# Graph Theory

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# Contents

# Chapter 1

# Continuity

## 1.1 Open and closed sets

**Definition 1** (Topological Space)**.** A set $X$ **topological space** is a topological space if for each $x$ of $X$, there is a nonempty collection of subsets of $X$, called neighbourhoods of $x$, which satisfy the following axioms:

(a) $x$ lies in each of its neighbourhoods.

(b) The intersection of two neighbourhoods of $x$ is itself a neighbourhood of $x$.

(c) If $N$ is a neighbourhood of $x$ and if $U$ is a subset of $X$ which contains $N$, then $U$ is a neighbourhood of $x$.

(d) If $N$ is a neighbourhood of $x$, then we denote **the interior** of $N$ as the set $\mathring{N} := \{z \in N \mid N \text{ is a neighbourhood of } z\}$. $\mathring{N}$ is a neighbourhood of $x$.

We say, if (a)-(d) are satisfied to each point $x \in X$, then there is a **topology** on the set $X$.

**Definition 2** (Map)**.** Let $X, Y$ be topological spaces. A function $f\colon X \to Y$ is **continuous** if for each point $x$ of $X$ and each neighbourhood $N$ of $f(x)$ in $Y$, the set $f^{-1}(N)$ is a neighbourhood of $x$ in $X$. Continuous functions are called **maps**.

**Definition 3** (Homeomorphism)**.** A function $h\colon X \to Y$ is a **homeomorphism** if it is one-one, onto, continuous, and has a continuous inverse. When such a function exists, $X$ and $Y$ are called **homeomorphic (or topologically equivalent) spaces**.

**Definition 4** (Surface)**.** A **surface** is a topological space in which each point has a neighbourhood homeomorphic to the plane, and for which any two distinct points possess disjoint neighbourhoods.

**Definition 5** (Open). Let $X$ be a topological space and call a subset $O$ of $X$ **open** if it is a neighbourhood of each of its points.

**Remark.** From axiom (c), the union of any collection of open sets is open and from axiom (b) the intersection of any *finite* number of open sets is open. Lastly, (d) shows that the interior of $N$ is an open set which lies inside $N$ and contains $x$.

**Definition 6** (New and Improved Topological Space). A topology on a set $X$ is a nonempty collection of subsets of $X$, which we call open sets, such that:

1. any union of open sets is itself open
2. any finite intersection of open sets is open
3. both the whole set $X$ and the empty set are open.

Given a point $x$ of $X$, we shall call a subset $N$ of $X$ a *neighbourhood* of $x$ if we can find an open set $O$ so $x \in O \subseteq N$. A set together with a topology on it is a topological space.

**Proof.** This set $X$ is a topological space because for each $x \in X$, $X$ is an open neighborhood of $x$ (a). This also confirms (c). If $N_1, N_2$ are neighbourhoods of $x$, we can find open sets $O_1, O_2$ so $x \in O_1 \subseteq N_1$ and $x \in O_2 \subseteq N_2$ such that $x \in O_1 \cap O_2 \subseteq N_1 \cap N_2$ Because $O_1 \cap O_2$ is open, $N_1 \cap N_2$ is a neighborhood of $x$ (b). If $N$ is a neighbourhood of $x$ then there is an open set $O \subseteq N$ so $x \in O$. By definition, $O$ is a neighborhood of each of its points. $\mathring{N}$ is the set of all points $z$ that $N$ is a neighbourhood of. Clearly, then, $O$ is contained in $\mathring{N}$. Thus, $\mathring{N}$ is a neighborhood of $x$. $\square$

**Definition 7** (Usual Topology on $\mathbb{E}^n$). A set $U$ is open if given $x \in U$, there exists $\varepsilon \in R^+$ so the ball with centre $x$ and radius $\varepsilon$ lies entirely in $U$.

**Definition 8** (Subspace/Induced Topology). Given a topological space $X$ and a subset $Y$ of $X$, the open sets of the **subspace/induced** topology on $Y$ are simply the intersection of all the open sets of $X$ with $Y$.

i.e. A subset $U$ of $Y$ is open in the subspace topology if there exists an open set $O$ of $X$ so $U = O \cap Y$.

A subspace $Y$ of a topological space $X$ implies that $Y$ is a subset of $X$ with the subspace topology.

