Math 51 Second Exam (Practice #1)

Name: ______________________________ **SUNet ID:** ____________ **ID #:** ___________

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- **You are being examined on mastery of methods taught in this course; solutions based on other techniques do not earn credit.** You may use any result discussed in class or the text, but clearly state the result before using it, and verify that the hypotheses are satisfied.
- Please check that your copy of this exam contains 7 pages of exam questions, *numbered* in the upper-right, and that it is adequately stapled.
- You may use 1 handwritten piece of $8.5'' \times 11''$ paper (both sides) with formulas and other notes as a "reference sheet". No electronic devices, including phones, headphones, or calculation aids, are permitted for any reason.
- **You have 2 hours.** The exam organizer will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to a member of teaching staff.
- Paper not provided by course staff (apart from your own reference sheet) is prohibited. If you need extra room for your answers, use one of the blank pages provided (those pages except for the one at the end are labeled at the bottom by lower-case Roman numerals, starting with "ii"), and **clearly indicate that that your answer continues there**. Do not unstaple or detach pages from this exam.
- It is your responsibility to look over your graded exam in a timely manner. You have until **two weeks after the date of this exam (5pm)**, to resubmit your exam for any regrade considerations; consult your section leader about the exact details of the submission process.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ________________________________

Please do not detach this page. If you use any of this space to continue your answer, please clearly indicate the problem number(s) and **indicate on the original page of the problem that your answer continues here**.

Please do not detach this page. If you use any of this space to continue your answer, please clearly indicate the problem number(s) and **indicate on the original page of the problem that your answer continues here**.

Please do not detach this page. If you use any of this space to continue your answer, please clearly indicate the problem number(s) and **indicate on the original page of the problem that your answer continues here**.

1. (10 points) Find the maximum and minimum values of $f(x,y) = x^2y - y^2 - 3x^2$ on the square $S = \{(x,y) \in \mathbf{R}^2 : -1 \le x, y \le 1\}$ shown in Figure 1, and the point(s) at which each is attained.

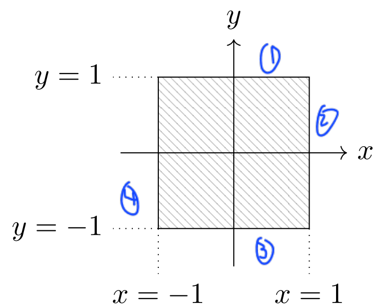


Figure 1: The square S consisting of points (x,y) with $-1 \le x, y \le 1$.

Interior

$\nabla f(x,y) = \begin{bmatrix} 2xy - 6x \\ x^2 - 2y \end{bmatrix} = \vec{0} \rightarrow$ $2xy - 6x = x(2y-6) = 0 \rightarrow x = 0,\ y = 3$

$x^2 - 2y = 0 \rightarrow x^2 = 2y$

$0^2 = 2y$, $y = 0$

$x^2 = 2(3)$, $x^2 = 6$, $x = \pm\sqrt{6}$

possible points are $(0,0)$, $\cancel{(-\sqrt{6}, 3)}$ not in S, and $\cancel{(\sqrt{6}, 3)}$ not in S

$f(x,y) = x^2y - y^2 - 3x^2$

Boundary

① $y = 1$, $-1 \le x \le 1$

$f(x,1) = x^2 - 1 - 3x^2 = -2x^2 - 1$

$\frac{d}{dx} f(x,1) = -4x = 0 \rightarrow x = 0$

possible point @ $(0,1)$

② $y = -1$, $-1 \le x \le 1$

$f(x,-1) = -x^2 - 1 - 3x^2 = -4x^2 - 1$

$\frac{d}{dx} f(x,-1) = -8x = 0 \rightarrow x = 0$

possible point @ $(0,-1)$

③ $x = 1$, $-1 \le y \le 1$

$f(1,y) = y - y^2 - 3$

$\frac{d}{dy} f(1,y) = 1 - 2y = 0 \rightarrow 1 = 2y \rightarrow y = \frac{1}{2}$

