MATH 113: Abstract Algebra

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Chapter 1

Introduction to Groups

1.1 Sets and Equivalence Relations

Note. $\mathbb{R}^*$ and $\mathbb{C}^*$ represent the set of all nonzero real and complex numbers. Zero is excluded from $\mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+$.

Note. When a set contains an element b that's algebraically or arithmetically equivalent to another element(s), our set can be partitioned into subsets $\overline{b}$ which denote all entitites equivalent to b. e.g. $\frac{2}{3} = \frac{4}{6}$.

Definition 1 (Parititon). A *partition* of a set is a decomposition of the set into subsests s.t. every element is in exactly one subset, or *cell*.

Definition 2 (Equivalence Relation). For a nonempty set S, $\sim$ is an equivalence relation between elements of S if for all $a, b, c \in S$, $(S, \sim)$ satisfies:

1. (Reflexive) $a \sim a$.
2. (Symmetric) $a \sim b \Rightarrow b \sim a$.
3. (Transitive) $a \sim b \wedge b \sim c \Rightarrow a \sim c$.

Non-equivalence relations usually use $\mathscr{R}$.

Note. All relations $\mathscr{R}$ are defined as $\{(a, b)$ for a $\in A, b \in B \mid a \mathscr{R} b\} \subseteq A \times B$. For equivalence relations, $\sim \subseteq S \times S$.

Remark (Natural Parition). $\sim$ yields a natural partition of S: $\overline{a} = \{x \in S \mid x \sim a\}$ for all $a \in S$.

Explanation. For any $a \in S$, $a \in \overline{a}$. So each element of S is in at least one cell. To show that a is in exactly one cell, let $a \in \overline{b}$ as well. We must show

$\overline{a} = \overline{b}$. $\Rightarrow$: If $x \in \overline{a}$ then $x \sim a$. From our assumption $a \sim b$ so by (3), $x \sim b$ so $x \in \overline{b}$ thus, $\overline{a} \subseteq \overline{b}$. $\Leftarrow$: If $x \in \overline{b}$, $x \sim b$. From our assumption, $a \sim b$ so, by (2), $b \sim a$ meaning $x \sim a$ via (3) implying $x \in \overline{a}$ s.t. $\overline{b} \subseteq \overline{a}$. This completes the proof.

Definition 3 (Equivalence Class). Each cell $\overline{a}$ in a natural partition given by an equivalence relation is called an equivalence class.

Definition 4 (Congruence Modulo n). Let h, k be distinct integers and $n \in \mathbb{Z}^+$. We say h *congruent* to k modulo n, written $h \equiv k \pmod{n}$ if $n \mid h - k$ s.t. $h - k = ns$ for some $s \in \mathbb{Z}$.

Definition 5 (Residue Classes Modulo). Equiva;ence calsses for congruence modulo n are *residue classes modulo n*.

Remark. Each residue class modulo $n \in \mathbb{Z}^+$ contains an infinite number of elements.

Definition 6 (Irreducible). An irreducible polynomial $h(x)$ is one that cannot be factored into polynomials in $\mathcal{P}(\mathbb{R})$ all of lower degree than $h(x)$.

1.2 Binary Operations

Definition 7 (Binary Operation). A *binary operation* * on a set S is a rule that assigns to each ordered pair (a, b) of elements of S another element of S generally denoted $a * b$ or formally $*(a, b)$. To be *well-defined*, $*$ must assign a value to every possible $a * b$.

Definition 8 (Closure under $*$). A set S is *closed under* $*$ if for all $a, b \in S$, $a * b \in S$. If a subset H of S is also closed under $*$, this is referred to as the *induced operation* $*$ on H.

Definition 9 (Commutative Operation). A binary operation $*$ on a set S is *commutative* iff $a * b = b * a$ for all $a, b \in S$.

Definition 10 (Associative operation). A binary operation $*$ on a set S is *associative* iff $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

Note. Associativty of function compostion follows.

Remark. A binary operation on a set, typically finite, can be represented as follows:

$*$	a	b	c
a	b	b	b
b	a	c	b
c	c	b	a

1.3 Groups

Definition 11 (Group). A group $\langle G, *\rangle$ is a set G combined with a binary operation $*$ on G which satisfies the following axioms:

($\mathscr{G}_1$) $*$ is associative.

($\mathscr{G}_2$) There exists a **unique** *identity* element e on G s.t. $e * x = x * e$ for all $x \in G$.

($\mathscr{G}_3$) For each $a \in G$, there exists an $a' \in G$ s.t. $a' * a = a * a' = e$. This a' is called the *inverse* of a with respect to the operation $*$.

($\mathscr{G}_4$) *(optional if part of binary operation definition)* G is closed under $*$.

Theorem 1 (Left/Right Cancellation). If G is a group with binary operation $*$, then the *left and right* cancellation laws hold s.t. $a * b = a * c \Rightarrow b = c$ and $b * a = c * a \Rightarrow b = c$ for all $a, b, c \in G$.

Proof. The right cancellation proof is identical to that below.

$$\begin{aligned} a * b &= a * c && \because \text{by supposition} \\ a' * (a * b) &= a' * (a * c) && \because \text{inverse axiom.} \\ (a' * a) * b &= (a' * a) * c && \because \text{associativity axiom} \\ e * b &= e * c && \because \text{inverse axiom} \\ b &= c && \square \text{ identity axiom} \end{aligned}$$

$\square$

Theorem 2. Trivially, in a group G, $(ab)' = b'a'$ for all $a, b \in G$.

Remark. Note that the solutions x, y to $a * x = b$ and $y * a = b$ have unique solutions in G for any $a, b \in G$. Similarly, e is unique.

Note (Idempotent for $*$). An element x of S is *idempotent for* $*$ if $x * x = x$. This is always in the identity element.

Definition 12 (Abelian Group). A group G is *abelian* if its binary operation is commutative.

Definition 13 (Roots of Unity)**.** Call the elements of the set $U_n := \{z \in \mathbb{C} \mid z^n = 1\}$ the n^{th} roots of unity, usually listed as $1 = \zeta^0, \zeta^1, \zeta^2, \ldots, \zeta^{n-1}$.

Remark. Let the unit circle $U := \{z \in \mathbb{C} \mid |z| = 1\}$. Clearly, for any $z_1, z_2 \in U$, $|z_1 z_2| = |z_1||z_2| = 1$ such that $z_1 z_2 \in U$ implying U is closed under $\cdot$. Note then that $\langle U, \cdot\rangle \simeq \langle R_{2\pi}, +_{2\pi}\rangle$. Similarly, $\langle U_n, \cdot\rangle \simeq \langle \mathbb{Z}_n, +_n\rangle$ for $n \in \mathbb{Z}^+$.

Definition 14 (Addition Modulo n)**.** We respecitvely write $\mathbb{Z}_n$ and $\mathbb{R}_c$ to denote $[0, 1, \ldots, n-1]$ and $[0, c]$. Addition modulo n/c is written $+_n$ or $+_c$.

1.4 Isomorphic Binary Structures

Definition 15 (Binary Algebraic Structures)**.** For two *binary algebraic structures* $\langle S, *\rangle$ and $\langle S', *'\rangle$ to be structurally alike, we would need a one-to-one correspondence between the elements $x \in S$ and $x' \in S'$ s.t. if $x \leftrightarrow x'$ and $y \leftrightarrow y'$ then $x * y \leftrightarrow x' *' y'$.

Remark (Homomorphism Property)**.** This last condition is called the *homorphism property*. If the function ϕ is NOT one-to-one, it is a homormorphism only.

Definition 16 (Isomorphism)**.** An *isomorphism* of S with S' is a one-to-one function ϕ mapping S onto S' such that $\phi(x * y) = \phi(x) *' \phi(y)$ for all $x, y \in S$.

If such a map exists, S and S' are called *isomorphic binary structures* denoted $S \simeq S'$.

Note (Show Binary Algebraic Structures are Isomorphic)**.**

(Step 1) Define the function ϕ which defines $\phi(s)$ for all $s \in S$ and gives the isomorphism from $S \to S'$.

(Step 2) Show ϕ is one-to-one.

(Step 3) Show ϕ is onto.

(Step 4) Show $\phi(x * y) = \phi(x) *' \phi(y)$ for all $x, y \in S$.

Example. Take the isomorphism $\phi\colon \mathbb{R} \to \mathbb{R}^+ : x \longmapsto e^x$ from $\langle \mathbb{R}, +\rangle$ to $\langle \mathbb{R}^+, \cdot\rangle$. Clearly, $\forall x \in \mathbb{R}, \phi(x) \in \mathbb{R}^+$ and ϕ is bijective. Last, for $x, y \in \mathbb{R}$, $\phi(x + y) = e^{x*y} = e^x e^y = \phi(x) \cdot \phi(y)$.

Definition 17 (Structural Property). A structural property is any property of a binary structure that is invariant to any isomorphic structure. These, like cardinality, are used to show no such isomorphism exists between structures.

Example. Although $\langle \mathbb{Q}, + \rangle$ and $\langle \mathbb{Z}, + \rangle$ both have cardinality $\aleph_0$ and have many one-to-one functions between them, the equation $x + x = c$ has a solution $x \in Q$ for all $c \in \mathbb{Q}$, but this is not true for $\mathbb{Z}$ if, say, $c = 3$. This structural propery distinguishes these binary structures and thus they are not isomorphic under the usual addition.

Theorem 3. Suppose $\langle S, * \rangle$ has an identity element e for $*$. If $\phi \colon S \to S'$ is an isomorphism to $\langle S', *' \rangle$ then $\phi(e)$ is an identity element for $*'$ on S'.

Proof. Because an isomorphism exists from $S \to S'$, for any element $s' \in S'$, there exists exactly one element $s \in S$ s.t. $\phi(s) = s'$. By the definition of an isomorphism $s' = \phi(s) = \phi(s * e) = \phi(s) *' \phi(e) = s' *' \phi(e)$ for an arbitary element s' of S. This implies $\phi(e)$ is the identity element for S'. □

1.5 More on Groups and Subgroups

Definition 18 (Semigroup). A semigroup is an algebraic structure combining a set with an associative binary oxperation.

Definition 19 (Monoid). A monoid is a semigroup that has an identity element corresponding to its binary operation.

Definition 20 (Subgroup). If a subset H of a group G is closed under the binary operation of G and is itself a group, H is a *subgroup* of G. This is denoted $H \leq G$. $H < G \Rightarrow H \neq G$.

Example. $\langle \mathbb{Z}, + \rangle < \langle \mathbb{R}, + \rangle$, but $\langle \mathbb{Q}, \cdot \rangle$ is *not* a subgroup of $\langle \mathbb{R}, - \rangle$.

Definition 21 (Proper and trivial subgroups). If G is a group, the subgroup consisting of G itself is the *improper subgroup* of G. All other subgroups are *proper subgroups*. The subgroup $\{e\}$ is the *trivial subgroup* of G and all other subgroups are nontrivial.

Theorem 4. A subset H of a group G is a subgroup of G if and only if:

1. H is closed under the binary operation of G.
2. the identity e of G is in H.

3. for all $a \in H$, $a^{-1} \in H$ also.

Proof. $\Rightarrow$: Let H be a subgroup of G. By definition, H is closed under G's binary operation (1). H must have an identity element because it is a group. Because $a * x = a$ and $y * a = a$ have unique solutions, H's identity element must be the same in H group as G group (2). (3) is trivial because H is a group.

$\Leftarrow$: Let (1), (2), (3) be true. Then H has a unique identity element on its binary operation ($\mathscr{G}_2$), each element of H has a unique inverse in H ($\mathscr{G}_3$), and H is closed under the binary operation of G (*optional* $\mathscr{G}_4$). To satisfy ($\mathscr{G}_1$), the binary operation on H must be associative s.t., for all $a, b, c \in H$, $(ab)c = a(bc)$. But this is clearly holds in G so ($\mathscr{G}_1$) is satisfied and H is a subgroup of G. □

1.6 Cyclic Groups

Theorem 5. Let G be a group and $a \in G$. Then

$$H = \{a^n \mid n \in \mathbb{Z}\}$$

is a subgroup of G and the *smallest* subgroup of G that contains a.

Proof. Let's first check H is indeed a subgroup of G. (1) For any $r, s \in \mathbb{Z}$, $a^r * a^s = \overbrace{(a * \cdots * a)}^{a\ r \text{ times}} * \overbrace{(a * \cdots * a)}^{a\ s \text{ times}} = a^{r+s} \in H$ so we have closure. (2) Let $e := a^0 \in H$ so for all $r \in \mathbb{Z}$, $a^r * a^0 = a^r$. (3) For all $r \in \mathbb{Z}$, $a^r \in H$ so $\exists a^{-r} \in H$ such that $a^r * a^{-r} = a^0 = e$. Thus, $H \leq G$.

Next, to show it's the smallest possible subgroup, just take the set $\{a\}$. To have closure, we must add a^n $\forall n \in Z^+$. To have inverses, we must have a^{-n} so our set becomes $\{a^n \mid n \in Z \setminus \{0\}\}$. To have an identity, we must have a^0 and this completes the proof. □

Definition 22 (Cyclic Subgroup of G). For any $a \in G$, define $\langle a \rangle$ to be the set $\{a^n \mid n \in \mathbb{Z}\}$. This is called the *cyclic subgroup of G generated by a*. An element a of a group G *generates* G and is a *generator for* G if $\langle a \rangle = G$.

