Jack Le - Math 51 Homework 3

Exercise 7.1. The vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 8 \\ -6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 23 \\ -19 \end{bmatrix}$ span a plane through the origin in $\mathbf{R}^4$.

(a) Compute an orthogonal basis for this plane.

(b) Using your answer to (a), compute the orthogonal projection of the point $\mathbf{x} = \begin{bmatrix} -26 \\ 16 \\ 15 \\ -18 \end{bmatrix}$ onto this plane. (Your answer should be a vector whose entries are integers.)

a) Suppose $V = \text{span}(\vec{v}, \vec{w})$. The orthogonal basis of the plane V is given by $\vec{v}$ and $\vec{w}' = \vec{w} - \text{Proj}_{\vec{v}}(\vec{w})$

To compute $\text{Proj}_{\vec{w}}(\vec{v})$, we will first compute a few dot products.

$$\vec{w} \cdot \vec{v} = 5 + 0 + 184 + 114 = 303$$

$$\vec{v} \cdot \vec{v} = 1 + 0 + 64 + 36 = 101$$

Using this, we can get:

$$\text{Proj}_{\vec{v}}(\vec{w}) = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{303}{101} \vec{v} = 3 \begin{bmatrix} 1 \\ 0 \\ 8 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 24 \\ -18 \end{bmatrix}$$

Therefore

$$\vec{w}' = \vec{w} - \text{Proj}_{\vec{v}}(\vec{w}) = \begin{bmatrix} 5 \\ 5 \\ 23 \\ -19 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 24 \\ -18 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \\ -1 \end{bmatrix}$$

We can check the orthogonality of the two vectors

$$\vec{v} \cdot \vec{w}' = \begin{bmatrix} 1 \\ 0 \\ 8 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \\ -1 \end{bmatrix} = 2 - 8 + 6 = 0$$

$$\boxed{\text{Thus, } \{\vec{v}, \vec{w}'\} \text{ is an orthogonal basis of the plane } V.}$$

b) We wish to find $\text{Proj}_V(\vec{x})$. Since $\{\vec{v}, \vec{w}'\}$ is an orthogonal basis of V, we can show:

$$\text{Proj}_V(\vec{x}) = \text{Proj}_{\vec{w}'}(\vec{x}) + \text{Proj}_{\vec{v}}(\vec{x})$$

$$= \frac{\vec{x} \cdot \vec{w}'}{\vec{w}' \cdot \vec{w}'} \vec{w}' + \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

To do this, we first compute some useful dot products:

$$\vec{x} \cdot \vec{w}' = \begin{bmatrix} -26 \\ 16 \\ 15 \\ -18 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \\ -1 \end{bmatrix} = 31$$

$$\vec{w}' \cdot \vec{w}' = 31$$

$$\vec{x} \cdot \vec{v} = 202$$

$$\vec{v} \cdot \vec{v} = 101$$

Plugging these into the equation above, we get

$$\begin{aligned} \mathrm{Proj}_V(\vec{x}) &= \mathrm{Proj}_{\vec{w}'}(\vec{x}) + \mathrm{Proj}_{\vec{v}}(\vec{x}) \\ &= \frac{\vec{x} \cdot \vec{w}'}{\vec{w}' \cdot \vec{w}'}\vec{w}' + \frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} \\ &= \frac{31}{31}\vec{w}' + \frac{202}{101}\vec{v} \\ &= \vec{w}' + 2\vec{v} \\ &= \begin{bmatrix} 2 \\ 5 \\ -1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 8 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 15 \\ -13 \end{bmatrix} \end{aligned}$$

Therefore, the orthogonal projection of the point $\vec{x}$ onto the plane V is the vector $\begin{bmatrix} 4 \\ 5 \\ 15 \\ -13 \end{bmatrix}$

Exercise 7.7. Consider the collection of 5 data points $(-3,-4)$, $(-2,0)$, $(-1,2)$, $(0,2)$, $(1,5)$. Suppose the line of best fit (in the least squares sense) is written as $y = mx + b$.

(a) Write down explicit 5-vectors $\mathbf{X}$ and $\mathbf{Y}$ so that for the 5-vector $\mathbf{1}$ whose entries are all equal to 1, the projection of $\mathbf{Y}$ into the plane $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ in $\mathbf{R}^5$ is $m\mathbf{X} + b\mathbf{1}$. (You are just being asked to write down such $\mathbf{X}$ and $\mathbf{Y}$, nothing more.)

(b) Compute an orthogonal basis of $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ having the form $\{\mathbf{1}, \mathbf{v}\}$ for a 5-vector $\mathbf{v}$, and find scalars t and s so that $\mathbf{Proj}_V(\mathbf{Y}) = t\mathbf{v} + s\mathbf{1}$. (If you build $\mathbf{v}$ by the method in the main text then you'll get t and s that are integers.)

(c) By expressing $\mathbf{v}$ from (b) as a linear combination of $\mathbf{X}$ and $\mathbf{1}$, use your answer to (b) to compute the equation $y = mx + b$ of the line of best fit. (The values of m and b are integers.)

(d) Draw by hand a plot of the data points and the line you found in (c). How well does it seem to fit the data?

a)

$$\vec{X} = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{Y} = \begin{bmatrix} -4 \\ 0 \\ 2 \\ 2 \\ 5 \end{bmatrix}$$

b) To compute the orthogonal basis, let $\vec{v} = \vec{X} - \mathrm{Proj}_{\vec{1}} \vec{X}$. This would allow $\vec{v}$ to be orthogonal to $\vec{1}$.

$$\begin{aligned} \vec{v} &= \vec{X} - \mathrm{Proj}_{\vec{1}} \vec{X} \\ &= \vec{X} - \frac{\vec{X} \cdot \vec{1}}{\vec{1} \cdot \vec{1}} \vec{1} \\ &= \vec{X} - \bar{x}\, \vec{1} \\ &= \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

We confirm orthogonality between $\vec{v}$ and $\vec{1}$

$$\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

Thus, $\{1, v\}$ is an orthogonal basis of V.

