综合题

1.若 $\lim\limits_{x\to+\infty}[(x^5+7x^4+2)^a-x]=b\neq 0$,求常数 a,b.

解 令 $t=\dfrac{1}{x}$,当 $x\to+\infty$ 时,由于

$$\lim_{x\to+\infty}[(x^5+7x^4+2)^a-x]=\lim_{t\to0^+}\left[(\frac{1}{t^5}+\frac{7}{t^4}+2)^a-\frac{1}{t}\right]$$

$$=\lim_{t\to0^+}\left[\frac{(1+7t+2t^5)^a}{t^{5a}}-\frac{1}{t}\right]=\lim_{t\to0^+}\frac{t^{1-5a}(1+7t+2t^5)^a-1}{t}=b\neq0,$$

于是 $\lim\limits_{t\to0^+}[t^{1-5a}(1+7t+2t^5)^a-1]=0$.知 $1-5a=0$,得 $a=\dfrac{1}{5}$,从而

$$原式=\lim_{t\to0^+}\frac{(1+7t+2t^5)^{\frac{1}{5}}-1}{t}=\lim_{t\to0^+}\frac{\frac{1}{5}(7t+2t^5)}{t}=\lim_{t\to0^+}\frac{1}{5}(7+2t^4)=\frac{7}{5}=b,$$

所以 $a=\dfrac{1}{5},b=\dfrac{7}{5}$.

2.若 $\lim\limits_{n\to+\infty}\dfrac{n^a}{n^b-(n-1)^b}=1995$,求常数 a,b.

解 $$\lim_{n\to\infty}\frac{n^a}{n^b-(n-1)^b}=\lim_{n\to\infty}\frac{n^a}{n^b\left[1-(1-\frac{1}{n})^b\right]}=\lim_{n\to\infty}\frac{n^a}{-n^b\left\{\left[1+(-\frac{1}{n})\right]^b-1\right\}}$$

$$=-\lim_{n\to\infty}\frac{n^a}{n^b\cdot b(-\frac{1}{n})}=\lim_{n\to\infty}\frac{n^{a+1}}{bn^b}=1995,则\ a+1=b,$$

于是原式 $=\lim\limits_{n\to\infty}\dfrac{n^{a+1}}{bn^{a+1}}=\dfrac{1}{b}=1995$,所以 $b=\dfrac{1}{1995}$, $a=-\dfrac{1994}{1995}$.

3.求下列极限:

(1) $\lim\limits_{x\to0}\dfrac{3\sin x+x^2\cos\frac{1}{x}}{(1+\cos x)\ln(1+x)}$;

解法一 $$\lim_{x\to0}\frac{3\sin x+x^2\cos\frac{1}{x}}{(1+\cos x)\ln(1+x)}=\lim_{x\to0}\frac{3\sin x+x^2\cos\frac{1}{x}}{2\cdot x}$$

$$=\lim_{x\to0}(\frac{3}{2}\cdot\frac{\sin x}{x}+\frac{1}{2}x\cos\frac{1}{x})=\frac{3}{2}.$$

(2) $\lim\limits_{x\to0}(\dfrac{a_1^x+a_2^x+\cdots+a_n^x}{n})^{\frac{1}{x}}$;($a_1,a_2,\cdots a_n$ 均为正数);

解 $\lim\limits_{x\to0}(\dfrac{a_1^x+a_2^x+\cdots+a_n^x}{n})^{\frac{1}{x}}$

$$=\lim_{x\to0}\left\{\left[1+(\frac{a_1^x+a_2^x+\cdots a_n^x}{n}-1)^{\frac{1}{x}}\right]^{\frac{1}{\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}-1}}\right\}^{\frac{1}{x}(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}-1)}$$

由于$\lim\limits_{x\to 0}\frac{1}{x}(\frac{a_1^x+a_2^x+\cdots a_n^x}{n}-1)=\lim\limits_{x\to 0}\frac{1}{n}(\frac{a_1^x-1}{x}+\frac{a_2^x-1}{x}+\cdots+\frac{a_n^x-1}{x})$

$=\frac{1}{n}(\ln a_1+\ln a_2+\cdots+\ln a_n)=\frac{1}{n}\ln a_1a_2\cdots a_n=\ln(a_1a_2\cdots a_n)^{\frac{1}{n}}$,

从而　原式 $=e^{\ln(a_1a_2\cdots a_n)^{\frac{1}{n}}}=(a_1a_2\cdots a_n)^{\frac{1}{n}}$.