possible point @ $(1, \frac{1}{2})$

④ $x = -1$, $-1 \le y \le 1$

$f(-1,y) = y - y^2 - 3$

$\frac{d}{dy} f(-1,y) = 1 - 2y = 0 \rightarrow 1 = 2y \rightarrow y = \frac{1}{2}$

possible point @ $(-1, \frac{1}{2})$

Corners

Possible points are $(-1,1)$, $(-1,-1)$, $(1,1)$, $(1,-1)$.

Test each point

$f(x,y) = x^2y - y^2 - 3x^2$

$f(-1,1) = 1 - 1 - 3 = -3$

$f(-1,-1) = -1 - 1 - 3 = -5$ min

$f(1,1) = 1 - 1 - 3 = -3$

$f(1,-1) = -1 - 1 - 3 = -5$ min

$f(0,1) = -1$

$f(0,-1) = -1$

$f(1, \frac{1}{2}) = \frac{1}{2} - \frac{1}{4} - 3 = -\frac{11}{4}$

$f(-1, \frac{1}{2}) = \frac{1}{2} - \frac{1}{4} - 3 = -\frac{11}{4}$

$f(0,0) = 0$ max

2. (10 points) For each of the following functions $f(x, y)$ and indicated points $\mathbf{a} \in \mathbf{R}^2$, (i) compute the best linear (really affine) approximation to f near $\mathbf{a}$ (expressed as a function in x, y); and (ii) give the unit vector in the direction that f is most rapidly *decreasing* away from $\mathbf{a}$.

(a) (3 points) $f(x, y) = 4\ln(1 + x^2 + y)$, $\mathbf{a} = (1, 2)$.

$$\nabla f(x,y) = \begin{bmatrix} \frac{4}{1+x^2+y}\cdot 2x \\ \frac{4}{1+x^2+y}\cdot 1 \end{bmatrix} = \begin{bmatrix} \frac{8x}{1+x^2+y} \\ \frac{4}{1+x^2+y} \end{bmatrix}$$

$$\nabla f(1,2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a})(\vec{x}-\vec{a})$$
$$= 4\ln(1+1+2) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$
$$f(x,y) \approx 4\ln(4) + 2(x-1) + (y-2)$$

most rapidly decreasing at $-\nabla f(1,2) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$$\|-\nabla f(1,2)\| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$$

unit vector is $-\nabla f(1,2) / \|-\nabla f(1,2)\|$

$$= \begin{bmatrix} -2 \\ -1 \end{bmatrix} / \sqrt{5}$$
$$= \begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

(b) (3 points) $f(x, y) = xe^{xy}$, $\mathbf{a} = (2, 1)$.

$$\nabla f(x,y) = \begin{bmatrix} e^{xy} + ye^{xy}x & xe^{xy}x \end{bmatrix} = \begin{bmatrix} e^{xy}(1+xy) & x^2e^{xy} \end{bmatrix}$$

$$\nabla f(2,1) = \begin{bmatrix} 3e^2 & 4e^2 \end{bmatrix}$$

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a})(\vec{x}-\vec{a})$$
$$f(x,y) \approx f(2,1) + \nabla f(2,1)\begin{bmatrix} x-2 \\ y-1 \end{bmatrix}$$
$$= 2e^2 + 3e^2(x-2) + 4e^2(y-1)$$

decreasing most at $-\nabla f(2,1) = \begin{bmatrix} -3e^2 & -4e^2 \end{bmatrix}$

$$\|-\nabla f(2,1)\| = \sqrt{9e^4 + 16e^4} = \sqrt{25e^4} = 5e^2$$

unit vector: $\begin{bmatrix} -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}$

(c) (4 points) $f(x, y) = \sqrt{1 + x^2 + xy}$, $\mathbf{a} = (1, 7)$.