We now find scalars t and s such that $\mathrm{Proj}_V (Y) = t\vec{v} + s\vec{1}$.

Since $\{1, v\}$ is an orthogonal basis of V,

$$\text{Proj}_V(\vec{y}) = \text{Proj}_{\vec{v}}(\vec{y}) + \text{Proj}_{\vec{1}}(\vec{y})$$

$$= \frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} + \frac{\vec{y}\cdot\vec{1}}{\vec{1}\cdot\vec{1}}\vec{1}$$

We first compute a few dot products

$$\vec{y}\cdot\vec{v} = 8+0+0+2+10 = 20$$

$$\vec{v}\cdot\vec{v} = 10$$

$$\vec{y}\cdot\vec{1} = 5$$

$$\vec{1}\cdot\vec{1} = 5$$

Hence, if we plug this back in

$$\text{Proj}_V Y = \frac{20}{10}\vec{v} + \frac{5}{5}\vec{1} = 2\vec{v} + \vec{1}$$

$\boxed{\text{Thus, } t = 2 \text{ and } s = 1}$

c) As previously defined, $\vec{v} = \vec{x} - \text{Proj}_{\vec{1}}\vec{x} = \vec{x} - \bar{x}\vec{1}$. We can then substitute $\vec{v}$ in the equation from (b) to get

$$\text{Proj}_V Y = 2(\vec{x} - \bar{x}\vec{1}) + \vec{1}$$

$$= 2\vec{x} - 2\bar{x}\vec{1} + \vec{1}$$

$$= 2\vec{x} - 2(-1)\vec{1} + \vec{1}$$

$$= 2\vec{x} + 3\vec{1}$$

$\boxed{\text{Therefore, the best fit line is } y = 2x + 3}$

d)

The line seems to fit the data very well. There looks to be a strong positive correlation in the data, and the line reflects that.

Exercise 7.10. The vectors $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix}$ span a plane $\mathcal{P}$ through the origin in $\mathbf{R}^4$. Let

$$L = \left\{ \begin{bmatrix} 4-t \\ 4+4t \\ 4-t \\ -7-2t \end{bmatrix} : t \in \mathbf{R} \right\}$$

be a line in $\mathbf{R}^4$.

(a) Consider the displacement vector $\mathbf{x}$ between any two different points of L (all such displacements are scalar multiples of each other since L is a *line*). Show that $\mathbf{x}$ belongs to $\mathcal{P}$; this is described in words by saying L is *parallel* to $\mathcal{P}$ (see Figure 7.6.1 below for an illustration of an analogous situation in $\mathbf{R}^3$).

Hint: you need to show that a 4-vector $\mathbf{x}$ belongs to $\mathcal{P} = \operatorname{span}(\mathbf{v}, \mathbf{w})$, and this becomes a system of 4 equations in 2 unknowns; solve 2 of those equations and check your solution also works for the other 2 equations.

(b) Whenever one has a linear subspace V of $\mathbf{R}^n$ and a line ℓ in $\mathbf{R}^n$ (possibly not through the origin) that is parallel to V, it is a fact (not difficult to show, but you may take it on faith) that all points in ℓ have the *same* distance to V; this is illustrated for a plane V in $\mathbf{R}^3$ in Figure 7.6.1 below. That is, for *every* point $\mathbf{y} \in \ell$ and the point $\mathbf{y}' \in V$ nearest to $\mathbf{y}$, the distance $\|\mathbf{y} - \mathbf{y}'\|$ is the same regardless of which $\mathbf{y}$ on ℓ we consider.

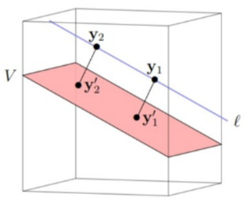


FIGURE 7.6.1. All points $\mathbf{y}_1$ and $\mathbf{y}_2$ on ℓ have the same distance to the linear subspace V.

Taking V and ℓ to be $\mathcal{P}$ and L above, compute the common distance $\|\mathbf{y} - \mathbf{y}'\|$ (since it is independent of $\mathbf{y}$, you may pick whatever you consider to be the most convenient point $\mathbf{y}$ in L to do the calculation). Your answer should have the form $\sqrt{A}$ for an integer A.

a) We begin by representing line L in parametric form

$$\vec{p} + t\vec{u} = L$$

$$= \begin{bmatrix} 4-t \\ 4+4t \\ 4-t \\ -7-2t \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -7 \end{bmatrix} + \begin{bmatrix} -t \\ 4t \\ -t \\ -2t \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -7 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ -1 \\ -2 \end{bmatrix}$$

Using this, we see that the difference vector $\vec{x}$ between point $\vec{p}$ and any point $\vec{l}$ on L will be $\vec{x} = \vec{l} - \vec{p} = t\vec{u}$. Since all displacements are scalar multiples of each other, it suffices to show that $\vec{u}$ belongs to $P = \operatorname{span}(\vec{v}, \vec{w})$.

