# Lecture 21 - Linear systems, Column Space, and Null Space

November 16, 2022

**Goals:** Define column/null space for any matrix, compute an orthogonal basis of the column space, and use it to determine if $A\mathbf{x} = \mathbf{b}$ has a solution.

We have discussed systems of equations $A\mathbf{x} = \mathbf{b}$ where $A$ is square to some extent; namely, we know such a system has only one solution precisely when $A$ is invertible, in which case $\mathbf{x} = A^{-1}\mathbf{b}$. Otherwise, how do we determine if is no solution or infinitely many? Moreover, how do we answer these if our $A$ is an $m \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$?

**Definition:** The **column space** of an $m \times n$ matrix $A$ is the span in $\mathbb{R}^m$ of the columns of $A$ (remember: columns of $A$ are $m$-vectors). It is denoted $C(A)$.

**Important:** This means that the system has a solution precisely when $\mathbf{b}$ is in $C(A)$ (it doesn't tell us how **many** solutions though. More on that later).

$\vec{b} = A\vec{x} \rightarrow$ linear combination of the columns of A

**Example 1:** Does the following system have a solution?

$$\vec{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -3 & 2 \\ 1 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\begin{aligned} 2x - 3y + 2z &= -1 \\ x + y - 4z &= 2 \\ y - 2z &= 1 \end{aligned}$$

$$C(A) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}\right) = x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$$

is $\vec{b} \in C(A)$?

note $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $\therefore$ $b \in C(A)$ so there's at least one solution

**Example 2:** Determine the column space of the following matrix:

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 1 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix}.$$

$$C(A) \stackrel{?}{=} \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}\right)$$

check lin dependence

① none are scalar mults

② check linear comb

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = x\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$

$2 = -3x + 2y \rightarrow 2 = -3x - 1 \rightarrow x = -1$

$1 = x - 4y \rightarrow -2y = 1 \rightarrow y = -\frac{1}{2}$

$0 = x - 2y \rightarrow x = 2y$

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$

$\therefore$ vectors are linearly dependent, drop one b/c dim = 2

$$C(A) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right)$$

or just any two of the vectors

In general, we can just apply Gram-Schmidt to the columns of $A$ to compute a basis for its span.

$\hookrightarrow$ find $\vec{w}_1, \ldots, \vec{w}_k \rightarrow C(A) = \text{span}(\vec{w}_1, \ldots, \vec{w}_k)$

**Definition:** For any two sets $V$ and $W$, the notation "$V \subset W$" read as "$V$ contained in $W$" or "$V$ is a subset of W" means every object in $V$ also belongs to $W$ (e.g. a line $V$ contained in a plane $W$).

$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

span of 3 2-vectors

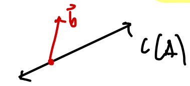
The following two propositions give a more geometric description about the solutions to a 2×3 or a 3×3 system.

**Proposition 21.2.5:** Let $A$ be a $2 \times 3$ matrix who columns are all nonzero. The subspace $C(A) \subset \mathbb{R}^2$ is a line when all columns are multiples of one another, or equivalently, "have the same slope"; if this is not the case, then $C(A) = \mathbb{R}^2$. For any such matrix $A$ and any $\mathbf{b} \in \mathbb{R}^2$, the linear system $A\mathbf{x} = \mathbf{b}$ of 2 equations in 3 unknowns has a solution precisely in the following circumstances:

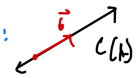
columns of $A$ (three are out) is a basis for $\mathbb{R}^2$

- if $C(A)$ is a line then there is a solution exactly when $\mathbf{b}$ lies on that line (so either $\mathbf{b} = \mathbf{0}$ or the slope $b_2/b_1$ is the same as that of all nonzero vectors in the line $C(A)$), or
- if $C(A) = \mathbb{R}^2$, then there is a solution for any $\mathbf{b}$.

no sol:



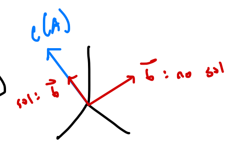

sol:



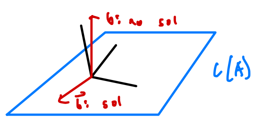

**Proposition 21.2.6:** Let $A$ be a $3 \times 3$ matrix whose columns are all nonzero. The subspace $C(A) \subset \mathbb{R}^3$ is a line when all columns are scalar multiples of each other, it is equal to $\mathbb{R}^3$ when the three columns are linearly independent, and in all other cases, it is a plane.
For such $A$ and any $\mathbf{b} \in \mathbb{R}^3$, the linear system $A\mathbf{x} = \mathbf{b}$ of 3 equations and 3 unknowns has a solution precisely in the following circumstances:

1. if $C(A)$ is a line, then there is a solution exactly when $\mathbf{b}$ lies in that line.
2. if $C(A)$ is a plane, then there is a solution exactly when $\mathbf{b}$ lies in that plane.
3. if $C(A) = \mathbb{R}^3$, then there is a solution for any $\mathbf{b}$.

1)




2)




More generally, how can we tell if $\mathbf{b} \in C(A)$ for any $m \times n$ matrix $A$? One way is as follows:

- if $\mathbf{b} \in C(A)$, then $\mathbf{Proj}_{C(A)}(\mathbf{b}) = \mathbf{b}$.

So calculating the projection will tell us exactly if there is a solution.

**Example 3:** Do the following systems have solutions?

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} x + z &= 1 \\ x + y + z &= 0 \\ y + z &= 0 \\ z &= 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} x + z &= 2 \\ x + y + z &= -2 \\ y + z &= 2 \\ z &= 6 \end{aligned} \qquad \vec{b}_2 = \begin{pmatrix} 2 \\ -2 \\ 2 \\ 6 \end{pmatrix}$$

step 1) find column space using Gram-Schmidt to find orthogonal basis

result: $\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}}_{w_2}, \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}}_{w_3} \right\}$, dim $C(A) = 3$

step 2) check projections

1) $\text{proj}_{C(A)} \vec{b}_1 = \frac{1}{2}\vec{w}_1 + -\frac{1}{6}\vec{w}_2 + \frac{1}{12}\vec{w}_3 = \begin{pmatrix} - \\ - \\ - \\ 1/4 \end{pmatrix} \neq \vec{b}_1 \Rightarrow$ system 1 has no solution bc $b \notin C(A)$

2) $\text{proj}_{C(A)} \vec{b}_2 = 0\vec{w}_1 + 0\vec{w}_2 + 2\vec{w}_3 = \begin{pmatrix} 2 \\ -2 \\ 2 \\ 6 \end{pmatrix} = \vec{b}_2 \Rightarrow$ system 2 has at least one solution

Recall that the **image** of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ is the collection of vectors $\mathbf{b} \in \mathbb{R}^m$ obtained as output of $f$, i.e. the vectors $\mathbf{b}$ such that $\mathbf{b} = f(\mathbf{x})$ for some $\mathbf{x}$.

$= A\vec{x}$

image$(f) = C(A)$

We have now discussed the question "When does $A\mathbf{x} = \mathbf{b}$ have a solution?" We will now address the questions "If there is a solution, is there more than one? How do we describe the set of all solutions?"

let $\vec{x}_1, \vec{x}_2$ be solutions to $A\vec{x} = \vec{b}$, this means $A\vec{x}_1 = \vec{b}$, $A\vec{x}_2 = \vec{b}$

$A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = 0 \longrightarrow A(\vec{x}_1 - \vec{x}_2) = \vec{0} \implies \vec{x}_1 - \vec{x}_2$ solves $A\vec{x} = \vec{0}$

① $A\vec{x} = \vec{0}$ has one solution $(\vec{x} = \vec{0}) \implies \vec{x}_1 - \vec{x}_2 = \vec{0} \implies \vec{x}_1 = \vec{x}_2$

② $A\vec{d} = \vec{0}$ for $\vec{d} \neq 0 \implies \vec{x}_1 - \vec{x}_2 = t\vec{d} \implies \vec{x}_1 = \vec{x}_2 + t\vec{d} \implies A\vec{x} = \vec{b}$ has $\infty$ solutions

$\vec{d} \in N(A)$

bc $\left.\begin{matrix} A(t\vec{d}) = \vec{0} \\ t(A\vec{d}) = \vec{0} \end{matrix}\right\}$ scalar mult of $\vec{d}$ still produces $\vec{0}$

second solution is just linear comb of first w/ $t\vec{d}$ can be anything

We see that the number of solutions to $A\mathbf{x} = \mathbf{b}$ is closely related to whether or not there is a nonzero solution to $A\mathbf{x} = \mathbf{0}$; we call this the **homogeneous system** associated with $A$.