$$\nabla f(x,y) = \begin{bmatrix} \frac{1}{2}(1+x^2+xy)^{-1/2}(2x+y) & \frac{1}{2}(1+x^2+xy)^{-1/2}(x) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2x+y}{2\sqrt{1+x^2+xy}} & \frac{x}{2\sqrt{1+x^2+xy}} \end{bmatrix}$$

$$\nabla f(1,7) = \begin{bmatrix} \frac{9}{2\sqrt{9}} & \frac{1}{2\sqrt{9}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{6} \end{bmatrix}$$

$$\|-\nabla f\| = \sqrt{(-3/2)^2 + (-1/6)^2}$$
$$= \sqrt{\frac{9}{4} + \frac{1}{36}}$$
$$= \sqrt{\frac{82}{36}} = \sqrt{\frac{41}{18}}$$

unit vector: $\begin{bmatrix} \frac{3/2}{\sqrt{41/18}} & \frac{1/6}{\sqrt{41/18}} \end{bmatrix}$

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a})(\vec{x}-\vec{a})$$
$$f(x,y) \approx f(1,7) + \nabla f(1,7)\begin{bmatrix} x-1 \\ y-7 \end{bmatrix}$$
$$= 3 + \begin{bmatrix} \frac{3}{2} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} x-1 \\ y-7 \end{bmatrix}$$
$$f(x,y) \approx 3 + \frac{3}{2}(x-1) + \frac{1}{6}(y-7)$$

3. (10 points) Let $f(x, y, z) = 6xy + z^3$.

(a) (3 points) Determine the tangent plane to the level surface $f(x, y, z) = 2$ through $\mathbf{a} = (1, -1, 2)$. Express your answer in two ways: write an *equation* involving x, y, z; and additionally give a parametric form (this has many possible answers).

$\nabla f(x,y,z) = \begin{bmatrix} 6y \\ 6x \\ 3z^2 \end{bmatrix}$

Equation

$\nabla f(x_1, y_1, z_1) \cdot \begin{bmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{bmatrix} = 0$

$\nabla f(1, -1, 2) \cdot \begin{bmatrix} x - 1 \\ y + 1 \\ z - 2 \end{bmatrix} = 0$

$\begin{bmatrix} -6 \\ 6 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y + 1 \\ z - 2 \end{bmatrix} = 0$

$-6(x-1) + 6(y+1) + 12(z-2) = 0$

$-6x + 6 + 6y + 6 + 12z - 24 = 0$

$-6x + 6y + 12z = 12$

$-x + y + 2z = 2$

Parametric

$\vec{p} = \vec{a} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\vec{Q} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{R} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$\vec{Q} - \vec{p} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$ $\quad \vec{R} - \vec{p} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + t' \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 - 3t - t' \\ -1 + t + 3t' \\ 2 - 2t - 2t' \end{bmatrix}$

(b) (7 points) Find the maximal and minimal values of f on the sphere $x^2 + y^2 + z^2 = 36$, and the points at which those extremal values are attained.

case 1) $\nabla g = 0$, $\nabla g = (2x, 2y, 2z) \Rightarrow x, y, z = 0 \rightarrow$ not in constraint

case 2) $\nabla f = \lambda \nabla g$, $\nabla f = \begin{pmatrix} 6y \\ 6x \\ 3z^2 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

$6y = \lambda 2x$
$6x = \lambda 2y$
$3z^2 = \lambda 2z$
$x^2 + y^2 + z^2 = 36$

4 equations, 4 unknowns

$\left.\begin{matrix} 6y = \lambda 2x \\ 6x = \lambda 2y \end{matrix}\right\}$ $6y\,\lambda 2y = 6x\,\lambda 2x$

case 1.1: $\lambda = 0 \rightarrow \nabla f = \lambda \nabla g = 0 \Rightarrow x, y, z = 0$ breaks constraint

case 1.2: $\lambda \neq 0$
$12y^2 = 12x^2$
$y^2 = x^2$

$\left.\begin{matrix} 6x = \lambda 2y \\ 3z^2 = \lambda 2z \end{matrix}\right\}$ $6x \cdot \lambda 2z = 3z^2 \cdot \lambda 2y$ ← we know $\lambda \neq 0$