In order for $\vec{u}$ to belong to $\operatorname{span}(\vec{v}, \vec{w})$, by the definition of a span, $\vec{u}$ must be able to be produced by a linear combination of $\vec{v}$ and $\vec{w}$.

$$\vec{u} = a\vec{v} + b\vec{w}$$

$$\begin{bmatrix} -1 \\ 4 \\ -1 \\ -2 \end{bmatrix} = a \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2a + 11b \\ -a + 5b \\ -a - 10b \\ a + b \end{bmatrix}$$

This can be written as a system of 4 equations

$$-1 = 2a + 11b$$
$$4 = -a + 5b$$
$$-1 = -a - 10b$$
$$-2 = a + b$$

We now solve the system of equations to find a and b

$$2a + 11b = -a - 10b \qquad -1 = 2a + 11b$$
$$3a = -21b \qquad -1 = 2(-7b) + 11b$$
$$a = -7b \qquad -1 = -3b$$
$$a = \frac{-7}{3} \qquad b = \frac{1}{3}$$

If we check this with all the equations

$$-1 = 2(-7/3) + 11(1/3)$$
$$= -14/3 + 11/3 = -3/3 = -1 \checkmark$$
$$4 = -(-7/3) + 5(1/3)$$
$$= 7/3 + 5/3 = 12/3 = 4 \checkmark$$
$$-1 = -(-7/3) - 10(1/3) =$$
$$= 7/3 - 10/3 = -3/3 = -1 \checkmark$$
$$-2 = -7/3 + 1/3 = -6/3 = -2 \checkmark$$

As such, $\vec{u} = -7/3\,\vec{v} + 1/3\,\vec{w}$ is contained in $\mathrm{span}(\vec{v}, \vec{w})$. Since $\vec{x}$ is a scalar multiple of $\vec{u}$, it is also contained in $\mathrm{span}(\vec{v}, \vec{w})$.

b) By definition, the point $\vec{y}' \in P$ nearest to $\vec{y}$ is $\mathrm{Proj}_P(\vec{y})$. As such, the common distance $\|y - y'\|$ is equivalent to the length of the displacement vector between y and the projection of y onto V.
In other words,

$$\|\vec{y} - \vec{y}'\| = \|\vec{y} - \mathrm{Proj}_P\,\vec{y}\|$$

To calculate this projection, we first consider a $\vec{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -7 \end{bmatrix} \in L$ and construct an orthogonal basis of P.

To make the orthogonal basis, we consider $\vec{v}$ and $\vec{w}' = \vec{w} - \mathrm{Proj}_{\vec{v}}(\vec{w})$.

$$\mathrm{Proj}_{\vec{v}}(\vec{w}) = \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \frac{28}{7}\,\vec{v} = 4\vec{v}$$

$$\vec{w}' = \vec{w} - 4\vec{v} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -6 \\ -3 \end{bmatrix}$$ Check: $\vec{w}' \cdot \vec{v} = 0$ ✓

Thus, $\{\vec{v}, \vec{w}'\}$ constitutes an orthogonal basis of P.

We now find $\text{Proj}_P(\vec{y})$. Since $\{\vec{v}, \vec{w}'\}$ is an orthogonal basis of P, $\text{Proj}_P(\vec{y}) = \text{Proj}_{\vec{v}}(\vec{y}) + \text{Proj}_{\vec{w}'}(\vec{y})$.

$$\text{Proj}_{\vec{v}}(\vec{y}) = \frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{-7}{7}\vec{v} = -\vec{v}$$

$$\text{Proj}_{\vec{w}'}(\vec{y}) = \frac{\vec{y}\cdot\vec{w}'}{\vec{w}'\cdot\vec{w}'}\vec{w}' = \frac{45}{135}\vec{w}' = \frac{1}{3}\vec{w}'$$

$$\text{Proj}_P(\vec{y}) = -\vec{v} + \frac{1}{3}\vec{w}' = -\begin{bmatrix}2\\-1\\-1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}3\\9\\-6\\-3\end{bmatrix} = \begin{bmatrix}-2\\1\\1\\-1\end{bmatrix} + \begin{bmatrix}1\\3\\-2\\-1\end{bmatrix} = \begin{bmatrix}-1\\4\\-1\\-2\end{bmatrix}$$

Using this, we can find the common distance $\|\vec{y} - \text{Proj}_P(\vec{y})\|$.

Firstly,

$$\vec{y} - \text{Proj}_P(\vec{y}) = \begin{bmatrix}4\\4\\4\\-7\end{bmatrix} - \begin{bmatrix}-1\\4\\-1\\-2\end{bmatrix} = \begin{bmatrix}5\\0\\5\\-5\end{bmatrix}$$

$$\boxed{\text{Therefore, } \|\vec{y} - \text{Proj}_P(\vec{y})\| = \|\vec{y} - \vec{y}'\| = \sqrt{5^2 + 5^2 + (-5)^2} = \sqrt{75}}$$

$\{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = 0\}$

Exercise 8.2. Consider the set $S = \{(x, y, z) \in \mathbf{R}^3 : x^3 + z^3 + 3y^2z^3 + 5xy = 0\}$.

(a) Give functions $f, h : \mathbf{R}^3 \to \mathbf{R}$ for which S is a level set of both $f(x, y, z)$ and $h(x, y, z)$.

a) $f(x,y,z) = x^3 + z^3 + 3y^2z^3 + 5xy$ at $a = 0$

$h(x,y,z) = \left(\frac{-5xy - x^3}{1+3y^2}\right)^{1/3} \cdot z^{-1}$ at $a = 1$

(b) By solving for z in terms of x and y, give a function $g : \mathbf{R}^2 \to \mathbf{R}$ for which S is the graph of g.

b) $x^3 + z^3 + 3y^2z^3 + 5xy = 0$

$z^3 + 3y^2z^3 = -5xy - x^3$

$z^3(1+3y^2) = -5xy - x^3$

$z^3 = \frac{-5xy - x^3}{1+3y^2}$

$z = \sqrt[3]{\frac{-5xy - x^3}{1+3y^2}}$

$$\boxed{g(x,y) = \left(\frac{-5xy - x^3}{1+3y^2}\right)^{1/3}}$$

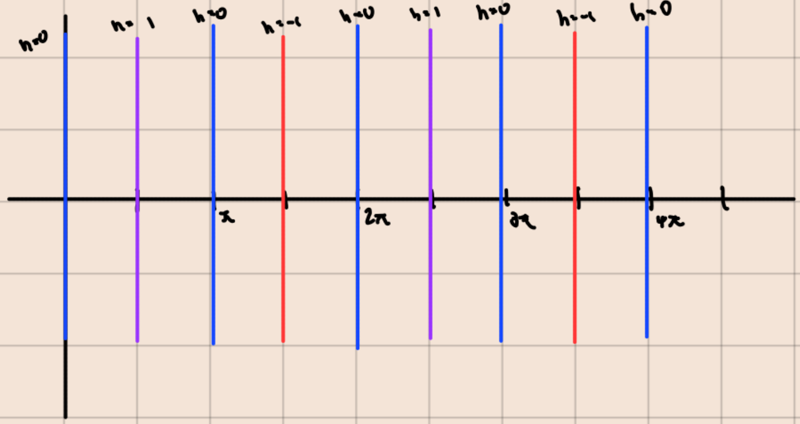
Exercise 8.7.

