Lecture 10 - Maxima, Minima, and Critical Points

October 17, 2022

Goals: Identify potential local max/min and use a contour plot to distinguish local extrema and saddle points. Calculate the maximum and minimum values of a function on given region.

1-dimension

f(a)

a

Definition:

- A function $f(x,y)$ achieves a **local maximum** at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) which are sufficiently close (a,b). In other words, if we move in any direction from (a,b), then as long as we stay nearby, $f(x,y)$ decreases or stays the same.
- A function $f(x,y)$ achieves a **local minimum** at (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) which are sufficiently close (a,b). In other words, if we move in any direction from (a,b), then as long as we stay nearby, $f(x,y)$ increases or stays the same.

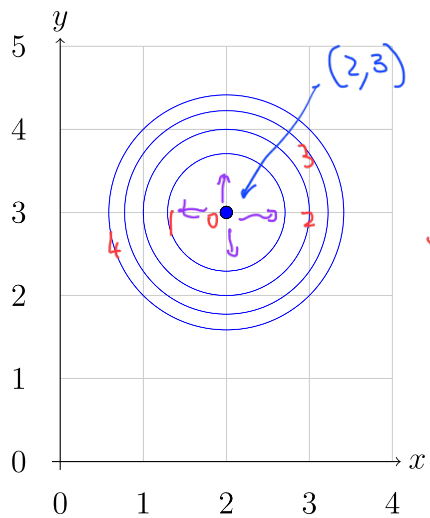
Theorem 10.2.2: Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that a point $\mathbf{a} \in \mathbb{R}^n$ is either a local maximum or a local minimum of f. Then **all** partial derivatives of f vanish at $\mathbf{x} = \mathbf{a}$, i.e. $f_{x_i}(\mathbf{a}) = 0$ for all $1 \leq i \leq n$.

first

if $f_x(2,1)=0$
$f_y(2,1)=-8$ $\to$ not a crit. pt.

Definition: If $f_{x_i}(\mathbf{a}) = 0$ for all $1 \leq i \leq n$, then we say that $\mathbf{a}$ is a **critical point for** f. In particular, every local max/min of $f : \mathbb{R}^n \to \mathbb{R}$ is a critical point.

Example 1: Find the critical points of the function $f(x,y) = 2(x-2)^2 + 2(y-3)^2$.



y is con

$f_x(x,y) = 4(x-2) = 0 \to x = 2$

x is cons

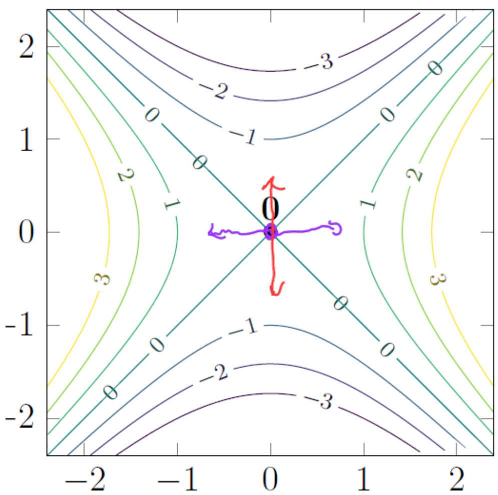
$f_y(x,y) = 4(y-3) = 0 \to y = 3$

(2, 3) is the only crit pt.

From the plot, if you move from (2,3) in any direction, it's increasing

So, (2,3) is a local minimum

Example 2: Find the critical points of $f(x, y) = x^2 - y^2$.



$f_x(x,y) = 2x = 0 \to x = 0$

$f_y(x,y) = -2y = 0 \to y = 0$

(0,0) is only critical point

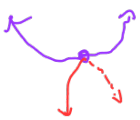
→ moving L/R, f is incr

→ moving U/D, f is decr

therefore (0,0) is neither local max or min

Note that if $\mathbf{a}$ is a critical point of f, then it could be neither a local max nor a local min (in the single variable case, think about $x = 0$ for the function $f(x) = x^3$).

