Math 51 Second Exam (Practice #2)

Name: ________________________________ **SUNet ID:** ______________ **ID #:** ____________

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- **You are being examined on mastery of methods taught in this course; solutions based on other techniques do not earn credit.** You may use any result discussed in class or the text, but clearly state the result before using it, and verify that the hypotheses are satisfied.
- Please check that your copy of this exam contains 8 pages of exam questions, *numbered* in the upper-right, and that it is adequately stapled.
- You may use 1 handwritten piece of $8.5'' \times 11''$ paper (both sides) with formulas and other notes as a "reference sheet". No electronic devices, including phones, headphones, or calculation aids, are permitted for any reason.
- **You have 2 hours.** The exam organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by course staff (apart from your own reference sheet) is prohibited. If you need extra room for your answers, use one of the blank pages provided (those pages except for the one at the end are labeled at the bottom by lower-case Roman numerals, starting with "ii"), and **clearly indicate that that your answer continues there**. Do not unstaple or detach pages from this exam.
- It is your responsibility to look over your graded exam in a timely manner. You have until **two weeks after the date of this exam (5pm)**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ____________________________________

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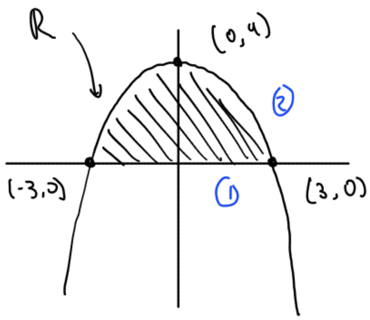
1. (10 points) Let R be the region $0 \le y \le 9 - x^2$, and define $f(x, y) = x^2y - y^2 - 11x^2$. Draw an approximate picture of R (it need not be to scale), and find the maximal and minimal values of f on R, and the points where they are attained.

$$\nabla f(x,y) = \begin{bmatrix} 2xy - 22x \\ x^2 - 2y \end{bmatrix}$$



Interior

$f_x = 0 = 2xy - 22x \rightarrow 2x(y-11) = 0 \rightarrow x = 0$ or $y = 11$

$f_y = 0 = x^2 - 2y \rightarrow x^2 = 2y \rightarrow 0^2 = 2y \Rightarrow y = 0$; $x^2 = 22$, $x = \pm\sqrt{22}$

possible points are $(0,0)$ and $\cancel{(\pm\sqrt{22}, 11)}$

Boundary

Region 1 : $y = 0$, $-3 \le x \le 3$

$f(x, 0) = x^2(0) - (0)^2 - 11x^2 = -11x^2$

$\frac{d}{dx} f(x, 0) = -22x = 0 \rightarrow x = 0$

possible point @ $(0,0)$

Region 2: $y = 9 - x^2$, $-3 \le x \le 3$

$f(x, 9-x^2) = x^2(9-x^2) - (9-x^2)^2 - 11x^2$

$\frac{d}{dx} f(x, 9-x^2) = 2x(9-x^2) + (-2x)(x^2) - 2(9-x^2)(-2x) - 22x = 0$

$= 18x - 2x^3 - 2x^3 + 36x - 4x^3 - 22x = 0$

$= -8x^3 + 32x = 0$

$-8x(x^2 - 4) = 0$

$x = 0, -2, 2$ so

$y = 9, 5, 5$

possible points @ $(0,9)$, $(-2,5)$, $(2,5)$

Corners:

possible points @ $(-3,0)$ and $(3,0)$

Check all with $f = x^2y - y^2 - 11x^2$

$f(0,0) = 0$

$f(0,9) = -81$

$f(\pm 2, 5) = 20 - 25 - 44 = -49$

$f(\pm 3, 0) = -99$

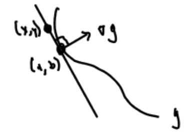
max at 0 @ $f(0,0)$

min at -99 @ $f(3,0)$ and $f(-3,0)$

2. (10 points) Let $g(x, y) = 9x^2 + 4xy + y^2$, and let C be the curve $g(x, y) = 54$ (this is a tilted ellipse with center at the origin, but that isn't needed to answer the questions below).

