**Exercise 12.3.** Let $f(x, y, z) = xz + yz$.

(a) Using the method of Lagrange multipliers, show that any global extrema for $f$ on the surface $x^2 + y^2 - 4z^2 = 1$ (called a "hyperboloid" because it cuts each plane $y = c$ in a hyperbola) must be among two possible points. (Be careful about division by 0, and don't forget to account for the possibility of points where the constraint equation has vanishing gradient.)

(b) Evaluating $f$ at both points you found in (a), exhibit a point on the hyperboloid where $f$ has a value larger than those two values, and another point where $f$ has a value smaller than those two values. Why does this imply that $f$ has *no* global extrema on the hyperboloid?

a) Let $g(x) = x^2 + y^2 - 4z^2$ :

$$\nabla f = \begin{bmatrix} z \\ z \\ x+y \end{bmatrix} \qquad \nabla g = \begin{bmatrix} 2x \\ 2y \\ -8z \end{bmatrix}$$

i) We first solve for when $\nabla g$ vanishes

$$\nabla g = 0 \rightarrow 2x = 0,\ x = 0\ ;\ 2y = 0,\ y = 0\ ;\ -8z = 0,\ z = 0$$

The gradient vanishes at $(0,0,0)$, but it's not on the curve $g(x,y,z) = 1$.

ii) We now solve for when $\nabla f = \lambda \nabla g$

$$\nabla f = \lambda \nabla g$$

$$\begin{bmatrix} z \\ z \\ x+y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ -8z \end{bmatrix} \rightarrow \begin{cases} z = \lambda(2x) \\ z = \lambda(2y) \\ x+y = \lambda(-8z) \end{cases}$$

These three equations give us the following triple equality

$$\lambda = \frac{z}{2x} = \frac{z}{2y} = \frac{x+y}{-8z}$$

Using the first equality, we can get

$$\frac{z}{2x} = \frac{z}{2y} \rightarrow (2y)z = (2x)z \rightarrow y = x$$

However, if we check the second equality with this, we get

$$\frac{z}{2x} = \frac{x+x}{-8z} \rightarrow -8z^2 = 4x^2 \rightarrow z^2 = -\frac{1}{2}x^2$$

This is impossible, since $x \neq 0$. Therefore, there are no points here.

We now check for when the denominators equal zero.

Case ①: $2x = 0 \rightarrow x = 0 \rightarrow y = 0$. Using the constraint with this,

$$g(0,0,z) = -4z^2 = 1 \rightarrow z^2 = -\frac{1}{4} \rightarrow \text{no real solutions}$$

Case ②: $2y=0 \to y=0 \to x=0$. Using the constraint with this,

$$g(0,0,z) = -4z^2 = 1 \to z^2 = -\tfrac{1}{4} \to \text{no real solutions}$$

Case ③: $-8z=0 \to z=0$. Therefore, $x+y=0 \to y=-x$.

$$g(x,-x,0) = x^2+(-x)^2 = 1 \to x^2 = \tfrac{1}{2} \to x = \pm\tfrac{1}{\sqrt{2}} = \pm\tfrac{\sqrt{2}}{2}$$

Since $y=-x$, possible points are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$

We now check the case where all denominators are non-zero.

$$\lambda = \frac{z}{2x} = \frac{z}{2y} = \frac{x+y}{-8z}$$

We found previously that $y=-x$. We substitute this in.

$$\lambda = \frac{z}{2x}, \quad \lambda = \frac{x-x}{-8z} = \frac{0}{-8z} = 0$$

So there are no points here.

$\boxed{\text{Therefore, the only critical points are } \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \text{ so any global extrema of } f \text{ will be among these two points.}}$

b) The two points were $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$.

$$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(0) = 0$$

$$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}\right)(0) + \left(\frac{\sqrt{2}}{2}\right)(0) = 0$$

We pick a point $(1,1,\frac{1}{2})$. This point is on the hyperboloid because

$$g(1,1,\tfrac{1}{2}) = (1)^2 + (1)^2 - 4\left(\tfrac{1}{2}\right)^2 = 2 - 4\left(\tfrac{1}{4}\right) = 2-1 = 1$$

If we plug this into $f$, we get $f(1,1,\frac{1}{2}) = 1$, which is greater than the value of $f$ at both of the points we found.

We pick another point $(1, 1, -\frac{1}{2})$. This point is on the hyperboloid because

$$g(1,1,-\tfrac{1}{2}) = (1)^2 + (1)^2 - 4(-\tfrac{1}{2})^2 = 2 - 4(\tfrac{1}{4}) = 2 - 1 = 1$$

If we plug this into $f$, we get $f(1,1,-\frac{1}{2}) = -1$, which is smaller than the value of $f$ at both of the points we found.

Therefore, since there exists points on the hyperboloid where $f$ attains values greater and smaller than all the critical points, which we found in (a) to be where all the global extrema must be among, $f$ has no global extrema on the hyperboloid.

**Exercise 12.6.** You have \$20 to buy some snacks at the on-campus market. Assume the market is so well-stocked with various sizes of its items that you can essentially buy any quantity (not necessarily integral) of your favorites, within budget of course. Among the choices are ice cream at \$7 per 16 fl.oz. tub, chips at \$3 per 1.5 oz. bag, and candy at \$4 per 3 oz. package. You want to get some of each of these, but you prefer ice cream, candy, and chips (in that order) to everything else available.

Your overall satisfaction with your purchases is measured by the function $U(x, y, z) = x^{4/7}y^{1/7}z^{2/7}$, where $x$ is the quantity (in fluid ounces) of ice cream you bought, $y$ is the quantity (in ounces) of chips you bought, and $z$ is the quantity (in ounces) of candy you bought. Use Lagrange multipliers to set up simultaneous equations (including any constraint equations!) whose solutions are the candidates to maximize your overall satisfaction, given your \$20 budget. You are not being asked to solve these equations. [You should assume $x, y, z > 0$, disregarding the cases $x, y, z = 0$ much as we are pretending that you can make fractional purchase amounts. This is an instance of the "Cobb–Douglas model" mentioned in Example 12.1.1.]

Remark: This is similar to Example 12.2.7, but with three products instead of two and explicit numbers for the prices/wealth. As mentioned in that example, the solution without explicit numbers is more generally useful and not significantly more complicated (although here there are 3 variables rather than 2). But for this problem we're just setting up the equations, not solving them.

We wish to maximize $U(x,y,z) = x^{4/7} y^{1/7} z^{2/7}$ following the constraint $\left(\frac{7}{16}\right)x + \left(\frac{3}{1.5}\right)y + \left(\frac{4}{3}\right)z = 20$.

Suppose $g(x,y,z) = \frac{7}{16}x + 2y + \frac{4}{3}z$. Our constraint is $g(x,y,z) = 20$.