Definition 23 (Cyclic Group). A group is *cyclic* if there is some element a in G that generates G.

Example. $\langle 1 \rangle = \langle 3 \rangle = \mathbb{Z}_4$ so Z_4 is cyclic and both 1 and 3 are generators.

Example. The group $\langle \mathbb{Z}, + \rangle$ is a cyclic group generated ONLY by 1 and -1.

Remark (Subgroup Diagrams)**.** Lattice, or *subgroup diagrams*, can be drawn such that lines run down from a group G to a group H if $H < G$.

Example. Take two group structures of order 4: $\mathbb{Z}_4$ and the Klein 4-group *Vierergruppe* defined as follows:

$\mathbb{Z}_4$:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

V :

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

We can map these as:

$$\begin{tikzcd} Z_4 \arrow[d, dash] \\ \{0,2\} \arrow[d, dash] \\ \{0\} \end{tikzcd}$$

and

$$\begin{tikzcd} & V \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \{e,a\} \arrow[dr, dash] & \{e,b\} \arrow[d, dash] & \{e,c\} \arrow[dl, dash] \\ & \{e\} & \end{tikzcd}$$

.

Definition 24 (Order)**.** If the cyclic subgroup $\langle a \rangle$ of G is finite, we say the *order* of a is the order $|\langle a \rangle|$. Otherwise, a is of *infinite order*.

Theorem 6. Every cyclic group is abelian.

Theorem 7 (Division Algortihm for $\mathbb{Z}$)**.** If $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, then there exist unique integers q, r such that

$$n = mq + r \quad \text{and} \quad 0 \le r < m.$$

Proof. From the archimedean property, there is a unique q such that $qm \le n < (q+1)m$. Then, $0 \le r = n - mq < m$ is unique. We regard q and r as the quotient and nonnegative remainder respectively when n is divided by m. $\square$

Theorem 8. A subgroup of a cyclic group is cyclic.

Proof. Take a cyclic group G with subgroup H. If $H = \langle e \rangle$, then H is cyclic and the proof is complete.

Otherwise, $H \neq \langle e \rangle$ so there exists $b \in H, b \neq e$. Because G is cyclic, there must exist $a \in G$ such that a generates G, i.e. for all $n \in \mathbb{Z}^+$, a^n spans every value of G including every element of H. Let $c := a^m$ where m is the least positive integer such that $c \in H$. Now, for all $b \in H$, take n such that $b = a^n$. From division algorithm, there exist integers q, r such that $n = mq + r$ so $a^n = a^{mq+r} = (a^m)^q a^r$ which implies, because $a^m \in H$ and

H is a group so $a^{-m} \in H$, $a^n(a^m)^{-q} = a^r$. H is a group so this implies $a^r \in H$. Because $0 \le r < m$ and m is the least positive integer such that $a^m \in H$, $r = 0$ such that $n = mq$ for all $b = a^n = (a^m)^q \in H$. $\langle c \rangle = H$ so H is cyclic. □

Definition 25 (Greatest Common Divsior)**.** The positive generator d of the cyclic group $H = \{nr + ms \mid n, m \in \mathbb{Z}\}$ under addition is called the *greatest common divisor* of r and s, written $d = \gcd(r, s)$.

Definition 26. Two integers are *relatively prime* if their gcd is 1.

Theorem 9. Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to $\langle \mathbb{Z}, + \rangle$. If G has finite order n, then G is instead isomorphic to $\langle \mathbb{Z}_n, +_n \rangle$.

Proof. Take the following two cases. **Case 1:** For all positive integers m, $a^m \ne e$. Suppose $a^h = a^k$ and $h > k$. Thus, $a^h a^{-k} = a^{h-k} = e$ which contradicts our assumption. Therfore, each element of G can be uniquely expressed as a^m for a unique $m \in \mathbb{Z}$. The map $\phi : G \to \mathbb{Z}$ defined as $\phi(a^i) = i$ is then well-defined and bijective on $\mathbb{Z}$. Last, $\phi(a^i a^j) = \phi(a^{i+j}) = i + j = \phi(a^i) + \phi(a^j)$ so the homomorphism property is satisfied and ϕ is an isomorphism to $\langle \mathbb{Z}, + \rangle$.

Case 2: $a^m = e$ for some $m \in \mathbb{Z}^+$. Let n be the smallest positive integer so $a^n = e$. If $s \in \mathbb{Z}$ and $s = q + r$ for $0 \le r < n$, then $a^s = a^{nq+r} = (a^n)^q a^r = a^r$. Like in case 1, if $0 < k < h < n$ and $a^h = a^k$, then $a^{h-k} = e$ and $0 < h - k < n$ contradicting our assumption that n is the smallest positive integer possible. Hence, $a^0, a^1, a^2, \ldots, a^{n-1}$ are all distinct and comprise all elements of G. We can then make the map $\psi : G \to \mathbb{Z}_n$ defined by $\psi(a^i) = i$ for $i = 0, 1, \ldots, n-1$ is well-defined and bijective on Z_n. Also, because $a^n = e$, $a^i a^j = a^k$ whenever $k = i +_n j$. Therefore, $psi(a^i a^j) = i +_n j = \psi(a^i) +_n \psi(a^j)$ satisfying the homomorphism property so ϕ is an isomorphism to $\langle \mathbb{Z}_n, +_n \rangle$. □

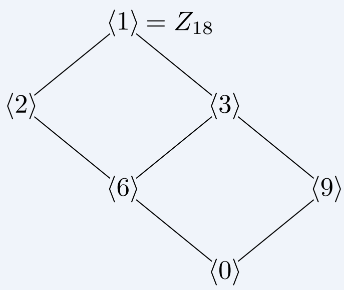
Theorem 10. Let G be a cyclic group generated by a with n elements. Let $b \in G$ and $b = a^s$. Then, b generates a cyclic subgroup H of G containing n/d elements where d is the greatest common divisor of n and s. Also, $\langle a^s \rangle = \langle a^n \rangle \Leftrightarrow \gcd(s, n) = \gcd(t, n)$.

Proof. We already know b generates a cyclic subgroup H of G. And that because it is finite, it has only as many elements as the smallest power m of b so $b^m = e$. This and $b = a^s$ implies $(a^s)^m = e$ if and only if n divides ms because $a^n = e$ because G is of finite order n. Let $d = \gcd(n, s)$ such that we want to find the smallest m so $\frac{ms}{n} = \frac{m(s/d)}{(n/d)}$ is an integer. This implies (n/d) divides m so the smallest m we can pick *is* n/d. Thus, H has order n/d.

We know G is isomorphic to $\langle \mathbb{Z}_n, +_n \rangle$ so taking cyclic subgroup $\langle d \rangle$ of $\mathbb{Z}_n$ where d divides n implies $\langle d \rangle$ has n/d elements and contains all positive integers m less than n such that $\gcd(m, n) = d$. Thus, there is only one subgroup of $\mathbb{Z}_n$ of order n/d. It immediately follows that $\langle a^s \rangle = \langle a^t \rangle$ if and only if $\gcd(s, n) = \gcd(t, n)$. $\square$

Corollary. If a is a generator of a finite cyclic group G of order n, then the other generators of G are the elements of the form a^r, where r is relatively prime to n.

Example. For instance, we can derive the subgroup diagram for Z_{18} as:



1.7 Generating Sets and Cayley Digraphs

Example. The Klein 4-group $V = \{e, a, b, c\}$ is generated by $\{a, b\}$ since $ab = c$. It is similarly generated by $\{a, c\}$, $\{b, c, \}$, and $\{a, b, c\}$.

Theorem 11. The intersection of some subgroups H_i of a group G for $i \in I$ is again a subgroup of G where I is the set of indices.

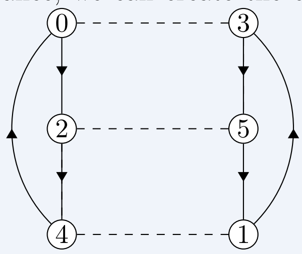
Proof. First, closure. For any $a, b \in \bigcap_{i \in I} H_i$, because each H_i has closure, $a, b \in H_i \Rightarrow ab \in H_i$ so $ab \in \bigcap_{i \in I} H_i$. Similarly, because the identity element of G is in H_i for all $i \in I$, $e \in \bigcap_{i \in I} H_i$. Last, for all $a \in H_i$, because H_i is a group, $a^{-1} \in H_i$. Thus, for any $a \in \bigcap_{i \in I} H_i$, $a \in H_i$ for all i so $a^{-1} \in H_i$ for all i so $a^{-1} \in \bigcap_{i \in I} H_i$. $\square$

Definition 27 (Subgroup generated by $\{a_i \mid i \in I\}$)**.** Let G be a group and $a_i \in G$ for $i \in I$. The smallest subgroup of G containing $\{a_i \mid i \in I\}$ is the *subgroup generated by* $\{a_i \mid i \in I\}$. If this subgroup is all of G then the set *generates* G and the a_i are the *generators of* G. If there is a finite set that generates G, we say G is *finitely generated.*

Definition 28 (Digraph)**.** A directed graph, abbreviated as *digraph*, consists of a fininite number of points, or *vertices* and some *arcs* denoted by an arrowhead joining them together.

Definition 29 (Cayley Digraphs). Cayley digraphs draw arcs of different types between each element of G demonstrating what each element is generated by. Of course, if $x \to y$ means $xa = y$ then $ya^{-1} = x$. Traveling opposite to arrow direction implies this second equality.

Example. For instance, we can create the digraph for Z_6 with generator set $S = \{2, 3\}$ as:



with solid (2) and dashed (3) lines. Dashed lines have no arrowhead because 3 is its own inverse.

Chapter 2

Permutations, Cosets, and Direct Products

2.1 Groups of Permutations

Definition 30 (Permutation of a set)**.** A *permutation of a set* A is a function $\phi\colon A \to A$ that is both one to one and onto.

Remark (Permutation Multiplication)**.** Function composition $\circ$ is a binary operation on the collection of all permutations of a set A. We call this operation *permutation multiplication.*

Remark. Let σ, τ be permutations of a set A so σ, τ are both one-to-one function mapping A onto A. then, $\sigma \circ \tau$, or simply $\sigma\tau$ is a permutation as long as it is one-to-one.

For any $a_1, a_2 \in A$, if $(\sigma\tau)(a_1) = (\sigma\tau)(a_2)$ gives $(\sigma(\tau(a_1))) = (\sigma()\tau(a_1))$. Because σ is injective, $\tau(a_1) = \tau(a_2)$. Because τ is injective, $a_1 = a_2$ so $\sigma\tau$ is injective.

For any $a \in A$, there exists some $b in A$ so $\sigma(b) = a$ because σ is onto A. Because τ is onto A, there exists some $c \in A$ so $\tau(c) = b$. Thus, $a = (\sigma\tau)(c)$ so $\sigma\tau$ is onto A.

Example. Given a set $A = \{1, 2, 3, 4, 5\}$, we can write a permutation σ as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix}.$$

so $\sigma(1) = 4$, etc.

Theorem 12. Let A be a nonempty set, and S_A be the collection of all permutations of A. Then, S_A is a group under permutation multiplication.

Proof. Because the composition of two permutations of A results in a permutation, we have closure under $\circ$. For any functions f, g, h, $((f \circ g) \circ h)(x) = (f(g)) \circ (h)(x) = f(g(h))(x) = f(g \circ h)(x)$ so $\mathscr{G}_1$ is easily satisfied. The permutation $\imath$ defined as $\imath(a) = a$ for all $a \in A$ is the identity ($\mathscr{G}_2$). Last, for any permutation σ, σ^{-1} reverse the direction of the mapping σ such that $\sigma^{-1}(a)$ is the element a' of A so $\sigma(a') = a$. This exists because σ is bijective. For any $a \in A$, $\imath(a) = a = \sigma(a') = \sigma(\sigma^{-1}(a')) = (\sigma\sigma^{-1})(a)$ and $\imath(a') = a' = \sigma^{-1}(a) = \sigma^{-1}(\sigma(a')) = (\sigma^{-1}\sigma)(a')$ satisfying $\mathscr{G}_3$. $\square$

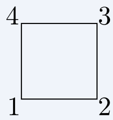
Remark. To define an isomorphism $\phi\colon S_A \to S_B$, we let $f\colon A \to B$ have one-to-one function mapping A onto B so A and B have the same cardinality so for $\sigma \in S_A$, let $\phi(\sigma) = \bar{\sigma} \in S_B$ so that for all $a \in A$, $\bar{\sigma}(f(a)) = f(\sigma(a))$.

Definition 31 (Symmetric Group on n Letters)**.** Let A be the finite set $\{1, 2, \ldots, n\}$. The group of all permutations of A is the *symmetric group on n letters* S_n. Note that S_n has $n!$ elements.

