综合题

求下列函数的不定积分：

1. $\int \frac{dx}{x^6(1+x^2)}$

解 $\int \frac{dx}{x^6(1+x^2)} = \int \frac{1+x^2-x^2}{x^6(1+x^2)}dx = \int \left[\frac{1}{x^6} - \frac{1+x^2-x^2}{x^4(1+x^2)}\right]dx$

$= \int \left[\frac{1}{x^6} - \frac{1}{x^4} + \frac{1+x^2-x^2}{x^2(1+x^2)}\right]dx = \int \left(\frac{1}{x^6} - \frac{1}{x^4} + \frac{1}{x^2} - \frac{1}{1+x^2}\right)dx$

$$= -\frac{1}{5}\cdot\frac{1}{x^5}+\frac{1}{3}\cdot\frac{1}{x^3}-\frac{1}{x}-\arctan x + C.$$

2. $\int \frac{x+2}{x^2\sqrt{1-x^2}}dx.$

解 设 $x=\sin t$，则 $dx=\cos t dt$，由 $-1<x<1$，于是限制 $-\frac{\pi}{2}<t<\frac{\pi}{2}$，从而有

$$\int \frac{x+2}{x^2\sqrt{1-x^2}}dx=\int\frac{(\sin t+2)\cos t dt}{\sin^2 t\,|\cos t|}=\int\frac{dt}{\sin t}+2\int\frac{dt}{\sin^2 t}$$

$$=\ln|\csc t-\cot t|-2\cot t+C=-\ln\frac{1+\sqrt{1-x^2}}{|x|}-\frac{2\sqrt{1-x^2}}{x}+C.$$

3. $\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}dx.$

解 $$\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}dx=\int\frac{(1+\sqrt{1-x^2})(1+\sqrt{1-x^2})}{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}dx$$

$$=\int\frac{2-x^2+2\sqrt{1-x^2}}{x^2}dx=-\frac{2}{x}-x-2\int\sqrt{1-x^2}\,d\left(\frac{1}{x}\right)$$

$$=-\frac{2}{x}-x-\frac{2}{x}\sqrt{1-x^2}-2\int\frac{dx}{\sqrt{1-x^2}}=-\frac{2+x^2}{x}-\frac{2}{x}\sqrt{1-x^2}-2\arcsin x+C.$$

4. $\int x\ln(4+x^4)dx.$

解 $$\int x\ln(4+x^4)dx=-\frac{1}{2}\int\ln(4+x^4)dx^2=\frac{1}{2}x^2\ln(4+x^4)-2\int\frac{x^5dx}{4+x^4}$$

$$=-\frac{1}{2}x^2\ln(4+x^4)-2\int\left(x-\frac{4x}{4+x^4}\right)dx=\frac{1}{2}x^2\ln(4+x^4)-x^2+2\arctan\left(\frac{x^2}{2}\right)+C.$$

5. $\int \frac{x\ln(1+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx.$

解 $$\int \frac{x\ln(1+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx=\int\ln(1+\sqrt{1+x^2})\,d\sqrt{1+x^2}$$

$$=\int\ln(1+\sqrt{1+x^2})\,d(1+\sqrt{1+x^2})=(1+\sqrt{1+x^2})\left[\ln(1+\sqrt{1+x^2})-1\right]+C$$

6. $\int\frac{\arctan x}{x^2(1+x^2)}dx.$

解 $$\int\frac{\arctan x}{x^2(1+x^2)}dx=\int\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)\arctan x\,dx$$

$$= \int \arctan x d\left(-\frac{1}{x}\right) - \int \frac{1}{1+x^2} \arctan x dx$$

$$= -\frac{1}{x}\arctan x + \int \frac{1}{x} \cdot \frac{1}{(1+x^2)} dx - \int \arctan x d \arctan x$$

$$= -\frac{1}{x}\arctan x + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx - \frac{1}{2}(\arctan x)^2$$

$$= -\frac{1}{x}\arctan x + \ln|x| - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}(\arctan x)^2 + C$$

$$= -\frac{1}{x}\arctan x + \frac{1}{2}\ln\frac{x^2}{1+x^2} - \frac{1}{2}(\arctan x)^2 + C.$$

7. $\int e^{2x}(\tan x + 1)^2 dx$.