We now compute gradients:

$$\nabla U = \begin{bmatrix} \frac{4}{7} y^{1/7} z^{2/7} x^{-3/7} \\ \frac{1}{7} x^{4/7} z^{2/7} y^{-6/7} \\ \frac{2}{7} x^{4/7} y^{1/7} z^{-5/7} \end{bmatrix} \qquad \nabla g = \begin{bmatrix} \frac{7}{16} \\ 2 \\ \frac{4}{3} \end{bmatrix}$$

Using Lagrange multipliers, we have $\nabla U = \lambda \nabla g$. This means

$$\begin{bmatrix} \frac{4}{7} y^{1/7} z^{2/7} x^{-3/7} \\ \frac{1}{7} x^{4/7} z^{2/7} y^{-6/7} \\ \frac{2}{7} x^{4/7} y^{1/7} z^{-5/7} \end{bmatrix} = \lambda \begin{bmatrix} \frac{7}{16} \\ 2 \\ \frac{4}{3} \end{bmatrix}$$

If we separate this into equations, we get as an equations

$$\frac{4}{7} x^{-3/7} y^{1/7} z^{2/7} = \lambda \left(\frac{7}{16}\right) \rightarrow \lambda = \frac{64 y^{1/7} z^{2/7}}{49 x^{3/7}}$$

$$\frac{1}{7} x^{4/7} y^{-6/7} z^{2/7} = \lambda (2) \rightarrow \lambda = \frac{x^{4/7} z^{2/7}}{14 y^{6/7}}$$

$$\frac{2}{7} x^{4/7} y^{1/7} z^{-5/7} = \lambda \left(\frac{4}{3}\right) \rightarrow \lambda = \frac{3 x^{4/7} y^{1/7}}{28 z^{5/7}}$$

and the constraint $\frac{7}{16} x + 2y + \frac{4}{3} z = 20$. The solutions to these four equations are candidates to maximize our overall satisfaction.

**Exercise 12.11.** Find the maximum and minimum of $f(x, y) = x + 2y$ on the circle $x^2 + y^2 = 5$, and where they are attained.

Let $g(x,y) = x^2 + y^2$. Our constraint is $g(x,y) = 5$.

$$\nabla f = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

We first check for points where $\nabla g = \vec{0}$ on the constraint $g(x,y) = 5$

$$\nabla g = \vec{0} \rightarrow \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \vec{0} \rightarrow \begin{cases} 2x = 0 \rightarrow x = 0 \\ 2y = 0 \rightarrow y = 0 \end{cases}$$

Therefore, $\nabla g = \vec{0}$ at the point $(0,0)$. However, $g(0,0)$ is not on the constraint.

We now check for when $\nabla f = \lambda \nabla g$.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$1 = \lambda 2x \rightarrow \lambda = \frac{1}{2x}$

$2 = \lambda 2y \rightarrow \lambda = \frac{1}{y}$

Case ①: Denominator is zero $\rightarrow 2x = 0 \rightarrow x = 0$

$g(0, y) = 5 \rightarrow 0^2 + y^2 = 5 \rightarrow y = \pm\sqrt{5}$

Possible points are $(0, \sqrt{5})$ and $(0, -\sqrt{5})$

Case ②: Denominator is zero $\rightarrow y = 0$

$g(x, 0) = 5 \rightarrow x^2 + 0^2 = 5 \rightarrow x = \pm\sqrt{5}$

Possible points are $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$

Case ③: Both denominators are non-zero $\rightarrow \frac{1}{2x} = \frac{1}{y} \rightarrow y = 2x$

$g(x, 2x) = 5 \rightarrow x^2 + (2x)^2 = 5 \rightarrow x^2 + 4x^2 = 5 \rightarrow x^2 = 1 \rightarrow x = \pm 1 \rightarrow y = \pm 2$

Possible points are $(-1, -2)$ and $(1, 2)$.

We now check all these points with $f$ to find extrema.

| $(x, y)$ | $f(x,y) = x + 2y$ |
|---|---|
| $(0, \sqrt{5})$ | $2\sqrt{5}$ |
| $(0, -\sqrt{5})$ | $-2\sqrt{5}$ |
| $(\sqrt{5}, 0)$ | $\sqrt{5}$ |
| $(-\sqrt{5}, 0)$ | $-\sqrt{5}$ |
| $(-1, -2)$ | $-5$ ← min |
| $(1, 2)$ | $5$ ← max |

$\boxed{\text{Therefore, the maximum of } f(x,y) \text{ on } g(x,y) = 5 \text{ is } 5 \text{ at } (1,2). \text{ The minimum is } -5 \text{ at } (-1,-2).}$

**Exercise 13.2.** Let $T_{a,b} : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear function given by $T_{a,b}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax \\ by \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ with $a, b > 0$. This exercise uses such linear functions to relate the geometry of a large class of "ellipse" equations to the unit circle.

(a) In terms of scaling, explain the geometric effect of applying $T_{2,1/3}$ to a vector $\mathbf{v} \in \mathbf{R}^2$. Check that this agrees with the outcome of applying $T_{2,1/3}$ to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by drawing some vectors. Then describe in words the effect of $T_{a,b}$ on $\mathbf{R}^2$ in general.

a) Suppose $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $T_{2,1/3}(\vec{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 1/3\, y \end{bmatrix}$. Using this, in terms of scaling, $T_{2,1/3}$ scales $\vec{v}$ by a factor of 2 in the x-direction and by a factor of 1/3 in the y-direction.

$T_{2,1/3}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1/3 \end{bmatrix}$ $\qquad$ $T_{2,1/3}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2/3 \end{bmatrix}$



In $\mathbb{R}^2$ in general, the effect of $T_{a,b}$ is that it turns any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ into $\begin{bmatrix} ax \\ by \end{bmatrix}$. In more general terms, it multiplies the first component of all vectors in $\mathbb{R}^2$ by a, scaling it by a factor of a in the x-direction. It also multiplies the second component of all vectors in $\mathbb{R}^2$ by b, scaling it by a factor of b in the y-direction

(b) Check that the composite functions $T_{1/a,1/b} \circ T_{a,b}$ and $T_{a,b} \circ T_{1/a,1/b}$ from $\mathbf{R}^2$ to itself are each equal to the identity function that keeps each 2-vector in place. (In other words, evaluate each composition on a general vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and check the final output coincides with the input.) This says that each of $T_{a,b}$ and $T_{1/a,1/b}$ undoes the effect of the other; explain in words why this should hold by using your answer to (a).

b) Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be a general vector in $\mathbb{R}^2$.

$$\begin{aligned} T_{1/a,1/b} \circ T_{a,b} &= T_{1/a,1/b}\left(T_{a,b}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) \\ &= T_{1/a,1/b}\left(\begin{bmatrix} ax \\ by \end{bmatrix}\right) \\ &= \begin{bmatrix} (1/a)\,ax \\ (1/b)\,by \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

$$\begin{aligned} T_{a,b} \circ T_{1/a,1/b} &= T_{a,b}\left(T_{1/a,1/b}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) \\ &= T_{a,b}\left(\begin{bmatrix} x/a \\ y/b \end{bmatrix}\right) \\ &= \begin{bmatrix} ax/a \\ by/b \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

In (a), we found that $T_{a,b}$ scales a vector by a in the x-direction and by b in the y-direction. The compositions above have us scaling the vector and then immediately scaling it by the inverse. In other words, a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is first scaled by either a or $1/a$ in the x-direction and b or $1/b$ in the y-direction, depending on which transformation comes first. It is then scaled by $1/a$ or a in the x-direction and $1/b$ or b in the y-direction. The combination of these produces a scaling factor of 1, as $(1/a)(a) = (1/b)(b) = 1$. As such, the vector is kept in place.