Remark. S_3 is also the 3rd dihedral group D_3 of *symmetries of an equilateral triangle* where ρ_i is rotations and μ_i is mirror images in bisectors of angles such that D_3 is made up of:

$$\begin{Bmatrix} \rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, & \mu_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \\ \rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, & \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \\ \rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, & \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}. \end{Bmatrix}$$

Definition 32 (nth Dihedral Group D_n)**.** The *nth dihedral group D_n* is the group of symmetries of the regular n-gon.



Example (Octic Group D_4)**.** Given a square: , D_4 is the set of:

$$\left\{ \begin{matrix} \rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, & \mu_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \\ \rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, & \mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \\ \rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, & \delta_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \\ \rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, & \delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}. \end{matrix} \right\}$$

where ρ_i, μ_i, δ_i represent rotations, mirror images in perpendicular bisectors of sides, and diagonal flips respectively.

Definition 33 (Image of H under f)**.** Let $f\colon A \to B$ be a function and H be a subset of A. The *image of H under f* is the set $\{f(h) \mid h \in H\}$ and is denoted $f[H]$.

Lemma 1. Let G, G' be groups and $\phi\colon G \to G'$ be a one-to-one function such that for all $x, y \in G$, $\phi(xy) = \phi(x)\phi(y)$. Thus $\phi[G]$ is a subgroup of G' and ϕ provides an isomorphism of G with $\phi[G]$.

Proof. We simply prove the subgroup requirements. For any $x', y' \in \phi[G]$, there exist $x, y \in G$ so $\phi(x) = x'$ and $\phi(y) = y'$. By hypothesis, $\phi(xy) = \phi(x)\phi(y)$ so $x'y' \in \phi[G]$ so $\phi[G]$ is closed under the operation of G'. Next, say e' is the identity of G'. Then, $e'\phi(e) = \phi(e) = \phi(ee) = \phi(e)\phi(e)$. Cancellation in G' shows $e' = \phi(e)$ so $e' \in \phi[G]$. Last, for any $x' \in \phi[G]$, $e' = \phi(e) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}) = x'\phi(x^{-1})$ implying $x'^{-1} = \phi(x^{-1}) \in \phi[G]$. Thus $\phi[G]$ is a subgroup of G'. We already showed ϕ is onto and therefore an isomorphism of G with $\phi[G]$. $\square$

Theorem 13 (Cayley's Theorem)**.** Every group is isomorphic to a group of permutations.

Proof. Let G be a group. We want to show G is isomorphic to a subgroup of S_G. By the previous lemma, we need only define a universal one-to-one function $\phi\colon G \to S_G$ with the homomorphism property. For any $x, g \in G$, let's define left multiplication by x via $\lambda_x\colon G \to G$ as $\lambda_x(g) = xg$. For all $c \in G$, $\lambda_x(x^{-}1c) = x(x^{-}1c) = c$ so clearly λ_x maps G onto G. Also, for any $a, b \in G$, $\lambda_x(a) = \lambda_x(b) \Rightarrow xa = xb \Rightarrow a = b$ through left cancellation. Thus, λ_x is one-to-one, onto, and a permutation of G. Now, we define $\phi\colon G \to S_G$ as $\phi(x) = \lambda_x$ for all $x \in G$.

To satisfy our lemma, we now only show ϕ is one-to-one and has the homomorphism property. Let e be the identity on G so that $\phi(x) = \phi(y)$ implies $\lambda_x = \lambda_y$ so $\lambda_x(e) = \lambda_y(e) \Rightarrow xe = ye \Rightarrow x = y$. Last, for any $x, y, g \in G$, $\lambda_{xy}(g) = (xy)g = x(yg) = \lambda_x(\lambda_y(g)) = \lambda_x\lambda_y(g)$ so $\phi(xy) = \phi(x)\phi(y)$ satisfying the homomorphism property. $\square$

Definition 34 (Left/Right Regular Representation). The map $\phi\colon G \to S_G$ defined as above is the *left regular represention* of G and the map $\mu\colon G \to S_G$ defined by $\mu(x) = \rho_{x^{-1}}$ where $\rho_x(g) = gx$ for all $x, g \in G$ is the *right regular representation* of G.

2.2 Orbits, Cycles, and the Alternating Groups

Definition 35 (Orbit of a under $\sigma \in S_A$). Let A be a set and $\sigma \in S_A$. For a fixed $a \in A$, the set $\mathcal{O}_{a.\sigma} = \{\sigma^n(a) \mid n \in \mathbb{Z}\}$ is the *orbit of a under σ*.

Remark. Let σ be a permutation of a set A. The equivalence classes in A are determined by the following equivalence class:

For $a, b \in A$, let $a \sim b$ if and only if $b = \sigma^n(a)$ for some $n \in \mathbb{Z}$.

These are called the *orbits of σ*.

Explanation. $\sim$ is an equivalence relation because it is:

1. **reflexive:** $a \sim a$ clearly because $a = \imath(a) = \sigma^0(a)$.
2. **symmetric:** If $a \sim b$, then $b = \sigma^n(a)$ for some $n \in \mathbb{Z}$ so $a = \sigma^{-n}(b)$ and $-n \in \mathbb{Z}$ so $b \sim a$.
3. **transitive:** If $a \sim b, b \sim c$, then $b = \sigma^n(a)$ and $c = \sigma^m(b)$ for some $n, m \in \mathbb{Z}$. This implies $c = \sigma^m(\sigma^n(a)) = \sigma^{n+m}(a)$ so $a \sim c$.

Example. The orbits of $\imath$ are the singleton subsets of A.

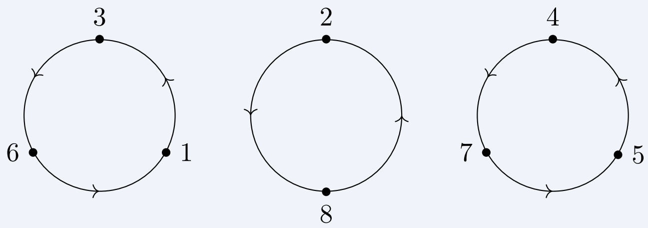
Example. Given the permutation σ of a finite set A defined as:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix},$$

the complete list of orbits of σ are

$$\{1, 3, 6\},\ \{2, 8\},\ \text{and}\ \{4, 5, 7\},$$

which we can map in the following way:



Definition 36. A permutation $\sigma \in S_n$ is a *cycle* if it has at most one orbit containing more than one element. The *length* of a cycle is the number of elements in its largest orbit.

Remark. We can use *cyclic notation* to simply denote $\mu = (1, 3, 6)$.

Remark. Cycles are *disjoint*. That is, no interger appears in the notations of 2 different cycles. Note that multiplication of disjoint cycles *is* commutative.

Theorem 14. Every permutation σ of a finte set is a product of disjoint cycles.

Proof. Let $B_1, B_2, \dots, B_r$ be the orbits of σ and define the cycle μ_i as:

$$\mu_i(x) = \begin{cases} \sigma(x) & x \in B_i \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\sigma = \mu_1 \mu_2 \cdots \mu_r$. Because the orbits $B_1, B_2, \dots, B_r$ are disjoint equivalence-classes, the cycles $\mu_1, \mu_2, \dots, \mu_r$ are disjoint also. □

Example. Take the disjoint cycles $\sigma = (1, 3, 5, 2)$ and $\tau = (2, 5, 6)$. To find $\sigma\tau$ (τ first), begin with 1 so $\sigma\tau = (1, \dots)$. τ doesn't map 1 but σ maps it to 3 so we get $(1, 3, \dots)$. Following this cycle, 3 isn't mapped anywhere by τ but is mapped to 5 so $(1, 3, 5, \dots)$. 5 is mapped to 6 but 6 isn't mapped anywhere so it stays fixed as $(1, 3, 5, 6, \dots)$. Beginning a new cycle, 2 is mapped to 5 then back to 2 so it becomes (1,3,5,6)(2). Finally, 4 isn't mapped anywhere by either so it stays as 4. Thus, $(1, 3, 5, 2)(2, 5, 6) = (1, 3, 5, 6)(2)(4) = (1, 3, 5, 6)$.

Definition 37 (Transposition). A cycle of length 2 is a *transposition*.

Corollary. Any permutation of a finite set of at least 2 elements is a product of transpositions. The identity, for S_n with $n \geq 2$ is $(1, 2)(1, 2)$.

Theorem 15. No permutation in S_n can be expressed both as a product of an even and odd number of transpositions.

Proof. (Linear Algebra) Recall $S_A \sim S_B$ if A, B have the same cardinality. Permutations work with n rows of the $n \times n$ I_n which has determinant 1. Interchanging any two rows changes the sign of the determinant. If C is a matrix obtained by some permutation σ of I_n and C could be obtained by an even and odd number of transpositions of rows, then its determinant would be both 1 and -1. $\square$

Proof. (Orbits) Let $\sigma \in S_n$ and $\tau = (i, j)$ be a transposition in S_n.

Case I: Suppose the orbits of σ and $\tau\sigma$ differ by 1. Suppose i, j are in different orbits of σ. Writing σ as a product of disjoint cycles with the first containing j and the second containing i, e.g. $(b, j, \times, \times, \times)(a, i, \times, \times)$ implies that $\tau\sigma = (i, j)\sigma = (i, j)(b, j, \times, \times, \times)(a, i, \times, \times)$ after calculating is $(a, j, \times, \times, \times, b, i, \times, \times)$. This is because a feeds into i now j feeds into $\times, \times, \times$ and b feeds into j now i into $\times, \times$. This is now a single orbit.

Case II: Suppose instead that i, j are in the same orbit of σ so σ can be written as the product of disjoint cycles so the first cycle is of form $(a, i, \times, \times, \times, b, j, \times, \times)$. $\tau\sigma = (i, j)\sigma$ gives $(a, j, \times, \times)(b, i, \times, \times, \times)$. This single orbit has been split into two.

These cases show the number of orbits of $\tau\sigma$ differs from the number of orbits of σ by 1. The identity permutation $\imath$ has exactly n orbits becasue each element is the only member of its orbit. So the orbits of a permutation $\sigma \in S_n$ must differ from n by an even or odd number. Each new transposition multiplied with the identity trying to create σ must then change that product's orbits by 1. So, there cannot be 2 sequences of different size because that would imply σ has different numbers of orbits. $\square$

Definition 38. Even/Odd Permutation A permutation of a finite set is known as *even or odd* depending on whether it can be written the product of an even or odd number of transpositions.

Example. The identity permutation $\imath \in S_n$ is even because it is $(1, 2)(1, 2)$.

Theorem 16. If $n \geq 2$, the collection of even permutations of $\{1, 2, 3, \ldots, n\}$ forms a subgroup of order $n!/2$ of the symmetric group S_n. Note the set of odd permutations is of the same size.

Proof. Take the set of even and odd (A_n and B_n) permutations in S_n Let τ be any fixed transposition in S_n. Because $n \geq 2$, we might as well suppose $\tau = (1, 2)$. Take the function $\lambda_\tau \colon A_n \to B_n$ defined as $\lambda_\tau(\sigma) = \tau\sigma$ for $\sigma \in A_n$. σ is even so $(1, 2)\sigma$ can be expressed as an odd number of transpositions so $\tau\sigma \in B_n$. Because S_n is a group, for any $\sigma, \mu \in A_n$, $\lambda_\tau(\sigma) = \lambda_\tau(\mu)$ implies $\sigma = \mu$ so λ_τ is injective. Note also that $\tau = \tau^{-1}$ so if $\rho \in B_n$, then $\tau^{-1}\rho \in A_n$ and $\lambda_\tau(\tau^{-1(\rho)}) = \tau(\tau^{-1}(\rho)) = \rho$ implying λ_τ is onto B_n. So B_n and A_n are of the same size because they are finite. The fact the set of even permutations is a subgroup is trivial. □

Definition 39 (Alternating Group A_n on n Letters)**.** The subgroup S_n consisting of the even permutations of n letters if the *altering group A_n on n letters.*

2.3 Cosets and the Theorem of Lagrange

Theorem 17. Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by

$$a \sim_L b \quad \text{if and only if} \quad a^{-1}b \in H.$$

Let $\sim_R$ be defined on G by

$$a \sim_R b \quad \text{if and only if} \quad ab^{-1} \in H.$$

Then $\sim_L, \sim_R$ are both equivalence relations on G.

Proof. (Just $\sim_L$) For any $a \in G$, $a^{-1}(a) = e \in H$ so $\sim_L$ is reflexive. For any $a, b \in G$, suppose $a^{-1}b \in H$. Because this is a subgroup, $(a^{-1}b)^-1 \in H$ so that $b^{-1}a \in H$ and thus $b \sim_L a$ so $\sim_L$ is symmetric. Lastly, if $a \sim_L b, b \sim_L c$ for some $a, b, c \in G$, then $a^{-1}b, b^{-1}c \in H$. By closure $a^{-1}bb^{-1}c = a^{-1}c \in H$ so $a \sim_L c$ implying $\sim_L$ is transitive. Thus, $\sim_L$ is an equivalence relation. □

Definition 40 (Left/Right Cosets)**.** Let H be a subgroup of group G. The subset $aH = \{ah \mid h \in H\}$ of G is the *left coset* of H containing a while the subset $Ha = \{ha \mid h \in H\}$ is the *right coset* of H containing a.

