [x\text{-vals}]$, $Y = [y\text{-vals}]$
2) find $\hat X = X - \text{Proj}_{\vec 1}(X) = X - \bar x(\vec 1)$
3) Project Y onto $V = \text{span}\{\hat X, \vec 1\}$
   $\text{Proj}_V(Y) = \text{Proj}_{\hat X}(Y) + \text{Proj}_{\vec 1}(Y)$
   $= m\hat X + \bar y(\vec 1)$
5) sub $X - \bar x(\vec 1)$ for $\hat X$
   $= m(X - \bar x(\vec 1)) + \bar y(\vec 1)$
   $= mX - m\bar x + \bar y(\vec 1)$
6) $y = mx + b$, $b = \bar y(\vec 1) - m\bar x$

Extrema on a region

1) find crit on interior ($\nabla f = 0$)
2) find crit on boundary
   ↳ single-var, set a variable
3) find crit on corners
   ↳ include intersection of boundary lines
4) plug in points

Linear Approximations

$f(\vec x) \approx f(\vec a) + \nabla f(\vec a) \cdot \begin{bmatrix}x_1 - a_1\\ \vdots \\ x_n - a_n\end{bmatrix}$

$f(\vec x) \approx f(\vec a) + Df(\vec a)(\vec x - \vec a)$

$f(\vec a + \vec h) \approx f(\vec a) + Df(\vec a)\vec h$

$(f\circ h)(\vec x) \approx (f\circ h)(\vec a) + (D(f\circ h)(\vec a))(\vec x - \vec a)$

Gradient Stuff

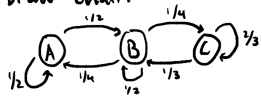
1) $-\nabla f(\vec a)/\|\nabla f(\vec a)\|$
   dir decr most rapidly
2) Tangent Plane: $\nabla f(a,b,c) \cdot \begin{bmatrix}x-a\\y-b\\z-c\end{bmatrix} = 0$
   Line: $\nabla f(a,b) \cdot \begin{bmatrix}x-a\\y-b\end{bmatrix} = 0$
3) Gradient Descent: $\vec a_0$ & t given
   $\vec a_1 = \vec a_0 + t\nabla f(\vec a_0)$
   $\vec a_2 = \vec a_1 + t\nabla f(\vec a_1)$

Functions & Transformations

1) Linear: $f(\vec x) = A\vec x$, $f(\vec 0) = \vec 0$
2) Affine: $f(\vec x) = A\vec x + \vec b$, $f(\vec 0) = \vec b$
3) $Df = \begin{bmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots \\ \partial f_2/\partial x_1 & \ddots & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} - \nabla f_1 - \\ - \nabla f_2 - \\ \vdots \end{bmatrix}$
4) $A_\theta = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$ CCW rotation in $\mathbb{R}^2$
4) Diagonal Matrices
   $AD \to$ scales col c_i by d_{ii}
   $DA \to$ scales row r_i by d_{ii}

Markov Matrices

1) Draw chain



2) Look arrows going in

$P_1 \begin{cases} A_1 = \frac12 A_0 + \frac14 B_0 + 0C_0 \\ B_1 = \frac12 A_0 + \frac12 B_0 + \frac13 C_0 \\ C_1 = 0A_0 + \frac14 B_0 + \frac23 C_0 \end{cases} \to \overset{M}{\begin{bmatrix}\frac12 & \frac14 & 0 \\ \frac12 & \frac12 & \frac13 \\ 0 & \frac14 & \frac23\end{bmatrix}} \overset{P_0}{\begin{bmatrix}A_0\\B_0\\C_0\end{bmatrix}}$

3) If M has all positive entries, it will stabilize and all columns are same
   $\vec P_k = M^k \vec P_0$

• Markov matrices always has an eigenvalue 1, and all other eigenvals are equal or smaller in magnitude.