(c) Let $C$ be the curve with equation $x^2 + y^2 = 1$ (unit circle with center at the origin), and for $a, b > 0$ let $E_{a,b}$ be the curve with equation $x^2/a^2 + y^2/b^2 = 1$ (an ellipse; the optional Appendix C shows the equivalence with the ancient Greek definition, for those who are curious).

Check that if a point $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ lies on $C$ then the point $T_{a,b}(\mathbf{v}) = \begin{bmatrix} av_1 \\ bv_2 \end{bmatrix}$ lies on $E_{a,b}$. Also check the reverse property that if $T_{a,b}(\mathbf{v}) \in E_{a,b}$ then $\mathbf{v} \in C$. (Put together, these say $E_{a,b}$ is exactly the output of applying the linear function $T_{a,b}$ to the circle $C$.)

c) If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ lies on $C$, then $v_1 = x$, $v_2 = y$, and $v_1^2 + v_2^2 = 1$.

Consider the point $T_{a,b}(\vec{v}) = \begin{bmatrix} av_1 \\ bv_2 \end{bmatrix}$. Suppose this point were on the curve $E_{a,b}$. We plug in the components to get

$$\frac{(av_1)^2}{a^2} + \frac{(bv_2)^2}{b^2} = 1 \longrightarrow \frac{a^2 v_1^2}{a^2} + \frac{b^2 v_2^2}{b^2} = 1$$

Since $a, b > 0$, we can cancel them to get $v_1^2 + v_2^2 = 1$. We know this equation is true since $\vec{v}$ lies on $C$. Therefore, if $\vec{v}$ is on $C$, then $T_{a,b}(\vec{v})$ is on $E_{a,b}$.

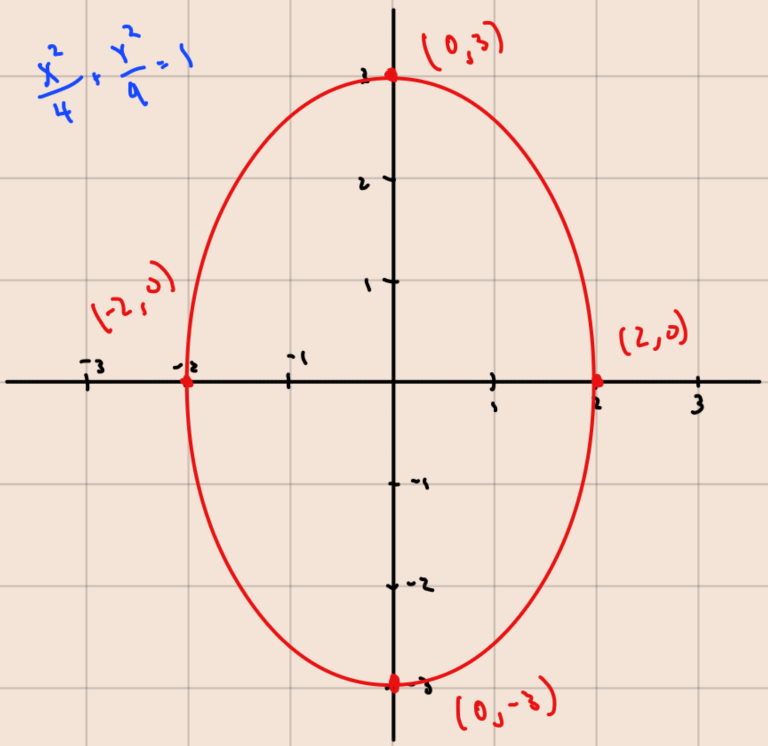
Similarly, if $T_{a,b}(\vec{v}) = \begin{bmatrix} av_1 \\ bv_2 \end{bmatrix}$ lies on $E_{a,b}$, then $x = av_1$, $y = bv_2$, and

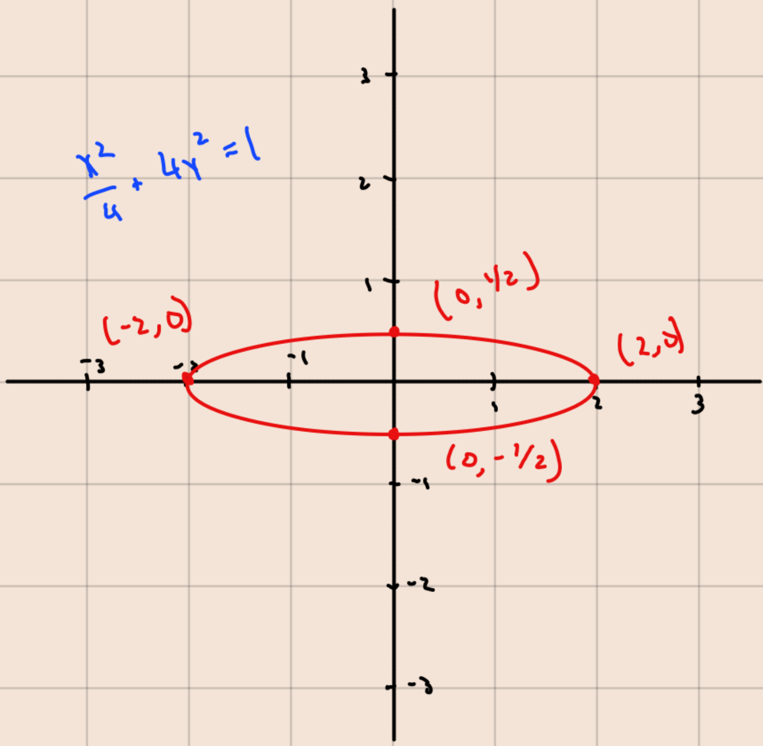
$$\frac{(av_1)^2}{a^2} + \frac{(bv_2)^2}{b^2} = 1 \longrightarrow \frac{a^2 v_1^2}{a^2} + \frac{b^2 v_2^2}{b^2} = 1 \longrightarrow v_1^2 + v_2^2 = 1.$$

Consider the point $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Suppose this point were on the curve $C$.

