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## 張筠

# 數字筆記本，普通

## NOTE

**A5**。深灰色。**65**張

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9/26/22

- p-set, pre-class questionaire (M/F, 8 AM) (only have to do 80%) ← submit to gradescope
- p-set due 9 AM wed
- have prereading wed
- OH, M: 2:30 - 3:30 (right after class) 381G
  T: virtual office hours, 11:00-1:00
  zwickham@stanford.edu

# LECTURE 1

# Lecture 1 - Vectors, Vector Addition, and Scalar Multiplication

September 26, 2022

**Goals:** Compute linear combinations and lengths of vectors.

**Definition:** For a whole number $n$, an $n$**-vector** is a list of $n$ real numbers. We denote by $\mathbb{R}^n$ the collection of all possible $n$-vectors.

**Notes:**

- In the text, boldface lowercase letters such as $\mathbf{x}, \mathbf{v}$ will refer to vectors and non-bold will refer to numbers (e.g. $a, b, x$).
- When handwriting, we use the notation $\vec{x}$ to avoid confusion between vectors and numbers. We will usually refer to numbers as **scalars**.
- Instead of saying " an $n$-vector $\mathbf{v}$," we may also say "a vector $\mathbf{v} \in \mathbb{R}^n$".

lazy $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbb{R}^3$ $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ // $\mathbb{R}^2$ $\begin{bmatrix} 1 \\ 0 \\ 3 \\ \pi \end{bmatrix} \in \mathbb{R}^4$

**Definition:** ★ $\vec{V}$ when writing

- The **sum** of two vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ in $\mathbb{R}^n$ is $\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}$

  addition n times add first comp together add 2nd comp, etc

  ↳ have to be same $\mathbb{R}^n$ (same # of comp)

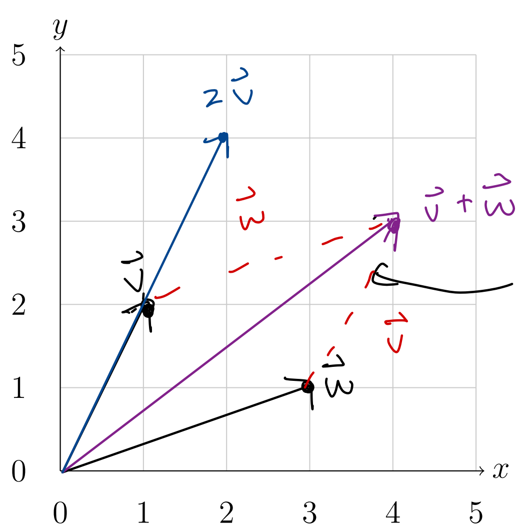
  scalar is #

- **scalar multiplication** between a scalar $c \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ is $c\mathbf{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$

  take each comp in vector & multiply by c

**Example 1:** Plot $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{v} + \mathbf{w}$, and $2\mathbf{v}$ on the same axes.

★ like connecting origin to this pt $\begin{bmatrix} x \\ y \end{bmatrix}$ direction




$\vec{v} + \vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

parallelogram law ★

$2\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

**Definition:** A **linear combination** $k$ vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k$ in $\mathbb{R}^n$ is a vector of the form $a_1\vec{\mathbf{v}}_1 + \cdots + a_k\vec{\mathbf{v}}_k$ for scalars $a_1, \ldots, a_k$.

$2\vec{v} + 8\vec{w} - \pi/2\,\vec{u}$ ← linear combo

specific LC

A linear combination is **convex** if $a_1 + \cdots + a_k = 1$. Note for two vectors, we think about a convex linear combination as $(1-t)\mathbf{v} + t\mathbf{w}$ where $0 \le t \le 1$.

**Example 2:** Compute the following linear combinations:

- $5\begin{bmatrix}-1\\4\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}-5\\20\end{bmatrix} + \begin{bmatrix}-2\\0\end{bmatrix} = \begin{bmatrix}-7\\20\end{bmatrix}$

- $-2\begin{bmatrix}2\\0\\4\end{bmatrix} + 3\begin{bmatrix}-1\\\pi\\\sqrt{2}\end{bmatrix}$

CONVEX

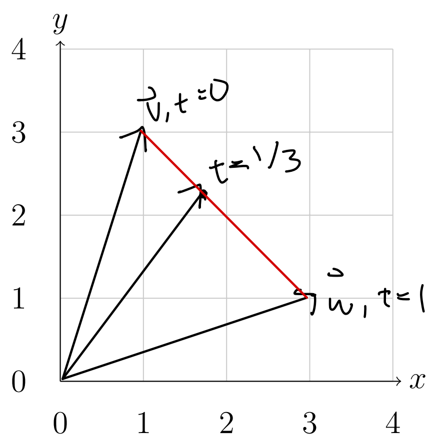
$0 \le t \le 1$

**Example 3:** Let $\mathbf{v} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}3\\1\end{bmatrix}$. Compute $(1-t)\vec{\mathbf{v}} + t\vec{\mathbf{w}}$ for $t = 0, \frac{1}{2}, \frac{1}{3}, 1$ and plot them on the same axes.

LC: $a_1\vec{v} + a_2\vec{w}$

$a_1 + a_2 = 1$

$a_1 = 1 - a_2$




$t=0$, $\vec{v} + 0\vec{w} = \begin{bmatrix}1\\3\end{bmatrix}$

$t=1$, $0\vec{v} + \vec{w} = \begin{bmatrix}3\\1\end{bmatrix}$

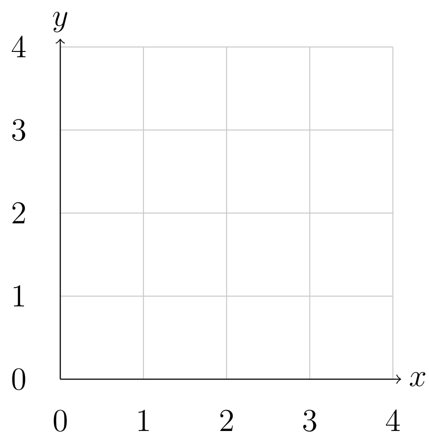
$t=1/3$, $\frac{2}{3}\vec{v} + \frac{1}{3}\vec{w} = \begin{bmatrix}2/3 + 1\\6/3 + 1/3\end{bmatrix} = \begin{bmatrix}5/3\\7/3\end{bmatrix}$

$t=1/2$, $\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} = \begin{bmatrix}1/2 + 3/2\\3/2 + 1/2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$

☆ convex combinations of $\vec{v}$ & $\vec{w}$ are on the line segment

☆ all convex comb form △ = "convex hull"

**Example 4:** If $\mathbf{v}$ and $\mathbf{w}$ are as in Example 3 and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then every convex linear combination is a point inside the triangle with vertices at $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.




**Example 5:** Since there is no extra credit, the grading scheme in the syllabus is a convex linear combination of your graded work.

The following theorem just tells us that the operations we've learned today are similar to the "usual" properties we have for regular numbers.

<u>**Theorem 1.5.2**</u>: For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, we have

- $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
- $(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$
- $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
- $(ab)\mathbf{v} = a(b\mathbf{v})$
- $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + b\mathbf{w}$

☆ you can add vectors in any order

<u>**Definition:**</u> The **length** or **magnitude** of a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, denoted $||\vec{\mathbf{v}}||$, is

$$\underbrace{||\mathbf{v}|| = \sqrt{v_1^2 + \cdots + v_n^2}.}\quad \geq 0$$

in $\mathbb{R}_2$ is pythag thm

**Note:** $\underbrace{||-\mathbf{v}|| = ||\mathbf{v}||}_{\text{same length}}$ and $\underbrace{||c\mathbf{v}|| = |c|\ ||\mathbf{v}||}_{\text{take length of vector \& multiply by abs val of } c}$ for any scalar $c$.

☆ use $||\vec{v}||$ notation

**Example 6:** For $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, compute $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, $\|-2\mathbf{v}\|$, and $\|\mathbf{v}-\mathbf{w}\|$.
(Note: $\mathbf{v}-\mathbf{w}$ is called the **displacement vector**, and $\|\mathbf{v}-\mathbf{w}\|$ is the **distance** between $v$ and $w$)

$$\|\vec{v}\| = \sqrt{1+4+0} = \sqrt{5}$$

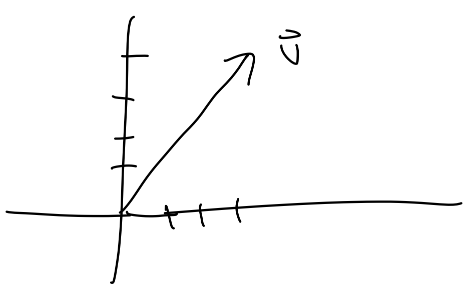
$\|-2\vec{v}\| = 2\sqrt{5}$ ← pos b/c length is pos #

The **zero vector** in $\mathbb{R}^n$ is $\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, and a **unit vector** is a vector with length 1.

**Example 7:** Find all unit vectors that are scalar multiples of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

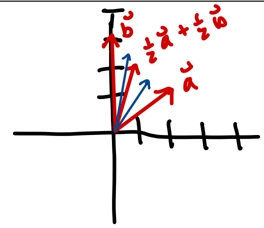
$\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\|\vec{v}\| = 5$




$$\boxed{\begin{array}{c} \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ -\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{array}}$$

☆ for unit vector, divide $\pm \|\vec{v}\|$




## Problem 1: Visualizing vectors and convex combinations in $\mathbb{R}^2$

Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

x y x y x y

(a) Compute $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$. Draw $\mathbf{a}$, $\mathbf{b}$ and $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ in a coordinate plane, and describe geometrically where the sum lies relative to $\mathbf{a}$ and $\mathbf{b}$. Do you expect such a geometric relationship is true for any 2-vectors $\mathbf{a}, \mathbf{b}$? How about for 3-vectors?

$\begin{bmatrix} 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

a) it's on the line segment of $\vec a$ + $\vec b$, mdpt of line seg

b) yes, becaun $\frac{1}{2} + \frac{1}{2} = 1$

c) will be on the plane btwn $\vec a$ + $\vec b$

(b) Compute $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$ and $\frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$, and plot these. Do you notice a pattern that should hold for any 2-vectors $\mathbf{a}$ and $\mathbf{b}$?

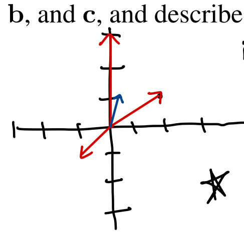
$\frac{1}{3}\vec a + \frac{1}{3}\vec b = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 7/3 \end{bmatrix}$

$\frac{3}{4}\vec a + \frac{1}{4}\vec b = \begin{bmatrix} 3/2 \\ 3/4 \end{bmatrix} + \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$

they're on the line segment of $\vec a$ + $\vec b$

(c) Compute $\frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$, plot this point and draw segments joining it to each of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, and describe geometrically where this lies relative to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

$\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

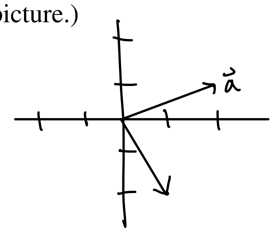


it's inside the $\Delta$ that $\vec a + \vec b + \vec c$ form convex null

★ WEIGHTED sum of vectors is inside triangle

(d) Find a nonzero vector that is perpendicular to $\mathbf{a}$. (Hint: draw a picture.)

$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\perp$ vectors dot product of have to be 0




## Problem 2: Linear combinations

(a) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$5\vec a + 4\vec b$

(b) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (This amounts to solving 2 equations in 2 unknowns.)

$2\vec v + \vec w$

$2x + 1y = 5$
$x + 2y = 4$
$2x + 4y = 8$

$3y = 3$
$y = 1$
$x = 2$

(c) Write a general 2-vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The coefficients of the linear combination will depend on $x$ and $y$. Make sure that for $x = 5$ and $y = 4$ it agrees with your answer to (b)!

$\begin{bmatrix} 2a + b \\ a + 2b \end{bmatrix}$ 2

$\boxed{\begin{matrix} 2a + b = x \\ a + 2b = y \end{matrix}}$

(d) (**Extra**) Draw a picture for each of (a) and (b). Then draw all points $n\mathbf{v} + m\mathbf{w}$ for integers $n$ and $m$ with $-2 \le n, m \le 3$, and draw lines through these parallel to each of $\mathbf{v}$ and $\mathbf{w}$, which should yield a tiling of the plane with copies of the parallelogram $P$ whose corners are at the tips of the vectors $\mathbf{0}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$.

Interpret geometrically (without any calculations) the meaning of the answer to (b) in terms of these parallelograms, and do the same for the result in (c) that such a linear combination always exists. (Hint: for (c), mark $(6, 5)$ and $(3, 4)$ and compare where they lie among the parallelograms with the general formula in (c) for these two cases.)

## Problem 3: Length and distance

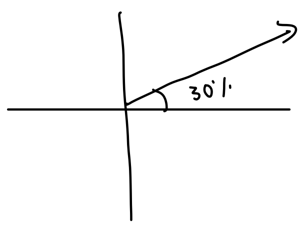
(a) Compute the distance between $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$. (The answer is an integer.)

$$\sqrt{(7-(-5))^2 + (-2-3)^2} = \sqrt{144+25} = \boxed{13}$$

(b) Compute the distance between $\begin{bmatrix} 4 \\ -1 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -6 \\ 1 \\ -3 \end{bmatrix}$. (The answer is an integer.)

$$\sqrt{3^2 + 5^2 + 1^2 + 1^2} = \sqrt{9+25+1+1} = \boxed{6}$$

(c) If a nonzero vector $\mathbf{v}$ lies at an angle 30° counterclockwise from the positive $x$-axis, what is the unit vector in the same direction as $\mathbf{v}$? (Draw a picture to get an idea.) What if 30° is replaced with a general angle $\theta$?




$$\sqrt{(a\sqrt{3})^2 + a^2} = 1$$

$3a^2 + a^2 = 1$

$4a^2 = 1 \quad a = \frac{1}{2}$

$$\begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

## Problem 4: Vector operations with data

Suppose there are three students in Math 51 with the following components for their course grades:
Student 1: 81/100 on homework, 83/100 on midterm A, 70/100 on midterm B, 75/100 on the final.
Student 2: 73/100 on homework, 75/100 on midterm A, 74/100 on midterm B, 88/100 on the final.
Student 3: 90/100 on homework, 95/100 on midterm A, 88/100 on midterm B, 92/100 on the final.

(a) Write down vectors $\mathbf{v}_{\mathrm{HW}}, \mathbf{v}_{\mathrm{A}}, \mathbf{v}_{\mathrm{B}}, \mathbf{v}_{\mathrm{Final}}$ (all in $\mathbf{R}^3$) representing respectively the grades as percentages on homework, midterm A, midterm B, and the final exam (e.g., for a score of $83/100$, the vector entry should be 83 rather than .83).

$$\vec{v}_{HW} = \begin{bmatrix} 81 \\ 73 \\ 90 \end{bmatrix} \quad \vec{v}_A = \begin{bmatrix} 83 \\ 75 \\ 95 \end{bmatrix} \quad \vec{v}_B = \begin{bmatrix} 70 \\ 74 \\ 88 \end{bmatrix} \quad \vec{v}_{final} = \begin{bmatrix} 75 \\ 88 \\ 92 \end{bmatrix}$$

(b) Give a general formula as a linear combination of those four vectors in $\mathbf{R}^3$ for a 3-vector $\mathbf{v}_{\mathrm{CG}}$ whose entries are the course grades of the three students in order from student 1 to student 3, assuming the breakdown of the total grade for the course is 16% homework, 36% final, 24% each midterm, and then compute it (you may use a calculator).

$$\vec{V}_{CG} = 0.16\,\vec{V}_{HW} + 0.24\,\vec{V}_A + 0.24\,\vec{V}_B + 0.36\,\vec{V}_{final}$$

$$\begin{bmatrix} 76.68 \\ 79.12 \\ 91.44 \end{bmatrix}$$

# CH 1 NOTES + QUESTIONS

1.

FIGURE 1.3.3: Blue segment of vectors $(1-t)\mathbf{v} + t\mathbf{w} = \mathbf{v} + t(\mathbf{w} - \mathbf{v})$ for $0 \le t \le 1$

For any $n$-vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, a *convex combination* of them means a linear combination

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$$

for which all $t_j \ge 0$ and the sum of the coefficients is equal to 1; that is, $t_1 + \cdots + t_k = 1$. When the $k$ coefficients are all equal, which is to say every $t_j$ is equal to $\frac{1}{k}$, this is the *average* (sometimes called the *centroid*) of the $k$ vectors. Working in $\mathbf{R}^2$, a convex combination such as

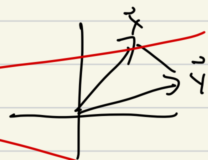
$$\frac{2}{3}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2 + \frac{1}{12}\mathbf{v}_3$$

is a point inside the polygon with vertices given by the $\mathbf{v}_j$'s, with its distance to each $\mathbf{v}_j$ weighted by the corresponding coefficient. If all coefficients are equal, it is the "center of mass" of the polygon. Two convex combinations are shown in Figure 1.3.4 below (see $\mathbf{u}$ and $\mathbf{w}$ there).

i don't get what this means

- point in $R_n$ means same as n-vector + vector
- net displacement
  - $(u_1 - u_2) + (u_2 - u_3) = u_1 - u_3$
- distance btwn vectors $= \|\vec{u} - \vec{v}\| = \sqrt{(\vec{u}_1 - \vec{v}_1)^2 + (\vec{u}_2 - \vec{v}_2)^2}$
  or length of difference vector
- TRIANGLE INEQUALITY (from geo)
  - distance of sum of 2 sides is larger than 3rd side

$$\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$$




? I don't get it, ~~shouldnt~~ they always be =

- $\|c\vec{v}\| = c\|\vec{v}\|$

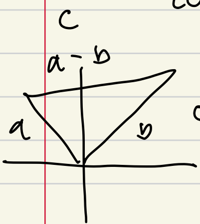
# CHAPTER 2 NOTES & QUESTIONS
## 2.1 - 2.1.7 , 2.2 - 2.2.2

## 2.1 ANGLES

vector form of law of cosines
$a_1b_1 + a_2b_2$ = dot product

· law of cosines

$$\cos\theta = \frac{a_1b_1 + a_2b_2}{\|a\|\,\|b\|} \quad \text{for } \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$




$$\cos\theta = \frac{A^2 + B^2 - C^2}{2AB} \quad \text{or} \quad C^2 = A^2 + B^2 - 2AB\cos\theta$$

· perpendicular

dot product

$$\cos 90° = 0 \quad \text{so} \quad \frac{u \cdot v}{\|u\|\,\|v\|} = 0 \rightarrow u \cdot v = 0$$

· 3 vectors

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\|a\|\,\|b\|}$$

· DOT PRODUCT is scalar, not vector

2.1.5 example

$$\cos\theta = \frac{3 + 0 - 8}{\sqrt{9+16} \cdot \sqrt{1+9+4}} = \frac{-5}{5\sqrt{14}} = \frac{-1}{\sqrt{14}} \rightarrow \theta = 105.5°$$

☆ dot product is scalar

2.1.6 consider n-vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

i) $x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n$
$= \sum_{i=1}^{n} x_i y_i$

ii) $\theta$ between 2 nonzero vectors $x, y$ is $\cos\theta = \dfrac{x \cdot y}{\|x\|\,\|y\|}$
x & y must be nonzero n-vectors for common n

iii) when $x \cdot y = 0$, $x$ & $y$ are $\perp$

Theorem 2.2.1 (properties of dot products)

for any n-vectors $v, w, w_1, + w_2$

i) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

ii) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

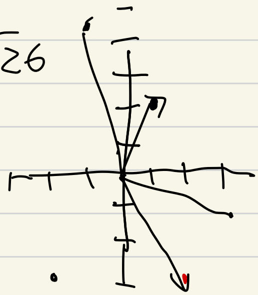
iii) $\vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$ for scalar $c$ + $\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v}\vec{w}_1 + \vec{v}\vec{w}_2$

iii') $\vec{v} \cdot (c_1\vec{w}_1 + c_2\vec{w}_2) = c_1(\vec{v} \cdot \vec{w}_1) + c_2(\vec{v} \cdot \vec{w}_2)$

Pre-reading questionaire

1) $\vec{v} - \vec{w} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ $\|\vec{v} - \vec{w}\| = \sqrt{1+25} = \sqrt{26}$

$\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\|\vec{v} + \vec{w}\| = \sqrt{10}$



# Lecture 2 - Vector Geometry in $\mathbb{R}^n$ and Correlation Coefficients

September 28, 2022

**Goals:** Be able to compute the dot product between two vectors, and use it to calculate angles between vectors and correlation coefficients.

