Jack Le - Math 51 - PSET 8

Exercise 20.1. Consider the following

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & -2 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Calculate:

(a) $\mathbf{x}^\top \mathbf{x}$
(b) $\mathbf{x}\mathbf{x}^\top$
(c) $\|A\mathbf{x}\|^2$
(d) $\mathbf{x}^\top A^\top A\mathbf{x}$.

a) $x^T = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$, $\quad x^T x = \overset{1\times 3}{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \boxed{\begin{bmatrix} 9 \end{bmatrix}}$

b) $x x^T = \overset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \overset{1\times 3}{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}} = \boxed{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}}$

c) $Ax = \overset{4\times 3}{\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & -2 & 1 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} = \begin{bmatrix} 2 \\ 6 \\ 9 \\ 0 \end{bmatrix}$

$$\|Ax\|^2 = \left(\sqrt{Ax_1^2 + \dots + Ax_n^2}\right)^2 = (Ax)\cdot(Ax) = \begin{bmatrix} 2 \\ 6 \\ 9 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \\ 9 \\ 0 \end{bmatrix} = \boxed{121}$$

d) $x^T A^T A x = \overset{1\times 3}{\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}} \overset{3\times 4}{\begin{bmatrix} 2 & 2 & 3 & 2 \\ 1 & 1 & 4 & -2 \\ -1 & 1 & -1 & 1 \end{bmatrix}} \overset{4\times 3}{\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & -2 & 1 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$

$$= \begin{bmatrix} 2 & 6 & 9 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 4 & -1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 43 & 44 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 121 \end{bmatrix}}$$

Alternatively, we can use the fact that $(AB)^T = B^T A^T$

$$x^T A^T A x = (Ax)^T Ax = \begin{bmatrix} 2 & 6 & 9 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 121 \end{bmatrix}$$

Exercise 20.2. For the following either find the quadratic form Q_A associated with the given symmetric matrix A or find the symmetric matrix A associated with the given quadratic form Q.

(a) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

(c) $Q(x, y, z) = x^2 + y^2 + 3z^2 + 2xy + 6xz$

(d) $Q(x_1, x_2, x_3, x_4) = x_1^2 - x_4^2 + x_2x_3$

a) $Q_A = v^T(Av), \quad v^T = [x \ y \ z]$

$$= [x \ y \ z]\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x + 2y + 2z \quad 2x + 2y + z \quad 2x + y + 3z]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x^2 + 2xy + 2xz + 2xy + 2y^2 + yz + 2xz + yz + 3z^2]$$

$$\boxed{= x^2 + 2y^2 + 3z^2 + 4xy + 4xz + 2yz}$$

d) $Q_A = v^T(Av), \quad v^T = [x_1 \ x_2 \ x_3 \ x_4], \quad Q_A = x_1^2 - x_4^2 + x_2 x_3$

Diagonal entries c_{ii} is coefficient for x_i^2, and $c_{ij} = c_{ji}$ multiplies $2x_i x_j$.

Thus, $\boxed{A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}$

Check:

$$= [x_1 \ x_2 \ x_3 \ x_4]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= [x_1 \quad 1/2\, x_3 \quad 1/2\, x_2 \quad -x_4]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= [x_1^2 \quad 1/2\, x_2 x_3 \quad 1/2\, x_2 x_3 \quad -x_4^2]$$

$$= x_1^2 - x_4^2 + x_2 x_3$$

Exercise 20.3. For the following matrices, decide if they are orthogonal. If they are, find their inverse.

(a) $A = (1/\sqrt{5})\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(c) The matrix B' which is given by dividing each column of B in (b) by its length.

(d) Use the matrix B' to calculate the inverse of B.

a) Since $m=n$ for A, to be orthogonal, $A^TA = I_2$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/5 + 4/5 & 2/5 - 2/5 \\ 2/5 - 2/5 & 4/5 + 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Since A is orthogonal, $A^{-1} = A^T$. Therefore, $\boxed{A^{-1} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}}$.

b) To be orthogonal, the columns of B are an orthonormal collection of vectors. Let the columns of B be denoted c_1, c_2, c_3, c_4.

$$B = \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & c_4 \end{matrix} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$\|c_1\| = 2$ $\quad \|c_3\| = 2$

$\|c_2\| = 2$ $\quad \|c_4\| = 2$

Since the columns are not unit vectors, they can't be an orthonormal collection. Thus, B is not orthogonal.

c)

$$B' = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$\|c_1\| = 1$ $\quad \|c_3\| = 1$

$\|c_2\| = 1$ $\quad \|c_4\| = 1$

$c_1 \cdot c_2 = -1/4 - 1/4 + 1/4 + 1/4 = 0$

$c_2 \cdot c_3 = 1/4 - 1/4 - 1/4 + 1/4 = 0$

$c_3 \cdot c_4 = -1/4 - 1/4 + 1/4 + 1/4 = 0$

$c_4 \cdot c_1 = 1/4 - 1/4 - 1/4 + 1/4 = 0$

Since the columns of B are unit vectors and orthogonal to each other, they are a collection of orthonormal vectors. Thus, B' is orthogonal with the inverse

$$\boxed{B'^{-1} = B'^{T} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}}$$

d) We know that $B' = \frac{1}{2}B$. We can rearrange this to get $2B' = B$. Assuming that the inverse of B exists, we can multiply both sides on the right to get $2B'B^{-1} = BB^{-1} = I_4$.