Example. Take the subgroup $3\mathbb{Z}$ of $\mathbb{Z}$. Using additive notation, the left coset of $3\mathbb{Z}$ containing m is $m+3\mathbb{Z}$. When $m = 0$, $3\mathbb{Z} = \{\cdots, -3, 0, 3, \cdots\}$ so $3\mathbb{Z}$ is itself such a left coset. Similarly, $1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}$ are left cosets. Together, these partition $\mathbb{Z}$. Because $\mathbb{Z}$ is abelian, left coset $m + 3\mathbb{Z}$ is the same as right coset $3\mathbb{Z} + m$.

Lemma 2. Take the one-one map $\phi\colon H \to gH$ so $\phi(h) = gh$ for each $h \in H$. This is onto gH by definition. Next, suppose $\phi(h_1) = \phi(h_2)$ for some $h_1, h_2 \in H$. Thus, $gh_1 = gh_2$ so by cancellation in G, $h_1 = h_2$ implying ϕ is bijective. If H is of finite order, then ϕ and a similar function for right cosets have equal numbers of elements to H.

Theorem 18 (Theorem of Lagrange)**.** Let H be a subgroup of a finite group G. Then the order of H is a divisor of the order of G.

Proof. Let n be the order of G and H have order m. Every coset (left or right) of a subgroup H of a group G has the same number of elements as H, namely m. Let G be partitioned into r left cosets of H so $n = rm$ implying m is a divisor of n. $\square$

Corollary. Every group of prime order is cyclic.

Proof. Let G be of prime order P and $a \in G, a \neq e$. Thus, $\langle a \rangle$ of G has at least 2 elements. But by Lagrange's Theorem, the order $m \geq 2$ of a must divide the prime p implying $m = p$ so $\langle a \rangle = G$ so G is cyclic. $\square$

Definition 41. Let H be a subgroup of a group G. The number of left cosets of H in G is the *index $(G : H)$ of H in G*. The index may be infinite or finite.

Theorem 19. Suppose H and K are subgroups of a group G so $K \leq H \leq G$ and suppose $(H : K)$ and $(G : H)$ are both finite. Then $(G : K) = (G : H)(H : K)$ is finite.

2.4 Finitely Generated Abelian Groups

Theorem 20 (Direct Product of Groups)**.** Let $G_1, G_2, \ldots, G_n$ be groups. For $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ in $\Pi_{i=1}^n G_i$. Define $(a_1, a_2, \ldots, a_n)$ times $(b_1, b_2, \ldots, b_n)$ as the element $(a_1 b_1, a_2 b2, \ldots, a_n b_n)$. This is the *direct product of the groups G_i* under this binary operation.

Proof. Closure is trivial. Take the element $(e_1, e_2, \ldots, e_n)$ as the identity. And for any $(a_1, a_2, \ldots, a_n)$, its inverse is $(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$. Thus, $\Pi_{i=1}^n G_i$ is a group. $\square$

Remark (Direct Sum of Groups)**.** In the case the binary operation of each G_i is commutative, we replace $\Pi_{i=1}^n G_i$ with the *direct sum of the groups G_i*, denoted $\oplus_{i=1}^n G_i$. We may also write it $G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

Example. The group $\mathbb{Z}_2 \times \mathbb{Z}_3$ obviously is of order 6. However, via the generator (1,1), we can show it is cyclic as:

- $1(1,1) = (1,1)$
- $2(1,1) = (0,2)$
- $3(1,1) = (1,0)$
- $4(1,1) = (0,1)$
- $5(1,1) = (1,2)$
- $6(1,1) = (0,0)$

Because there is only one cyclic group structure of a given order, we see $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

In contrast, however, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is a group of 9 elements but every 3 operationsd generates the identity and thus it is not cyclic. The same goes for $\mathbb{Z}_2 \times \mathbb{Z}_2$ which must be isomorphic, then, to the Klein 4-group.

Theorem 21. The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and isomorphic to $\mathbb{Z}_{mn}$ if and only if m, n are relatively prime.

Proof. $\Rightarrow$: Consider the cyclic subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by (1,1). Clearly, the smallest number that is a multiple of both m and n will be mn if and only if $\gcd(m, n) = 1$. It is at this number of summands that (1,1) yields the identity and implies mn is the order of $\mathbb{Z}_m \times \mathbb{Z}_n$ and $\mathbb{Z}_{mn}$. Because $\langle(1,1)\rangle$ is cyclic, they are isomorphic.

$\Leftarrow$: Suppose $\gcd(m, n) = d > 1$. Then, mn/d is divisible by both m and n so for any $(r, s) \in \mathbb{Z}_m \times \mathbb{Z}_n$, □

Corollary. The group $\Pi_{i=1}^n \mathbb{Z}_{m_i}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \cdots m_n}$ if and only if any two of the numbers m_i for $i = 1, \ldots, n$ are coprime.

Example. Thus, if $n = (p_1)^{n_1}(p_2)^{n_2} \cdots (p_r)^{n_r}$ for distinct primes, then $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{(p_1)^{n_1}} \times \mathbb{Z}_{(p_2)^{n_2}} \times \cdots \times \mathbb{Z}_{(p_r)^{n_r}}$. In particular, $\mathbb{Z}_7 2$ is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_9$.

Example. The order of (8,4,10) in $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$ is the least common multiple of $(\frac{12}{\gcd(8,12)}, \frac{60}{\gcd(4,60)}, \frac{24}{\gcd(10,24)}) = 3 \cdot 5 \cdot 4 = 60$.

Theorem 22. Let $(a_1, a_2, \ldots, a_n) \in \Pi_{i=1}^n G_i$. If a_i is of finite order r_i in G_i, then the order of $(a_1, a_2, \ldots a_n)$ in $\Pi_{i=1}^n G_i$ is equal to the least common multiple of all the r_i.

Proof. Only for the power $\operatorname{lcm}(r_1, r_2, \ldots, r_n)$ does $(a_1, a_2, \ldots, a_n)$ give the identity $(e_1, e_2, \ldots, e_n)$. □

Theorem 23 (Fundamental Theorem of Finitely Generated Abelian Groups)**.** Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in form

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \mathbb{Z}$$

where p_i are primes, not necessarily distinct, and $r_i \in \mathbb{Z}^+$. The direct product is unique except for possible rearrangement. In other words, the *Betti number* of G of factors $\mathbb{Z}$ is unique and the prime power $(p_i)^{r_i}$ are unique.

We call the left part the *torsion part* and *free part*.

Example. We can decompose every group of order $360 = 2^3 3^2 5$ through separating groups into groups of coprime orders. Then, $\mathbb{Z}_4\mathbb{Z}_6\mathbb{Z}_{15}$ is equivalent to $\mathbb{Z}_4\mathbb{Z}_2\mathbb{Z}_3\mathbb{Z}_3\mathbb{Z}_5 = \mathbb{Z}_3\mathbb{Z}_{12}\mathbb{Z}_{10}$.

Definition 42 (Decomposable)**.** A group G is *decomposable* if it is isomorphic to a direct product of two proper nontrivial subgroups. Otherwise G is *decomposable.*

Theorem 24. The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.

Proof. $\Rightarrow$: Let G be a finite indecomposable abelian group. Thus, G is isomorphic to a direct product of cyclic groups of a prime power. Since G is indecomposable, this direct product must consist of just one cyclic group whose order is a power of a prime number.

$\Leftarrow$: Let p be a prime number so $\mathbb{Z}_{p^r}$ is indecomposable such that if $\mathbb{Z}_{p^r}$ were isomorphic to $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ where $i + j = r$, then every element would have an order at most $p^{\max(i,j)} < p^r$. □

Theorem 25. If m divides the order of a finite abelian group G, then G has a subgroup of order m.

Proof. G finite so it can be written as $\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}}$ where not all primes p_i need be distinct. This implies $(p_1)^{r_1}(p_2)^{r_2}\cdots(p_n)^{r_n}$ is the order of G. So $m = (p_1)^{s_1}(p_2)^{s_2}\cdots(p_n)^{s_n}$ where $0 \leq s_i \leq r_i$. This implies $(p_i)^{r_i - s_i}$ generates a cyclic subgroup of $\mathbb{Z}^{(}_{(p_i)}r_i)$ of order $(p_i)^{s_i}$. This implies that $\langle (p_1)^{r_1 - s_1}\rangle \times \langle (p_2)^{r_2 - s_2}\rangle \times \cdots \times \langle (p_n)^{r_n - s_n}\rangle$ is the required subgroup of order m. □

Theorem 26. If m is a square free interger, that is, m is not divisible by the square of any prime, then every abelian group of order m is cyclic.

Proof. Let G be an abelian group of square free order m so G finite and isomorphic to $\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}}$ where $m = (p_1)^{r_1}(p_2)^{r_2}\cdots(p_n)^{r_n}$. Because m is square free, all $r_i = 1$ and all p_i distinct primes implying G isomorphic to $\mathbb{Z}_{p_1 p_2 \ldots p_n}$ so G cyclic. □

Chapter 3

Homormorphisms and Factor Groups

3.1 Homomorphisms

Definition 43 (Homomorphism)**.** A map ϕ of a group G into a group G' is a *homomorphism* if the homomorphism property that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$ holds.

Remark (Trivial Homomorphism)**.** There is at least always the homomorphism $\phi\colon G \to G'$ defined as $\phi(g) = e'$ for all $g \in G$ is called the *trivial homomorphism.*

Example. Let S_n be the symmetric group on n letters and let $\phi\colon S_n \to \mathbb{Z}_n$ be defined by:

$$\phi(\sigma) = \begin{cases} 0 & \sigma \text{ even permutation} \\ 1 & \sigma \text{ odd permutation.} \end{cases}$$

Clearly, σ is a homormorphism.

Example (Evaluation Homomorphism)**.** Let F be the additive group of all functions mapping R into R and R be the additive group of all reals and $c \in \mathbb{R}$. Then, $\phi_c\colon F \to \mathbb{R}$ is the *evaluation homomorphism* defined as $\phi_c(f) = f(c)$ for $f \in F$.

Example. The *projection map* $\pi_i\colon G \to G_i$ where $G = G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n$ and $\pi_i(g_1, g_2, \cdots, g_i, \cdots, g_n) = g_i$ for each $i = 1, 2, \cdots, n$.

Definition 44 (Image, Range, Preimage)**.** Let ϕ be a mapping on a set X into a set Y and $A \subseteq X, B \subseteq Y$. The *image* $\phi[A]$ *of* A *in* Y *under* ϕ is

$\{\phi(a) \mid a \in A\}$.

The set $\phi[X]$ is the *range of* ϕ.

The *inverse image* $\phi^{-1}[B]$ *of* B *in* X is $\{x \in X \mid \phi(x) \in B\}$.

Theorem 27. Let ϕ be a homomorphism of a group G into a group G'. Then,

1. If e is the identity element in G, $\phi(e)$ is the identity element $e' \in G'$.
2. If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$.
3. If H is a subgroup of G, then $\phi[H]$ is a subgroup of G'.
4. If K' is a subgroup of G', then $\phi^{-1}[K']$ is a subgroup of G.

Definition 45 (Kernel)**.** Let $\phi\colon G \to G'$ be a homomorphism of groups. The subgroup $\phi^{-1}[\{e'\}] = \{x \in G \mid \phi(x) = e'\}$ is the *kernel of* ϕ, denoted by $\ker(\phi)$.

Theorem 28. Let $\phi\colon G \to G'$ be a group homomorphism and $H = \ker(\phi)$. For $a \in G$, the set

$$\phi^{-1}[\{\phi(a)\}] = \{x \in G \mid \phi(x) = \phi(a)\}$$

is the left coset aH and right coset aH of H. Thus, the partitions of G into left cosets and right cosets are the same.

Proof. We want to show $\{x \in G \mid \phi(x) = \phi(a)\} = aH$, i.e. they are subsets of one another.

$\subseteq$: If $\phi(x) = \phi(a)$, then $e' = \phi(a)^{-1}\phi(x) = \phi(a^{-1})\phi(x) = \phi(a^{-1}x)$ so $a^{-1}x \in H = \ker(\phi)$. Thus, $a^{-1}x = h$ for some $h \in H$ so $x = ah \in aH$ so $\{x \in G \mid \phi(x) = \phi(a) = aH\}$.

$\supseteq$: Say $y \in aH$ so $y = ah$ for some $h \in H$. Thus, $\phi(y) = \phi(ay) = \phi(a)\phi(h) = \phi(a)e' = \phi(a)$ so $y \in \{x \in G \mid \phi(x) = \phi(a)\}$. $\square$

Corollary. A group homomorphism $\phi\colon G \to G'$ is injective $\Leftrightarrow \ker(\phi) = \{e\}$.

Proof. $\Rightarrow$: If $\ker(\phi) = \{e\}$, then the elements mapped to $\phi(a)$ are exactly the elements of the left coset $a\{e\} = \{e\}$ showing that ϕ is injective. $\Leftarrow$: If ϕ is injective, then simply e can be the only element mapped to e'. $\square$

Note (Show $\phi\colon G \to G'$ Is an Isomorphism)**.**

(Step 1) Show ϕ homormorphism.