解法二　原式 $=\lim\limits_{x\to 0}e^{\frac{1}{x}\ln(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n})}=e^{\lim\limits_{x\to 0}\frac{1}{x}\ln\left[1+(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}-1)\right]}$

$=e^{\lim\limits_{x\to 0}\frac{1}{x}(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}-1)}=e^{\lim\limits_{x\to 0}\frac{1}{n}(\frac{a_1^x-1}{x}+\frac{a_2^x-1}{x}+\cdots+\frac{a_n^x-1}{x})}$

$=e^{\frac{1}{n}(\ln a_1+\ln a_2+\cdots+\ln a_n)}=(a_1a_2\cdots a_n)^{\frac{1}{n}}$

(3) $\lim\limits_{x\to+\infty}x^{\frac{3}{2}}[\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}]$;

解　$\lim\limits_{x\to+\infty}x^{\frac{3}{2}}[\sqrt{x+2}-2\sqrt{x+1}+\sqrt{x}]\overset{(令\frac{1}{x}=t)}{=\!=\!=}$

$\lim\limits_{t\to 0^+}(\frac{1}{t})^{\frac{3}{2}}\left[\sqrt{\frac{1}{t}+2}-2\sqrt{\frac{1}{t}+1}+\sqrt{\frac{1}{t}}\right]$

$=\lim\limits_{t\to 0^+}\frac{\sqrt{1+2t}-2\sqrt{1+t}+1}{t^2}=\lim\limits_{t\to 0^+}\frac{(\sqrt{1+2t}+1)^2-4(1+t)}{t^2(\sqrt{1+2t}+1+2\sqrt{1+t})}$

$=\lim\limits_{t\to 0^+}\frac{2\sqrt{1+2t}-2-2t}{4t^2}=\lim\limits_{t\to 0^+}\frac{\sqrt{1+2t}-(1+t)}{2t^2}$

$=\lim\limits_{t\to 0^+}\frac{(1+2t)-(1+t)^2}{2t^2[\sqrt{1+2t}+1+t]}=\lim\limits_{t\to 0^+}\frac{-t^2}{2t^2\cdot 2}=-\frac{1}{4}$.

(4) $\lim\limits_{x\to 0}(\frac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c})^{\frac{1}{x}}(a>0,b>0,c>0)$;

解　由于$\lim\limits_{x\to 0}(\frac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c})^{\frac{1}{x}}$

$$=\lim_{x\to 0}\left\{\left[1+(\frac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}-1)\right]^{\frac{1}{(\frac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}-1)}}\right\}^{\frac{1}{x}(\frac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}-1)}$$

而且$\lim\limits_{x\to 0}\frac{1}{x}(\frac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}-1)$

$=\frac{1}{a+b+c}(a\ln a+b\ln b+c\ln c)=\ln(a^ab^bc^c)^{\frac{1}{a+b+c}}$,

于是，原式 $=e^{\ln(a^ab^bc^c)^{\frac{1}{a+b+c}}}=(a^ab^bc^c)^{\frac{1}{a+b+c}}$.

(5) $\lim\limits_{x\to a}\dfrac{a^{a^x}-a^{x^a}}{a^x-x^a}(a>0)$;

解 $\lim\limits_{x\to a}\dfrac{a^{a^x}-a^{x^a}}{a^x-x^a}=\lim\limits_{x\to a}a^{x^a}\cdot\dfrac{a^{a^x-x^a}-1}{a^x-x^a}(x\to a,a^x-x^a\to 0)=a^{a^a}\ln a.$

(6) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\cdots+\dfrac{n}{n^2+n+n}\right)$;

解 由于$\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\cdots+\dfrac{n}{n^2+n+n}$

$<\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+1}+\cdots+\dfrac{n}{n^2+n+1}=\dfrac{n(n+1)}{2(n^2+n+1)}$,

$\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\cdots+\dfrac{n}{n^2+n+n}$

$>\dfrac{1}{n^2+n+n}+\dfrac{2}{n^2+n+n}+\cdots+\dfrac{n}{n^2+n+n}=\dfrac{n(n+1)}{2(n^2+2n)}=\dfrac{n+1}{2(n+2)}$,

而且$\lim\limits_{n\to\infty}\dfrac{n(n+1)}{2(n^2+n+1)}=\dfrac{1}{2}$, $\lim\limits_{n\to\infty}\dfrac{n+1}{2(n+2)}=\dfrac{1}{2}$,

根据夹逼定理知$\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\cdots+\dfrac{n}{n^2+n+n}\right)=\dfrac{1}{2}$.

