Exercise 18.1. In this exercise you will get some practice with matrix algebra involving inverses. Assume that all matrices in this exercise have size $n \times n$.

(a) Suppose A, B, C are invertible. Show that ABC is invertible by finding an explicit expression for its inverse in terms of A^{-1}, B^{-1}, and C^{-1} (you should check your answer by multiplying it by ABC separately on the left and on the right, verifying that you get I_n each way). Hint: try to adapt the pattern in the case of a product of two such matrices in Example 18.4.4.

(b) Let D be an $n \times n$ matrix (not necessarily invertible). Simplify $(ADA^{-1})^{12}$ as much as you can. (Hint: $(ADA^{-1})^2 = ADA^{-1}ADA^{-1} = AD\mathrm{I}_nDA^{-1} = AD^2A^{-1}$; can you continue the pattern?) This illustrates part of a (very useful) method for quickly computing large powers of matrices in Chapter 24.

a) Let $T_A = A\vec{x}$, $T_B = B\vec{x}$, and $T_C = C\vec{x}$. $T_A \circ T_B \circ T_C = A(B(C\vec{x})) = (ABC)\vec{x}$.

To invert this process, we reverse $A(B(C\vec{x}))$, then $B(C\vec{x})$, then $C\vec{x}$. Since A, B, C are invertible, this is equal to $T_C^{-1} \circ T_B^{-1} \circ T_A^{-1} = C^{-1}B^{-1}A^{-1}\vec{x}$.

Therefore, $\boxed{(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.}$

① $(C^{-1}B^{-1}A^{-1})(ABC) = C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}\mathrm{I}_nBC$
$= C^{-1}B^{-1}BC = C^{-1}\mathrm{I}_nC = C^{-1}C = \mathrm{I}_n$ ✓

② $(ABC)(C^{-1}B^{-1}A^{-1}) = ABCC^{-1}B^{-1}A^{-1} = AB\mathrm{I}_nB^{-1}A^{-1}$
$= ABB^{-1}A^{-1} = A\mathrm{I}_nA^{-1} = AA^{-1} = \mathrm{I}_n$ ✓

b) Casework: $(ADA^{-1})^3 = (ADA^{-1})^2(ADA^{-1}) = AD^2A^{-1}ADA^{-1} = AD^2\mathrm{I}_nDA^{-1} = AD^3A^{-1}$
$(ADA^{-1})^4 = (ADA^{-1})^3(ADA^{-1}) = AD^3A^{-1}ADA^{-1} = AD^3\mathrm{I}_nDA^{-1} = AD^4A^{-1}$

Proof: We claim $(ADA^{-1})^n = AD^nA^{-1}$. We prove this by induction.

Base Case: $n=1 \rightarrow (ADA^{-1})^1 = ADA^{-1}$ and $AD^1A^{-1} = ADA^{-1}$

Thus completing our base case.

Inductive Case: Assuming $(ADA^{-1})^n = AD^nA^{-1}$, we find $(ADA^{-1})^{n+1}$

$(ADA^{-1})^{n+1} = (ADA^{-1})^n(ADA^{-1}) = AD^nA^{-1}ADA^{-1} = AD^n\mathrm{I}_nDA^{-1} = AD^nDA^{-1}$
$= AD^{n+1}A^{-1}$

Thus completing our inductive case and our proof.

Using the above, we can conclude that $(ADA^{-1})^{12} = AD^{12}A^{-1}$

Exercise 18.4.

(a) Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ 2 & 5 & 13 \end{bmatrix}$. Verify that $A \begin{bmatrix} 1 & 1 & -1 \\ 10 & -3 & -2 \\ -4 & 1 & 1 \end{bmatrix} = \mathrm{I}_3$ (so by Theorem 18.1.8, A is invertible and $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 10 & -3 & -2 \\ -4 & 1 & 1 \end{bmatrix}$).