Chain Rule

$D(f\circ g)(\vec x) = Df(g(\vec x)) \cdot Dg(\vec x)$

↳ $[(f\circ g)_x \ (f\circ g)_y \ (f\circ g)_z]$

$= [f_x(g(\vec x)) \ f_y(g(\vec x)) \ f_z(g(\vec x))] \begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \\ g_{3x} & g_{3y} & g_{3z}\end{bmatrix}$

$D(f\circ g\circ h)(\vec x) = Df((g\circ h)(\vec x)) \cdot Dg(h(\vec x)) \cdot Dh(\vec x)$

Alternate Form:

$$\frac{\partial f}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_j}$$

Ex: $h(r,\theta) = f(\overset{x}{r\cos\theta}, \overset{y}{r\sin\theta})$

$\frac{\partial h}{\partial r} = \frac{\partial}{\partial r} f(r\cos\theta, r\sin\theta)$
$= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$
$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$

Inverses $(AB)^{-1} = B^{-1}A^{-1}$

Conditions for invertibility

① unique input $\vec x$ for output $\vec b$
② $\det(A) = ad - bc \neq 0$
③ A is square $(n \times n)$
④ $N(A) = \{\vec 0\}$

$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b \\ -c & a\end{bmatrix}$

If A not invertible, no soln or infinitely many solns.

Newton's Method

Goal: Find zeros of $f(\vec x)$, start at $f(\vec a_0) \neq 0$

$\vec a_{k+1} = \vec a_k - (Df(\vec a_k))^{-1} f(\vec a_k)$

Linear Independence

1) $c_1\vec v_1 + c_2\vec v_2 + \dots + c_k\vec v_k = \vec 0$ only when $c_1 = c_2 = \dots = c_k = 0$
2) $N(A) = \{\vec 0\} \Rightarrow$ cols of A lin. ind.
3) $V = \text{span}\{\vec v_1, \dots, \vec v_k\}$, $\dim(V) = k$
4) All $\vec w_i$ (orthog. basis of V) from Gram-Schmidt nonzero
5) Collection does not contain $\vec 0$

Orthogonal

• Gr-Sc: $\vec w_1 = \vec v_1$; $\vec w_2 = \vec v_2 - \text{Proj}_{\vec v_1}(\vec v_2)$

• Orthogonal Complement: $V^\perp$; orthog to V; $\dim V^\perp = n - \dim V$ for $\mathbb{R}^n$

• Orthogonal Matrix $(n\times n)$
   1) $A^TA = I_n$, $A^{-1} = A^T$
   2) Columns are orthonormal $\vec v$

Quadratic Form

$q_A = \vec v^T A \vec v$

$n=2$: $ax^2 + by^2 + 2uxy$

$n=3$: $ax^2 + by^2 + cz^2 + 2uxy + 2vxz + 2wyz$

Matrix Decompositions

$A = LU$: lower & upper

$LU\vec x = \vec b \to L\vec y = \vec b \to U\vec x = \vec y$

$A = QR$: orthog & upper

$\overset{Q}{\begin{bmatrix} \frac{\vec w_1}{\|\vec w_1\|} & \frac{\vec w_2}{\|\vec w_2\|}\end{bmatrix}} \overset{R}{\begin{bmatrix}\|\vec w_1\| & c\|\vec w_1\| \\ 0 & \|\vec w_2\|\end{bmatrix}}$

Q: orthogonal gram-sch

R: write v_i as lin comb of w_i'

Second Derivative Test

1) If $Hf(a)$ pos-def or all eigenvals pos, local min
2) If $Hf(a)$ neg-def or all eigenvals neg, local max
3) If $Hf(a)$ indef or eigenvals pos & neg, saddle

