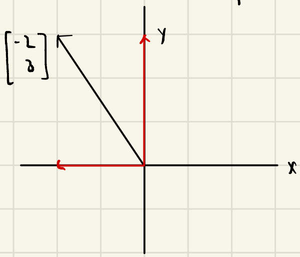
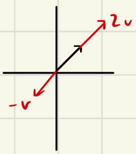
Essence of Linear Algebra

1) Vectors

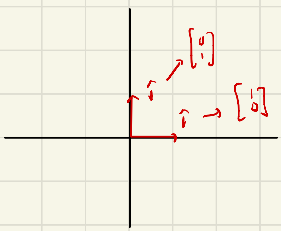
Vectors are arrows in space and ordered lists of numbers



Scaling → scalar multiplying, stretching vectors



2) Linear Combinations, Spans, and Basis



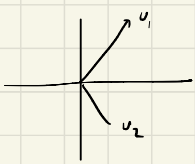
$\{\hat{i}, \hat{j}\}$ is standard basis

Linear combination

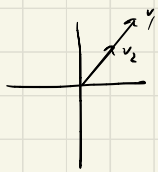
$$a\vec{v} + b\vec{w}$$

Span

set of all possible linear combinations, every vector you can reach



span is $\mathbb{R}^2$



span is a line

For $\mathbb{R}^3$, if 3^{rd} vector is in span of first two, the span is still just the span of first two

↳ linearly dependent → $v_3 \in \text{span}\{v_1, v_2\}$

↳ linearly independent → each vector adds a new dimension to span

Basis

Set of linearly independent vectors that span the full space

3) Linear Transformations & Matrices

Linear Transformation → fancy word for function

$\underset{\text{input}}{\vec{v}_1} \xrightarrow{\text{transform}} \underset{\text{output}}{\vec{v}_2}$

Linear → origin remains fixed $(f(0) = 0)$

lines remain lines $(c_1x_1 + c_2x_2 + c_3x_3 + \dots)$

↳ grid lines parallel & evenly spaced

To find matrix for linear transformation, find $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ & $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

Each column of matrix is where each standard basis vector lands after being transformed

4) Matrix Multiplication

For a composition of transformations, resulting matrix is where standard basis vectors land after all transformations

read right to left

$$\underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}_{\text{Shear}}\left(\underbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}_{\text{Rotation}}\begin{bmatrix}x\\y\end{bmatrix}\right) = \underbrace{\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}}_{\text{Composition}}\begin{bmatrix}x\\y\end{bmatrix}$$

Note: $\begin{bmatrix}3 & 1\\2 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}3\\2\end{bmatrix} + y\begin{bmatrix}1\\-2\end{bmatrix}$

When multiplying, imagine where the standard basis vectors $\hat{\imath}$ and $\hat{\jmath}$ lands.

After rotation, $\hat{\imath}$ is at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\hat{\jmath}$ is at $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then, we shear $\hat{\imath}$ and $\hat{\jmath}$ to get composition

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{final } \hat{\imath}} \qquad \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\text{final } \hat{\jmath}} \qquad \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{final matrix}}$$

5) Three-dimensional linear transformations

Similar to 2 dimension, transform based on standard basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.

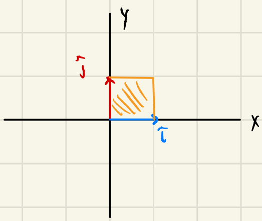
$$\begin{array}{c} \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \hat{\imath} & \hat{\jmath} & \hat{k} \end{array} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + y\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + z\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

6) The Determinant

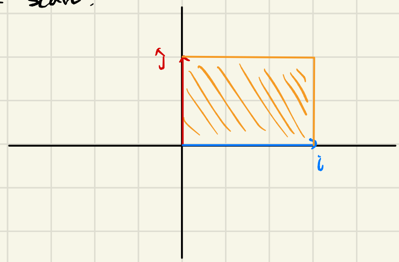
Exactly how much are things being stretched?

How much does the area of a given region scale?