(a) Sketch a contour map in the xy-plane for the function $h(x, y) = \sin(x)$, labeling level sets by the values of h. (Hint: $\sin(x)$ is periodic in x with period 2π, so each level set should have a "2π-periodic" repeating pattern.)

(b) Sketch a contour map in the xy-plane for the function $f(x, y) = \sin(x - 3y)$, labeling level sets by the values of f. (Hint: This should be a "tilted" version of your picture in (a).)

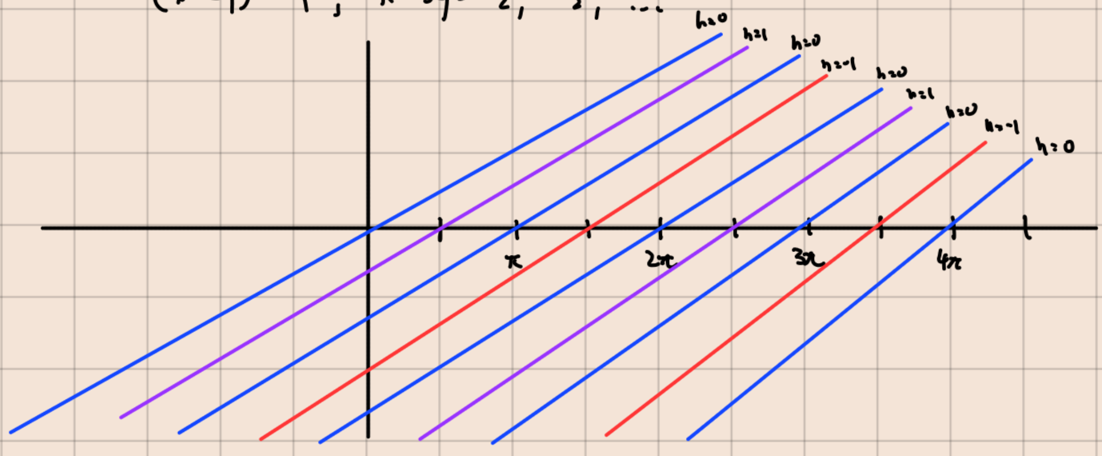
a) $h(x,y) = \sin(x) = a$

* $a = 0$: $\sin(x) = 0$, $x = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$
* $a = -1$: $\sin(x) = -1$, $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$
* $a = 1$: $\sin(x) = 1$, $x = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$



a) $h(x,y) = \sin(x-3y) = c$

* $a = 0$: $\sin(x-3y) = 0$, $x-3y = 0, \pi, 2\pi, 3\pi, 4\pi$
* $a = -1$: $\sin(x-3y) = -1$, $x-3y = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$
* $a = 1$: $\sin(x-3y) = 1$, $x-3y = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$



Exercise 8.8. Let $f : \mathbf{R}^2 \to \mathbf{R}^3$ be given by

$$f(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

and let $g : \mathbf{R}^3 \to \mathbf{R}$ be given by

$$g(x, y, z) = x^2 + y^2 + z^2.$$

(a) Calculate $g \circ f : \mathbf{R}^2 \to \mathbf{R}$.

a) $g \circ f : \mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}$

$g \circ f\,(\theta, \phi) = g\left(f(\theta, \phi)\right)$

$f(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$

$g\left(f(\theta,\phi)\right) = g(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$

$= (\cos\theta \sin\phi)^2 + (\sin\theta \sin\phi)^2 + (\cos\phi)^2$

$= \cos^2\theta \sin^2\phi + \sin^2\theta \sin^2\phi + \cos^2\phi$

$= \sin^2\phi\,(\cos^2\theta + \sin^2\theta) + \cos^2\phi$

$= \sin^2\phi + \cos^2\phi$

$\boxed{g(f(\theta,\phi)) = 1}$

(b) Explain using (a) why each point $f(\theta, \phi)$ lies on the unit sphere in $\mathbf{R}^3$ centered at the origin. It turns out that *every* point in the unit sphere is in the output of f, but we are not asking you to show this.

(Google "spherical coordinate system" to see the geometric meaning of f, with θ an angle in the xy-plane. Beware that the notational conventions in math and in physics/engineering for θ and ϕ are swapped; this may be because in German mathematical writing the usual notation for an angle is ϕ rather than θ and early 20th-century physics was dominated by German scientists.)

b) The unit sphere in $\mathbb{R}^3$ is $x^2 + y^2 + z^2 = 1$. This means that a point $(x, y, z) \in \mathbb{R}^3$ lies on the unit sphere if $g(x, y, z) = 1$. We saw in (a) that for any input (θ, ϕ) that we give to f, $g(f(\theta, \phi))$ is always equal to 1. As such, all outputs (a, b, c) of $f(\theta, \phi)$ lies on the unit sphere in $\mathbb{R}^3$.

Exercise 8.9. Let $g : \mathbf{R}^2 \to \mathbf{R}$ be given by

$$g(v, w) = v^2 - w^2.$$

This exercise works out the contour plot of g via visual reasoning; later it will be an important special case for the study of what are called "saddle points" in the multivariable second derivative test.