Linear Systems

1) Column Space: span in $\mathbb{R}^m$ of columns
   ↳ set of all possible outputs for $A\vec x$
   ↳ Rank: $\dim(C(A))$, # of dim in output
   ↳ For $f(\vec x) = A\vec x$, image(f) = C(A)
2) Null Space: all solutions to $A\vec x = \vec 0$, $\vec x \in \mathbb{R}^n$ input
   ↳ $N(A) = \{\vec 0\} \Rightarrow$ A is invertible
   ↳ All $\vec v \in N(A)$; $\vec v \neq \vec 0$ are eigenvectors with $\lambda = 0$
3) For $m\times n$ matrix $\dim C(A) + \dim N(A) = n$
4) Solving system $A\vec x = \vec b$
   • if $\vec b \in C(A)$, $\text{Proj}_{C(A)}(\vec b) = \vec b$
   • no solution if $\vec b \notin C(A)$
   • one soln if $\vec b \in C(A)$ and $N(A) = \{\vec 0\}$
   • infinite soln if $\vec b \in C(A)$ and $N(A) \neq \{\vec 0\}$
5) For $m \times n$ A, m equations in n unknowns
   • overdetermined $(m > n)$: often no solution
   • underdetermined $(m < n)$: if ∃ soln, infinitely many

Eigenvectors ($A\vec v = \lambda\vec v$, $A^n\vec v = \lambda^n\vec v$)

• 0 is eigenval of A when A is not invertible
• λ is eigenval when $(A - \lambda I_n)$ not invertible
• for diag/lower/upper, λ are diagonal entries
• to find λ, solve $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$
• to find $\vec v$, solve $(A - \lambda I_n)\vec v = \vec 0$

Spectral Theorem, Quadratic Form, Definiteness

• An $n\times n$ symmetric matrix has n real eigenvalues and eigenvectors form an orthogonal basis

• Diagonalization: $q_A(\vec v) = \sum_{i=1}^{n} \lambda_i (w_i \cdot w_i) t_i^2$

$q_A(\vec v) = \lambda_1 t_1^2 + \lambda_2 t_2^2$, t_1 along $\vec w_1$, t_2 along $\vec w_2$

• Definiteness: pos $q_A(\vec v) > 0\ \forall \vec v \neq \vec 0$, pos-semi ≥ 0
neg $q_A(\vec v) < 0\ \forall \vec v \neq \vec 0$, neg-semi ≤ 0, indefinite if none,
pos all $\lambda > 0$, neg all $\lambda < 0$, pos-semi $\lambda \geq 0$ and $\lambda = 0$,
neg-semi $\lambda \leq 0$ and $\lambda = 0$, indefinite ∃ pos & neg λs

• $A = WDW^T$, columns of W are $\frac{\vec w_i}{\|\vec w_i\|}$, D_{ii} is λ_i

• $A^m = WD^mW^T$, $A^m \approx \left(\frac{\lambda_i^m}{\vec w_i \cdot \vec w_i}\right)\vec w_i \vec w_i^T$ (λ_i is largest λ)

Hessian, Quadratic Approx, Local Extrema

$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix}$ $f(\vec a + \vec h) \approx f(\vec a) + \nabla f(\vec a)\cdot \vec h + \frac12 \vec h^T(Hf(\vec a))\vec h$
$\approx f(\vec a) + \nabla f(\vec a) \cdot \vec h + \frac12 q_{Hf(\vec a)}(\vec h)$

If there is a critical pt at $\vec a$, find $Hf(\vec a)$ and $q_{Hf(\vec a)}(t_1, t_2) = \lambda_1 t_1^2 + \lambda_2 t_2^2$

1) λ_1 & λ_2 same sign → ellipse, longer along eigenvalue w/ smaller absolute value
2) λ_1 & λ_2 opp sign → hyperbola, asymptotes closer along λ w/ smaller abs value
   ↳ $c > 0$ opens to pos λ, $c < 0$ opens to neg λ
4) Inconclusive if semi-def or $\lambda = 0$

For 2x2 matrices

1) If $\det(A) = \lambda_1\lambda_2 < 0$, saddle
2) $\text{tr}(A) = \lambda_1 + \lambda_2$ tells pos (min) or neg (max)

Lagrange Multipliers

$g(x,y)=c$

Max/min $f(\vec{x}) = x^2 y$ subject to $2x^2+y^2=3$

1) check $\nabla g = 0 \to \begin{bmatrix}4x\\2y\end{bmatrix} = \vec{0} \to x=0, y=0$ but $g(0,0) \neq 3$