解 $\int e^{2x}(\tan x + 1)^2 dx = \int e^{2x}\sec^2 x dx + 2\int e^{2x}\tan x dx$

$$= e^{2x}\tan x - \int 2e^{2x}\tan x dx + 2\int e^{2x}\tan x dx = e^{2x}\tan x + C.$$

8. $\int \frac{dx}{\sin x \sqrt{1+\cos x}}$.

解 设 $\sqrt{1+\cos x} = t$，则 $\sin x = t\sqrt{2-t^2}$，$dx = -\frac{2}{\sqrt{2-t^2}}dt$，于是

$$\int \frac{dx}{\sin x \sqrt{1+\cos x}} = -\int \frac{2dt}{t^2(2-t^2)} = -\int \left(\frac{1}{t^2} + \frac{1}{2-t^2}\right) dt$$

$$= \frac{1}{t} - \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + C = \frac{1}{\sqrt{1+\cos x}} - \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}-\sqrt{1+\cos x}}\right| + C.$$

9. $\int \frac{x\arctan x}{\sqrt{1+x^2}} dx$.

解 $\int \frac{x\arctan x}{\sqrt{1+x^2}} dx = \int \arctan x d\sqrt{1+x^2}$

$$= \sqrt{1+x^2}\arctan x - \int \frac{1}{\sqrt{1+x^2}} dx = \sqrt{1+x^2}\arctan x - \ln(x + \sqrt{1+x^2}) + C.$$

10. $\int \frac{\sin 2x}{\sqrt{1+\cos^4 x}} dx$.

解 $\int \frac{\sin 2x}{\sqrt{1+\cos^4 x}} dx = -\int \frac{\frac{1}{2}d(1+\cos 2x)}{\sqrt{1+\frac{1}{4}(1+\cos 2x)^2}} = -\int \frac{d(1+\cos 2x)}{\sqrt{(1+\cos 2x)^2+4}}$

$$= -\ln\left[1+\cos 2x + \sqrt{(1+\cos 2x)^2+4}\right] + C$$

$$= -\ln(\cos^2 x + \sqrt{1+\cos^4 x}) + C_1 (C_1 = C - \ln 2).$$

11. $\int \frac{x\ln x}{(1+x^2)^2}dx$.

解 $\int \frac{x\ln x}{(1+x^2)^2}dx = \frac{1}{2}\int \frac{\ln x}{(1+x^2)^2}d(1+x^2) = -\frac{1}{2}\int \ln x d(\frac{1}{1+x^2})$

$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int \frac{1}{1+x^2}\cdot\frac{1}{x}dx = -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int (\frac{1}{x} - \frac{x}{1+x^2})dx$

$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\ln|x| - \frac{1}{4}\ln(1+x^2) + C = -\frac{\ln x}{2(1+x^2)} + \frac{1}{4}\ln\frac{x^2}{1+x^2} + C.$

12. $\int \sqrt{1-x^2}\arcsin x dx$.

解 $\int \sqrt{1-x^2}\arcsin x dx = x\sqrt{1-x^2}\arcsin x - \int x(1 - \frac{x}{\sqrt{1-x^2}}\arcsin x)dx$

$= x\sqrt{1-x^2}\arcsin x - \frac{1}{2}x^2 - \int \sqrt{1-x^2}\arcsin x dx + \int \frac{\arcsin x}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2}\arcsin x - \frac{x^2}{2} - \int \sqrt{1-x^2}\arcsin x dx + \frac{1}{2}(\arcsin x)^2$，于是

$\int \sqrt{1-x^2}\arcsin x dx = \frac{x}{2}\sqrt{1-x^2}\arcsin x - \frac{x^2}{4} + \frac{1}{4}(\arcsin x)^2 + C.$

13. 设 $f(x^2-1) = \ln\frac{x^2}{x^2-2}$，且 $f(\varphi(x)) = \ln x$，求 $\int \varphi(x)dx$.

