Lecture 24 - The Spectral Theorem, Quadratic Forms, and Matrix Powers

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Eigenvalues: $A\vec{v} = \overset{\text{eigenvalue}}{\lambda}\vec{v}$, $\vec{v} \neq \vec{0}$

$A^T = A$

Goals: Use eigenvalues and eigenvectors to analyze definiteness of symmetric matrices and to draw conclusions about powers of a matrix A.

Theorem 24.1.1: Let A be an $n \times n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_r$ a collection of eigenvectors for A with respective eigenvalues $\lambda_1, \ldots, \lambda_r$, so $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for all j.

1. If the r eigenvalues are pairwise different then the collection of r vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is linearly independent, so $r \leq n$. Thus, an $n \times n$ matrix cannot have more than n different eigenvalues.
2. If A is symmetric and $\lambda_i \neq \lambda_j$, then $\mathbf{v}_i \cdot \mathbf{v}_j = 0$. In other words, the eigenvectors for different eigenvalues of a symmetric $n \times n$ matrix are orthogonal to each other.

Theorem 24.1.4: (Spectral Theorem) Let A be a symmetric $n \times n$ matrix. There is an orthogonal basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of $\mathbb{R}^n$ consisting of eigenvectors for A. The corresponding eigenvalues are all of the eigenvalues for A (so if $\mathbf{w}_j$ has eigenvalue λ_j, then any eigenvalue of A equals some λ_j).

The collection of eigenvalues of a square matrix is called its **spectrum**.

Recall that we can apply the Fourier formula to an orthogonal basis to find the coefficients of a vector $\mathbf{v}$ with respect to that basis. So if $\{\mathbf{w}_j\}$ is an orthogonal basis of eigenvectors of a symmetric $n \times n$ matrix A, then

$$\boxed{\mathbf{v} = \sum_{i=1}^{n} \left(\frac{\mathbf{v} \cdot \mathbf{w}_i}{\mathbf{w}_i \cdot \mathbf{w}_i} \right) \mathbf{w}_i.}$$

We will use this to study quadratic forms.

Example 1: Let $A = \begin{bmatrix} 13 & -3 \\ -3 & 5 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A and compute the quadratic form $q_A(x, y)$.

$\lambda^2 - \text{tr}(A)\lambda + \det(A)$

1) Find eigenvalues: $\det(A - \lambda I_n) = \det\left(\begin{bmatrix} 13-\lambda & -3 \\ -3 & 5-\lambda \end{bmatrix}\right) = (13-\lambda)(5-\lambda) - 9 = \lambda^2 - 18\lambda + 56$

$= (\lambda - 14)(\lambda - 4) = 0$

$\to \boxed{\lambda = 4, \ 14}$

2) $\lambda = 14$-eigenspace: $N(A - 14I_n) = N\left(\begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix}\right) \to$ solve $\begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix}\vec{x} = \vec{0}$

sol $\to \vec{x} = x_2\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ eigenvector: $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda = 4$-eigenspace: $N(A - 4I_n) = N\left(\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\right) \to$ solve $\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\vec{x} = \vec{0}$

sol $\to \vec{x} = x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ eigenvector: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

3) Find Quadratic

$q_A = 13x^2 + 5y^2 - 6xy$

If $\mathbf{w}_1$ and $\mathbf{w}_2$ are the eigenvectors from Example 1, then we have that $\{\mathbf{w}'_1, \mathbf{w}'_2\}$, where $\mathbf{w}'_j = \mathbf{w}_j / \|\mathbf{w}_j\|$ is an *orthonormal* basis of $\mathbb{R}^2$. This basis is then a rotation of the standard basis of $\mathbb{R}^2$. Let's see how this affects our quadratic form:

Example 2: Write $\mathbf{v} \in \mathbb{R}^2$ in terms of the orthonormal basis $\{\mathbf{w}_1', \mathbf{w}_2'\}$ and recalculate $q_A(\mathbf{v})$. Then, sketch a contour plot of $q_A(x, y)$.