(Step 2) Show $\ker(\phi) = \{e\}$.

(Step 3) Show ϕ is surjective.

Definition 46 (Normal Subgroup)**.** A subgroup H of a group H is normal if its left and right cosets coincide, that is, if $gH = Hg$ for all $g \in G$. Normal subgroups are denoted as $H \lhd G$.

Note. All subgroups of abelian groups are normal.

Corollary. If $\phi\colon G \to G'$ is a group homomorphism, then $\ker(\phi)$ is a normal subgroup of G.

3.2 Factor Groups

Theorem 29. Let $\phi\colon G \to G'$ be a group homomorphism with kernel H. Then the cosets of H form a *factor group* G/H where $(aH)(bH) = (ab)H$. Also, the map $\mu\colon G/H \to \phi[G]$ defined by $\mu(aH) = \phi(a)$ is an isomorphism.

A factor group G/H is also called the *factor group of G modulo H* and elements in the same coset are said to be *congruent modulo H*.

Example. The isomorphism $\mu\colon \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}_5$ assigns to each coset of $5\mathbb{Z}$ its smallest nonnegative element, i.e. $\mu(5\mathbb{Z}) = 0, \mu(1 + 5\mathbb{Z}) = 1$, etc.

Theorem 30. Let H be a subgroup of a group G. Then left coset multiplication is well defined by $(aH)(bH) = (ab)H$ if and only if H is a normal subgroup of G.

Proof. $\Rightarrow$: Suppose $(aH)(bH) = (ab)H$ is a well-defined operation on left cosets. Then, we want to show aH and Ha are the same set. Let $x \in aH$. Picking representatives $x \in aH$ and $a^{-1} \in a^{-1}H$, we get $(xH)(a^{-1}H) = (xa^{-1})H$. This must be equal to $(aH)(a^{-1}H) = (eH) = H$ so $xa^{-1} = h \in H$ Thus, $x = ha \Rightarrow x \in Ha$ so $aH \subseteq Ha$. The symmetric proof is also true so $aH = Ha$.

$\Leftarrow$: Suppose H is a normal subgroup of G. Take $a, ah_1 \in aH, b, bh_2 \in bH$ so $h_1 b \in Hb = bH$ so $h_1 b = bh_3$ for some $h_1, h_2, h_3 \in H$. Thus,

$$(ah_1)(bh_2) = a(h_1 b)(h_2) = a(bh_3)h_2 = (ab)(h_3 h_2) \in (ab)H.$$

Going the other direction, if $x \in (ab)H \Rightarrow x = abh = (ae)(bh) \in (aH)(bH)$. $\square$

Definition 47 (Factor/Quotient Group)**.** Let $H \lhd G$. Then the cosets of H form a group G/H under the binary operation $(aH)(bH) = (ab)H$. This group is called the *factor, or quotient, group of G by H*.

Example. Because $\mathbb{Z}$ is an abelian group, $n\mathbb{Z}$ is a normal subgroup so $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$.

Theorem 31. Let $H \lhd G$. Then $\gamma: G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.

Proof. Let $x, y \in G$. Clearly, $\gamma(a)\gamma(b) = (aH)(bH) = (ab)H = \gamma(ab)$ so it is a homomorphism. Plus, if $\gamma(x) \in eH$, then $xH = eH$ so clearly $x \in H$. Thus, $\ker(\gamma) = H$. □

Theorem 32 (The Fundamental Homomorphism Theorem)**.** Let $\phi: G \to G'$ be a group homomorphism with kernel H. Then $\phi[G]$ is a group and $\mu: G/H \to \phi[G]$ given by $\mu(gH) = \phi(g)$ is an isomorphism. If $\gamma: G \to G/H$ is the homormorphism given by $\gamma(g) = gH$, then $\phi(g) = \mu\gamma(g)$ for each $g \in G$. μ is the *natural, or canonical isomorphism.*

Theorem 33. The following are 3 equivalent conditions for a subgroup H of a group G to be a normal subgroup of G:

1. $ghg^{-1} \in H$ for all $g \in G, h \in G$.
2. $gHg^{-1} = H$ for all $g \in G$
3. $gH = Hg$ for all $g \in G$.

Definition 48 ((Inner) Automorphism)**.** An isomorphism $\phi: G \to G$ of a group G with itself is a *automorphism of* G. The automorphism $i_g: G \to G$ where $i_g(x) = gxg^{-1}$ for all $x \in G$ is the *inner automorphism of* G *by* g.

Definition 49 (Conjugate Subgroup)**.** Performing i_g on x is called the *conjugation of* x *by* g. A subgroup K of G is a *conjugate subgroup* of H if $K = i_g[H]$ for some $g \in G$.

3.3 Simple Groups

Remark. For a normal subgroup N of G, the factor group G/N collapses N to a single element, namely the identity.

Example. The trivial subgroup $N = \{0\}$ of $\mathbb{Z}$ is obviously normal and has factor group isomorphic to $\mathbb{Z}$.

Example. We can show the falsity of the converse of Lagrange's Theorem. That is, A_4 has order 12 yet has no subgroup of order 6.

Suppose $H < A_4$ and H was of order 6. It would follow that H is a normal subgroup of A_4 so A_4/H would only have 2 elements, H and σH for some $\sigma \in A_4/H$. Because it's a group of order 2, the square of this element but be the identity so $(\sigma H)(\sigma H) = H$. Thus, the square of every element in A_4 must be in H. However, this is 8 elements so H cannot have order 6.

Theorem 34. Let $G = H \times K$ be the direct product of groups H and K. Then $\bar{H} = \{(h, e) \mid h \in H\} \lhd G$. Also, $G/\bar{H} \simeq K$ and $G/\bar{K} \simeq H$ in natural ways.

Proof. Take the homomorphism $\pi_2 \colon H \times K \to K$ where $\pi_2(h, k) = k$. Because $\ker(\pi_2) = \bar{H}$, $\bar{H} \lhd H \times K$. Because π_2 is onto K, $(H \times K)/\bar{H} \simeq K$. □

Theorem 35. A factor group of a cyclic group is cyclic.

Proof. Let G be a cyclic group generated by a with normal subgroup N. To compute all powers of aN means computing all powers of the representative a which gives all elements in G such that aN gives all cosets of N such that G/N is cyclic. □

Example. To find the factor group of $\mathbb{Z}_4 \times \mathbb{Z}_6/\langle(2, 3)\rangle$, note that $\langle(2, 3)\rangle$ has order 2 and $\mathbb{Z}_4 \times \mathbb{Z}_3$ has order 24 implying the factor group has order 12 which is either of form, up to isomorphism, $\mathbb{Z}_4 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. However, note that $(1, 0) + \langle(2, 3)\rangle$ is of order 4 in the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2, 3)\rangle$ so the group must be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_3$ or equivalently $\mathbb{Z}_1 2$.

Definition 50 (Simple Groups)**.** A group is *simple* if it is nontrivial and has no proper nontrivial normal subgroups.

Remark. The alternating group A_n is simple for $n \geq 5$.

Theorem 36. Let $\phi \colon G \to G'$ be a group homomorphism. If $N \lhd G$, then $\phi[N] \lhd \phi[G]$. Also, if N' is a normal subgroup of $\phi[G]$, then $\phi^{-1}[N'] \lhd G$. Note that $\phi[N]$ may not be normal in G'.

Definition 51 (Maximal Normal Subgroup of a Group G)**.** A *maximal normal subgroup of a group* G is a normal subgroup M not equal to G such taht there is no proper normal subgroup N of G properly containing M.

Theorem 37. M is a *maximal normal subgroup* of G if and only if G/M is simple.

Proof. $\Rightarrow$: Let M be a maximal normal subgroup of G. Take the canonical homomorphism $\gamma \colon G \to G/M$ Now, γ^{-1} of any nontrivial proper normal subgroup of G/M is a proper normal subgroup of G properly containing M. But M is maximal so this isn't possible. So G/M is simple.

$\Leftarrow$: If N is a normal subgroup of G properly containing M, then $\gamma[N]$ is normal in G/M. If $N \neq G$, then $\gamma[N] \neq G/M$ and $\gamma[N] \neq \{M\}$. If G/M is simple, no such $\gamma[N]$ and thus no such N can exist so M is maximal. $\square$

Definition 52 (Center of G). Every nonabelian group has a *center* $Z(G)$ such that

$$Z(G) = \{z \in G \mid zg = gz \forall g \in G\}.$$

The center always contains the identity, but if it only contains this then it is trivial.

Definition 53 (Commutator). To abelianze G, we will find all elements such that $ab = ba$, or that $aba^{-1}b^{-1} = e$. This element is a *commutator of the group*.

Theorem 38. Let G be a group. The set of all commutators $aba^{-1}b^{-1}$ for $a, b \in G$ generates the *commutator subgroup* C of G. This is a normal subgroup of G. Furthermore, if N is a normal subgroup of G, then G/N is abelian if and only if $C \leq N$.

Proof. The commutators surely generate a subgroup C from e, inverses, and closure. Now, for any $x \in C$, and any $g \in G$, $x = cdc^{-1}d^{-1}$ for some $c, d \in G$ such that $g^{-1}xg = (g^{-1}cdc^{-1})(e)(d^{-1}g)$. This becomes $(g^{-1}cdc^{-1})(gd^{-1}g^{-1})(d^{-1}g) = [(g^{-1}c)d(g^{-1}c)^{-1}d^{-1}][dg^{-1}d^{-1}g] \in C$ so $C \lhd G$.

Next, if $N \lhd G$, then $\Rightarrow$: If G/N abelian, $(a^{-1}N)(b^{-1}N) = (b^{-1}N)(a^{-1}N)$ so $aba^{-1}b^{-1}N = N$ so $aba^{-1}b^{-1} \in N \Rightarrow C \leq N$.

$\Leftarrow$: If $C \leq N$, then $(aN)(bN) = abN = ab(b^{-1}a^{-1}ba)N = baN = (bN)(aN)$. $\square$

3.4 Group Action on a Set

Definition 54 (Group Action). Let X be a set and G a group. An *action of G on X* is a map $* \colon G \times X \to X$ such that:

1. $ex = x$ for all $x \in X$
2. $(g_1g_2)(x) = g_1(g_2x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

In this case, we call X a *G-set*.

Theorem 39. Let X be a G-set. For each $g \in G$, the function $\sigma_g \colon X \to X$ defined by $\sigma_g(x) = gx$ for $x \in X$ is a permutation of X. Also, the map $\phi \colon G \to S_x$ defined by $\phi(g) = \sigma_g$ is a homomorphism with the property that $\phi(g)(x) = gx$.

Proof. To show σ_g is a permutation of X, we must show it is bijective. (i) If $\sigma_g(x_1) = \sigma_g(x_2)$, then $gx_1 = gx_2$ so $g^{-1}(gx_1) = g^{-1}(gx_2)$ so $(g^{-1}g)(x_1) = (g^{-1}g)(x_2)$ from the second condition of group actions so $e(x_1) = e(x_2)$ so $x_1 = x_2$ from the first condition. Next, for any $x \in X$, $\sigma_g(g^{-1}x) = g(g^{-1})x = (gg^{-1})x = ex = x$ so σ_g is onto and 1-1 making it a permutation.

Next, $\phi \colon G \to S_X$ defined by $\phi(g) = \sigma_g$ is a homomorphism because $\sigma(g_1g_2)(x) = (g_1g_2)x = g_1(g_2x) = g_1\sigma_{g_2}(x) = \sigma_{g_1}(\sigma_{g_2}(x)) = (\sigma_{g_1} \circ \sigma_{g_2})(x) = (\sigma_{g_1}\sigma_{g_2})(x) = (\phi(g_1)\phi(g_2))(x)$. ☐

Definition 55 (Acting Faithfully, Transitive)**.** Note that the subset of G leaving each element of X fixed is a normal subgroup N of G. We say G *acts faithfully* on X if $N = \{e\}$.

We say G is transitive on a G-set X if and only if the subgroup $\phi[G]$ of S_X is transitive on X, that is, if for each $x_1, x_2 \in X$, there exists some $g \in G$ so that $gx_1 = x_2$.

Remark. Every group is itself a $G - set$.

Theorem 40. Let X be a G-set. Then G_x is a subgroup of G for each $x \in X$.

Note. As notation, for G-set X with $x \in X, g \in G$, we say

$$X_g = \{x \in X \mid gx = x\} \quad \text{and} \quad G_x = \{g \in G \mid gx = x\}.$$

Proof. Let $x \in X$, $g_1, g_2 \in G_x$. So $g_1x = g_2x = x$ so $g_1x = x = g_2x$. Thus, $(g_1g_2)x = g_1(g_2x) = g_1x = x$ so $g_1, g_2 \in G_x$ and G_x closed under the induced operation of G. Clearly, $e \in G_x$. And $g \in G_x$ implies $x = ex = (g^{-1}g)x = g^{-1}(gx) = g^{-1}x$ so $g^{-1} \in G_x$. Thus $G_x \leq G$. ☐

Definition 56 (Isotropy Subgroup)**.** Let X be a G-set and $x \in X$. The subgroup G_x is the *isotropy subgroup of* x.