**Proposition 2.1.1:** The angle $0° \le \theta \le 180°$ (or $0 \le \theta \le \pi$ radians) between two non-zero vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ satisfies $\mathbb{R}^2$ dot product

$$\cos\theta = \frac{\overbrace{a_1b_1 + a_2b_2}}{||\mathbf{a}||\,||\mathbf{b}||}.$$

magnitude

**Note:** This is just the law of cosines in disguise. See the book for further explanation if interested.

**Example 1:** Find a non-zero vector perpendicular to $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\cos(90°) = 0 = \frac{-3a + 4b}{5\,||\vec{v}||}$$

$\sqrt{9+16} = 5$

$-3a+4b=0$

$a = \frac{4}{3}b$ $\quad \vec{v} = \begin{bmatrix} 4/3\, b \\ b \end{bmatrix} = b\begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$

$b=1$ $\begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$

$b=3$ $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

let b be anything you want

**Definition:** Consider $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$. → $\mathbb{R}^1$ = usual multiplication

- The **dot product** of $\mathbf{x}$ and $\mathbf{y}$ is the scalar

$$\boxed{\mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i.}$$

- The **angle** between $\mathbf{x}$ and $\mathbf{y}$ (both non-zero) ~~is~~ satisfies

$$\boxed{\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}}$$

- When $\mathbf{x}\cdot\mathbf{y} = 0$ (i.e. $\theta = 90°$ or $\pi/2$ radians), we say they are **perpendicular** or **orthogonal**.

**Example 2:** Find the angle between $\overset{\vec{x}}{\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}}$ and $\overset{\vec{y}}{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}}$.

$\vec{x} \cdot \vec{y} = 8 + 0 - 1 = 7$

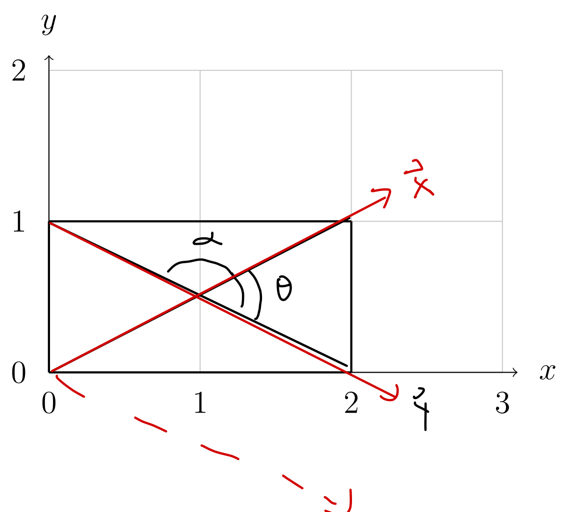
$\|\vec{x}\| = \sqrt{16+9+1} = \sqrt{26}$

$\|\vec{y}\| = \sqrt{4+1} = \sqrt{5}$

$$\cos\theta = \frac{8 - 0 - 1}{\sqrt{16+9+1} \cdot \sqrt{4+0+1}} = \frac{7}{\sqrt{26} \cdot \sqrt{5}} = \frac{7}{\sqrt{130}}$$

$\boxed{\theta \approx 52.13^\circ}$ (0.91 rads)

**Example 3:** Find the acute angle between the main diagonals of a rectangle with vertices $(0,0)$, $(0,1)$, $(2,0)$, and $(2,1)$.




$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{y}' = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\vec{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

☆ ending pt − starting pt

$$\cos\theta = \frac{4 - 1}{\sqrt{5} \cdot \sqrt{5}} = \frac{3}{5}$$

$\theta \approx 53.13^\circ$

what if went ← this way → calc $\alpha$

$$\cos\alpha = \frac{\vec{x} \cdot \vec{y}'}{5} = \frac{-3}{5} \qquad \alpha = 126.87^\circ$$

Fact: $\cos(\theta) > 0 \Rightarrow$ acute

$\cos(\theta) < 0 \Rightarrow \theta$ obtuse

**Example 4:** Find all unit vectors that form a $45^\circ$ angle with $\begin{bmatrix} \cancel{-1} \\ 1 \end{bmatrix}$. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \|\vec{v}\| = 1$

$$\cos(45^\circ) = \frac{\sqrt{2}}{2} = \frac{\vec{v} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\|\vec{v}\| \cdot \sqrt{2}} = \frac{a+b}{\sqrt{2}}$$

$2 = 2(a+b) \quad a + b = 1, \quad b = 1 - a \rightarrow \sqrt{a^2 + (1-a)^2} = 1$

$a^2 + 1 - 2a + a^2 = 1 \qquad 2a^2 - 2a = 0 \qquad 2a(a-1) = 0 \qquad a = 0, 1$

$b = 1, 0$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

**Theorem 2.2.1:** (Properties of the dot product) For any vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{w}_1$, and $\mathbf{w}_2$ in $\mathbb{R}^n$, and scalars $c, c_1, c_2 \in \mathbb{R}$, we have

(i) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

(ii) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$ $\quad \vec{v} = \begin{bmatrix} a \\ y \end{bmatrix} \quad \vec{v} \cdot \vec{v} = a^2 + y^2 = ||\vec{v}||^2$

(iii) $\mathbf{v} \cdot (c\mathbf{w}) = c(\mathbf{v} \cdot \mathbf{w})$

(iv) $\mathbf{v} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \cdot \mathbf{w}_1 + \mathbf{v} \cdot \mathbf{w}_2$

(v) $\mathbf{v} \cdot (c_1\mathbf{w}_1 + c_2\mathbf{w}_2) = c_1(\mathbf{v} \cdot \mathbf{w}_1) + c_2(\mathbf{v} \cdot \mathbf{w}_2)$

**Correlation Coefficients** (application)

$y = mx + b$

Given data points $(x_1, y_1), \ldots, (x_n, y_n)$, we often would like a "line of best fit" to describe the data since lines are relatively simple functions. With the equation of the line, we can compute predictions with no trouble at all.

Finding the line is called **linear regression**, which we will discuss in lecture 7. The question we will address today is: "Is a line a good choice for the data set, i.e., is there a linear relationship among the points?"

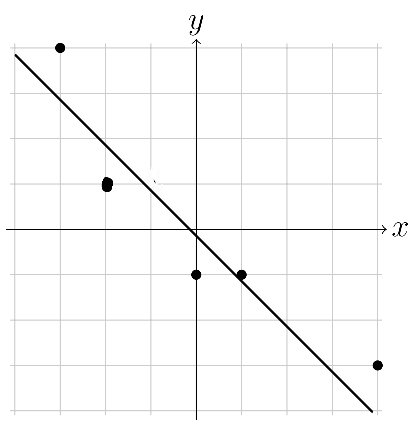
The **correlation coefficient** measures the "strength" of the linear relationship.

We will give an outline on how to proceed via an example.

**Example 5:** Consider 5 data points: $(-3, 4)$, $(-2, 1)$, $(0, -1)$, $(1, -1)$, and $(4, -3)$.

Collect all the $x$-coordinates into a single $n$-vector $\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and all $y$-coordinates into a single vector $\mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

$$\vec{x} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 4 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 4 \\ 1 \\ -1 \\ -1 \\ -3 \end{bmatrix}$$




NOTE: Average of $\vec{X} = \bar{X}$

$$\bar{X} = \frac{1}{5}(-3 - 2 + 0 + 1 + 4) = 0$$

$$\bar{Y} = \frac{1}{5}(0) = 0$$

if not 0, calc avg $\vec{x}, \vec{x} - \begin{bmatrix} \bar{x} \\ \bar{x} \\ \bar{x} \end{bmatrix} = \hat{x}$

ignore

**Assumptions:** We will make the following assumptions for now.

- The data points do **not** lie on a single horizontal or vertical line.
- The averages of **X** and **Y** are 0.

**Definition:** Under the above assumptions, the **correlation coefficient** $r$ is the cosine of the angle between **X** and **Y**, i.e.

$$\boxed{r = \frac{\mathbf{X} \cdot \mathbf{Y}}{\|\mathbf{X}\| \, \|\mathbf{Y}\|}}$$

**Example 5 continued:**

$$r = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \cdot \|\vec{y}\|} = \frac{-27}{\sqrt{30} \cdot \sqrt{28}} \approx -0.9316$$

strength of LR (linear relationship)

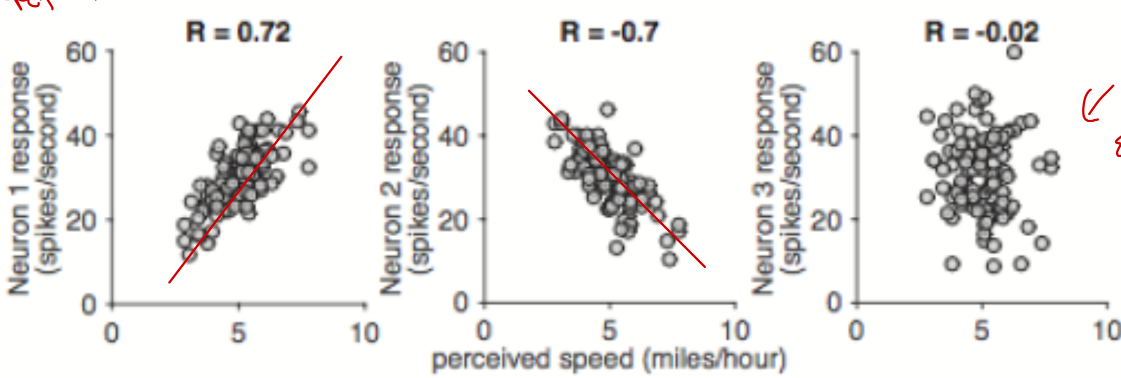
$r \approx 1 \rightarrow$ positive

$r \approx -1 \rightarrow$ negative

**Theorem 2.4.5:** The correlation coefficient $r$ always lies between $-1$ and $1$. When $r$ is close to 1 or -1, the data points are close to a line of positive or negative slope, respectively. When $r$ is close to 0, there *does not* appear to be a strong linear relationship between the data points. Here are some images to illustrate this point:

always

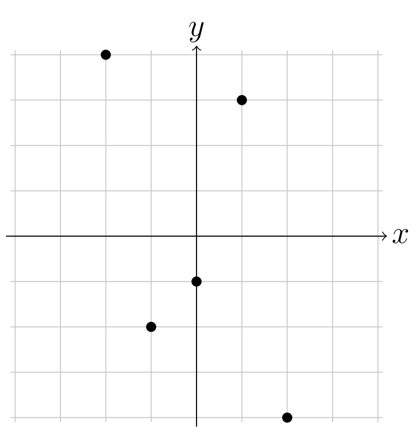
$r \neq$ slope, tells us if there's linear relationship




not as strong linear relationship

**Example 6:** Compute the correlation coefficient of the data points:

(-2,4), (-1,-2), (0,-1), (1,3), and (2,-4)




$$\vec{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 4 \\ -2 \\ -1 \\ 3 \\ -4 \end{bmatrix}$$

$$r = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \cdot \|\vec{y}\|} = \frac{-8 + 2 - 1 + 3 - 8}{\sqrt{4+1+1+4} \cdot \sqrt{16+4+1+9+16}}$$

$$= \frac{-11}{\sqrt{10} \cdot \sqrt{46}} = \frac{-11}{\sqrt{460}} = \frac{-11}{2\sqrt{115}} \approx \boxed{-0.51}$$

## Problem 1: Geometry with dot products

(a) Using that perpendicularity is governed by the dot products being equal to 0, find a nonzero vector in $\mathbf{R}^3$ that is perpendicular to $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Next, find a *unit* vector with the same property. Finally, find a vector perpendicular to $\mathbf{v}$ that is *not* a scalar multiple of the first.

1) $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

2) $2a - b + c = 0$

3) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$2a - b + c = 0$

$a = 1,\ b = 1,\ c = -1$

$\sqrt{a^2 + b^2 + c^2} = 1$

$\vec{x} \cdot \|\vec{x}\| =$

$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$

$\vec{v}$

(b) Find an equation in $x, y, z$ that characterizes when $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. What does this collection of vectors look like?

$\cos 90 = 0$

$0 = \dfrac{2x - y + z}{\|\vec{v}\| \cdot \sqrt{4 + 1 + 1}}$

$\boxed{2x - y + z = 0}$

an infinite collection

(c) (**Extra**) What does the collection of nonzero vectors $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$ making an angle of at most $60°$ against $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ look like? Using the relation of dot products and cosines to describe this region with a pair of conditions of the form $ax^2 + bxy + cy^2 \geq 0$ and $y \leq (3/4)x$ (away from the origin).

## Problem 2: Algebra with dot products

(a) For $\mathbf{a} = \begin{bmatrix}4\\-2\\3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}1\\5\\-2\end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix}6\\-4\\-1\end{bmatrix}$ show that $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c}$.

$$\vec{b} - \vec{c} = \begin{bmatrix}1-6\\5-(-4)\\-2-(-1)\end{bmatrix} = \begin{bmatrix}-5\\9\\-1\end{bmatrix}$$

$$\vec{a}\cdot(\vec{b}-\vec{c}) = -20-18-3 = -41$$

$$\vec{a}\cdot\vec{b} = 4-10-6 = -12$$

$$\vec{a}\cdot\vec{c} = 24+8-3 = 29$$

$$\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = -12-29 = -41$$

$=$

(b) Give an example of 2-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ for which $(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \neq (\mathbf{a}\cdot\mathbf{c})\mathbf{b}$. (Hint: what if $\mathbf{b}$ and $\mathbf{c}$ are not on the same line through the origin?)

$$\vec{a} = \begin{bmatrix}2\\1\\2\end{bmatrix} \quad \vec{b} = \begin{bmatrix}3\\2\\1\end{bmatrix} \quad \vec{c} = \begin{bmatrix}5\\7\\2\end{bmatrix}$$

$$\vec{a}\cdot\vec{b} = 6+2+2 = 10 \quad (\vec{a}\cdot\vec{b})\cdot\vec{c} = \begin{bmatrix}50\\70\\20\end{bmatrix}$$

$$\vec{a}\cdot\vec{c} = 10+7+4 = 21 \quad (\vec{a}\cdot\vec{c})\cdot\vec{b} = \begin{bmatrix}63\\42\\21\end{bmatrix}$$

$\neq$

(c) (**Extra**) Explain in terms of variables why $\mathbf{v}\cdot(\mathbf{w}_1+\mathbf{w}_2) = \mathbf{v}\cdot\mathbf{w}_1 + \mathbf{v}\cdot\mathbf{w}_2$ for any 3-vectors $\mathbf{v} = \begin{bmatrix}x\\y\\z\end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}$, and $\mathbf{w}_2 = \begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}$. If you then replace $\mathbf{v}$ with $\mathbf{v}_1+\mathbf{v}_2$ for 3-vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and apply another instance of the same general identity, why does it follow without any extra work with algebra in vector entries that

$$(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{w}_1+\mathbf{w}_2) = \mathbf{v}_1\cdot\mathbf{w}_1 + \mathbf{v}_1\cdot\mathbf{w}_2 + \mathbf{v}_2\cdot\mathbf{w}_1 + \mathbf{v}_2\cdot\mathbf{w}_2?$$

(This is showing the analogue for vectors of the fact for numbers that the distributive law $r(s+t) = rs+rt$ is what makes the identity $(a+b)(c+d) = ac+ad+bc+bd$ hold, since $(a+b)(c+d) = (a+b)c+(a+b)d = ac+bc+ad+bd$.)

Do your arguments work for $n$-vectors for any $n$?

$$\vec{w}_1 = \begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}$$

(d) For $n$-vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, verify that $\|\mathbf{w}_1+\mathbf{w}_2\|^2 = \|\mathbf{w}_1\|^2 + 2(\mathbf{w}_1\cdot\mathbf{w}_2) + \|\mathbf{w}_2\|^2$ by using the relation $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w}$ and general properties of dot products as stated in (c) (even if you didn't do (c)), *not* by writing out big formulas for lengths and dot products in terms of vector entries.

$$\vec{w}_1 + \vec{w}_2 = \begin{bmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\end{bmatrix}$$

$$\|\vec{w}_1\|^2 = a_1^2+b_1^2+c_1^2$$

$$\vec{w}_1\cdot\vec{w}_2 = a_1\cdot a_2 + b_1b_2 + c_1c_2$$

$$\|\vec{w}_2\|^2 = a_2^2+b_2^2+c_2^2$$

$$\|w_1+w_2\|^2 = (a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2$$

$$\|w_1\|^2 + 2(w_1+w_2) + \|w_2\|^2$$

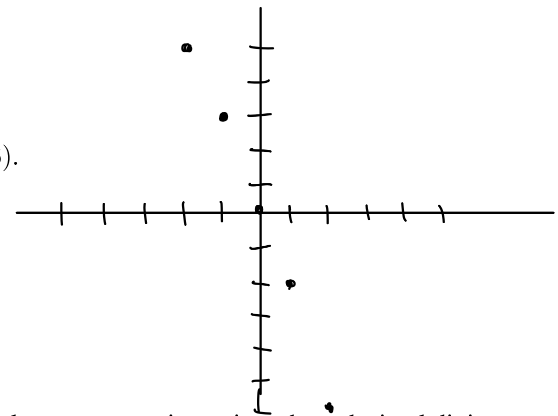
$$= a_1^2+b_1^2+c_1^2+2a_1a_2+2b_1b_2+2c_1c_2 + a_2^2+b_2^2+c_2^2$$

$$= (a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2$$

## Problem 3: A correlation coefficient

Consider the collection of 5 data points: $(-2, 5)$, $(-1, 3)$, $(0, 0)$, $(1, -2)$, $(2, -6)$.