$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = x\vec{e}_1 + y\vec{e}_2 \to$ by Fourier formula, $\vec{v} = t_1\vec{w}_1' + t_2\vec{w}_2'$

$t_i = \dfrac{\vec{v}\cdot\vec{w}_i'}{\underbrace{\vec{w}_i'\cdot\vec{w}_i'}_{=1}} = \vec{v}\cdot\vec{w}_i'$

$\vec{w}_1' = \frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\lambda = 14$

$\vec{w}_2' = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\lambda = 4$

$A\vec{w}_1' = 14\vec{w}_1'$

$A\vec{w}_2' = 4\vec{w}_2'$

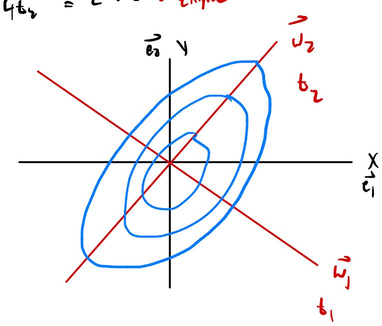
$q_A(x,y) = 13x^2 + 5y^2 - 6xy$

$$\begin{aligned} q_A(\vec{v}) = q_A(x,y) &= \vec{v}\cdot(A\vec{v}) \\ &= \vec{v}\cdot\Big(A\big(\overbrace{t_1\vec{w}_1'}^{\text{eigenvector}} + t_2\vec{w}_2'\big)\Big) \\ &= \vec{v}\cdot\big(14t_1\vec{w}_1' + 4t_2\vec{w}_2'\big) \\ &= 14t_1(\vec{v}\cdot\vec{w}_1') + 4t_2(\vec{v}\cdot\vec{w}_2') \\ &= 14t_1(t_1) + 4t_2(t_2) \\ &= 14t_1^2 + 4t_2^2 \end{aligned}$$

Contour plot

$14t_1^2 + 4t_2^2 = 0$ gives a point

$14t_1^2 + 4t_2^2 = c > 0$ gives an ellipse



The takeaway from the previous example / remark is that writing a quadratic form q_A in terms of the orthonormal basis of eigenvectors of A greatly simplifies its expression (i.e. no cross terms). This is especially helpful for $n \times n$ symmetric matrices for large n.

Definition: For an $n \times n$ symmetric matrix A and the associated quadratic form $q_A(\mathbf{x}) = \mathbf{x} \cdot (A\mathbf{x})$, we say A and q_A are:

- **positive-definite** if $q_A(\mathbf{v}) > 0$ for all $\mathbf{v} \neq \mathbf{0}$, and **positive-semidefinite** if $q_A(\mathbf{v}) \geq 0$ for all $\mathbf{v} \neq \mathbf{0}$.
- **negative-definite** if $q_A(\mathbf{v}) < 0$ for all $\mathbf{v} \neq \mathbf{0}$, and **negative-semidefinite** if $q_A(\mathbf{v}) \leq 0$ for all $\mathbf{v} \neq \mathbf{0}$.
- **indefinite** if $q_A(\mathbf{v})$ takes both positive and negative values for different $\mathbf{v}$.

Example 3: Determine the definiteness of $A = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$.

$q_A = -3x^2 - 4y^2 < 0 \ \forall \vec{v} \neq \vec{0}$, negative-definite

$q_B = x^2 + 9y^2 - 6xy = (x - 3y)^2 \geq 0 \ \forall \vec{v} \neq \vec{0}$, positive-semidefinite

In general, it is difficult to look at a quadratic form to determine its definiteness. However, we can determine it much easier if we rewrite it using the ideas from Example 2. If $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are the orthogonal eigenvectors of A with associated eigenvalues $\lambda_1, \ldots, \lambda_n$, then we can write any $\mathbf{v} = \sum_{i=1}^{n} t_i \mathbf{w}_i$ via the Fourier Formula. Repeating as in Example 2, we arrive at the **diagonalization formula**

$$\boxed{q_A(\mathbf{v}) = \sum_{i=1}^{n} \lambda_i (\mathbf{w}_i \cdot \mathbf{w}_i) t_i^2.}$$

Written this way, we have the following proposition.

Proposition 24.2.10: A symmetric $n \times n$ matrix A is:

1. positive-definite when its eigenvalues are all positive;
2. negative-definite when its eigenvalues are all negative;
3. indefinite when some eigenvalue is positive and some eigenvalue is negative;
4. positive-semidefinite when all eigenvalues are ≥ 0 **and** 0 is an eigenvalue;
5. negative-semidefinite when all eigenvalues are ≤ 0 **and** 0 is an eigenvalue.

