Lecture 12 - Constrained Optimization via Lagrange Multipliers

October 24, 2022

Goals: Setup the equations for the Lagrange multiplier method for a constrained optimization problem, and use them to determine local extrema.

The types of problems we will be considering are optimization problems of the form: maximize or minimize a function $f : \mathbb{R}^n \to \mathbb{R}$ subject to a **constraint** $g(\mathbf{x}) = c$. In other words, if we restrict our set of points to some level set of another function, what is the max/min of f?

Some examples include a company maximizing revenue subject to certain resource constraints or minimizing the cost for a construction company to build something subject to a budget constraint.

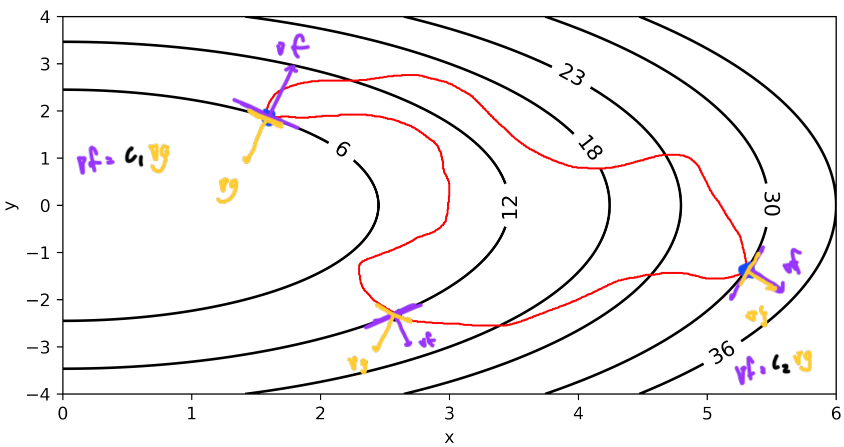
Here's some motivation as to why the method we will discuss works:

contour map of $f(x,y)$ $\qquad$ $g(x,y) = c$

→ gradient of f

→ gradient of g

grad points in direction of maximal increase



what is max val of f that lies on $g(x,y) = c$?

max = 30

min = 6

∇f is only scalar mult of ∇g at extremes

max & min points satisfy $\nabla f = \lambda \nabla g$ (λ is a constant)

<u>**Theorem 12.2.1**</u>: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are functions, and consider the problem of finding a local maximum (or local minimum) of f on the region where $g(\mathbf{x}) = c$ (the constraint region). If a local extremum of f on the constraint region occurs at the point $\mathbf{a}$, then either

lagrange multiplier

- $\nabla g(\mathbf{a}) = \mathbf{0}$, or
- $\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$, where λ is a constant that depends on $\mathbf{a}$.

The result of Theorem 12.2.1 tells us that if we want to solve a constrained optimization problem (i.e. "maximize/minimize f subject to $g = c$"), then we can get a list of possible points by solving

1) • $\nabla g = \mathbf{0}$ for points on the constraint $g = c$, and ∇g vanishes

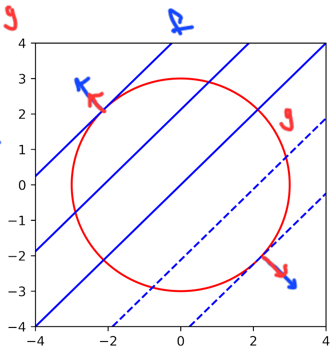
2) • $\nabla f = \lambda \nabla g$.

This is referred to as the method of **Lagrange Multipliers**.

Example 1: Find the maximum value of the function $f(x, y) = -2x + 2y$ subject to $x^2 + y^2 = 9$.

only one curve g

looking at whole contour plot of f



$g(x,y) = x^2 + y^2$, constraint $g(x,y) = 9$

$\nabla f = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $\nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

1) $\nabla g = 0$ @ $(0,0)$, but it's not on curve $g(x,y) = 9$

2) setup equations

$\nabla f = \lambda \nabla g$

$\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \rightarrow \begin{cases} -2 = \lambda 2x \\ 2 = \lambda 2y \end{cases} \rightarrow \begin{cases} -1 = \lambda x \\ 1 = \lambda y \end{cases}$

$\rightarrow -x\lambda = y\lambda$

$\rightarrow \underline{-x = y}$

if $\lambda = 0$, then $\nabla f = 0 \cdot \nabla g = 0$, which can't happen bc ∇f never vanishes

$\therefore \lambda \neq 0$, so we can divide it out

↳ ☆ plug this into constraints

$g(x, -x) = 2x^2 = 9 \rightarrow x = \pm \frac{3}{\sqrt{2}}$

we get points $\underline{(3/\sqrt{2}, -3/\sqrt{2})}$ and $\underline{(-3/\sqrt{2}, 3/\sqrt{2})}$ as possible extrema

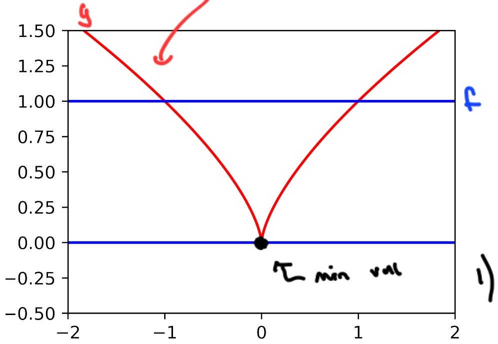
☆ plug into f

$f(3/\sqrt{2}, -3/\sqrt{2}) = -6\sqrt{2}$, $f(-3/\sqrt{2}, 3/\sqrt{2}) = 6\sqrt{2}$

$\boxed{\text{max @ } f = 6\sqrt{2}}$

Example 2: Find the minimum value of $f(x, y) = y$ subject to $x^2 - y^3 = 0$.