(b) Using (a), find all simultaneous solutions to the following system of equations, and verify directly that all solutions you find really do work:

$$\begin{cases} x + 2y + 5z = 4 \\ 2x + 3y + 8z = 3 \\ 2x + 5y + 13z = 2 \end{cases}.$$

a) $$A \begin{bmatrix} 1 & 1 & -1 \\ 10 & -3 & -2 \\ -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ 2 & 5 & 13 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 10 & -3 & -2 \\ -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+20-20 & 1-6+5 & -1-4+5 \\ 2+30-32 & 2-9+8 & -2-6+8 \\ 2+50-52 & 2-15+13 & -2-10+13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I_3}$$

$$\boxed{\text{Therefore, } A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 10 & -3 & -2 \\ -4 & 1 & 1 \end{bmatrix}}$$

b) Suppose the equation $A\vec{x} = \vec{b}$. If A is invertible, we multiply the left of both sides by A^{-1} to get $A^{-1}A\vec{x} = A^{-1}\vec{b} \rightarrow I_n \vec{x} = A^{-1}\vec{b} \rightarrow \vec{x} = A^{-1}\vec{b}$.

We can represent the system of equations as

$$A\vec{x} = \vec{b} : \underset{A}{\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ 2 & 5 & 13 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}$$

We found in (a) that A is invertible and $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 10 & -3 & -2 \\ -4 & 1 & 1 \end{bmatrix}$.

Therefore, to find the solutions $\vec{x}$, we solve $\vec{x} = A^{-1}\vec{b}$.

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 10 & -3 & -2 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+3-2 \\ 40-9-4 \\ -16+3+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 27 \\ -11 \end{bmatrix}$$

We check this with the system:

① $x + 2y + 5z = 4$

$5 + 2(27) + 5(-11) = 59 - 55 = 4$ ✓

② $2x + 3y + 8z = 3$

$2(5) + 3(27) + 8(-11) = 91 - 88 = 3$ ✓

③ $2x + 5y + 13z = 2$

$2(5) + 5(27) + 13(-11) = 145 - 143 = 2$ ✓

Thus, the solution to the system is $\boxed{x = 5, \quad y = 27, \quad z = -11}$

Exercise 18.7.

(a) Check that for any $a \in \mathbf{R}$ whatsoever, $M_a = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible with inverse $M_{-a} = \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Check that for any $b \in \mathbf{R}$ whatsoever, $N_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ is invertible with inverse $N_{-b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$.

(c) Show that for any $a, b \in \mathbf{R}$ whatsoever, $\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ is invertible by giving an explicit expression for its inverse. (Hint: write it as a product of matrices from parts (a) and (b), and be careful with the order.)

a) To be invertible, $M_a M_a^{-1} = M_a M_{-a} = I_3$.

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a+a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

We see that the matrix multiplication $M_a M_{-a}$ gives us directly the identity matrix. Therefore, regardless of what a is, we still get the identity matrix. In other words, for all $a \in \mathbb{R}$, M_a is invertible with inverse M_{-a}.

b) To be invertible, $N_b N_b^{-1} = N_b N_{-b} = I_3$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b+b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

We see that the matrix multiplication $N_b N_{-b}$ gives us directly the identity matrix. Therefore, regardless of what b is, we still get the identity matrix. In other words, for all $b \in \mathbb{R}$, N_b is invertible with inverse N_{-b}.

c) The matrix is equivalent to $N_b M_a$: $N_b M_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$

To find the inverse, we find $(N_b M_a)^{-1}$. Since we know that N_b and M_a are invertible, $(N_b M_a)^{-1} = M_a^{-1} N_b^{-1}$.

$$M_a^{-1} N_b^{-1} = M_{-a} N_{-b} = \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}$$

We check that this is the inverse by the fact that $AA^{-1} = I_n$.

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a+a & ab-ab \\ 0 & 1 & -b+b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Therefore, for any $a, b \in \mathbb{R}$, the given matrix is invertible with the inverse

$$\boxed{M_{-a} N_{-b} = \begin{bmatrix} 1 & -a & ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}.}$$

Exercise 18.11. Let A, B, C be $n \times n$ matrices for which $A = BC$. Assume that A is invertible.

(a) Show that B is also invertible by finding an expression for B^{-1} in terms of A, B, C, and A^{-1}. (Hint: use the equation $A = BC$ to "solve" for B^{-1} assuming it exists, then show that your expression is valid. To save effort, use Theorem 18.1.8.)