(a) (3 points) Compute an *equation* for the tangent line to C at the point $(1, 5)$ that lies on C (since $g(1, 5) = 54$). (Your equation should involve x, y and explicit numbers that you compute.)

$$\nabla g(x,y) = \begin{bmatrix} 18x + 4y \\ 4x + 2y \end{bmatrix}$$



The tangent line to g at a point (a,b) is found by

$$\nabla g(a,b) \cdot \begin{bmatrix} x-a \\ y-b \end{bmatrix} = 0 \longrightarrow \nabla g(1,5) \cdot \begin{bmatrix} x-1 \\ y-5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 38 \\ 14 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-5 \end{bmatrix} = 0$$

$$38(x-1) + 14(y-5) = 0$$

$g(x,y) = 9x^2 + 4xy + y^2 = 54$

(b) (8 points) Find the maximal and minimal values for $f(x, y) = xy$ on the curve C, indicating all points where each extremum is attained.

$$\nabla g(x,y) = \begin{bmatrix} 18x + 4y \\ 4x + 2y \end{bmatrix} \qquad \nabla f(x,y) = \begin{bmatrix} y \\ x \end{bmatrix}$$

1) $\nabla g = 0 \longrightarrow \begin{cases} 18x + 4y = 0 \\ 4x + 2y = 0 \end{cases} \longrightarrow \begin{matrix} 4y = -18x \\ 2y = -4x \end{matrix} \longrightarrow \begin{matrix} y = -\frac{9}{2}x \\ y = -2x \end{matrix} \longrightarrow -\frac{9}{2}x = -2x \to$ only true if $x = 0$, so $y = 0$

possible point at $(0,0)$, but it's not on C as $g(0,0) \neq 54$, so we ignore it.

2) $\nabla f = \lambda \nabla g$

$$\begin{bmatrix} y \\ x \end{bmatrix} = \lambda \begin{bmatrix} 18x + 4y \\ 4x + 2y \end{bmatrix} \longrightarrow \left.\begin{matrix} y = \lambda(18x + 4y) \\ x = \lambda(4x + 2y) \\ 9x^2 + 4xy + y^2 = 54 \end{matrix}\right\}$$ we now solve this system of equations

We cross multiply the first two equations

$y\lambda(4x + 2y) = x\lambda(18x + 4y)$

Case 1: $\lambda = 0 \to \begin{matrix} y = 0(18x+4y) = 0 \\ x = 0(4x+2y) = 0 \end{matrix} \to (0,0)$ not on C

Case 2: $\lambda \neq 0 \longrightarrow 4xy + 2y^2 = 18x^2 + 4xy$

$y^2 = 9x^2$

$y = \pm 3x$

We've now eliminated λ, so our equations to solve is

$y = \pm 3x$

$9x^2 + 4xy + y^2 = 54$

case 1: $y = 3x$

$9x^2 + 4x(3x) + (3x)^2 = 54$

$9x^2 + 12x^2 + 9x^2 = 54$

$30x^2 = 54$

$x^2 = \frac{54}{30} = \frac{9}{5}$

$x = \pm \frac{3}{\sqrt{5}}, \ y = \pm \frac{9}{\sqrt{5}}$

case 2: $y = -3x$

$9x^2 + 4x(-3x) + (-3x)^2 = 54$

$9x^2 - 12x^2 + 9x^2 = 54$

$6x^2 = 54$

$x^2 = 9$

$x = -3, 3$

$y = 9, -9$

check points

$f(3/\sqrt{5}, 9/\sqrt{5}) = \frac{27}{5}$

$f(-3/\sqrt{5}, -9/\sqrt{5}) = \frac{27}{5}$

$f(-3, 9) = -27$

$f(3, -9) = -27$

3. (10 points)

(a) (4 points) For $f(x, y, z) = x^y + yz^2$ (with $x > 0$, so x^y makes sense: it means $e^{y\ln(x)}$), compute ∇f and write the best linear approximation to f near $(1, 2, 3)$. (Your approximation should be expressed as a function in x, y, z, involving explicit numbers that you compute.)