$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$



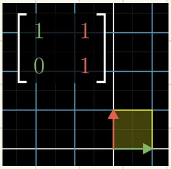
$\rightarrow$



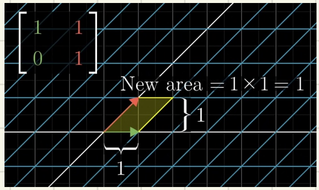
$\det(A) = 6$

Linear transformation scaled area of orange region by 6

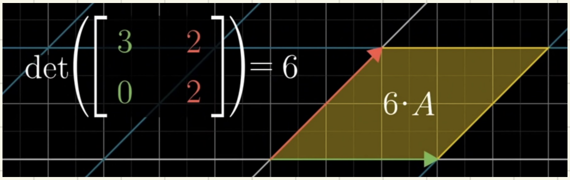
$\det(A) = 1$



$\rightarrow$

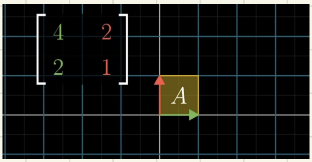


Determinant is the scaling factor of any given area



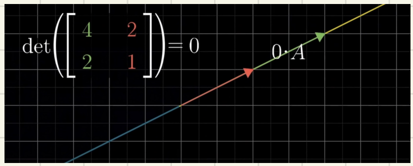
Determinant of 2D transformation is 0 if it squishes space into a line

Before



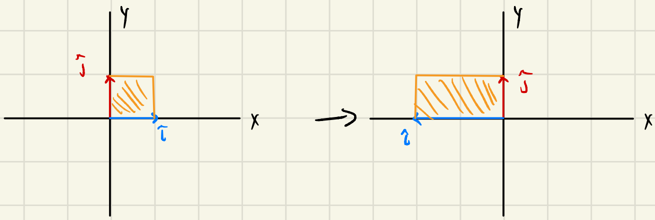
$\rightarrow$

After



A negative determinant means the orientation of space has been inverted

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$$



$\det(A) = -2$

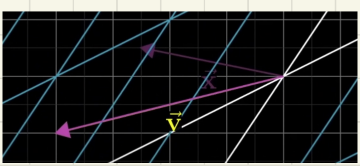
↑ space gets flipped over, and area is scaled by 2

When the determinant is zero, the columns are linearly dependent
↳ a basis vector after transformation is the linear combination of the other

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

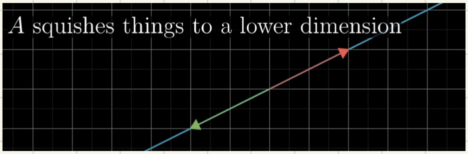
7) Inverse Matrices, Column Space, Null space

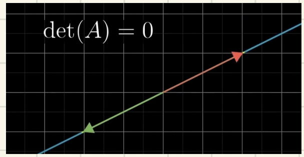
For $A\vec{x} = \vec{v}$, we solve for $\vec{x}$ where A applied to $\vec{x}$ brings it to $\vec{v}$



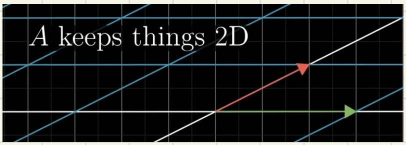
There are two cases for the solution of $A\vec{x} = \vec{v}$ (for example, $A \in \mathbb{R}^2$)

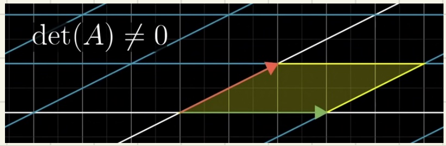
If A squishes everything into a lower dimension, $\det(A) = 0$





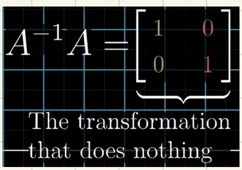
Or if it leaves everything spanning the two dimensions when it started, $\det(A) \neq 0$