解 因为 $f(x^2-1) = \ln\frac{(x^2-1)+1}{(x^2-1)-1}$，所以 $f(x) = \ln\frac{x+1}{x-1}$

又 $f(\varphi(x)) = \ln\frac{\varphi(x)+1}{\varphi(x)-1} = \ln x$，从而 $\frac{\varphi(x)+1}{\varphi(x)-1} = x$ 或 $\varphi(x) = \frac{x+1}{x-1}$.

于是 $\int \varphi(x)dx = \int \frac{x+1}{x-1}dx = \int (1 + \frac{2}{x-1})dx = x + 2\ln|x-1| + C.$

14. $\int \frac{\arcsin e^x}{e^x}dx$.

解 $\int \frac{\arcsin e^x}{e^x}dx = -\int \arcsin e^x d e^{-x} = -e^{-x}\arcsin e^x + \int \frac{dx}{\sqrt{1-e^{2x}}}$

$= -e^{-x}\arcsin^x - \int \frac{de^{-x}}{\sqrt{e^{-2x}-1}} = -e^{-x}\arcsin e^x - \ln(e^{-x} + \sqrt{e^{-2x}-1}) + C$

$= x - e^{-x}\arcsin e^x - \ln(1 + \sqrt{1-e^{2x}}) + C.$

15. $\int x^x(1+\ln x)dx$.

解 $\int x^x(1+\ln x)dx = \int e^{x\ln x}d(x\ln x) = e^{x\ln x} + C = x^x + C.$

16. $\int \frac{1+\sin x}{1+\cos x}e^x dx$.

解 $\int \frac{1+\sin x}{1+\cos x} e^x dx = \int \left(\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) e^x dx = \int \frac{e^x}{2\cos^2\frac{x}{2}} dx + \int e^x \tan\frac{x}{2} dx$

$= \int e^x d\left(\tan\frac{x}{2}\right) = \int \tan\frac{x}{2} de^x = e^x \tan\frac{x}{2} - \int \tan\frac{x}{2} de^x + \int \tan\frac{x}{2} de^x = e^x \tan\frac{x}{2} + C.$

17. $\int |x| dx$.

解 由于 $|x| = \begin{cases} -x, & x \leqslant 0, \\ x, & x > 0, \end{cases}$ 于是 $\int |x| dx = \begin{cases} -\frac{x^2}{2} + C_1, & x \leqslant 0, \\ \frac{x^2}{2} + C_2, & x > 0. \end{cases}$

由原函数可导,则原函数连续,当然在点 $x = 0$ 处也连续,有 $0 + C_1 = 0 + C_2$,即 $C_1 = C_2$,因此

$$\int |x| dx = \begin{cases} -\frac{x^2}{2} + C_1, & x \leqslant 0, \\ \frac{x^2}{2} + C_1, & x > 0 \end{cases} = (\operatorname{sng} x)\frac{1}{2}x^2 + C_1 = \frac{x|x|}{2} + C.$$

18. $\int [|1+x| - |1-x|] dx$.

解 $\int [|1+x| - |1-x|] dx = \int |1+x| d(1+x) + \int |1-x| d(1-x)$

$= \frac{(1+x)|1+x|}{2} + \frac{(1-x)|1-x|}{2} + C.$

19. $\int e^{-|x|} dx$.

解 由于 $e^{|x|} = \begin{cases} e^{-x}, & x \leqslant 0, \\ e^x, & x > 0. \end{cases}$ 于是 $\int e^{-|x|} dx = \begin{cases} e^x + C_1, & x \leqslant 0, \\ -e^{-x} + C_2, & x > 0. \end{cases}$

又原函数在点 $x = 0$ 处连续,有 $e^0 + C_1 = -e^0 + C_2$,得 $C_1 = -2 + C_2, C_1 + 1 = -1 + C_2 = C$.因此

$$\int e^{-|x|} dx = \begin{cases} e^x - 1 + C, & x \leqslant 0, \\ 1 - e^{-x} + C, & x > 0. \end{cases}$$

