Final Exam: Tuesday, 12/13 @ 12:15 - 3:15
Hewlett 200 & 201

Zac OH Next Week: Monday, 11-12 on Zoom

Spectral Theorem

Given $n \times n$ symmetric A, it can be written as $A = WDW^T$

W is an orthogonal matrix of eigenvectors

D is a diagonal matrix of eigenvalues

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

1) Find eigval

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda_1 = 3, \lambda_2 = -1$

2) Find eigvects

3-eigspace: $(A - 3I_2)\vec{x} = 0 \rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \rightarrow x = y \rightarrow \vec{x} = \begin{bmatrix} x \\ x \end{bmatrix}$

$N(A - 3I_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \quad \vec{w_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

-1-eigspace: $(A + 1I_2)\vec{x} = 0 \longrightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \rightarrow x = -y \rightarrow \vec{x} = \begin{bmatrix} -y \\ y \end{bmatrix}$

$N(A + 1I_2) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}, \quad \vec{w_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Thus, $W = \begin{bmatrix} | & | \\ \frac{\vec{w_1}}{\|w_1\|} & \frac{\vec{w_2}}{\|w_2\|} \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow q_A = 3t_1^2 - t_2^2$

$A = WDW^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

$A^3 = WD^3W^T$

Graphing w/ spectral

Suppose $Hf(\vec{a}) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad \lambda_1 = 3, \quad \lambda_2 = 1$

$\vec{w_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{w_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

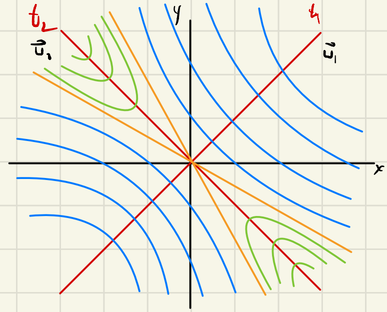
$q_{A}(\vec{x}) = 3t_1^2 - t_2^2$

$3t_1^2 - t_2^2 = c = 0$ ← asymptote closer to t_2

$t_2 = \pm\sqrt{3}\, t_1$

$3t_1^2 - t_2^2 = c = 1 > 0$ ← opens to t_1

$3t_1^2 - t_2^2 = c = -1 < 0$ ← opens to t_2



Null space

$N(A) = \{0\} \Rightarrow \lambda \neq 0$ bc $A\vec{v} = 0\vec{v}$ is not possible $\Rightarrow A$ is not semidefinite

Column space is output

Null space is input

Rank Nullity

For A is $m \times n$ matrix, $\dim(C(A)) + \dim(N(A)) = n$ ← input dimension

output ↑ ↑ input dim

Suppose A is 2×3, $\dim(C(A)) + \dim(N(A)) = 3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{input}} = \underbrace{\begin{bmatrix} 6 \\ 15 \end{bmatrix}}_{\text{output}}$$

Systems

$A\vec{x} = \vec{b}$ →

1) if $\vec{b} \notin C(A)$, no solution ← $\vec{b} = \text{Proj}_{C(A)}\vec{b}$

2) if $\vec{b} \in C(A)$:

   a) if $N(A) = \{0\}$, one soln

   b) if $\dim(N(A)) > 0$, infinite soln ← nonzero vecs