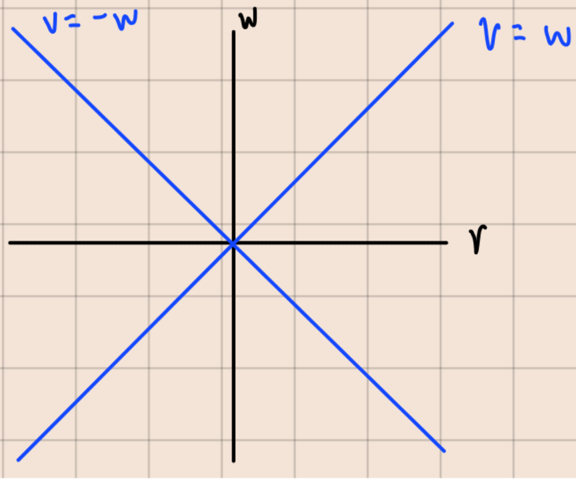
(a) Sketch the level set $g(v, w) = 0$.

a) $g(v,w) = 0$

$v^2 - w^2 = 0$

$v^2 = w^2$

$v = \pm w$



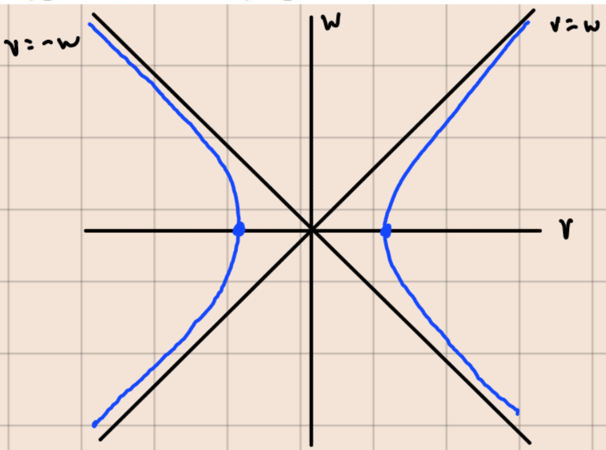
(b) The level set $g(v, w) = 1$ is a hyperbola (with two "branches"). Mark where it cuts either of the coordinate axes, and explain why for $|v|$ or $|w|$ large we have $v^2 \approx w^2$ on this level set, so $v \approx \pm w$. Sketch the resulting hyperbola, with asymptotes $v = \pm w$.

b) $g(v,w) = 1$

$v^2 - w^2 = 1$

$v^2 = w^2 + 1$

when $w = 0$, $v = \pm 1$



For large $|v|$ or $|w|$, the 1 term in $v^2 = w^2 + 1$ becomes neglible. As an example, suppose $w = 10{,}000$. Plugging this in and solving for v, we get $v = 10000.00005$, which is very close to w.

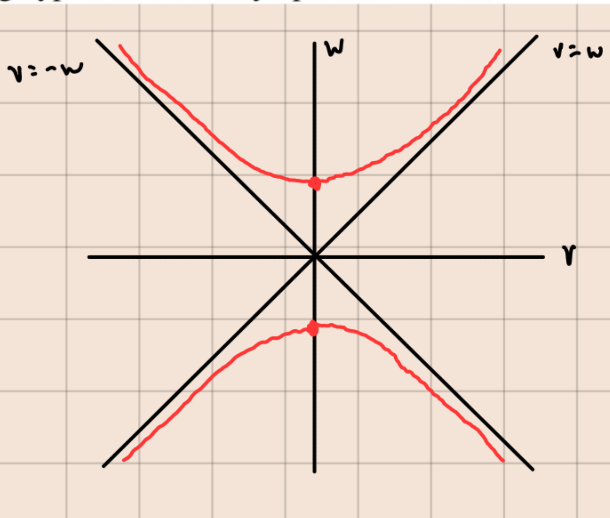
(c) The level set $g(v, w) = -1$ is a hyperbola (with two "branches"). Mark where it cuts either of the coordinate axes, and explain why for $|v|$ or $|w|$ large we have $v^2 \approx w^2$ on this level set, so $v \approx \pm w$. Sketch the resulting hyperbola, with asymptotes $v = \pm w$.

c) $g(v,w) = -1$

$v^2 - w^2 = -1$

$w^2 = v^2 + 1$

when $v = 0$, $w = \pm 1$



For large $|v|$ or $|w|$, the 1 term in $w^2 = v^2 + 1$ becomes neglible. As an example, suppose $v = 10{,}000$. Plugging this in and solving for w, we get $w = 10000.00005$, which is very close to v.

(d) For $c > 0$, check that $g(v/\sqrt{c}, w/\sqrt{c}) = 1$ precisely when $g(v, w) = c$. Using this and (b), explain why the level set $g(v, w) = c$ is the same as the scaling up by the factor $\sqrt{c}$ of the level set in (b).

d)
$$g(v,w) = c$$
$$v^2 - w^2 = c$$
$$\frac{v^2 - w^2}{c} = 1$$
$$\frac{v^2}{c} - \frac{w^2}{c} = 1$$

Since $c > 0$, we can simplify this equation to

$$\left(\frac{v}{\sqrt{c}}\right)^2 - \left(\frac{w}{\sqrt{c}}\right)^2 = 1$$

Since $g(v,w) = v^2 - w^2$,

$$g\left(\frac{v}{\sqrt{c}}, \frac{w}{\sqrt{c}}\right) = \left(\frac{v}{\sqrt{c}}\right)^2 - \left(\frac{w}{\sqrt{c}}\right)^2$$

This was shown previously to equal 1.
Therefore, when $g(v,w) = c$,

$$g\left(\frac{v}{\sqrt{c}}, \frac{w}{\sqrt{c}}\right) = 1$$

To avoid confusion, we denote v, w from (b) as v_b, w_b.
The level set in (b) was $g(v_b, w_b) = v_b^2 - w_b^2 = 1$. If we scale this level set up by a factor of $\sqrt{c}$, we get

$$g(\sqrt{c}\, v_b, \sqrt{c}\, w_b) = (\sqrt{c}\, v_b)^2 - (\sqrt{c}\, w_b)^2 = c v_b^2 - c w_b^2$$
$$= c(v_b^2 - w_b^2) = c(1) = c$$

Therefore, the level set $g(v,w) = c$ is the same as the scaled up level set $g(\sqrt{c}\, v_b, \sqrt{c}\, w_b) = c$.