2) check $\nabla f = \lambda \nabla g \to \begin{bmatrix}2xy\\x^2\end{bmatrix} = \lambda \begin{bmatrix}4x\\2y\end{bmatrix}$

3) solve the following system

$\nabla f = \lambda \nabla g$: $2xy = \lambda 4x \to \lambda = xy/2x$

$x^2 = \lambda 2y \to \lambda = x^2/2y$

$g(x)$: $2x^2+y^2=3$

4) check when denom equal 0

make sure to check with all equations

Case 1: $2x=0 \to x=0 \to g(0,y)=3$

$\to 2(0)^2+y^2=3 \to y=\pm\sqrt{3}$

possible points at $(0,\sqrt{3})$ and $(0,-\sqrt{3})$

Case 2: $2y=0 \to y=0 \to x^2=\lambda 2(0)$

$\to x=0$, but $g(0,0)\neq 3$

5) check when denom nonzero

$\lambda = xy/2x = x^2/2y \to \frac{y}{2} \times \frac{x^2}{2y} \to 2y^2=2x^2$

$\to$ plug into $g \to 2x^2+x^2=3 \to x^2=1$

$\to x=\pm 1 \to y=\pm 1$

possible points at $(1,1), (1,-1), (-1,1), (-1,-1)$

6) Plug all possible points into f

Contour Plot

$q_A(x,y) = -x^2+12xy-y^2 \to A=\begin{bmatrix}-1&6\\6&-1\end{bmatrix} \to \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$

$\lambda^2+2\lambda-35 = (\lambda-5)(\lambda+7)=0$

$\boxed{\lambda_1 = 5, \quad \lambda_2 = -7}$

5-eigenspace: Find $N(A-5I_2)$

$(A-5I_2)\vec{x}=\vec{0} \to \begin{bmatrix}-6&6\\6&-6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \to \begin{matrix}-6x+6y=0\\6x-6y=0\end{matrix} \to y=x \to \vec{x}=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\x\end{bmatrix}=x\begin{bmatrix}1\\1\end{bmatrix}$

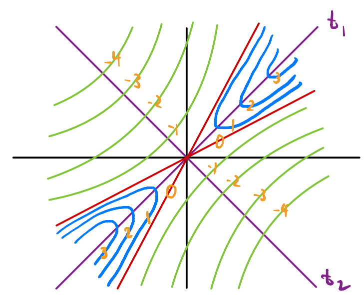
$N(A-5I_2)=\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, $\boxed{\vec{w}_1=\begin{bmatrix}1\\1\end{bmatrix}}$

-7-eigenspace: Find $N(A+7I_2)$

$(A+7I_2)\vec{x}=\vec{0} \to \begin{bmatrix}6&6\\6&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \to 6x+6y=0 \to y=-x \to \vec{x}=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\-x\end{bmatrix}=x\begin{bmatrix}1\\-1\end{bmatrix}$

$N(A+7I_2)=\text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$, $\boxed{\vec{w}_2=\begin{bmatrix}1\\-1\end{bmatrix}}$

Diagonalization formula, $q_A(t_1,t_2)=5t_1^2-7t_2^2$



Level sets:

$q_A = 5t_1^2-7t_2^2 = 0$ line ~

$q_A = 5t_1^2-7t_2^2 = -1 < 0$ opens to t_2 ~

$q_A = 5t_1^2-7t_2^2 = 1 > 0$ opens to t_1 ~

Orthogonal Projection Theorem

Given basis $\{\vec{v}_1, \dots, \vec{v}_n\}$,

① find orthog basis using Gram-Schmidt

② $\text{Proj}_V(\vec{x}) = \text{Proj}_{\vec{w}_1}(\vec{x}) + \text{Proj}_{\vec{w}_2}(\vec{x}) + \dots$

$= A\vec{x} = \begin{bmatrix}| & |\\ \text{Proj}_V(\vec{e}_1) & \text{Proj}_V(\vec{e}_2)\\ | & |\end{bmatrix}$