We plug in the components to get $v_1^2 + v_2^2 = 1$. We know this equation is true since $T_{a,b}(\vec{v}) = \begin{bmatrix} av_1 \\ bv_2 \end{bmatrix}$ lies on $E_{a,b}$, and plugging it in produces the same equation. Therefore, $\vec{v}$ must also lie on $C$.

(d) Using (c) and the effect of $T_{a,b}$ as described in part (a), sketch the curves $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ ($a = 2$, $b = 3$) and $\dfrac{x^2}{4} + 4y^2 = 1$ ($a = 2$, $b = 1/2$). Indicate where each crosses the coordinate axes.




**Exercise 13.3.** This exercise uses algebra and thinking (more instructive than a computer) to determine some geometry of an ellipse from its equation $Ax^2 + By^2 = C$. This will be useful later when relating contour plots to the multivariable second derivative test. You do *not* need a calculator for this exercise; human brainpower is sufficient!

(a) For each ellipse in (i)-(iv) below, compute: where it crosses the $x$-axis (corresponding to $y = 0$), where it crosses the $y$-axis (corresponding to $x = 0$), and which consecutive integers each axis intercept lies between. (Note: if a number lies between consecutive squares $n^2$ and $(n+1)^2$ then its square root lies between $n$ and $n+1$.)

(i) $x^2 + 6y^2 = 10$
(ii) $3x^2 + 5y^2 = 13$
(iii) $7x^2 + 2y^2 = 18$
(iv) $5x^2 + y^2 = 21$

a) (i) $x^2 + 6(0)^2 = 10 \rightarrow x^2 = 10 \rightarrow x = \pm\sqrt{10}$

↳ x-intercept lies between $-4$ and $-3$ and $3$ and $4$.

$(0)^2 + 6y^2 = 10 \rightarrow y^2 = \frac{5}{3} \rightarrow y = \pm\sqrt{5/3}$

↳ y-intercept lies between $-2$ and $-1$ and $1$ and $2$.

(iii) $7x^2 + 2(0)^2 = 18 \rightarrow x^2 = \frac{18}{7} \rightarrow x = \pm\sqrt{\frac{18}{7}}$

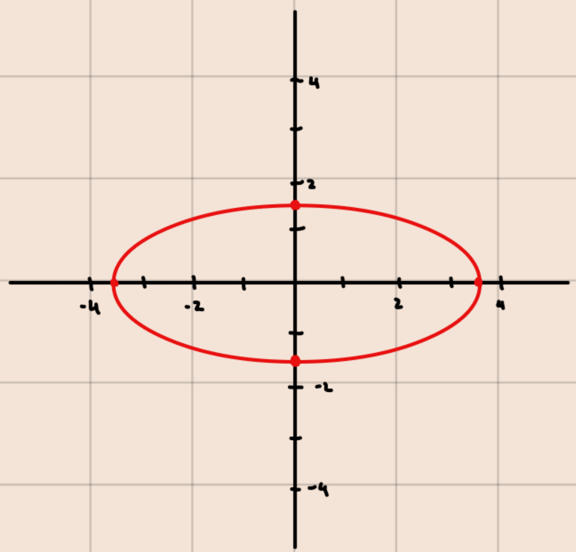
↳ x-intercept lies between $-2$ and $-1$ and $1$ and $2$.

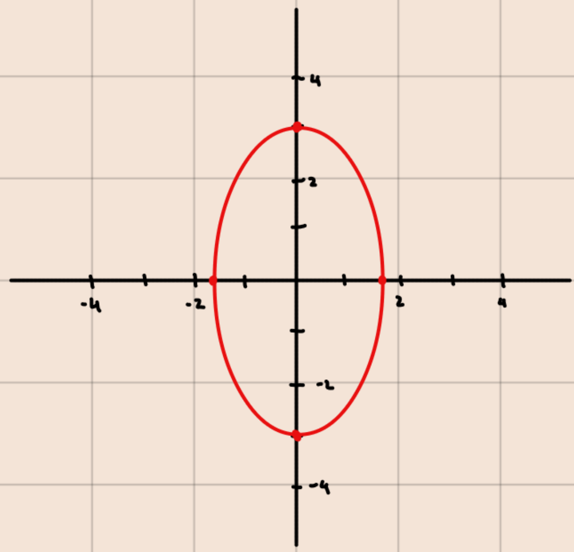
$7(0)^2 + 2y^2 = 18 \rightarrow y^2 = 9 \rightarrow y = \pm 3$

↳ y-intercept lies on $-3$ and $3$

(b) Use the information found in (a) to approximately draw each ellipse on a coordinate grid, indicating along which coordinate axis ($x$ or $y$) the ellipse is longer (a qualitatively correct picture is sufficient).

b) $x^2 + 6y^2 = 10$ $\qquad\qquad$ $7x^2 + 2y^2 = 18$




(c) For a general ellipse $Ax^2 + By^2 = C$ with $A, B, C > 0$, relate which is bigger among $A$ or $B$ to the coordinate direction ($x$ or $y$) along which the ellipse is longer. You are not asked to justify the general pattern, but rather to find one consistent with your pictures in (b).

c) If A is bigger, the ellipse is longer along the y-direction. We see this in the equation $7x^2+2y^2=18$ from part (b). Similarly, if B is bigger, the ellipse is longer along the x-direction. We see this in the equation $x^2+6y=10$ from part (b).

**Exercise 13.6.** For each of the following functions $f : \mathbf{R}^n \to \mathbf{R}^m$, compute $(Df)(\mathbf{x})$ as an $m \times n$ matrix whose entries are functions of $\mathbf{x} \in \mathbf{R}^n$.

(a) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x+y \\ xe^y + y \end{bmatrix}$

$$(Df)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y & x \\ 1 & 1 \\ e^y & xe^y + 1 \end{bmatrix}$$

(b) $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x\sin(y) + ze^x \\ x^2y + yz^3 + z^5x \end{bmatrix}$

$$(Df)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \sin(y) + ze^x & x\cos(y) & e^x \\ 2xy + z^5 & x^2 + z^3 & 3z^2y + 5z^4x \end{bmatrix}$$

**Exercise 13.8.** For each function $f$ below, compute its derivative matrix in general and then at the specified point $\mathbf{a}$. Use the latter to compute the best linear approximation to the function at $\mathbf{a}$ in at least one of two forms: $f(\mathbf{a}+\mathbf{h})$ for $\mathbf{h}$ near $\mathbf{0}$, and $f(\mathbf{x})$ for $\mathbf{x}$ near $\mathbf{a}$.

