Graph Theory

Jack Lipson

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Chapter 0

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0.1 Definitions and Concepts

Using source: On the existence of triangulated spheres in 3-graphs and related Problems by Erdös and Brown.

Definition 1. Define an *r-graph* as $H^{(r)}$ as the pair of sets $V(H^r)$ of *vertices* and a class $E(H^{(r)})$ of r-subsets of V. If we follow $H^{(r)}$ with (n) or $(n;k)$, this denotes the r-graph has exactly n vertices and at least k r-tuples.

Remark. When $r = 2$ we omit the superscript and refer to it simply as a *graph*.

Definition 2. The letter G is reserved for all r-graphs with the properties appended, i.e. $G^{(r)}, G^{(r)}(n), and G^{(r)}(n;k)$.

Note. It is a well known property of graphs that any $G(n;n)$ contains a polygon.

Definition 3. For any fixed family of r-graphs, let $\text{ex}(n;H)$ or $\text{ex}(n;H^{(r)})$ denote the largest integer k for which there exists a $G^{(r)}(n;k)$ containing none of the members of H as a sub-r-graph.

Remark. For s less than r, the s-tuples of an r-graph will be *any* set of s vertices. The *star* of an s-tuple S in a $G^{(r)}$ is the $(r-s)$-graph which has vertices of $V(G^{(r)}) - V(S)$.

Definition 4. The *valency* of an s-tuple is the number of $(r-s)$ tuples in its star.

Definition 5. The *product* of an r-graph $A^{(r)}$ and an s-graph $B^{(s)}$ will be an $(r+s)$-graph whose vertex set is $V(A^{(r)} \cup V(B^{(r)}))$ and whose $(r+s)$-tuples are all unions of an r-tuple of A and a disjoint s-tuple of the second.

Definition 6. In particular, a *cone* over A^r is a product of A with a disgoint $G^{(1)}(1;1)$.

Definition 7. A *double pyramid* is a product of a polygon (graph) with a disjoint $G^{(1)}(2;2)$.

Note. It will be helpful to use geometrical language to interpret the triples of a 3-graph as the 2-simplexes of a simplical 2-complex (which contains all possible 1-simplexes). A wheel will be a cone over a polygon. An octahedron will be a double pyramid over a 4-gon.

Explanation. A simplical complex is a set composed of

Note. A simplex is a generalization of the simplest possible polytope in any given dimension. I.e. a point, line segment, triangle, tetrahedron, and 5-cell. A k-simplex has $k+1$ vertices.