20. $\int f(x) dx$,其中 $f(x) = \begin{cases} 1, & x < 0, \\ x + 1, & 0 \leqslant x < 1, \\ 2x, & x \geqslant 1. \end{cases}$

解 $\int f(x)dx = \begin{cases} x + C_1, & x \leqslant 0, \\ \dfrac{x^2}{2} + x + C_2, & 0 \leqslant x < 1, \\ x^2 + C_3, & x \geqslant 1. \end{cases}$

由原函数在点 $x = 0, x = 1$ 处连续,有 $C_1 = C_2$ 且 $\frac{1}{2} + 1 + C_2 = 1 + C_3$,得 $C_1 = C_2, C_3 = \frac{1}{2} + C_2$ 于是

$$\int f(x)dx = \begin{cases} x + C_2, & x < 0, \\ \dfrac{x^2}{2} + x + C_2, & 0 \leqslant x < 1, \\ x^2 + C_3, & x \geqslant 1. \end{cases}$$

21. 设 $f'(x^2) = \frac{1}{x}, (x > 0)$,求 $f(x)$.

解 设 $x^2 = t$,由 $x > 0$,有 $x = \sqrt{t}$,则 $f'(t) = \frac{1}{\sqrt{t}}$ 或 $f'(x) = \frac{1}{\sqrt{x}}$. 于是

$f(x) = \int f'(x)dx = \int \frac{1}{\sqrt{x}}dx = 2\sqrt{x} + C.$

22. $\int \max\{x^2, x^3\}dx$.

解 由于 $\max\{x^2, x^3\} = \begin{cases} x^2, & x \leqslant 1, \\ x^3, & x > 1. \end{cases}$

于是 $\int \max\{x^2, x^3\}dx = \begin{cases} \dfrac{1}{3}x^3 + C_1, & x \leqslant 1, \\ \dfrac{1}{4}x^4 + C_2, & x > 1. \end{cases}$ 由原函数在点 $x = 1$ 处连续,知 $\frac{1}{3} + C_1 = \frac{1}{4} + C_2$,得 $C_2 = \frac{1}{12} + C_1$,因此

$$\int \max\{x^2, x^3\}dx = \begin{cases} \dfrac{1}{3}x^3 + C_1, & x \leqslant 1, \\ \dfrac{1}{4}x^4 + \dfrac{1}{12} + C_1, & x > 1. \end{cases}$$

23. $\int \left[\frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{f'^3(x)}\right]dx$.

解 $\int \left[\frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{f'^3(x)}\right]dx = \int \frac{f(x)}{f'(x)}dx - \int \frac{f^2(x)}{f'^3(x)}df'(x)$

$= \int \frac{f(x)}{f'(x)}dx + \frac{1}{2}\int f^2(x)d\,\frac{1}{f'^2(x)}$

$$= \int \frac{f(x)}{f'(x)}dx + \frac{f^2(x)}{2f'^2(x)} - \frac{1}{2}\int \frac{1}{f'^2(x)} \cdot 2f(x)f'(x)dx = \frac{f^2(x)}{f'^2(x)} + C.$$

24. $\int \frac{f'(\ln x)}{x\sqrt{f(\ln x)}}dx$.

解 $\int \frac{f'(\ln x)}{x\sqrt{f(\ln x)}}dx = \int \frac{f'(\ln x)}{\sqrt{f(\ln x)}}d\ln x = \int \frac{1}{\sqrt{f(\ln x)}}df(\ln x) = 2\sqrt{f(\ln x)} + C.$

25. 推导下列递推公式

(1) 若 $I_n = \int \sin^n x dx$，则 $I_n = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2} \quad (n \in N)$.

(2) 若 $I_n = \int \cos^n x dx$，则 $I_n = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}I_{n-1} \quad (n \in N)$.

(3) 若 $I_n = \int \tan^n dx$，则 $I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2} \quad (n \in N)$.