Example 4: The matrix $A = \begin{bmatrix} 16 & -2 & -6 \\ -2 & 19 & -3 \\ -6 & -3 & 27 \end{bmatrix}$ has eigenvectors $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$ with eigenvalues 12, 20, and 30 respectively. Calculate $q_A(x, y, z)$, its diagonalized form, and determine its definiteness. Verify that this matches the conclusion of Proposition 24.2.10.

$$q_A(x,y,z) = 16x^2 + 14y^2 + 27z^2 - 4xy - 12xz - 6yz$$

first row, second col   first row, third col   second row, third col

Diagonal: $q_A(\vec{v}) = 12(\vec{w}_1 \cdot \vec{w}_1)t_1^2 + 20(\vec{w}_2 \cdot \vec{w}_2)t_2^2 + 30(\vec{w}_3 \cdot \vec{w}_3)t_3^2$

Fourier - formula
$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t_1\vec{w}_1 + t_2\vec{w}_2 + t_3\vec{w}_3$

$= 72t_1^2 + 100t_2^2 + 900t_3^2$ > 0 $\forall \vec{v} \neq 0$ $\therefore$ positive-definite

All eigenvalues > 0, $\therefore$ positive-definite

Matrix Powers

Example 5: If $\mathbf{v}$ is an eigenvector of A with eigenvalue 42, what is $A^2\mathbf{v}$, $A^4\mathbf{v}$, and $A^{42}\mathbf{v}$?

$A\vec{v} = 42\vec{v}$ $A^{42}\vec{v} = (42)^{42}\vec{v}$

Thus, $A^2\vec{v} = A(42\vec{v}) = 42(42\vec{v}) = 42^2\vec{v}$

$A^4\vec{v} = (42)^4\vec{v}$

In general, if $A\mathbf{v} = \lambda\mathbf{v}$, then $A^r\mathbf{v} = \lambda^r\mathbf{v}$ for any $r \geq 1$.

Note also that if A is invertible, then $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$, so we also have $(A^{-1})^r\mathbf{v} = \lambda^{-r}\mathbf{v}$.

0 is not eigenvalue

We can use this idea to calculate high powers of certain $n \times n$ matrices. We have the following interpretation of the spectral theorem.

Theorem 24.4.1: (Spectral Theorem via matrix decomposition) Let A be a symmetric $n \times n$ matrix with orthogonal eigenvectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Let W be the $n \times n$ matrix whose columns are the unit vectors $\mathbf{w}_j / \|\mathbf{w}_j\|$.

Then $W^\top = W^{-1}$ (i.e. W is orthogonal), and

$$\boxed{A = WDW^\top = WDW^{-1},}$$

change of basis

where D is the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$.

Important: The order of the eigenvalues in D **must** match the order of the eigenvectors in W.

Example 6: Let A be as in Example 1. Find the decomposition $A = WDW^\top$ and describe what happens to A^m for large m.

$$A = \begin{bmatrix} 13 & -3 \\ -3 & 5 \end{bmatrix}, \quad \lambda_1 = 14, \quad \lambda_2 = 4, \qquad \vec{w}_1' = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad \vec{w}_2' = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$A = WDW^T = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 14 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$A^2 = (WDW^T)(WDW^T) = WDI_nDW^T = WD^2W^T$$

$$A^m = W \begin{bmatrix} 14^m & 0 \\ 0 & 4^m \end{bmatrix} W^T = \frac{1}{\sqrt{10}} \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 14^m & 0 \\ 0 & 4^m \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} -3(14)^m & 4^m \\ 14^m & 3(4)^m \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9(14)^m + 4^m & -3(14^m) + 3(4)^m \\ -3(14)^m + 3(4^m) & 14^m + 9(4^m) \end{bmatrix}$$

$$= \frac{14^m}{10} \begin{bmatrix} 9 + \left(\frac{2}{7}\right)^m & -3 + 3\left(\frac{2}{7}\right)^m \\ -3 + 3\left(\frac{2}{7}\right)^m & 1 + 9\left(\frac{2}{7}\right)^m \end{bmatrix} \approx \frac{14^m}{10} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

The last example says that if A has a largest (in absolute value) eigenvalue, it will control the behavior of high powers of A.

Proposition 24.4.2: For a symmetric $n \times n$ matrix A, if there is an eigenvalue whose absolute value exceeds that of all other eigenvalues (called the **dominant eigenvalue**) and if the solutions to $A\mathbf{x} = \lambda\mathbf{x}$ constitute a line (which happens whenever there are n different eigenvalues), then for large m, we have

$$A^m \approx \left(\frac{\lambda^m}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}\mathbf{w}^\top,$$

for any eigenvector $\mathbf{w}$ of A with eigenvalue λ.

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$M = WDW^T$ ← diagonal

↑ orthogonal

$M^n = WD^nW^{-1} = WD^nW^T \to D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, W = \left(\frac{\vec{w}_1}{\|\vec{w}_1\|} \cdots \frac{\vec{w}_n}{\|\vec{w}_n\|} \right)$

λ_1 is val for $\vec{w}_1$

If M is an $n \times n$ symmetric matrix, then that implies that there are n real eigenvalues.