$2xz = z^2 y$

Set of equations to solve (we've eliminated λ)

eq1 $y^2 = x^2 \rightarrow y = \pm x$

eq2 $2xz = z^2 y$

eq3 $x^2 + y^2 + z^2 = 36$

case i) $y = x = 0$] eq1
$0^2 + 0^2 + z^2 = 36$] eq3
$z = \pm 6$
$f(0, 0, \pm 6) = \pm 6^3 = \pm 216$

case ii) $y = x \neq 0$] eq1
$2z = z^2$] eq2
$z = 0$ or 2

$x^2 + y^2 + 0^2 = 36$
$x^2 = 18$
$x = \pm\sqrt{18}$

$x^2 + x^2 + 2^2 = 36$] eq3
$2x^2 = 32$
$x = \pm 4$

case iii) $y = -x \neq 0$] eq1
$-2z = z^2$
$z^2 + 2z = 0$
$z(z + 2) = 0$
$z = 0$ or -2] eq2

$x^2 + (-x)^2 + 0 = 36$
$2x^2 = 36$
$x = \pm\sqrt{18}$

$x^2 + (-x)^2 + (-2)^2 = 36$] eq3
$2x^2 = 32$
$x = \pm 4$

Steps
1) check $\nabla g = 0$
2) check $\nabla f = \lambda \nabla g$
3) try to get equations that don't depend on λ
↳ do this by doing cross multiplication of the system of equations
4) solve remaining equations w/ constraint equation

4. (10 points) Let V be the plane $x+y+z=0$ in $\mathbf{R}^3$ through the origin, so V has an orthogonal basis $\{\mathbf{v},\mathbf{w}\}$ for $\mathbf{v}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}1\\1\\-2\end{bmatrix}$. Let $L:\mathbf{R}^3\to\mathbf{R}^3$ be the function $L(\mathbf{x})=\mathbf{Proj}_V(\mathbf{x})$.

(a) (4 points) Compute the 3×3 matrix A for L; the entries should be fractions with denominator 3. (Hint: what is the meaning of each column?)

$$\mathrm{Proj}_V(\vec{x}) = \mathrm{Proj}_{\vec{v}}(\vec{x}) + \mathrm{Proj}_{\vec{w}}(\vec{x})$$

$$= \frac{\vec{v}\cdot\vec{x}}{\vec{v}\cdot\vec{v}}\vec{v} + \frac{\vec{w}\cdot\vec{x}}{\vec{w}\cdot\vec{w}}\vec{w}$$

matrix for L

$\mathbb{R}^3 \to \mathbb{R}^3$: three columns

$L\begin{pmatrix}1\\0\\0\end{pmatrix}$ is first column $\longrightarrow \left(\frac{1}{2}\right)\vec{v} + \left(\frac{1}{6}\right)\vec{w} = \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix} + \begin{pmatrix}1/6\\1/6\\-1/3\end{pmatrix} = \begin{pmatrix}2/3\\-1/3\\-1/3\end{pmatrix}$

$L\begin{pmatrix}0\\1\\0\end{pmatrix}$ is second column $\rightarrow \begin{pmatrix}-1/3\\2/3\\-1/3\end{pmatrix}$

$L\begin{pmatrix}0\\0\\1\end{pmatrix}$ is third column $\rightarrow \begin{pmatrix}-1/3\\-1/3\\2/3\end{pmatrix}$

matrix of L is $\begin{bmatrix}2/3 & -1/3 & -1/3\\ -1/3 & 2/3 & -1/3\\ -1/3 & -1/3 & 2/3\end{bmatrix}$

