Elementary Algebraic Topology

Jack Lipson

October 15, 2023

Contents

Chapter 1

Continuity

1.1 Open and closed sets

Definition 1 (Topological Space). A set X **topological space** is a topological space if for each x of X, there is a nonempty collection of subsets of X, called neighbourhoods of x, which satisfy the following axioms:

(a) x lies in each of its neighbourhoods.

(b) The intersection of two neighbourhoods of x is itself a neighbourhood of x.

(c) If N is a neighbourhood of x and if U is a subset of X which contains N, then U is a neighbourhood of x.

(d) If N is a neighbourhood of x, then we denote **the interior** of N as the set $\mathring{N} := \{z \in N \mid N \text{ is a neighbourhood of } z\}$. $\mathring{N}$ is a neighbourhood of x.

We say, if (a)-(d) are satisfied to each point $x \in X$, then there is a **topology** on the set X.

Definition 2 (Map). Let X, Y be topological spaces. A function $f\colon X \to Y$ is **continuous** if for each point x of X and each neighbourhood N of $f(x)$ in Y, the set $f^{-1}(N)$ is a neighbourhood of x in X. Continuous functions are called **maps**.

Definition 3 (Homeomorphism). A function $h\colon X \to Y$ is a **homeomorphism** if it is one-one, onto, continuous, and has a continuous inverse. When such a function exists, X and Y are called **homeomorphic (or topologically equivalent) spaces**.

Definition 4 (Surface). A **surface** is a topological space in which each point has a neighbourhood homeomorphic to the plane, and for which any two distinct points possess disjoint neighbourhoods.

Definition 5 (Open)**.** Let X be a topological space and call a subset O of X **open** if it is a neighbourhood of each of its points.

Remark. From axiom (c), the union of any collection of open sets is open and from axiom (b) the intersection of any *finite* number of open sets is open. Lastly, (d) shows that the interior of N is an open set which lies inside N and contains x.

Definition 6 (New and Improved Topological Space)**.** A topology on a set X is a nonempty collection of subsets of X, which we call open sets, such that:

1. any union of open sets is itself open
2. any finite intersection of open sets is open
3. both the whole set X and the empty set are open.

Given a point x of X, we shall call a subset N of X a *neighbourhood* of x if we can find an open set O so $x \in O \subseteq N$. A set together with a topology on it is a topological space.

Proof. This set X is a topological space because for each $x \in X$, X is an open neighborhood of x (a). This also confirms (c). If N_1, N_2 are neighbourhoods of x, we can find open sets O_1, O_2 so $x \in O_1 \subseteq N_1$ and $x \in O_2 \subseteq N_2$ such that $x \in O_1 \cap O_2 \subseteq N_1 \cap N_2$ Because $O_1 \cap O_2$ is open, $N_1 \cap N_2$ is a neighborhood of x (b). If N is a neighbourhood of x then there is an open set $O \subseteq N$ so $x \in O$. By definition, O is a neighborhood of each of its points. $\mathring{N}$ is the set of all points z that N is a neighbourhood of. Clearly, then, O is contained in $\mathring{N}$. Thus, $\mathring{N}$ is a neighborhood of x. $\square$

Definition 7 (Usual Topology on $\mathbb{E}^n$)**.** A set U is open if given $x \in U$, there exists $\varepsilon \in R^+$ so the ball with centre x and radius ε lies entirely in U.

Definition 8 (Subspace/Induced Topology)**.** Given a topological space X and a subset Y of X, the open sets of the **subspace/induced** topology on Y are simply the intersection of all the open sets of X with Y.

i.e. A subset U of Y is open in the subspace topology if there exists an open set O of X so $U = O \cap Y$.

A subspace Y of a topological space X implies that Y is a subset of X with the subspace topology.

Definition 9 (Discrete Topology)**.** The largest possible topology on a given set X is the **discrete topology** wherein every subset of X is an open set.

If X has the discrete topology, we call it a discrete space.

Example. If we take the set of points of $\mathbb{E}^n$ which have integer coordinates, and give it the subset topology, the result is a discrete space.