(b) Show that C is also invertible by checking that $A^{-1}B$ works. (Theorem 18.1.8 will again save effort.)

a) Since A is invertible, we can multiply both sides of $A = BC$ on the right by A^{-1} to get $AA^{-1} = BCA^{-1}$. This simplifies to $I_n = BCA^{-1}$. We can group this as $I_n = B(CA^{-1})$. By theorem 18.1.8, since B and (CA^{-1}) satisfy $B(CA^{-1}) = I_n$, B is invertible and (CA^{-1}) is its inverse. Therefore, $\boxed{B^{-1} = CA^{-1}}$.

b) Since A is invertible, we can multiply both sides of $A = BC$ on the left by A^{-1} to get $A^{-1}A = A^{-1}BC$. This simplifies to $I_n = A^{-1}BC$. We can group this as $I_n = (A^{-1}B)C$. By theorem 18.1.8, since $(A^{-1}B)$ and C satisfy $(A^{-1}B)C = I_n$, $(A^{-1}B)$ is invertible and C is its its inverse, and automatically, $C(A^{-1}B) = I_n$ holds. Therefore, C is also invertible and $(A^{-1}B)$ is its inverse. Thus, $\boxed{C^{-1} = A^{-1}B}$.

Alternatively, we assume $A^{-1}B = C^{-1}$. Therefore, we wish to show $A^{-1}BC = I_n$

$$(A^{-1}B)C = A^{-1}(BC) = A^{-1}A = I_n \quad \checkmark$$

By theorem 18.1.8, $C^{-1} = A^{-1}B$ and C is invertible.

Exercise 19.2. Define the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

(a) Let $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be the output of the Gram-Schmidt process applied to $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Check that $\mathbf{w}_3 = \mathbf{0}$ whereas $\mathbf{w}_1, \mathbf{w}_2$ are nonzero, so V is 2-dimensional (i.e., a plane in $\mathbf{R}^3$ through the origin) with both $\{\mathbf{w}_1, \mathbf{w}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ as bases. Use the definitions of $\mathbf{w}_2$ and $\mathbf{w}_3$ to write $\mathbf{w}_2$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with nonzero coefficients and to write $\mathbf{w}_3$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ with *nonzero* coefficients.

(b) Use the work in the Gram-Schmidt process to express each of $\mathbf{v}_2$ and $\mathbf{v}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, and compute each such linear combination explicitly to confirm that you indeed recover $\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

(c) Use the work in (a) to discover the relation $2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 = \mathbf{0}$ (this will come from the vanishing of $\mathbf{w}_3$), and find scalars a, b and a', b' so that $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$ and $\mathbf{v}_1 = a'\mathbf{v}_2 + b'\mathbf{v}_3$. Verify the correctness of the scalars by computing each right side to check it recovers the left side.

a) We apply the Gram-Schmidt process to V.

V_1: $\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

V_2: $\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - \text{Proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \dfrac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1$

$$= \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \frac{-5}{5}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

V_3: $\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}(\vec{v}_3) = \vec{v}_3 - \left(\text{Proj}_{\vec{w}_1}(\vec{v}_3) + \text{Proj}_{\vec{w}_2}(\vec{v}_3)\right)$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \left(\frac{-5}{5}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{9}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that $\vec{w}_1$ and $\vec{w}_2$ is nonzero, while $\vec{w}_3 = \vec{0}$. Therefore, $\dim(V) = 2$.

As linear combinations of $\vec{v}_1$ and $\vec{v}_2$,

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - \text{Proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \vec{v}_2 - (-1)\vec{w}_1$$

$$\boxed{\vec{w}_2 = \vec{v}_2 + \vec{v}_1}$$

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}(\vec{v}_3) = \vec{v}_3 - \left(\text{Proj}_{\vec{w}_1}(\vec{v}_3) + \text{Proj}_{\vec{w}_2}(\vec{v}_3)\right) = \vec{v}_3 - \left(-\vec{w}_1 + \frac{1}{3}\vec{w}_2\right)$$

$$\boxed{\vec{w}_3 = \vec{v}_3 + \vec{v}_1 - \frac{1}{3}(\vec{v}_2 + \vec{v}_1) = \frac{2}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2 + \vec{v}_3}$$

b) (1) $\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

(2) $\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - \text{Proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \vec{v}_2 - (-1)\vec{w}_1 = \vec{v}_2 + \vec{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$$\boxed{\vec{v}_2 = \vec{w}_2 - \vec{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}} \checkmark$$

