Graph Theory

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Contents

Chapter 1

Introduction

1.1 Basic Definitions/Concepts

Definition 1 (Topological Space)**.** A set X **topological space** is a topological space if for each x of X, there is a nonempty collection of subsets of X, called neighbourhoods of x, which satisfy the following axioms:

(a) x lies in each of its neighbourhoods.

(b) The intersection of two neighbourhoods of x is itself a neighbourhood of x.

(c) If N is a neighbourhood of x and if U is a subset of X which contains N, then U is a neighbourhood of x.

(d) If N is a neighbourhood of x, then we denote **the interior** of N as the set $\mathring{N} := \{z \in N \mid N \text{ is a neighbourhood of } z\}$. $\mathring{N}$ is a neighbourhood of x.

We say, if (a)-(d) are satisfied to each point $x \in X$, then there is a **topology** on the set X.

Definition 2 (Map)**.** Let X, Y be topological spaces. A function $f\colon X \to Y$ is **continuous** if for each point x of X and each neighbourhood N of $f(x)$ in Y, the set $f^{-1}(N)$ is a neighbourhood of x in X. Continuous functions are called **maps**.

Definition 3 (Homeomorphism)**.** A function $h\colon X \to Y$ is a **homeomorphism** if it is one-one, onto, continuous, and has a continuous inverse. When such a function exists, X and Y are called **homeomorphic (or topologically equivalent) spaces**.

Definition 4 (Surface)**.** A **surface** is a topological space in which each point has a neighbourhood homeomorphic to the plane, and for which any two distinct points possess disjoint neighbourhoods.

Definition 5 (Open)**.** Let X be a topological space and call a subset O of X **open** if it is a neighbourhood of each of its points.

Remark. From axiom (c), the union of any collection of open sets is open and from axiom (b) the intersection of any *finite* number of open sets is open. Lastly, (d) shows that the interior of N is an open set which lies inside N and contains x.

Definition 6 (New and Improved Topological Space)**.** A topology on a set X is a nonempty collection of subsets of X, which we call open sets, such that:

1. any union of open sets is itself open
2. any finite intersection of open sets is open
3. both the whole set X and the empty set are open.

Given a point x of X, we shall call a subset N of X a *neighbourhood* of x if we can find an open set O so $x \in O \subseteq N$. A set together with a topology on it is a topological space.

Proof. This set X is a topological space because for each $x \in X$, X is an open neighborhood of x (a). This also confirms (c). If N_1, N_2 are neighbourhoods of x, we can find open sets O_1, O_2 so $x \in O_1 \subseteq N_1$ and $x \in O_2 \subseteq N_2$ such that $x \in O_1 \cap O_2 \subseteq N_1 \cap N_2$ Because $O_1 \cap O_2$ is open, $N_1 \cap N_2$ is a neighborhood of x (b). If N is a neighbourhood of x then there is an open set $O \subseteq N$ so $x \in O$. By definition, O is a neighborhood of each of its points. $\mathring{N}$ is the set of all points z that N is a neighbourhood of. Clearly, then, O is contained in $\mathring{N}$. Thus, $\mathring{N}$ is a neighborhood of x. □

Definition 7 (Usual Topology on $\mathbb{E}^n$)**.** A set U is open if given $x \in U$, there exists $\varepsilon \in R^+$ so the ball with centre x and radius ε lies entirely in U.

Definition 8 (Subspace/Induced Topology)**.** Given a topological space X and a subset Y of X, the open sets of the **subspace/induced** topology on Y are simply the intersection of all the open sets of X with Y.

i.e. A subset U of Y is open in the subspace topology if there exists an open set O of X so $U = O \cap Y$.

A subspace Y of a topological space X implies that Y is a subset of X with the subspace topology.

Definition 9 (Discrete Topology)**.** The largest possible topology on a given set X is the **discrete topology** wherein every subset of X is an open set. If X has the discrete topology, we call it a discrete space.

Example. If we take the set of points of $\mathbb{E}^n$ which have integer coordinates, and give it the subset topology, the result is a discrete space.

Definition 10 ("Larger" Topologies)**.** If one topology contains all the open sets of another, we say it is **larger**.

Definition 11 (Closed)**.** A subset of a topological space is closed if its complement is open.

Example. The following subsets of $\mathbb{E}^2$ are closed: the unit circle, the unit disk ($\{(x, y) \mid x^2 + y^2 \leq 1\}$), $y = e^x$, and $\{(x, y) \mid x \geq y^2\}$.

The set of all points (x, y) where $x \geq 0, y > 0$ is neither open nor closed.

The space X of all points (x, y) where $x \geq 1, y \leq -1$ with the topology induced from $\mathbb{E}^2$ is both open and closed in X (and notably not open in $\mathbb{E}^2$).