Theorem 41. Let X be a G-set. For $x_1, x_2 \in X$, say $x_1 \sim x_2$ if and only if there exists $g \in G$ so $gx_1 = x_2$. $\sim$ is an equivalence relation on X.

Proof. (i)For any $x \in X$, $ex = x$ so $x \sim x$. (ii) For any $x_1, x_2 \in X$, if $x_1 \sim x_2$, then $gx_1 = x_2$ so $g^{-1}x_2 = g^{-1}(gx_1) = (g^{-1}g)(x_1) = ex_1 = x_1$ so $x_2 \sim x_1$. (iii) Last, if $x_1 \sim x_2, x_2 \sim x_3$, then $g_1x_1 = x_2, g_2x_2 = x_3$ for some $g_1, g_2 \in G$ so $(g_2 g_1)(x_1) = g_2(g_1 x_1) = g_2(x_2) = x_3$ so $x_1 \sim x_3$. □

Definition 57 (Orbit of x). Let X be a G-set. Each cell in the partition of the equivalence relation is described as a *orbit in* X *under* G. For $x \in X$, the cell containing x is the *orbit of* x. This is Gx.

Theorem 42. Let X be a G-set, $x \in X$. Then $|Gx| = (G\colon G_x)$. If $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.

Proof. Let's define the 1-1 map ψ from G_X onto the collection of left cosets of G_x in G. Let $x_1 \in Gx$. Then, there exists $g_1 \in G$ so $g_1 x = x_1$. Say $\psi(x_1)$ is the left coset $g_1 G_x$ of G_x. To show this is well defined, if $g_1' x = x_1$, then $g_1 x = g_1' x$ so $g_1^{-1}(g_1 x) = g_1^{-1}(g_1' x)$ implying $x = g(g_1^{-1} g_1')x$ so $g_1^{-1} g_1' \in G_x$ so $g_1' \in g_1 G_x$ and $g_1 G_x = g_1' G_x$.

To show ψ is one-one, $x_1, x_2 \in Gx$ gives $\psi(x_1) = \psi(x_2)$ so there exists $g_1, g_2 \in G$ so $x_1 = g_1 x, x_2 = g_2 x$ where $g_2 \in g_1 Gx$ giving $g_2 = g_1 g$ for some $g \in G_x$. Thus, $x_2 = g_2 x = g_1(gx) = g_1 x = x_1$.

To show it's onto, for any left coset of G_x $g_1 G_x$ in G, if $g_1 x = x_1$ then $g_1 G_x = \psi(x_1)$. This map is bijective so $|Gx| = (G\colon G_x)$. If $|G|$ finite, then clearly, $|Gx|$ divides $|G|$. □

Chapter 4

Rings and Fields

4.1 Rings and Fields

Definition 58 (Ring)**.** A *ring* $\langle R, +, \cdot \rangle$ is a set R together with two binary operations $+$ and $\cdot$ which we call *addition* and *multiplication* defined on R such that the following are satisfied:

($\mathscr{R}_1$) $\langle R, + \rangle$ is an abelian group.

($\mathscr{R}_2$) Multiplication is associative.

($\mathscr{R}_3$) For all $a, b, c \in R$, the *left and right distributive laws* – $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ hold.

Example. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all rings with addition and multiplication. In fact, these axioms hold for any subset of the complex numbers that is a group under addition and closed under multiplication.

Example. For any ring R, the collection of all $n \times n$ matrices having elements of R as entries, $M_n(R)$, is an abelian additive group. Note, in particular, that (matrix) multiplicaton is not commutative for these.

Theorem 43. If R is a ring with additive identity 0, then for any $a, b \in R$, we have:

1. $0a = a0 = 0$.
2. $a(-b) = (-a)b = -(ab)$.
3. $(-a)(-b) = ab$.

Proof. (i) $a0 + a0 = a(0 + 0) = a0 = 0 + a0$ so $a0 = 0$. (ii) $a(-b) + ab = a(0) = 0$ so $a(-b) = -(ab)$. The same goes for $(-a)b$. (iii) $-(a(-b)) = -(-(ab))$ so $(-a)(-b) = ab$. □

Definition 59 (Ring Homomorphism). For rings R and R', a map $\phi\colon R \to R'$ is a homomorphism if both $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$. ϕ is one-to-one if and only if its kernel ($\{a \in R \mid \phi(a) = 0'\}$) is just the subset $\{0\}$ of R. This gives rise to a factor group as well as a factor ring.

Definition 60 (Ring Isomorphism). A ring isomorphism is a homomorphism $\phi\colon R \to R'$ that is bijective. Group isomorphisms do not necessarily extend to ring isomorphisms.

Definition 61 (Unity). A ring with a multiplicative identity element, denoted by 1, is a *ring with unity*. 1 is the "unity."

Definition 62 (Commutative Ring). A ring in which multiplication is commutative is a *commutative ring*.

Example. For intergers r, s where $\gcd(r, s) = 1$, the rings $\mathbb{Z}_{rs}$ and $\mathbb{Z}_r \times \mathbb{Z}_s$ are isomorphic. $\phi\colon Z_{rs} \to \mathbb{Z}_r \times \mathbb{Z}_s$ defined by $\phi(n \cdot 1) = n \cdot (1, 1)$ is an additive group isomorphism. Also, $\phi(nm) = (nm) \cdot (1, 1) = [n \cdot (1, 1)][m \cdot (1, 1)] = \phi(n)\phi(m)$ so it is a ring isomorphism as well.

Definition 63 (Multiplicative Inverse). A *multiplicative inverse* of an element a in a ring R with unity $1 \neq 0$ is an element $a^{-1} \in R$ so $aa^{-1} = a^{-1}a = 1$.

Remark. Only the ring $\{0\}$ has both the multiplicative and additive inverse as the same element.

Definition 64 (Unit, Division Rings). Let R be a ring with $1 \neq 0$. An element $u \in$ is a *unit* of R if it has a multiplicative inverse in R. If every nonzero element is a unit, then R is a *division ring* or *skew field*.

Definition 65 (Field). A *field* is a commutative division ring. A noncommutative division ring is a *strictly skew field*.

Definition 66 (Subring and Subfield). A *subring* is a subset of a ring with under induced operations. A subfield is defined similarly.

Note. *Unit* denotes an element with a multiplicative inverse and *unity* denotes the actual multiplicative identity element 1.

4.2 Integral Domains

Definition 67 (Divisors of 0)**.** If a and b are two nonzero elements of a ring R so that $ab = 0$, then a and b are *divisors of 0.*

Theorem 44. In the ring $\mathbb{Z}_n$, the divisors of 0 are the nonzero elements that are *not* relatively prime to n.

Proof. Let $m \in Z_n, m \neq 0$ and $d = gcd(m, n) \neq 1$. Thus, $m\left(\frac{n}{d}\right) = \left(\frac{m}{d}\right)n$ so $(\frac{m}{d})n$ is 0 in $\mathbb{Z}_n$ so $m(n/d)$ is 0 in $\mathbb{Z}_n$ also but neither $m, n/d = 0$ so m is a divisor of 0.

On the other hand, if $m \in \mathbb{Z}_n, \gcd(m, n) = 1$ and $ms = 0$ for some $s \in Z_n$, then $n \mid ms$. But, $\gcd(m, n) = 1$ so $n \mid s$ but $s \in Z_n$ so $s = 0$ in $\mathbb{Z}_n$ meaning m is not a divisor. $\square$

Corollary. If p is prime, then $\mathbb{Z}_p$ has no divisors of 0.

Theorem 45. The multiplicative cancellation laws hold in a ring R if and only if R has no divisors of 0.

Proof. Say R is a ring with cancellation laws and $ab = 0$ for some $a, b \in R$. If $a \neq 0$, then $ab = a0$ implies $b = 0$ via cancellation, WLOG. Conversely, if R has no divisors of 0 and $ab = ac, a \neq 0$ for any $a, b, c \in R$, then $0 = ab - ac = a(b - c)$. $a = 0$ and R has no divisors of 0 so $b = c$ and we can do cancellation. The same goes for right cancellation. $\square$

Definition 68 (Integral Domain)**.** A *integral domain* D is a commutative ring with unity $1 \neq 0$ that has *NO* divisors of 0.

Theorem 46. Every field F is an integral domain.

Proof. For any $a, b \in F$, if $a \neq 0$ and $ab = 0$ then $b = 1b = (a^{-1}a)b = a^{-1}0 = 0$. So no divisors of 0 in F exist (from commutativtity for other direction). $\square$

Theorem 47. Every finite integral domain is a field.

Proof. Take the finite domain D with finite elements $0, 1, a_1, \ldots, a_n$. We must show that for any $a \in D, a \neq 0, \exists b \in D$ so $ab = 1$. If all elements of D are distinct and all nonzero (no divisors of 0), then we find $a1, aa_1, \ldots aa_n$ can contain no 0 elements but must all be distinct as if they weren't, by cancellation laws, $aa_i = aa_j \Rightarrow a_i = a_j$. Thus, this must be some permutation of $0, 1, a_1, \ldots, a_n$ so some a_k must be the multiplicative inverse of 1. $\square$

Corollary. If p is prime, then $\mathbb{Z}_p$ is a field.

Definition 69 (Characteristic of a Ring). The *characteristic of a ring* R is the least positive integer $\min\{n \in \mathbb{Z}^+ \mid n \cdot a = 0 \text{ for all } a \in R\}$. If none exists, the characteris of R is 0.

Example. The ring $\mathbb{Z}_n$ has characteristic n while $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ all have characteristic 0.

Theorem 48. Let R be a unital ring. If $n \cdot 1 \neq 0$ for all $n \in \mathbb{Z}^+$, then R has characteristic 0. But, if $n \cdot 1 = 0$ then the smallest such integer n is the characteristic of R.

Proof. If $n \cdot 1 \neq 0$ for all $n \in \mathbb{Z}^+$, then surely we cannot have $n \cdot a = 0$ for all positive integers n so R has characteristic 0. Otherwise, if $n \cdot 1 = 0$ for some $n \in \mathbb{Z}^+$, then for any $a \in R$, $n \cdot a = a + \cdots + a = a(1 + \cdots + 1) = a(n \cdot 1) = a0 = 0$. □

4.3 Fermat's and Euler's Theorems

Remark. For any field, the nonzero elements form a group under the field multiplication.

Theorem 49. Fermat's Little Theorem If $a \in \mathbb{Z}$ and p is a prime *not* dividing a, then p divides a^{p-1} so $a^{p-1} \equiv 1(\text{mod } p)$ for $a \neq 0(\text{mod} p)$.

Corollary. If $a \in \mathbb{Z}$, then for any prime p, $a^p \equiv a(\text{mod } p)$.

Example. $8^{103} \div 13$ gives $(8^{12})^8(8^7) \equiv (1^8)(8^7) \equiv (-5)^7 \equiv (25)^3(-5) \equiv (-1)^3(-5) \equiv 5(\text{mod } 13)$.

Theorem 50. The set G_n of nonzero elements of $\mathbb{Z}_n$ that are not 0 divisors forms a group under multiplication modulon n.

Proof. For any $a, b \in G_n$, if $ab \notin G_n$ then there would exist some $c \neq 0$ in $\mathbb{Z}_n$ so $(ab)c = 0$. But, this implies $a(bc) = 0$. Because b is not a 0 divisor, $bc \neq 0$ but then $a \notin G_n$. Contradiction, so $ab \in G_n$ so G_n has closure. $1 \in G_n$ obviously and multiplication mod n is associative.

To show the existence of an inverse, we can use a proof by counting. For any $a \in G_n$, given distinct elements of G_n: $1, a_1, \ldots, a_r$, the elements $a1, aa_1, \ldots, aa_r$ must also be distinct as $aa_i = aa_j \Rightarrow a(a_i - a_j) = 0$ but a is not a divisor of 0 so $a_i = a_j$. Because of closure, these products must cover G_n so there exists some a_k so $aa_k = 1$. □

Remark (Euler's Totient/Phi Function $\phi(n)$). $\phi(n)$ is equal to the number of positive intergers less than or equal to n and relatively prime to n. Note $\phi(1) = 1$. This is equal to the number of nonzero elements of $\mathbb{Z}_n$ that are not divisors of 0.

Theorem 51 (Euler's Theorem). If a is an integer relatively prime to n, then $a^{\phi(n)} - 1$ is divisible by n. I.e. $a^{\phi(n)} \equiv 1 (\text{mod } n)$.

Proof. If a is coprime with n then the coset $a + n\mathbb{Z}$ of $n\mathbb{Z}$ contaning a contains an integer $b < n$ also coprime to n. Because multiplication mod n of representatives is well-defined, $a^{\phi(n)} \equiv b^{\phi(n)} (\text{mod } n)$. b can then be viewed as an element of G_n of order $\phi(n)$ consisting of the $\phi(n)$ elements of $\mathbb{Z}_n$ coprime to n so $b^{\phi(n)} \equiv 1 (\text{mod} n)$. □

Theorem 52. Let $m \in \mathbb{Z}^+, a \in \mathbb{Z}_m$ so $\gcd(a, m) = 1$. For each $b \in \mathbb{Z}_m$, the equation $ax = b$ has a unique solution in $\mathbb{Z}_m$.