(a) Plot the points to see if they look close to a line.



38
36

(b) Compute the compute correlation coefficient exactly. Plug that into a calculator to approximate it to three decimal digits to see if its nearness to $\pm 1$ fits well with the visual quality of fit of the line to the data plot in (a).

$\vec{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -2 \\ -6 \end{bmatrix}$

$$R = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \cdot \|\vec{y}\|} = \frac{-10 - 3 - 2 - 8}{\sqrt{4+1+1+4} \cdot \sqrt{25+9+4+36}}$$

$$= \frac{-23}{\sqrt{10} \cdot \sqrt{74}} = \boxed{\frac{-23}{\sqrt{740}}} = -0.845$$

## Problem 4: More convex combinations (Extra)

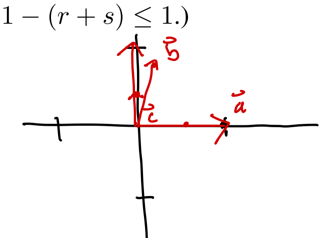
(a) For the 2-vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, describe the set of all possible vectors $r\mathbf{a} + s\mathbf{b} + t\mathbf{c}$ where $r + s + t = 1$ with $0 \leq r, s, t \leq 1$. Which points in your description correspond to the case $t = 0$. How about $s = 0$? Or $r = 0$? (Hint: plot points for a variety of triples $(r, s, t) = (r, s, 1 - r - s)$ with $0 \leq r, s, 1 - (r + s) \leq 1$.)

| | 1) | 2) | 3) |
|---|---|---|---|
| r | 0 | 1/2 | 1/4 |
| s | 1/3 | 0 | 3/4 |
| t | 2/3 | 1/2 | 0 |

1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$

2) $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

3) $\begin{bmatrix} 1/4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$




(b) Try the same using the 3-vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (Hint: first sketch the points you get with $t = 0$, then with $s = 0$, then with $r = 0$, and finally with $r = s = t = 1/3$.)

(c) Can you explain why your description in (a) applies to any three 2-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ not on a common line? Use whatever physical or mathematical idea comes to mind. (Here is one approach: for $0 \leq t < 1$ check the equality $r\mathbf{a} + s\mathbf{b} + t\mathbf{c} = (1 - t)\mathbf{d}_{r,s} + t\mathbf{c}$ with a convex combination on the right where $\mathbf{d}_{r,s}$ is defined to be the convex combination $(r/(r + s))\mathbf{a} + (s/(r + s))\mathbf{b}$; this algebra works because $r + s = 1 - t > 0$. Interpret these convex combinations geometrically.)

(d) Is there a version for a triple of 3-vectors not all on a common line in space? Can you explain why it works?

# CHAPTER 3 NOTES & QUESTIONS

## 3.1 Planes in $\mathbb{R}^3$, line of sight in computer graphics

The collection of points $(x, y, z)$ in $\mathbf{R}^3$ satisfying $3x - 4y + 2z = 10$ is a plane, and more generally the collection of points $(x, y, z)$ in $\mathbf{R}^3$ satisfying an equation of the form

$$ax + by + cz = d,$$

with at least one of the constants $a$, $b$, or $c$ nonzero, is a *plane* in $\mathbf{R}^3$. In particular, although the equation $x = 0$ on $\mathbf{R}^2$ defines a line (the $y$-axis, consisting of points $(0, y)$), the "same" equation $x = 0$ on $\mathbf{R}^3$ defines a *plane*, namely the vertical $yz$-plane consisting of points $(0, y, z)$.

Pre-reading questonnaire

1) $a + 2b = 0 \quad 2a - 3b = 0$

$2a + 4b = 0$

$7b = 0 \quad b = 0, a = 0$

2) $x + 2y + 4 = 6 \quad x + 2y = 2$

3) $x + 2y + z > 6 \quad x + 2y + z < 6$

$1 + 8 - 1 = 8$

$1 + 4 + 2 = 7$

# Lecture 3 - Planes in $\mathbb{R}^3$

September 30, 2022

**Goals:** Learn about the various forms of a plane in $\mathbb{R}^3$ as well as the parametric equation of a line in $\mathbb{R}^3$.

**Definition:** The collection of points $(x, y, z) \in \mathbb{R}^3$ satisfying an equation of the form

$$\boxed{ax + by + cz = d,}$$

$\mathbb{R}^2 \to ax + by = d \to$ line

where at least one of the of the constants $a$, $b$, or $c$ is non-zero is a **plane** in $\mathbb{R}^3$. We call this the **equational form** of a plane.

many pts contain 2 pts

Let's discuss some other ways to describe a plane.

**1.** Three points in $\mathbb{R}^3$ that do not all lie on the same line define a unique plane. Think about these points forming a triangle; the resulting plane will be an extension of this triangle.

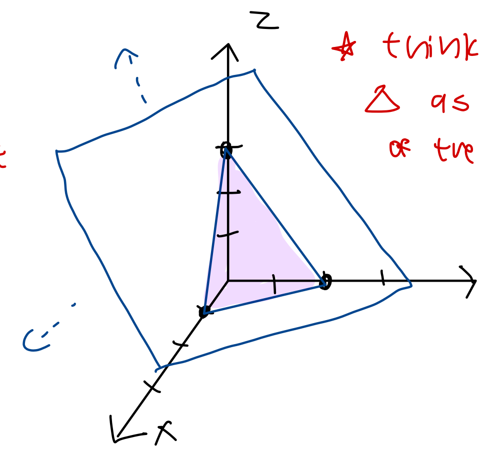
3 pt = unique plane

**Example 1:** Graph the plane $6x + 3y + 2z = 6$. Find 3 pts on plane

$y = z = 0$ : $6x + 0 + 0 = 6 \to (1, 0, 0)$ x-int
$x = 1$

$x = z = 0$ : $0 + 3y + 0 = 6 \to (0, 2, 0)$ y-int
$y = 2$

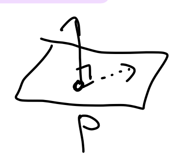
$x = y = 0$ : $0 + 0 + 2z = 6 \to (0, 0, 3)$ z-int
$z = 3$




* think of △ as tilt of the plane

**2.** Another way to determine a plane is with a point on the plane $P$ and a **normal vector** $\mathbf{n}$ that is perpendicular to the plane.

Given $ax + by + cz = d$, it turns out that $\underset{\text{normal vector}}{\mathbf{n}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and $d$ will depend on the point $P$.



**Example 2:** Find the equation of the plane that passes through the point (1, −1, 0) and is perpendicular to $\begin{bmatrix}1\\2\\3\end{bmatrix}$

displacement vector

$(1,-1,0) \longrightarrow (x,y,z)$

1) fancy way

displacement vector: $\begin{bmatrix}x\\y\\z\end{bmatrix} - \begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}x-1\\y+1\\z\end{bmatrix}$

$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}x-1\\y+1\\z\end{bmatrix} = 0 \qquad x-1+2y+2+3z=0$

$\boxed{x+2y+3z=-1}$

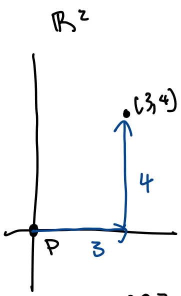
2) easy/normal way

$x+2y+3z=d$

$1-2+0=d \quad d=-1$

$\boxed{x+2y+3z=-1}$

scalar multiples of this also works




$3\begin{bmatrix}1\\0\end{bmatrix}+4\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}$ ← shifts over P/starting pt

**3.** Given a point $P$ and two **displacement vectors e** and **e′** that are not scalar multiples of each other, we can also describe a plane by the **parametric equation**

$e \nearrow e'$ ← live, not helpful

any scalars

can't for fact might not contain origin

$$\boxed{P + t\mathbf{e} + t'\mathbf{e}',}$$

where $t$ and $t'$ are scalars. This is the **parametric form** of a plane. You can think of **e** and **e′** being the directions of the $t$-axis and $t'$-axis respectively.

**Example 3:** Find a parametric form of the plane in Example 1.
(Note: It is *a* parametric form since different choices of $P$ and the displacement vectors will result in a different parametrization of the same plane).

$6x+3y+2z=6$ → find 3 pts: $(1,0,0)$, $(0,2,0)$, $(0,0,3)$

1. find p + 2 displacement vector
2. pick one to be p → $p=(1,0,0)$
3. $\vec{e} = \begin{bmatrix}0\\2\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix}$

   $\vec{e}' = \begin{bmatrix}0\\0\\3\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\3\end{bmatrix}$

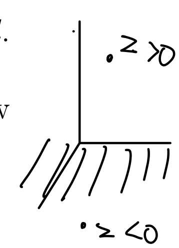
P.F.

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}-1\\2\\0\end{bmatrix} + t'\begin{bmatrix}-1\\0\\3\end{bmatrix}$$

$t, t' \in \mathbb{R}$ ☆

**Question:** Given a plane and two points, how do we determine whether they lie on the same side of the plane?

Idea: The plane $ax + by + cz = d$ splits $\mathbb{R}^3$ into two parts: $ax + by + cz > d$ and $ax + by + cz < d$.




**Example 4:** Do (1, 1, 2) and (−2, 1, 3) lie on the same side of the plane $x + 2y + 3z = 4$? How about (3, 1, −1) and (1, 1, 1)?

1) $(1,1,2)$: $1+2+6=9>4$
   $(-2,1,3)$: $-2+2+9=9>4$
   yes, lie on same side

2) $(3,1,-1)$: $3+2-3=2<4$
   $(1,1,1)$ = $1+2+3=6>4$
   NO, don't lie on same side

The next example will summarize the ways to think about planes.

**Example 5:** Given the points $(0, 1, 1)$, $(0, 2, 3)$, and $(1, 3, 2)$:

(i) Find the parametric form of the plane determined by these points.

(ii) Find the normal vector to the plane.

(iii) Find the equational form of the plane.

i) $p = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{e} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$\vec{e}\,' = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\boxed{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t' \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad t, t' \in \mathbb{R}}$$

ii) $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\quad \vec{n} \cdot \vec{e} = 0 \Rightarrow b + 2c = 0$

$\vec{n} \cdot \vec{e}\,' = 0 \Rightarrow a + 2b + c = 0$

$b = -2c \Rightarrow a - 4c + c = 0$

$a - 3c = 0$

$a = 3c$

$\vec{n} = \begin{pmatrix} 3c \\ -2c \\ c \end{pmatrix} = c \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

any vector multiple of $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ will work

iii) $3x - 2y + z = d$

$(0, 1, 1) \Rightarrow 0 - 2 + 1 = -1 = d$

$$\boxed{3x - 2y + z = -1}$$

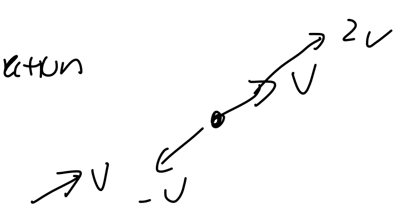
We can also describe lines in $\mathbb{R}^3$ via a parametric equation. Two parameters ($t$ and $t'$) gives us a plane. One parameter will give us a line. Think of them as "degrees of freedom".

Here are two ways to do this:

1. Line through a point $P$ with direction $\mathbf{v}$: only 1 displacement/direction

$$\Rightarrow \boxed{P + t\mathbf{v}, \quad t \in \mathbb{R}}$$

$\cdot P$ $\quad \nearrow v$




2. Passing through two points $P$ and $Q$:

$$\boxed{t\mathbf{P} + (1 - t)\mathbf{Q}, \quad t \in \mathbb{R}}$$

similar to convex but now $t$ can be any #

many ans

**Example 6:** Find the equation of the line through $(1, 0, 1)$ and $(-1, 2, 3)$. Also, determine the line's direction vector $\mathbf{v}$ (Note: There are many answers to the second part, but they will all be non-zero scalar multiples of one another).

$$\boxed{t\begin{bmatrix}1\\0\\1\end{bmatrix} + (1-t)\begin{bmatrix}-1\\2\\3\end{bmatrix}} = t\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}-1\\2\\3\end{bmatrix} - t\begin{bmatrix}-1\\2\\3\end{bmatrix}$$

$$= \begin{bmatrix}-1\\2\\3\end{bmatrix} + t\left(\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}-1\\2\\3\end{bmatrix}\right)$$

$$= \begin{bmatrix}-1\\2\\3\end{bmatrix} + t\begin{bmatrix}2\\-2\\-2\end{bmatrix}$$

$$\boxed{\vec{v} = \begin{bmatrix}2\\-2\\-2\end{bmatrix}}$$

**Example 7:** Find the equation of the line through $(1, 1, 1)$ that is perpendicular to the plane $-x + 3y + 2z = 2$.

point + direction

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \vec{n} = \begin{bmatrix}-1\\3\\2\end{bmatrix}$$

$$\boxed{\begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}-1\\3\\2\end{bmatrix}, \quad t \in \mathbb{R}}$$

# CH 4 NOTES + QUESTIONS
4.1 - 4.1.8

## Ch 4. span, subspaces, + dimension
- take geometric concepts (length or angle) + generalize them to n-vectors for any n

The theme of this chapter is *linear subspaces* of $\mathbf{R}^n$. By the end of the chapter, you should be able to:

- define span, linear subspace, and dimension in $\mathbf{R}^n$;
- recognize linear subspaces of $\mathbf{R}^3$ and their dimensions;
- relate orthogonality to 1 or 2 given vectors in $\mathbf{R}^n$ to a linear subspace.

## 4.1 span + linear subspaces
- plan P in $\mathbb{R}^3$ passing thru $0 = (0,0,0)$
  - P is flat w/ 2 degrees of freedom
  - choose 2 other pts in P that don't lie on a common line thru 0
  - get any pt on P by starting at 0 + walking some dist in v-direction then w-direction

$$P = \{\text{all vectors of the form } a\vec{v} + b\vec{w} \text{ for scalars } a, b\}$$

  - expand plane P, encoding the flatness of P w/ 2 degrees of freedom
  - any vector we can obtain from v + w repeatedly using vector operations is in form $a\vec{v} + b\vec{w}$

$$2\vec{v} - 3\vec{w} + 7\vec{v} + 11\vec{w} = 9\vec{v} + 8\vec{w}$$

4.1.3

**Definition 4.1.3.** The *span* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbf{R}^n$ is the collection of all vectors in $\mathbf{R}^n$ that one can obtain from $\mathbf{v}_1, \ldots, \mathbf{v}_k$ by repeatedly using addition and scalar multiplication. In symbols,

$$\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{\text{all } n\text{-vectors } \mathbf{x} \text{ of the form } c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k\}$$

where $c_1, \ldots, c_k$ are arbitrary scalars.

In $\mathbf{R}^3$, for $k = 2$ and nonzero $\mathbf{v}_1, \mathbf{v}_2$ not multiples of each other, this recovers the parametric form (3.2.3) of a plane through $P = \mathbf{0}$. In general, the span of a collection of finitely many $n$-vectors is the collection of *all* the $n$-vectors one can reach from those given $n$-vectors by forming linear combinations in every possible way.

This is a very new kind of concept – considering such a collection of $n$-vectors all at the same time. But it is ultimately no different than how we may visualize a plane in our head yet it consists of a lot of different points. **The span of two nonzero $n$-vectors that are not scalar multiples of each other should be visualized as a "plane" through 0 in $\mathbf{R}^n$**; with practice you'll get accustomed to thinking about general spans as an extension of that visualization to larger collections of $n$-vectors.

4.1.4 $\mathbb{R}^2$, if $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then span$(\vec{v})$ = set of all scalar multiples $c\vec{v} = \begin{bmatrix} c \\ 2c \end{bmatrix}$ => line thru $(0,0)$ w/ equation $y = 2x$

4.1.5

- in $\mathbb{R}^3$, the span of nonzero vectors $\vec{v}$ & $\vec{w}$ can be either a line or a plane thru $0$
  - otherwise, their span is a plane thru $0$

4.1.6

- set of vectors in $\mathbb{R}^n$ $\perp$ to any fixed nonzero vector in $\mathbb{R}^n$ is a span of $n-1$ nonzero $n$-vectors

Ex

$\vec{v} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \perp$ to $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 7 \end{bmatrix}$

$2x + 3y + z + 7w = 0$

$w = -\frac{2}{7}x - \frac{3}{7}y - \frac{1}{7}z$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -(2/7)x - (3/7)y - z/7 \end{bmatrix} = x\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2/7 \end{bmatrix}}_{\mathbf{a}} + y\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -3/7 \end{bmatrix}}_{\mathbf{b}} + z\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1/7 \end{bmatrix}}_{\mathbf{c}}$$

$V$ is the span of $a, b, c$ in $\mathbb{R}^4$

- a line in $\mathbb{R}^2$ or $\mathbb{R}^3$ passing thru 0 & a plane in $\mathbb{R}^3$ passing thru 0 each arise as a span of one or two vectors
  - but lines & planes not passing thru 0 are NOT a span of any collection of vectors BC span of any collection of n-vectors ALWAYS CONTAINS 0
- 4.1.7, LINEAR SUBSPACE of $\mathbb{R}^n$ is subset $\mathbb{R}^n$ that is the span of finite collection of vectors in $\mathbb{R}^n$ (SUBSPACE), if $\vec{V}$ is a linear subspace of $\mathbb{R}^n$, a SPANNING SET for $\vec{V}$ is a collection of n-vectors $v_1, \ldots v_k$ whose span equals $\vec{V}$ (linear subspace has LOTS of spanning sets)
- LINEAR SUBSPACE vs SPAN
  - no diff but span emphasizes input whereas linear subspace emphasizes output collection $\vec{V}$ of n-vectors w/o specific $v_1 \ldots v_k$

4.1.8

PRQ

1) $x = 1+t$ $\quad 1 + \cancel{t} - 2 - \cancel{4t} + 1 + \cancel{3t} = 1$
$y = 1 + 2t$ $\quad 0 = 1 \rightarrow //$ because never touches
$z = 1 + 3t$

2) $\text{span}(\vec{v}, \vec{w}) = \{$all 4-vectors x of the form $c_1 \vec{v}_1 + c_2 \vec{w}_2\}$, let $t_1 = 2$, let $t_2 = 3$

$\mathbb{R}^4 \quad k = 2$

$$t_1 \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 9 \\ -9 \\ 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 11 \\ -13 \\ 10 \\ 5 \end{bmatrix}$$

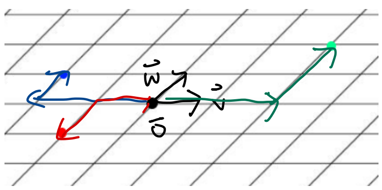
3) $\text{span}(\vec{v}, \vec{w})$ $\mathbb{R}^3$

# Lecture 4 - Span, Subspaces, Dimension

October 3, 2022

**Goals:** Define span, linear subspace, and dimension in $\mathbb{R}^n$ with specific examples in $\mathbb{R}^3$.

Consider a plane through the origin. Describe how you can reach the three colored points using combinations of $\mathbf{v}$ and $\mathbf{w}$.