(3) $\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}(\vec{v}_3) = \vec{v}_3 - \left(\text{Proj}_{\vec{w}_1}(\vec{v}_3) + \text{Proj}_{\vec{w}_2}(\vec{v}_3)\right) = \vec{v}_3 - \left(-\vec{w}_1 + \frac{1}{3}\vec{w}_2\right)$

$\vec{v}_3 = \vec{w}_3 - \vec{w}_1 + \frac{1}{3}\vec{w}_2$ since $\vec{w} = \vec{0}$, we can leave it out

$$\boxed{\vec{v}_3 = \frac{1}{3}\vec{w}_2 - \vec{w}_1 = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}} \checkmark$$

c) From (a), we know that $\vec{w}_3 = \vec{v}_3 + \vec{v}_1 - \frac{1}{3}(\vec{v}_2 + \vec{v}_1) = \frac{2}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2 + \vec{v}_3$. We also know that $\vec{w}_3 = \vec{0}$. Therefore, $\frac{2}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2 + \vec{v}_3 = \vec{0}$. If we multiply both sides by the scalar 3, we get $\boxed{2\vec{v}_1 - \vec{v}_2 + 3\vec{v}_3 = 0.}$

We rearrange this equation to get find a, b, a', and b'.

$\vec{v}_3 = a\vec{v}_1 + b\vec{v}_2$

$3\vec{v}_3 = -2\vec{v}_1 + \vec{v}_2$

$\vec{v}_3 = -\frac{2}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_2$

$= -\frac{2}{3}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ ✓

$$\boxed{a = -\frac{2}{3},\ b = \frac{1}{3}}$$

$\vec{v}_1 = a'\vec{v}_2 + b'\vec{v}_3$

$2\vec{v}_1 = \vec{v}_2 - 3\vec{v}_3$

$\vec{v}_1 = \frac{1}{2}\vec{v}_2 - \frac{3}{2}\vec{v}_3$

$= \frac{1}{2}\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ ✓

$$\boxed{a' = \frac{1}{2},\ b' = -\frac{3}{2}}$$

Exercise 19.3. Consider the following three 4-vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 10 \\ 7 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 10 \\ -6 \\ 10 \\ 4 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to verify that the $\mathbf{v}_i$'s are linearly independent by finding an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for their span V. (The vectors $\mathbf{w}_i$ that you compute should all have integer entries and be nonzero vectors; as a safety-check on your work you may wish to verify by direct computation that they are pairwise orthogonal.)

(b) Use the work in (a) to express each $\mathbf{w}_i$ as a linear combination of the $\mathbf{v}_j$'s, and to then express each $\mathbf{v}_j$ as a linear combination of the $\mathbf{w}_i$'s. Verify the correctness of your expressions for $\mathbf{w}_3$ and $\mathbf{v}_3$ by direct computation of the corresponding linear combination.

(c) Give an orthonormal basis of V.

a) We apply the Gram-Schmidt process to $V = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$

V_1: $\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}$

V_2: $\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - \text{Proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \dfrac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1$

$$= \begin{bmatrix} -2 \\ 10 \\ 7 \\ -1 \end{bmatrix} - \frac{42}{14}\begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 7 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix}$$

V_3: $\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}(\vec{v}_3) = \vec{v}_3 - \left(\text{Proj}_{\vec{w}_1}(\vec{v}_3) + \text{Proj}_{\vec{w}_2}(\vec{v}_3)\right)$

$$= \vec{v}_3 - \left(\frac{14}{14}\vec{w}_1 + \frac{-56}{28}\vec{w}_2\right)$$

$$= \begin{bmatrix} 10 \\ -6 \\ 10 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \\ 9 \end{bmatrix}$$

