Lecture 16 - Applications of Matrix Algebra

November 2, 2022

Goals: Apply matrix algebra to Markov chains.

Definition: A square matrix where the entries in each column are non-negative and sum to 1 is called a **Markov matrix**.

More generally, a Markov chain is a model which describes the evolution of a system governed at each stage by

- knowledge of the previous stage and no previous ones ("memory less" process)
- A Markov matrix of probabilities for transitioning among different possibilities.

Example 1: (Bird migration) Consider the following migration pattern of birds inhabiting three islands (A, B, and C):

- All the birds on island A migrate to another island each year. Half migrate to island B and the other half to island C.
- All the birds on island B migrate to another each year. Half migrate to island A and the other half to island C
- One-third of the birds on island C stay in place each year, one-third migrate to island A, and one-third to island B.

Suppose there are 10,000 birds on each island initially. Assuming no births/deaths, what happens to the population on each island after 20 years?

$\vec{P}_n = \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} \longrightarrow$ population on each island after n migrations (n years)

$\vec{P}_0 =$ initial state $= \begin{bmatrix} 10\,000 \\ 10\,000 \\ 10\,000 \end{bmatrix}$

How do I compute $\vec{P}_{n+1}$ if I know $\vec{P}_n$?

$$\vec{P}_{n+1} = \begin{pmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \end{pmatrix} = \begin{pmatrix} 0A_n + \frac12 B_n + \frac13 C_n \\ \frac12 A_n + 0B_n + \frac13 C_n \\ \frac12 A_n + \frac12 B_n + \frac13 C_n \end{pmatrix} = \begin{bmatrix} 0 & \frac12 & \frac13 \\ \frac12 & 0 & \frac13 \\ \frac12 & \frac12 & \frac13 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = M\vec{P}_n$$

$\downarrow \ \downarrow \ \downarrow$

$1 \quad 1 \quad 1$

Markov matrix bc all columns add to 1 and all entries non-negative

$$\vec{P}_1 = M\vec{P}_0 = \begin{bmatrix} 8333.33 \\ 8333.33 \\ 13333.33 \end{bmatrix}, \quad \vec{P}_2 = M(\vec{P}_1) = M(M\vec{P}_0) = M^2\vec{P}_0$$

$$\vec{P}_3 = M\vec{P}_2 = M(M^2\vec{P}_0) = M^3\vec{P}_0 \longrightarrow \boxed{\vec{P}_{20} = M^{20}\vec{P}_0} \approx \begin{bmatrix} .286 & .286 & .286 \\ .286 & .286 & .286 \\ .429 & .429 & .429 \end{bmatrix}\vec{P}_0$$

sums to 1 $\downarrow \ \downarrow \ \downarrow$

$1 \quad 1 \quad 1$

$$\vec{P}_{20} = \begin{bmatrix} 8580 \\ 8580 \\ 12870 \end{bmatrix}$$

* Powers of Markov matrices are also Markov matrices

It turns out that $M^r \approx M^{20}$ for $r \geq 20$. Therefore, it essentially "stabilizes"

Suppose $\vec{p}'_0 = \begin{pmatrix} 5000 \\ 5000 \\ 20000 \end{pmatrix}$, then $M^{20}\vec{p}'_0 = \begin{pmatrix} 8580 \\ 8580 \\ 12870 \end{pmatrix}$

As we saw in the previous example, our bird population **stabilized**; after a certain amount of time, the columns of our matrix remained the same, and any further powers remain roughly the same with high accuracy. We call this the **steady-state** of the system.

Because of this, our initial distribution of the birds was actually irrelevant to the problem! It turns out that Markov matrices with all positive entries will stabilize. As we just saw, a Markov matrix can still stabilize with 0 entries (and some assumptions which we will not specify), but this does not always happen.

Example 2: (An observation about stabilization) Suppose we had a system modeled by Markov matrix

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

! !

Does this system reach a steady-state?

$M^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ← identity matrix

Therefore, $M^3 = M^2 M = I^2 M = M$ bc M^2 is identity

$\Rightarrow \underbrace{M^{2k}}_{\text{even}} = \underbrace{I_2}_{2\times 2 \text{ identity}}$, $\underbrace{M^{2k+1}}_{\text{odd}} = M$

It continuously swaps back & forth.

↳ so system doesn't stabilize

Example 3: (Cauliflower Breeding) Suppose the color of a cauliflower is controlled by a single allele with two possibilities: r and R. Each cauliflower has the allele twice with three possible color outcomes:

$$RR \text{ - Purple,} \qquad Rr \text{ - Orange,} \qquad rr \text{ - green}$$

When two cauliflowers mate, their offspring inherits one allele from each parent at random.

(a) If we mate a purple cauliflower with an orange cauliflower, what are the chances that their offspring will be a certain color?

	R	R
R	RR	RR
r	Rr	Rr

1/2 chance of Purple
1/2 chance of Orange
0 chance of Green

(b) If we mate two orange cauliflowers, what are the chances that their offspring will be a certain color?

