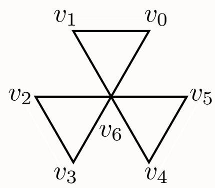
Page 183, Problem 11 (ab) (Worked with David LaRoche)

Proof. (a) Join together three copies of the boundary of a triangle at a single vertex as shown below:

K 

$|K|$ is connected so

$$H_0(K) = \mathbb{Z}^1 \quad \text{[Theorem 8.2]}.$$

K has fundamental group of a 3-bouquet F_3 generated by $z_1 = (v_6v_1) + (v_1v_0) + (v_0v_6)$, $z_2 = (v_6v_5) + (v_5v_4) + (v_4v_6)$, and $z_3 = (v_6v_3) + (v_3v_2) + (v_2v_6)$. Thus, its abelianization

$$\langle z_1, z_2, z_3 \mid z_1z_2z_1^{-1}z_2^{-1}, z_1z_3z_1^{-1}z_3^{-1}, z_2z_3z_2^{-1}z_3^{-1} = e\rangle$$

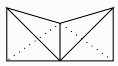
is equal to

$$H_1(K) \cong \mathbb{Z}^3 \quad \text{[Theorem 8.3]}.$$

There are no 2 - or higher - simplexes so $B_{\geq 2}(K), Z_{\geq 2}(K) = \{0\}$ and

$$H_q(K) = \{0\} \quad q \geq 2.$$

(b) We can glue two hollow tetrahedra along an edge to get the following:

K 

$|K|$ is connected so

$$H_0(K) = \mathbb{Z}^1 \quad \text{[Theorem 8.2]}.$$

Note that each empty tetrahedra is homeomorphic to S^2 so they have the same homotopy type. Because they're path-connected, they have isomorphic fundamental groups $\{e\}$. Now, by Van Kampen's theorem, K has the same fundamental group as the one point union of 2-spheres and thus has fundamental group equal to the free product of each trivial group which is itself the trivial group. Thus, $\pi_1(K)$'s abelianization gives

$$H_1(K) = \{0\} \quad \text{[Theorem 8.3]}.$$

There are no complexes of dimension 3 or greater so $B_2(K)$, $B_3(K)$, $\cdots = \{0\}$ and $Z_3(K), Z_4(K), \cdots = \{0\}$. We can therefore show $H_2(K) = Z_2(K)/\{0\} = \text{Ker } \partial\colon C_2(K) \to C_1(K) \cong \mathbb{Z}^2$ as follows.

Pick any 2-simplex to start. To annihilate the boundary of any 2-chain starting at this triangle, we MUST orient it compatibly with another 2-simplex on the same tetrahedron in order to kill their shared edge. This requires another 2-simplex on the same tetrahedron by the same argument and then the last one, all on one tetrahedron. This has already killed the boundary so its clear any 2-cycle can be decomposed into scalar multiples of these isolated 2-cycles on each tetrahedredon. Therefore, they together generate $Z(K)$. Addition of these cycles is commutative so the abelianization gives $\mathbb{Z}(K) \cong \mathbb{Z}^2$. Thus,

$$H_2(K) = Z_2(K)/\{0\} \cong \mathbb{Z}^2.$$

Given what we said about $Z_3(K), Z_4(K), \cdots$ and $B_3(K), B_4(K), \cdots$,

$$H_q(K) = \{0\} \quad q \geq 3.$$

□

Page 183, Problem 13

Proof. Take any (path-connected) graph Γ with maximal tree T. $G(\Gamma, T)$ by construction includes only the generators g_{ij} $(i < j)$ of edges of $\Gamma - T$ each 'representing' exactly one non-2-simplex triangle (a triangle made of 3 1-simplices which is not filled in). Recall every such triangle also has only this generator. T is clearly contractible and can be thought of as a single vertex from which each edge generator is independently attached so that $G(\Gamma, T)$ is isomorphic to the free group on n generators where n is the number of such edges. (Recall specifically there are no relations in $G(\Gamma, T)$ as T is maximal and an edge path contained entirely within T can always be found from the basepoint to some generator and back implying the generators are indpendent.) Last, Γ is path-connected so $G(\Gamma, T)$ is isomorphic to $\pi_1(\Gamma, v)$ regardless of choice of v.

This same result can also be achieved through repeatedly applying Van Kampen's theorem on T and one empty triangle at a time which would each time give just the generating loop of that triangle with the single relation that that loop 0 times is the identity, i.e. the trivial relation and therefore a free group on β_1 generators.