$$\nabla f = \begin{bmatrix} e^{y\ln x} \cdot \frac{y}{x} \\ e^{y\ln x} \cdot \ln x + z^2 \\ 2zy \end{bmatrix} = \begin{bmatrix} \frac{x^y y}{x} \\ x^y \ln x + z^2 \\ 2zy \end{bmatrix}$$

$$f(x,y,z) \approx f(1,2,3) + \nabla f(1,2,3) \left(\begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} \right)$$

$$= 1 + 18 + \begin{bmatrix} 2 \\ 9 \\ 12 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix}$$

$$f(x,y,z) \approx 19 + 2(x-1) + 9(y-2) + 12(z-3)$$

(b) (4 points) If $g(x, y)$ is some function which satisfies $g(1, 2) = 4$, $g(1.1, 2.1) = 4.1$, and $g(.9, 2.2) = 3.8$, give estimates for $g_x(1, 2)$ and $g_y(1, 2)$. (Hint: define $a = g_x(1, 2)$ and $b = g_y(1, 2)$ and apply linear approximation for g at $(1, 2)$ with the given numerical values to get two simultaneous equations in a and b that you can then solve. This idea underlies how computers approximate partial derivatives via numerical methods.)

$$g(1.1, 2.1) = 4.1 \approx g(1,2) + g_x(1,2)(1.1-1) + g_y(1,2)(2.1-2)$$
$$4.1 \approx 4 + g_x(1,2)(0.1) + g_y(1,2)(0.1)$$
$$0.1 \approx g_x(1,2)(0.1) + g_y(1,2)(0.1)$$
$$g(0.9, 2.2) = 3.8 \approx g(1,2) + g_x(1,2)(0.9-1) + g_y(1,2)(2.2-2)$$
$$3.8 \approx 4 + g_x(1,2)(-0.1) + g_y(1,2)(0.2)$$
$$-0.2 \approx g_x(1,2)(-0.1) + g_y(1,2)(0.2)$$

$$-0.2 \approx g_x(1,2)(-0.1) + g_y(1,2)(0.2) \quad +$$
$$0.1 \approx g_x(1,2)(0.1) + g_y(1,2)(0.1)$$
$$\overline{-0.1 \approx g_y(1,2)(0.3)}$$
$$g_y(1,2) \approx -\frac{1}{3}$$
$$\frac{1}{10} \approx g_x(1,2)\left(\frac{1}{10}\right) + \left(-\frac{1}{3}\right)\left(\frac{1}{10}\right)$$
$$\frac{4}{30} \approx g_x(1,2)\left(\frac{1}{10}\right) \rightarrow g_x(1,2) \approx \frac{4}{3}$$

(c) (3 points) If $g : \mathbf{R}^2 \to \mathbf{R}^3$ is a vector-valued function which satisfies

$$g(1,2) = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \quad (Dg)(1,2) = \begin{bmatrix} 0 & 4 \\ 1 & 2 \\ -3 & 6 \end{bmatrix}$$

then use linear approximation for g to estimate $g(1.2, 1.9)$; simplify your answer as much as possible.

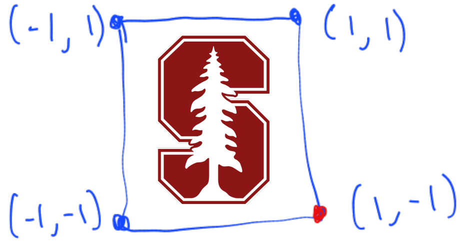
$$g(1.2, 1.9) \approx g(1,2) + (Dg)(1,2)\left(\begin{bmatrix} 1.2-1 \\ 1.9-2 \end{bmatrix}\right)$$

$$\approx \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 1 & 2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.4 \\ 0.2 - 0.2 \\ -0.6 - 0.6 \end{bmatrix}$$

$$\approx \begin{bmatrix} -1.4 \\ 3 \\ -0.2 \end{bmatrix}$$

4. (12 points) (a) (6 points) Consider the effect a linear transformation on the Stanford emblem.



For each of the following matrices M, identify which picture shows the output when M is applied to the Stanford emblem above. 1 point per answer; no justification required and no partial credit.