Definition 10 ("Larger" Topologies)**.** If one topology contains all the open sets of another, we say it is **larger**.

Definition 11 (Closed)**.** A subset of a topological space is closed if its complement is open.

Example. The following subsets of $\mathbb{E}^2$ are closed: the unit circle, the unit disk ($\{(x, y) \mid x^2 + y^2 \leq 1\}$), $y = e^x$, and $\{(x, y) \mid x \geq y^2\}$.

The set of all points (x, y) where $x \geq 0, y > 0$ is neither open nor closed.

The space X of all points (x, y) where $x \geq 1, y \leq -1$ with the topology induced from $\mathbb{E}^2$ is both open and closed in X (and notably not open in $\mathbb{E}^2$).

Remark. The intersection of any family of closed sets is closed. As is the union of any *finite* family of closed sets.

Definition 12 (Limit Point)**.** Let A be a subset of a topological space X. A point $p \in X$ is a **limit point** (or accumulation point) of A if every neighbourhood of p contains at least one point of $A - \{p\}$.

Example. Give the set of all real numbers X the *finite-complement topology* where a set is open if its complement is finite or all of X. If we take A to be an infinite subset of X, then every point of X is a limit point of A. Conversely, a finite subset of X has no limit points in this topology.

Explanation. To be a neighbourhood N of any $p \in X$, N must be open implying its complement is either finite or X. If $N^C = X$, $N = \emptyset$ so N cannot be a neighbourhood of p (this definition simply ensures $\emptyset$ is open so this is indeed a topology). Thus, to be a neighbourhood, N must be infinite with a finite complement.

If A is an infinite subset of X, it must then share some infinite points with N implying N contains points of $A - \{p\}$. Because this is the case for all N of p and all $p \in X$, p is a limit point of A.

If A is a finite subset, there exists neighbourhoods such that $N^C = A$ so every neighbourhood of p certainly does not contain one point of $A - \{p\}$ implying no point of X is a limit point of A.

Theorem 1. A set is closed if and only if it contains all its limit points.

Proof. $\Rightarrow$: If A is closed, then its complement $X - A$ is open so $X - A$ is a neighbourhood of each of its points. Clearly, if a limit point p of A were in $X - A$ then $X - A$ must contain a point of $A - \{p\}$ of which there are none. So A contains all its limit points.

$\Leftarrow$: Suppose A contains all its limit points. If $x \in X - A$, x is not a limit point of A so there exists a neighbourhood N of x which contains no point of A implying $N \subseteq X - A$ such that $X - A$ is also a neighbourhood of x for all $x \in X - A$ so $X - A$ is a neighbourhood of all of its points so it is open meaning A is closed. □

Definition 13 (Closure)**.** The union of A and all its limit points is called the **closure** of A and is written $\overline{A}$.

Theorem 2. The closure of A is the smallest closed set containg A. i.e. the closure is the intersection of all closed sets containing A.

Proof. The closure of A is closed because if $x \in X - \overline{A}$ then x cannot be a limit point of A so there exists an open neighbourhood N of x such that it contains no points of A. Because N is an open set, it is a neighbourhood of all of its points so none of its points are limit points of A either. Thus, $N \subseteq X - A$ so $X - A$ is a neighbourhood of x so $X - A$ is a neighbourhood of each of its points so $X - \overline{A}$ is open so $\overline{A}$ is closed. Because $\overline{A}$ is closed, contains A, and is contained in every closed set containing A, it must be the intersection of all such sets.

NOTE: If we just said there exists a neighbourhood N of x, this neighbourhood may contain a limit point of A even if it does not contain a point of A. Thus, it is meaningful to prove none of its points can be limit points of A by saying the neighbourhood is open. □

Corollary. A set is closed if and only if it is equal to its closure.

Definition 14 (Dense)**.** A set whose closure is the whole space is said to be **dense** in the space. A dense set meets every nonempty open subset of the space.

Definition 15 (Interior)**.** The **interior** $\mathring{A}$ of a set A is the union of all open sets contained in A. A point x lies in the interior of A if and only if A is a neighbourhood of x. Also, an open set is its own interior.