Since all w_i's are nonzero, $\dim(V) = 3$ and the collection of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent.

b) ① $\vec{w}_1 = \vec{v}_1$

② $\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}(\vec{v}_2) = \vec{v}_2 - \text{Proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \dfrac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \vec{v}_2 - 3\vec{w}_1$

$= \vec{v}_2 - 3\vec{v}_1$

③ $\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}(\vec{v}_3) = \vec{v}_3 - \left(\text{Proj}_{\vec{w}_1}(\vec{v}_3) + \text{Proj}_{\vec{w}_2}(\vec{v}_3)\right)$

$= \vec{v}_3 - \left(\vec{v}_1 - 2(\vec{v}_2 - 3\vec{v}_1)\right) = \vec{v}_3 - (\vec{v}_1 - 2\vec{v}_2 + 6\vec{v}_1) = -7\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3$

$$= -7\begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 10 \\ 7 \\ -1 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \\ 9 \end{bmatrix} \checkmark$$

(1) $\vec{v}_1 = \vec{w}_1$

(2) $\vec{w}_2 = \vec{v}_2 - \mathrm{Proj}_{v_1}(\vec{v}_2) = \vec{v}_2 - \mathrm{Proj}_{\vec{w}_1}(\vec{v}_2) = \vec{v}_2 - \dfrac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1$

$= \vec{v}_2 - 3\vec{w}_1$

$\vec{v}_2 = \vec{w}_2 + 3\vec{w}_1$

(3) $\vec{w}_3 = \vec{v}_3 - \mathrm{Proj}_{v_2}(\vec{v}_3) = \vec{v}_3 - \left(\mathrm{Proj}_{\vec{w}_1}(\vec{v}_3) + \mathrm{Proj}_{\vec{w}_2}(\vec{v}_3)\right) = \vec{v}_3 - (\vec{w}_1 - 2\vec{w}_2)$

$= \vec{v}_3 - \vec{w}_1 + 2\vec{w}_2$

$\vec{v}_3 = \vec{w}_3 + \vec{w}_1 - 2\vec{w}_2$

$$= \begin{bmatrix} 6 \\ 0 \\ 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 10 \\ 4 \end{bmatrix} \checkmark$$

c) We know that $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthogonal basis of V. To find an orthonormal basis, we divide each by its length.

Orthonormal basis : $\left\{ \dfrac{\vec{w}_1}{\|\vec{w}_1\|}, \dfrac{\vec{w}_2}{\|\vec{w}_2\|}, \dfrac{\vec{w}_3}{\|\vec{w}_3\|} \right\}$

$\|\vec{w}_1\| = \sqrt{w_1 \cdot w_1} = \sqrt{14}$ $\quad \|\vec{w}_2\| = \sqrt{w_2 \cdot w_2} = \sqrt{28} = 2\sqrt{7}$ $\quad \|\vec{w}_3\| = \sqrt{126} = 3\sqrt{14}$

An orthonormal basis of V is :

$$\boxed{\left\{ \frac{1}{\sqrt{14}}\begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{7}}\begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}}$$

Exercise 19.6. Consider the 5-vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -12 \\ 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 13 \\ -11 \\ 6 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Verify the two linear dependence relations

$$5\mathbf{v}_1 - \mathbf{v}_2 + 4\mathbf{v}_3 - 2\mathbf{v}_4 - \mathbf{v}_5 = \mathbf{0}, \quad 3\mathbf{v}_1 + 5\mathbf{v}_2 - 13\mathbf{v}_3 + 3\mathbf{v}_4 - 2\mathbf{v}_5 = \mathbf{0}.$$

(b) By forming suitable linear combinations of these relations to eliminate $\mathbf{v}_5$, express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Likewise express $\mathbf{v}_5$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(c) Check the correctness of your answer in (b) by evaluating explicitly the linear combinations that you obtained and checking that they recover $\mathbf{v}_4$ and $\mathbf{v}_5$.

