Jack Le - (713)-449-5966
call if found

Lecture 1 (Vectors)

1. Vector sum: $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}$
2. Scalar Multiplication: $c\vec{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$
3. Linear Combination: $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$
4. Convex Combination: Lin comb when $a_1 + a_2 + \dots + a_n = 1$
   ↳ for two vectors, convex is $(1-t)\vec{v} + t\vec{w}$, $0 \le t \le 1$
5. Vector Properties:
   → $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ → $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
   → $(ab)\vec{v} = a(b\vec{v})$ → $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$
   → $(a+b)\vec{v} = a\vec{v} + b\vec{v}$
6. Magnitude: $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2}$
   $\|-\vec{v}\| = \|\vec{v}\|$ and $\|c\vec{v}\| = |c|\,\|\vec{v}\|$
7. Displacement: $\vec{v} - \vec{w}$ is displacement vector, $\|\vec{v} - \vec{w}\|$ is distance btwn $\vec{v}$ & $\vec{w}$

Lecture 2 (Dot Products)

1. $\cos\theta = \dfrac{\vec{v} \cdot \vec{w}}{\|v\|\,\|w\|}$ (angle between $\vec{v}$ and $\vec{w}$)
2. Dot prod: $\vec{x} \cdot \vec{y} = x_1y_1 + \dots + x_ny_n = \sum_{i=1}^{n} x_iy_i$
3. Orthogonal: $\vec{x} \cdot \vec{y} = 0$
4. Dot prod properties:
   → $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ → $\vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w})$
   → $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ → $a\vec{v} \cdot b\vec{w} = ab(\vec{v} \cdot \vec{w})$
   → $\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2$
   → $\vec{v} \cdot (c_1\vec{w}_1 + c_2\vec{w}_2) = c_1(\vec{v} \cdot \vec{w}_1) + c_2(\vec{v} \cdot \vec{w}_2)$
5. Correlation Coefficient: $r = \dfrac{\vec{x} \cdot \vec{y}}{\|x\|\,\|y\|}$, $-1 \le r \le 1$

Lecture 3 (Planes)

To find side of plane of point, plug in point and compare to d.

Equation form: $ax + by + cz = d$ — $ax+by+cz > d$, $ax+by+cz < d$ are opp sides

Normal vector: find point P and $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ ← orthogonal to plane

Parametric: $P + te + t'e'$

Equation ⟷ Normal

1. coefficients of equation is $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
2. use ~~displacement from $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to point~~ or plug in point to equation to find d.

Equation → Parametric

1. find 3 points from equation, (x, y, z)
2. pick one to be point P
3. find displacement $\vec{e} = \vec{Q} - \vec{P}$, $\vec{e}' = \vec{R} - \vec{P}$
4. Put into parametric form

Parametric → Normal

1. solve for $\vec{n}$, $\vec{n} \cdot \vec{e} = 0$, $\vec{n} \cdot \vec{e}' = 0$
2. use point to find plane

Lecture 4 (Subspaces)

1. Span: $\text{span}(\vec{v}_1, \dots, \vec{v}_k) = \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \,;\, c_1, \dots, c_k \in \mathbb{R}\}$
   ↳ $\vec{0}$ is always in span, $\text{span}(\vec{0}) = \{\vec{0}\}$
   ↳ spans are not unique. Multiple vectors can make same span
   ↳ linear subspace is same thing as span; must contain origin
      ↳ all linear combs of vectors in subspace are also in subspace
2. Dimension: for subspace V, $\dim(V)$ = smallest # of vectors to span V.
   For $V = \text{span}(v_1, \dots, v_k)$, $\dim(V) = k$ if each v_i is not LC of other vectors.