- $2\vec{v} + 2\vec{w}$ · to get to green pt
- $-2\vec{v} + \vec{w}$ to get to blue point
- $-\vec{v} - \vec{w}$ to get to red pt

linear combos of $\vec{v}$ & $\vec{w}$

If a plane contains $\mathbf{0}$, then you can describe *any* point on the plane as $a\mathbf{v} + b\mathbf{w}$ (where $\mathbf{v}$ and $\mathbf{w}$ are two vectors in the plane that are not multiples of each other).

**Example 1:** Recall the parametric form of a plane is $P + a\mathbf{e} + b\mathbf{e}'$. If $\mathbf{0}$ is on the plane, then you can let $P = \mathbf{0}$. Find the parametric form of the plane $x - 2y + z = 0$.

1, find 3 pts on the plane

$P = (0,0,0)$, $(1,1,1)$ & $(1,2,3)$

2, find displacement from pt to pt

$\vec{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \vec{0}$ $\quad \vec{e}' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \vec{0}$

P.F. $\boxed{a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ a,b \in \mathbb{R}}$

Note: If we only looked at multiples of a single direction $\mathbf{v}$, we would get a line through the origin.

$\vec{p} \xrightarrow{\vec{v}}$ $\quad \cancel{\vec{p}} + t\vec{v} \Rightarrow t\vec{v}$ = line through origin

These are examples of a more general concept:

**Definition:** The **span** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, i.e.

$$\boxed{\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + \cdots c_k\mathbf{v}_k\},}$$

where $c_1, \ldots, c_k$ are scalars.

**Fact:** $\mathbf{0}$ is always in the span of any vectors, and $\text{span}\{\mathbf{0}\} = \{\mathbf{0}\}$. ← because if you make scalar 0 → go thru origin

**Example 2:** What is the span of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$?

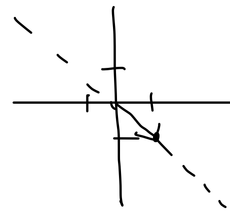
1) algebraically

$$\text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \left\{c\begin{bmatrix} 1 \\ -1 \end{bmatrix}, c \in \mathbb{R}\right\}$$

2) geometrically

a line thru the origin

$y = -x$



**Example 3:** From Example 1, $x - 2y + z = 0$ is the span of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Can you find a different set of vectors that spans this plane?

yes, can find diff set of vectors that spans the plane

$(0,1,2)$ $(-1,0,1)$, $(0,0,0)$

$\vec{e} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{e}\,' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad a\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $a,b \in \mathbb{R}$

plane $= \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$= \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

Thus, **the span of a set of vectors is not unique**. Different sets of vectors can have the same span.

**Example 4:** Let $U$ be the set of 4-vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ ($\mathbb{R}^4$) perpendicular to $\vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$. ~~Find a spanning set for $U$~~.

$U = \text{span}\{?\}$, find vectors that span $U$

all vectors in $U$'s dot product w/ $\vec{v}$ is 0

if $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in U$, $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = -x + 2y + 3z + w = 0$

1) pick one variable & write it in terms of the other 3 variables

ex, $x = 2y + 3z + w$ $\quad$ ☆ trying to find relationships amongst comp

☆ break into parts

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2y+3z+w \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3z \\ 0 \\ z \\ 0 \end{bmatrix} + \begin{bmatrix} w \\ 0 \\ 0 \\ w \end{bmatrix}$$

$$= y\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $U$ can be described as span

$$\boxed{U = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

★ For span, you have to specify spanning set

synonym of span

**Definition:** A **linear subspace** of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that is the span of a *finite* set of $n$-vectors. If $V$ is a subspace of $\mathbb{R}^n$, a **spanning set** for $V$ is a collection of vectors such that their span equals $V$. ← vectors in span, inputs

U is a linear subspace of $\mathbb{R}^4$ because it was found to be a span

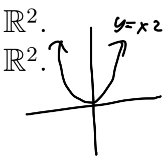
Note: There is no difference between a subspace and span, but span emphasizes a specific spanning set (e.g. $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\3\end{bmatrix}$) whereas subspace just means we know we can find a spanning set if we want, but at the moment we don't need to look at one.

**Example 5:** Lines and planes **through the origin** are subspaces of $\mathbb{R}^3$. The set $U$ from Example 4 is a subspace of $\mathbb{R}^4$ since we were able to find a spanning set.

**Proposition 4.1.11:** If $V$ is a linear subspace of $\mathbb{R}^n$, then for any vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m \in V$ ← in that subspace and scalars $c_1, \ldots, c_m$, the linear combination $c_1\mathbf{x}_1 + \cdots c_m\mathbf{x}_m$ is also in $V$.

In other words, all linear combinations of vectors in $V$ are also in $V$.

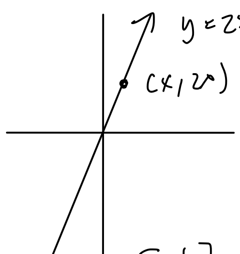
**Example 6:** Confirm the previous proposition for the subspace described by $y = 2x$ in $\mathbb{R}^2$. Also, use the previous proposition to explain why $y = x^2$ does **not** describe a subspace of $\mathbb{R}^2$.





$y=2x$ is a line thru origin ($\vec{0}$), so it describes a subspace

subspace is span$\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$





$\begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ in span

$\begin{bmatrix}3\\6\end{bmatrix} + \begin{bmatrix}\pi\\2\pi\end{bmatrix} = \begin{bmatrix}3+\pi\\6+2\pi\end{bmatrix} = 3+\pi\begin{bmatrix}1\\2\end{bmatrix}$ → vector is in span

$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} + c_3\begin{bmatrix}1\\2\end{bmatrix} = (c_1+c_2+c_3)\begin{bmatrix}1\\2\end{bmatrix}$

$y = x^2$

pts on parabola $= \left\{\begin{bmatrix}x\\x^2\end{bmatrix}, x \in \mathbb{R}\right\}$

$\begin{bmatrix}-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$ not in set

★ Found 2 vectors whose LC don't end up back in subspace

To elaborate more on the previous example, Proposition 4.1.11 gives us the following useful fact: (checklist to see if subspace of $\mathbb{R}^n$)

$V$ is a subspace if and only if

- $\mathbf{0} \in V$, contain $\vec{0}$ → lines/planes not thru origin are not subspaces of $\mathbb{R}^3$
- For any $\mathbf{v}, \mathbf{w} \in V$, we have $a\mathbf{v} + b\mathbf{w} \in V$ for any scalars $a$ and $b$, all LC end up back in V

**Definition:** Let $V$ be a nonzero subspace of $\mathbb{R}^n$. The **dimension** of $V$, denoted $\dim(V)$, is the smallest number of vectors needed to span $V$. We define $\dim(\{\mathbf{0}\}) = 0$.

**Theorem 4.2.5:** For $k \geq 2$, consider $V = \text{span}\{\mathbf{v}_1 \ldots, \mathbf{v}_k\}$. $\dim(V) = k$ whenever there is "no redundancy," i.e. each $\mathbf{v}_i$ is **not** a linear combination of the others.

# of vectors in span

Equivalently, $\dim(V) < k$ if there **is** redundancy.

$a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$

* solve sys of eq
- if sol → redundancy

**Example 7:** Let $V = \text{span}\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\}$. What is the dimension of $V$?

1) check for redundancy (are one of the vectors a combo of others?)

$\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix} \rightarrow \begin{bmatrix}-1\\0\end{bmatrix}$ doesn't add new info, is redundant

$V = \text{span}\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\}$ $\quad \dim V = 2$

* doesn't matter which one is redundant, what's important is how many are redundant

**Example 8:** What is the dimension of $\mathbb{R}^3$? Of $\mathbb{R}^n$?

$\mathbb{R}^3$ is span $\begin{bmatrix}x\\y\\z\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow$ no redundancies

so $\dim(\mathbb{R}^3)$ is 3

so $\dim(\mathbb{R}^n)$ is $n$

**Example 9:** Do $\begin{bmatrix}3\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\2\\-1\end{bmatrix}$, and $\begin{bmatrix}2\\-4\\3\end{bmatrix}$ span $\mathbb{R}^3$?

No, it spans a plane

$\mathbb{R}^2 \rightarrow$ need 2 vectors $t\begin{bmatrix}a\\b\end{bmatrix} + t'\begin{bmatrix}c\\d\end{bmatrix}$

$\mathbb{R}^3 \rightarrow$ need 3 vectors $t\begin{bmatrix}a\\b\\c\end{bmatrix} + t'\begin{bmatrix}c\\d\\e\end{bmatrix} + t''\begin{bmatrix}f\\g\\h\end{bmatrix}$

$a\begin{bmatrix}0\\2\\-1\end{bmatrix} + b\begin{bmatrix}2\\-4\\3\end{bmatrix} = \begin{bmatrix}3\\-1\\2\end{bmatrix}$

$2b = 3$
$2a - 4b = -1$
$-a + 3b = 2$

$\boxed{b = 3/2, \ a = 5/2}$ → so $\begin{bmatrix}3\\-1\\2\end{bmatrix}$ is redundant

if have only 2 vectors → can't span $\mathbb{R}^3$ (can only span $\mathbb{R}^2$/plane)

**Example 10:** Describe the intersection of $x - 2y + z = 0$ and $3x + y = 0$.

intersect 2 planes ⇒ line

$y = -3x$

$x + 6x + z = 7x + z = 0$

$z = -7x$, $\quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x\\-3x\\-7x\end{bmatrix}$ $\quad$ intersection = span $\{\begin{bmatrix}1\\-3\\-7\end{bmatrix}\}$

In fact, the intersection of two subspaces will also be a subspace.

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \begin{bmatrix}3\\2\\1\end{bmatrix}$$

## Problem 1: Parametric and equational forms of a plane in $\mathbb{R}^3$

Let $P$ be the plane in $\mathbf{R}^3$ containing the points $(1, 1, 1)$, $(1, 2, 3)$, and $(3, 2, 1)$. (Why is there a unique such plane?) $t, t' \in \mathbb{R}$

(a) Find a parametric representation of $P$. (Extra: can you write down many other parametrizations?)

$$\vec{e} = \begin{bmatrix}0\\1\\2\end{bmatrix} \quad \vec{e}' = \begin{bmatrix}2\\1\\0\end{bmatrix} \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\2\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix}$$

(b) Use the dot product to find a normal vector to $P$. (Hint: Think about why it is the same as a vector perpendicular to two different "directions" within the plane, and then form some displacement vectors.)

$\vec{n}$ = normal vector $\quad \vec{e}\cdot\vec{n} = 0 \quad \vec{e}'\cdot\vec{n} = 0$

$\vec{n} = \begin{bmatrix}a\\b\\c\end{bmatrix}$

$b + 2c = 0 \quad c = -\frac{1}{2}b$

$2a + b = 0$

$a = -\frac{1}{2}b$

$\vec{n} = \begin{bmatrix}-\frac{1}{2}b\\b\\-\frac{1}{2}b\end{bmatrix} = b\begin{bmatrix}-\frac{1}{2}\\1\\-\frac{1}{2}\end{bmatrix} \quad \vec{n} = \begin{bmatrix}-\frac{1}{2}\\1\\-\frac{1}{2}\end{bmatrix}$

(c) Find an equation for $P$ of the form $ax + by + cz = d$ for some $a, b, c, d$ in $\mathbf{R}$.

$-\frac{1}{2}x + y - \frac{1}{2}z = d$

$(1,1,1)$: $-\frac{1}{2} + 1 - \frac{1}{2} = d \quad d = 0$

$$\boxed{-\frac{1}{2}x + y - \frac{1}{2}z = 0}$$

## Problem 2: Determining if points are on the same or opposite sides of a plane, and parallel planes

Let $P$ be the plane in $\mathbf{R}^3$ given by the equation $11x - 3y + 2z = 12$, and let $Q$ be the plane parallel to $P$ passing through the point $\mathbf{v} = \begin{bmatrix}2\\1\\0\end{bmatrix}$.

A: $11 + 2 = 13$

B: $11 - 6 - 2 = 4$

C: $33 - 24 - 2 = 7$

Consider the following points: $A = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $B = \begin{bmatrix}1\\2\\-1\end{bmatrix}$, $C = \begin{bmatrix}3\\8\\-1\end{bmatrix}$ (check that none of these lie on $P$!).

(a) For each pair among the three points (i.e. $A$ and $B$, $A$ and $C$, $B$ and $C$), determine whether the two points are on the same side of $P$ or on opposite sides.

A: $13 > 12$

B: $4 < 12$

C: $7 < 12$

A on opp side

B & C on same side

(b) Work out an equation for $Q$, and then repeat part (a) but for the plane $Q$.

$11x - 3y + 2z = d$

$22 - 3 + 0 = d$

$d = 19$

$$\boxed{11x - 3y + 2z = 19}$$

A & B & C are all on same side

(c) Note that, because $P$ and $Q$ are parallel to one another and distinct, there is a region of $\mathbf{R}^3$ that is *between* $P$ and $Q$. Determine which (if any) of $A$, $B$, and $C$ lie between $P$ and $Q$.

A lies between P & Q

## Problem 3: Visualizing a span

For each collection of vectors in $\mathbf{R}^2$, sketch its span: is it a point, a line, or all of $\mathbf{R}^2$?

(a) $\begin{bmatrix}1\\0\end{bmatrix}$ (b) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ (c) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$ (d) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$ (e) $\begin{bmatrix}0\\0\end{bmatrix}$

(a) x-axis

(b) entirety of $\mathbb{R}^2$

$x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

(c) x-axis

(d) x-axis

(e) origin

For each collection of vectors in $\mathbf{R}^3$ sketch its span: is it a point, a line, a plane, or all of $\mathbf{R}^3$?

(f) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ (g) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ (h) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \cancel{\begin{bmatrix}1\\1\\0\end{bmatrix}}$ (i) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$

(f) x-y plane

(g) entirety of $\mathbb{R}^3$

(h) xy plane

(i) x-axis

## Problem 4: What sets can be linear subspaces, and what cannot?

For each of the following subsets of $\mathbf{R}^2$ or $\mathbf{R}^3$, write down a collection of finitely many vectors whose span is that set or explain why there is no such collection.

(a) The line $x + y = 1$ (b) The line $x + y = 0$ (c) The unit disk $x^2 + y^2 \leq 1$ (d) $\{\mathbf{0}\}$ (e) The plane $x + y + z = 0$

(a) 0, not on line
not span

(b) line through origin

$a\begin{bmatrix}1\\-1\end{bmatrix}$

(c) cannot be span

(d) span of $\{0\}$

(e) $z = -x-y$

$\begin{bmatrix}x\\y\\-x-y\end{bmatrix} = \begin{bmatrix}x\\0\\-x\end{bmatrix} + \begin{bmatrix}0\\y\\-y\end{bmatrix}$

$x\begin{bmatrix}1\\0\\-1\end{bmatrix} + y\begin{bmatrix}0\\1\\-1\end{bmatrix}$

span of $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$

## Problem 5: Multiple descriptions as a span (Extra)

Let $\mathbf{v}, \mathbf{w}$ be two vectors in $\mathbf{R}^{12}$. Show that $\text{span}(\mathbf{v}, \mathbf{w}) = \text{span}(\mathbf{v} + \mathbf{w}, \mathbf{v} - \mathbf{w})$. (Hint: You can show that two sets $S$ and $T$ are equal in two steps: everything belong to $S$ also belong to $T$, and everything belonging to $T$ also belongs to $S$.)

# Lecture 5 - Basis and Orthogonality

October 5, 2022

**Goals:**

- Determine a basis and dimension for a linear subspace of $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Verify whether a collection of vectors in $\mathbb{R}^n$ is orthogonal or orthonormal.

**Definition:** A **basis** for a non-zero subspace $V$ is spanning set of exactly $\dim(V)$ vectors.

**Example 1:** $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of $\mathbb{R}^3$. There are a lot (infinitely many!) of different bases for $\mathbb{R}^3$:

↳ standard basis

$$\vec{e}_i = \begin{cases} 1 & \text{in the ith slot} \\ 0 & \text{else} \end{cases}$$

$$\underbrace{\vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{basis}}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-2\\0\end{pmatrix}$$

↑ another basis for $\mathbb{R}^3$, make sure have no redundancies

## Dimension Criterion:

Here is how to determine the dimension of a span of 1, 2, or 3 (non-zero) vectors in $\mathbb{R}^n$:

* basis $\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$ dim = 2

**One vector:** The span of a single vector is always dimension one. → $\text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ dim = 1

* not basis bc too many vectors $\text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}\right\}$ dim = 1 * can throw out $\begin{pmatrix}2\\2\end{pmatrix}$ bc scalar multiple

**Two vectors:** For two non-zero $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we have $\dim(\text{span}\{\mathbf{v}, \mathbf{w}\}) = 2$ *except* for when the vectors are scalar multiples of each other, in which case the dimension is 1.

**Three vectors:** For three non-zero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ with span $V$, we have $\dim(V) = 3$ *except* for the following cases: * if have too much info, can throw them out

(i) all three vectors are scalar multiples of each other, in which case $\dim(V) = 1$.
(ii) exactly two of the vectors are scalar multiples of each other, in which case $\dim(V) = 2$.
(iii) no $\mathbf{v}_i$ is a scalar multiple of another, but some $\mathbf{v}_i$ is a linear combination of the other 2, in which case $\dim(V) = 2$

**Example 2:** Here are some examples for the span of three vectors cases above.

$$(i) = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \cancel{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}}, \cancel{\begin{pmatrix}\pi/2\\\pi/2\\\pi/2\end{pmatrix}} \right\} \quad \text{dim span} = 1$$

* doesn't matter which one you throw out
* 2 are redundant

$$(ii) \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \cancel{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\} \quad \text{dim span} = 2$$

$$(iii) \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \cancel{\begin{pmatrix}2\\1\\2\end{pmatrix}} \right\} \quad \text{dim span} = 2$$

$$\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix} \rightarrow \text{no new info}$$

**Example 3:** Last time, we saw that $U = \left\{ \mathbf{v} \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = 0 \right\}$ was the span of the vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. What is the dimension of $U$?