a) $$5\vec{v}_1 - \vec{v}_2 + 4\vec{v}_3 - 2\vec{v}_4 - \vec{v}_5 = 5\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} + 4\begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -12 \\ 6 \\ -2 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 13 \\ -11 \\ 6 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -10 \\ 5 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -12 \\ 12 \\ 0 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ 4 \\ -2 \\ -12 \end{bmatrix} - \begin{bmatrix} 13 \\ -11 \\ 6 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\vec{0}}$$

$$3\vec{v}_1 + 5\vec{v}_2 - 13\vec{v}_3 + 3\vec{v}_4 - 2\vec{v}_5 = 3\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} - 13\begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -12 \\ 6 \\ -2 \\ 1 \\ 6 \end{bmatrix} - 2\begin{bmatrix} 13 \\ -11 \\ 6 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -6 \\ 3 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 20 \\ 5 \\ 15 \\ 10 \\ -10 \end{bmatrix} + \begin{bmatrix} 39 \\ -39 \\ 0 \\ -13 \\ -13 \end{bmatrix} + \begin{bmatrix} -36 \\ 18 \\ -6 \\ 3 \\ 18 \end{bmatrix} + \begin{bmatrix} -26 \\ 22 \\ -12 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\vec{0}}$$

Since these linear combinations evaluate to zero with at least one non-zero coefficient, $\{v_1, v_2, v_3, v_4, v_5\}$ is linearly dependent.

b) $2\left(5\vec{v}_1 - \vec{v}_2 + 4\vec{v}_3 - 2\vec{v}_4 - \vec{v}_5 = \vec{0}\right)$, $-1\left(3\vec{v}_1 + 5\vec{v}_2 - 13\vec{v}_3 + 3\vec{v}_4 - 2\vec{v}_5 = \vec{0}\right)$

$$10\vec{v}_1 - 2\vec{v}_2 + 8\vec{v}_3 - 4\vec{v}_4 - 2\vec{v}_5 = \vec{0}$$

$+$

$$-3\vec{v}_1 - 5\vec{v}_2 + 13\vec{v}_3 - 3\vec{v}_4 + 2\vec{v}_5 = \vec{0}$$

$$7\vec{v}_1 - 7\vec{v}_2 + 21\vec{v}_3 - 7\vec{v}_4 = 0$$

$$\boxed{\vec{v}_4 = \vec{v}_1 - \vec{v}_2 + 3\vec{v}_3}$$

$$3\left(5\vec{v}_1 - \vec{v}_2 + 4\vec{v}_3 - 2\vec{v}_4 - \vec{v}_5 = \vec{0}\right), \quad 2\left(3\vec{v}_1 + 5\vec{v}_2 - 13\vec{v}_3 + 3\vec{v}_4 - 2\vec{v}_5 = \vec{0}\right)$$

$$15\vec{v}_1 - 3\vec{v}_2 + 12\vec{v}_3 - 6\vec{v}_4 - 3\vec{v}_5 = \vec{0}$$

$+$

$$6\vec{v}_1 + 10\vec{v}_2 - 26\vec{v}_3 + 6\vec{v}_4 - 4\vec{v}_5 = \vec{0}$$

$$21\vec{v}_1 + 7\vec{v}_2 - 14\vec{v}_3 - 7\vec{v}_5 = 0$$

$$\boxed{\vec{v}_5 = 3\vec{v}_1 + \vec{v}_2 - 2\vec{v}_3}$$

c) $\vec{v}_4 = \vec{v}_1 - \vec{v}_2 + 3\vec{v}_3$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \\ -2 \\ 1 \\ 6 \end{bmatrix} \checkmark$$

$\vec{v}_5 = 3\vec{v}_1 + \vec{v}_2 - 2\vec{v}_3$

$$= 3\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ -11 \\ 6 \\ 0 \\ -1 \end{bmatrix} \checkmark$$

Exercise 19.8. This exercise explores how linear dependence and linear independence impact the behavior of linear combinations. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbf{R}^n$ are n-vectors that satisfy the linear dependence relation

$$-3\mathbf{v}_1 + 7\mathbf{v}_2 - 5\mathbf{v}_3 + 2\mathbf{v}_4 = \mathbf{0}.$$

(a) Verify that for any scalars a_1, a_2, a_3, a_4 and any scalar t,

$$(a_1 - 3t)\mathbf{v}_1 + (a_2 + 7t)\mathbf{v}_2 + (a_3 - 5t)\mathbf{v}_3 + (a_4 + 2t)\mathbf{v}_4 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4.$$

The lesson (using $t \neq 0$) is that in the presence of linear dependence, two *different* choices of coefficients for the $\mathbf{v}_i$'s can yield linear combinations that are *equal* as vectors.