$\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}1\\-1\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix} \to \begin{bmatrix}1\\1\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$: 3

$\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}-1\\-1\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix} \to \begin{bmatrix}-1\\1\end{bmatrix}$ $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$: 8

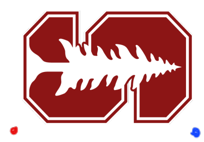
$\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}2\\-1\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix} \to \begin{bmatrix}2\\1\end{bmatrix}$ $\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$: 5

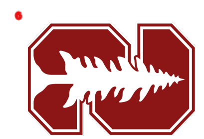
$\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}1\\-1\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix} \to \begin{bmatrix}-1\\-1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$: 2

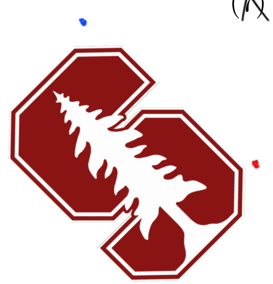
$\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}0\\2\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix} \to \begin{bmatrix}2\\0\end{bmatrix}$ $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$: 6

$\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}2\\1\end{bmatrix}$ $\begin{bmatrix}1\\-1\end{bmatrix} \to \begin{bmatrix}0\\-1\end{bmatrix}$ $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$: 7

(1)

(2)

(~~3~~)

(4)

(~~5~~)

(~~6~~)

(~~7~~)

(~~8~~)


know how to derive this rotation matrix

(b) (6 points) Compute the 2×2 matrix corresponding to the linear transformation obtained by first applying a 60-degree counterclockwise rotation of $\mathbf{R}^2$ around $\mathbf{0}$ (reminder: $\sin(60°) = \sqrt{3}/2$ and $\cos(60°) = 1/2$) and then applying the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that multiplies vectors on the x-axis by -1 and multiplies vectors on the y-axis by 2.

rotation matrix

1) $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \to \begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}$

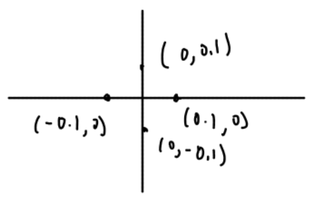
$\begin{bmatrix}-1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix} = \begin{bmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3} & 1\end{bmatrix}$

2) $T(e_1): \begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}-1\\0\end{bmatrix}$ $\to \begin{bmatrix}-1 & 0\\0 & 2\end{bmatrix}$

$T(e_2): \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\2\end{bmatrix}$

5. (8 points) **True or False**
For each of the following statements, circle either TRUE (meaning, "always true") or FALSE (meaning, "not always true"), and briefly and convincingly justify your answer. 1 point for correct choice, and 3 points for convincing justification.

(a) The critical point $(0, 0)$ of $f(x, y) = 3x^2 - 4xy + 4y^2$ is a saddle.
Circle one, and justify below: TRUE   FALSE



$f_x = 6x - 4y$ $f_y = -4x + 8y$

$f_x(0.1, 0) = 0.6$ $f_y(0, 0.1) = 0.8$

$f_x(-0.1, 0) = -0.6$ $f_y(0, -0.1) = -0.8$

local min on x   local min on y

f_y is a local minimum. Therefore, it is not a saddle point.

other answer

$f(x,y) = 3x^2 - 4xy + 4y^2$
$= 2x^2 + x^2 - 4xy + 4y^2$
$= 2x^2 + (x - 2y)^2$

$f(0,0) = 0$

$f(x,y) = 2x^2 + (x-2y)^2$ is always greater than zero if x or y is nonzero bc it is the sum of squares. Therefore, all points (x,y) will have $f(x,y) > f(0,0)$. Therefore, $(0,0)$ is a global minimum.

(b) Let M be an $n \times n$ Markov matrix, then for large n, all columns of M^n are equal.
Circle one, and justify below: TRUE   FALSE

The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a Markov matrix because all entries are non-negative and columns sum to zero. For large n, we see that $I_2^n = I_2$, since $AI_2 = A$ for all 2x2 matrices A. The columns of I_2 are not equal.

6. (10 points) Define the functions $g : \mathbf{R}^3 \to \mathbf{R}^2$ and $f : \mathbf{R}^2 \to \mathbf{R}$ by the formulas

$$g(x,y,z) = \begin{bmatrix} x^2z - e^{xy} \\ \ln(1+x^2z^2) \end{bmatrix}, \quad f(u,w) = u^2e^w,$$

so $g(1,0,-1) = (-2, \ln(2))$ and $f(-2,\ln(2)) = 8$.