~~(i)~~ no scalar multiples

~~(ii)~~

(iii) pick one of them: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

If solution: it is a LC $\Rightarrow$ redundant & can throw away vector

Else: keep it

$$\left.\begin{aligned} 1 &= 2a + 3b \\ 0 &= a \\ 0 &= b \\ 1 &= 0 \end{aligned}\right) \text{ no solution} \rightarrow$$

so all 3 vectors necessary to describe $U$

so $\boxed{\dim(U) = 3}$

spans plane instead

**Example 4:** Last time, we showed $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$ did not span $\mathbb{R}^3$. What is the dimension of their span?

pick one

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = a \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + b \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$$

$3 = 2b \quad b = \frac{3}{2}$

$$\left.\begin{aligned} -1 &= 2a - 4b & -1 &= 2a - 6 & a &= \tfrac{5}{2} \\ 2 &= -a + 3b & 2 &= -a + \tfrac{9}{2} & a &= \tfrac{5}{2} \end{aligned}\right) \checkmark$$

* check that system is consistent, solve for a w/ both eq

so $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \frac{5}{2} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$

so I don't need it

so $\text{span}\left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \right\} \Rightarrow \boxed{\dim = 2}$

**Definition:** A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is called **orthogonal** if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. That is, when the vectors are all perpendicular to one another.

**Theorem 5.2.2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an orthogonal collection of **nonzero** vectors in $\mathbb{R}^n$ then it is a basis for span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. In particular, span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ then has dimension $k$ and we call $\mathbf{v}_1, \ldots, \mathbf{v}_k$ an **orthogonal basis** for its span. → there is no redundancy, dim span = k

review

**Example 5:** Find an orthogonal basis for the span of the vectors in Example 4.

basis for span: $\left\{ \underbrace{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}}_{\vec{v}_2} \right\}$ $\quad \vec{v}_1 \cdot \vec{v}_2 = -11 \neq 0$ → basis is not orthogonal

How do we find orthogonal basis? → find $\vec{w}$ that $\vec{w} \cdot \vec{v}_1 = 0$ and $\vec{w} \in \text{span}\left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \right\}$

$\vec{w} \in \text{span} \to \vec{w} = a\vec{v}_1 + b\vec{v}_2$

$0 = \vec{w} \cdot \vec{v}_1 = (a\vec{v}_1 + b\vec{v}_2) \cdot \vec{v}_1 = a\|\vec{v}_1\|^2 + b(\vec{v}_1 \cdot \vec{v}_2)$

$= 5a - 11b \Rightarrow b = \frac{5}{11}a$

so gram-schmidt

$\vec{w} = a\vec{v}_1 + \frac{5}{11}a\vec{v}_2$

$\vec{w} = \frac{a}{11}(11\vec{v}_1 + 5\vec{v}_2) = \frac{a}{11}\left[\begin{pmatrix} 0 \\ 22 \\ -11 \end{pmatrix} + \begin{pmatrix} 10 \\ -20 \\ 15 \end{pmatrix}\right]$

$= \frac{2a}{11}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} \approx \boxed{\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}}$

$\text{span}\left\{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}\right\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$

**Theorem 5.2.5:** Every nonzero linear subspace of $\mathbb{R}^n$ has an orthogonal basis.

**Definition:** A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called **orthonormal** if they are orthogonal *and* they are all unit vectors.

**Example 6:** The **standard basis** for $\mathbb{R}^n$ is an orthonormal basis.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ orthonormal basis of $\mathbb{R}^2$

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ orthogonal basis, not orthonormal

↳ $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ orthonormal basis of $\mathbb{R}^2$

**Theorem 5.3.6:** (**Fourier Formula**). For any orthogonal collection of nonzero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, $\quad \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ → LC

$$\mathbf{v} = \sum_{i=1}^{k} \left( \frac{\mathbf{v} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} \right) \mathbf{v}_i. \qquad \boxed{c_i = \frac{\vec{v} \cdot \vec{v}_i}{\|\vec{v}_i\|^2}}$$

If the $\mathbf{v}_i$'s are unit vectors (orthonormal), then $\mathbf{v} = \sum (\mathbf{v} \cdot \mathbf{v}_i)\mathbf{v}_i$.

$\leftarrow \|\vec{v}_i\|^2 = 1$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

**Example 7:** Consider the orthogonal basis $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Express $\begin{bmatrix}3\\-1\\5\end{bmatrix}$ and $\begin{bmatrix}5\\1\\3\end{bmatrix}$ as linear combinations of this basis.

$$\begin{bmatrix}3\\-1\\5\end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \Rightarrow c_i = \frac{\vec{v}\cdot\vec{v}_i}{\|\vec{v}_i\|^2}$$

$$c_1 = \frac{\begin{pmatrix}3\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{2} = \frac{3-1}{2} = 1$$

$$c_2 = \frac{\begin{pmatrix}3\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{2} = \frac{4}{2} = 2$$

$$c_3 = \frac{\begin{pmatrix}3\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{1} = 5$$

$$\boxed{\begin{pmatrix}3\\-1\\5\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\-1\\0\end{pmatrix} + 5\begin{pmatrix}0\\0\\1\end{pmatrix}}$$

check

$$\begin{pmatrix}5\\1\\3\end{pmatrix} = 3\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$$

**Example 8 (if time):** Find an orthogonal basis for the plane in $\mathbb{R}^3$ defined by the equation $2x - 3y - z = 0$.

2 ways: parametric form $a\vec{e} + b\vec{e}'$

$\{\vec{e}, \vec{e}'\}$ is a basis $\to$ proceed as Ex 5

OR: E.g. $\begin{pmatrix}1\\0\\2\end{pmatrix}$ is on plane,

$\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \vec{w}\right\}$ $\qquad \vec{w}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix} = 0 \qquad \vec{w}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix} = 0$ $\quad \vec{w}\cdot\begin{pmatrix}2\\3\\-1\end{pmatrix} = 0??$

orthogonal

* make sure $\vec{w}$ is on plane

$$\vec{w} = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

$a + 2c = 0 \quad a = -2c \Longrightarrow -4c - 3b - c = -3b - 5c = 0$

$2a - 3b - c = 0$

$b = -\frac{5}{3}c$

$$\vec{w} = \begin{pmatrix}-2c\\-\frac{5}{3}c\\c\end{pmatrix} \to c = -3 \quad \boxed{\vec{w} = \begin{pmatrix}6\\5\\-3\end{pmatrix}}$$

?

✓

## Problem 1: Determining the nature of a span

For each collection of 3-vectors, determine whether its span is a point, a line, a plane, or all of $\mathbf{R}^3$. Give a basis of the span in each case. (Keep in mind that if a vector in the collection is a linear combination of others then it can be dropped without affecting the span.)

(a) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}$

plane

$$\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix} \right\}$$

(b) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \cancel{\begin{bmatrix}-1\\1\\-1\end{bmatrix}}$

$$\begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

plane

$$\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix} \right\}$$

(c) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}$

all of $\mathbb{R}^3$

$$\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix} \right\}$$

## Problem 2: Linear subspaces and orthogonality (computations)

span?

Let $V$ be the set of vectors in $\mathbf{R}^4$ orthogonal to both $\begin{bmatrix}1\\0\\4\\2\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$. Find a pair of vectors that span $V$, so it is a linear subspace.

$$\vec{n} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}-4c-2d\\-c-d\\c\\d\end{bmatrix} = \begin{bmatrix}-8\\-3\\1\\2\end{bmatrix}$$

$a + 4c + 2d = 0$

$b + c + d = 0$

$a = -4c - 2d$

$b = -c - d$

let $c = 1, d = 2$

let $c = 2, d = 1$

$$\begin{bmatrix}-10\\-3\\2\\1\end{bmatrix}$$

$$\boxed{V = t\begin{bmatrix}-8\\-3\\1\\2\end{bmatrix} + t'\begin{bmatrix}-10\\-3\\2\\1\end{bmatrix} \quad t, t' \in \mathbb{R}}$$

## Problem 3: More recognizing and describing linear subspaces

* intersection of linear subspace is lin subspace

Which of the following subsets $S$ of $\mathbf{R}^3$ are linear subspaces? If a set $S$ is a linear subspace, exhibit it as a span. If it is not a linear subspace, describe it geometrically and explain why it is not a linear subspace.

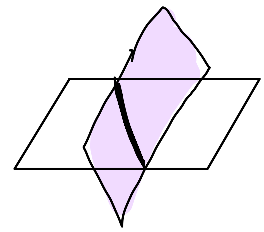
intersection of planes

(a) The set $S_1$ of points $(x, y, z)$ in $\mathbf{R}^3$ with <u>both</u> $z = x + 2y$ and $z = 5x$.

is a linear subspace

$5x = x + 2y \qquad 4x = 2y \qquad y = 2x$

$$\begin{bmatrix} x \\ 2x \\ 5x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \qquad \text{span}\{S_1\} = \left\{ c \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \ c \in \mathbb{R} \right\}$$



(b) The set $S_2$ of points $(x, y, z)$ in $\mathbf{R}^3$ with <u>either</u> $z = x + 2y$ <u>or</u> $z = 5x$.

z planes

$0 = x + 2y - z \qquad 0 = 5x - z \leftarrow$ plane on y-axis

not a linear subspace

2 planes ?

their addition moves out of $S_2$

✓ point

S is closed under linear combination if for all $v_1, v_2 \in S$, any linear combo $a\vec{v}_1 + b\vec{v}_2 \in S$ as well

(c) The set $S_3$ of points $(x, y, z)$ in $\mathbf{R}^3$ of the form $t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t' \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ for some scalars $t$ and $t'$ (which are allowed to be anything, depending on the point $(x, y, z)$).

$t + 2t' = -3 \rightarrow -1 + 2t' = -3 \qquad t' = -1$

$2t + t' = -3 \qquad -2 - 1 = -3$

$3t = -3$

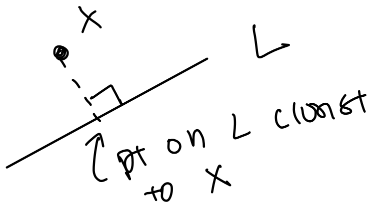
$t = -1$

is a linear subspace, goes through 0.

$$\text{span}\{S_3\} = \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \ c_1, c_2 \in \mathbb{R} \right\}$$

## Problem 4: Building another orthogonal vector (Extra)

If $\{\mathbf{v}, \mathbf{w}\}$ is a pair of nonzero orthogonal vectors in $\mathbf{R}^3$ then we can always enlarge it to an orthogonal basis $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ of $\mathbf{R}^3$ by taking $\mathbf{u}$ to be a nonzero normal vector to the plane $\text{span}(\mathbf{v}, \mathbf{w})$. If $n > 3$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ are mutually orthogonal nonzero vectors in $\mathbf{R}^n$ then can we always find a nonzero $\mathbf{v}_n$ orthogonal to those (so $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis of $\mathbf{R}^n$)?




# Lecture 6 - Projections

October 7, 2022

**Goals:** Compute the projection into a subspace $V$ when given an orthogonal basis, and compute the point on a line in $\mathbb{R}^n$ through 0 that is closest to a given point in $\mathbb{R}^n$.

**Proposition 6.1.1:** Let $L = \text{span}(\mathbf{w}) = \{c\mathbf{w} : c \in \mathbb{R}\}$ be a 1-dimensional linear subspace of $\mathbb{R}^n$ (i.e. $\mathbf{w} \neq 0$), a "line". Choose any point $\mathbf{x} \in \mathbb{R}^n$. Then there is exactly one point in $L$ closest to $\mathbf{x}$, and it is given by the scalar multiple

$\text{Proj}_{\vec{w}} \vec{x} =$ scalar

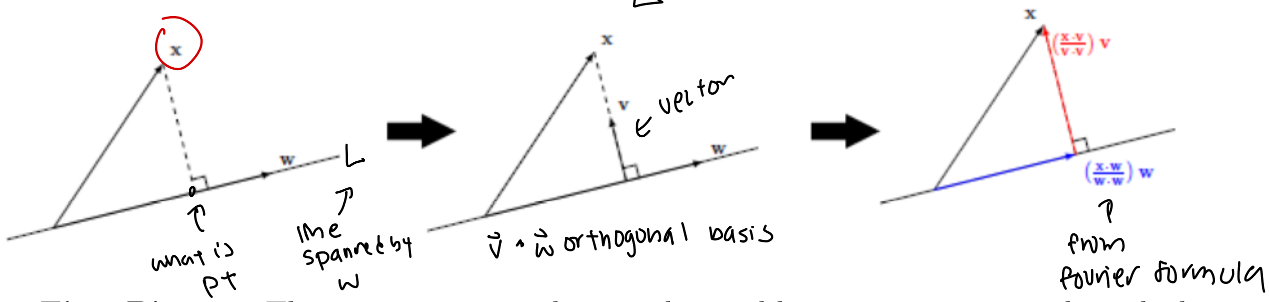
$$\boxed{\underbrace{\left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)}\mathbf{w}}$$

$\vec{x}\cdot\vec{w} = \#$

$\vec{w}\cdot\vec{w} = \|\vec{w}\|^2 = \#$

of $\mathbf{w}$. This is called "the projection of $\mathbf{x}$ onto $\text{span}(\mathbf{w})$"; we denote it by the symbol $\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x}$.

↳ $L$



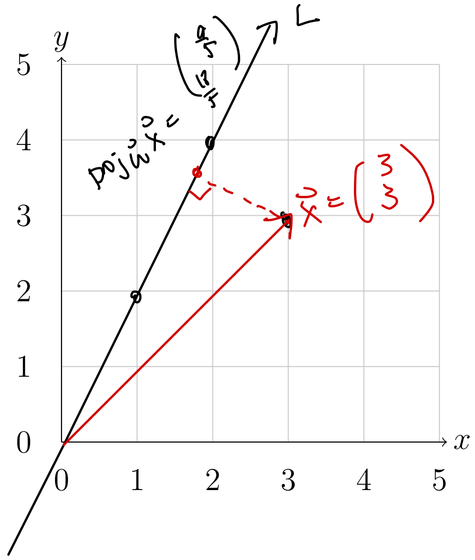

- **First Picture:** The unique point is the one obtained by creating a 90° angle with the line.
- **Second Picture:** Picking any vector $\mathbf{v}$ along the dashed line creates a the orthogonal basis for the plane containing $\mathbf{x}$ and $\mathbf{w}$. Thus:
- **Third Picture:** We can represent $\mathbf{x}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ via the Fourier Formula from last lecture.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

**Example 1:** Let $L$ be the line $y = 2x$ in $\mathbb{R}^2$. Describe $L$ as the span of a single vector $\mathbf{w}$, and compute $\mathbf{Proj}_{\mathbf{w}}\,\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

projection x onto w

$L = \text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, $\vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$




$$\text{Proj}_{\vec{w}}\,\vec{x} = \frac{\begin{pmatrix} 3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\|^2} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{3+6}{5} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ \frac{18}{5} \end{pmatrix}$$

**Example 2:** Find the closest point to $\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ on the line span$(\mathbf{w})$, where $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$L = \text{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$ $\quad \text{Proj}_{\vec{w}}\,\vec{x} = \frac{\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{2+4}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}}$

Projections onto lines through the origin satisfy a very nice algebraic property, which can be shown via the distribution of dot products (page 109 for more details):

$$\mathbf{Proj}_{\mathbf{w}}\left(c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k\right) = c_1\mathbf{Proj}_{\mathbf{w}}\left(\mathbf{x}_1\right) + \cdots + c_k\mathbf{Proj}_{\mathbf{w}}\left(\mathbf{x}_k\right). \quad = \text{Proj}_{\vec{w}}\,\vec{x}$$

In particular, if you need to project many points, it might help to compute the projection of the standard basis vectors first and then use the above property.

$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2 \begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\mathbb{R}^3$

**Example 3:** Compute the projections of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ onto the line spanned by $\mathbf{w} = \begin{bmatrix}1\\2\\0\end{bmatrix}$.

$\text{proj}_{\vec{w}} \vec{e}_1 = \dfrac{\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1\\2\\0\end{pmatrix}}{5} \begin{pmatrix}1\\2\\0\end{pmatrix} = \boxed{\frac{1}{5}\begin{pmatrix}1\\2\\0\end{pmatrix}}$

$\text{proj}_{\vec{w}} \vec{e}_2 = \frac{2}{5}\begin{pmatrix}1\\2\\0\end{pmatrix}$

$\boxed{\vec{e}_3 = \vec{0}}$

**Example 4:** Using the previous example, compute the projections of $\begin{bmatrix}-2\\-1\\3\end{bmatrix}$ and $\begin{bmatrix}4\\3\\1\end{bmatrix}$ onto the line spanned by $\mathbf{w}$.