Proof. a is a unit in $\mathbb{Z}_m$ by the previous theorem so $s = a^{-1}b$ is a solution of this equation. multiplying both sides of $ax = b$ by a^{-1} reveals this indeed is the only solution. □

Theorem 53. Let $m \in \mathbb{Z}^+$ and $a, b \in \mathbb{Z}_m$. Let $d = \gcd(a, m)$. The equation $ax = b$ has a solution in $\mathbb{Z}_m$ iff $d \mid b$. If so, the equation has exactly d solutions in $\mathbb{Z}_m$.

Proof. Suppose $s \in Z_m$ is a solution to $ax = b$. Then $as - b = qm$ for some $q \in \mathbb{Z}$ so $b = as - qm$. d divides a, m so d must also divide the LHS so a solution s only exists if $d \mid b$.

Next, if $d \mid b$, let $a = a_1 d, b = b_1 d, m = m_1 d$ so $as - b = qm$ can be rewritten as $d(a_1 s - b_1) = dqm_1$ so $as - b$ is a multiple of m if and only if $a_1 s - b_1$ is also a multiple of m_1. This yelds the solutions $s \in \mathbb{Z}_m$ of $ax = b$ as precisely $s, s + m_1, s + 2m_1, \ldots, s + (d - 1)m_1$. Thus, d solutions to the equation exist in $\mathbb{Z}_m$. □

Example. Take the congruence $12x \equiv 27 (\text{mod } 18)$. The greatest common divisor of 12 and 18 is 6, but 6 is not a divisor of 27 so no such solutions exist.

For $15x \equiv 27 (\text{mod } 18)$, however, their gcd is 3 which divides 27 so this has 3 solutions, $3 + 18\mathbb{Z}, 9 + 18\mathbb{Z}, 15 + 18\mathbb{Z}$.

4.4 The Field of Quotients of an Integral Domain

Remark. Let's think of the rationals as the formal quotient (a, b) within $D \times D$ for integral domain $D = \mathbb{Z}$.

Definition 70 (Equivalent). Let $S = \{(a, b) \mid a, b \in D, b \neq 0\}$ for an integral domain D. Two elements (a, b) and (c, d) in S are *equivalent*, denoted as $(a, b) \sim (c, d)$ if and only if $ad = bc$.

Lemma 3. The relation $\sim$ on the set S is an equivalence relation. (i) $ab = ba \Rightarrow (a, b) \sim (b, a)$. (ii) $(a, b) \sim (c, d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c, d) \sim (b, a)$. (iii) $(a, b) \sim (c, d), (c, d) \sim (e, f) \Rightarrow ad = bc, cf = ed$ so $acf/e = bc$ so $af = be$ implying $(a, b) \sim (e, f)$. (Note division is simply shorthand for cancellation which is allowed because of integral domain).

Note. This chapter discusses the formation of field F from $D \times D$. Proof shows addition and multiplication well defined and has field axioms and contains D.

Lemma 4. To show that F contains D, we simply construct an isomorphism $i \colon D \to F$ as given by $i(a) = [(a, 1)]$ with a subring of F.

Proof. For any $a, b \in D$, $i(a + b) = [(a + b, 1)] = [(a1 + b1, 1)] = [(a, 1)] + [(b, 1)] = i(a) + i(b)$. Also, $i(ab) = [(ab, 1)] = [(a, 1)][(b, 1)] = i(a)i(b)$. Thus, i is a ring homomorphism. Next, if $i(a) = i(b)$, then $[(a, 1)] = [(b, 1)] \Rightarrow (a, 1) \sim (b, 1) \Rightarrow a1 = 1b \Rightarrow a = b$ so i is injective. Because it is of the same size as D, this is an isomorphism of D with $i[D]$. So $i[D]$ is a subdomain of F. □

Theorem 54 (Field of Quotients of D). Any integral domain D can be enlarged to or embedded in a field F so each element of F can be expressed as a quotient of two elements of D. Here, a field F is a *field of quotients of* D.

Proof. $[(a, b)] = [(a, 1)][(1, b)] = [(a, 1)]/[(b, 1)] = i(a)/i(b)$. □

Theorem 55. Let F be a field of quotients of D and L be any field containing D. Then, there exists a map $\psi \colon F \to L$ which gives an isomorphism of F with a subfield of L so $\psi(a) = a$ for all $a \in D$.

Proof. Proof omitted. □

Corollary. Every field L containing an integral domain D contains a field of quotients of D.

Corollary. Any two fields of quotients of an integral domain D are isomorphic.

4.5 Rings of Polynomials

Note. We call x an *indeterminate* rather than a variable in the ring $\mathbb{Z}[x]$.

Definition 71 (Polynomial $f(x)$ with Coefficients in R)**.** Let R be a ring. A *polynomial* $f(x)$ with coefficients in R is an infinite formal sum $\Sigma_{i=0}^{\infty} a_i x^i$ where $a_u \in R$ and $a_i = 0$ for all but a finite number of values of i. The largest such value of i is the *degree* of $f(x)$ while a_i are the coefficients.

Note. An element of R is a *constant polynomial*.

Theorem 56. The set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication. If R is commutative, then so is $R[x]$ and R has unity $1 \neq 0$ so 1 is also a unity for $R[x]$.

Proof. Clearly $\langle R[x], +\rangle$ is an abelian group. The associative law for multiplication and the distributive laws are clear as well. $\square$

Example. In $\mathbb{Z}_2[x]$, $(x+1)^2 = (x+1)(x+1) = x^2 + (1+1)x + 1 = x^2 + 1$ while $(x+1) + (x+1) = 0x$.

Example. We can even form the ring $(R[x])[y]$, i.e. the ring of polynomials in y with coefficents that are polynomials in x. This is naturally isomorphic to $(R[y])[x]$. Thus, we can denote the ring $R[x, y]$ as the ring of of polynomials in two indeterminates x and y with coefficients in R. In fact the ring $R[x_1, \ldots, x_n]$ of polynomials in n indeterminates x_i with coefficients in R is similarly defined.

Given integral domain D, $D[x]$ is also an integral domain. If F is a field, $F[x]$ is a field but *not* a field as x is not a unit in $F[x]$. However, we can do same goes for $F(x_1, \ldots, x_n)$ or the field of rational functions with n indeterminates over field F.

Theorem 57 (The Evaluation Homomorphisms for Field Theory)**.** Let F be a subfield of a field E, $\alpha \in E$, and x be the indeterminate. Define the map $\phi_\alpha : F \to E$ as $\phi_\alpha(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ is a homomorphism of $F[x]$ into E. Note that $\phi_\alpha(x) = \alpha$ so ϕ_α maps F isomorphically by the identity map such that $\phi_\alpha(a) = a$ for any $a \in F$. This map is the *evaluation homomorphism at* α.

Proof. This map is obviously well-defined. Next, for any $f(x) = a_0 + a_1 x + \cdots + a_n x^n, g(x) = b_0 + b_1 x + \cdots + b_m x^m$, let $h(x) = f(x) + g(x) = c_0 + c_1 x + \cdots + c_r x^r$. Thus, $\phi_\alpha(f(x) + g(x)) = \phi_\alpha(h(x)) = c_0 + c_1\alpha + \cdots + c_r\alpha^r = a_0 + a_1 x + \cdots + a_n x^n + b_0 + b_1 x + \cdots + b_m x^m = \phi_\alpha(f(x)) + \phi_\alpha(g(x))$. Multiplication works similarly by definition of polynomial multiplication $d_j = \Sigma_{i=0}^{j} a_i b_{j-i}$. Thus, ϕ_α is a homomorphism. $\square$

Example. Let $F = \mathbb{Q}, E = \mathbb{R}$ and apply the evaluation homomorphism $\phi_0 \colon Q[x] \to \mathbb{R}$ such that each polynomial is mapped onto its constant term.

Example. Let $F = \mathbb{Q}, E = \mathbb{C}$, we can apply the evaluation homomorphism from $Q[x] \to \mathbb{C}$ at i so $\phi(x^2 + 1) = 0$ so $x^2 + 1$ is in the kernel of ϕ_i.

Remark. A more interesting example uses the same evaluation homomorphism from $\mathbb{Q}[x] \to \mathbb{R}$ but at π. Because π is transcendental, no algebraic solution exists for $a_0 + a^1\pi + \cdots + a_n\pi^n = 0$ as this implies $a_i = 0$ so the kernel of ϕ_π is $\{0\}$ implying it is an injective map and thus ring isomorphic to $\mathbb{Q}[x]$.

Definition 72 (Zero of $f(x)$)**.** Take subfield F of field E, $\alpha \in E$ and let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in F[x]$. Given evaluation homomorphism $\phi_\alpha \colon F[x] \to E$, we say α is a *zero* of $f(x)$ if $f(\alpha) = 0$.

Theorem 58. The polynomial $x^2 - 2$ has no zeroes in the rational numbers. Thus $\sqrt{2} \notin \mathbb{Q}$.

Proof. Take m/n for $m, n \in \mathbb{Z}$ such that $(m/n)^2 = 2$ and we simplify so that $\gcd(m, n) = 1$. Then, $m^2 = 2n^2$ but this implies 2 is a factor of $2n^2$ and therefore must be a factor of m^2 as well. But, if this is the case, m^2 is a multiple of 4 so n^2 must have a multiple of 2 as well. But this implies their greatest common divisor is not 1. Contradiction. □

4.6 Factorization of Polynomials over a Field

Theorem 59 (Division Algorithm for $F[x]$)**.** Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0$ be two elements of $F[x]$ with nonzero $a_n, b_m \in F, m > 0$. Then there exist unique polynomials $q(x), r(x)$ in $F[x]$ so $f(x) = g(x)q(x) + r(x)$ where either $r(x) = 0$ or its degree is less than the degree m of $g(x)$.

Theorem 60 (Factor Theorem)**.** An element $a \in F$ is a zero of $f(x) \in F[x]$ if and only if $x - a$ is a factor of $f(x)$ in $F[x]$.

Proof. $\Rightarrow$: Suppose that $f(a) = 0$ for some $a \in F$. Then, there exists a $q(x), r(x) \in F[x]$ so $f(x) = (x - a)q(x) + r(x)$ where $r(x) = 0$ or the degree of $r(x) < 1$. hus, $r(x)$ must equal c for $c \in F$ such that $f(x) = (x - a)q(x) + c$. Applying the evaluation homomorphism $\phi_a \colon F[x] \to F$, we get $0 = f(a) = 0q(a) + c$ implying $c = 0$. Therefore, $x - a \mid f(x)$.

$\Leftarrow$: If $x - a$ is a factor of $f(x) \in F[x]$, then clearly, $f(x) = (x - a)q(x)$ for $q(x) \in F[x]$ so $f(a) = (0)q(a) = 0$ □

Corollary. A nonzero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in a field F.

Corollary. If G is a finite subgroup of the multiplicative group $(F^*, \cdot)$ for a field F, then G is cyclic.

Proof. If G is a finite abelian group, it must be isomorphic to $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ where each d_i is a power of a prime. Thinking of each $\mathbb{Z}_{d_i}$ as a multiplicative cyclic group, take $m = \text{lcm}(d_1, d_2 \ldots, d_r)$ so $m \leq d_1 d_2 \cdots d_r$. Note that, for any $\alpha \in G, \alpha^m = 1$ so every element of G is a zero of $x^m - 1$. Because G has $d_1 d_2 \cdots d_r$ elements yet $x^m - 1$ has at most m zeros, $m \geq d_1 d_2 \cdots d_r$ so $m = d_1 d_2 \cdots d_r$. Therefore, the primes involved in the prime powers are distinct implying the group G is isomorphic to the cyclic group Z_m. □

Definition 73 (Irreducible Polynomial in $F[x]$). A nonconstant polynomial $f(x) \in F[x]$ is *irreducible over* F if $f(x) = g(x)h(x)$ for $g, h \in F[x]$ both of lower degree than $f(x)$. Otherwise $f(x)$ is *reducible over* F.

Example. Note that $x^2 - 2$ has no zeros in $\mathbb{Q}$ and is therefore not irreducible over $\mathbb{Q}$ but clearly has roots in $\mathbb{R}$ over which it is reducible.

Theorem 61. Let $f(x) \in F[x]$ and let $f(x)$ have degree 2 or 3. Then, it is reducible over F if and only if it has a zero in F.

Proof. If $f(x)$ is reducible and therefore $f(x) = g(x)h(x)$, we can say $g(x)$, WLOG, has degree 1. Thus, $g(x)$ is of the form $x - a$ so $g(a) = 0$ so $f(a) = 0$ implying $f(x)$ indeed must have3 a zero in F. Conversely, if $f(a) = 0$ for some $a \in F$, then $x - a \mid f(x)$ making $f(x)$ reducible. □

Theorem 62. If $f(x) \in \mathbb{Z}[x]$, then $f(x)$ factors into a product of two polynomial of lower degrees $r, s \in \mathbb{Q}[x]$ if and only if it has such a factorization with polynomials of the same degree $r, s \in \mathbb{Z}[x]$.