We know from (c) that $B'^{-1} = B'^T$. Thus, $B'B'^{-1} = B'B'^T = I_4$. This tells us that $2B'B^{-1} = B'B'^T$. We can move the scalar to get $B'(2B^{-1}) = B'B'^T = I_4$.

Since the inverse is unique, it is safe to say that $2B^{-1} = B'^T$. Thus, $B^{-1} = \frac{1}{2}B'^T$.

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix}$$

Check:

$$BB^{-1} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

Exercise 20.10. A quadratic form $q : \mathbf{R}^n \to \mathbf{R}$ is called *positive-definite* if $q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, *negative-definite* if $q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$, and *indefinite* if q takes on both positive and negative values. For each of the following quadratic forms, determine which of the preceding three types it is.

(a) $q(x, y, z) = -2x^2 - y^2 - 15z^2$
(b) $q(x, y, z) = -19x^2 + 6y^2 - 237z^2$
(c) $q(x, y) = -3xy$
(d) $q(x, y, z) = x^2 + 7y^2 + 10z^2$

a) q is negative definite: x^2, y^2, and z^2 will always be positive. Since the coefficients of all three are negative, the result will always be a negative number for all $\vec{x} = \{x, y, z\}$, $\vec{x} \neq 0$.

b) q is indefinite: x^2, y^2, and z^2 will always be positive. Since the coefficients are a mix of positive and negative, the sign of q depends on the values of $\vec{x}$. If $6y^2 > 19x^2 + 237x^2$, then q will be positive. Otherwise, q will be negative or zero.

c) q is indefinite: if x is negative and y is positive, or vice versa, q is positive. If both is negative or both is positive, then q is negative.

d) q is positive definite: x^2, y^2, and z^2 will always be positive. Since the coefficients of all three are positive, the result will always be a positive number for all $\vec{x} = \{x, y, z\}$, $\vec{x} \neq 0$.

Exercise 21.2. For each of the matrices

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 1 \\ 2 & 5 & -1 & 0 \\ 3 & 2 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$$

$A: \mathbb{R}^2 \to \mathbb{R}^2$, $B: \mathbb{R}^4 \to \mathbb{R}^3$, $C: \mathbb{R}^2 \to \mathbb{R}^2$

do the following:

(a) Determine the space $\mathbf{R}^d$ to which the null space belongs (write down the value of d for each matrix).
(b) Write down a homogeneous system of linear equations whose set of solutions is the null space of the matrix.
(c) Find all solutions to the system you give in (b), and thereby give a concise description (e.g., a basis if it is nonzero) for the null space. (Hint for B: the third row is the sum of the first two rows.)

A: a) For any $m \times n$ matrix A, $N(A) \subset \mathbb{R}^n$. Thus, for the given 2×2 matrix A, the null space belongs to $\mathbb{R}^2$, $d = 2$.

b) The null space is the set of solutions to $A\vec{x} = 0$.

$$A\vec{x} = \vec{0} \to \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \to \boxed{\begin{matrix} x - 2y = 0 \\ 2x - 4y = 0 \end{matrix}}$$

c) We wish to solve (b): $\begin{matrix} x - 2y = 0 \\ 2x - 4y = 0 \end{matrix} \longrightarrow \begin{matrix} y = 1/2\, x \\ y = 1/2\, x \end{matrix} \longrightarrow$ solutions: $\vec{x} = \begin{bmatrix} x \\ 1/2\, x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

$$\boxed{\text{Thus, } N(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}\right\}. \text{ Basis of } N(A) \text{ is } \left\{\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}\right\}.}$$

B: a) For any $m \times n$ matrix A, $N(A) \subset \mathbb{R}^n$. Thus, for the given 3×4 matrix B, the null space belongs to $\mathbb{R}^4$, $d = 4$.

b) The null space is the set of solutions to $B\vec{x} = \vec{0}$.

$$B\vec{x} = \vec{0} \to \begin{bmatrix} 1 & -3 & 0 & 1 \\ 2 & 5 & -1 & 0 \\ 3 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} \to \boxed{\begin{matrix} x_1 - 3x_2 + x_4 = 0 \\ 2x_1 + 5x_2 - x_3 = 0 \\ 3x_1 + 2x_2 - x_3 + x_4 = 0 \end{matrix}}$$

c) We wish to solve the system from (b).

