Lecture 19 - Linear Independence and the Gram-Schmidt Process

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Goals: Use the Gram-Schmidt process to compute an orthogonal basis of V and compute $\dim V$.

When discussing a collection of vectors throughout the course, we often used the words "redundancy" or "no redundancy." We now make this more precise:

Definition: For $k > 1$, a collection of vectors $\{\mathbf{v}_1, \ldots, vv_k\}$ in $\mathbb{R}^n$ is called **linearly dependent** if some $\mathbf{v}_i$ belongs to the span of the others. Otherwise, the collection is called **linearly independent** (so no $\mathbf{v}_i$ belongs to the span of the others, or there is "no redundancy").

In the case of a single vector, $\{\mathbf{v}\}$ is linearly dependent if $\mathbf{v} = \mathbf{0}$, otherwise it is linearly independent.

Linear independence has a nice alternative meaning:

Theorem 19.1.5: A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is linearly independent precisely when the **only** collection of scalars $a_1, \ldots, a_k$ for which

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = 0$$

is $a_1 = a_2 = \cdots = a_k = \mathbf{0}$.

Equivalently, the collection of $\mathbf{v}_i$'s is linearly dependent precisely when there is some collection of coefficients not all equal to 0 for which $\sum_{i=1}^{k} a_i\mathbf{v}_i = \mathbf{0}$ (note, we don't need **all** the a_i to be nonzero, just some a_i).

Example 1: Determine whether or not the given collection of vectors is linearly dependent or linearly independent.

- $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

$\hookrightarrow \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = 0 \to a_1 = a_2 = a_3 = 0 \to$ linearly independent

- $\mathbf{v}_1 = \begin{bmatrix}2\\1\\3\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\2\\1\end{bmatrix}$

$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

$\begin{pmatrix}2a_1\\a_1 + a_2 + 2a_3\\3a_1 + a_3\end{pmatrix} = 0 \to \begin{matrix}a_1 = 0\\a_2 + 2a_3 = 0\\a_3 = 0\end{matrix} \to a_2 = 0 \to a_1 = a_2 = a_3 = 0$

$\hookrightarrow$ linearly independent

- $\mathbf{v}_1 = \begin{bmatrix}2\\0\\1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-1\\3\\0\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}-2\\0\\0\\1\end{bmatrix}$

$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$

$\begin{bmatrix}2a_1 - a_2 - 2a_4\\3a_2\\a_1 + a_3\\a_3 + a_4\end{bmatrix} = 0 \to a_2 = 0 \to 2a_1 = 2a_4 \to a_3 = -a_4$

$a_1 = -a_3 \to$

$\to$ you can pick any a_4, so infinitely many solutions

linearly dependent bc there exists $a_4 \neq 0$

$\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} = \begin{bmatrix}a_4\\0\\-a_4\\a_4\end{bmatrix} = a_4\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$

- If A is invertible, then this system has a *unique* solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$. In particular, $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as its only solution.

Example 2: How does invertibility of an $n \times n$ matrix relate to linear dependence/independence?

$\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n \longrightarrow a_1 v_1 + \dots + a_n v_n = 0$

For $A\vec{x} = 0$, A is invertible when there is one solution

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \vec{0}$$

$\Rightarrow$ A is invertible precisely when its columns are linearly independent

Orthogonal Bases and Gram-Schmidt

In Chapter 7, we learned how to find an orthogonal basis for span$(\mathbf{v}_1, \mathbf{v}_2)$ when $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of one another. We now go over the procedure for producing an orthogonal basis $\mathbf{w}_1, \mathbf{w}_2, \dots$ for the span of any number of vectors $\mathbf{v}_1 \dots, \mathbf{v}_k$.