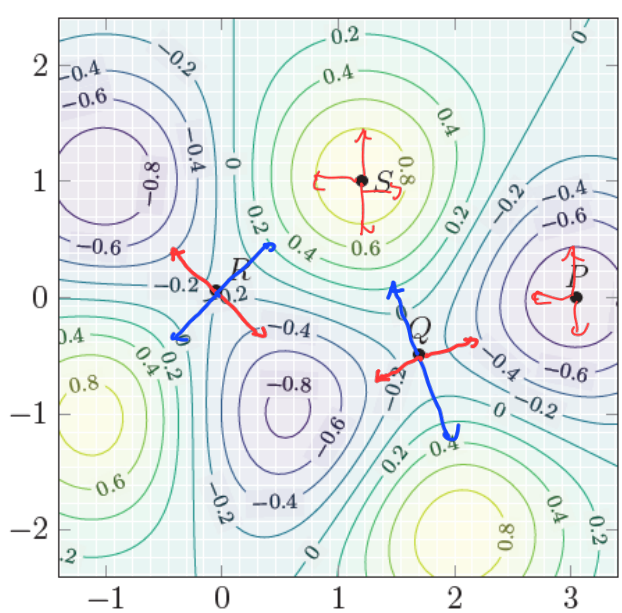
← a point that's neither



Definition: A critical point $\mathbf{a} \in \mathbb{R}^n$ of $f : \mathbb{R}^n \to \mathbb{R}$ is a **saddle point** if

- as we move away from $\mathbf{a}$ in one direction, then f increases nearby (so $\mathbf{a}$ looks like it might be a local min along the line) **and**
- as we move away from $\mathbf{a}$ in some other direction, then f decreases nearby (so $\mathbf{a}$ looks like it might be a local max along the line).

Example 3: Here's a contour plot of a function $f(x, y)$. Determine whether the critical points P, Q, R and S are local extrema or saddle points.



if the crit point is enclosed in a contour in the contour plot, it's an indicator that it's a local extrema

P: local min (f incr in all dir, nearby)

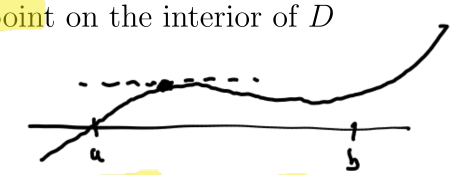
S: local max (f decr in all dir, nearby)

R: saddle point (→ f incr → f decr)

Q: saddle point (→ f decr → f incr)

Classifying local extrema / saddle points will be covered later in single-var, this is the interval on the line

Theorem 10.4.6: For a function $f : \mathbb{R}^n \to \mathbb{R}$ and a region D inside $\mathbb{R}^n$, suppose the function $f : D \to \mathbb{R}$ considered on D has a local extremum at a point $\mathbf{a} \in D$. Then the point $\mathbf{a}$ must be a critical point of f when $\mathbf{a}$ is in the *interior* of D.
In particular, any local extremum of $f : D \to \mathbb{R}$ is either a critical point on the interior of D or is a boundary point of D.

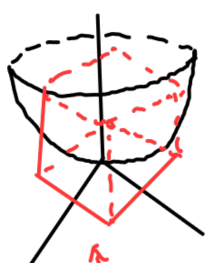


extremum at crit point or boundary value

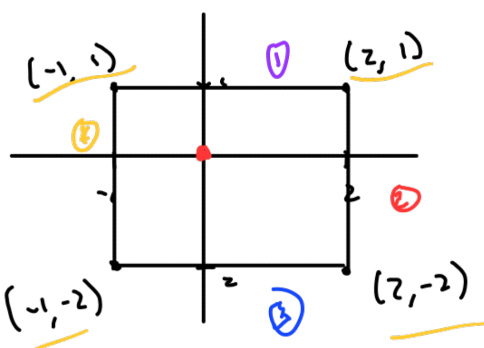
Steps for finding extrema for $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$.

- Find possible extrema by using the first partials to locate critical points of f in D. ← in the region) interval
- Look at the restriction of the function to the boundary curves. On each curve, the function can be $\boxed{\text{reduced}}$ to a single-variable function, and then you can use techniques from previous calculus courses to find points that are possible extrema.
- Plug all of the points you found into f. The largest value is the maximum and the smallest is the minimum (note, it is possible that the max/min values appears multiple times; **this is okay**).