① $x_1 - 3x_2 + x_4 = 0$, ② $2x_1 + 5x_2 - x_3 = 0$, ③ $3x_1 + 2x_2 - x_3 + x_4 = 0$

$\downarrow$ $\quad x_4 = 3x_2 - x_1$

$\downarrow$ $\quad x_3 = 2x_1 + 5x_2$

$\downarrow$
$$3x_1 + 2x_2 - (2x_1 + 5x_2) + (3x_2 - x_1)$$
$$3x_1 - 2x_1 - x_1 + 2x_2 - 5x_2 + 3x_2 = 0$$
$$0 = 0 \checkmark$$

$$\text{Solutions: } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 + 5x_2 \\ -x_1 + 3x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$

$$\boxed{\text{Thus, } N(B) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 3 \end{bmatrix}\right\}. \text{ Basis of } N(B) \text{ is } \left\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 3 \end{bmatrix}\right\}.}$$

C: a) For any $m \times n$ matrix A, $N(A) \subset \mathbb{R}^n$. Thus, for the given 2×2 matrix C, the null space belongs to $\mathbb{R}^2$, $d=2$.

b) The null space is the set of solutions to $C\vec{x}=0$.

$$A\vec{x}=\vec{0} \rightarrow \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \rightarrow \boxed{\begin{matrix} 3x+y=0 \\ -4x-y=0 \end{matrix}}$$

c) We wish to solve (b): $\begin{matrix} 3x+y=0 \\ -4x-y=0 \end{matrix} \xrightarrow{\text{add}} 3x-4x=0,\ -x=0,\ x=0 \longrightarrow \begin{matrix} 3(0)+y=0 \\ -4(0)-y=0 \end{matrix} \rightarrow y=0$

Thus, $\boxed{N(C)=\{\vec{0}\}}$.

Exercise 21.3. For each of the following linear systems, say whether it is overdetermined or underdetermined. Use your answer to give a "rule of thumb" prediction about the number of solutions (do not perform any computations).

(a) $\underbrace{\begin{bmatrix} 2 & 2 & 5 \\ 0 & -1 & 3 \end{bmatrix}}_{}^{2\times3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

(b) $\overset{3\times2}{\begin{bmatrix} 33 & 25 \\ -23 & 12 \\ 5 & 7 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Does the fact that this system is homogeneous give you any information about the correctness of the rule of thumb in this case (again, without doing any computations)?

(c) $\overset{2\times4}{\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. By comparing the first and second rows, do you get any information about the correctness of the rule of thumb in this case? (Write out the equations and stare at them.)

a) The dimensions of the matrix are 2×3. Since $m<n$, the system is underdetermined. Because of this, if there is a solution, it will automatically have infinitely many solutions.

a) The dimensions of the matrix are 3×2. Since $m>n$, the system is overdetermined. Because of this, there will likely be no solutions.

However, the fact that the system is homogeneous tells us that the rule of thumb is incorrect, because $\vec{x}=\vec{0}$ is always a solution to homogeneous systems.

a) The dimensions of the matrix are 2×4. Since $m<n$, the system is underdetermined. Because of this, if there is a solution, it will automatically have infinitely many solutions.

However, because the first and second rows are identical and the first and second entries of $\vec{b}$ are not identical, there are no solutions to the system. Hence, the rule of thumb does not apply.

$$\begin{matrix} w+y+2z=1 \\ w+y+2z=-1 \end{matrix} \rightarrow 1=-1 \therefore \text{no solution}$$

Exercise 21.7. Consider a linear system $A\mathbf{x} = \mathbf{b}$ for

$$A = \overset{3\times 4}{\begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 13 & 17 & -10 \\ -1 & -3 & -3 & 2 \end{bmatrix}}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbf{R}^4, \quad \mathbf{b} \in \mathbf{R}^3.$$

This exercise uses the projection method from Example 21.2.7 to analyze the solutions $\mathbf{x}$ for some $\mathbf{b}$.

(a) Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 13 \\ -3 \end{bmatrix}$ respectively denote the first and second columns of A. Observe that the third column $\mathbf{a}_3$ equals $-3\mathbf{a}_1 + 2\mathbf{a}_2$ and the fourth column $\mathbf{a}_4$ equals $\mathbf{a}_1 - \mathbf{a}_2$ (such relations among columns could be found by running Gram–Schmidt on the collection of columns, for example). Build an orthogonal basis for $C(A)$ of the form $\{\mathbf{a}_1, \mathbf{a}_2'\}$ (in particular, $\dim C(A) = 2$, so $\dim N(A) = 4 - 2 = 2$ by the Rank–Nullity Theorem), and explain why the preceding linear relations among the columns express that $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \in N(A)$ (so these two vectors in $N(A)$, visibly linearly independent, must constitute a basis for $N(A)$ since $\dim N(A) = 2$).

(b) Using the orthogonal basis for $C(A)$ found in (a) to compute $\mathbf{Proj}_{C(A)}(\mathbf{b})$ for each of the following 3-vectors $\mathbf{b}$, determine if $A\mathbf{x} = \mathbf{b}$ has a solution and when it does then use your projection work to find a solution and *verify your solution works*: $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$.

(c) In the cases in (b) for which you found a solution, use the basis of $N(A)$ in (a) to describe all solutions in a "parametric form".

a) ① We find an orthogonal basis for $C(A)$. We keep $\vec{a}_1$.

$$\vec{a}_1 \cdot \vec{a}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 13 \\ -3 \end{bmatrix} = 44$$

$$\vec{a}_1 \cdot \vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 11$$

$$\vec{a}_2' = \vec{a}_2 - \mathrm{Proj}_{\vec{a}_1}(\vec{a}_2) = \vec{a}_2 - \frac{\vec{a}_1 \cdot \vec{a}_2}{\vec{a}_1 \cdot \vec{a}_1}\vec{a}_1 = \begin{bmatrix} 2 \\ 13 \\ -3 \end{bmatrix} - \frac{44}{11}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ -3 \end{bmatrix} - \begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Thus, $\boxed{\{\vec{a}_1, \vec{a}_2'\} = \left\{\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\} \text{ is an orthogonal basis for } C(A).}$

② The null space is defined as the set of solutions to $A\vec{x} = 0$.