Example. In $\mathbb{E}^2$, denote the unit disk D and the unit circle C. D's interior is $D - C$ while C's interior is $\emptyset$.

Definition 16 (Frontier). The **frontier** of a set A is the intersection of $\overline{A}$ with $\overline{X - A}$. This is equivalent to the points of X neither in the interior of A nor the interior of $X - A$.

Example. In $\mathbb{E}^2$, the unit disc D, its interior $\mathring{D}$, and the unit circle C all have the same frontier C.

The froniter of the set of points in $\mathbb{E}^3$ which have rational coordinates is all of $\mathbb{E}^3$. In this case, the frontier is the whole space.

Definition 17 (Base/Basis). Given a topology on a set X, a collection β of open sets is called a **base/basis** for the topology if every open set is a union of members of β. Elements of β are called *basic open sets.*

Equivalently, given any point $x \in X$ and a neighbourhood N of x, there is always an element B of β so $x \in B \subseteq N$.

Theorem 3. Let β be a nonempty collection of subsets of a set X. If the intersection of any finite number of members of β is always in β, and if $\bigcup \beta = X$, then β is a base for a topology on X.

Proof. Let the collection of all possible unions of members of β be open sets. This then immediately satisfies our new definition for a topological space. $\square$

Remark. A space whose topology has a countable base is called a **second countable space**. A space which contains a countable dense subset is said to be **separable**.

1.2 Continuous functions

Note. Let X and Y be topological spaces.

Remark (Old Idea of Continuity). A function $f\colon X \to Y$ is continuous if for each point x of X and each neighbourhood N of $f(x)$ in Y the set $f^{-1}(N)$ is a neighbourhood of x in X.

Theorem 4 (Continuity). A function from X to Y is continuous if and only if the inverse image of each open set of Y is open in X.

Proof. $\Leftarrow$: Suppose f is continuous. If O is an open subset of Y then O is a neighbourhood of each of its points and therefore $f^{-1}(O)$ must be a neighbourhood of each of its points in X. So $f^{-1}(O)$ is open in X.

$\Rightarrow$: Suppose the inverse image of each open set of Y is open in X. For any x in X, let the open subset O of Y contain $f(x)$. Because O is open, it is a neighbourhood of all of its points. Thus the inverse image of O is open in X □

Definition 18 (Map). A continuous function is often called a **map** for short.

Theorem 5. The composition of two maps is a map.

Proof. Suppose $f\colon X \to Y, g\colon Y \to Z$ are continuous. Say O is an open set in Z. Then, $g^-1(O)$ is open in Y. Thus, $f^{-1}(g^{-1}(O))$ is open in X. So $g \circ f$ is continuous. □

Theorem 6. Suppose $f\colon X \to Y$ is continuous, and let $A \subseteq X$ have the subspace topology. Then the restriction $f|_A\colon A \to Y$ is continuous.

Proof. Let O be an open set in Y. f is continuous so $f^{-1} \cap (O)$ is open in X. By subspace topology, $f|_A^{-1}(O) = A \cap f^{-1}(O)$ is open in the subspace toplogy on A. Thus $f|_A$ is continuous. □

Definition 19 (Identity Map 1_X). The map from X to X which sends each point x to itself is called the **idenity map of** X, and written 1_X. If we restrict 1_X to a subspace A of X, we obtain the **inclusion map** $i\colon A \to X$.

Theorem 7. The following are equivalent:

(a) $f\colon X \to Y$ is a map.

(b) If β is a base for the topology of Y, the inverse image of every member of β is open in X.

(c) $f(\bar{A}) \subseteq \overline{f(A)}$ for any subset A of X.

(d) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any subset B of Y.

(e) The inverse image of each closed set in Y is closed in X.