(b) $f(x,y) = \begin{bmatrix} e^x(x-y)^2 \\ 3xy^2 \end{bmatrix}$, $\mathbf{a} = (0,1)$.

$$(Df)(x,y) = \begin{bmatrix} e^x(x-y)^2 + 2(x-y)e^x & 2e^x(x-y)(-1) \\ 3y^2 & 6xy \end{bmatrix}$$

$$= \begin{bmatrix} e^x(x-y)^2 + 2e^x(x-y) & -2e^x(x-y) \\ 3y^2 & 6xy \end{bmatrix}$$

$$(Df)(0,1) = \begin{bmatrix} e^0(0-1)^2 + 2e^0(0-1) & -2e^0(0-1) \\ 3(1)^2 & 6(0)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$$

Linear Approximation:

$$f(0,1) = \begin{bmatrix} e^0(0-1)^2 \\ 3(0)(1)^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

i) $f(\vec{x}) \approx f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a})$

$$\approx f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\boxed{f(\vec{x}) \approx \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix}}$$

ii) $f(\vec{a}+\vec{h}) \approx f(\vec{a}) + Df(\vec{a})\vec{h}$

$$\boxed{f(\vec{a}+\vec{h}) \approx \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}\vec{h}}$$

**Exercise 13.9.**

(a) Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be the affine function $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for $A = \begin{bmatrix} -2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$. Verify the equality $(Df)(\mathbf{c}) = A$ for every $\mathbf{c} \in \mathbf{R}^3$.

(b) For a general $2 \times 2$ matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and general 2-vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, verify that the function $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ satisfies $(Df)(\mathbf{c}) = A$ for every $\mathbf{c} \in \mathbf{R}^2$ by explicitly computing partial derivatives of the component functions of $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

a) $f(\vec{x}) = A\vec{x} + \vec{b} = \begin{bmatrix} -2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}\vec{x} + \begin{bmatrix} 5 \\ -7 \end{bmatrix}$

Since $\vec{x} \in \mathbb{R}^3$, let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$f(\vec{x}) = \begin{bmatrix} -2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -2x + 3y + z \\ -4x + 0y + 2z \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -2x + 3y + z + 5 \\ -4x + 2z - 7 \end{bmatrix}$$

Suppose $\vec{c} \in \mathbb{R}^3$. Let $\vec{c} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$(Df)(\vec{c}) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$$

We see that this is equal to $A$. Therefore, for any $\vec{c} \in \mathbb{R}^3$, $(Df)(\vec{c}) = A$.

b) $f(\vec{x}) = A\vec{x} + \vec{b} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$= \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a_{11}x + a_{12}y + b_1 \\ a_{21}x + a_{22}y + b_2 \end{bmatrix}$$

Suppose $\vec{z} \in \mathbb{R}^2$. Let $\vec{z} = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$(Df)(\vec{z}) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\boxed{\text{We see that this is equal to } A. \text{ Therefore, for any } \vec{z} \in \mathbb{R}^2, (Df)(\vec{z}) = A.}$

**Exercise 14.2.** For the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, compute the products:

(a) $AB$
(b) $AC$
(c) $(BC)A$ and $B(CA)$.

a)

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1-2 & -1+0 \\ 3-3 & -3+0 \\ 2-1 & -2+0 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & -1 \\ 0 & -3 \\ 1 & -2 \end{bmatrix}}$$

c)

$$(BC)A = \left( \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \left( \begin{bmatrix} 1-1 & 0-2 & 1-2 \\ -1+0 & 0+0 & -1+0 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-6-2 & 0-6-1 \\ -1+0-2 & -2+0-1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -8 & -7 \\ -3 & -3 \end{bmatrix}}$$

$$B(CA) = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1+0+2 & 2+0+1 \\ 1+6+4 & 2+6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 11 & 10 \end{bmatrix}$$

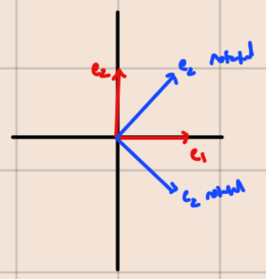
$$= \begin{bmatrix} 3-11 & 3-10 \\ -3+0 & -3+0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -8 & -7 \\ -3 & -3 \end{bmatrix}}$$

**Exercise 14.3.** The operation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates vectors **clockwise** by $45°$ and *then* stretches by 2 along the $x$-direction (i.e., doubles the $x$-coordinate) carries parallelograms to parallelograms, or more specifically is compatible with the parallelogram law, and it interacts equally well with scaling vectors by any scalar, so $T$ is linear. (This is the same as the reasoning in the main text for why any rotation around the origin is linear and hence is given by a $2 \times 2$ matrix.)

Compute in two ways the $2 \times 2$ matrix $A$ that corresponds to $T$:

(a) Find the columns of $A$ directly by using that if $f : \mathbf{R}^2 \to \mathbf{R}^2$ is linear then $f(\mathbf{e}_i)$ is the $i$th column of the corresponding matrix (Proposition 13.4.5), where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbf{R}^2$.
(b) Compute the matrix $R$ for the $45°$ clockwise rotation around the origin and the matrix $M$ for stretching by 2 along the $x$-direction, and then multiply $R$ and $M$ *in the appropriate order*. This should agree with (a).
(c) Explain in words why composing the rotation and stretching operations in the other order should give a different outcome. Multiply $R$ and $M$ in the other order to see what you get.

a) 


Suppose $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A 45° clockwise rotation of $e_1$ brings it to $\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$. A stretch by 2 in the x-direction then brings this to $\begin{bmatrix} \sqrt{2} \\ -\sqrt{2}/2 \end{bmatrix}$.

Now suppose $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A 45° clockwise rotation of $e_2$ brings it to $\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$. A stretch by 2 in the x-direction then brings this to $\begin{bmatrix} \sqrt{2} \\ \sqrt{2}/2 \end{bmatrix}$.

Since $T$ respects the origin, it is a linear function. Following the proposition and our above results, $T(e_1) = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2}/2 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} \sqrt{2} \\ \sqrt{2}/2 \end{bmatrix}$. Therefore, the matrix $A = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$.

b) The rotation matrix for $\mathbb{R}^2$ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. A 45° clockwise rotation is the same as a 315° counterclockwise rotation. We plug this into the equation to get $R = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$. The stretch by 2 in the x-direction makes the matrix $M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. The appropriate order for these is $MR$.

$$MR = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

c) Suppose the vector we want to transform is $\vec{x}$. We first want to rotate this, giving us $R\vec{x}$. Then, we want to stretch it, giving us $M(R\vec{x})$. This is the same as $(MR)\vec{x}$. Therefore, $A = MR$ is the correct order. If we did it in the other order, we would see a stretch first then a rotation, which is not what we want.

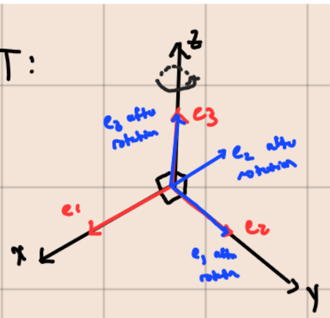
$$RM = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2}/2 \end{bmatrix}$$

This does not agree with (a).