Case 1 $\det(A) \neq 0$

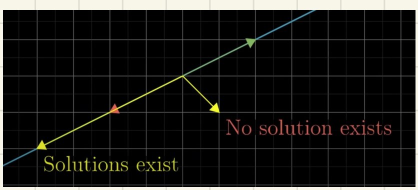
There will be 1 and only 1 vector $\vec{x}$ that will land on $\vec{v}$ after applying A. We do this by playing the transformation in reverse, starting from $\vec{v}$ and moving back to $\vec{x}$



$$A\vec{x} = \vec{v}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{v}$$
$$\vec{x} = A^{-1}\vec{v}$$

Case 2 $\det(A) = 0$

If it squishes it all into a line, we can't reverse this. We've lost information. A solution for $A\vec{x} = \vec{v}$ only exists if $\vec{v}$ lives on the line after A is applied



Rank: The number of dimensions in the output

↳ If A squishes it into a line, $\text{rank}(A) = 1$

↳ If A squishes it into a plane, $\text{rank}(A) = 2$

a) Set of all possible outputs for $A\vec{v}$ $\xleftrightarrow{\text{same as}}$ "column space" $\longleftrightarrow$ span of columns

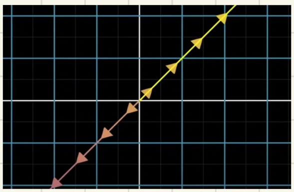
b) rank $\xleftrightarrow{\text{same as}}$ dim(column space)

c) full rank $\longleftrightarrow$ dim(column space) $= n$ $\rightarrow$ no linear dependence

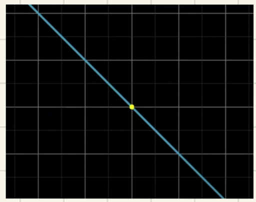
For full rank matrices, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only solution to $A\vec{x} = \vec{0}$

For rank < full rank, we can have multiple vectors as solutions to $A\vec{x} = \vec{0}$

e.g. If A squishes 2 dim to 1 dim, we have a whole line of vectors that map to $\vec{0}$



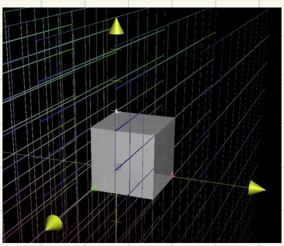
$\longrightarrow$



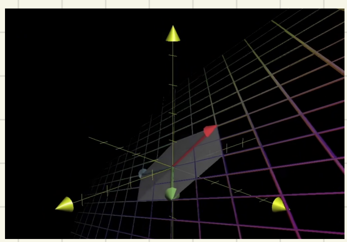
The set of vectors that land on the origin is the "null space"

↳ space of all vectors that become null

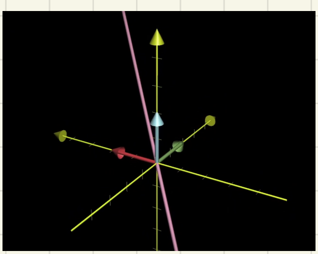
Ex. If 3D transformation A squishes space into plane,



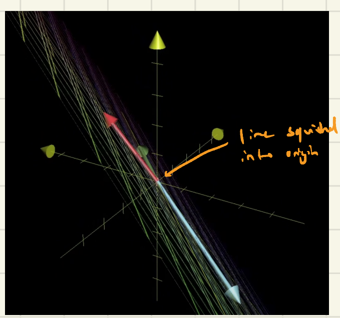
$\xrightarrow{A}$



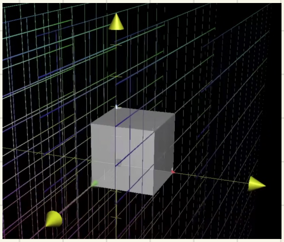
a whole line of vectors gets squished to the origin