Using the first linear relation $\vec{a}_3 = -3\vec{a}_1 + 2\vec{a}_2$, we can rearrange this to get

$$-3\vec{a}_1 + 2\vec{a}_2 - \vec{a}_3 = A\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \vec{0}.$$ Thus, the vector $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ must be contained in $N(A)$.

Using the second linear relation $\vec{a}_4 = \vec{a}_1 - \vec{a}_2$, we can rearrange this to get

$$\vec{a}_1 - \vec{a}_2 - \vec{a}_4 = A\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \vec{0}.$$ Thus, the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ must be contained in $N(A)$.

b) Using the orthogonal basis from (a), $\mathrm{Proj}_{C(A)}(\vec{b}) = \mathrm{Proj}_{\vec{a}_1}(\vec{b}) + \mathrm{Proj}_{\vec{a}_2'}(\vec{b})$

① $\vec{b}_1 = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$, $\mathrm{Proj}_{C(A)}(\vec{b}_1) = \frac{11}{11}\vec{a}_1 + \frac{12}{6}\vec{a}_2' = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$, $\vec{b}_1 \neq \mathrm{Proj}_{C(A)}(\vec{b}_1)$, therefore there is no solution.

② $\vec{b}_2 = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$, $\mathrm{Proj}_{C(A)}(\vec{b}_1) = \frac{-11}{11}\vec{a}_1 + \frac{12}{6}\vec{a}_2' = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$, $\vec{b}_2 = \mathrm{Proj}_{C(A)}(\vec{b}_2)$, therefore there is a solution.

Using the work above, $-\vec{a}_1 + 2\vec{a}_2' = -\vec{a}_1 + 2(\vec{a}_2 - 4\vec{a}_1) = -9\vec{a}_1 + 2\vec{a}_2 = \vec{b}_2$.

Thus, $\boxed{\vec{x} = \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix}}$ is a solution: $A\vec{x} = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 13 & 17 & -10 \\ -1 & -3 & -3 & 2 \end{bmatrix}\begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -9+4 \\ -27+26 \\ 9-6 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} = \vec{b}_2$ ✓

③ $\vec{b}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$, $\text{Proj}_{C(A)}(\vec{b}) = \frac{-11}{11}\vec{a}_1 + \frac{-6}{6}\vec{a}_2' = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$, $\vec{b}_3 = \text{Proj}_{C(A)}(\vec{b}_3)$, therefore there is a solution.

Using the work above, $-\vec{a}_1 - \vec{a}_2' = -\vec{a}_1 - (\vec{a}_2 - 4\vec{a}_1) = 3\vec{a}_1 - \vec{a}_2 = \vec{b}_3$

Thus, $\boxed{\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}}$ is a solution: $A\vec{x} = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 13 & 17 & -10 \\ -1 & -3 & -3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 9-13 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = \vec{b}_3$ ✓

c) We know from (a) that the basis of $N(A)$ is $\left\{\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

The solutions for $A\vec{x} = \vec{b}_2$ are $\boxed{\vec{x} = \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t'\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}}$.

The solutions for $A\vec{x} = \vec{b}_3$ are $\boxed{\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t'\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}}$.

Exercise 22.3. Consider the underdetermined linear system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 3 & 4 & 7 & 1 & 20 \\ 0 & 2 & 5 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

Solve this in terms of x_4 and x_5 (i.e., move all x_4-terms and x_5-terms to the right side, use the last equation to solve for x_3 in terms of x_4 and x_5, and keep back-substituting). Check your answer works.

In equation form, the system above is

③ $3x_1 + 4x_2 + 7x_3 + x_4 + 20x_5 = -1$

② $2x_2 + 5x_3 + 2x_4 + x_5 = 1$

① $x_3 - 2x_4 + x_5 = -1$

We solve this from the bottom up using back-substitution.

① $x_3 - 2x_4 + x_5 = -1$

$\longrightarrow x_3 = 2x_4 - x_5 - 1$

② $2x_2 + 5x_3 + 2x_4 + x_5 = 1$

$2x_2 + 5(2x_4 - x_5 - 1) + 2x_4 + x_5 = 1$

$2x_2 + 10x_4 - 5x_5 - 5 + 2x_4 + x_5 = 1$

$2x_2 = -12x_4 + 4x_5 + 6$

$\longrightarrow x_2 = -6x_4 + 2x_5 + 3$

(3) $3x_1 + 4x_2 + 7x_3 + x_4 + 20x_5 = -1$

$$3x_1 + 4(-6x_4 + 2x_5 + 3) + 7(2x_4 - x_5 - 1) + x_4 + 20x_5 = -1$$

$$3x_1 - 24x_4 + 8x_5 + 12 + 14x_4 - 7x_5 - 7 + x_4 + 20x_5 = -1$$

$$3x_1 - 9x_4 + 21x_5 + 5 = -1$$

$$3x_1 = 9x_4 - 21x_5 - 6$$

$$\longrightarrow x_1 = 3x_4 - 7x_5 - 2$$

Thus, the solution to the system in terms of x_4 and x_5 is:

$$\boxed{\vec{x} = \begin{bmatrix} 3x_4 - 7x_5 - 2 \\ -6x_4 + 2x_5 + 3 \\ 2x_4 - x_5 - 1 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -6 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