Lecture 5 (Basis / Orthogonality)

1. Basis: basis of subspace V is spanning set & $\dim(V)$ vectors (no redundant vectors)
   ↳ $\{e_1, e_2, e_3\}$ is basis of $\mathbb{R}^3$; there are many more
2. Dimension Criterion: ① span of one vector has $\dim(V) = 1$
   ② $\dim(V) < k$ if vectors are scalar mults or linear combos ← pick a vector & show it's a linear comb of others
   ↳ remove all scalar mults & linear combos, remaining vectors is dimension
3. Orthogonal Basis: if $\vec{v}_1, \dots, \vec{v}_k$ is orthogonal collection (all ⊥ to each other), then it is a basis for $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$. $\dim(\text{span}(\vec{v}_1, \dots, \vec{v}_k)) = k$ → orthogonal basis
4. Orthonormal basis: orthogonal basis w/ unit vectors → $\dfrac{v}{\|v\|}$
5. Standard basis: for $\mathbb{R}^3$, it's → $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$
6. Fourier formula: for orthogonal collection $v_1, \dots, v_k \in \mathbb{R}^n$ and $v \in \text{span}(v_1, \dots, v_k)$
   $\vec{v} = \sum_{i=1}^{k} \left(\dfrac{v \cdot v_i}{v_i \cdot v_i}\right) v_i$ → $c_i = \dfrac{v \cdot v_i}{v_i \cdot v_i}$ → if orthonormal, $v = \sum (v \cdot v_i) v_i$

Lecture 6 (Projections)

1. $\text{Proj}_{\vec{w}}\,\vec{x} = \left(\dfrac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$ ✓ point in $L = \text{span}(\vec{w})$ closest to $\vec{x}$.
2. Projecting onto lines using identity:
   $\text{Proj}_{\vec{w}}(c_1x_1 + \dots + c_kx_k) = c_1\text{Proj}_{\vec{w}}(x_1) + \dots + c_k\text{Proj}_{\vec{w}}(x_k)$
   $= \text{Proj}_{\vec{w}}(\vec{x})$
   ↳ compute $\text{Proj}_{\vec{w}}(\vec{e}_1), \dots, \text{Proj}_{\vec{w}}(\vec{e}_k)$
   ↳ then $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_k\vec{e}_k$
3. Orthogonal Projection Theorem: (subspace V)
   ↳ for orthogonal basis $\vec{v}_1, \dots, \vec{v}_k$ of V
   $\text{Proj}_V(\vec{x}) = \text{Proj}_{\vec{v}_1}(\vec{x}) + \dots + \text{Proj}_{\vec{v}_k}(\vec{x})$
4. Orthogonal Projection Theorem (Ver. 2): $\vec{x} = \vec{v} + \vec{v}'$, where $\vec{v} = \text{Proj}_V(\vec{x})$ and $v' = x - \text{Proj}_V(\vec{v})$ [V is lin subspace, $\vec{x} \in V$]

Lecture 7 (Orthogonal Basis)

1. $\vec{y}$ and $\vec{x}' = \vec{x} - \text{Proj}_{\vec{y}}(\vec{x})$ is orthogonal basis of $\text{span}(\vec{x}, \vec{y})$
2. Linear Regression Steps:
   1) find X and Y
   2) find $\hat{X} = X - \text{Proj}_{\mathbf{1}}(X) = X - \bar{x}\mathbf{1}$
   3) use $\{\hat{X}, \mathbf{1}\}$ as orthog basis for the space $V = \text{span}(X, \mathbf{1})$
   4) Project Y onto V
      $\text{Proj}_V Y = \text{Proj}_{\hat{X}} Y + \text{Proj}_{\mathbf{1}} Y$
      $= a\hat{X} + \bar{y}\mathbf{1}$
   5) sub $(X - \bar{x}\mathbf{1})$ for $\hat{X}$
      $= a(X - \bar{x}) + \bar{y}$
      $= aX - a\bar{x} + \bar{y}$
   6) $y = mx + b$, $m = a$, $b = \bar{y} - a\bar{x}$

Lecture 8 (Level Sets)

1. Scalar-valued function: $f: \mathbb{R}^n \to \mathbb{R}$
2. Vector-valued function: $f: \mathbb{R}^n \to \mathbb{R}^m$ } same
   ↳ component functions: $f_1, \dots, f_m: \mathbb{R}^n \to \mathbb{R}$
3. Composition: $(f \circ g)(x) = f(g(x))$ $(\underbrace{\mathbb{R}^n \to \mathbb{R}^p}_{g} \to \mathbb{R}^m)$, $f: \mathbb{R}^p \to \mathbb{R}^m$
   ↳ output size of g match input of f
4. Graph: $f: \mathbb{R}^n \to \mathbb{R}$
   $\text{Graph}(f) = \{(x_1, \dots, x_n, z) \in \mathbb{R}^{n+1} : z = f(x_1, \dots, x_n)\}$
5. Level Sets: for $f: \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$, level set of f @ level c is pts $\in \mathbb{R}^n$ s.t. $f(x_1, \dots, x_n) = c$
6. Contour Plot: depicts level sets in z plane for many values of c in $\mathbb{R}^2$.
   let $f(x, y) = x^2 + y^2$
   $c = 0$: $0 = x^2 + y^2$
   $c = 1$: $1 = x^2 + y^2$
   you can add constants to get 2 different functions w/ same level set