$\begin{pmatrix}-2\\-1\\3\end{pmatrix} = -2\vec{e}_1 - \vec{e}_2 + 3\vec{e}_3$

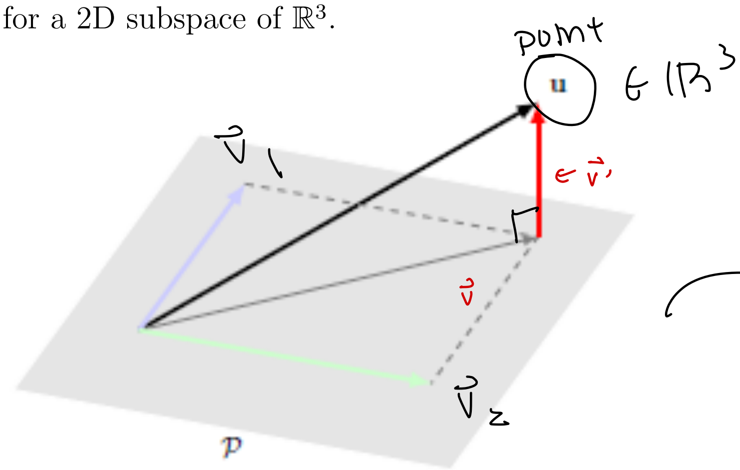
$\text{proj}_{\vec{w}} \begin{pmatrix}-2\\-1\\3\end{pmatrix} = -2\,\text{proj}_{\vec{w}}\vec{e}_1 - \text{proj}_{\vec{w}}\vec{e}_2 + 3\,\text{proj}_{\vec{w}}\vec{e}_3$

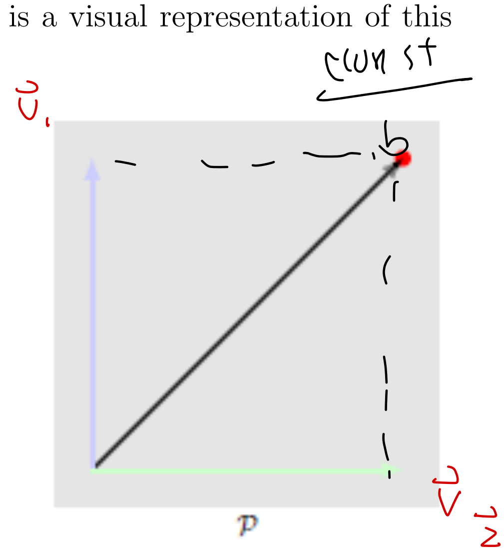
$= -\frac{2}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} - \frac{2}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} + \vec{0} = -\frac{4}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} = \boxed{\begin{pmatrix}-4/5\\-8/5\\0\end{pmatrix}}$

Check: $\text{proj}_{\vec{w}} \begin{pmatrix}4\\3\\1\end{pmatrix} = 4\,\text{proj}_{\vec{w}}\vec{e}_1 + 3\,\text{proj}_{\vec{w}}\vec{e}_2 + \text{proj}_{\vec{w}}\vec{e}_3 = \begin{pmatrix}2\\4\\0\end{pmatrix}$

✱ do it yourself

We can expand the notion of projection onto a one dimensional subspace (line) to general subspaces $V$ of $\mathbb{R}^n$. Namely, given a point in $\mathbb{R}^n$ and a subspace in $\mathbb{R}^n$, there is a unique vector in that subspace that is closest to the point. Here is a visual representation of this for a 2D subspace of $\mathbb{R}^3$.




$\mathcal{P} = V = \text{span}\{\vec{v}_1, \vec{v}_2\}$

**Theorem 6.2.1:** **(Orthogonal Projection Theorem, Version 1)** For any $\mathbf{x} \in \mathbb{R}^n$ and linear subspace $V$ of $\mathbb{R}^n$, there is a unique vector $\mathbf{v} \in V$ that is closest to $\mathbf{x}$. This $\mathbf{v}$ is called the **projection of x onto** $V$, and is denoted $\mathbf{Proj}_V(\mathbf{x})$.

- $\mathbf{Proj}_V(\mathbf{x})$ is the only vector $\mathbf{v} \in V$ with the property that the displacement $\mathbf{x} - \mathbf{v}$ is perpendicular to every vector in $V$.
- If $V$ is nonzero, then for any *orthogonal basis* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of $V$, we have

$$\mathbf{Proj}_V(\mathbf{x}) = \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \cdots + \mathbf{Proj}_{\mathbf{v}_k}(\mathbf{x})$$

$V = \text{span}\{\vec{v}_1, \vec{v}_2\}$ $(\dim V = 2)$ dim = # of elements/vectors in span

**Example 5:** Let $V$ be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, find the point $\mathbf{v} \in V$ closest to $\mathbf{x}$.

* need orthogonal basis to use formula

check $\vec{v}_1 \cdot \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = 1 - 1 = 0$ ✓ → can apply formula

$$\text{proj}_V \vec{x} = \text{proj}_{\vec{v}_1} \vec{x} + \text{proj}_{\vec{v}_2} \vec{x}$$

$$= \frac{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}}{7} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$= -\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \frac{14}{7} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} 3/2 \\ 5/2 \\ 2 \\ 4 \end{pmatrix}}$$

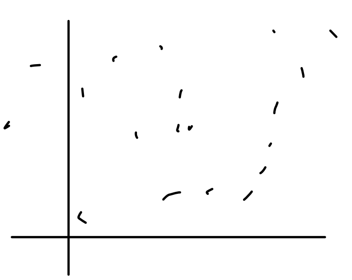
think of it like dot

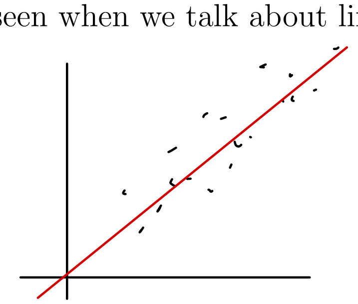
**Theorem 6.2.4:** **(Orthogonal Projection Theorem, Version 2)** If $V$ is a linear subspace of $\mathbb{R}^n$ then every vector $\mathbf{x} \in \mathbb{R}^n$ can be uniquely expressed as a sum

$$\boxed{\mathbf{x} = \mathbf{v} + \mathbf{v}'}$$

with $\mathbf{v} \in V$ and $\mathbf{v}'$ orthogonal to everything in $V$. Explicitly, $\mathbf{v} = \mathbf{Proj}_V(\mathbf{x})$ and $\mathbf{v}' = \mathbf{x} - \mathbf{Proj}_V(\mathbf{x})$.

The use of this version will be seen when we talk about linear regression next lecture.





what is $y = mx + b$? ↑ ↑ orthogonal proj will help

$y = mX + b\begin{bmatrix} : \end{bmatrix}$

Ch 7 - PRQ

$$\left(\frac{w \cdot v}{v \cdot v}\right) v$$

$$\frac{2+2+2}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{distance} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \sqrt{3}$$

$$\vec{x} = 2\vec{v} + 3(\vec{u} - 2\vec{v}) = 2\vec{v} + 3\vec{u} - 6\vec{v} = -4\vec{v} + 3\vec{u}$$

| $y$ | $x+1$ | |
|---|---|---|
| 3 | 1 | $\rightarrow 4$ |
| 0 | 2 | $\rightarrow 4$ |
| 3 | 3 | $\rightarrow 0$ |

# Lecture 7 - Orthogonal Bases of Planes and Linear Regression

$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{w} \cdot \vec{v}}{\vec{w} \cdot \vec{w}} \vec{w}$

October 10, 2022

**Goals:** Produce a an orthogonal basis for a 2D subspace in $\mathbb{R}^n$, and apply the result to linear regression problems.

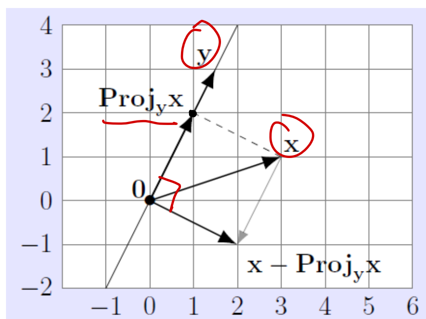
We start by introducing a systematic technique to produce an orthogonal basis of a 2D subspace (compare this to Example 5 from lecture 5).

**Theorem 7.1.1:** Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are nonzero and not scalar multiples of each other. The vectors

$$\boxed{\mathbf{y} \quad \text{and} \quad \mathbf{x}' = \mathbf{x} - \mathbf{Proj_y}\, \mathbf{x}}$$

constitute an orthogonal basis of span$(\mathbf{x}, \mathbf{y})$. (In particular, this span is 2-dimensional).

- pick a vector
- project the other onto it, then subtract projection from vector




Note, we can instead use $\mathbf{x}$ and $\mathbf{y}' = \mathbf{y} - \mathbf{Proj_x}\, \mathbf{y}$ if you would rather include $\mathbf{x}$ in your orthogonal basis.

$\text{proj}_V \vec{x} = \text{proj}_{\vec{v}_1} \vec{x} + \dots \text{proj}_{\vec{v}_k} \vec{x}$

↳ have to be orthogonal?

**Example 1:** Consider the subspace $V$ spanned by $\underset{\vec{x}}{\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}}$ and $\underset{\vec{y}}{\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}}$ from Lecture 5. Compute an orthogonal basis of $V$ using the above theorem. **Then,** find the projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $V$.

a) keep $\vec{x} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$. replace $\vec{y}$:

$\boxed{\text{Proj}_{\vec{x}} \vec{y} = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}} \vec{x}}$ $\quad \frac{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \frac{0 - 8 - 3}{5} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \frac{-11}{5} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$\vec{y}' = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} - \left( \frac{-11}{5} \right) \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ $\quad \boxed{\left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \cancel{\tfrac{2}{5}} \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} \right\}}$ → orthogonal

don't need the constant

check: $\left\{ \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}, \frac{2}{29} \begin{pmatrix} 11 \\ 7 \\ 2 \end{pmatrix} \right\}$

also orthogonal

b) $\text{proj}_V \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \text{Proj}_{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \text{proj}_{\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

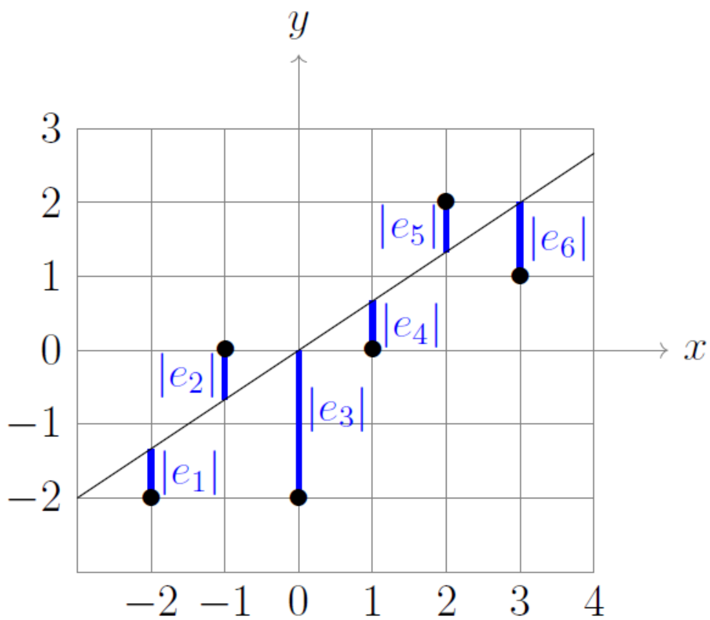
$= \frac{1}{3} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$ ← point in subspace V that is closest to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

? need $\frac{1}{3}$ bc is specific pt

## Application: Linear Regression

Given $n$ data points $(x_1, y_1), \ldots, (x_n, y_n)$, we would like to calculate the line which bests fits the data. That is, we seek a function $f(x) = mx + b$ which will help us make predictions using this data.

How do we do this? Given $x_i$ (our input), the actual output value is $y_i$. On the other hand, $f(x_i) = mx_i + b$ is our estimated or predicted value. The error of our estimate is thus actual - estimate $= y_i - (mx_i + b)$.




So to clarify, by "best fit," we want the line which makes all of these errors collectively small. While there are a few ways one might choose to do this, we will focus on the **least squares method**, which involves minimizing the sum of the square of the errors:

· gets rid of negatives
· $|x|$ not differentiable @ 0 but $x^2$ is
·

$$\sum_{i=1}^{n} (y_i - (mx_i + b))^2 . = \text{SSE}$$

sum of squares of errors + add it up

We can use our knowledge of previous lectures to rephrase this problem. Collect all $x_i$ and $y_i$ into vectors $\mathbf{X}$ and $\mathbf{Y}$ respectively,

$$\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

vector w/ all 1's

Define the vector $\mathbf{1}$ to be the vector of all ones (the size of $\mathbf{1}$ will be clear from the context of the problem; in this case, it has $n$ ones).

$$1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1-(mx_1+b) \\ y_2-(mx_2+b) \\ \vdots \\ y_n-(mx_n+b) \end{pmatrix}$$

Then in fact, we have

$$\boxed{\sum_{i=1}^{n}(y_i-(mx_i+b))^2 = \underbrace{\|\mathbf{Y}-(m\mathbf{X}+b\mathbf{1})\|^2}.}$$

LC of $X, 1$

We can recognize this as the length of the displacement between the vectors $\mathbf{Y}$ and $m\mathbf{X}+b\mathbf{1}$, where the latter vector lies in $V = \mathrm{span}(\mathbf{X},\mathbf{1})$. Thus, we can rephrase the error minimization problem as: **find the vector in span($\mathbf{X},\mathbf{1}$) that is closest to $\mathbf{Y}$.** Last lecture, we determined that this vector is $\mathbf{Proj}_V\,\mathbf{Y}$. Before we can calculate this vector, we need an orthogonal basis of $V$. From the theorem above, this is $\{\hat{\mathbf{X}},\mathbf{1}\}$, where

estimate

$$\hat{\mathbf{X}} = \mathbf{X} - \underbrace{\mathbf{Proj}_{\mathbf{1}}\,\mathbf{X}}_{\text{proj } x \text{ onto } 1} = \mathbf{X} - \bar{x}\mathbf{1}.$$

$$\frac{x\cdot 1}{\|1\|^2}\cdot 1 = \underbrace{\frac{x_1+x_2+\cdots x_n}{n}}_{\bar{x}\ \uparrow\ \text{avg}}\cdot 1$$

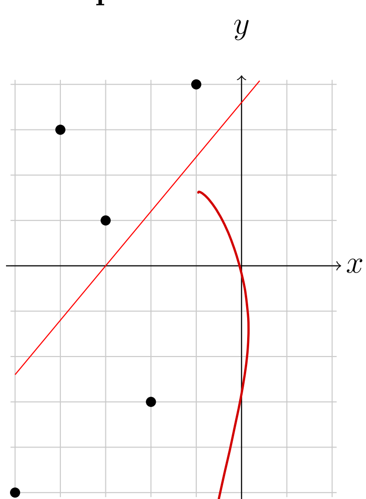
Finally, we conclude

$$\underline{\mathbf{Proj}_V\,\mathbf{Y}}_{= m\hat{x}+b1} = \mathbf{Proj}_{\hat{\mathbf{X}}}\,\mathbf{Y} + \mathbf{Proj}_{\mathbf{1}}\,\mathbf{Y} = \left(\frac{\hat{\mathbf{X}}\cdot\mathbf{Y}}{\hat{\mathbf{X}}\cdot\hat{\mathbf{X}}}\right)\overbrace{\hat{\mathbf{X}}}^{=(\hat{x}-\bar{x}\cdot 1)} + \bar{y}\mathbf{1} = \overbrace{\left(\frac{\hat{\mathbf{X}}\cdot\mathbf{Y}}{\hat{\mathbf{X}}\cdot\hat{\mathbf{X}}}\right)}^{m}\mathbf{X} + \underbrace{\left(\bar{y} - \bar{x}\left(\frac{\hat{\mathbf{X}}\cdot\mathbf{Y}}{\hat{\mathbf{X}}\cdot\hat{\mathbf{X}}}\right)\right)}_{b}\mathbf{1} = m\mathbf{X}+b\mathbf{1}.$$

combine like terms

We will summarize finding $m$ and $b$ with the following example.

**Example 2:** Find the best fit line for the data $(-5,-5)$, $(-4,3)$, $(-3,1)$, $(-2,-3)$, $(-1,4)$.




$$X = \begin{pmatrix} -5 \\ -4 \\ -3 \\ -2 \\ -1 \end{pmatrix},\quad Y = \begin{pmatrix} -5 \\ 3 \\ 1 \\ -3 \\ 4 \end{pmatrix}$$

$$\bar{x} = \frac{-5-4-3-2-1}{5} = -3$$

$$\hat{x} = x - (-3)\mathbf{1} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\mathrm{proj}_V\, Y = \mathrm{proj}_{\hat{x}}\, Y + \bar{y}\,\underline{1}$$

$$\bar{y} = \frac{-5+3+1-3+4}{5} = 0$$

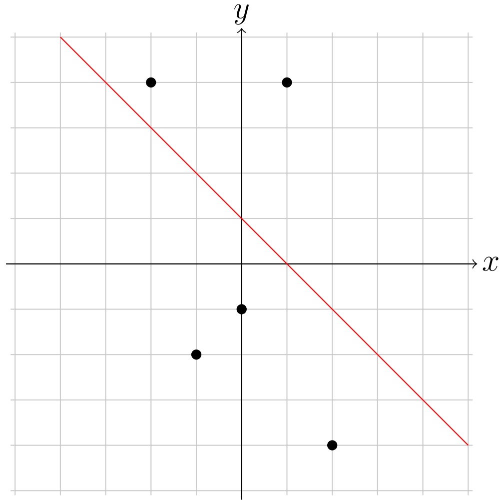
$$= \frac{\hat{x}\cdot y}{\hat{x}\cdot\hat{x}}\,\hat{x} + 0\cdot\mathbf{1}$$

$$= \frac{6}{5}\hat{x} = \frac{6}{5}(x+3\mathbf{1}) = \frac{6}{5}x + \frac{18}{5}\mathbf{1}$$

Line: $\boxed{y = \frac{6}{5}x + \frac{18}{5}}$

$V = \text{span}\{\hat{x}, \mathbb{1}\}$

**Example 3:** Calculate the best fit line for the data $(-2, 4), (-1, -2), (0, -1), (1, 4), (2, -4)$.




$$X = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \qquad Y = \begin{pmatrix} 4 \\ -2 \\ -1 \\ 4 \\ -4 \end{pmatrix}$$

$\bar{x} = 0 \rightarrow \hat{x} = \vec{x}$

↳ if $\bar{x} = 0$, then $\hat{x} = \vec{x}$ → already orthogonal prediction

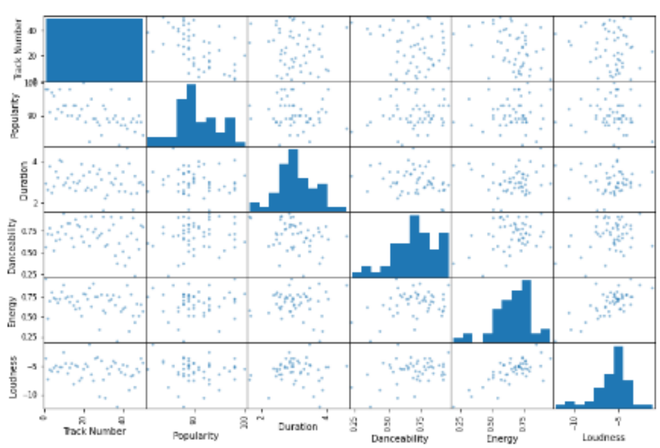
$$\text{proj}_V Y = \frac{\hat{x} \cdot y}{\hat{x} \cdot \hat{x}} \hat{x} + \bar{y}\mathbb{1} = \frac{-10}{10}\hat{x} + \frac{1}{5}\mathbb{1}$$

$$= \boxed{\underbrace{-\vec{x}}_{m} + \underbrace{\tfrac{1}{5}\mathbb{1}}_{b}} \rightarrow \boxed{\begin{array}{l} -x + \frac{1}{5} = y \\ y = -x + \frac{1}{5} \end{array}}$$