We check this answer with the original equation $A\vec{x} = \vec{b}$ to ensure it works:

$$A\vec{x} = \vec{b} \rightarrow A\left(\begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -6 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) = A \begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + A\left(x_4 \begin{bmatrix} 3 \\ -6 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right) + A\left(x_5 \begin{bmatrix} -7 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= A \begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_4 \left(A \begin{bmatrix} 3 \\ -6 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right) + x_5 \left(A \begin{bmatrix} -7 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3 & 4 & 7 & 1 & 20 \\ 0 & 2 & 5 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 & 4 & 7 & 1 & 20 \\ 0 & 2 & 5 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 & 4 & 7 & 1 & 20 \\ 0 & 2 & 5 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -7 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \vec{b}$$

Thus, our solution works for any x_4 and x_5 as it satisfies $A\vec{x} = \vec{b}$.

Exercise 22.7.

(a) Verify the LU-decomposition

$$\overset{L}{\begin{bmatrix}1&0&0\\5&2&0\\3&-4&-1\end{bmatrix}}\overset{U}{\begin{bmatrix}3&2&-2\\0&1&1\\0&0&-4\end{bmatrix}}=\overset{LU}{\begin{bmatrix}3&2&-2\\15&12&-8\\9&2&-6\end{bmatrix}}.$$

(b) For L and U as in (a), compute L^{-1} and U^{-1} and check that each work.

(c) Compute $U^{-1}L^{-1}$ and check that this is inverse to the matrix $A = LU$ computed on the right side in (a) (multiply your computation of $U^{-1}L^{-1}$ against A on one side or the other; we know it isn't necessary to check both sides).

a) $$\begin{bmatrix}1&0&0\\5&2&0\\3&-4&-1\end{bmatrix}\begin{bmatrix}3&2&-2\\0&1&1\\0&0&-4\end{bmatrix}=\begin{bmatrix}3+0+0&2+0+0&-2+0+0\\15+0+0&10+2+0&-10+2+0\\9+0+0&6-4+0&-6-4+4\end{bmatrix}=\begin{bmatrix}3&2&-2\\15&12&-8\\9&2&-6\end{bmatrix}\ \checkmark$$

b) Suppose $L^{-1}=\begin{bmatrix}d_1&0&0\\a&d_2&0\\b&c&d_3\end{bmatrix}$. We know $LL^{-1}=I_3$.

$$LL^{-1}=\begin{bmatrix}1&0&0\\5&2&0\\3&-4&-1\end{bmatrix}\begin{bmatrix}d_1&0&0\\a&d_2&0\\b&c&d_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

row 1: $\underline{d_1=1}$

row 2: $5d_1+2a=0\to 2a=-5\to \underline{a=-5/2}$

$2d_2=1\to \underline{d_2=1/2}$

row 3: $3d_1-4a-b=0\to 3-4(-5/2)=b\to \underline{b=13}$

$-4d_2-c=0\to -4(1/2)=c\to \underline{c=-2}$

$-d_3=1\to \underline{d_3=-1}$

$$\boxed{\text{Thus, } L^{-1}=\begin{bmatrix}1&0&0\\-5/2&1/2&0\\13&-2&-1\end{bmatrix}.}$$

We check our answer:

$$\begin{bmatrix}1&0&0\\5&2&0\\3&-4&-1\end{bmatrix}\begin{bmatrix}1&0&0\\-5/2&1/2&0\\13&-2&-1\end{bmatrix}=\begin{bmatrix}1+0+0&0+0+0&0+0+0\\5-5+0&0+1+0&0+0+0\\3+10-13&0-2+2&0+0+1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3\checkmark$$

Suppose $U^{-1}=\begin{bmatrix}d_1&a&b\\0&d_2&c\\0&0&d_3\end{bmatrix}$. We know $UU^{-1}=I_3$.

$$UU^{-1}=\begin{bmatrix}3&2&-2\\0&1&1\\0&0&-4\end{bmatrix}\begin{bmatrix}d_1&a&b\\0&d_2&c\\0&0&d_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

row 1: $3d_1=1\to \underline{d_1=1/3}$

$3a+2d_2=0\to 3a+2=0\to \underline{a=-2/3}$

$3b+2c-2d_3=0\to 3b+2/4+2/4=0\to \underline{b=-1/3}$

row 2: $\underline{d_2=1}$

$c+d_3=0\to c-1/4=0\to \underline{c=1/4}$

row 3: $-4d_3=1\to \underline{d_3=-1/4}$

$$\boxed{\text{Thus, } U^{-1}=\begin{bmatrix}1/3&-2/3&-1/3\\0&1&1/4\\0&0&-1/4\end{bmatrix}.}$$

