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Proof. Take the square $[0,1] \times [0,1]$ such that two opposite edge pairs can be identified to create the torus T^2. Once the torus is wrapped together, imagine a circular puncture poked directly on the seam of the cylinder which made the torus but not on the seam of where the cylinder's boundary circles were identified, i.e., say $x = 1/2$ along the cylinder's length but half around $y = 0$ and half around $y = 1$. After being unwrapped, this will essentially have created a square with 2 small rectangular divets, say, of width 2δ, at the top and bottom at the halfway point (like a fat 'H' shape). We can smoothly homotopy each divet taller and taller as well as the sides of the 'H' thinner and thinner until we reach a very thin H with a skinny segment connecting it. This could be done via something like $F((x,y),t) = \begin{cases} (1-t)(x,y) + t(1/2-\delta, y) & \text{for } x \leq 1/2 - \delta \\ (1-t)(x,y) + t(x, 1/2) & \text{for } |x - 1/2| < \delta \\ (1-t)(x,y) + t(1/2+\delta, y) & \text{for } x \geq 1/2 + \delta \end{cases}$.

Note this modified square is still TOTALLY homeomorphically equivalent to the square which produces the punctured torus.

Applying the first identification connects the leftmost ends to each other and the right most ends to each other making 'o-o' where the two circles lie in a plane perpendicular to the segment connecting them. Doing the second identification identifies each circle together (imagine holding the connection of a pair of handcuffs down fixed and lifting up and gluing each cuff together). This will form a new loop via the small segment. Because the line segment and each circle were only connected at a single point based on the above homotopy, just rotating one loop will form 2 circles with union of a single point. This completes our deformation retract. □

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Proof. The question asks for just descriptions so these aren't ultra-rigorous!

(a) Take the boundary circle C embedded in the möbius strip S. Any basepoint on the circle C will have fundamental group $\mathbb{Z}$ representing the number and direction of winds around it. Therefore, we will take the homotopy class of a single loop α covering just $0 \to 2\pi$ to generate the group.

Now, take the rectangle $[0,10] \times [0,1]$ which, when its edges are inversely identified, creates the möbius strip. We can apply the continuous homotopy $F((x,y),t) = (1-t)(x,y) + t(x,1/2)$ to turn this pre-identification möbius strip into the boundary circle C implying after identification, it will simply have the fundamental group $\mathbb{Z}$ as well, using the same basepoint as C. More visually, any jumble of a loop on the pre-identication strip can be straightened to the identity because the set is convex. But, post-identification, a loop can be formed by traveling straight from $(0, 1/2$ along the x-axis until reaching $x = 1$ which is the same point. I.e. following a path paralleling the edge of the mobius strip will eventually bring you to the same point. This implies the möbius strip has the same generator α. Thus the *identity group isomorphism* between cyclic groups $\pi_1(S^1) \to \pi_1(S^1)$ or really $\mathbb{Z} \to \mathbb{Z}$ satisfies our homomorphism. This final result is directly analogous to the solution for (c).

(b) Take the diagonal circle $C = \{(x, y) \in T^2 \mid x = y\}$ embedded in $S = S^1 \times S^1$. To visualize C with basepoint $(1/2, 1/2)$, go back to the $[0, 1]^2$ pre-torus and draw the line $y = x$. Clearly, after both identifications, this will be a loop as $(0, 0) = (1, 1)$. However, note that this loop, say γ, loops horizontally *and* vertically. After the first identification, the line will travel from $(0, 0)$ at its leftmost bottom point around the back of the cylinder up until the top of the cylinder at $x = 1/2$ and then back down around the front to $(1, 1) = (0, 0)$. Therefore, after the final identification, it will form a loop that is both around the torus's perimeter (x-axis) and its girth (y-axis). As a result, performing this loop multiple times will perform both inseparable 'components' simultaneously that many times. $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$ is generated by these two loops – one around its ring (from $\alpha(t) = (1/2, t)$ on the square) and another around its outside $\beta(t) = (t, 1/2)$). These two loops generate one copy of $\mathbb{Z}$ that together form fundamental group $\mathbb{Z}^2$. Because each iteration of C's generator always does α, β once each, we can form the homomorphism $\gamma_* \colon \pi_1(C, (1/2, 1/2)) \to \pi_1(S^1 \times S^1, (1/2, 1/2))$ defined via $\langle \gamma^n \rangle \mapsto \langle (\alpha^n, \beta^n) \rangle$ where $\pi_1(C) = \mathbb{Z}$ so really this function sends $k \in \mathbb{Z}$ to $(k, k) \in \mathbb{Z}^2$ which is indeed a homomorphism using addition.

(c) Take the boundary circle $S^1 \times \{0\}$ embedded in the cylinder $S^1 \times [0, 1]$. Once again, we can homotopy the cylinder via $F(x, y)$ where x, y refers to the $[0, 1]^2$ square which can have its top and bottom edges identified to create the cylinder. The continuous homotopy $F((x, y), t) = (1 - t)(x, y) + t(0, y)$ tells us the cylinder is homotopically equivalent to the boundary circle C or S^1. Both consequently have fundamental groups $\mathbb{Z}$ regardless of their shared basepoint and we can utilize the identity group isomorphism once more because the fundamental groups as well as the surfaces themselves are identical. For a more visual example, however, any loop winding around the surface of the cylinder will be slowly squashed via the homotopy until it is just looping around the circle. Because this homomorphism is from $\mathbb{Z} \to \mathbb{Z}$, addition in $\mathbb{Z}$ again satisfies the homomorphism property and is well-defined.

$\square$

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Proof.

(a) The 2-sphere has the continuous and onto map $f(x) = -x$, swapping antipodal points. By construction no point is fixed so the 2-sphere does NOT have the fixed-point property.

(b) Imagine taking infinitely many slices of the torus like you would a bundt cake. Each S^1 circle does not have the fixed-point property as shown by a simple rotation by π or maybe another antipodal swap map, both of which are continuous. Doing this for each infitesimal slice around the torus completes our proof. This is equivalent to rotating the pre-boundary-circle-identification cylinder and then identifying the boundaries which leaves no points fixed. Rigorously, on the $S^1 \times S^1$ formulation, define $f(x, y) = f(x, -y)$. This means it also does NOT have the fixed-point property.

(c) Recall the open disc is homeomorphic to the euclidean plane $\mathbb{E}^2$ via the stereographic projection h. A simple translational shift $g \colon \mathbb{E}^2 \to \mathbb{E}^2$ that sends $(x, y) \mapsto (x + 1, y + 1)$ leaves no fixed point and is continuous. Because h is a homeomorphism then, $h^{-1} \circ g \circ h$ is a continuous function from the open disc to itself with no fixed point. Thus, the open disc does NOT have the fixed-point property.

(d) According to the book, a continuous function from X to itself which does not have a fixed point need not be onto. Hence, we can use the antipodal map for all points of one circle and send the other circle to wherever the shared point p was sent by the first circle's map. By gluing lemma, because both circles are closed in their union and both are continuous on their restrictions, and they agree on their intersection, this combined function is continuous and by construction leaves no fixed points. (*Note the 2nd circle mapping to the single point is continuous because any open subset containing wherever p is mapped to will have preimage some open stretch on the 1st circle unioned with the open line segment of the 2nd circle excluding its shared point.*) Thus, this space does NOT have the fixed-point property either.

Alternatively, in light of #109.25, you could unidentify the boundary circles of a punctured torus with a single point missing into a cylinder, rotate it, and then identify its boundary circles again. This is definitely continuous and leaves no fixed points.

□