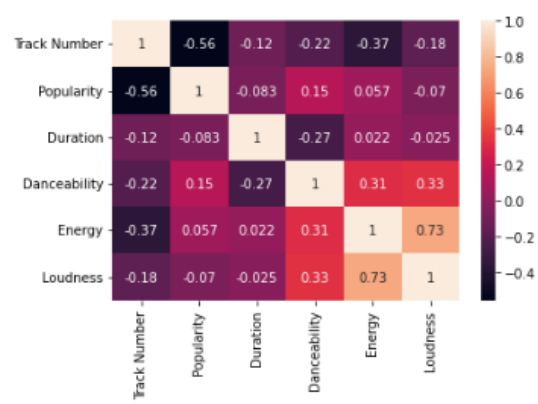
**Example 4:** The data below consists of information from the "Today's Top Hits" playlist on Spotify (10/9/2022). The first 10 songs are shown.

| | Name | Track Number | Popularity | Duration | Danceability | Energy | Loudness |
|---|---|---|---|---|---|---|---|
| **0** | I'm Good (Blue) | 1 | 98 | 2.921 | 0.561 | 0.965 | -3.673 |
| **1** | Unholy (feat. Kim Petras) | 2 | 98 | 2.616 | 0.714 | 0.472 | -7.375 |
| **2** | CUFF IT | 3 | 91 | 3.756 | 0.780 | 0.689 | -5.668 |
| **3** | I Ain't Worried | 4 | 97 | 2.475 | 0.704 | 0.797 | -5.927 |
| **4** | As It Was | 5 | 96 | 2.788 | 0.520 | 0.731 | -5.338 |
| **5** | STAR WALKIN' (League of Legends Worlds Anthem) | 6 | 89 | 3.510 | 0.637 | 0.715 | -4.971 |
| **6** | I Like You (A Happier Song) (with Doja Cat) | 7 | 92 | 3.214 | 0.733 | 0.670 | -6.009 |
| **7** | Tití Me Preguntó | 8 | 98 | 4.062 | 0.650 | 0.715 | -5.198 |
| **8** | Calm Down (with Selena Gomez) | 9 | 91 | 3.989 | 0.801 | 0.806 | -5.206 |
| **9** | Super Freaky Girl | 10 | 93 | 2.850 | 0.950 | 0.891 | -2.653 |

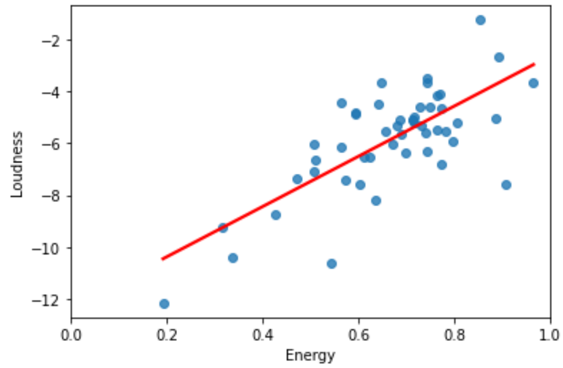
Here is a plot of scatter plots comparing all numerical data. On the right, we have a correlation heatmap, which shows all the correlation coefficients between two features.




r - coeff



The loudness and energy features seem to have the highest correlation (Spotify uses loudness as part of the calculation of the energy stat, so some correlation between them makes sense). Below is their plot zoomed in along with their line of best fit.




Finally, here is some code to calculate the coefficients of the line like we did today, along with how one would do it "on the job."

```
#Compute coefficients
X = np.array(data['Energy'])
Y = np.array(data['Loudness'])

X_hat = X-X.mean()

m = X_hat.dot(Y) / X_hat.dot(X_hat)
b = Y.mean() - X.mean()*m

print(m,b)
```

```
9.66606757985904 -12.300587686834524
```

```
#Let the computer do the work for you
reg = LinearRegression().fit(X[:,np.newaxis],Y[:,np.newaxis])
print (reg.coef_, reg.intercept_)
```

```
[[9.66606758]] [-12.30058769]
```

## Problem 1: Orthogonality and projections

(a) In the span of $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix}$ find a non-zero vector $\mathbf{v}$ orthogonal to $\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$.

$\vec{v} = a\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + b\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix}$ $\quad \vec{v}\cdot\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = 0$

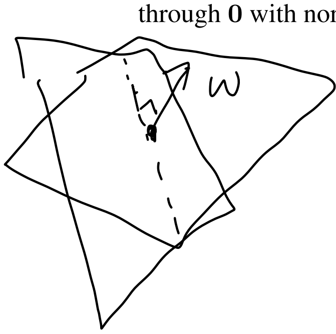
$a(1+2+3-4) + b(1-2+3+4) = 0$

$a(2) + b(6) = 0 \quad 6b = -2a \quad a = -3b$

$\vec{v} = -3b\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + b\begin{bmatrix}1\\-2\\3\\-4\end{bmatrix} = b\begin{bmatrix}-3+1\\-6-2\\-9+3\\-12-4\end{bmatrix}$

$\boxed{\begin{bmatrix}-2\\-8\\-6\\-16\end{bmatrix}}$

(b) Here is a geometric analogue to the algebra in (a): for a plane $P$ through the origin in $\mathbf{R}^3$ and a nonzero 3-vector $\mathbf{w}$ not orthogonal to $P$, why should there always be nonzero vectors in $P$ orthogonal to $\mathbf{w}$? (Hint: visualize the plane $W$ through $\mathbf{0}$ with normal vector $\mathbf{w}$, and think about how it meets the plane $P$).




any vector on the line that's the intersection btwn the planes

(c) Find a nonzero vector $\mathbf{u} \in \mathbf{R}^3$ for which the projections of $\mathbf{v} = \begin{bmatrix}3\\0\\4\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}2\\2\\1\end{bmatrix}$ onto $\mathbf{u}$ are equal. (Recall that the projection of $\mathbf{x}$ onto a nonzero vector $\mathbf{u}$ is given by the formula $\left(\frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$.) There are many answers. Informally, the condition says that $\mathbf{v}$ and $\mathbf{w}$ make the same "shadow" onto the line spanned by $\mathbf{u}$.

$\left(\frac{v\cdot u}{u\cdot u}\right)u = \left(\frac{w\cdot u}{u\cdot u}\right)u$

$\frac{v\cdot u}{\|u\|^2} = \frac{w\cdot u}{\|u\|^2} \quad v\cdot u = w\cdot u \quad \vec{u} = \begin{bmatrix}a\\b\\c\end{bmatrix} \quad 3a+4c = 2a+2b+c$

$a = 2b - 3c \rightarrow$ find random # & it will work

## Problem 2: An orthogonal basis

Let $V$ be the set of vectors $\mathbf{v} \in \mathbf{R}^3$ satisfying $\mathbf{v} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ (this says that both of these explicit 3-vectors have the same projection onto $\mathbf{v}$, or in other words make the same "shadow" onto the line spanned by $\mathbf{v}$).

(a) Express $V$ as the collection of 3-vectors orthogonal to a single nonzero 3-vector.

$$V \cdot \left( \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) = 0 \qquad V \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{0}$$

?. I'm confused

(b) By fiddling with orthogonality equations, build an orthogonal basis of $V$. There are many possible answers.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad a+b+c=0$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 : \begin{aligned} a+b+c&=0 \\ a-b&=0 \\ b&=a \\ c&=-2a \end{aligned} \qquad \begin{pmatrix} a \\ a \\ -2a \end{pmatrix} = a \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

(c) Use your answer to (b) to give an orthonormal basis for $V$.

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$$

## Problem 3: Subspaces defined by orthogonality, orthogonal bases, and shortest distances in $\mathrm{R}^3$

(a) For each linear subspace $V_i$ in $\mathbf{R}^3$ given below, exhibit the set

$$V_i' = \{\mathbf{x} \in \mathbf{R}^3 \mid \mathbf{x} \text{ is orthogonal to every vector in } V_i\}$$

as the span of a finite collection of vectors (so, as a linear subspace), and give a basis for $V_i'$.

(i) $V_1 = \text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right)$

$a+2b+3c=0$

$4a+5b+6c=0$

$2a+b=0$

$b=-2a$

$a+2(-2a)+3c=0$

$-3a+3c=0$

$c=a$

$\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\-2a\\a\end{bmatrix}$

$\left\{ \begin{bmatrix}1\\-2\\1\end{bmatrix} \right\}$

(ii) $V_2$ is the set of solutions in $\mathbf{R}^3$ to the pair of equations $\begin{cases} x_1 + 2x_2 + 3x_3 = 0, \\ 4x_1 + 5x_2 + 6x_3 = 0. \end{cases}$ (*Hint:* relate this to $V_1$ and think geometrically.)

$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$ ? redo this

(b) For each of the two $V_i$'s given above, compute an orthogonal basis for it and *set up* how you'd find the distance from the point $\begin{bmatrix}7\\0\\0\end{bmatrix}$ to $V_i$ (i.e. the minimal distance from $\begin{bmatrix}7\\0\\0\end{bmatrix}$ to a point in $V_i$) using such a basis. Finally, compute each distance.

(*Hint for computation:* first treat the case of $V_2$. For the case of the plane $V_1$, use projections to compute an orthogonal basis and to give an expression for a vector whose length is the distance you want. It gets cumbersome to carry out that distance calculation by hand, so instead compute the distance to $V_1$ by relating it to the distance to $V_2$. Try drawing a picture of an orthogonal line and plane to get an idea.)

# Lecture 8 - Multivariable Functions, Level Sets, and Contour Plots

October 12, 2022

**Goals:** Determine component functions of vector-valued functions, recognize the level sets of a multivariable function, and interpret some basic features of a 2-variable function from its contour plot.

inputs output

**Notation:** $f : \mathbb{R}^n \to \mathbb{R}^m$ means $f$ is a function that takes vectors in $\mathbb{R}^n$ as inputs and gives vectors in $\mathbb{R}^m$ as output.

**Definition:** A **scalar-valued function** is a function $f : \mathbb{R}^n \to \mathbb{R}$, i.e. it gives real number outputs.

**Definition:** A **vector-valued function** is a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, with $m \geq 1$. Such an $\mathbf{f}$ can be expressed in terms of $m$ scalar-valued **component functions** (also referred to as **coordinate functions**) $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\boxed{\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix} = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})).}$$

All of the following notations are equivalent:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \mathbf{f}(x_1, \ldots, x_n)$$

**Example 1:** For each of following functions, identify the size of the vector inputs/outputs.

3 inputs → 1 output

**(a)** $f(x, y, z) = 2x - 3y + z$

$f : \mathbb{R}^3 \to \mathbb{R}$

(scalar-value)

4-vector input

3-vector output

**(b)** $\mathbf{f}(w, x, y, z) = \begin{bmatrix} xw - z \\ y^3 - 4w \\ e^{x+w} \end{bmatrix}$

$f : \mathbb{R}^4 \to \mathbb{R}^3$

(vector-valued)

1 input → 4-vector

**(c)** $f(a) = \begin{bmatrix} 12 \\ \pi \\ a^2 + 2a + 1 \\ |a| \end{bmatrix}$

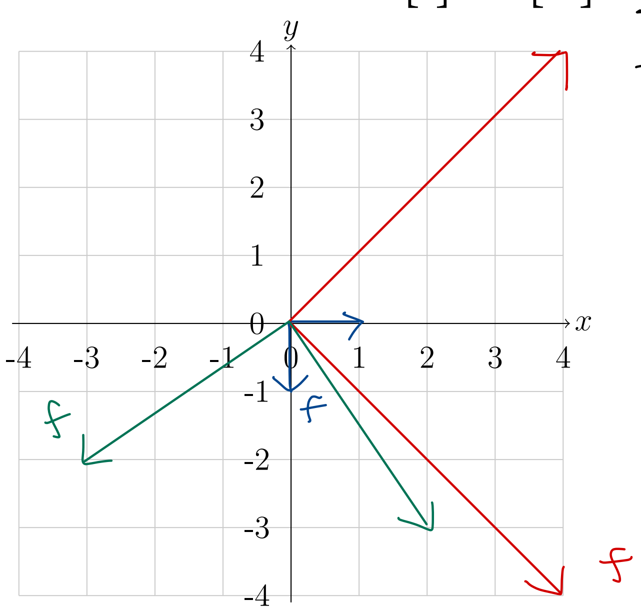
$f : \mathbb{R} \to \mathbb{R}^4$

(vector-valued)

**Example 2:** Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$\mathbf{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}.$$

Calculate $\mathbf{f}$ at the vectors $\mathbf{e}_1$, $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. What is a geometric description of $\mathbf{f}$?




$\vec{f}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\vec{f}\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

$\vec{f}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

If at each input, you draw it's output → slope field

move 90° clockwise

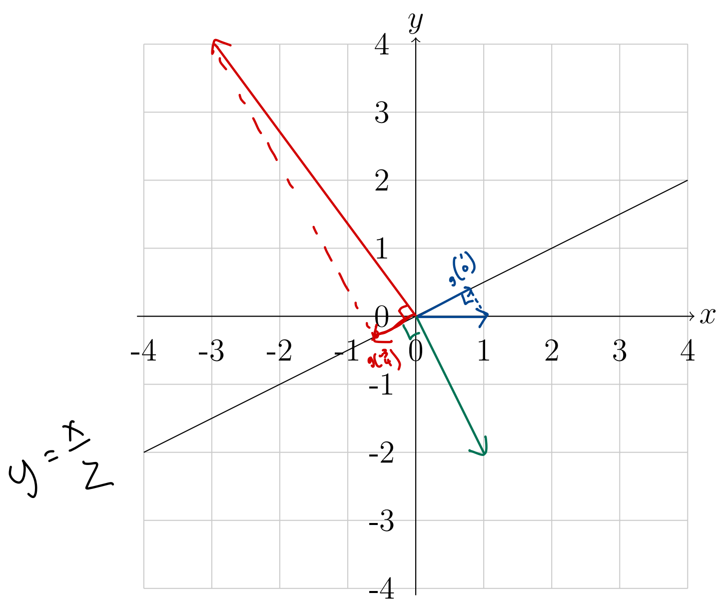
$\vec{f}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

f rotates nonzero vectors 90° clock wise

**Example 3:** Let $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Define the function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$\mathbf{g}(\mathbf{x}) = \mathbf{Proj}_{\mathbf{v}}(\mathbf{x}).$$

Write an explicit formula for $\mathbf{g}$ and evaluate $\mathbf{g}$ at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.




$g(x, y) = \dfrac{\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$= \dfrac{2x+y}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \dfrac{2+0}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \dfrac{2}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix}$

$g\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \dfrac{-6+4}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \dfrac{-2}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4/5 \\ -2/5 \end{pmatrix}$

$g\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \vec{0}$

Much like single variable functions $f : \mathbb{R} \to \mathbb{R}$, we can define **composite functions** for vector-valued functions as well.

fog not defined, got $\mathbb{R}^4 \to \mathbb{R}^3$

fog not desired

$\mathbb{R}^4 \to \mathbb{R}^2$ $\quad$ $\mathbb{R}^2 \to \mathbb{R}^3$

**Definition:** Let $\mathbf{f} : \mathbb{R}^p \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^p$. The **composition function** is denoted by $\mathbf{f} \circ \mathbf{g}$ : $\mathbb{R}^n \to \mathbb{R}^m$

$$(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x}))$$

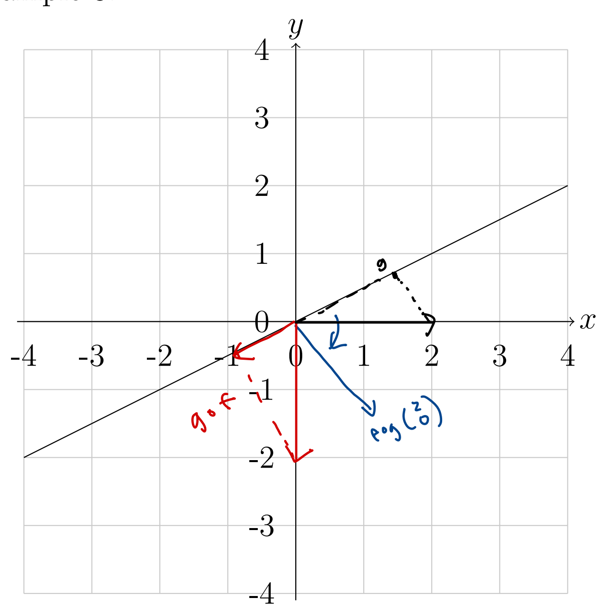
plug g into f

• output of inside must equal input size of outside

Note: The order of composition matters; if the output size of $\mathbf{g}$ doesn't match the input size of $\mathbf{f}$, then $\mathbf{f} \circ \mathbf{g}$ is not even defined.

feeling agreed

**Example 4:** Compute $\mathbf{f} \circ \mathbf{g}$ and $\mathbf{g} \circ \mathbf{f}$, where $\mathbf{f}$ is from Example 2, and $\mathbf{g}$ is from Example 3.




$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$, $g(x,y) = \frac{2x+y}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ $\mathbb{R}^2 \to \mathbb{R}^2$

• $f \circ g(x,y) = f\left(\frac{4x+2y}{5}, \frac{2x+y}{5}\right)$

$\begin{pmatrix} \frac{2x+y}{5} \\ -\left(\frac{4x+2y}{5}\right) \end{pmatrix}$ $\quad$ x = first component

$g \circ f(x,y) = g(y,-x) = \frac{2y-x}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$f \circ g\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -8/5 \end{pmatrix}$ / $g \circ f\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/5 \\ -2/5 \end{pmatrix}$

**Example 5:** We noted that $\mathbf{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}$ from Example 2 rotated vectors by 90° clockwise. What does $\mathbf{f} \circ \mathbf{f}$ do? How many compositions will it take to reach where we started (assuming we started anywhere but the origin)?

f∘f : rotates 180° cw

f∘f∘f = rotates 270° cw

f∘f∘f∘f : does nothing, rotates twice

$f \circ f \circ f \circ f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

**Example 6:** If $f(x,y,z) = (x+y)^2 + z$ and $\mathbf{g}(x,y) = \begin{bmatrix} -x \\ y \\ -(x-y)^2 \end{bmatrix}$, determine which of $\mathbf{f} \circ \mathbf{g}$ and $\mathbf{g} \circ \mathbf{f}$ is defined and compute it.

f∘g is defined

$f : \mathbb{R}^3 \to \mathbb{R}$ $\quad$ $g : \mathbb{R}^2 \to \mathbb{R}^3$

$f \circ g(x,y) = f\begin{pmatrix} -x \\ y \\ -(x-y)^2 \end{pmatrix} = (-x+y)^2 - (x-y)^2 = \boxed{0}$ ✓

$f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, then f∘g & g∘f are defined

You have likely graphed functions $f : \mathbb{R} \to \mathbb{R}$ in some previous math class. Such visual representations take place in $\mathbb{R}^2 = \mathbb{R}^{1+1}$, i.e. one dimension higher than the function's inputs. We can generalize this to the following:

graph(f) = $\{(x, x^2)\}$ $x \in \mathbb{R}$

$f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$

$(x, x^2)$

→ scalar-valued

**Definition:** The **graph of f** : $\mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+1}$ is defined as

$$\text{Graph}(f) = \{(x_1, \ldots, x_n, z) \in \mathbb{R}^{n+1} : z = f(x_1, ..., x_n)\}.$$

↑ such that

Of course, for $n > 2$, it is very challenging to draw these graphs (let me know if you can!), so we will focus on the case $n = 2$; such graphs live in $\mathbb{R}^3$. Even still, it is difficult to draw them without the aid of a computer, so we will instead use a 2D plot to describe features of the graph.