**Definition 9** (Discrete Topology). The largest possible topology on a given set $X$ is the **discrete topology** wherein every subset of $X$ is an open set.

If $X$ has the discrete topology, we call it a discrete space.

**Example.** If we take the set of points of $\mathbb{E}^n$ which have integer coordinates, and give it the subset topology, the result is a discrete space.

**Definition 10** ("Larger" Topologies)**.** If one topology contains all the open sets of another, we say it is **larger**.

**Definition 11** (Closed)**.** A subset of a topological space is closed if its complement is open.

**Example.** The following subsets of $\mathbb{E}^2$ are closed: the unit circle, the unit disk ($\{(x, y) \mid x^2 + y^2 \leq 1\}$), $y = e^x$, and $\{(x, y) \mid x \geq y^2\}$.

The set of all points $(x, y)$ where $x \geq 0, y > 0$ is neither open nor closed.

The space $X$ of all points $(x, y)$ where $x \geq 1, y \leq -1$ with the topology induced from $\mathbb{E}^2$ is both open and closed in $X$ (and notably not open in $\mathbb{E}^2$).

**Remark.** The intersection of any family of closed sets is closed. As is the union of any *finite* family of closed sets.

**Definition 12** (Limit Point)**.** Let $A$ be a subset of a topological space $X$. A point $p \in X$ is a **limit point** (or accumulation point) of $A$ if every neighbourhood of $p$ contains at least one point of $A - \{p\}$.

**Example.** Give the set of all real numbers $X$ the *finite-complement topology* where a set is open if its complement is finite or all of $X$. If we take $A$ to be an infinite subset of $X$, then every point of $X$ is a limit point of $A$. Conversely, a finite subset of $X$ has no limit points in this topology.

**Explanation.** To be a neighbourhood $N$ of any $p \in X$, $N$ must be open implying its complement is either finite or $X$. If $N^C = X$, $N = \emptyset$ so $N$ cannot be a neighbourhood of $p$ (this definition simply ensures $\emptyset$ is open so this is indeed a topology). Thus, to be a neighbourhood, $N$ must be infinite with a finite complement.

If $A$ is an infinite subset of $X$, it must then share some infinite points with $N$ implying $N$ contains points of $A - \{p\}$. Because this is the case for all $N$ of $p$ and all $p \in X$, $p$ is a limit point of $A$.

If $A$ is a finite subset, there exists neighbourhoods such that $N^C = A$ so every neighbourhood of $p$ certainly does not contain one point of $A - \{p\}$ implying no point of $X$ is a limit point of $A$.

**Theorem 1.** A set is closed if and only if it contains all its limit points.

**Proof.** $\Rightarrow$: If $A$ is closed, then its complement $X - A$ is open so $X - A$ is a neighbourhood of each of its points. Clearly, if a limit point $p$ of $A$ were in $X - A$ then $X - A$ must contain a point of $A - \{p\}$ of which there are none. So $A$ contains all its limit points.

$\Leftarrow$: Suppose $A$ contains all its limit points. If $x \in X - A$, $x$ is not a limit point of $A$ so there exists a neighbourhood $N$ of $x$ which contains no point of $A$ implying $N \subseteq X - A$ such that $X - A$ is also a neighbourhood of $x$ for all $x \in X - A$ so $X - A$ is a neighbourhood of all of its points so it is open meaning $A$ is closed. $\square$

**Definition 13** (Closure)**.** The union of $A$ and all its limit points is called the **closure** of $A$ and is written $\overline{A}$.

**Theorem 2.** The closure of $A$ is the smallest closed set containg $A$. i.e. the closure is the intersection of all closed sets containing $A$.

**Proof.** The closure of $A$ is closed because if $x \in X - \overline{A}$ then $x$ cannot be a limit point of $A$ so there exists an open neighbourhood $N$ of $x$ such that it contains no points of $A$. Because $N$ is an open set, it is a neighbourhood of all of its points so none of its points are limit points of $A$ either. Thus, $N \subseteq X - A$ so $X - A$ is a neighbourhood of $x$ so $X - A$ is a neighbourhood of each of its points so $X - \overline{A}$ is open so $\overline{A}$ is closed. Because $\overline{A}$ is closed, contains $A$, and is contained in every closed set containing $A$, it must be the intersection of all such sets.