$\text{Proj}_{\vec{v}}\vec{x} = \frac{\vec{x}\cdot\vec{v}}{\underbrace{\vec{v}\cdot\vec{v}}_{\|\vec{v}\|^2}}\vec{v} = \frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T\vec{x} = \frac{1}{6}\begin{bmatrix}1&1&2\\1&1&2\\2&2&4\end{bmatrix}\vec{x}$

2. (10 points) Consider the function $f(x,y)=x^{1/3}y^{2/3}$ on the first quadrant: $x,y>0$. On the portion

of the ellipse $x^2+2y^2=15$ *in the first quadrant*, f attains a maximal value at exactly one point $P=(a,b)$, and that value has the form $\sqrt{m}$ for a whole number m.

Find the point P, as well as the value $f(P)$ written as $\sqrt{m}$ for a whole number m (not as some messier expression).

Letting $g(x,y)=x^2+2y^2$, the point $P=(a,b)$ of interest is a local maximum for f on the curve $g=15$, so either $(\nabla g)(P)=\mathbf{0}$ or there is a scalar λ for which $(\nabla f)(P)=\lambda(\nabla g)(P)$. We first compute the gradients:

$$(\nabla g)(x,y)=\begin{bmatrix}2x\\4y\end{bmatrix}, \quad (\nabla f)(x,y)=\begin{bmatrix}(1/3)(y/x)^{2/3}\\(2/3)(x/y)^{1/3}\end{bmatrix}.$$

Since we are working in the first quadrant, which is to say $x,y>0$, the expression for ∇g shows that it is non-vanishing on this region. Hence, there must be a scalar λ for which $(\nabla f)(a,b)=\lambda(\nabla g)(a,b)$. In terms of the explicit vector expressions, this says:

$$\begin{bmatrix}(1/3)(b/a)^{2/3}\\(2/3)(a/b)^{1/3}\end{bmatrix}=\lambda\begin{bmatrix}2a\\4b\end{bmatrix}=\begin{bmatrix}2\lambda a\\4\lambda b\end{bmatrix}.$$

Equating corresponding vector entries, this says:

$$\frac{1}{3}\left(\frac{b}{a}\right)^{2/3}=2\lambda a, \quad \frac{2}{3}\left(\frac{a}{b}\right)^{1/3}=4\lambda b.$$

Since $a,b>0$, there are no division-by-zero issues: we can get two expressions for λ by division in the usual way, obtaining

$$\frac{b^{2/3}}{6a^{5/3}}=\lambda=\frac{2a^{1/3}}{12b^{4/3}}=\frac{a^{1/3}}{6b^{4/3}}.$$

Cross-multiplying yields $6b^2=6a^2$, so $b^2=a^2$. Since $a,b>0$, this implies $b=a$ (no issue with negative square roots!).

Now we go back to the constraint $15=g(a,b)=g(a,a)=3a^2$, so $a=\sqrt{5}$ and hence $b=a=\sqrt{5}$. In other words, $P=(\sqrt{5},\sqrt{5})$. The value $f(P)$ is $\sqrt{5}^{1/3}\sqrt{5}^{2/3}=\sqrt{5}^{1/3+2/3}=\sqrt{5}$.

Matrix Forms

$f(\vec{x}) = A\vec{x} = \begin{bmatrix}| & |\\ f(\vec{e}_1) & f(\vec{e}_2)\\ | & |\end{bmatrix}\vec{x} = \begin{bmatrix}| & |\\ A\vec{e}_1 & A\vec{e}_2\\ | & |\end{bmatrix}\vec{x}$

$(A^T)^T = A$

$(A+B)^T = A^T+B^T$

$(AB)^T = B^TA^T$

$(A^{-1})^T = (A^T)^{-1}$

$\vec{v}\cdot\vec{w} = \vec{v}^T\vec{w}$

$(A\vec{x})\cdot y = \vec{x}\cdot(A^T\vec{y})$

Symmetric if $A^T=A$

For any matrix, M^TM and MM^T are symmetric

$f:\mathbb{R}^n \to \mathbb{R}^m$

$m \times n$ matrix

i love you ♥
your so smart
you got this!!!