# Midterm 2: Ch 10 → Ch 17

**Ch 10:** Critical points, max/min on a region

**ex1:** $f(x,y) = x^3 - 3x^2 - 6xy + 9x + 3y^2$

To find crit point, $f_x = 0$ & $f_y = 0$

$f_x = 0 = 3x^2 - 6x - 6y + 9$ — plug in

$f_y = 0 = -6x + 6y \rightarrow x = y$

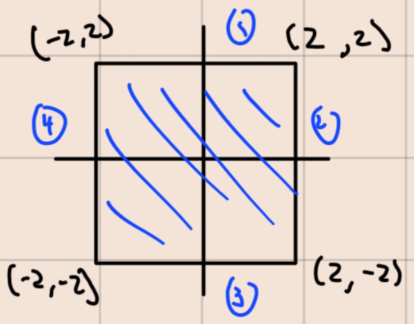
$$3x^2 - 6x - 6x + 9 = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 0$$

$$3(x-3)(x-1) = 0$$

$$x = 3, 1$$

→ crit points are $(3,3)$, $(1,1)$

**ex2:** using F from above, find max/min of F on R




crit points

i) interior: $(1,1)$ from ex1

ii) region 1: $y = 2$, $-2 \le x \le 2$

$$f(x,2) = x^3 - 3x^2 - 12x + 9x + 12$$

$$\frac{d}{dx} f(x,2) = 3x^2 - 6x - 3$$

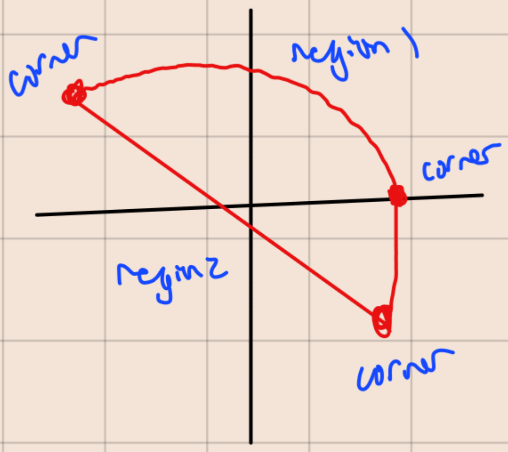
$$= 3(x^2 - 2x - 1) = 0$$

$$x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)} = 1 \pm \sqrt{2}$$

not in R

possible crit points are $\cancel{(1+\sqrt{2}, 0)}$ and $(1-\sqrt{2}, 0)$

repeat for region 2, 3, and 4, and check corners, then plug points into f : largest = max, smallest = min




if this is the region, we have to check

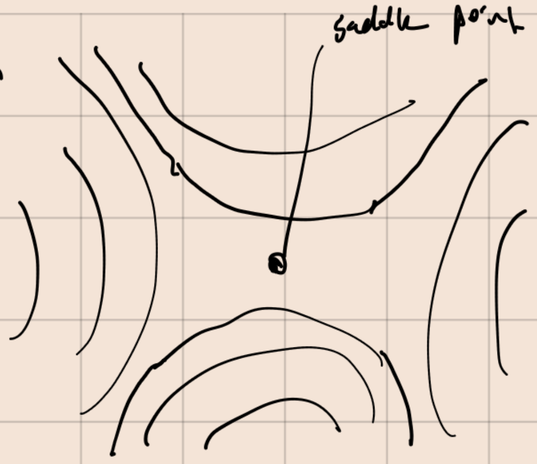
1) interior
2) region 1 (arc)
3) region 2 (line)
4) corners

↳ since region 1 is split into two, there's another corner where the split is

↳ it's split into two bc its a sqrt function, needs a second to show negative part

Ex 3

contour plot


Ch 11: gradient $f: \mathbb{R}^n \to \mathbb{R}$

$$\nabla f : \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{bmatrix}$$

Suppose $f(x,y) = c$ is the level curve. The tangent line to the curve at $(a,b)$ is

$$\nabla f(a,b) \cdot \begin{bmatrix} x-a \\ x-b \end{bmatrix} = 0$$


$$f(x,y) - f(a,b) \approx 0.$$

Linear approx: $f(x,y) \approx f(a,b) + \nabla f(a,b) \cdot \begin{bmatrix} x-a \\ x-b \end{bmatrix}$

Ex 1

$f: \mathbb{R}^3 \to \mathbb{R}$ $\qquad f(x,y,z) = c \to$ surface, not curve

Tangent plane to the surface @ $(a, b, c)$ is

$$\nabla f(a,b,c) \cdot \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix} = 0$$

# Ch 12: lagrange multipliers

max / min $f(\vec{x})$ subject to $g(\vec{x}) = c$ occurs at either

$$\nabla g = 0 \quad \text{or}$$

$$\nabla f = \lambda \nabla g$$

Using this, if $f$ is $\mathbb{R}^n$, you get a system of $n+1$ equations.

