Lecture 18 - Matrix Inverses and Multivariable Newton's Method

November 7, 2022

Goals: Computing inverses of 2×2 matrices and using them to run one step of Newton's method

Some functions have an "unambiguous inverse" function which allows you to go backwards from the output to the input. For example, $f(x) = x + 1$, then $f^{-1}(x) = x - 1$. You can check that $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ (another way to think about this is f^{-1} "undoes" what f does). Some functions do not have unambiguous inverses. For example $f(x) = x^2$. Would $f^{-1}(4)$ be 2 or -2? undo each other

ambiguous

The same holds true for linear functions; some will have inverses, others will not.

Proposition 18.1.5: For any $n \times n$ matrix A the following two conditions on A are equivalent:

- B1 The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ ($A\vec{x}$) is invertible. Explicitly, for every (output) $\mathbf{b} \in \mathbb{R}^n$, there is a **unique** (input) $\mathbf{x} \in \mathbb{R}^n$ that solves $\boxed{A\mathbf{x} = \mathbf{b}}$.
- B2 There is an $n \times n$ matrix B for which $AB = I_n = BA$ (in which case the functions T_A and T_B are inverse functions).

When these conditions hold, B is **uniquely determined** and is denoted A^{-1}.

Definition: Any A satisfying the above is called **invertible**, and B is called the inverse matrix of A (likewise, A is then the inverse matrix of B).

Example 1: Verify $A = \begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3/4 & 1/2 \end{bmatrix}$ are inverses of each other. Also, verify algebraically that $T_A \circ T_B$ and $T_B \circ T_A$ are the identity maps.

B2)

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 2 & -1 \\ -3/4 & 1/2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\Rightarrow$ A and B are inverses of each other.

B1)

$$T_A \circ T_B(x,y) = T_A\left(T_B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = T_A\left(B\begin{bmatrix} x \\ y \end{bmatrix}\right) = T_A\left(\begin{bmatrix} 2x - y \\ -3/4x + 1/2y \end{bmatrix}\right) = \begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 2x - y \\ -3/4x + 1/2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow T_A \circ T_B$ is the identity map (maps input to input)

$$T_B \circ T_A(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

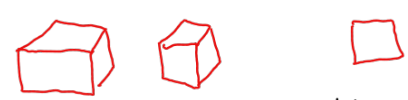
$\Rightarrow T_B \circ T_A$ is the identity map

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{\mathbb{R}^3 \to}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{bmatrix}}_{\mathbb{R}^2}$$

2, 3

can't revert this with a matrix to get back to $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\mathbb{R}^3$ $\mathbb{R}^2$



you lose info between dimensions, can't go back

Note: Inverses of matrices, when they exist, are only defined for *square matrices.*

It turns out that we do not need to check both conditions $AB = I_n$ and $BA = I_n$ to determine if $B = A^{-1}$.

Theorem 18.1.8: If A and B are $n \times n$ matrices that satisfy $AB = I_n$, then A is invertible and B is its inverse; i.e. we have $BA = I_n$ automatically.

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = ad-bc \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For a 2×2 matrix, we have the following useful property:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc \neq 0$, then it turns out A is invertible with formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \rightarrow$$

1) swap the main diagonal

2) negate the off-diagonal

(**Note:** If $ad - bc = 0$, then A is not invertible. See Remark 18.2.3 for a proof) → bc there'll be two solutions to $A\vec{x} = 0$

Definition: For a 2×2 matrix, the **determinant of** A (denoted $\det(A)$) is the scalar $ad - bc$. Thus, if $\det(A) = 0$, A is not invertible. This is actually true for all $n \times n$ matrices, however, we will not define nor compute the determinant of a matrix larger than 2×2 for now.

$\text{Area}(R) \xrightarrow{T_A} \underbrace{|\det(A)|}_{\text{scales, essentially}} \text{Area}(R)$

Example 2: Compute the determinant of the following matrices and determine whether they are invertible. If they are, find their inverse matrix.

- $\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$, $\det(A) = ad - bc = (-1)(2) - (-1)(2) = 0$

  not invertible → (rows/columns are multiples ⇒ not invertible)

- $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $\det(A) = 10 - 9 = 1$

  invertible

  $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}}$

- $\begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix}$, $\det(A) = -3 - 4 = -7$

  $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \boxed{-\frac{1}{7}\begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}}$

Example 3: Recall the matrix A_θ which rotates a 2-vector by angle θ counterclockwise. Show that A_θ is always invertible and find its inverse. What does its inverse do geometrically?

$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\det(A_\theta) = \underset{ad}{\cos^2\theta} - \underset{bc}{(-\sin^2\theta)} = \cos^2\theta + \sin^2\theta = 1$ always invertible

Geometrically, $(A_\theta)^{-1}$ rotates a vector θ clockwise.