0 level curve of $g(x,y)$



$g(x,y) = x^2 - y^3$, constraint $g(x,y) = 0$

$\nabla f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\nabla g = \begin{bmatrix} 2x \\ -3x^2 \end{bmatrix}$

1) ∇g vanishes @ $(0,0)$ which is on the curve

2) $\nabla f = \lambda \nabla g$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ -3x^2 \end{bmatrix} \longrightarrow \begin{cases} 0 = 2\lambda x \\ 1 = -3y^2\lambda \end{cases}$ → either $x = 0$, $g(0,y) = -y^3 = 0$, $y = 0$ ↓ $1 = -3y^2\lambda$, $1 = 0$ or $\lambda = 0$ then $\nabla f = 0 \cdot \nabla g = 0$ not possible bc $\nabla f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, never vanishes ↓ $1 = 0$

* therefore, there is no λ s.t. $\nabla f = \lambda \nabla g$

Check: only pt is origin → how do you know max or min?
$f(0,0) = 0$
↳ pick another point on $x^2 - y^3 = 0$
↳ $(1,1) \to f(1,1) = 1 > 0$ ∴ 0 is min

Example 3: Find the points on the curve (constraint) $8y^2 - 4x^3 + x^4 = 0$ closest to the point $(3, 0)$, and then compute that minimal distance.

* $g(x,y) = 8y^2 - 4x^3 + x^4$, constraint $g(x,y) = 0$

→ minimizing distance between $(3,0)$ and a point (x,y)

→ minimizing $d(x,y) = \sqrt{(x-3)^2 + y^2}$ (distance formula)

↳ fact! max/min d^2 gets same point as d) ← is this for all functions? all functions w/ sqrt?

* $\left(\sqrt{f(x)}\right)' = \dfrac{f'(x)}{2\sqrt{f(x)}} = 0$ ✓

→ we minimize $f(x,y) = (x-3)^2 + y^2$

$\nabla f = \begin{bmatrix} 2(x-3) \\ 2y \end{bmatrix}$, $\nabla g = \begin{bmatrix} -12x^2 + 4x^3 \\ 16y \end{bmatrix}$

$\nabla g = 0 \to 16y = 0 \to y = 0$

$\to -12x^2 + 4x^3 = 0 \to 4x^2(x-3) = 0 \to x = 0, 3$

→ vanish @ $(0,0)$ and $\cancel{(3,0)}$ → not on curve

$$\nabla f = \lambda \nabla g \rightarrow \begin{cases} 2x-6 = \lambda\,(4x^3 - 12x^2) \\ 2y = \lambda\,(16y) \end{cases}$$

Systematic solution: solve for λ

$$\lambda = \frac{2(x-3)}{4x^3-12x^2} = \frac{2(x-3)}{4x^2(x-3)}\,, \qquad \lambda = \frac{2y}{16y}$$

case ①: $4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$\boxed{x = 0,\ 3}$ plug into constraint

$g(0,y) = 8y^2 = 0$

$y = 0$

$g(3,y) = 8y^2 - 27 = 0$

$y = \pm\sqrt{27/8}$

possible points

$\boxed{(3, \sqrt{27/8})}$

$\boxed{(3, -\sqrt{27/8})}$

$(0,0)$

case ②: denominator is zero

$16y = 0$

$y = 0$

$g(x,0) = x^4 - 4x^3 = 0$

$x^3(x-4) = 0$

$x = 0, 4$

points

$(0,0)$, $\boxed{(4,0)}$

case ③: both denominators are non zero

$$\lambda = \frac{2(x-3)}{4x^2(x-3)} = \frac{1}{2x^2}\,, \qquad \lambda = \frac{2y}{16y} = \frac{1}{8}$$

$$\frac{1}{8} = \frac{1}{2x^2} \rightarrow 4 = x^2 \rightarrow x = \pm 2$$

$g(2,y) = 8y^2 - 16 = 0$, $y = \pm\sqrt{2}$

$g(-2,y) = 8y^2 + 48 = 0$, no solution

possible points

$\boxed{(2, \sqrt{2})}$

$\boxed{(2, -\sqrt{2})}$

Now, check all possible points w/ f to find min value

$f(2, \sqrt{2})$ $\quad f(0,0)$ $\quad f(3, \sqrt{27/8})$

$f(2, -\sqrt{2})$ $\quad f(4,0)$ $\quad f(3, -\sqrt{27/8})$

★ min @ point $(4,0)$

Q: for a given curve g, how do we know which direction the gradient points in?

↳ w/ f, we do direction of maximal increase

Q: still confused how setting denominator to zero gives us possible points. Does it solve for x? What is the reason?

Q: fact: max / min d^2 gets same point as d)

↑ is this for all functions? all functions w/ sqrt?

☆ $\left(\sqrt{f(x)}\right)' = \frac{f'(x)}{2\sqrt{f(x)}} = 0$ ✓