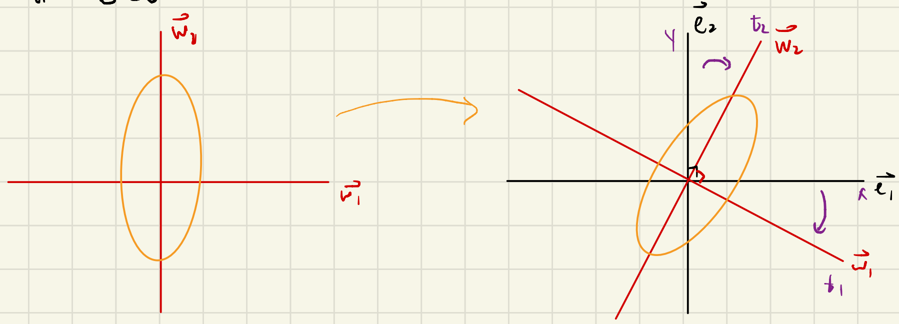
Diagonalization → can only guarantee diagonalization exists for symmetric

$q = ax^2 + by^2 + cxy$ → graphing using standard basis

$q = ab_1^2 + bb_2^2$ → graphing using two orthogonal eigenvectors $\vec{w}_1, \vec{w}_2$

change of basis



For symmetric matrices, eigenvectors are always orthogonal bc eigenspaces are orthog

Symmetric matrices has n-real eigenvalues

Newton's Method

Goal: find zeros of $f(x)$, $f: \mathbb{R}^n \to \mathbb{R}^n$

start at $f(a_0) \neq 0$

$f(a_0) \neq 0$, we want $f(a_0+h) = 0$

$$f(a_0+h) \approx f(a_0) + Df(a_0)\cdot h = 0$$

$$\Rightarrow Df(a_0)\cdot h = -f(a_0)$$

$$\Rightarrow h = -(Df(a_0))^{-1} f(a_0)$$

→ but $f(a+h) \neq 0$, so let $a_1 = a+h$

$$f(a_1+h) \approx f(a_1) + Df(a_1)\cdot h = 0$$

$$h = -(Df(a_1))^{-1} f(a_1)$$

$$a_2 = a_1 - [(Df)(a_1)]^{-1} f(a_1)$$

$$a_{k+1} = a_k - [(Df)(a_k)]^{-1} f(a_k)$$

Diagonalization example

matrix will be symmetric

$$M = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \\ 4 & 1 & 3 \end{bmatrix}$$

it will give λ if $n > 2$

To find eigenvectors, solve $(M - \lambda_i I_3)\vec{x} = \vec{0}$

For upper/lower triangular & diagonal matrices, λ are the diagonal entries

$A = WDW^{-1}$

look at lec 24

$$D^k = \begin{bmatrix} a_1^k & & \\ & \ddots & \\ & & a_n^k \end{bmatrix} \qquad D^{k+1} = D^k D = \begin{bmatrix} a_1^{k+1} & & \\ & \ddots & \\ & & a_n^{k+1} \end{bmatrix}$$ ✓ induction

For upper/lower triangular & diagonal matrix, if 0 is a diagonal entry, then 0 is an eigenvalue and the matrix is not invertible.

To invert upper/lower/diagonal, the diagonal entries are reciprocal, find the rest of vals.

For a matrix $M: \mathbb{R}^n \to \mathbb{R}^m$

null space is the inputs $M\vec{x} = 0$

column space is the space of possible outputs $M\vec{x} = \vec{b}$

Null space tells us about linear dependency of column space

Ex: $\begin{bmatrix}1\\1\end{bmatrix}$ lives in $N(M)$, w/ $M = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}$.

This tells us that $M\vec{x} = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1c_1 + 1c_2 = 0$

$\Rightarrow c_2 = -c_1$

$\Rightarrow$ cols not linearly independent

Columns are linearly dependent when null space has non-zero

Matrix is non-invertible if null space has non-zero

All vectors in the null space are eigenvectors with $\lambda = 0$, but not all eigenvectors are in the null space.