The Gram Schmidt Process: Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be **non-zero** n-vectors with span V in $\mathbb{R}^n$. Define:

define subspaces $\longrightarrow$ $V_1 = \text{span}(\mathbf{v}_1)$, $V_2 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$, $V_3 = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), \dots$, $V_k = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$

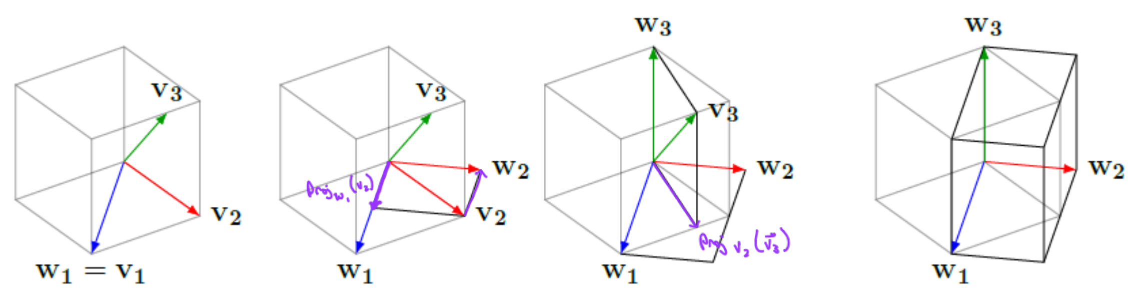
The following algorithm gives an orthogonal basis for the span V of all the $\mathbf{v}_j$'s.

- Let $\mathbf{w}_1 = \mathbf{v}_1$ and define $\mathcal{B}_1$ to be $\{\mathbf{w}_1\}$, which is an orthogonal basis for V_1.
- Let $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{V_1}(\mathbf{v}_2)$.
  If $\mathbf{w}_2 \neq \mathbf{0}$, then $\mathcal{B}_2 := \{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis for V_2.
  If $\mathbf{w}_2 = \mathbf{0}$, then $\mathbf{v}_2 \in V_1$, so $\mathcal{B}_2 := \mathcal{B}_1$ is an orthogonal basis for V_2. $\text{Proj}_{v_1}(v_2) = v_2$, so $B_2 = B_1$, so B_1 is basis for V_2
  ↳ "defined to be"
- Let $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{V_2}(\mathbf{v}_3)$.
  If $\mathbf{w}_3 \neq \mathbf{0}$, then $\mathcal{B}_3 := \mathcal{B}_2 \cup \{\mathbf{w}_3\}$ is an orthogonal basis for V_3.
  If $\mathbf{w}_3 = \mathbf{0}$, then $\mathbf{v}_3 \in V_2$, so $\mathcal{B}_3 := \mathcal{B}_2$ is an orthogonal basis for V_3.

$\text{Proj}_{V_2}(v_3)$ is either $\text{Proj}_{w_1}(v_3) + \text{Proj}_{w_2}(v_3)$, $\text{Proj}_{w_1}(v_3)$ or $\text{Proj}_{w_1}(v_3)$

- Continue this process of considering $\mathbf{w}_j := \mathbf{v}_j - \mathbf{Proj}_{V_{j-1}}(\mathbf{v}_j)$ and whether to add $\mathbf{w}_j$ to $\mathcal{B}_{j-1}$ or not.

After k steps, $\mathcal{B}_k$ will be an orthogonal basis for V. $\longrightarrow$ w_j end up with an orthogonal basis of span$(v_1, \dots, v_k)$



$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}(\vec{v}_2)$
$= \vec{v}_2 - \text{Proj}_{\vec{w}_1}(\vec{v}_2)$

$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}(\vec{v}_3)$
$\text{Proj}_{V_2}(\vec{v}_3) = \text{Proj}_{\vec{w}_1}(\vec{v}_3) + \text{Proj}_{\vec{w}_2}(\vec{v}_3)$

Example 3: Apply the Gram-Schmidt process to the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}.$$

Then, make an orthonormal basis of the span of the $\mathbf{v}_i$'s using your output from Gram-Schmidt.

$$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - \operatorname{Proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - \operatorname{Proj}_{\vec{w}_1}(\vec{v}_2)$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \text{ nonzero}$$

$$\vec{w}_3 = \vec{v}_3 - \operatorname{Proj}_{V_2}(\vec{v}_3) = \vec{v}_3 - \left(\operatorname{Proj}_{\vec{w}_1}(\vec{v}_3) + \operatorname{Proj}_{\vec{w}_2}(\vec{v}_3)\right)$$

$$= \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - \left(\frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 + \frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2 \right)$$

$$= \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \text{ nonzero}$$

Orthogonal basis: $\{w_1, w_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$

orthonormal: $\left\{ \frac{1}{\sqrt{2}} \vec{w}_1, \frac{1}{2} \vec{w}_2, \frac{1}{\sqrt{8}} \vec{w}_3 \right\}$