(a) (5 points) Compute $(Dg)(1,0,-1)$ and $(Df)(-2,\ln(2))$ (the entries in these two matrices are all integers), and use the Chain Rule to compute $(D(f \circ g))(1,0,-1)$. Using this information, identify the value of $\dfrac{\partial(f\circ g)}{\partial z}(1,0,-1)$.

$$Dg(x,y,z) = \begin{bmatrix} 2xz - ye^{xy} & -xe^{xy} & x^2 \\ \dfrac{2xz^2}{1+x^2z^2} & 0 & \dfrac{2zx^2}{1+x^2z^2} \end{bmatrix}$$

$$Dg(1,0,-1) = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$Df(u,w) = \begin{bmatrix} 2ue^w & u^2e^w \end{bmatrix}$$

$$Df(-2,\ln(2)) = \begin{bmatrix} -8 & 8 \end{bmatrix}$$

$$\begin{aligned} D(f\circ g)(1,0,-1) &= Df(g(1,0,-1)) \cdot Dg(1,0,-1) \\ &= Df(-2,\ln 2) \cdot Dg(1,0,-1) \\ &= \begin{bmatrix} -8 & 8 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 8 & -16 \end{bmatrix} \end{aligned}$$

(b) (2 points) Use the matrix $(D(f\circ g))(1,0,-1)$ from part (a) to compute the *equation* of the tangent plane to the surface $f(g(x,y,z)) = 8$ at $(1,0,-1)$. Write your answer as equation involving x, y, z, and explicit numbers that you compute.

tangent plane: $\nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$

$\nabla f(g(\vec{a})) \cdot (\vec{x} - g(\vec{a})) = 0$

$\nabla f(g(1,0,-1)) \cdot \begin{bmatrix} x-1 \\ y-0 \\ z+1 \end{bmatrix} = 0$

$\begin{bmatrix} 24 \\ 8 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-0 \\ z+1 \end{bmatrix} = 0$

$24(x-1) + 8y + 10(z+1) = 0$

(c) (3 points) Use $(Dg)(1, 0, -1)$ from (a) and the approximation $\ln(2) \approx .7$ (to the nearest tenth) to estimate the 2-vector $g(1.2, 0.1, -1.1)$; simplify your answer as much as possible.

$$g(1.2, 0.1, -1.1) \approx g(1, 0, -1) + (Dg)(1,0,-1) \cdot \begin{bmatrix} 1.2-1 \\ 0.1-0 \\ -1.1-(-1) \end{bmatrix}$$

$$\approx \begin{bmatrix} -2 \\ \ln 2 \end{bmatrix} + \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.1 \\ -0.1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -2 \\ 0.7 \end{bmatrix} + \begin{bmatrix} -0.4 - 0.1 - 0.1 \\ 0.2 + 0 + 0.1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -2.6 \\ 1.0 \end{bmatrix}$$

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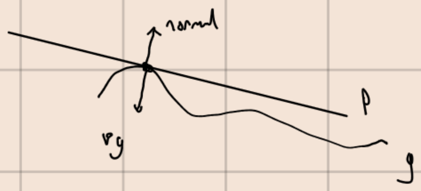
Plane P: $x+y+z=3$

Let $g(x,y,z) = xz - y^2 = 3$

Find (x,y,z) where P is tangent plane at (x,y,z) to $g(x,y,z)=3$.



gradient is scalar mult of normal

Normal of P is $\begin{bmatrix}1\\1\\1\end{bmatrix}$

$$\nabla g = \begin{bmatrix} z \\ -2y \\ x \end{bmatrix}$$

$$\begin{bmatrix} z \\ -2y \\ x \end{bmatrix} = \lambda \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$z = \lambda$

$-2y = \lambda \rightarrow y = \frac{\lambda}{-2}$

$x = \lambda$

$x + y + z = 3$

$\lambda - \frac{\lambda}{2} + \lambda = 3$

$2\lambda - \frac{\lambda}{2} = 3$

$\frac{3\lambda}{2} = 3$

$3\lambda = 6$

$\lambda = 2$

$$\begin{bmatrix} z \\ -2y \\ x \end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

$z = 2$

$y = -1$

$x = 2$