Proof. We will use (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

- For $(a) \Rightarrow (b)$, if f is a map, it is continuous so the inverse image of every open set in Y is open in X. Each member of β is open so in Y so their inverse image is open in X.
- For $(b) \Rightarrow (c)$, if A is a subset of X, every point of $f(A)$ lies inside $\overline{f(A)}$ so we must show if $x \in \overline{A} - A$ and $f(x) \notin f(A)$, then $f(x)$ is a limit point of $f(A)$. For x to be a limit point of $f(A)$, every neighbourhood of $f(x)$ must contain a point of $f(A) - \{x\}$. If N is a neighbourhood of $f(x)$ in Y, we can find a basic open set B in β so $f(x) \in B \subseteq N$. From (b), we know $f^{-1}(B)$ is open in X and therefore a neighbourhood of x. Because x is a limit point of A,

$f^{-1}(B)$ must contain a point of A. Thus, B, and N, must contain a point of $f(A)$ implying $f(x)$ is a limit point of $f(A)$.

- For $(c) \Rightarrow (d)$, assume $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset A of X. Let $x \in \overline{f^{-1}(B)}$. If $x \in f^{-1}(B)$, then $x \in f^{-1}(\overline{B})$. Otherwise, $x \in \overline{f^{-1}(B)} - f^{-1}(B)$. From our assumption, $f(x) \in f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq B \subseteq \overline{B}$. Thus, $x \in f^{-1}(\overline{B})$.
- For $(d) \Rightarrow (e)$, if B is a closed subset of Y then $\overline{B} = B$. (d) then implies $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$ so $f^{-1}(B)$. Clearly, $f^{-1}(B) \subseteq \overline{f^{-1}(B)}$ so $f^{-1}(B) = \overline{f^{-1}(B)}$ so $f^{-1}(B)$ is closed in X.
- For $(e) \Rightarrow (a)$, suppose the inverse image of each closed set in Y is closed in X. Then, for any open set B in Y, $Y - B$ is closed so, by our assumption, $A := f^{-1}(Y - B)$ must be closed such that $X - A$ is open. If $y \in B$, $y \notin A$ because this would imply $f(y) \in Y - B$ so $y \in X - A$. Thus, the inverse of any B, namely $X - f^{-1}(Y - b)$ is open suggesting $f \colon X \to Y$ is continuous and therefore a map.

□

Example. Let C denote the unit circle in the complex plane, taken with the subspace topology and give the interval $[0, 1)$ the induced topology from the real line. Define $f \colon [0, 1) \to C$ by $f(x) = e^{2\pi i x}$. Let the set of all open segments of the circle be a base for the topology on C. If S is in the base and $1 \notin S$, then $f^{-1}(S)$ is an open interval (a, b) where $0 < a < b < 1$. Thus, $f^{-1}(S)$ is open in $[0, 1)$. If some segment S' does contain 1, f^-1S' has the form $[0, a) \cup (b, 1)$ where $0 < a < b < 1$. This is the intersection of the open set $(-1, a) \cup (b, 1)$ of the real line with $[0, 1)$ and thus S' is open also. Part (b) from the last theorem shows f is then continuous.

Despite f being bijective, its inverse is NOT continuous. To show this, we need only make an open set O of $[0, 1)$ so $(f^{-1})^{-1}(O) = f(O)$ is not open in C. For instance, take O to be the interval $[0, \frac{1}{2})$ which is open in $[0, 1)$. But its image under the exponential map consists of complex numbers z in C for which $0 \leq arg(z) < \pi$. This is not open in C.

Definition 20 (Homeomorphism)**.** A **homeomorphism** $h \colon X \to Y$ is a function which is continuous, one-one, onto, and has a continuous inverse.

Example. EXAMPLE ON PAGE 34 HERE!!!!!!!!! all the way until lemma 2.10

Lemma 1. Any homeomorphism from the boundary of a disc to itself can be extended to a homeomorphism of the whole disc.

Proof. Let A be a disc and hoose a homeomorphism $h \colon A \to D$. Given a □

FOR EXAMPLE AFTER TFAE THEOREM, DRAW FIGURE 2.1 FROM

PAGE 33.

TAKES NOTE ON EXAMPLE ON PAGE 34. (linked above where it should go)