4. 已知$\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax+b\right)=3$，求常数 a,b.

解 $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax+b\right)=\lim\limits_{x\to\infty}\dfrac{(1-a)x^2+(b-a)x+b+1}{x+1}=3$，由题意知 $1-a=0,b-a=3$，解得 $a=1,b=4$.

5. 证明不等式：

(1) $(1+\dfrac{1}{n})^n<\mathrm{e}<(1+\dfrac{1}{n})^{n+1}\quad n=1,2,3\cdots$;

证 由$\left\{(1+\dfrac{1}{n})^n\right\}$严格递增，且$\lim\limits_{n\to\infty}(1+\dfrac{1}{n})^n=\mathrm{e}$，知$(1+\dfrac{1}{n})^n<\mathrm{e}$.

由于$(n+1)^2=n^2+2n+1>n^2+2n$，得$\dfrac{1}{n}>\dfrac{n+2}{(n+1)^2}$ (1),

设 $b>a>0$，有$\dfrac{b^{n+1}-a^{n+1}}{b-a}=b^n+b^{n-1}a+\cdots+a^n>(n+1)a^n$，得

$b^{n+1}>(b-a)(n+1)a^n+a^{n+1}=a^n[a+(n+1)(b-a)]$,

令 $b=1+\dfrac{1}{n}$, $a=1+\dfrac{1}{n+1}$，有 $b>a$ 代入上式得

$(1+\dfrac{1}{n})^{n+1}>(1+\dfrac{1}{n+1})^n(1+\dfrac{1}{n+1}+\dfrac{1}{n})$(由(1)知)

$>(1+\dfrac{1}{n+1})^n\left[1+\dfrac{1}{n+1}+\dfrac{n+2}{(n+1)^2}\right]$

$$= (1+\frac{1}{n+1})^n\left[1+\frac{2}{n+1}+\frac{1}{(n+1)^2}\right] = (1+\frac{1}{n+1})^{n+2}.$$

知$\left\{(1+\frac{1}{n})^{n+1}\right\}$严格递减，又$\lim\limits_{n\to\infty}(1+\frac{1}{n})^{n+1} = \lim\limits_{n\to\infty}(1+\frac{1}{n})^n\cdot(1+\frac{1}{n}) = e$,

所以$(1+\frac{1}{n})^{n+1} > e$，故$(1+\frac{1}{n})^n < e < (1+\frac{1}{n})^{n+1}$, $n = 1,2,3\cdots$.

(2) $\frac{1}{n+1} < \ln(1+\frac{1}{n}) < \frac{1}{n}$　　$n = 1,2,3\cdots$.

由(1)的结论$(1+\frac{1}{n})^n < e < (1+\frac{1}{n})^{n+1}$，得$n\ln(1+\frac{1}{n}) < \ln e$或$\ln(1+\frac{1}{n}) < \frac{1}{n}$,

$1 = \ln e < (n+1)\ln(1+\frac{1}{n})$或$\ln(1+\frac{1}{n}) > \frac{1}{n+1}$，所以$\frac{1}{n+1} < \ln(1+\frac{1}{n}) < \frac{1}{n}$.

6. 设$x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n$, $n = 1,2,\cdots$，证明$\{x_n\}$收敛(利用第5题).

证　由于$x_{n+1}-x_n = \frac{1}{n+1}-\ln(n+1)+\ln x = \frac{1}{n+1}-\ln(1+\frac{1}{n}) < 0$，知$\{x_n\}$严格递减，又

$$x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n > \ln(1+1)+\ln(1+\frac{1}{2})+\cdots+\ln(1+\frac{1}{n})-\ln n$$

$= \ln(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n+1}{n})-\ln n = \ln(n+1)-\ln n > 0$，知$\{x_n\}$有界. 根据单调有界定理知$\{x_n\}$收敛.