The homeomorphic reduction of 2-simplices first into one of their respective line-segment components and then all of T's 'spikes' down into just one point leaving only empty triangles implies Γ shares the exact homotopy type as a bouquet of circles. This also implies their isomorphic fundamental groups.

Now by Theorem 8.3, the abelianization of F_n gives $\overbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}^{n \text{ times}} \cong \mathbb{Z}^n$ as the first homology group of Γ. $H_1(\Gamma)$ has no torsion element so the first Betti number of Γ is the rank of $\mathbb{Z}^n$, i.e. n, i.e. the number of not-filled-in triangles in the graph $= \beta_1$.

□

Page 188, Problem 20

Proof. Take chain maps $\phi\colon C(K) \to C(L)$, $\psi\colon C(L) \to C(M)$. Because $\partial \circ \phi_q = \phi_{q-1} \circ \partial$ and $\partial \circ \psi_q = \psi_{q-1} \circ \partial$, we know that

$$\partial \circ (\psi_q \circ \phi_q) = (\partial \circ \psi_q) \circ \phi_q = (\psi_{q-1} \circ \partial) \circ \phi_q = (\psi_{q-1} \circ \phi_{q-1}) \circ \partial.$$

(Generically, we can show our induced map is well-defined via the following. For normal subgroups $N \lhd G, M \lhd H$ and $f\colon G \to H$, if $f(N) \subseteq M$ then $f_*\colon G/N \to H/M$ is well-defined as for any cosets $g_1 + N = g_2 + N$, $f(g_1) + M = f(g_1) + f(N) + M = f(g_1 + N) + M = f_*(g_1 + N) = f_*(g_2 + N) = f(g_2 + N) + M = f(g_2) + f(N) + f(M) = f(g_2) + M$. This is trivially a homomorphism too.)

Thus, to show $\psi \circ \phi$ induces a homomorphism $(\psi \circ \phi)_*\colon H_q(K) \to H_q(M)$, we need only show $\psi_q \circ \phi_q(Z_q(K)) \subseteq Z_q(M)$ and $\psi_q \circ \phi_q(B_q(K)) \subseteq B_q(M)$. This is implied (via the proof on page 185) by the fact that both ψ, ϕ are chain maps and therefore

$$\psi_q \circ \phi_q(Z_q(K)) \subseteq \psi_q(Z_q(L)) \subseteq Z_q(M),$$
$$\psi_q \circ \phi_q(B_q(K)) \subseteq \psi_q(B_q(L)) \subseteq B_q(M).$$

Thus, $\psi \circ \phi$ is a chain map and by the above containments, for any $z \in Z_q(K)$, we get $(\psi \circ \phi)_*(z + B_q(K)) = (\psi \circ \phi)(z + B_q(K)) + B_q(M) = (\psi \circ \phi)(z) + \psi \circ \phi(B_q(K)) + B_q(M) = (\psi \circ \phi)(z) + B_q(M) = (\psi \circ \phi)(z) + \psi(B_q(L)) + B_q(M) = \psi(\phi(z) + B_q(L)) + B_q(M) = \psi_*(\phi(z) + B_q(L)) = \psi_*(\phi(z) + \phi(B_q(K)) + B_q(L)) = \psi_*(\phi(z + B_q(K)) + B_q(L)) = \psi_*(\phi_*(z + B_q(K))) = \psi_* \circ \phi_*(z + B_q(K))$. z is arbitrary so $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. □

Page 192, Problem 25

Proof. Take similicial maps $s, t\colon |K| \to |L|$. Suppose there exists a homomorphism $d_q\colon C_q(K) \to C_{q+1}(L)$ for all q so that

$$d_{q-1}\partial + \partial d_q = t - s\colon C_q(K) \to C_{q+1}(L).$$

Then for any $z \in Z_q(K)$, $\partial(z) = 0$ and $d_q(z) \in C_{q+1}(L) \Rightarrow \partial d_q(z) \in B_q(L)$. This means $t(z) - s(z) = d_{q-1}\partial(z) + \partial(d_q(z)) = 0 + \partial d_q(z) \in B_q(L)$ and consequently that $t_*(z) - s_*(z) = [0]$ so finally $t_*(z) = s_*(z)$ for all z. Thefore s, t induce the same homomorphism $s_* = t_*\colon H_q(K) \to H_q(L)$. □