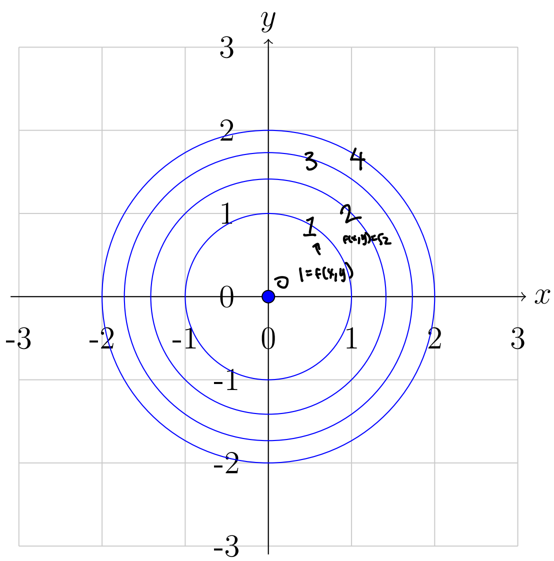
**Definition:** For a function $f : \mathbb{R}^n \to \mathbb{R}$ and any $c \in \mathbb{R}$, the **level set of $f$ at level $c$** is the set of points in $\mathbb{R}^n$ for which $f(x_1, \ldots, x_n) = c$.

ex. $x^2 = 4$ → set of pts is $\{2, -2\}$

In particular, for $f : \mathbb{R}^2 \to \mathbb{R}$, a **contour plot** of $f$ is a picture in $\mathbb{R}^2$ that depicts the level sets of $f$ for many different values of $c$ (typically with a common difference like $c = 0, 1, 2, 3, \ldots$ or $c = .5, 1, 1.5, \ldots$).

**Example 7:** Let $f(x, y) = x^2 + y^2$. (Parabaloid)

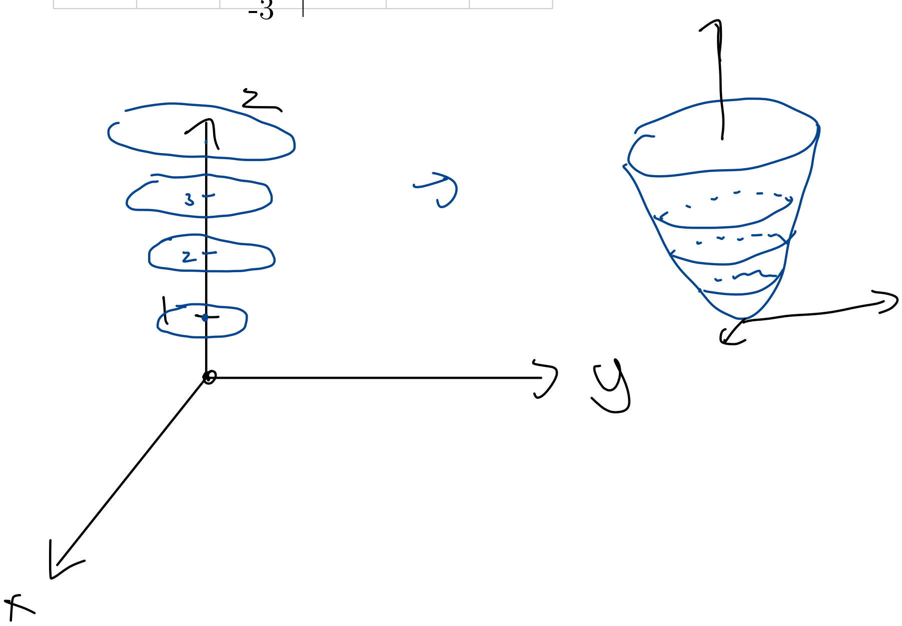
circles get closer & closer together



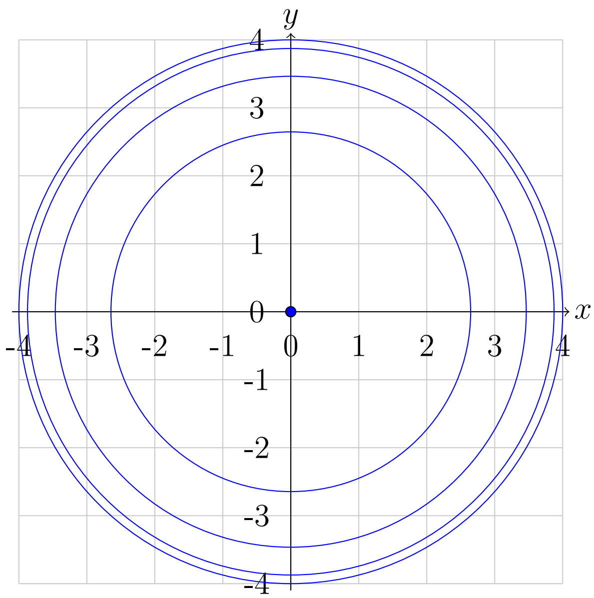

$c \in \mathbb{R}$

$f(x,y) = x^2 + y^2 = c$

- $c < 0$: $x^2 + y^2 = c$ → no pts $\emptyset$
- $c = 0$: $x^2 + y^2 = 0$ → $(0,0)$
- $c > 0$: $x^2 + y^2 = c$ → circle of radius $\sqrt{c}$



**Example 8 (If time):** $f(x,y) = \sqrt{16 - x^2 - y^2}$. (Top half of sphere)




$f(x,y) = C$

- $C < 0 \to$ no solution
- $C = 0 \to f(x,y) = 0 = \sqrt{16-x^2-y^2}$

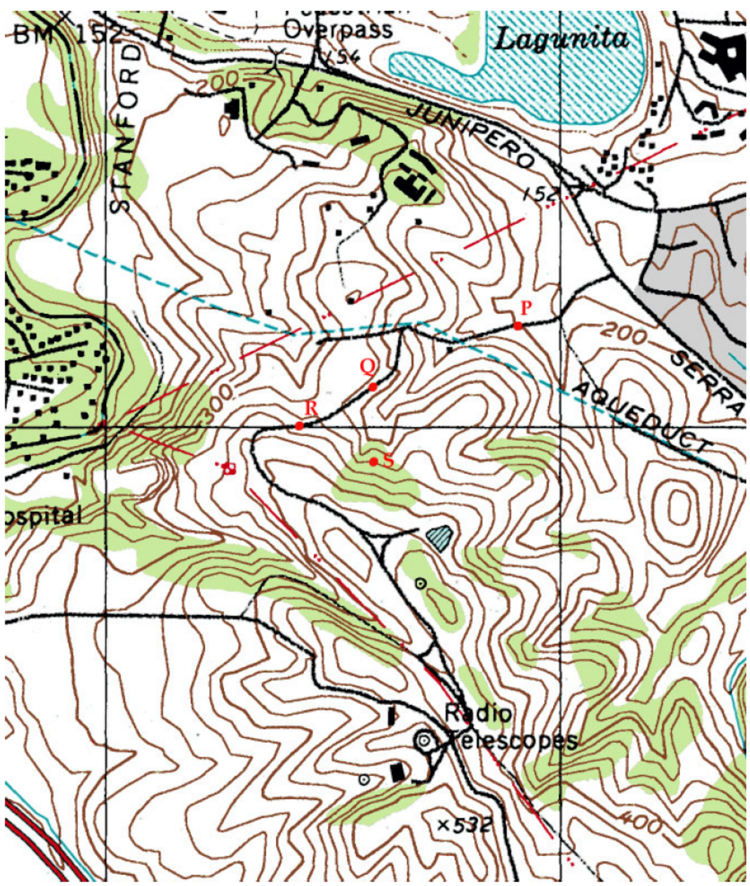
  $\to 0 = 16 - x^2 - y^2$

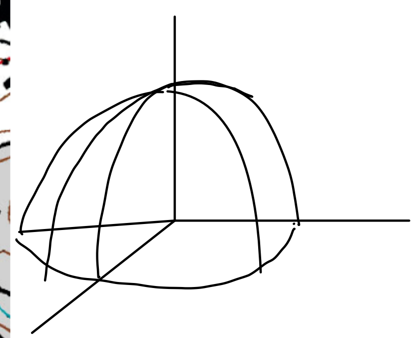
  $\to x^2 + y^2 = 16$

  $\to$ circle w/ $r = 4$
- $C = 4 \to f(x,y) \to 16 - x^2 - y^2 = 4^2$

  $\to (0,0)$

**For fun:** Here's a contour plot of the Stanford Dish.



## Problem 1: A best fit line

The collection of 5 data points $(-1, 6)$, $(0, 3)$, $(1, 0)$, $(2, -3)$, $(3, -4)$ lies close to a line of negative slope; see Figure 1. We are going to compute that line.

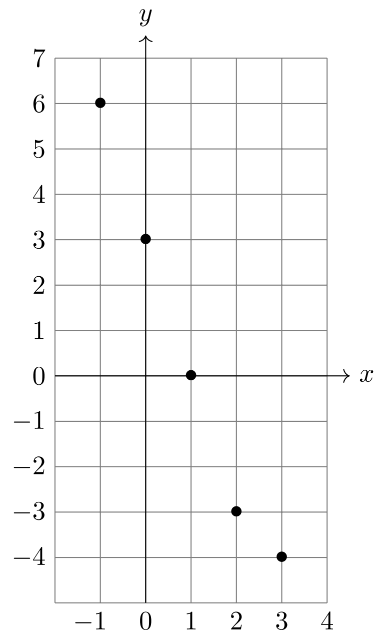


Figure 1: Five data points: $(-1, 6)$, $(0, 3)$, $(1, 0)$, $(2, -3)$, $(3, -4)$.

Suppose the line of best fit (in the least squares sense) is written as $y = mx + b$.

(a) Write down explicit 5-vectors $\mathbf{X}$ and $\mathbf{Y}$ so that for the 5-vector $\mathbf{1}$ whose entries are all equal to 1, the projection of $\mathbf{Y}$ into the plane $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ in $\mathbf{R}^5$ is $m\mathbf{X} + b\mathbf{1}$.

$$X = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad Y = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -3 \\ -4 \end{bmatrix}$$

(b) Compute an orthogonal basis of $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ having the form $\{\mathbf{1}, \mathbf{v}\}$ for a 5-vector $\mathbf{v}$, and find scalars $t$ and $s$ so that $\mathbf{Proj}_V(\mathbf{Y}) = t\mathbf{v} + s\mathbf{1}$.

$$\mathrm{proj}_{1}\vec{x} = \frac{-1+1+2+3}{5}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\vec{x}'$

$$\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \qquad \left\{ 1, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$\mathrm{proj}_V(Y) = \mathrm{Proj}_{1}(\vec{y}) + \mathrm{Proj}_{\vec{v}}(\vec{y}) = \frac{\vec{y}\cdot 1}{5} 1 + \frac{\begin{bmatrix} 6 \\ 3 \\ 0 \\ -3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}}{4+1+1+4} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$s = \frac{6+3-3-4}{5} = \frac{2}{5} \qquad t = \frac{-12-3-3-8}{10} = \frac{-26}{10} = \frac{-13}{5}$$

(c) By expressing $\mathbf{v}$ from (b) as a linear combination of $\mathbf{X}$ and $\mathbf{1}$, use your answer to (b) to find $m$ and $b$ so that the equation $y = mx + b$ gives the line of best fit. (As a safety check on your computations, you may want to plot your line on the above figure to see that it is a good fit for the data.)

$$\frac{-26}{10}\vec{x}' + \frac{2}{5}\vec{1} = \frac{-26}{10}\vec{x} + \frac{26}{10}\vec{1} + \frac{2}{5}\vec{1} = \frac{-26}{10}\vec{x} + 3\vec{1}$$

## Problem 2: Level sets of multivariable functions

(a) Describe and sketch the level sets of $\ln(y - x^2)$ on the region where $y > x^2$, relating each level set to the parabola $y = x^2$.

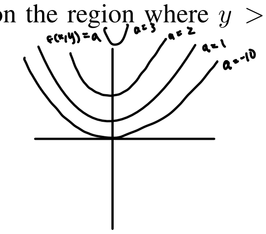
$f(x, y) = \ln(y - x^2)$ $\quad f : \mathbb{R}^N \to \mathbb{R}$

$f(x, y) = a$

$a = 1 \quad \ln(y - x^2) = 1 \quad y - x^2 = e$

$a = 2 \quad \ln(y - x^2) = 2 \quad y - x^2 = e^2$

$a = -10 \quad \ln(y - x^2) = -10 \quad y - x^2 = e^{-10}$



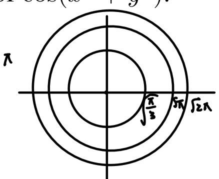

(b) Describe and sketch the level sets of $\cos(x^2 + y^2)$.

$f(x, y) = a$

$a = 1 \quad \cos(x^2 + y^2) = 1 \quad x^2 + y^2 = 0, 2\pi, 4\pi$

$a = \frac{1}{2} \quad \cos(x^2 + y^2) = \frac{1}{2} \quad x^2 + y^2 = \frac{\pi}{3}$

$a = -1 \quad \cos(x^2 + y^2) = -1 \quad x^2 + y^2 = \pi$




(c) Express the surface graph of $f(x, y) = x^2 + y^2$ in $\mathbf{R}^3$ as a level set of a function $h(x, y, z)$.

$f(x, y) = a$

$f(x, y) = z$

graph $f(x, y) = x^2 + y^2$

$z = x^2 + y^2$

$g(x, y, z) = x^2 + y^2 - z \quad a = 0$

$g(x, y, z) = \frac{x^2 + y^2}{z}$ at $a = 1$

(d) (Extra) By using polar coordinates, describe the part of the graph of $f(x, y) = x^2 + y^2$ from (c) that lies over a line in the $xy$-plane through the origin, and use that to sketch the actual surface graph. (Don't "cheat" by looking on a computer; the point is to learn for yourself how to use restriction over well-chosen lower-dimensional subspaces, such as lines through the origin in $\mathbf{R}^2$, to build up a mental model of what happens over the entire domain.)

## Problem 3: Computations with vector-valued functions

For the functions $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ and $\mathbf{g} : \mathbf{R}^m \to \mathbf{R}^p$ below, compute $\mathbf{g} \circ \mathbf{f} : \mathbf{R}^n \to \mathbf{R}^p$ by working out its component functions; in each part also state the values of $n$, $m$, and $p$.

(a) $\mathbf{f}(x, y) = (e^x \cos(y), e^x \sin(y))$, $\mathbf{g}(v, w) = (v^2 - w^2, 2vw)$

(b) $\mathbf{f}(x, y) = (x^2 - y^2, 2xy)$, $\mathbf{g}(v, w) = (e^v \cos(w), e^v \sin(w))$

(c) $\mathbf{f}(t) = (1 - t^2, 2t, 1 + t^2)$, $\mathbf{g}(x, y, z) = x^2 + y^2 - z^2$

$f : \mathbb{R} \to \mathbb{R}^3 \qquad g : \mathbb{R}^3 \to \mathbb{R}$

$g \circ f : \mathbb{R} \to \mathbb{R}$

$g \circ f(t) = g(1 - t^2, 2t, 1 + t^2) = (1 - t^2)^2 + (2t)^2 - (1 + t^2)^2 = 0$

## Problem 4: A linear mathematical model via closest vector and dot products (Extra)

A researcher measures the basal metabolic rate[1], height, and weight for 100 people and expresses the result as vectors:

$$\mathbf{B}, \mathbf{W}, \mathbf{H} \in \mathbf{R}^{100}$$

Here, the $i$th entry of $\mathbf{H}$ is the height of the $i$th person in inches, and similarly for $\mathbf{B}$ (basal metabolic rate in kilocalories per day) and $\mathbf{W}$ (weight in pounds).

The researcher would like to work out a linear formula to estimate the basal metabolic rate in terms of height and weight. In mathematical terms, she would like to find $a, b \in \mathbf{R}$ for which

$$a\mathbf{H} + b\mathbf{W} \text{ is as close to } \mathbf{B} \text{ as possible.}$$

(a) Suppose that the vectors were in $\mathbf{R}^3$ rather than $\mathbf{R}^{100}$. Draw a picture to explain why the $a, b$ we are looking for must satisfy

$$\mathbf{B} - (a\mathbf{H} + b\mathbf{W}) \text{ is perpendicular to } \mathbf{H}, \mathbf{W}.$$

(We know this is true in $\mathbf{R}^{100}$ by the Orthogonal Projection Theorem; the point is to understand it intuitively with a picture in $\mathbf{R}^3$.)

(b) Use the orthogonality as discussed in (a) (which must hold for 100-vectors) and dot products to write down a system of linear equations for $a, b$ (whose coefficients involve dot products among 100-vectors).

(c) The researcher computes that $\mathbf{H} \cdot \mathbf{H} = 1/2, \mathbf{W} \cdot \mathbf{W} = 3$ and $\mathbf{H} \cdot \mathbf{W} = 3/2$; also $\mathbf{B} \cdot \mathbf{W} = 3$ and $\mathbf{B} \cdot \mathbf{H} = 2$. Using the vanishing of dot products against $\mathbf{H}$ and $\mathbf{W}$ arising from (a), solve for $a$ and $b$. (In the real world, such dot products would usually be "ugly" numbers; we made them clean, as we do on exams, so the answer comes out cleanly without using a calculator.)

Observe that the solution did not require knowledge of the 100-element vectors– just knowledge about their dot products! (Of course, to *compute* those dot products one has to know the 100-vectors, but the point is that the *only* way the knowledge of the 100-vectors is relevant is solely to compute those dot products.)

[1] rate at which the body uses energy, measured in kilocalories per day, if the person is at rest