Suppose f : linear map $\mathbb{R}^3 \to \mathbb{R}^3$

$A\vec{x} \to \begin{pmatrix} | & | & | \\ \vec{v}_1, & \vec{v}_2, & \vec{v}_3 \\ | & | & | \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\vec{v}_1 = f\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\vec{v}_2 = f\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\vec{v}_3 = f\begin{pmatrix}0\\0\\1\end{pmatrix}$

(b) (3 points) For $\mathbf{a}=\begin{bmatrix}1\\3\\5\end{bmatrix}$, compute $\mathbf{Proj}_V(\mathbf{a})$ in two ways: using the orthogonal basis $\{\mathbf{v},\mathbf{w}\}$ for V, and using the matrix-vector product against your answer in (a). (You should get the same answer both ways, a vector with integer entries.)

1) $\mathrm{Proj}_V(a) = \mathrm{Proj}_{\vec{v}}(a) + \mathrm{Proj}_w(a)$

$$= \frac{v\cdot a}{v\cdot v}v + \frac{w\cdot a}{w\cdot w}w$$

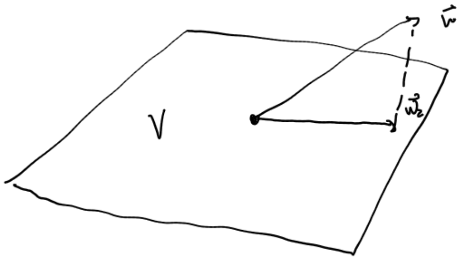
$$= \left(\frac{-2}{2}\right)\vec{v} + \left(\frac{-6}{6}\right)\vec{w}$$

$$= -\vec{v} - \vec{w} = \boxed{\begin{pmatrix}-2\\0\\2\end{pmatrix}}$$

2) $$\begin{bmatrix}2/3 & -1/3 & -1/3\\ -1/3 & 2/3 & -1/3\\ -1/3 & -1/3 & 2/3\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}-6\\0\\6\end{bmatrix} = \boxed{\begin{bmatrix}-2\\0\\2\end{bmatrix}}$$

(c) (3 points) The geometric definition of $\mathbf{Proj}_V$ gives that its output lies in V, on which $\mathbf{Proj}_V$ has no effect, so $\mathbf{Proj}_V \circ \mathbf{Proj}_V = \mathbf{Proj}_V$. Check that your answer A in (a) satisfies the corresponding matrix equality $A^2 = A$. (Hint: if you write $A = (1/3)B$ for a matrix B with integer entries then the calculation will be cleaner.)



$\mathrm{Proj}_V\, \vec{w} = \vec{w_2}$

$\mathrm{Proj}_V\, \vec{w_2} = \vec{w_2}$

$\mathrm{Proj}_V\, (\mathrm{Proj}_V\, (\vec{w})) = \vec{w_2}$

$\mathrm{Proj}_V \circ \mathrm{Proj}_V\, (\vec{w}) = \vec{w_2}$

$$A = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \frac{1}{3}^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \frac{1}{3}\left(\frac{1}{3}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= A$$

5. (10 points) For each of the 30 weeks of the academic year at a certain university, first-year undergraduates are in one of two types: those who plan to major in computer science, and everyone else. Let's call these two types of students "CS" and "non-CS", and assume that during each week a student changes their type at most once (and may change again in subsequent weeks). Let $\mathbf{p}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ be the 2-vector whose entries x_n and y_n are the proportions of such students of type non-CS and CS respectively at the end of the nth week (so $x_n + y_n = 1$ always).

