Lecture 15 - Matrix Algebra

October 31, 2022

Goals: Perform various operations between two matrices

$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

Definition: For an $n \times n$ matrix A (such a matrix is called a **square matrix**), its **diagonal** (or **main diagonal**) consists of the entries a_{ii} in position (i,i) for all $1 \le i \le n$. This is the diagonal going from upper left to lower right. The **anti-diagonal** is the diagonal from lower left to upper right.

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

More generally, for an $m \times n$ matrix B for any $m, n \ge 1$, its **diagonal** consists of its entries b_{ii}. An $m \times n$ matrix is called **diagonal** if all entries away from the diagonal vanish.

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ are diagonal

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

The $n \times n$ **identity matrix**, denoted I_n, is defined to be the diagonal matrix

only for square matrices

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$I_n \vec{x} = \vec{x}$

whose diagonal entries are equal to 1.

- The corresponding linear transformation $T_{I_n} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $T_{I_n}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.
- For any $m \times n$ matrix A, we have $I_m A = A = A I_n$. acts like the number 1 in multiplication

One of the critical facts from linear algebra (which we will see in Chapter 27) is that every $m \times n$ matrix A can be expressed as $A = RDR'$ where D is an $m \times n$ diagonal matrix with non-negative diagonal entries, and R and R' are $m \times m$ and $n \times n$ matrices which express rigid motions of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Example 1: Perform the following matrix multiplications. How does it affect the matrix with letters?

- $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ -g & -h & -i \end{bmatrix}$

  - $R_1 \to 2R_1$ (1-1 entry of diag)
  - $R_2 \to 3R_2$ (2-2 entry)
  - $R_3 \to -1R_3$ (3-3 entry)

  Multiplying by diagonal matrix on the left scales the rows of the matrix on the right

- $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a-d & b-e & c-f \\ d & e & f \\ g & h & i \end{bmatrix}$

  - $R_1 \to R_1 - 1 \cdot R_2$
  - R_2 & R_3 unchanged

  "called a row operation"
  - -1 entry is at $1,2$ $\to$ R_1 replaced with $R_1 + (-1)R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{bmatrix} A \to m$ is $(3,2) \to R_3$ becomes $R_3 + mR_2$

3 is at (2, 1) → C_1 becomes $C_1 + 3C_2$

Example 1 ctd.

- $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3b & b & c \\ d+3e & e & f \\ g+3h & h & i \end{bmatrix}$

- $C_1 \to C_1 + 3C_2$
- C_2 & C_3 unchanged

Multiplying by diagonal matrix on the right scales the cols of the matrix on the left

Example 2: Compute both AB and BA for A and B below.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$BA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

so A and B commute

implies A and B are inverses bc their multiplication is the identity matrix

The non-lettered matrices in the second and third parts of Example 1, as well as the ones in Example 2, have special names. **Upper triangular matrices** are matrices where all entries below the main diagonal (so the ij-entry for $i > j$) are 0; note that other entries on or above the diagonal can be 0 as well. Similarly, **lower triangular matrices** are those which are 0 above the main diagonal (so the ij-entry for $i < j$).

Example 3: The product of upper/lower triangular matrices is always an upper/lower triangular matrix.

- $\begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

- $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 3 \end{bmatrix}$

Consider a scalar c and two $m \times n$ matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn}. \end{bmatrix}$$

<u>**Definition:**</u> The **matrix sum** $A + B$ is defined to be the $m \times n$ matrix

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

component-wise addition

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

and the **scalar multiple** cA is defined to be the $m \times n$ matrix

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

$$2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

<u>**Proposition 15.2.4:**</u> The linear transformations T_{A+B} and T_{cA} for the matrix sum and the scalar multiple respectively satisfy $T_{A+B}(\mathbf{x}) = T_A(\mathbf{x}) + T_B(\mathbf{x})$ and $T_{cA}(\mathbf{x}) = cT_A(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

Properties of Matrix Algebra

n-vector

- (MM1) If A is an $m \times n$ matrix, and $\mathbf{x} \in \mathbb{R}^n$ is thought of as an $n \times 1$ matrix, then the matrix-matrix product $A\mathbf{x}$ is the same as the matrix-vector product.

B+C both m×n

A is p×m

- (MM2) $A(B + C) = AB + AC$, and $(A' + B')C' = A'B' + B'C'$ (Distributive laws)
- (MM3) $A(BC) = (AB)C$, and $A(cB) = (cA)B = c(AB)$ for any scalar c. Note that this implies $\boxed{(AB)\mathbf{x} = A(B\mathbf{x}).}$
- (MM4) If A is an $m \times n$ matrix, then $I_m A = A = AI_n$.

Example 4: Note that the polynomial $t^2 + 2t - 8$ factors into $(t-2)(t+4)$. Verify that this identity is true for any square matrix A in place of t. In other words, show

$$A^2 + 2A - 8I_n = (A - 2I_n)(A + 4I_n)$$

like $4 \cdot 1$

for every $n \times n$ matrix A.

$$(A - 2I_n)(A + 4I_n) = (A - 2I_n)A + (A - 2I_n)(4I_n) \quad \text{(MM3)}$$
$$= A^2 - 2I_nA + 4I_nA - 8I_nI_n \quad \text{(MM2)}$$
$$= A^2 - 2A + 4A - 8I_n \quad \text{(MM4)}$$
$$= A^2 + 2A - 8I_n$$

Some polynomial identities also hold for the corresponding matrix identity

In multiplication, $ab = ac$ implies $b = c$ if $a \neq 0$ by "cancelling" a from both sides. This is **not true in general for matrices.** In particular, the zero product property **does not hold for matrices** in general: $AB = \mathbf{0}$ does not mean one of A and B is the zero matrix.

Example 5: Show that for the following matrices, $AB = AC$, but $B \neq C$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

so $AB = AC$, but $B \neq C$

Example 6: Show that $AB = 0$ for the two non-zero matrices below

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{"}0\text{"}$$

Extra

1) $\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$

$A^2 = A \rightarrow$ idempotent matrix

2) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^2 = 0 \rightarrow$ nilpotent matrix

$\hookrightarrow A^n = 0$ for some n