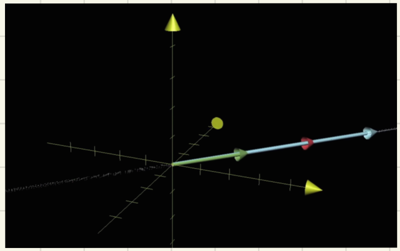
$\longrightarrow$



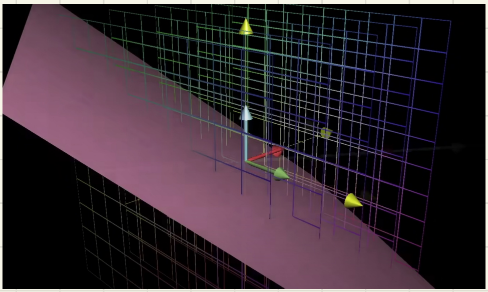
Ex. If 3D transformation A squishes space into line,



$\xrightarrow{A}$



then there is a whole plane of vectors that land on the origin



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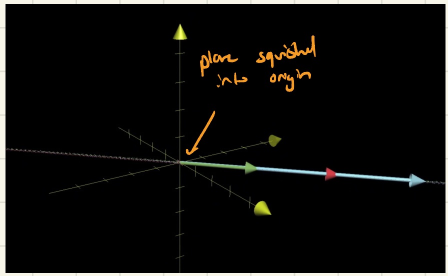


8) Linear Transformations of Non-square matrices → between dimensions

To find matrix, find coordinates of landing spot as column of the matrix

where $\hat{i}$ lands

$$A\vec{x} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -x+y \\ -2x+y \end{bmatrix}$$

where $\hat{j}$ lands

$f: \mathbb{R}^2 \to \mathbb{R}^3$ is a 3×2 matrix

$f: \mathbb{R}^n \to \mathbb{R}^m$ is an $m \times n$ matrix

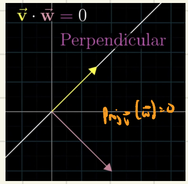
Column space of A is a 2D plane living in 3D space. However, the matrix is still full rank, because column space is same dimension as input space

For $m \times n$, $n = \dim(C(A)) + \dim(N(A))$

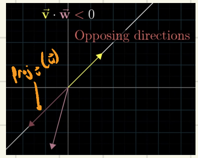
output ↑ ↑ input space

9) Dot products and duality

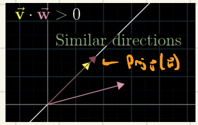
$\vec{v} \cdot \vec{w} = 0 \rightarrow \vec{v}$ & $\vec{w}$ perpendicular



$\vec{v} \cdot \vec{w} < 0 \rightarrow \vec{v}$ & $\vec{w}$ opposite dir

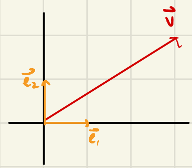


$\vec{v} \cdot \vec{w} > 0 \rightarrow \vec{v}$ & $\vec{w}$ same dir



13) Change of basis

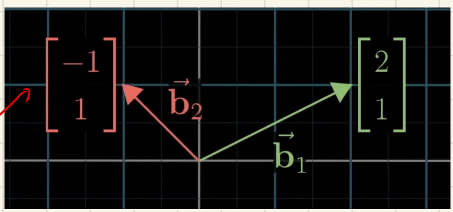
$\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is same as $3\vec{e}_1 + 2\vec{e}_2$, with basis $\{\vec{e}_1, \vec{e}_2\}$



For the standard coordinate system, $\{\vec{e}_1, \vec{e}_2\}$ are the basis vectors

We can create new coordinate systems by changing these basis vectors

coordinate in standard basis

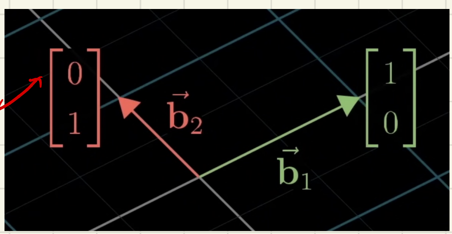


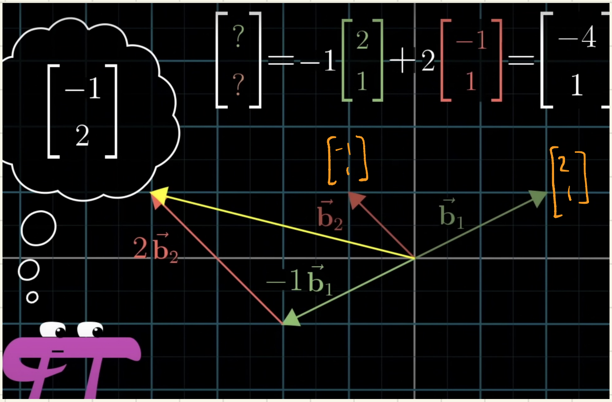
$\{\vec{b}_1, \vec{b}_2\}$ is the new basis