证 (1) $I_n = \int \sin^n x dx = \int \sin^{n-1} x d(-\cos x)$

$$= -\sin^{n-1}x\cos x + \int \cos x(n-1)\sin^{n-2}x\cos x dx$$

$$= -\sin^{n-1}x\cos x + \int (n-1)(1-\sin^2 x)\sin^{n-2}x dx$$

$$= -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

故 $I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}$.

(2) $I_n = \int \cos^n x dx = \int \cos^{n-1} x d\sin x = \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x dx$

$$= \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n$$

故 $I_n = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}$.

(3) $I_n = \int \tan^n x dx = \int \tan^{n-2}x(\sec^2 x - 1)dx = \int \tan^{n-2}x d\tan x - I_{n-2}$

$$= \frac{1}{n-1}\tan^{n-1}x - I_{n-2}.$$

26. 设 $I_n = \int \ln^n x dx \quad (n \in N)$.

(1) 证明：$I_n = x\ln^n x - nI_{n-1}$；

(2) 证明：

$$I_n = x[\ln^n x - n\ln^{n-1}x + n(n-1)\ln^{n-2}x - \cdots + (-1)^{n-1}n(n-1)\cdots 3 \cdot 2\ln x + (-1)^n n!] + C.$$

证　(1) $I_n = \int \ln^n dx = x\ln^n x - \int xn(\ln^{n-1} x) \cdot \frac{1}{x} dx = x\ln^n x - nI_{n-1}$

(2) 由(1)知

$I_n = x\ln^n x - nI_{n-1} = x\ln^n x - n[x\ln^{n-1} x - (n-1)I_{n-2}]$

$= x\ln^n x - nx\ln^{n-1} x + n(n-1)I_{n-2}$

$= x\ln^n x - nx\ln^{n-1} x + n(n-1)[x\ln^{n-2} x - (n-2)I_{n-3}]$

$= x\ln^n x - nx\ln^{n-1} x + n(n-1)x\ln^{n-2} x - n(n-1)(n-2)I_{n-3}$

$= \cdots = x\ln^n x - nx\ln^{n-1} x + n(n-1)x\ln^{n-2} x$

$+ \cdots + [(-1)^{n-1} n(n-1)\cdots 3 \cdot 2\ln x + (-1)^n n!] + C.$

27. 设 $I_{k,m} = \int x^k \ln^m x dx \quad (k \neq 1, k, m \in N)$.

求证 $I_{k,m} = \frac{1}{k+1} x^{k+1} \ln^m x - \frac{m}{k+1} I_{k,m-1}$

证　$I_{k,m} = \int x^k \ln^m x dx = \int \ln^m x d\, \frac{1}{k+1} x^{k+1}$

$= \frac{1}{k+1} x^{k+1} \ln^m x - \frac{m}{k+1} \int x^{k+1} \cdot \ln^{m-1} x \cdot \frac{1}{x} dx$

$= \frac{1}{k+1} x^{k+1} \ln^m x - \frac{m}{k+1} \int x^k \ln^{m-1} x dx = \frac{1}{k+1} x^{k+1} \ln^m x - \frac{m}{k+1} I_{k,m-1}.$

28. 设函数 $y = f(x)$ 在某区间具有连续的导数，且 $f'(x) \neq 0$，$x = f^{-1}(y)$ 是它的反函数，试证明：

(1) $\int f(x) dx = xf(x) - \int f^{-1}(y) dy$;

(2) $\int f^{-1}(x) dx = xf^{-1}(x) - F(f^{-1}(x)) + C$.

其中 $F(x)$ 是 $f(x)$ 的一个原函数.

证　(1) $\int f(x) dx = xf(x) - \int x df(x) = xf(x) - \int f^{-1}(y) dy$.

(2) $\int f^{-1}(x) dx = xf^{-1}(x) - \int x df^{-1}(x)$ (设 $f^{-1}(x) = y$)

$= xf^{-1}(x) - \int f(y) dy = xf^{-1}(x) - F(y) + C = xf^{-1}(x) - F(f^{-1}(x)) + C$.