$A\vec{v} = 0\vec{v} = \vec{0} \Rightarrow \vec{v}$ is eigenvector, $\vec{v} \in N(A)$

$$\text{Proj}_{\vec{v}}\vec{x} = \frac{\vec{x}\cdot\vec{v}}{\underbrace{\vec{v}\cdot\vec{v}}_{\|\vec{v}\|^2}}\vec{v} = \frac{1}{\|\vec{v}\|^2}\vec{v}\vec{v}^T\vec{x} = \frac{1}{6}\begin{bmatrix}1 & 1 & 2\\1 & 1 & 2\\2 & 2 & 4\end{bmatrix}\vec{x}$$

(b) (7 points) Find the maximal and minimal values of f on the sphere $x^2+y^2+z^2=36$, and the points at which those extremal values are attained.

The sphere is $g(x,y,z)=36$ for $g(x,y,z)=x^2+y^2+z^2$, and

$$\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

only vanishes at the origin, which is not on the sphere. Hence, ∇g is non-vanishing on the sphere, so by the theorem of Lagrange multipliers any extremum $\mathbf{a}$ for f on the sphere must satisfy $(\nabla f)(\mathbf{a}) = \lambda(\nabla g)(\mathbf{a})$ for some scalar λ. Written out explicitly for $\mathbf{a}=(x,y,z)$, this vector equality says

$$\begin{bmatrix} 6y \\ 6x \\ 3z^2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \begin{bmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{bmatrix}.$$

Equating corresponding entries, we arrive at three scalar equations

$$6y = 2\lambda x, \;\; 6x = 2\lambda y, \;\; 3z^2 = 2\lambda z$$

along with the constraint equation $x^2+y^2+z^2=36$.

Solving for λ in each of those equations, we get

$$\frac{3y}{x} = \lambda, \;\; \frac{3x}{y} = \lambda, \;\; \frac{3z^2}{2z} = \lambda$$

with the understanding that each such equation only makes sense when its denominator is *nonzero*. So first we address cases with a vanishing denominator. If $x=0$ then $6y=2\lambda x=0$, so $y=0$; likewise, if $y=0$ then $6x=2\lambda y=0$, so $x=0$. In such cases with $x=0$ and $y=0$, the constraint $g=36$ forces $z=\pm 6$, so we have the candidate points $(0,0,\pm 6)$. Setting that aside for now, we may assume $x,y\neq 0$, so

$$\frac{3y}{x} = \lambda = \frac{3x}{y},$$

and cross-multiplying (and cancelling 3) gives $y^2=x^2$, so $y=\pm x$. Hence, $\lambda = 3y/x = \pm 3$ with same sign as for the relation $y=\pm x$. The final multiplier equation $3z^2 = 2\lambda z = \pm 6z$ then gives that either $z=0$ or (upon cancelling a *nonzero* z) $3z=\pm 6$, so $z=\pm 2$ with the same sign as for the relation $y=\pm x$. To summarize, when $x,y\neq 0$ the point has the form $(x,\pm x,0)$ or

$(x,\pm x,\pm 2)$ where in the latter case the signs are *the same*. We need to figure out the possibilities for x.

Bringing in the constraint equation $g=36$, the point $(x,\pm x,0)$ must satisfy $2x^2=36$, so $x^2=18$ or equivalently $x=\pm 3\sqrt{2}$. In other words, we get the four points $(3\sqrt{2},\pm 3\sqrt{2},0)$ and $(-3\sqrt{2},\pm 3\sqrt{2},0)$. If instead we're at $(x,\pm x,\pm 2)$ (with the same sign) then the constraint equation $g=36$ forces $2x^2+4=36$, or equivalently $x^2=16$, so $x=\pm 4$. Hence, we get the points $(4,4,2),(4,-4,-2),(-4,4,-2),(-4,-4,2)$.

Finally, we evaluate $f(x,y,z)=6xy+z^3$ at each of our candidates and thereby identity the biggest and smallest values. We have $f(0,0,\pm 6)=\pm 6^3=\pm 216$, and with unrelated signs $f(\pm 3\sqrt{2},\pm 3\sqrt{2},0)=\pm 6(3\sqrt{2})^2=\pm 108$. Finally,

$$f(4,4,2) = 6(16)+8 = 96+8 = 104, \;\; f(-4,-4,2) = 96+8 = 104,$$

$$f(4,-4,-2) = -96-8 = -104, \;\; f(-4,4,-2) = -96-8 = -104.$$

Inspecting these results, the largest and smallest values are 216 and -216 respectively, attained at the points $(0,0,6)$ and $(0,0,-6)$ respectively.