*NOTE: If we just said there exists a neighbourhood $N$ of $x$, this neighbourhood may contain a limit point of $A$ even if it does not contain a point of $A$. Thus, it is meaningful to prove none of its points can be limit points of $A$ by saying the neighbourhood is open.* $\square$

**Corollary.** A set is closed if and only if it is equal to its closure.

**Definition 14** (Dense)**.** A set whose closure is the whole space is said to be **dense** in the space. A dense set meets every nonempty open subset of the space.

**Definition 15** (Interior)**.** The **interior** $\mathring{A}$ of a set $A$ is the union of all open sets contained in $A$. A point $x$ lies in the interior of $A$ if and only if $A$ is a neighbourhood of $x$. Also, an open set is its own interior.

**Example.** In $\mathbb{E}^2$, denote the unit disk $D$ and the unit circle $C$. $D$'s interior is $D - C$ while $C$'s interior is $\emptyset$.

**Definition 16** (Frontier). The **frontier** of a set $A$ is the intersection of $\overline{A}$ with $\overline{X - A}$. This is equivalent to the points of $X$ neither in the interior of $A$ nor the interior of $X - A$.

**Example.** In $\mathbb{E}^2$, the unit disc $D$, its interior $\mathring{D}$, and the unit circle $C$ all have the same frontier $C$.

The froniter of the set of points in $\mathbb{E}^3$ which have rational coordinates is all of $\mathbb{E}^3$. In this case, the frontier is the whole space.

**Definition 17** (Base/Basis). Given a topology on a set $X$, a collection $\beta$ of open sets is called a **base/basis** for the topology if every open set is a union of members of $\beta$. Elements of $\beta$ are called *basic open sets.*

Equivalently, given any point $x \in X$ and a neighbourhood $N$ of $x$, there is always an element $B$ of $\beta$ so $x \in B \subseteq N$.

**Theorem 3.** Let $\beta$ be a nonempty collection of subsets of a set $X$. If the intersection of any finite number of members of $\beta$ is always in $\beta$, and if $\bigcup \beta = X$, then $\beta$ is a base for a topology on $X$.

**Proof.** Let the collection of all possible unions of members of $\beta$ be open sets. This then immediately satisfies our new definition for a topological space. $\square$

**Remark.** A space whose topology has a countable base is called a **second countable space**. A space which contains a countable dense subset is said to be **separable**.

## 1.2 Continuous functions

**Note.** Let $X$ and $Y$ be topological spaces.

**Remark** (Old Idea of Continuity). A function $f\colon X \to Y$ is continuous if for each point $x$ of $X$ and each neighbourhood $N$ of $f(x)$ in $Y$ the set $f^{-1}(N)$ is a neighbourhood of $x$ in $X$.

**Theorem 4** (Continuity). A function from $X$ to $Y$ is continuous if and only if the inverse image of each open set of $Y$ is open in $X$.

**Proof.** $\Leftarrow$: Suppose $f$ is continuous. If $O$ is an open subset of $Y$ then $O$ is a neighbourhood of each of its points and therefore $f^{-1}(O)$ must be a neighbourhood of each of its points in $X$. So $f^{-1}(O)$ is open in $X$.

$\Rightarrow$: Suppose the inverse image of each open set of $Y$ is open in $X$. For any $x$ in $X$, let the open subset $O$ of $Y$ contain $f(x)$. Because $O$ is open, it is a neighbourhood of all of its points. Thus the inverse image of $O$ is open in $X$ □

**Definition 18** (Map). A continuous function is often called a **map** for short.

**Theorem 5.** The composition of two maps is a map.

**Proof.** Suppose $f: X \to Y, g: Y \to Z$ are continuous. Say $O$ is an open set in $Z$. Then, $g^-1(O)$ is open in $Y$. Thus, $f^{-1}(g^{-1}(O))$ is open in $X$. So $g \circ f$ is continuous. □

**Theorem 6.** Suppose $f: X \to Y$ is continuous, and let $A \subseteq X$ have the subspace topology. Then the restriction $f|_A: A \to Y$ is continuous.