v_1, v_2, v_3 is linearly independent

Example 4: Compute an orthogonal basis for the span of the following three vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

$$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{w}_2$ points in same dir as $5\vec{w}_2$

$$\vec{w}_2 = \vec{v}_2 - \operatorname{Proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - \operatorname{Proj}_{\vec{w}_1}(\vec{v}_2)$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 1 \\ 4/5 \end{pmatrix} \to \vec{w}_2 = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} \text{ nonzero}$$

$$\vec{w}_3 = \vec{v}_3 - \operatorname{Proj}_{V_2}(\vec{v}_3) = \vec{v}_3 - \left(\operatorname{Proj}_{\vec{w}_1}(\vec{v}_3) + \operatorname{Proj}_{\vec{w}_2}(\vec{v}_3)\right)$$

$$= \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \left(\frac{4}{5} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{9}{45} \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ zero, throw out}$$

Orthogonal basis of $\operatorname{span}(v_1, v_2, v_3)$ is $\{w_1, w_2\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} \right\}$

v_1, v_2, v_3 is linearly dependent

Since the Gram-Schmidt process produces an orthogonal basis for a subspace V, we have the following theorem:

Theorem 19.2.3: If $V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ for k nonzero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$, then $\dim(V)$ equals the number of nonzero $\mathbf{w}_i$'s obtained from the Gram-Schmidt process. Moreover, the following conditions are equivalent:

- $\dim(V) = k$
- all $\mathbf{w}_i$'s are nonzero,
- the collection $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

if $\dim(V)=k$, all $\vec{w}_i$ are nonzero and $\{v_1, \ldots, v_k\}$ are linearly independent

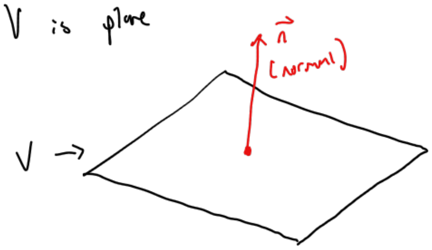
In other words: "basis" and "linearly independent spanning set" for a nonzero subspace V of $\mathbb{R}^n$ mean **exactly the same thing**. Moreover, we now have a way of determining the dimension of the span of more than 3 vectors.

Theorem 19.2.5: if V is a linear subspace of $\mathbb{R}^n$ then the collection $V^\perp$ of n-vectors orthogonal to everything in V. We call $V^\perp$ the **orthogonal complement** of V. It is a linear subspace of $\mathbb{R}^n$ and $\dim V^\perp = n - \dim V$.

$\dim(\mathbb{R}^n)$

Example 5: What is the orthogonal complement of a plane passing through $\mathbf{0}$ in $\mathbb{R}^3$? What is the orthogonal complement of $\mathbb{R}^n$, for any n?

V is plane



$\vec{n}$ is orthogonal to all vectors in V

$\Rightarrow V^\perp = \{0 + t\vec{n},\ t \in \mathbb{R}^2\}$

line w/ direction $\vec{n}$ passing through origin

$\dim V^\perp = n - \dim V = 3 - 2 = 1$

only 1 vector in $V^\perp$

Ex

In $\mathbb{R}^n$: $\vec{v} \cdot \vec{x} = 0$ for any $\vec{x} \in \mathbb{R}^n$

implies $\vec{v} = \vec{0}$

so $(\mathbb{R}^n)^\perp = \{\vec{v}\} = \{\vec{0}\}$

$\dim \{\vec{0}\} = n - n = 0$

$V = \mathbb{R}^n$, so $\dim(V) = n$

$\mathbb{R}^n$ dim in n

Ex 6: $\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ $\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$

$\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$

$$\vec{w}_1 = \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{\vec{w}_1} \vec{v}_2 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix}$$

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{v_2}(\vec{v}_3) = \begin{pmatrix}1\\0\\1\\1\end{pmatrix} - \left(\frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1/2}{3/2}\begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix} \right)$$

$$= \begin{pmatrix}1\\0\\1\\1\end{pmatrix} + \begin{pmatrix}-1/2+1/6\\-1/2-1/6\\-1/3\\0\end{pmatrix}$$

$$= \begin{pmatrix}2/3\\-2/3\\2/3\\1\end{pmatrix}$$