Dimension Criterion

One Vector: $\text{span}(\vec{v})$ has $\dim = 1$

Two Vectors: $\dim(\text{span}(\vec{v}, \vec{w})) = 2$ if $\vec{v}$ and $\vec{w}$ not scalar mult, else $\dim = 1$

Three Vectors: $\dim(V) = 3$ <u>except</u>

① all three vectors scalar mult → $\dim = 1$

② two vectors scalar mult → $\dim = 2$

③ no scalar mult, but one v_i is linear comb of other two → $\dim = 2$

↳ every v_i will be linear comb of other two if one is

Ex: Span $\mathbb{R}^3$ subspace in $\mathbb{R}^4$

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = 0 \right\}$$

$-x + 2y + 3z + w = 0$

$x = 2y + 3z + w$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2y+3z+w \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2y \\ y \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3z \\ 0 \\ z \\ 0 \end{bmatrix} + \begin{bmatrix} w \\ 0 \\ 0 \\ w \end{bmatrix}$$

$$= y\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$U = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Ex: find 3 5-vectors where $\text{span}(v_1, v_2, v_3) = U$

$U = \{ x \in \mathbb{R}^5 : x \cdot v_1 = 0,\ x \cdot v_2 = 0 \}$

$x \cdot v_1 = 0$, $v_1 = (1, -2, 0, -4, 3)$

$x_1 - 2x_2 - 4x_4 + 3x_5 = 0$

$x_1 = 2x_2 + 4x_4 - 3x_5$

$x \cdot v_2 = 0$, $v_2 = (0, 5, -1, 2, 2)$

$5x_2 - x_3 + 2x_4 + 2x_5 = 0$

$x_3 = 5x_2 + 2x_4 + 2x_5$

$$x = \begin{bmatrix} 2x_2 + 4x_4 - 3x_5 \\ x_2 \\ 5x_2 + 2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

so, $v_1 = \uparrow$, $v_2 = \uparrow$, $v_3 = \uparrow$

Ex: line as span of single vector

$y = 2x$

↳ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$L = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

To show that three vectors are not on a line, get displacement vector $\vec{PQ}$ and $\vec{PR}$ and make sure they are not scalar multiples.

Two var, all 2nd partials

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Lecture 9 (Partial Derivatives)

1. Partial Notation: $\frac{\partial f}{\partial x_i}(a,b)$, $\left.\frac{\partial f}{\partial x_i}\right|_{(a,b)}$, $f_{x_i}(a,b)$ ← all three are same
   ← move in x_i dir, all others constant
2. Definition: $\frac{\partial f}{\partial x_i}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$
3. Symbolic: Don't plug constants until final calculation
   ↳ think of other x_j's as constant
4. Numerical: replace each x_j with constants before differentiating
5. Partial Derivatives on Contour plot
   ↳ $f_x(a,b)$ is slope experienced walking on $z = f(x,y)$ from W to E at pt (a,b)
   ↳ $f_y(a,b)$ is slope walking on $z = f(x,y)$ from S to N
6. Second partial: $\frac{\partial^2 f}{\partial x_i^2 \partial x_j^2} = \frac{\partial f}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$

   $\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = f_{yx} = (f_y)_x = (f_x)_y$
7. Clairaut-Schwarz: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \to f_{xy} = f_{yx}$
8. Functions that satisfy $f_{xx} + f_{yy} = 0$ are called harmonic
9. Chain Rule: $\frac{\partial}{\partial x}(F^2) = \frac{\partial}{\partial F}\frac{\partial F}{\partial x} = 2F\frac{\partial F}{\partial x} = 2FF_x$