**Exercise 14.5.** Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be rotation around the $z$-axis corresponding to a $90°$ counterclockwise rotation in the $xy$-plane, and $T' : \mathbf{R}^3 \to \mathbf{R}^3$ the rotation around the $x$-axis corresponding to a $90°$ counterclockwise rotation in the $yz$-plane.

(a) Compute the $3 \times 3$ matrix $A$ for $T$ and $A'$ for $T'$. (Hint: the columns are the effect of $T$ and $T'$ on the standard basis; think visually.)

a) T:

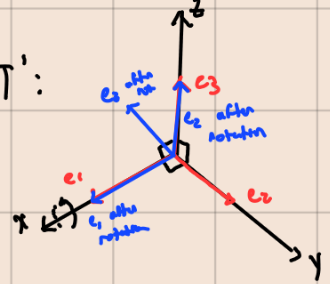



$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}-1\\0\\0\end{bmatrix} \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\0\\1\end{bmatrix}$$

The matrix $A$ for $T$ is

$$\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

T':




$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{T'} \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{T'} \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{T'} \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

The matrix $A$ for $T'$ is

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}$$

(b) Determine the $3 \times 3$ matrix for $T' \circ T$ in two ways: matrix multiplication, and finding its columns by evaluating $(T' \circ T)(\mathbf{e}_j) = T'(T(\mathbf{e}_j))$ for each $j$ via thinking in terms of rotations. Your answers should agree!

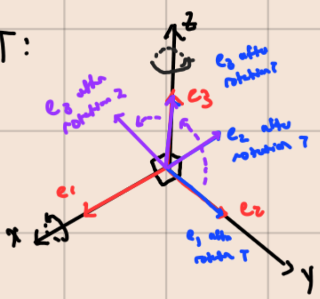
[It turns out that $T' \circ T$ is a $120°$ rotation around the line spanned by $(1, -1, 1) \in \mathbf{R}^3$ (check for yourself with an actual ball, and a pencil or chalk), so your matrix should carry $(1, -1, 1)$ to itself; we aren't asking you to verify this, but that is a safety check on your work.]

b) i) $T' \circ T \to T'T$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0+0+0 & -1+0+0 & 0+0+0\\0+0+0 & 0+0+0 & 0+0-1\\0+1+0 & 0+0+0 & 0+0+0\end{bmatrix}$$

$$T' \circ T = \begin{bmatrix}0 & -1 & 0\\0 & 0 & -1\\1 & 0 & 0\end{bmatrix}$$

ii) T:




$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{T'} \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}-1\\0\\0\end{bmatrix} \xrightarrow{T'} \begin{bmatrix}-1\\0\\0\end{bmatrix} \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{T'} \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

The matrix $A$ for $T' \circ T$ is

$$\begin{bmatrix}0 & -1 & 0\\0 & 0 & -1\\1 & 0 & 0\end{bmatrix}$$

(c) Determine the $3 \times 3$ matrix for $T \circ T'$ in two ways: matrix multiplication, and finding its columns by evaluating $(T \circ T')(\mathbf{e}_j) = T(T'(\mathbf{e}_j))$ for each $j$ via thinking in terms of rotations. Your answers should agree!

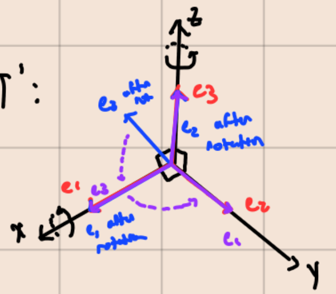
[It turns out that $T \circ T'$ is a $120°$ rotation around the line spanned by $(1,1,1) \in \mathbf{R}^3$ (check for yourself with an actual ball, and a pencil or chalk), so your matrix should carry $(1,1,1)$ to itself; we aren't asking you to verify this, but that is a safety check on your work.]

c) i) $T \circ T' \rightarrow TT'$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+1+0 \\ 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+1 & 0+0+0 \end{bmatrix}$$

$$T \circ T' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

ii) $T'$:




$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{T'} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{T'} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{T'} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The matrix $A$ for $T \circ T'$ is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

**Exercise 14.9.** This exercise uses a linear transformation to relate the curve $H_\pm$ defined by $x^2 - y^2 = \pm 1$ to the hyperbola defined by $xy = \pm 1/2$ (i.e., $y = \pm 1/(2x)$), with a single choice of sign ($\pm$) throughout. Additional linear transformations are then used to work out the geometry of other specific equations of the form $Ax^2 - By^2 = \pm 1$ with $A, B > 0$.

(a) Let $R = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$; the effect of $T_R : \mathbf{R}^2 \to \mathbf{R}^2$ is **clockwise** rotation by $45°$ around the origin. For a point $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, show that $\mathbf{v}$ lies on the hyperbola $xy = 1/2$ precisely when $R\mathbf{v}$ lies on the curve $H_+$ defined by $x^2 - y^2 = 1$. Also show that $\mathbf{v}$ lies on the hyperbola $xy = -1/2$ precisely when $R\mathbf{v}$ lies on the curve $H_-$ defined by $x^2 - y^2 = -1$. (Put together, these say $H_\pm$ is exactly the output of applying the rotation $T_R$ to the graph of the function $\pm 1/(2x)$.)

a) $$R\vec{v} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1/\sqrt{2} + v_2/\sqrt{2} \\ -v_1/\sqrt{2} + v_2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} (v_1+v_2)/\sqrt{2} \\ (v_2-v_1)/\sqrt{2} \end{bmatrix}$$

When $R\vec{v}$ lies on $H_+ : x^2 - y^2 = 1$, we see $\left((v_1+v_2)/\sqrt{2}\right)^2 - \left((v_2-v_1)/\sqrt{2}\right)^2 = 1$

$$\frac{1}{2}\left(v_1^2 + 2v_1v_2 + v_2^2\right) - \frac{1}{2}\left(v_1^2 - 2v_1v_2 + v_2^2\right) = 1$$

$$\cancel{v_1^2} + 2v_1v_2 + \cancel{v_2^2} - \cancel{v_1^2} + 2v_1v_2 - \cancel{v_2^2} = 2$$

$$4v_1v_2 = 2 \longrightarrow v_1v_2 = \frac{1}{2}$$

Therefore, when $R\vec{v}$ lies on $H_+$, $\vec{v}$ lies on $xy = 1/2$ because $v_1v_2 = 1/2$.