We check our answer:

$$\begin{bmatrix}3&2&-2\\0&1&1\\0&0&-4\end{bmatrix}\begin{bmatrix}1/3&-2/3&-1/3\\0&1&1/4\\0&0&-1/4\end{bmatrix}=\begin{bmatrix}1+0+0&-2+2+0&-1+1/2+1/2\\0+0+0&0+1+0&0+1/4-1/4\\0+0+0&0+0+0&0+0+1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3\checkmark$$

c) $$U^{-1}L^{-1}=\begin{bmatrix}1/3&-2/3&-1/3\\0&1&1/4\\0&0&-1/4\end{bmatrix}\begin{bmatrix}1&0&0\\-5/2&1/2&0\\13&-2&-1\end{bmatrix}=\begin{bmatrix}1/3+5/3-13/3&0-1/3+2/3&0+0+1/3\\0-5/2+13/4&0+1/2-1/2&0+0-1/4\\0+0-13/4&0+0+1/2&0+0+1/4\end{bmatrix}=\boxed{\begin{bmatrix}-7/3&1/3&1/3\\3/4&0&-1/4\\-13/4&1/2&1/4\end{bmatrix}}$$

Since $A=LU$, then $A^{-1}=U^{-1}L^{-1}$ and $AA^{-1}=I_3$. Here we check $A(U^{-1}L^{-1})=I_3$.

$$A(U^{-1}L^{-1})=\begin{bmatrix}3&2&-2\\15&12&-8\\9&2&-6\end{bmatrix}\begin{bmatrix}-7/3&1/3&1/3\\3/4&0&-1/4\\-13/4&1/2&1/4\end{bmatrix}=\begin{bmatrix}-7+3/2+13/2&1+0-1&1-1/2-1/2\\-35+9+26&5+0-4&5-3-2\\-21+3/2+39/2&3+0-3&3-1/2-5/2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3\ \checkmark$$

Exercise 22.8. For this exercise, you should do all work with exact numbers (do *not* use a calculator with decimal approximations). This involves just basic manipulations with square roots, nothing too ugly.

(a) Verify the QR-decomposition

$$\overset{Q}{\begin{bmatrix} 1/\sqrt{3} & \sqrt{2/3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}} \overset{R}{\begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}} = \overset{QR}{\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix}}.$$

(You're *not* asked to check that Q is orthogonal, though you may wish to verify this for yourself in private anyway.)

(b) Compute R^{-1}, checking it works.

(c) For $A = QR$ computed in (a), use your answer to (b) to verify that A^{-1} $(= R^{-1}Q^{-1} = R^{-1}Q^{\top})$ is given by

$$A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and check that this really is inverse to A (i.e., multiply it against A on the left or right to check that the product is I_3).

Aside: $\sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{18}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{9}} = \frac{1}{3}$

a)

$$\begin{bmatrix} 1/\sqrt{3} & \sqrt{2/3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1+0+0 & 4/3 + 2/3 + 0 & -5/3 - 1/3 + 0 \\ 1+0+0 & 4/3 - 1/3 + 0 & -5/3 + 1/6 - 1/2 \\ 1+0+0 & 4/3 - 1/3 + 0 & -5/3 + 1/6 + 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 4/3 + 2/3 + 0 & -5/3 - 1/3 + 0 \\ 1+0+0 & 4/3 - 1/3 + 0 & -5/3 + 1/6 - 1/2 \\ 1+0+0 & 4/3 - 1/3 + 0 & -5/3 + 1/6 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix} \checkmark$$

b) Suppose $R^{-1} = \begin{bmatrix} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{bmatrix}$. We know $RR^{-1} = I_3$.

$$RR^{-1} = \begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1

$\sqrt{3}\, d_1 = 1 \longrightarrow \underline{d_1 = 1/\sqrt{3}}$

$\sqrt{3}\, a + 4 d_2/\sqrt{3} = 0 \longrightarrow \sqrt{3}\, a + \frac{4\frac{\sqrt{3}}{\sqrt{2}}}{\sqrt{3}} = \sqrt{3}\, a + \frac{4}{\sqrt{2}} = 0 \rightarrow \underline{a = \frac{-4}{\sqrt{6}}}$

$\sqrt{3}\, b + 4c/\sqrt{3} - 5d_3/\sqrt{3} = 0 \longrightarrow \sqrt{3}\, b + 4/\sqrt{6} - 5\sqrt{2}/\sqrt{3} = 0 \longrightarrow \sqrt{3}\, b = \frac{5(2)}{\sqrt{6}} - \frac{4}{\sqrt{6}} = \frac{6}{\sqrt{6}} \rightarrow b = \frac{6}{\sqrt{18}} = \frac{6}{3\sqrt{2}} = \frac{2}{\sqrt{2}} = \underline{\sqrt{2}}$

row 2

$\sqrt{2/3}\, d_2 = 1 \longrightarrow \underline{d_2 = \sqrt{3/2}}$

$\sqrt{2/3}\, c - d_3/\sqrt{6} = 0 \longrightarrow \frac{\sqrt{2}}{\sqrt{3}} c = \frac{\sqrt{2}}{\sqrt{6}} \rightarrow \underline{c = \frac{1}{\sqrt{2}}}$

row 3

$d_3/\sqrt{2} = 1 \longrightarrow \underline{d_3 = \sqrt{2}}$

$$\boxed{\text{Thus, } R^{-1} = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{6} & \sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}}$$