(b) Using (a), give two different 4-tuples (b_1, b_2, b_3, b_4) and (c_1, c_2, c_3, c_4) that are both different from $(-2, 4, 3, -5)$ and yet satisfy

$$b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + b_4\mathbf{v}_4 = -2\mathbf{v}_1 + 4\mathbf{v}_2 + 3\mathbf{v}_3 - 5\mathbf{v}_4 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4.$$

(There are many possible answers; you don't have to know *anything* about the $\mathbf{v}_i$'s beyond their linear dependence relation, which is all that is used in (a).)

(c) Now consider four vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 \in \mathbf{R}^n$ that are linearly *independent*. The problem of "non-unique coefficients" for a linear combination as in (b) never happens for such $\mathbf{w}_i$'s! Namely, verify that the only way there can be an equality of vectors

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 + a_4\mathbf{w}_4 = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3 + b_4\mathbf{w}_4$$

for scalars a_i and b_j is when they match term-by-term: $a_1 = b_1, \ldots, a_4 = b_4$. This is an important general feature of linear independence, done here for four vectors only for specificity; the method here is completely general, working for any number of linearly independent n-vectors. (Hint: subtract the right side from the left side and combine terms.)

a) We first expand all of the terms via distribution

$$(a_1 - 3t)\vec{v}_1 + (a_2 + 7t)\vec{v}_2 + (a_3 - 5t)\vec{v}_3 + (a_4 + 2t)\vec{v}_4$$

$$= a_1\vec{v}_1 - 3t\vec{v}_1 + a_2\vec{v}_2 + 7t\vec{v}_2 + a_3\vec{v}_3 - 5t\vec{v}_3 + a_4\vec{v}_4 + 2t\vec{v}_4$$

$$= a_1\vec{v}_1 + a_2\vec{v}_3 + a_3\vec{v}_3 + a_4\vec{v}_4 - 3t\vec{v}_1 + 7t\vec{v}_2 - 5t\vec{v}_3 + 2t\vec{v}_4$$

$$= a_1\vec{v}_1 + a_2\vec{v}_3 + a_3\vec{v}_3 + a_4\vec{v}_4 + t\left(-3\vec{v}_1 + 7\vec{v}_2 - 5\vec{v}_3 + 2\vec{v}_4\right)$$

We know that the vectors satisfy the linear dependence relation

$-3\vec{v}_1 + 7\vec{v}_2 - 5\vec{v}_3 + 2\vec{v}_4 = 0$. Therefore, the above equation is equal to

$$= a_1\vec{v}_1 + a_2\vec{v}_3 + a_3\vec{v}_3 + a_4\vec{v}_4 + t(0)$$

$$= a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4$$

Thus, for any scalars a_1, a_2, a_3, a_4, and t,

$$(a_1 - 3t)\vec{v}_1 + (a_2 + 7t)\vec{v}_2 + (a_3 - 5t)\vec{v}_3 + (a_4 + 2t)\vec{v}_4$$

$$= a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4$$

because the vectors are linearly dependent with $-3\vec{v}_1 + 7\vec{v}_2 - 5\vec{v}_3 + 2\vec{v}_4 = 0$

b) From (a), we know the following

$$(a_1 - 3t)\vec{v}_1 + (a_2 + 7t)\vec{v}_2 + (a_3 - 5t)\vec{v}_3 + (a_4 + 2t)\vec{v}_4 = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4$$

In this case of $-2\vec{v}_1 + 4\vec{v}_2 + 3\vec{v}_3 - 5\vec{v}_4$: $a_1 = -2$, $a_2 = 4$, $a_3 = 3$, $a_4 = -5$

To find our set of tuples b and c, we simply select different values of t, $t \neq 0$ for the following equation to find our new coefficients.