Example 4: Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ on the rectangle $D = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 2, -2 \le y \le 1\}$.



checking extremum in region



crit points in D

$f_x : 2x = 0$

$f_y : 2y = 0$

(0,0) is crit point

↑ it's in the region

points on boundary

pt	f
(0,0)	0
(-1,1)	2
(2,1)	5
(-1,-2)	5
(2,-2)	8
(2,0)	4
(0,-2)	4
(-1,0)	1
(0,1)	1

1 line is $y = 1$, $-1 \le x \le 2$

$f(x,1) = x^2 + 1$, $f_x = 2x = 0$, $x = 0$

↳ (0,1) is a crit point

↳ (-1,1) and (2,1) is end point

2 line is $x = 2$, $-2 \le y \le 1$

$f(2,y) = 4 + y^2$, $f_y = 2y$, $y = 0$

↳ (2,0) is a crit point

↳ (2,1) and (2,-2) is endpt

3 line is $y = -2$, $-1 \le x \le 2$

$f(x,-2) = x^2 + 4 \to f_x = 2x \to x = 0$

↳ (0,-2) is crit

↳ (-1,-2) & (2,-2) is end point

4 line is $x = -1$, $-2 \le y \le 1$

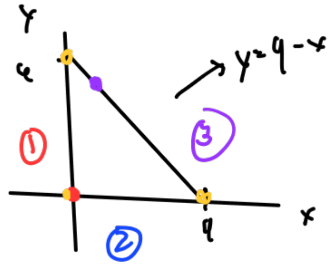
$f(-1,y) = 1 + y^2 \to f_y = 2y = 0 \to y = 0$

↳ (-1,0) is crit

↳ (-1,1) & (-1,-2) is end point

Max at $(2,-2)$, $f = 8$ | Min at $(0,0)$, $f = 0$

Example 5: Find the maximum and minimum values of $f(x, y) = x(y + 6) + x^2 + y^2$ on the region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.



Critical pts in D

$f_y = y + 6 + 2x = 0$

$f_y = x + 2y = 0$

$x = -2y$

$y + 6 - 4y = 0 \rightarrow y = 2, \; x = -4$

$(-4, 2)$

↑ not in D, don't consider

① $x = 0, \; 0 \le y \le 9$

$f(0, y) = y^2, \; f_y = 2y \rightarrow y = 0$ ✓ in interval

$\rightarrow (0, 0)$

② $y = 0, \; 0 \le x \le 9$

$f(x, 0) = 6x + x^2, \; f_x(x, 0) = 6 + 2x = 0$

$x = -3$ ✗ not in interval

③ $y = 9 - x, \; 0 \le x \le 9$

$f(x, 9-x) = x(15 - x) + x^2 + (9-x)^2$

$f_x(x, 9-x) = 15 - 2x + 2x - 2(9-x) = 15 + 2x - 18$

$x = \frac{3}{2}$ ✓ in interval $\rightarrow y = 9 - \frac{3}{2} = 15/2$

$(3/2, 15/2)$ is crit

Check

(x, y)	$f(x,y)$
$(0, 0)$	0
$(0, 9)$	81
$(9, 0)$	135
$(3/2, 15/2)$	78.75

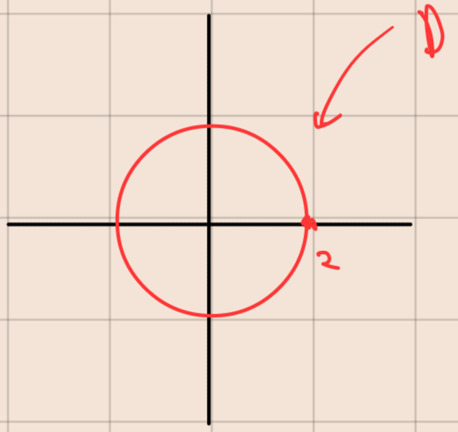
→ 0

→ max

Another example

$$f(x,y) = x^2 - y^2$$

$$D: \quad x^2 + y^2 \le 4$$



boundary curves

upper semicircle

$$y = \sqrt{4-x^2}$$

$$-2 \le x \le 2$$

lower semicircle

$$y = -\sqrt{4-x^2}$$

$$-2 \le x \le 2$$

ans: max $= 4$ @ $(\pm 2, 0)$

min $= -4$ @ $(0, \pm 2)$