(a) (3 points) Among those of type CS at the start of each week suppose 90% remain that way at the end of the week but 10% switch to non-CS. Among those of type non-CS at the start of each week suppose 85% remain that way at the end of the week but 15% switch to CS. Write down an explicit 2×2 Markov matrix M for which $\mathbf{p}_{n+1} = M\mathbf{p}_n$ for all n.

$$\mathbf{p}_{n+1} = \begin{bmatrix} 0.85x_n & 0.1y_n \\ 0.15x_n & 0.9y_n \end{bmatrix} = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$M = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix}$$

(b) (3 points) Using your answer to (a), what proportion of students who are type CS at the end of a given week are also type CS two weeks later (they may have switched to non-CS and back in the meantime)?

$$M^2 = \begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix}\begin{bmatrix} 0.85 & 0.1 \\ 0.15 & 0.9 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & 0.015 + 0.81 \end{bmatrix}$$

0.825 %

week 1 — 1

week 2 — 90% cs, 10% non cs

week 3 — 81% cs + 15% of the 10% non-cs

(c) (4 points) If you computed M correctly then it turns out that to an accuracy of two decimal digits for all $m \geq 17$ we have

$$M^m \approx \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}.$$

Interpret in words what this means, and also interpret in words the fact (verified by direct calculation) that for *any* $0 \leq x \leq 1$ we have

$$\begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}\begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}.$$

i) After 17 years, regardless of initial pop, 40% of students will be non-cs and 60% of students will be cs.

6. (10 points) Let $F : \mathbf{R}^2 \to \mathbf{R}^3$ be the function $F(x, y) = \begin{bmatrix} xy + y^2 \\ \cos(\pi xy) \\ xy^2 + xy \end{bmatrix}$ and suppose $G : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $G(1, 2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $(DG)(1, 2) = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$.

(a) (4 points) Compute $(DF)(x, y)$ and $(D(F \circ G))(1, 2)$.

$$DF(x,y) = \begin{bmatrix} y & x+2y \\ -\pi y\sin(\pi xy) & -\pi x\sin(\pi xy) \\ y^2+y & 2xy+x \end{bmatrix}$$

$$(D(F\circ G))(1,2) = DF(G(1,2)) \cdot DG(1,2)$$
$$= DF(1,1) \cdot \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}}_{3\times 2} \cdot \underbrace{\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 4 & 13 \\ 0 & 0 \\ 5 & 17 \end{bmatrix}$$

(b) (4 points) Estimate the 3-vector $(F \circ G)(1.1, 1.9)$.

$$(F\circ G)(1.1, 1.9) \approx (F\circ G)(1,2) + D(F\circ G)(1,2)\begin{bmatrix} 1.1-1 \\ 1.9-2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 & 13 \\ 0 & 0 \\ 5 & 17 \end{bmatrix}\begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.4 - 1.3 \\ 0 \\ 0.5 - 1.7 \end{bmatrix} = \begin{bmatrix} 1.1 \\ -1 \\ 0.8 \end{bmatrix}$$

(c) (2 points) Estimate $\begin{bmatrix} h \\ k \end{bmatrix}$ for which $G(1 + h, 2 + k) = \begin{bmatrix} 0.8 \\ 0.9 \end{bmatrix}$.

$$G(1+h, 2+k) \approx G(1,2) + DG(1,2)\begin{bmatrix} (1+h)-1 \\ (2+k)-2 \end{bmatrix}$$
$$\begin{bmatrix} 0.8 \\ 0.9 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} h+4k \\ h+3k \end{bmatrix}$$
$$\begin{bmatrix} -0.2 \\ -0.1 \end{bmatrix} \approx \begin{bmatrix} h+4k \\ h+3k \end{bmatrix} \qquad \begin{bmatrix} h \\ k \end{bmatrix} \approx \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix}$$

$$\begin{aligned} h+4k &= -0.2 \\ h+3k &= -0.1 \\ \hline k &= -0.1 \end{aligned} \qquad \begin{aligned} h+3(-0.1) &= -0.1 \\ h &= 0.2 \end{aligned}$$

Please do not detach this page. If you use any of this space to continue your answer, please clearly indicate the problem number(s) and **indicate on the original page of the problem that your answer continues here**.