When $R\vec{v}$ lies on $H_- : x^2 - y^2 = -1$, we see $\left((v_1+v_2)/\sqrt{2}\right)^2 - \left((v_2-v_1)/\sqrt{2}\right)^2 = -1$

$$\frac{1}{2}\left(v_1^2+2v_1v_2+v_2^2\right) - \frac{1}{2}\left(v_1^2-2v_1v_2+v_2^2\right) = -1$$

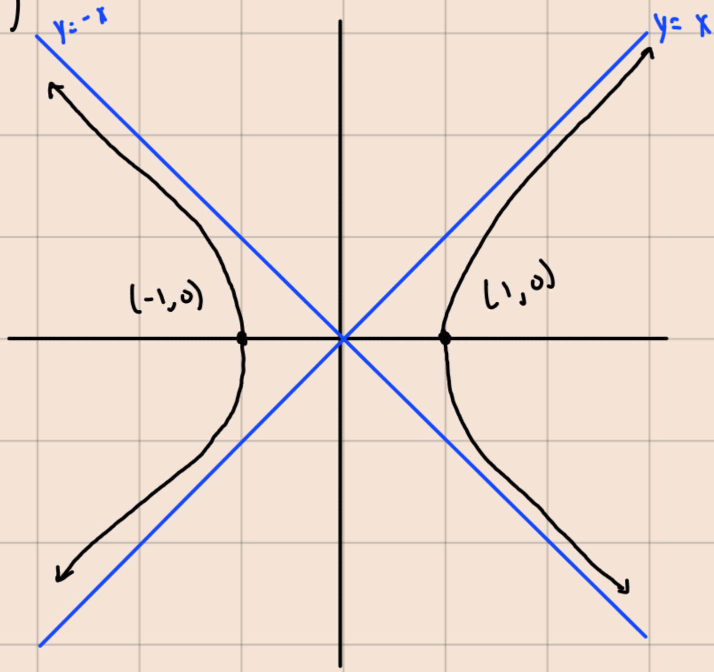
$$\cancel{v_1^2}+2v_1v_2+\cancel{v_2^2} - \cancel{v_1^2}+2v_1v_2-\cancel{v_2^2} = -2$$

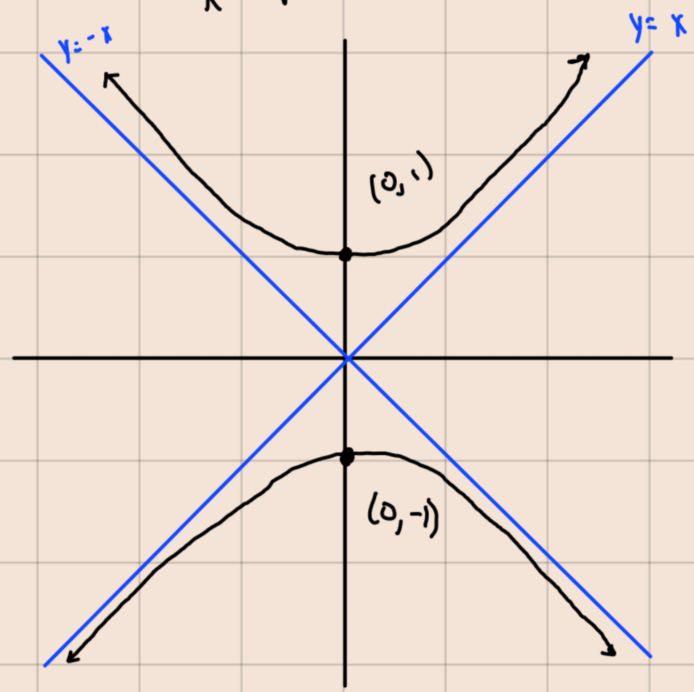
$$4v_1v_2=-2 \longrightarrow v_1v_2=-\frac{1}{2}$$

Therefore, when $R\vec{v}$ lies on $H_-$, $\vec{v}$ lies on $xy=-\frac{1}{2}$ because $v_1v_2=-\frac{1}{2}$.

(b) Use (a) to sketch $H_+$ and $H_-$ on separate grids, indicating where each crosses the coordinate axes. Also use that the coordinate axes are the asymptotes for the graphs of $\pm 1/(2x)$ to explain why the lines $y=\pm x$ are the asymptotes of $H_+$ and also are the asymptotes of $H_-$.

b) $x^2-y^2=1$ $\qquad\qquad$ $x^2-y^2=-1$



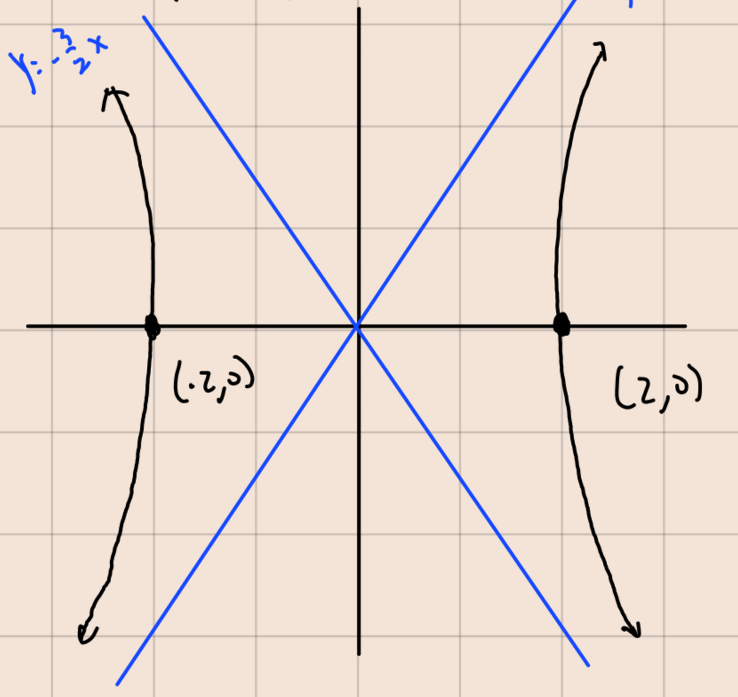

$H_\pm$: $x^2-y^2=1$ is the $45°$ clockwise rotation of $y=\pm 1/(2x)$. Therefore, the asymptotes of $H_\pm$ is also the $45°$ clockwise rotation of the asymptotes of $y=\pm 1/(2x)$.

The $45°$ clockwise rotation of the axes $y=0$, $x=0$ is $y=x$ and $y=-x$. Therefore, $y=\pm x$ are the asymptotes of $H_\pm$.