**Proof.** Let $O$ be an open set in $Y$. $f$ is continuous so $f^{-1} \cap (O)$ is open in $X$. By subspace topology, $f|_A^{-1}(O) = A \cap f^{-1}(O)$ is open in the subspace toplogy on $A$. Thus $f|_A$ is continuous. □

**Definition 19** (Identity Map $1_X$). The map from $X$ to $X$ which sends each point $x$ to itself is called the **idenity map of** $X$, and written $1_X$. If we restrict $1_X$ to a subspace $A$ of $X$, we obtain the **inclusion map** $i: A \to X$.

**Theorem 7.** The following are equivalent:

(a) $f: X \to Y$ is a map.

(b) If $\beta$ is a base for the topology of $Y$, the inverse image of every member of $\beta$ is open in $X$.

(c) $f(\bar{A}) \subseteq \overline{f(A)}$ for any subset $A$ of $X$.

(d) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any subset $B$ of $Y$.

(e) The inverse image of each closed set in $Y$ is closed in $X$.

**Proof.** We will use (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

- For $(a) \Rightarrow (b)$, if $f$ is a map, it is continuous so the inverse image of every open set in $Y$ is open in $X$. Each member of $\beta$ is open so in $Y$ so their inverse image is open in $X$.
- For $(b) \Rightarrow (c)$, if $A$ is a subset of $X$, every point of $f(A)$ lies inside $\overline{f(A)}$ so we must show if $x \in \overline{A} - A$ and $f(x) \notin f(A)$, then $f(x)$ is a limit point of $f(A)$. For $x$ to be a limit point of $f(A)$, every neighbourhood of $f(x)$ must contain a point of $f(A) - \{x\}$. If $N$ is a neighbourhood of $f(x)$ in $Y$, we can find a basic open set $B$ in $\beta$ so $f(x) \in B \subseteq N$. From (b), we know $f^{-1}(B)$ is open in $X$ and therefore a neighbourhood of $x$. Because $x$ is a limit point of $A$,

$f^{-1}(B)$ must contain a point of $A$. Thus, $B$, and $N$, must contain a point of $f(A)$ implying $f(x)$ is a limit point of $f(A)$.

- For $(c) \Rightarrow (d)$, assume $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset $A$ of $X$. Let $x \in \overline{f^{-1}(B)}$. If $x \in f^{-1}(B)$, then $x \in f^{-1}(\overline{B})$. Otherwise, $x \in \overline{f^{-1}(B)} - f^{-1}(B)$. From our assumption, $f(x) \in f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B} \subseteq \overline{B}$. Thus, $x \in f^{-1}(\overline{B})$.
- For $(d) \Rightarrow (e)$, if $B$ is a closed subset of $Y$ then $\overline{B} = B$. (d) then implies $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$ so $f^{-1}(B)$. Clearly, $f^{-1}(B) \subseteq \overline{f^{-1}(B)}$ so $f^{-1}(B) = \overline{f^{-1}(B)}$ so $f^{-1}(B)$ is closed in $X$.
- For $(e) \Rightarrow (a)$, suppose the inverse image of each closed set in $Y$ is closed in $X$. Then, for any open set $B$ in $Y$, $Y - B$ is closed so, by our assumption, $A := f^{-1}(Y - B)$ must be closed such that $X - A$ is open. If $y \in B$, $y \notin A$ because this would imply $f(y) \in Y - B$ so $y \in X - A$. Thus, the inverse of any $B$, namely $X - f^{-1}(Y - b)$ is open suggesting $f : X \to Y$ is continuous and therefore a map.

$\square$

**Example.** Let $C$ denote the unit circle in the complex plane, taken with the subspace topology and give the interval $[0, 1)$ the induced topology from the real line. Define $f : [0, 1) \to C$ by $f(x) = e^{2\pi i x}$. Let the set of all open segments of the circle be a base for the topology on $C$. If $S$ is in the base and $1 \notin S$, then $f^{-1}(S)$ is an open interval $(a, b)$ where $0 < a < b < 1$. Thus, $f^{-1}(S)$ is open in $[0, 1)$. If some segment $S'$ does contain 1, $f^-1S'$ has the form $[0, a) \cup (b, 1)$ where $0 < a < b < 1$. This is the intersection of the open set $(-1, a) \cup (b, 1)$ of the real line with $[0, 1)$ and thus $S'$ is open also. Part $(b)$ from the last theorem shows $f$ is then continuous.