(e) For $c < 0$, similarly relate the level set $g(v, w) = c$ to what you drew in (c) using a scaling factor of $\sqrt{|c|}$.

e) $g(v,w) = c \rightarrow v^2 - w^2 = c \rightarrow \frac{v^2 - w^2}{c} = 1 \rightarrow \frac{v^2}{c} - \frac{w^2}{c}$

Since $c < 0$, we can rearrange the equation to get

$$\frac{w^2}{|c|} - \frac{v^2}{|c|} = \left(\frac{w}{\sqrt{|c|}}\right)^2 - \left(\frac{v}{\sqrt{|c|}}\right)^2 = 1$$

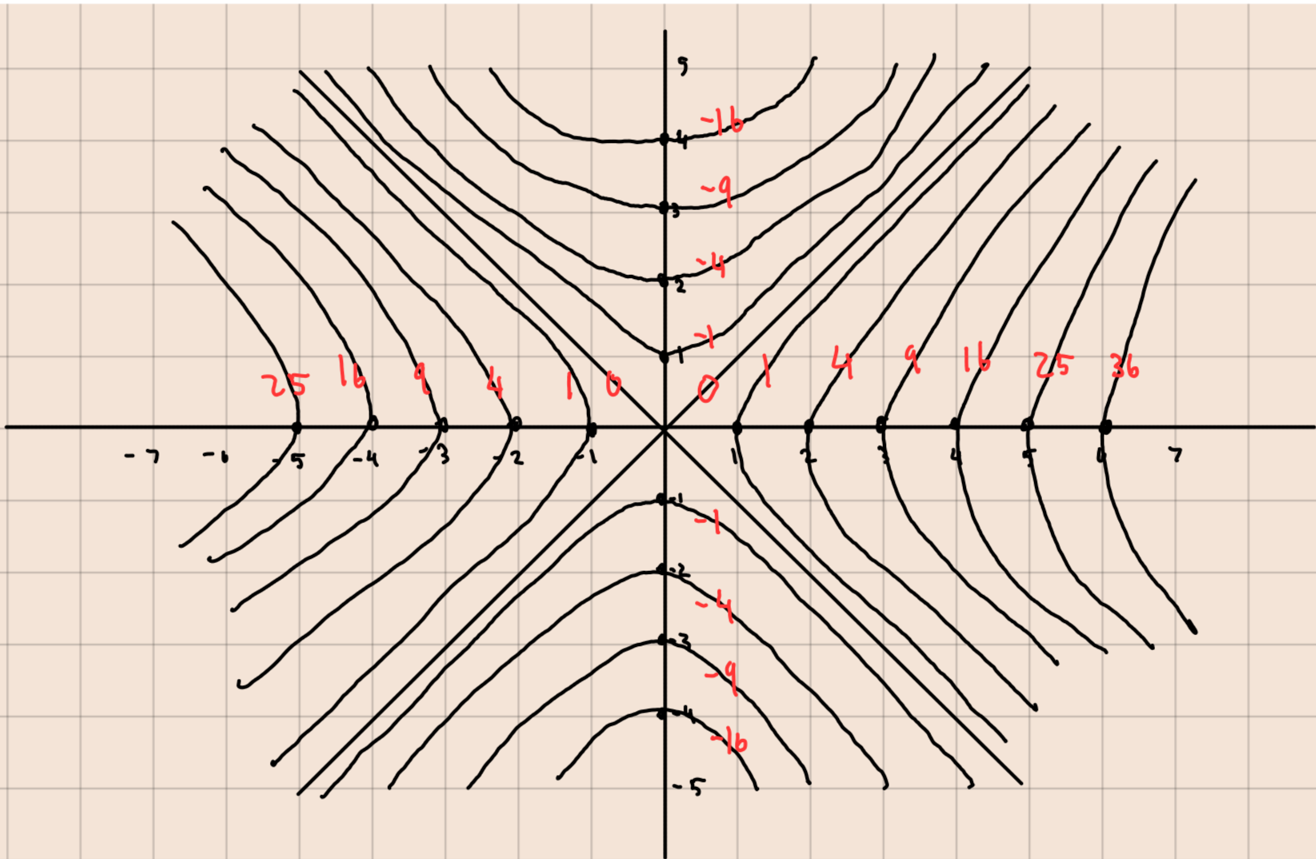
We represent v and w from (c) as v_c and w_c. (c) says

$$g(v_c, w_c) = v_c^2 - w_c^2 = -1$$

$$g(\sqrt{|c|}\,v_c, \sqrt{|c|}\,w_c) = |c|v_c^2 - |c|w_c^2$$
$$= |c|(v_c^2 - w_c^2)$$
$$= |c|(-1)$$
$$= -|c|$$

Since $c < 0$, $-|c| = c$. Therefore, $g(\sqrt{|c|}\,v, \sqrt{|c|}\,w) = c$. Therefore, the level set $g(v,w) = c$ is the same as the scaled up level set $g(\sqrt{|c|}\,v_c, \sqrt{|c|}\,w_c) = c$.

(f) Sketch a contour map for g in the vw-plane, labeling the level sets.



Exercise 9.2. Compute the following partial derivatives "symbolically" (i.e. using Method 1 from Example 9.3.1).

(a) $\dfrac{\partial f}{\partial z}(1,2,3)$, where $f(x,y,z) = z^2\tan(\pi x/4) + yz$.

(b) $\left.\dfrac{\partial g}{\partial x}\right|_{(5,8)}$, where $g(x,y) = 12e^{\cos y} + y^3$.