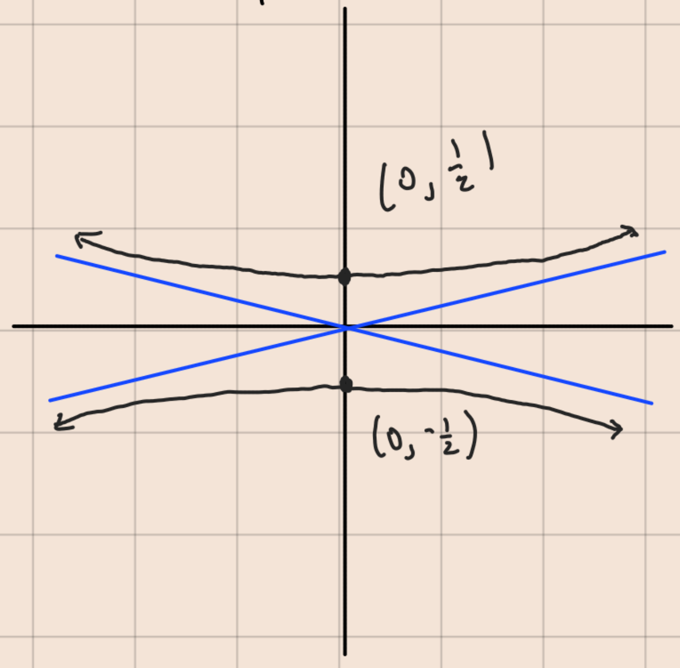
(c) Using considerations with $T_{a,b}$'s as in Exercise 13.2, sketch the graphs of $\frac{x^2}{4}-\frac{y^2}{9}=1$ ($a=2$, $b=3$) and $\frac{x^2}{4}-4y^2=-1$ ($a=2$, $b=1/2$). Indicate where each crosses the coordinate axes by applying suitable $T_{a,b}$'s to your sketches in (b) of $H_+$ and $H_-$ respectively. Also determine and draw the asymptotes for each. (Hint: $T_{a,b}$ carries asymptotes to asymptotes.)

**Remark**. The same method shows that for any $a,b>0$, the curve $x^2/a^2-y^2/b^2=\pm 1$ is obtained from $xy=\pm 1/2$ via applying a $45°$ rotation and suitable scaling in the coordinate directions. (These are hyperbolas in the sense of ancient Greek geometry; for those who are curious, a proof is given in Appendix C.)

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$




$$\frac{x^2}{4} - 4y^2 = -1$$




**Exercise 14.10.** Let $T$ be the triangular region (including interior) with vertices $\mathbf{u} = (0,0)$, $\mathbf{v} = (1,0)$, $\mathbf{w} = (1,2)$ as shown in Figure 14.7.1.

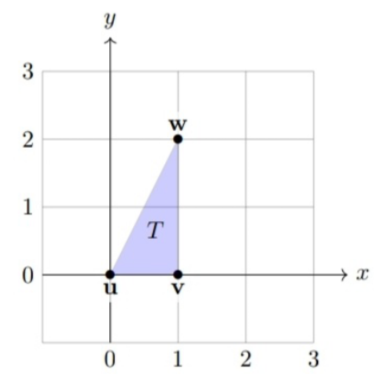



FIGURE 14.7.1. The triangle $T$

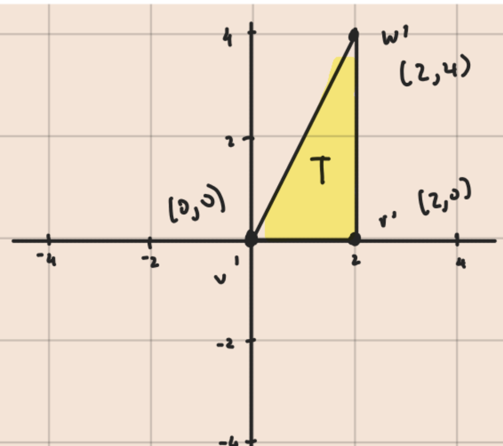
For the linear transformations $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by each of the following matrices, draw the "image" $f(T)$ (meaning the collection of all points $f(\mathbf{x})$ for $\mathbf{x} \in T$; it is the entire output of $f$ on points of the triangle and its interior).

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}$.

Hint: think about how edges and interior points are described via convex combinations of two or three vertices, and then how linear transformations interact with convex combinations.

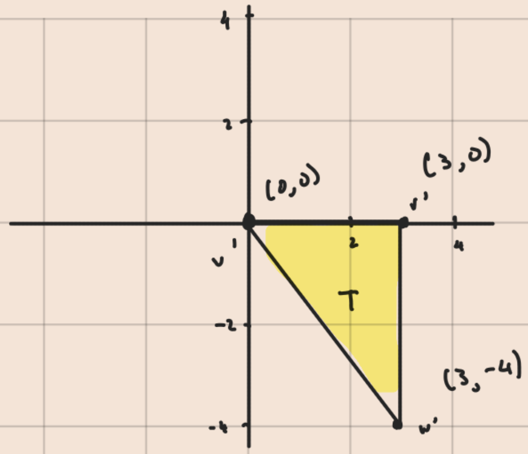
a) $A\vec{w} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \leftarrow w'$

$A\vec{v} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \leftarrow v'$



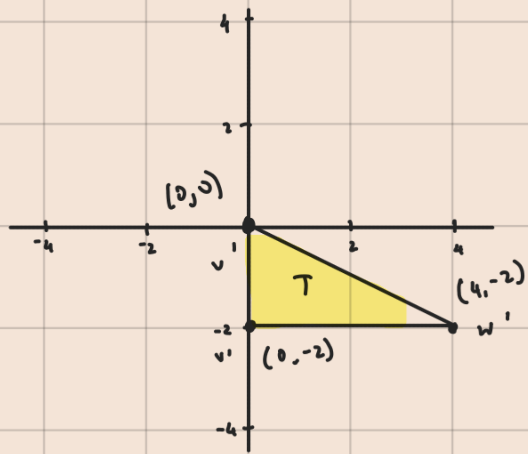
b) $A\vec{w} = \begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}3\\ -4\end{bmatrix} \swarrow w'$

$A\vec{v} = \begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}3\\ 0\end{bmatrix} \swarrow v'$



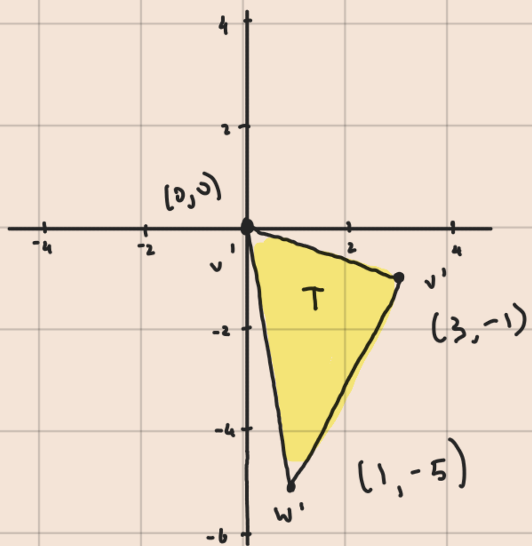

c) $A\vec{w} = \begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}4\\ -2\end{bmatrix} \swarrow w'$

$A\vec{v} = \begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ -2\end{bmatrix} \swarrow v'$




d) $A\vec{w} = \begin{bmatrix}3 & -1\\ -1 & -2\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}1\\ -5\end{bmatrix} \swarrow w'$

$A\vec{v} = \begin{bmatrix}3 & -1\\ -1 & -2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}3\\ -1\end{bmatrix} \swarrow v'$




Time Spent:

- Attending class: 3 hours
- Homework: 9 hours
- Studying: 4 hours

I spent about 16 hours on this class this week.