↳ $n$ equations from $\nabla f = \lambda \nabla g$

↳ 1 equation from constraint: $g(\vec{x}) = c$

ex 1

find max / min of $f(x,y) = x^2 y$ subject to $\overbrace{2x^2 + y^2 = 3}^{g(\vec{x}) = c}$

$$\nabla f = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}, \quad \nabla g = \begin{pmatrix} 4x \\ 2y \end{pmatrix}$$

1) $\nabla g = 0 \Rightarrow \begin{pmatrix} 4x \\ 2y \end{pmatrix} = 0 \rightarrow x = 0,\ y = 0$, but $g(0,0) \neq 3$ so no point here

2) $\nabla f = \lambda \nabla g \quad \begin{pmatrix} 2xy \\ x^2 \end{pmatrix} = \lambda \begin{pmatrix} 4x \\ 2y \end{pmatrix}$

Solve these equations

$$\begin{cases} 2xy = \lambda 4x & \longrightarrow \lambda = \dfrac{2xy}{4x} \\ x^2 = \lambda 2y & \longrightarrow \lambda = \dfrac{x^2}{2y} \\ 2x^2 + y^2 = 3 \end{cases}$$

constraint

1) Check when denominator equal 0 with constraint

case 1: $4x = 0 \Rightarrow x = 0$, $g(0, y) = 2(0)^2 + y^2 = 3 \rightarrow y = \pm\sqrt{3}$

$$\boxed{\text{possible points are } (0, \sqrt{3}) \ \& \ (0, -\sqrt{3})}$$

case 2: $2y = 0 \to y = 0 \to x^2 = 2\lambda y \to x^2 = 2\lambda(0) \to x = 0$

but $g(0,0) \neq 3$ so no possible points

2) check when denom nonzero

case 3:

$\lambda = \frac{2xy}{4x} = \frac{x^2}{2y} \to \frac{y}{2} = \frac{x^2}{2y} \to 2y^2 = 2x^2 \to y^2 = x^2$ plug into constraint

$2x^2 + y^2 = 3 \to 2x^2 + x^2 = 3 \to x^2 = 1 \to x = \pm 1$

$y = \pm 1$

$\boxed{\text{possible points are } (1,1), (-1,1), (1,-1), (-1,-1)}$

3) plug all points into $f = x^2 y$

$f(1,1) = 1$ $\quad f(0, \sqrt{3}) = 0$

$f(-1,1) = 1$ $\quad f(0, -\sqrt{3}) = 0$

$f(1,-1) = -1$

$f(-1,-1) = -1$

$$\boxed{\begin{array}{l} \text{max} = 1 \text{ @ } f(1,1),\ f(-1,1) \\ \text{min} = -1 \text{ @ } f(1,-1),\ f(-1,-1) \end{array}}$$

Cross multiplication method

2) $\nabla f = \lambda \nabla g$ $\quad \begin{pmatrix} 2xy \\ x^2 \end{pmatrix} = \lambda \begin{pmatrix} 4x \\ 2y \end{pmatrix}$

solve these equations

$$\begin{cases} 2xy = \lambda 4x \\ x^2 = \lambda 2y \\ 2x^2 + y^2 = 3 \end{cases}$$

constraint

$\to 2xy\lambda 2y = x^2 \lambda 4x$

case 1: $\lambda = 0 \to \nabla f = \lambda \nabla g = 0$

$x = 0,\ y = 0$, but $g(0,0) \neq 3$

so breaks constraint

case 2: $\lambda \neq 0$

$2xy2y = x^2 4x$

$4xy^2 = 4x^3$

$y^2 = x^2$ plug into constraint

$2x^2 + y^2 = 3 \to 2x^2 + x^2 = 3 \to x^2 = 1 \to x = \pm 1$

$y = \pm 1$

possible points are $(1,1), (-1,1), (1,-1), (-1,-1)$

## Ch 13-15: Matrix Algebra

$T: \mathbb{R}^n \to \mathbb{R}^m$ is linear

$T(\vec{x}) = A\vec{x}$

$m \times n$ $\quad$ $n \times 1$

**Find A:**

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$

- to find A, get each column
- each column of A is the application of T onto the standard basis vectors

$$T(e_1) = A(e_1) = A\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \text{first column of } A$$

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$

steps of T:
1) multiply 1st component by $-1$ $(\mathbb{R}^2 \to \mathbb{R}^2)$
2) attach 0 in third component $(\mathbb{R}^2 \to \mathbb{R}^3)$

**Find A**

Method 1:

$$T(e_1): \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, the matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Method 2: let matrix B be step 1, and C be step 2

B) apply $e_1$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, apply $e_2$: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

C) apply $e_1$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, apply $e_2$: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Then $A\vec{x} = C(B\vec{x})$

$$CB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = A$$

## Ch 16: Markov Matrices

Suppose we have $P_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in year $n$, $P_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x_n + \frac{2}{3}y_n \\ \frac{2}{3}x_n + \frac{1}{3}y_n \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}}_{M} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

if $M^n$ stays the same for large $n$,

the markov matrix stabilizes $\rightarrow M^n P_0$

## Ch 17 chain rule

$f: \mathbb{R}^n \to \mathbb{R}^m \qquad f = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix}$

$$Df = \begin{bmatrix} \text{——} & \nabla f_1 & \text{——} \\ & \vdots & \\ \text{——} & \nabla f_m & \text{——} \end{bmatrix}$$

$g: \mathbb{R}^p \to \mathbb{R}^n \quad , \quad f \circ g: \mathbb{R}^p \to \mathbb{R}^m$

$$\boxed{D(f \circ g)(\vec{x}) = Df(g(\vec{x})) \cdot Dg(\vec{x})}$$

ex 1

$$f(x,y,z) = \begin{pmatrix} xy+z \\ yz \end{pmatrix}, \quad g(x,y) = \begin{pmatrix} x+y \\ x-y \\ xy \end{pmatrix}$$

Find $D(f \circ g)(1,1)$

$$Df = \begin{bmatrix} y & x & 1 \\ 0 & z & y \end{bmatrix} \qquad Dg = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ y & x \end{bmatrix}$$

$$g(1,1) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$D(f \circ g)(1,1) = Df(g(1,1)) \cdot Dg(1,1)$$

$$= Df(2,0,1) \cdot Dg(1,1)$$

$$= \underbrace{\begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{2 \times 3} \cdot \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}}_{3 \times 2} = \underbrace{\boxed{\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}}}_{2 \times 2}$$