(c) $h_{x_3}(2,-1,3,0,5)$, where $h(x_1,x_2,x_3,x_4,x_5) = 2x_1x_2^2 + 3x_1x_3x_5 + 9x_4^2x_5 - x_2 + x_3 + 8$.

a) $f_z(x,y,z) = 2z\tan(\pi x/4) + y$

$f_z(1,2,3) = 2(3)\tan(\pi(1)/4) + (2)$

$= 6\tan(\pi/4) + 2 = 6+2$

$= \boxed{8}$

b) $g_x(x,y) = 0$

$g_x(5,8) = \boxed{0}$

c) $h_{x_3}(x_1, x_2, x_3, x_4, x_5) = 3x_1x_5 + 1$

$h_{x_3}(2, -1, 3, 0, 5) = 3(2)(5) + 1 = \boxed{31}$

Exercise 9.4. Use partial derivatives to approximate the following values of functions. (You may leave your answer as a fraction or in terms of e; if you want to use a calculator to turn that into a decimal approximation then that is fine, but if so then do *not* plug in such a decimal approximations until the very last step. We want to see that you understand how to perform basic algebraic manipulations with derivatives.)

(a) $f(-1.2, 5)$, where $f(x_1, x_2) = \sqrt{x_1 + 2x_2^2}$.

a) $f(a,b) = f(x_1,x_2) + f_{x_1}(x_1,x_2) * (a - x_1) + f_{x_2}(x_1,x_2) * (b - x_2)$

$f_{x_1}(x_1,x_2) = \frac{1}{2}(x_1 + 2x_2^2)^{-1/2} = 1/2\sqrt{x_1 + 2x_2^2}$

$f_{x_2}(x_1,x_2) = \frac{1}{2}(x_1 + 2x_2^2)^{-1/2} * 4x_2 = 2x_2/\sqrt{x_1 + 2x_2^2}$

$f(-1.2, 5) \approx f(-1,5) + f_{x_1}(-1,5) * (-1.2 - (-1)) + f_{x_2}(-1,5) * (5 - 5)$

$\approx \sqrt{-1 + 2(5)^2} + \left(\frac{1}{2\sqrt{-1+2(5)^2}}\right)(-0.2) + \left(\frac{2(5)}{\sqrt{-1+2(5)^2}}\right)(0)$

$\approx \sqrt{49} + \left(\frac{1}{2\sqrt{49}}\right)(-0.2)$

$\approx 7 + \left(\frac{1}{14}\right)\left(\frac{-1}{5}\right)$

$\approx 7 - \frac{1}{70}$

$\approx \boxed{\frac{489}{70}}$

Exercise 9.6. Consider the contour plot in Figure 9.8.1 of a 2-variable function $f(x, y)$.

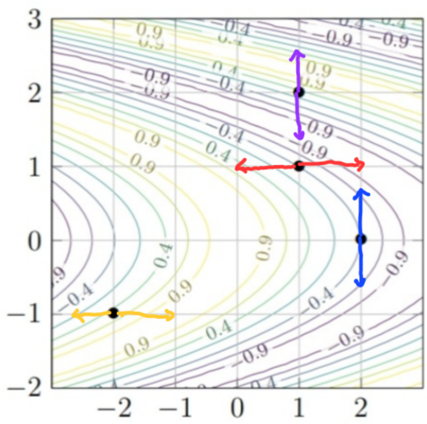


FIGURE 9.8.1. Contour plot of a function $f(x, y)$.

Determine whether each of the following partial derivatives is clearly positive, clearly negative, or close to zero. Give an explanation for each of your answers.

(a) $f_x(1,1)$.

(b) $\dfrac{\partial f}{\partial y}(2,0)$.

(c) $f_y(1,2)$.

(d) $\dfrac{\partial f}{\partial x}(-2,-1)$.

a) $f_x(1,1)$ is clearly negative. The partial derivative w.r.t. x means we consider the contour plot in the x direction (i.e. west to east) with y held constant. Starting from smaller values of x to larger (i.e. west to east), the level sets are $0.9 \to 0.4 \to -0.4 \to -0.9$. As such, $f(x,1)$ is clearly decreasing near $x=1$, so $f_x(1,1)$ is negative.

b) $\dfrac{\partial f}{\partial y}(2,0)$ is close to zero. To begin, the line $x=2$ is tangent to the level set 0.4. In addition, moving from smaller values of y to larger, the level sets are $-0.9 \to 0.4 \to -0.9$. Therefore, $f(2,y)$ is increasing then decreasing again, with the change near point $(2,0)$. As such, $f_y(2,0)$ is close to zero.

c) $f_y(1,2)$ is clearly positive. Moving from south to north, the level sets are $-0.9 \to -0.4 \to 0.4 \to 0.9$. Therefore, $f(1,y)$ is increasing near the point $(1,2)$. As such, $f_y(1,2)$ is clearly positive.

d) $\dfrac{\partial f}{\partial x}(-2,-1)$ is clearly positive. Moving from east to west, the level sets are $-0.4 \to 0.4 \to 0.9$. Therefore, $f(x,-1)$ is increasing near the point $(-2,-1)$. As such, $f_x(-2,-1)$ is clearly positive.

Exercise 9.7. Consider the contour plot in Figure 9.8.2 of a 2-variable function $f(x, y)$.

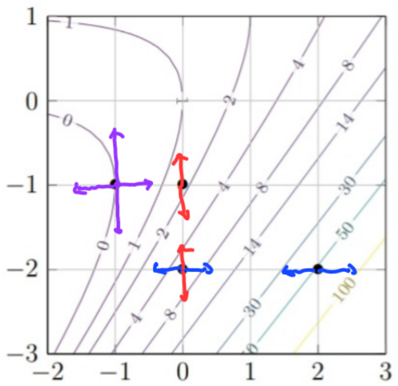


FIGURE 9.8.2. Contour plot of a function $f(x, y)$.