Despite $f$ being bijective, its inverse is NOT continuous. To show this, we need only make an open set $O$ of $[0, 1)$ so $(f^{-1})^{-1}(O) = f(O)$ is not open in $C$. For instance, take $O$ to be the interval $[0, \frac{1}{2})$ which is open in $[0, 1)$. But its image under the exponential map consists of complex numbers $z$ in $C$ for which $0 \le arg(z) < \pi$. This is not open in $C$.

**Definition 20** (Homeomorphism)**.** A **homeomorphism** $h : X \to Y$ is a function which is continuous, one-one, onto, and has a continuous inverse.

**Example.** EXAMPLE ON PAGE 34 HERE!!!!!!!!! all the way until lemma 2.10

**Lemma 1.** Any homeomorphism from the boundary of a disc to itself can be extended to a homeomorphism of the whole disc.

**Proof.** Let $A$ be a disc and hoose a homeomorphism $h : A \to D$. Given a $\square$

FOR EXAMPLE AFTER TFAE THEOREM, DRAW FIGURE 2.1 FROM

PAGE 33.

TAKES NOTE ON EXAMPLE ON PAGE 34. (linked above where it should go)

## Page 35, Problem 17

Let $X$ denote the set of all real numbers with the finite-complement topology. Define $f \colon \mathbb{E} \to X$ by $f(x) = x$.

**Proof.** For any open set $A \subseteq X$, $X - A$ must be $X$ or finite.

- If $X - A = X$, $A = \emptyset$ so $f^{-1}(A) = \emptyset$ which is open in $\mathbb{E}^1$.
- Let $B := X - A$. If $B = \emptyset$, $A = X$ so $f^{-1}(A) = \mathbb{E}^1$ which is open in itself. (I'm not sure if I need this...)
- If $B$ is a nonempty, finite subset of $X$, then for all $x_i \in E^1 - f^{-1}(B)$, let $d_i := \frac{1}{2}\min(\{|b_i - x| : b_i \in B\})$. Let $N := B_d(x_i)$ be an open neighbourhood of $x$ so $N \cap B = \emptyset$. Thus, $x$ is not a limit point of $f^{-1}(B)$. This is valid for all $x \in E^1 - f^{-1}(B)$ so $E^1 - f^{-1}(B)$ has no limit points of $f^{-1}(B)$ so $f^{-1}(B)$ must contain all its limit points implying it is closed and $f^{-1}(A) = E^1 - f^{-1}(B)$ is open whenver $A$ is open. Thus, $f$ is continuous.

Let $A = (0, 1)$ so $A$ is open in $E^1$. So, $(f^{-1})^{-1}(A) = f(A) = \{x \in \mathbb{R} \colon 0 < x < 1\}$. But, $X - f(A)$ is infinite and not equal to $X$ so $f(A)$ is not open. Thus $f^{-1}$ is not continuous so f is not a homeomorphism. □

## Page 41, Problem 28

Let $A, B$ be disjoint closed subsets of a metric space.

**Proof.** Let $d(x, y)$ be the distance metric on some topological space $X$. If $A$ and $B$ are both closed, $A = \overline{A}, B = \overline{B}$. So if $A \cap B = \emptyset$, then $\overline{A} \cap \overline{B} = \emptyset$. (Not sure if this is needed, just want to emphasize they are far away from each other.)

For each $a_i \in A$, let $d_i := \frac{1}{2}\min(\{d(b, a_i) \mid b \in B\})$. (I'm not confident about whether I can assume the neighbourhoods are open in the next part because they're balls on a metric space or not). Let $N_i := B_i(a_i)$ be an open neighbourhood of $a_i$. And $U_i = N_i \cap A$. Clearly, $a_i \in U_i$ so $U_i \neq \emptyset$. Let $U = \bigcup_{i=1}^{\infty} N_i$ (or just up until some $k$ if $A$ is finite) so $U \cap B = \emptyset$ and $U \subseteq A$. Do the same for $B$ for a final union $V \subseteq B$. $U, V$ are both unions of open sets so they are each open in $X$, $U \subseteq A$ and $V \subseteq B$, and $U \cap V = \emptyset$. □