$$(-2 - 3t)\vec{v}_1 + (4 + 7t)\vec{v}_2 + (3 - 5t)\vec{v}_3 + (-5 + 2t)\vec{v}_4 = -2\vec{v}_1 + 4\vec{v}_2 - 3\vec{v}_3 - 5\vec{v}_4$$

(1) Let $t = 1$

$$(-2 - 3)\vec{v}_1 + (4 + 7)\vec{v}_2 + (3 - 5)\vec{v}_3 + (-5 + 2)\vec{v}_4$$

$$= -2\vec{v}_1 + 4\vec{v}_2 + 3\vec{v}_3 - 5\vec{v}_4 + (-3\vec{v}_1 + 7\vec{v}_2 - 5\vec{v}_3 + 2\vec{v}_4)$$

$$= -2\vec{v}_1 + 4\vec{v}_2 + 3\vec{v}_3 - 5\vec{v}_4 \checkmark$$

$$(-2 - 3)\vec{v}_1 + (4 + 7)\vec{v}_2 + (3 - 5)\vec{v}_3 + (-5 + 2)\vec{v}_4$$

$$= -5\vec{v}_1 + 11\vec{v}_2 - 2\vec{v}_3 - 3\vec{v}_4$$

$$\boxed{\text{Therefore, } b = (-5, 11, -2, -3)}$$

(1) Let $t = 2$

$$(-2 - 3(2))\vec{v}_1 + (4 + 7(2))\vec{v}_2 + (3 - 5(2))\vec{v}_3 + (-5 + 2(2))\vec{v}_4$$

$$= -2\vec{v}_1 + 4\vec{v}_2 + 3\vec{v}_3 - 5\vec{v}_4 + 2(-3\vec{v}_1 + 7\vec{v}_2 - 5\vec{v}_3 + 2\vec{v}_4)$$

$$= -2\vec{v}_1 + 4\vec{v}_2 + 3\vec{v}_3 - 5\vec{v}_4 \checkmark$$

$$(-2 - 3(2))\vec{v}_1 + (4 + 7(2))\vec{v}_2 + (3 - 5(2))\vec{v}_3 + (-5 + 2(2))\vec{v}_4$$

$$= -8\vec{v}_1 + 18\vec{v}_2 - 7\vec{v}_3 - \vec{v}_4$$

$$\boxed{\text{Therefore, } c = (-8, 18, -7, -1)}$$

c) We are given the following equation, and the fact that $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$ are linearly independent.

$$a_1\vec{w}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 + a_4\vec{w}_4 = b_1\vec{w}_1 + b_2\vec{w}_2 + b_3\vec{w}_3 + b_4\vec{w}_4$$

$$a_1\vec{w}_1 - b_1\vec{w}_1 + a_2\vec{w}_2 - b_2\vec{w}_2 + a_3\vec{w}_3 - b_3\vec{w}_3 + a_4\vec{w}_4 - b_4\vec{w}_4 = 0$$

$$(a_1 - b_1)\vec{w}_1 + (a_2 - b_2)\vec{w}_2 + (a_3 - b_3)\vec{w}_3 + (a_4 - b_4)\vec{w}_4 = 0$$

By the definition of linear independence, the only collection of scalars where $c_1\vec{w}_1 + c_2\vec{w}_2 + c_3\vec{w}_3 + c_4\vec{w}_4 = 0$ is $c_1 = c_2 = c_3 = c_4 = 0$.

Thus, we use the equation and this fact to find the following:

$$c_1 = a_1 - b_1 = 0 \rightarrow a_1 = b_1$$

$$c_2 = a_2 - b_2 = 0 \rightarrow a_2 = b_2$$

$$c_3 = a_3 - b_3 = 0 \rightarrow a_3 = b_3$$

$$c_4 = a_4 - b_4 = 0 \rightarrow a_4 = b_4$$

Therefore, given that $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$ is linearly independent, the only time when $a_1\vec{w}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 + a_4\vec{w}_4$ equals $b_1\vec{w}_1 + b_2\vec{w}_2 + b_3\vec{w}_3 + b_4\vec{w}_4$ is when $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, and $a_4 = b_4$.