Ex: $M = \begin{bmatrix} 3/5 & 1/5 & 1/5 \\ 1/5 & 3/5 & 1/5 \\ 1/5 & 1/5 & 3/5 \end{bmatrix}$ given: $\lambda_1 = 1$, $\lambda_2 = \frac{2}{5}$

only 2 bc multiple eigenvectors can share eigenvalues

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ given: $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is eigenvector

need to find out which value is duplicated

- do so by calculating dimension of eigenspace of each eigenvalue

eigenspace: span of eigenvectors w/ that eigenvalue

1) $M\vec{w}_1 = M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \therefore \vec{w}_1 \in$ 1-eigenspace

Are there more eigenvectors in 1-eigenspace?

$\to M\vec{x} = \lambda\vec{x} \to M\vec{x} = 1\vec{x}$, find $\vec{x}$

$M\vec{x} - \vec{x} = \vec{0}$

$(M - I_3)\vec{x} = \vec{0}$ ← null space

$\boxed{\therefore \text{ the 1-eigenspace } = N(M - I_3)}$

So you find $\dim(N(M - I_3))$

Rank-Nullity: For $m \times n$, $n = \dim(C(A)) + \dim(N(A))$

$$M - I_3 = \begin{array}{c} \begin{bmatrix} -2/5 & 1/5 & 1/5 \\ 1/5 & -2/5 & 1/5 \\ 1/5 & 1/5 & -2/5 \end{bmatrix} \\ \begin{matrix} c_1 & \quad c_2 & \quad c_3 \end{matrix} \end{array}$$

Find $\dim C(M - I_3)$

$\vec{c}_1 + \vec{c}_2 + \vec{c}_3 = \vec{0}$ $\therefore$ cols are lin dependent

$\therefore$ remove one column

$\therefore \{\vec{c}_1, \vec{c}_2\}$ is basis of $C(M - I_3)$

$\therefore \dim C(M - I_3) = 2$

$\therefore \dim N(M - I_3) = 1$ by rank-nullity

$\therefore \dim(\text{1-eigenspace}) = 1$, 1-eigenspace = span($\vec{w}_1$)

$\therefore$ only 1 copy of λ_1 in set of eigenvalues bc $\vec{w}_1$ is the eigenvector for it

$\therefore$ there's two copies of $\lambda_2 \to \lambda_3 = \lambda_2$

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{bmatrix}$

Now we find W.

We know $\vec{w}_1$ is eigenvec for λ_1, $\vec{w}_2$ is eigenvec for λ_2, $\vec{w}_3$ is eigenvec for λ_2

$$W = \begin{bmatrix} | & | & | \\ \frac{\vec{w}_1}{\|\vec{w}_1\|} & \frac{\vec{w}_2}{\|\vec{w}_2\|} & \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ | & | & | \end{bmatrix}$$

We know $\vec{w}_1$. Now we calculate $\vec{w}_2$ using $\lambda = \frac{2}{5}$

$M\vec{x} = \lambda\vec{x} \rightarrow M\vec{x} = \frac{2}{5}\vec{x}$

$\rightarrow (M - \frac{2}{5}I_n)\vec{x} = 0$

$$\begin{bmatrix} 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightarrow \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0$$

$x + y + z = 0$

$x = -y - z$

$$\vec{x} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Thus, $\frac{2}{5}$-eigenspace $= \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

We transform this into an orthogonal basis bc we want an orthonormal basis for W.

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad \|\vec{v}_1\| = 2$

keep $\vec{v}_1$, find $\vec{v}_2'$

$\vec{v}_2' = \vec{v}_2 - \text{Proj}_{\vec{v}_1}(\vec{v}_2)$

$$= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}(\vec{v}_1) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$\vec{v}_2' = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ $\qquad \|\vec{v}_2'\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$

Thus, an orthogonal basis of $2/5$ eigenspace is $\{\vec{v}_1, \vec{v}_2'\}$

thus $\vec{w}_2 = \vec{v}_1$, $\vec{w}_3 = \vec{v}_2'$

$$W = \begin{bmatrix} | & | & | \\ \frac{\vec{w}_1}{\|\vec{w}_1\|} & \frac{\vec{w}_2}{\|\vec{w}_2\|} & \frac{\vec{w}_3}{\|\vec{w}_3\|} \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$M^{100} = W D^{100} W^T \approx W \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

$$\left(\frac{2}{5}\right)^{100} \approx 0$$

$$W \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

A Markov matrix A always has an eigenvalue 1. All other eigenvalues are in absolute value smaller or equal to 1.