**Definition:** The **null space** of $A$, denoted $N(A)$, is the set of all solutions in $\mathbb{R}^n$ to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

In the context of linear transformations, $N(A)$ is everything that $T_A$ sends to $\mathbf{0}$. So if $T_A$ is invertible, $N(A) = \{\vec{0}\}$

**Example 4:** Let $V$ be a linear subspace of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ be the projection of a vector onto $V$. What is $N(A)$, where $A$ is the matrix associated with $T$? null space is $T(\vec{x}) = \vec{0}$

$T(\vec{x}) = \text{Proj}_V(\vec{x}) = \sum_{i=1}^{k} \text{Proj}_{\vec{w}_i}(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{x}\cdot\vec{w}_i}{\vec{w}_i\cdot\vec{w}_i}\vec{w}_i \implies x \cdot w_i = 0 \rightarrow x$ is orthogonal to all vectors in $V$

(orthog basis)

Therefore, if $T(\vec{x}) = \vec{0}$, then $\vec{x} \in V^\perp$.
(orthogonal complement)

Conversely, if $\vec{x} \in V^\perp$, then $T(\vec{x}) = \vec{0}$

$\implies \boxed{N(A) = V^\perp}$

**Proposition 21.3.5:** For any $m \times n$ matrix $A$, $N(A) \subset \mathbb{R}^n$ contains $\mathbf{0}$. Also, if $\mathbf{x}_1, \ldots, \mathbf{x}_k \in N(A)$, then any linear combination $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k \in N(A)$.

contains $\vec{0}$ $\quad \cdot\ A\vec{0} = \vec{0} \implies \vec{0} \in N(A)$

respects addition and scalar multiplication $\quad \cdot\ A(c_1\vec{x}_1 + \ldots + c_k\vec{x}_k) = c_1 A\vec{x}_1 + \ldots + c_k A\vec{x}_k = c_1(\vec{0}) + \ldots + c_k(\vec{0}) = \vec{0}$

**Proposition 21.3.6:** The null space $N(A)$ is a linear subspace of $\mathbb{R}^n$.

So in general, to find $N(A)$, one needs to solve $A\mathbf{x} = \mathbf{0}$ and express $\mathbf{x}$ as a linear combination of vectors.

☆ **Theorem 21.3.7:** If $V \subset \mathbb{R}^n$ satisfies (i) $\mathbf{0} \in V$ and (ii) for any $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$ and all scalars $c_1, \ldots, c_k$, $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k \in V$, then $V$ is a linear subspace of $\mathbb{R}^n$.

**Proposition 21.3.10:** For any $m \times n$ matrix $A$ and $\mathbf{b} \in \mathbb{R}^m$ for which the vector equation $A\mathbf{x} = \mathbf{b}$ has some solution $\mathbf{x}_0 \in \mathbb{R}^n$, the solutions to $A\mathbf{x} = \mathbf{b}$ are precisely the vectors of the form $\mathbf{x}_0 + \mathbf{d}$, where $\mathbf{d} \in N(A)$. There are infinitely many solutions whenever $N(A)$ contains a nonzero vector.

$A(\vec{x}_0 + \vec{d}) = A\vec{x}_0 + A\vec{d} = \vec{b} + \vec{0} = \vec{b}$

**Example 5:** How many solutions does the following system of linear equations?

$$\begin{aligned} x + y &= 2 \\ x - y + z &= 2 \\ 2y - z &= 0 \end{aligned} \quad \longrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

More generally, how can we tell if $\mathbf{b} \in C(A)$ for any $m \times n$ matrix $A$? One way is as follows:

- if $\mathbf{b} \in C(A)$, then $\mathbf{Proj}_{C(A)}(\mathbf{b}) = \mathbf{b}$.

So calculating the projection will tell us exactly if there is a solution.