	R	r
R	RR	Rr
r	Rr	rr

1/4 chance of Purple
1/2 chance of Orange
1/4 chance of Green

(c) If we mate an orange cauliflower with a green cauliflower, what are the chances that their offspring will be a certain color?

	r	r
R	Rr	Rr
r	rr	rr

0 chance of Purple
1/2 chance of Orange
1/2 chance of Green

Suppose we start an experiment where we take a cauliflower of any color and mate it with an orange cauliflower. Then, we mate their offspring with another orange cauliflower, etc.

(d) For the nth generation, let

1/2 chance of being purple if other parent is purple (one parent is orange)
(from (a))

$$\mathbf{p}(n) = \begin{bmatrix} \text{probability that offspring is purple } (RR) \\ \text{probability that offspring is orange } (Rr) \\ \text{probability that offspring is green } (rr) \end{bmatrix}.$$

Compute the Markov matrix M associated with this system (we also call this the *transition matrix*).

$$\vec{p}\,(n+1) = \begin{bmatrix} \frac{1}{2}(RR) + \frac{1}{4}(Rr) + 0(rr) \\ \frac{1}{2}(RR) + \frac{1}{2}(Rr) + \frac{1}{2}(rr) \\ 0(RR) + \frac{1}{4}(Rr) + \frac{1}{2}(rr) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} RR \\ Rr \\ rr \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix} \vec{p}\,(n)$$

other parent purple   other parent orange   other parent green

(e) Suppose we know that

$$M^{10} \approx \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.25 \end{bmatrix}.$$

After the 10th generation, what is the probability we end up with a green cauliflower if we had started with a purple cauliflower? What if we started with an orange cauliflower? What if we started with a green cauliflower?

1) $\vec{p}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow M^{10}\vec{p}_0 = M^{10}\vec{e}_1 = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$ 0.25 chance of ending up with green starting with purple

2) $\vec{p}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longrightarrow M^{10}\vec{p}_0 = M^{10}\vec{e}_2 = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$ 0.25 chance of ending up with green starting with orange

3) $\vec{p}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow M^{10}\vec{p}_0 = M^{10}\vec{e}_3 = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$ 0.25 chance of ending up with green starting with green

Example 4: (Gambler's ruin) Suppose Alice has \$2 and Bob has \$3. They play a game with an "unfair" coin: it has probability 1/3 to land on heads and 2/3 to land on tails. Alice gives \$1 to Bob if the coin comes up heads, and Bob gives \$1 to Alice if the coin comes up tails. The game ends when either Alice or Bob gets all \$5. What is the probability that Alice wins?

$$\vec{p}(n) = \begin{bmatrix} \text{probability that Alice is at} & \$0 \\ & \$1 \\ & \$2 \\ & \$3 \\ & \$4 \\ & \$5 \end{bmatrix} = \begin{bmatrix} p_0(n) \\ p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \\ p_5(n) \end{bmatrix}$$

Initial

$$\vec{p}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{p}(1) = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 2/3 \\ 0 \\ 0 \end{bmatrix}$$

coin lands on heads, she loses \$1

coin lands on tails, she gains \$1

$$p_0(n+1) = \text{she start at \$1 and loses or she already had \$0} = p_1(n)\cdot\frac{1}{3} + \overbrace{p_0(n)\cdot 1}^{\text{end state of game}}$$

$$p_1(n+1) = p_2(n)\cdot\frac{1}{3} + \cancel{p_0(n)\cdot\frac{2}{3}}$$

you can't gain coins at \$0, bc that's end state

$$p_2(n+1) = p_1(n)\cdot\frac{2}{3} + p_3(n)\cdot\frac{1}{3}$$

$$p_3(n+1) = p_2(n)\cdot\frac{2}{3} + p_4(n)\cdot\frac{1}{3}$$

$$p_4(n+1) = p_3(n)\cdot\frac{2}{3}$$

$$p_5(n+1) = p_4(n)\cdot\frac{2}{3} + p_5(n)\cdot 1$$

$$\vec{p}(n+1) = \begin{bmatrix} p_1(n)\cdot\frac{1}{3} + p_0(n)\cdot 1 \\ p_2(n)\cdot\frac{1}{3} \\ p_1(n)\cdot\frac{2}{3} + p_3(n)\cdot\frac{1}{3} \\ p_2(n)\cdot\frac{2}{3} + p_4(n)\cdot\frac{1}{3} \\ p_3(n)\cdot\frac{2}{3} \\ p_4(n)\cdot\frac{2}{3} + p_5(n)\cdot 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 1 \end{bmatrix}}^{M} \vec{p}(n)$$

M^{28} stabilizes, $M^{28}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} .226 \\ \vdots \\ \vdots \\ .773 \end{pmatrix}$

alice original state

alice loses

alice wins