↓

This creates a new coordinate system

coordinate in new basis





Suppose we had $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in the new basis,

This is thus

$\vec{v} = -1 b_1 + 2 b_2$

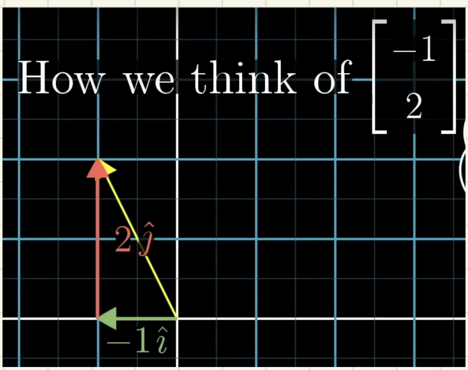
If $\vec{v}$ is in new basis, it is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

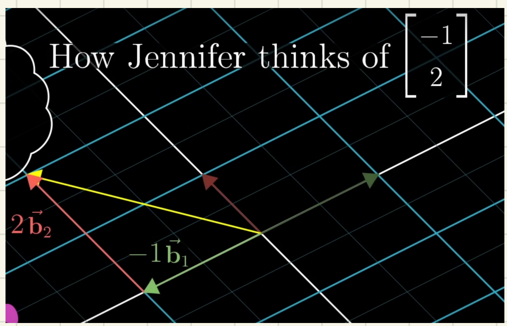
If it is in standard basis, it is

$-1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

We can represent change of basis as a matrix, with each column being the basis vectors.

Using previous example, this is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, and to find location of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in new basis in the standard basis, we have $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$





move $\hat{\imath}$ to $\vec{b}_1$, and $\hat{j}$ to $\vec{b}_2$

14) Eigenvectors and Eigenvalues

To find eigenvector, $\det(A - \lambda I_n) = 0$ bc it squishes space into a smaller dimension

$$(A - \lambda I_n)\vec{v} = 0$$

A nonzero vector can only become zero if $(A - \lambda I_n)$ squishes space into a lower dimension, bring $\vec{v}$ to zero.

Eigenbasis

If we change our basis to one made up of eigenvectors

Suppose $\hat{i}$ is scaled by -1 and $\hat{j}$ is scaled by two. This gives the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

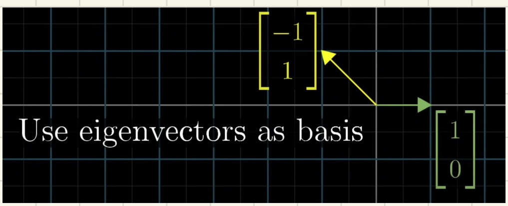
This is a diagonal matrix. All the basis vectors are eigenvectors: $\hat{i}$ w/ $\lambda = -1$, and $\hat{j}$ w/ $\lambda = 2$

You can change a matrix into an eigenbasis to compute powers

made up of eigenvectors

contains eigenvectors

$$W^{-1} A W = D$$



$$\rightarrow W = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Suppose $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$$W^{-1} A W = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}^{\text{diagonal w/ eigenvalues}}$$

$$W^{-1} A W = D$$
$$W W^{-1} A W = W D$$
$$A W = W D$$
$$A W W^{-1} = W D W^{-1}$$
$$A = W D W^{-1}$$

15) <u>Trick for computing eigenvalues</u>

1) $\frac{1}{2} \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{a+d}{2} = \frac{\lambda_1 + \lambda_2}{2} = m$ (mean)

2) $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \lambda_1 \lambda_2 = p$ (product

3) $\lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}$