We check our answer:

$$RR^{-1} = \begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{6} & \sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & -4/\sqrt{2} + 4/\sqrt{2} & 6/\sqrt{3}\sqrt{2} + 4/\sqrt{3}\sqrt{2} - 10/\sqrt{3}\sqrt{2} \\ 0+0+0 & 0+1+0 & 0 + 1/\sqrt{3} - 1/\sqrt{3} \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

c) Since $A = QR$, $A^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T$

$$A^{-1} = R^{-1}Q^T = \begin{bmatrix} 1/\sqrt{3} & -4/\sqrt{6} & \sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3 - 4/3 + 0 & 1/3 + 4/6 - 1 & 1/3 + 4/6 + 1 \\ 0 + 1 + 0 & 0 - 1/2 - 1/2 & 0 - 1/2 + 1/2 \\ 0 + 0 + 0 & 0 + 0 - 1 & 0 + 0 + 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

We check that $AA^{-1} = I_3$, since A^{-1} should be the inverse.

$$AA^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & -2+2 & 2-2 \\ -1+1 & -1+2 & 2-2 \\ -1+1 & -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \checkmark$$

Exercise 22.9. Consider the linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 6 \\ x_1 + x_2 - 2x_3 &= -2 \\ x_1 + x_2 - x_3 &= 1 \end{aligned}$$

(a) Use A^{-1} given in the statement of Exercise 22.8(c) to solve the given linear system, and check that your answer works.

(b) For the QR-decomposition of A given in the statement of Exercise 22.8(a), write out the upper triangular system $R\mathbf{x} = Q^\top\mathbf{b}$ that encodes the same solution(s) as the original linear system, and solve the upper triangular system by back-substitution. You should get the same solution as in (a).

a) The system can be represented as $A\vec{x} = \vec{b}$ as follows

$$\begin{array}{ccc} A & \vec{x} & \vec{b} \\ \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = & \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \end{array}$$

Since A is invertible (from Exercise 22.8), we can find $\vec{x}$ using $\vec{x} = A^{-1}\vec{b}$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 3 \end{bmatrix} \longrightarrow \boxed{\begin{aligned} x_1 &= -4 \\ x_2 &= 8 \\ x_3 &= 3 \end{aligned}}$$

We check our answers with the system

$(-4) + 2(8) - 2(3) = 6$ ✓

$(-4) + (8) - 2(3) = -2$ ✓

$(-4) + (8) - (3) = 1$ ✓

b)

$$R\vec{x} = Q^T\vec{b} \rightarrow \begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \sqrt{3} & 4/\sqrt{3} & -5/\sqrt{3} \\ 0 & \sqrt{2/3} & -1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{3} - 2/\sqrt{3} + 1/\sqrt{3} \\ 6\sqrt{2}/\sqrt{3} + 2/\sqrt{6} - 1/\sqrt{6} \\ 0 + 2/\sqrt{2} + 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{3} \\ 13/\sqrt{6} \\ 3/\sqrt{2} \end{bmatrix}$$

$$x_3/\sqrt{2} = 3/\sqrt{2} \rightarrow \underline{x_3 = 3}$$

$$\sqrt{2}x_2/\sqrt{3} - x_3/\sqrt{6} = 13/\sqrt{6} \rightarrow \left(\sqrt{2}x_2/\sqrt{3} - 3/\sqrt{6} = 13/\sqrt{6}\right)\sqrt{6}$$

$$2x_2 - 3 = 13 \rightarrow 2x_2 = 16$$

$$\underline{x_2 = 8}$$

$$\sqrt{3}x_1 + 4x_2/\sqrt{3} - 5x_3/\sqrt{3} = 5/\sqrt{3} \rightarrow \left(\sqrt{3}x_1 + 32/\sqrt{3} - 15/\sqrt{3} = 5/\sqrt{3}\right)\sqrt{3}$$

$$3x_1 + 32 - 15 = 5 \rightarrow 3x_1 = -12$$

$$\underline{x_1 = -4}$$

Thus, the solution to $R\vec{x} = Q^T\vec{b}$ is $\boxed{\begin{bmatrix} -4 \\ 8 \\ 3 \end{bmatrix}}$.

Exercise 22.10. Consider the following matrix A and its column vectors:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & -1 & -2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}.$$

If you run Gram-Schmidt on these, you get the following:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \mathbf{v}_2 - \frac{1}{3}\mathbf{w}_1 = \frac{1}{3}\begin{bmatrix} 5 \\ -1 \\ -4 \end{bmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{5}{7}\mathbf{w}_2 - \frac{2}{3}\mathbf{w}_1 = \frac{6}{7}\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

Using this information, construct the QR-decomposition of A and check that your answer is correct (by computing the matrix product QR directly to see that it is equal to A).

We first make an orthonormal collection of vectors from $\vec{w}_1, \vec{w}_2, \vec{w}_3$.