This plot does *not* use uniform increments in f-values: the gaps between f-values for successive level curves grow by a lot as we move to the right. Pay attention to those f-values in what follows. For each of the following pairs of partial derivatives, explain which one is greater. Also explain which one is greater in magnitude (i.e., in absolute value).

(a) $f_x(0, -2)$ or $f_x(2, -2)$?

(b) $\frac{\partial f}{\partial y}(0, -2)$ or $\frac{\partial f}{\partial y}(0, -1)$?

(c) $f_y(-1, -1)$ or $f_x(-1, -1)$?

a) To estimate f_x, we say $f_x \approx \frac{\text{change in } z}{\text{change in } x}$, where $z = f(x, y)$.

$f_x(0, -2)$ using estimated values near it:

$x_1 = -0.5$, $z_1 \approx 3$

$x_2 = 0.5$, $z_2 \approx 10$

$f_x \approx \frac{z_2 - z_1}{x_2 - x_1} = \frac{10-3}{0.5-(-0.5)} = 7$

$f_x(2, -2)$ using nearby estimates:

$x_1 = 1.5$, $z_1 \approx 30$

$x_2 = 2.5$, $z_2 \approx 100$

$f_x \approx \frac{100 - 30}{2.5 - 1.5} = 70$

$\boxed{\text{Therefore, } f_x(2,-2) \text{ is both greater and greater in magnitude.}}$

b) $\frac{\partial f}{\partial y} \approx \frac{\text{change in } z}{\text{change in } y}$

$\frac{\partial f}{\partial y}(0, -2)$ using nearby estimates:

$y_1 = -2.5$, $z_1 \approx 10$

$y_2 = -1.5$, $z_2 \approx 3$

$\frac{\partial f}{\partial y} \approx \frac{3-10}{-1.5-(-2.5)} = \frac{-7}{1} = -7$

$\frac{\partial f}{\partial y}(0, -1)$ using nearby estimates:

$y_1 = -1.5$, $z_1 \approx 3$

$y_2 = -0.5$, $z_2 \approx 1.5$

$\frac{\partial f}{\partial y} \approx \frac{1.5-3}{-0.5-(-1.5)} = \frac{-1.5}{1} = -1.5$

Therefore, $\frac{\partial f}{\partial y}(0,-1)$ is greater, and $\frac{\partial f}{\partial y}(0,-2)$ is greater in magnitude.

c) $f_y \approx \frac{\text{change in } z}{\text{change in } y}$ and $f_x \approx \frac{\text{change in } z}{\text{change in } x}$

$f_y(-1,-1)$ using nearby estimates:

$y_1 = -1.5$, $z_1 \approx 0.5$ $\quad f_x \approx \frac{0.2-0.5}{-0.5-(-1.5)} = \frac{-0.3}{1} = -0.3$

$y_2 = -0.5$, $z_2 \approx 0.2$

$f_x(-1,-1)$ using nearby estimates:

$x_1 = -1.5$, $z_1 \approx -0.5$ $\quad f_x \approx \frac{0.7-(-0.5)}{-0.5-(-1.5)} = \frac{1.2}{1} = 1.2$

$x_2 = -0.5$, $z_2 \approx 0.7$

Therefore, $f_x(-1,-1)$ is both greater and greater in magnitude.

Exercise 9.13.

(a) For $f(x,y) = \ln(ax^2 + bxy + cy^2)$, show that $xf_x + yf_y = 2$.

$$f_x(x,y) = \frac{1}{ax^2+bxy+cy^2}\left(2ax+by\right) = \frac{2ax+by}{ax^2+bxy+cy^2}$$

$$f_y(x,y) = \frac{1}{ax^2+bxy+cy^2}\left(bx+2cy\right) = \frac{2cy+bx}{ax^2+bxy+cy^2}$$

$$xf_x = x\left(\frac{2ax+by}{ax^2+bxy+cy^2}\right) = \frac{2ax^2+bxy}{ax^2+bxy+cy^2}$$

$$yf_y = y\left(\frac{2cy+bx}{ax^2+bxy+cy^2}\right) = \frac{2cy^2+bxy}{ax^2+bxy+cy^2}$$

$$xf_x + yf_y = \frac{2ax^2+bxy}{ax^2+bxy+cy^2} + \frac{2cy^2+bxy}{ax^2+bxy+cy^2}$$

$$= \frac{2ax^2+2bxy+2cy^2}{ax^2+bxy+cy^2}$$

$$= \frac{2(ax^2+bxy+cy^2)}{ax^2+bxy+cy^2} = \boxed{2}$$

Exercise 9.15. For each of the following function $f(x, y)$, compute all second partial derivatives. (As a safety check on your work, you may wish to compute f_{xy} in both possible ways and confirm that your answers coincide.)

(a) $f(x, y) = e^{xy} + y\ln(x)$.

$$\frac{\partial f}{\partial x} = ye^{xy} + \frac{1}{x}y = ye^{xy} + \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = xe^{xy} + \ln(x)$$

① $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = y^2e^{xy} + (-yx^{-2})$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = y^2e^{xy} - \frac{y}{x^2}}$$

② $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = x^2e^{xy}$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = x^2e^{xy}}$$

③ $$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = e^{xy} + xe^{xy}y + \frac{1}{x}$$

$$\boxed{\frac{\partial^2 f}{\partial y\,\partial x} = e^{xy} + xye^{xy} + \frac{1}{x}}$$

④ $$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = e^{xy} + ye^{xy}x + \frac{1}{x}$$

$$\boxed{\frac{\partial^2 f}{\partial x\,\partial y} = e^{xy} + xye^{xy} + \frac{1}{x}}$$

Time Spent:

I joined this class from Math 61DM, so I had a lot of catch-up reading to complete.

Attending class: 5 hours

Homework: 8 hours

Reading: 20 hours

In total, I spent about 33 hours this week on this course.