Remark. The intersection of any family of closed sets is closed. As is the union of any *finite* family of closed sets.

Definition 12 (Limit Point)**.** Let A be a subset of a topological space X. A point $p \in X$ is a **limit point** (or accumulation point) of A if every neighbourhood of p contains at least one point of $A - \{p\}$.

Example. Give the set of all real numbers X the *finite-complement topology* where a set is open if its complement is finite or all of X. If we take A to be an infinite subset of X, then every point of X is a limit point of A. Conversely, a finite subset of X has no limit points in this topology.

Explanation. To be a neighbourhood N of any $p \in X$, N must be open implying its complement is either finite or X. If $N^C = X$, $N = \emptyset$ so N cannot be a neighbourhood of p (this definition simply ensures $\emptyset$ is open so this is indeed a topology). Thus, to be a neighbourhood, N must be infinite with a finite complement.

If A is an infinite subset of X, it must then share some infinite points with N implying N contains points of $A - \{p\}$. Because this is the case for all N of p and all $p \in X$, p is a limit point of A.

If A is a finite subset, there exists neighbourhoods such that $N^C = A$ so every neighbourhood of p certainly does not contain one point of $A - \{p\}$ implying no point of X is a limit point of A.

Theorem 1. A set is closed if and only if it contains all its limit points.

Proof. $\Rightarrow$: If A is closed, then its complement $X - A$ is open so $X - A$ is a neighbourhood of each of its points. Clearly, if a limit point p of A were in $X - A$ then $X - A$ must contain a point of $A - \{p\}$ of which there are none. So A contains all its limit points.

$\Leftarrow$: Suppose A contains all its limit points. If $x \in X - A$, x is not a limit point of A so there exists a neighbourhood N of x which contains no point of A implying $N \subseteq X - A$ such that $X - A$ is also a neighbourhood of x for all $x \in X - A$ so $X - A$ is a neighbourhood of all of its points so it is open meaning A is closed. $\square$

Definition 13 (Closure)**.** The union of A and all its limit points is called the **closure** of A and is written $\overline{A}$.

Theorem 2. The closure of A is the smallest closed set containg A. i.e. the closure is the intersection of all closed sets containing A.

Proof. The closure of A is closed because if $x \in X - \overline{A}$ then x cannot be a limit point of A so there exists an open neighbourhood N of x such that it contains no points of A. Because N is an open set, it is a neighbourhood of all of its points so none of its points are limit points of A either. Thus, $N \subseteq X - A$ so $X - A$ is a neighbourhood of x so $X - A$ is a neighbourhood of each of its points so $X - \overline{A}$ is open so $\overline{A}$ is closed. Because $\overline{A}$ is closed, contains A, and is contained in every closed set containing A, it must be the intersection of all such sets.

NOTE: If we just said there exists a neighbourhood N of x, this neighbourhood may contain a limit point of A even if it does not contain a point of A. Thus, it is meaningful to prove none of its points can be limit points of A by saying the neighbourhood is open. $\square$

Corollary. A set is closed if and only if it is equal to its closure.

Definition 14 (Dense)**.** A set whose closure is the whole space is said to be **dense** in the space. A dense set meets every nonempty open subset of the space.

Definition 15 (Interior)**.** The **interior** $\mathring{A}$ of a set A is the union of all open sets contained in A. A point x lies in the interior of A if and only if A is a neighbourhood of x. Also, an open set is its own interior.

Example. In $\mathbb{E}^2$, denote the unit disk D and the unit circle C. D's interior is $D - C$ while C's interior is $\emptyset$.

Definition 16 (Frontier)**.** The **frontier** of a set A is the intersection of $\overline{A}$ with $\overline{X - A}$. This is equivalent to the points of X neither in the interior of A nor the interior of $X - A$.

Example. In $\mathbb{E}^2$, the unit disc D, its interior $\mathring{D}$, and the unit circle C all have the same frontier C.

The froniter of the set of points in $\mathbb{E}^3$ which have rational coordinates is all of $\mathbb{E}^3$. In this case, the frontier is the whole space.

Definition 17 (Base/Basis)**.** Given a topology on a set X, a collection β of open sets is called a **base/basis** for the topology if every open set is a union of members of β. Elements of β are called *basic open sets.*

Equivalently, given any point $x \in X$ and a neighbourhood N of x, there is always an element B of β so $x \in B \subseteq \beta$.

Theorem 3. Let β be a nonempty collection of subsets of a set X. If the intersection of any finite number of members of β is always in β, and if $\bigcup \beta = X$, then β is a base for a topology on X.

Proof. Let the collection of all possible unions of members of β be open sets. This then immediately satisfies our new definition for a topological space. □

Remark. A space whose topology has a countable base is called a **second countable space**. A space which contains a countable dense subset is said to be **separable**.