① $\|\vec{w}_1\| = \sqrt{\vec{w}_1 \cdot \vec{w}_1} = \sqrt{3} \longrightarrow \underline{\vec{w}_1'} = \vec{w}_1/\|\vec{w}_1\| = \vec{w}_1/\sqrt{3} \rightarrow \vec{w}_1' = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

② $\|\vec{w}_2\| = \sqrt{\vec{w}_2 \cdot \vec{w}_2} = \sqrt{\frac{1}{9}(42)} = \frac{\sqrt{42}}{3} \longrightarrow \underline{\vec{w}_2'} = \vec{w}_2/\|\vec{w}_2\| = \vec{w}_2/\left(\frac{\sqrt{42}}{3}\right) = \vec{w}_2 \times \frac{3}{\sqrt{42}} \rightarrow \vec{w}_2' = \begin{bmatrix} 6/\sqrt{42} \\ -1/\sqrt{42} \\ -4/\sqrt{42} \end{bmatrix}$

③ $\|\vec{w}_3\| = \sqrt{\vec{w}_3 \cdot \vec{w}_3} = \sqrt{\frac{36}{49}(14)} = \frac{6}{7}\sqrt{14} \longrightarrow \underline{\vec{w}_3'} = \vec{w}_3/\|\vec{w}_3\| = \vec{w}_3/\left(\frac{6}{7}\sqrt{14}\right) = \frac{7}{6\sqrt{14}}\vec{w}_3 \rightarrow \vec{w}_3' = \begin{bmatrix} -1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$

$\vec{w}_1', \vec{w}_2', \vec{w}_3'$ is now an orthonormal collection spanning the same space as $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Thus,

$$\boxed{Q = \begin{bmatrix} | & | & | \\ \vec{w}_1' & \vec{w}_2' & \vec{w}_3' \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 5/\sqrt{42} & -1/\sqrt{14} \\ 1/\sqrt{3} & -1/\sqrt{42} & 3/\sqrt{14} \\ 1/\sqrt{3} & -4/\sqrt{42} & -2/\sqrt{14} \end{bmatrix}}$$

We now find $\vec{v}_1, \vec{v}_2, \vec{v}_3$ as a linear combination of $\vec{w}_1', \vec{w}_2', \vec{w}_3'$.

(1) $\vec{w}_1 = \vec{v}_1 \rightarrow \vec{w}_1' = \vec{w}_1 / \sqrt{3} = \vec{v}_1/\sqrt{3} \rightarrow \vec{v}_1 = \sqrt{3}\,\vec{w}_1'$

(2) $\vec{w}_2 = \vec{v}_2 - \frac{1}{3}\vec{w}_1 \longrightarrow \vec{v}_2 = \frac{1}{3}\vec{w}_1 + \vec{w}_2 = \frac{1}{3}\left(\sqrt{3}\,\vec{w}_1'\right) + \left(\frac{\sqrt{42}}{3}\vec{w}_2'\right)$

$$\vec{v}_2 = \frac{\sqrt{3}}{3}\vec{w}_1' + \frac{\sqrt{42}}{3}\vec{w}_2'$$

(3) $\vec{w}_3 = \vec{v}_3 - \frac{5}{7}\vec{w}_2 - \frac{2}{3}\vec{w}_1 \longrightarrow \vec{v}_3 = \frac{2}{3}\vec{w}_1 + \frac{5}{7}\vec{w}_2 + \vec{w}_3 = \frac{2}{3}\left(\sqrt{3}\,\vec{w}_1'\right) + \frac{5}{7}\left(\frac{\sqrt{42}}{3}\vec{w}_2'\right) + \left(\frac{6\sqrt{14}}{7}\right)\left(\vec{w}_3'\right)$

$$\vec{v}_3 = \frac{2\sqrt{3}}{3}\vec{w}_1' + \frac{5\sqrt{42}}{21}\vec{w}_2' + \frac{6\sqrt{14}}{7}\vec{w}_3'$$

We know that the coefficient of $\vec{w}_i'$ of the linear combination for $\vec{v}_j$ is r_{ij}. Using the above, we have that

$$\boxed{R = \begin{bmatrix} \sqrt{3} & \frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} \\ 0 & \frac{\sqrt{42}}{3} & \frac{5\sqrt{42}}{21} \\ 0 & 0 & \frac{6\sqrt{14}}{7} \end{bmatrix}}$$

Thus, the QR-decomposition of A is the two matrices above. We check the to make sure it is true.

$$QR = \begin{bmatrix} 1/\sqrt{3} & 5/\sqrt{42} & -1/\sqrt{14} \\ 1/\sqrt{3} & -1/\sqrt{42} & 3/\sqrt{14} \\ 1/\sqrt{3} & -4/\sqrt{42} & -2/\sqrt{14} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} \\ 0 & \frac{\sqrt{42}}{3} & \frac{5\sqrt{42}}{21} \\ 0 & 0 & \frac{6\sqrt{14}}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/3 + 5/3 & 2/3 + 25/21 - 6/7 \\ 1 & 1/3 - 1/3 & 2/3 - 5/21 + 18/7 \\ 1 & 1/3 - 4/3 & 2/3 - 20/21 - 12/7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6/3 & 14/21 + 25/21 - 18/21 \\ 1 & 0 & 14/21 - 5/21 + 54/21 \\ 1 & -3/3 & 14/21 - 20/21 - 36/21 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & -1 & -2 \end{bmatrix} = A \quad \checkmark$$