7. 求极限$\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n})$(利用第6题)

解　由第6题知$\{x_n\}$收敛，设$\lim\limits_{n\to\infty}x_n = a$，则$\lim\limits_{n\to\infty}x_{2n} = a$. 于是

$\lim\limits_{n\to\infty}(x_{2n}-x_n) = a-a = 0$.

$$x_{2n}-x_n = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}-\ln 2n+\ln n = \frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{2n}-\ln 2,$$

由$\lim\limits_{n\to\infty}(x_{2n}-x_n) = 0$. 得$\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}-\ln 2) = 0$,

所以$\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}) = \ln 2$.

8. 证明：若$\lim\limits_{n\to\infty}a_n = a$，则$\lim\limits_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n} = a$.

证　由$\lim\limits_{n\to\infty}a_n = a$，于是，对于任意给定的$\varepsilon > 0$, $\exists N_1$，当$n > N_1$时，都有$|a_n-a| < \frac{\varepsilon}{2}$，从而，$|a_{N_1+1}-a| < \frac{\varepsilon}{2}$, $|a_{N_1+2}-a| < \frac{\varepsilon}{2}$, $\cdots|a_n-a| < \frac{\varepsilon}{2}$. 对于给定的$N_1$，由于$|(a_1-a)+(a_2-a)+\cdots+(a_{N1}-a)|$(为常数) $= c$，知$\lim\limits_{n\to\infty}\frac{c}{n} = 0$.

对上述的 $\varepsilon>0$, $\exists N_2$, 当 $n>N_2$ 时, 都有 $\frac{c}{n}<\frac{\varepsilon}{2}$, 取 $N=\max\{N_1,N_2\}$, 当 $n>N$ 时, 都有 $\left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right|$

$$=\left|\frac{(a_1-a)+(a_2-a)+\cdots+(a_{N_1}-a)}{n}+\frac{(a_{N_1+1}-a)+\cdots+(a_n-a)}{n}\right|$$

$$\leqslant\left|\frac{(a_1-a)+(a_2-a)+\cdots+(a_{N_1}-a)}{n}+\frac{(a_{N_1+1}-a)+\cdots+(a_n-a)}{n}\right|$$

$$<\frac{\varepsilon}{2}+\frac{(n-N_1)}{n}\frac{\varepsilon}{2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

由数列极限定义知 $\lim\limits_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n}=a$.

9. 证明:若 $\lim\limits_{n\to\infty}a_n=a$, 且 $a_n>0$, 则 $\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}=a$(利用第 8 题)

解 $\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\cdots a_n}=\lim\limits_{n\to\infty}e^{\ln(a_1a_2\cdots a_n)^{\frac{1}{n}}}$

$$=e^{\lim\limits_{n\to\infty}\frac{\ln a_1+\ln a_2+\cdots+\ln a_n}{n}}(\text{由第 8 题结论})=e^{\ln a}=a.$$

10. 证明:若 $a_n>0$, $n=1,2,3\cdots$, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=a$, 则 $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=a$(利用第 9 题).

证 设 $b_n=\frac{a_{n+1}}{a_n}$, 知 $\lim\limits_{n\to\infty}b_n=a\geqslant 0$, 于是

$\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}}\cdot a_1}=\lim\limits_{n\to\infty}\sqrt[n]{a_1}\cdot\sqrt[n]{b_1\cdot b_2\cdots b_n}$(由第9题结论得)$=1\cdot a=a$.

11. 求下列极限:

(1) $\lim\limits_{n\to\infty}\frac{\sqrt{2}+\sqrt[3]{2}+\cdots+\sqrt[n]{2}}{n}$(利用第 8 题);

解 由于 $\lim\limits_{n\to\infty}\sqrt[n]{2}=1$, 由第 8 题结论得 $\lim\limits_{n\to\infty}\frac{\sqrt{2}+\sqrt[3]{2}+\cdots+\sqrt[n]{2}}{n}=1$.

(2) $\lim\limits_{n\to\infty}\frac{n}{\sqrt[n]{n!}}$(利用第 10 题);

解 设 $a_n=\frac{n^n}{n!}$, 由于

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)!}\div\frac{n^n}{n!}\right]=\lim_{n\to\infty}(1+\frac{1}{n})^n=e,$$

因此, 第 10 题结论知

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}=\lim_{n\to\infty}\sqrt[n]{a_n}=e.$$