$(A_\theta)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Example 4: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \cancel{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}$. Compute A^{-1}, B^{-1} and $(AB)^{-1}$. What do you notice?

$\det(A) = ad - bc = 1 - 0 = 1$, $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\det(B) = ad - bc = 1 - 0 = 1$, $B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\boxed{\det(AB)} = 2 - 1 = 1 = \boxed{\det(A) \cdot \det(B)}$

$(AB)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = B^{-1} \cdot A^{-1}$

$A^{-1}B^{-1} \neq (AB)^{-1}$

$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

AB applies B then A to $\vec{x}$ $(AB\vec{x})$

to reverse, you reverse A then B

Matrix inverses are *very* important to solving linear systems. Here is a result which summarizes how being invertible affects the number of solutions to a system.

$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I_n\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

Theorem 18.3.3: Let A be an $n \times n$ vector, and consider the system of n equations in n unknowns $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is a given n-vector.

- If A is invertible, then this system has a *unique* solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$. In particular, $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as its only solution.
- If A is not invertible, then the system has either no solution or infinitely many solutions. In particular, for non-invertible A, the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Thus, A is invertible precisely when $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as its only solution, and A is non-invertible precisely when $A\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Example 5: Solve the following system of linear equations

$$\begin{aligned} 2x + y - z &= 4 \\ x - 2y + z &= 3 \\ x - y - z &= 1 \end{aligned}$$

by using the fact that the inverse of $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ is $\frac{1}{7}\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & -3 \\ 1 & 3 & -5 \end{bmatrix}$.

$A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$\vec{x} = A^{-1}\vec{b}$

solution is $$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\frac{1}{7}\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & -3 \\ 1 & 3 & -5 \end{bmatrix}}_{A^{-1}}\underbrace{\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}}_{\vec{b}} = \boxed{\frac{1}{7}\begin{bmatrix} 17 \\ 2 \\ 8 \end{bmatrix}}$$

Example 6: Consider the two systems of equations

$$\begin{aligned} -x - y &= 3 \\ 2x + 2y &= 4 \end{aligned}$$

$$\underbrace{\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\vec{b}}$$

A is not invertible

$-x - y = 3 \longrightarrow x + y = -3$

$2x + 2y = 4 \longrightarrow x + y = 2$

not possible for both to be true

$\boxed{\text{no solution}}$

$$\begin{aligned} -x - y &= 2 \\ 2x + 2y &= -4 \end{aligned}$$

$$\underbrace{\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\vec{b}}$$

A is not invertible.

$-x - y = 2 \longrightarrow x + y = -2$

$2x + 2y = -4 \longrightarrow x + y = -2$

system is now basically one equation w/ two unknowns

$\boxed{\text{Infinitely many solutions}}$

$y = -x - 2$

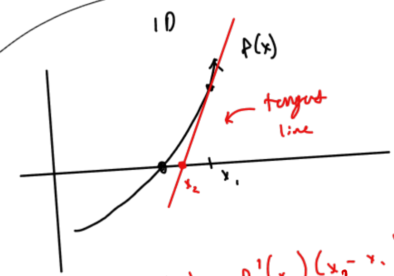
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x-2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

for any x

Newton's Method for approximating zeros of non-linear functions: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non-linear function, and let $\mathbf{a}$ be an initial guess for a solution to $f(\mathbf{x}) = \mathbf{0}$, with $Df(\mathbf{a})$ invertible. Then if we have chosen $\mathbf{a}$ reasonably, the sequence of vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots \in \mathbb{R}^n$ defined by $\mathbf{a}_1 = \mathbf{a}$ and

$$\mathbf{a}_{k+1} = \mathbf{a}_k - \underbrace{(Df(\mathbf{a}_k))^{-1}} f(\mathbf{a}_k), \qquad k \geq 1$$

are defined and converges rapidly to a solution of $f(\mathbf{x}) = \mathbf{0}$.



$$TL @ x_1 = f(x_1) + f'(x_1)(x_2 - x_1) = 0$$

$$\frac{-f(x_1)}{f'(x_1)} = x_2 - x_1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - (f'(x_1))^{-1} f(x_1)$$

Example 7: Suppose we wish to solve the system

$$\begin{aligned} x^2 + 2y - 2 &= 0 \\ x^3 - 2xy + 1 &= 0 \end{aligned}$$

$$f(x,y) = \begin{bmatrix} x^2 + 2y - 2 \\ x^3 - 2xy + 1 \end{bmatrix}$$

$$Df(x,y) = \begin{bmatrix} 2x & 2 \\ 3x^2 - 2y & -2x \end{bmatrix}$$

Start at $\vec{a}_1 = (2,0)$

check $\vec{a}_2 = \begin{bmatrix} 27/20 \\ 3/10 \end{bmatrix}$