Is there a solution?

Gram-Schmidt: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$, dim $C(A) = 2$

$\text{Proj}_{C(A)} \vec{b} = 2\vec{w}_1 + 0\vec{w}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \vec{b}$ ✓ $\therefore$ there's at least 1 solution

How many solutions?

Calculate $N(A)$ $\longrightarrow$ $A\vec{x} = \vec{0}$ $\longrightarrow$

$$\begin{aligned} x + y &= 0 \to x = -y \to y = -x \\ x - y + z &= 0 \to x + y - 2x = 0 \ ✓ \\ 2y - z &= 0 \to z = 2y = -2x \end{aligned}$$

$A\vec{x} = 0$ at $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ -2x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $\therefore N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$, dim $N(A) = 1$

Since $N(A)$ contains a nonzero vector, $A\vec{x} = \vec{b}$ has infinitely many solutions

solutions: $\underbrace{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}_{\vec{x}_0} + t \underbrace{\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}}_{\vec{d}}$, $t \in \mathbb{R}$

To summarize:

**Theorem 21.3.14:** Let $A$ be an $m \times n$ matrix. The equation $A\mathbf{x} = \mathbf{b}$ has

- no solution if $\mathbf{b} \notin C(A)$.
- exactly one solution if $\mathbf{b} \in C(A)$ **and** $N(A)$ consists of only $\mathbf{0}$.
- infinitely many solutions if $\mathbf{b} \in C(A)$ and $N(A)$ contains a **nonzero** vector.

★ **Theorem 21.3.16 (Rank-Nullity Theorem):** For every $m \times n$ matrix $A$,

$$\underbrace{\dim(C(A))}_{\text{rank}} + \underbrace{\dim(N(A))}_{\text{nullity}} = n \quad \longrightarrow \quad \underbrace{(\vec{x}_1, \dots, \vec{x}_k)}_{N(A)}, \overbrace{\vec{x}_{k+1}, \dots, \vec{x}_n}^{n-k \text{ vectors}}$$

$A\vec{x}_i = \vec{0}$

C(A)

These results give a sort of "guideline" to determining the number of solutions to over/under determined systems:

Suppose $A\mathbf{x} = \mathbf{b}$ is a system with $m$ equations in $n$ unknowns (so $A$ is $m \times n$).

- If the system is **overdetermined** ( $m > n$), then it often fails to have any solution since there are "too many equations" happening simultaneously.
- If the system is **underdetermined** $(m < n)$, then if there is a solution, there is automatically infinitely many since there are 'too few equations" so there is not enough information to pin down exactly one solution.

Again, these are not always the case; they are simply a guideline.

PCRQ:

Depending on $\vec{b}$ in $\mathbb{R}^3$, how many solution?

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$x+y=1$
$y=0$ no solution
$x+y=2$

$x+y=1$
$y=0$ one solution
$x+y=1$

$x+y=1$
$y=-1$ no solution
$x+y=2$

$C(A) = \text{span}\left(\left\{\begin{matrix}1\\0\\1\end{matrix}\right\}, \left\{\begin{matrix}1\\1\\1\end{matrix}\right\}\right)$ plane

$A\vec{x} = 0$
$x+y=0$
$y=0$ $\rightarrow$ $x=0$, $y=0$
$x+y=0$
$N(A) = \{\vec{0}\}$

$x + a_{12}y + a_{13}z = 2$
$-y + a_{23}z = -5$
$z = 3$ integer

$-y + 3a_{23} = -5$
$y = 3a_{23} + 5$, $a_{23} \in \mathbb{Z}$ integer

$x + a_{12}(3a_{23}+5) + a_{13}(3) = 2$
$x + 3a_{12}a_{23} + 5a_{12} + 3a_{13} = 2$
$x = 2 - 3a_{12}a_{23} + 5a_{12} + 3a_{13}$ integer

scalar multiple of $\mathbf{v}$). The null space $N(A)$ consists of vectors $\mathbf{x} \in \mathbf{R}^3$ that are orthogonal to each row of $A$ (by the meaning of $A\mathbf{x}$ and of $N(A)$), but those rows are multiples of $\mathbf{w}^\top$, so this