Corollary. If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with $a_0 \neq 0$, and if $f(x)$ has a zero in $\mathbb{Q}$, then it has a zero m in $\mathbb{Z}$ and m must divide a_0.

Theorem 63 (Einstein Criterion). Let $p \in \mathbb{Z}$ be a prime. Suppose $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ and $a_n \neq 0 (\text{mod } p)$, but $a_i = 0 (\text{mod } p)$ for all $i < n$, with $a_0 \neq 0 (\text{mod } p^2)$. Then $f(x)$ is irreducible over $\mathbb{Q}$.

Proof. We need only show $f(x)$ does not factor into polynomials of lower degree in $\mathbb{Z}[x]$. If $f(x) = (b_r x^r + \cdots + b_0)(c_s x^s + \cdots + c_0)$ is such a factorization with $b_r, c_s \neq 0$ and $r, s < n$, then $a_0 \neq 0 (\text{mod } p^2)$ implies b_0, c_0 are not both congruent to 0 mod p. Supposing $b_0 \neq 0 (\text{mod } p)$ but

$c_0 = 0(\text{mod } p)$. This then implies, because $a_n \neq 0(\text{mod } p)$, that $b_r, c_s \neq 0(\text{mod } p)$. Because $a_n = b_r c_s$, if m is the smallest value of K so $c_k \neq 0(\text{mod } p)$, then $a_m = b_0 + b_1 c_{m-1} + \cdots + \begin{cases} b_m c_0 & \text{if } r \geq m \\ b_r c_{m-r} & \text{if } r < m \end{cases}$. The fact neither b_0 nor c_m are congruent to 0 modulo p while $c_{m-1}, \cdots, c_0$ are all congruent to 0 modulo p implies that $a_m \neq 0(\text{mod } p)$ so $m = n$. Hence, $s = n$ so s is not less than n against our asssumption meaning this factorization was nontrivial. □

Corollary (p^{th} Cyclotomic Polynomial)**.** The polynomial $\Phi_p(x) = \frac{x^p-1}{x-1} = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$ for any prime p. $\Phi_p(x)$ is the p^{th} *cyclotomic polynomial.*

Theorem 64. Let $p(x)$ be an irreducible polynomial in $F[x]$ If $p(x)$ divides $r(x)s(x)$ for $r, s \in F[x]$, then either $p(x)$ divides $r(x)$ or $s(x)$

Theorem 65. If F is a field, then every nonconstant polynomial $f(x) \in F[x]$ can be factored in $F[x]$ into a product of irreducible polynomials which are unique except for order and for unit (nonzero constant) factors in F.

Chapter 5

Ideals and Factor Rings

5.1 Homomorphisms and Factor Rings

Theorem 66. Let ϕ be a homomorphism of a ring R into a ring R'. These qualities follow: (a) If 0 is the additive identity in R, then $\phi(0) = 0'$ is the additive identity in R'. (b) If $a \in R$, then $\phi(-a) = -\phi(a)$. (c) If S is a subring of R, then $\phi[S]$ is a subgring of R'. (d) If S' is a subring of R' then $\phi^{-1}[S']$ is a subring of R. Finally, if R has unity 1, then $\phi(1)$ is the unity for $\phi[R]$.

Theorem 67. For ring homomorphism $\phi\colon R \to R'$ with kernel H, if $a \in R$, then $\phi^{-1}[\phi(a)] = a + H = H + a$ where $a + H = H + a$ is the coset containing a of the commutative additive group $\langle H, +\rangle$.

Remark. Ring homomorphism $\phi\colon R \to R'$ is injective iff $\ker(\phi) = \{0\}$.

Theorem 68. Given ring homomorphism $\phi\colon R \to R'$ with kernel H, the additive cosets of H form a ring R/H which have addition and multiplication defined with

$$(a + H) + (b + H) = (a + b) + H, \quad (a + H)(b + H) = (ab) + H.$$

Also the map $\mu\colon R/H \to \phi[R]$ defined via $\mu(a + H) = \phi(a)$ is an isomorphism.

Proof. Addition of cosets is well-defined from group theory. For multiplication, given $h_1, h_2 \in H$, $a + h_1 \in a + H, b + h_2 \in b + H$ say $c = (a + h_1)(b + h_2) = ab + ah_2 + h1_b + h_1h_2$. c will lie in $ab + H$ if $\phi(c) = \phi(ab)$ where $ab + H = \phi^{-1}[\phi(ab)]$. Because $\phi(h) = 0'$ for $h \in H$, we get $\phi(c) = \phi(ab) + \phi(ah_2) + \phi(h_1b) + \phi(h_1h_2) = \phi(ab)$, making multiplication well-defined.

We are left to show R/H is a ring. This requires associative property for multiplication and the distributive laws which follow from the representatives of R. An earlier theorem then shows μ is well-defined and bijective onto $\phi[R]$ and satisfies the multiplicative property of a homomorphism. Multiplicatively, $\mu[(a+H)(b+H)] = \mu(ab+H) = \phi(a)\phi(b) = \mu(a+H)\mu(b+H)$. So μ is an isomorphism. □

Theorem 69. Given subring H of ring R, multiplication of additive cosets of H is well-defined ($(a+H)(b+H) = ab+H$) if and only if $ah \in H$ and $hb \in H$ for all $a, b \in R, h \in H$.

Definition 74 (Ideal). An additive subgroup N of a ring R for which $aN \subseteq N$ and $Nb \subseteq N$ for all $a, b \in R$ is an *ideal*.

Example. $n\mathbb{Z}$ is an ideal for the ring $\mathbb{Z}$.

Corollary. Let N be an ideal of ring R. Then the additive cosets of N form a ring R/N with binary operations $(a+N)+(b+N) = (a+b)+N$ and $(a+N)(b+N) = ab+N$.

Definition 75 (Factor Ring). The ring R/N is the *factor ring*, or *quotient ring of R by N*.

Theorem 70 (Fundamental Homomorphsim Theorem). Given ring homomorphism $\phi\colon R \to R'$ with kernel N, $\phi[R]$ is a ring and the map $\mu\colon R/N \to \phi[R]$ given by $\mu(x+N) = \phi(x)$ is an isomorphism. Moreover, if $\gamma\colon R \to R/N$ is the homomorphism given by $\gamma(x) = x+N$, then for all $x \in R$, $\phi(x) = \mu\gamma(x)$.

Proof. This follows from previous theorems. □

Example. As an example, take ideal $n\mathbb{Z}$ of $\mathbb{Z}$ so we can take the factor ring $\mathbb{Z}/n\mathbb{Z}$. We therefore have the ring homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Z}_n$ where $\phi(m)$ is the remainder of m mod n such that $\ker(\phi) = n\mathbb{Z}$. This implies $\mu\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}_n$ where $\mu(m+n\mathbb{Z})$ is the remainder of m mod n is well-defined and an isomorphism.

Remark. An ideal in ring theory is analogous to a normal subgroup in group theory. Both structures allow us to form a factor structure like R/N which give rise to a certain homomorphism.

Similarly, $\phi[N]$ is an ideal of $\phi[R]$ though not necessarily of R' and if N' is an ideal of either $\phi[R]$ or R' then $\phi^{-1}[N']$ is indeed an ideal of R.

5.2 Prime and Maximal Ideals

Example. Take the following examples:

(a) The ring $\mathbb{Z}_p$ is a field for prime p implying a factor ring $(\mathbb{Z}/p\mathbb{Z})$ of an integral domain may be a field.

(b) While $\mathbb{Z} \times \mathbb{Z}$ is not an integral doman as $(0,1)(1,0) = (0,0)$, $N = \{(0,n) \mid n \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z} \times \mathbb{Z}$ where $(\mathbb{Z} \times \mathbb{Z})/N$ is isomorphic to $\mathbb{Z}$. This implies a factor ring of a ring may be an integral domain even though the original ring isn't.

(c) The subset $N = \{0, 3\} \subset \mathbb{Z}_6$ is an ideal and has factor ring of 3 elements. Thus, even if R is not an integral domain, R/N can still be a field.

(d) Finally, $\mathbb{Z}$ is an integral domain but $\mathbb{Z}/6\mathbb{Z}$ isn't so a factor ring isn't necessarily 'better.'

Remark (Improper/Trivial Ideals)**.** Every nonzero ring has the *improper ideal* R itself and the trivial ideal $\{0\}$. These have factor rings isomorphic to $\{0\}$ and R itself.

Theorem 71. Given unital ring R, if its ideal N contains a unit, then $N = R$.

Proof. With unit $u \in N$, the condition $rN \subseteq N$ for all $r \in R$ so taking $r = u^{-1}$ implies $1 = u^{-1}u \in N$ meaning $rN \subseteq N$ for all $r \in R$ so $N = R$. $\square$

Corollary. A field contains no proper nontrivial ideals.

Definition 76. A *maximal ideal of a ring* R is an ideal M different from R such that there is no proper ideal N of R properly containing M.

Theorem 72. Given unital commutative ring R, M is a maximal ideal of R if and only if R/M is a field.

Proof. $\Rightarrow$: Suppose M is a maximal ideal of R. If R is commutative with unity, then R/M is also a nonzero commutative ring with unity. Now, we must show every nonzero element is a unit. Since $M \neq R$ because M maximal, say $(a + M) \in R/M$ with $a \notin M$ so $a + M$ is not the additive identity element of R/M. If $a + M$ has no multiplicative inverse, then the set $(R/M)(a+M)$ does not contain $1+M$. It's then clear, $R/M(a+M)$ is an ideal of R/M. It's nontrivial because $a \notin M$ and proper because it doesn't contain $1+M$. Thus, if $\gamma\colon R \to R/M$ is the canonical homomorphism, then $\gamma^{-1}[(R/M)(a + M)]$ is a proper ideal of R properly containing M making M not the maximal ideal so $a+M$ must indeed have a multiplicative inverse in R/M, making R/M a field.

$\Leftarrow$: Conversely, if R/M is a field and N is an ideal of R, then $M \subset N \subset R$ by canonical homomorphism γ of R onto R/M. This implies $\gamma[N]$ is an ideal of R/M not equal to R/M but larger than than $\{0 + M\}$. But this contradicts the earlier corollary that R/M contains no proper nontrivial ideals so if R/M is a field, then M must be maximal. ☐

Example. Since $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$ and $\mathbb{Z}_n$ is a field iff n is prime, the maximal ideals of $\mathbb{Z}$ are precisely the ideals $p\mathbb{Z}$ for prime p.

Corollary. A commutative unital ring if a field iff it has no proper nontrivial ideals.

Proof. The earlier corollary shows a field has no proper nontrivial ideals. Conversely, if a commutative ring R with unity has no proper nontrivial ideals, then $\{0\}$ is a maximal ideal and $R/\{0\}$ isomorphic to R must be a field. ☐

Remark. The factor ring R/N will be an integral domain if and only if $(a + N)(b - N) = N$ implies $a + N = N$ or $b + N = N$, i.e. R/N has no divisors of 0. This condition amounts to saying $ab \in N \Rightarrow a \in N \vee b \in N$.

Definition 77 (Prime Ideal)**.** An ideal $N \neq R$ in a commutative ring R is a *prime ideal* if $ab \in N$ implies either $a \in N$ or $b \in N$ for $a, b \in R$. Note $\{0\}$ is a prime ideal in any integral domain.

Theorem 73. Let R be a commutative unital ring so $N \neq R$ is an ideal in R. Then R/N is an integral domain if and only if N is a prime ideal in R.

Corollary. Every maximal ideal in a commutative ring R with unity is a prime ideal.

Remark. We can summarize the above with the following: for a commutative unital ring R:

1. An ideal M or R is maximal iff R/M is a field.
2. An ideal N of R is prime iff R/N is an integral domain.
3. Every maximal ideal of R is a prime ideal.

Theorem 74. If R is a ring with unity 1, then there exists a homomorphism $\phi\colon \mathbb{Z} \to R$ given by $\phi(n) = n \cdot 1$ for $n \in \mathbb{Z}$.

Proof. $\phi(n + m) = (n + m) \cdot 1 = (n \cdot 1) + (m \cdot 1) = \phi(n) + \phi(m)$. Next,

$\phi(nm) = (nm) \cdot 1 = (n \cdot 1)(m \cdot 1) = \phi(n)\phi(m)$. □

Corollary. If R is a unital ring with characteristic $n > 1$, then R contains a subring isomorphic to $\mathbb{Z}_n$. If R has characteristic 0, then R contains a subring isomorphic to $\mathbb{Z}$.

Proof. The homomorphism $\phi\colon \mathbb{Z} \to R$ given by $\phi(m) = m \cdot 1$ for $m \in \mathbb{Z}$ has kernel of form $s\mathbb{Z}$ ideal in $\mathbb{Z}$ for some $s \in \mathbb{Z}$. If R has characteristic $n > 0$, then the kernel of ϕ is $n\mathbb{Z}$ with image $\phi[\mathbb{Z}] \leq R$ isomorphic to $\mathbb{Z}/n\mathbb{Z} \sim \mathbb{Z}_n$. If R has characteristic 0, then $m \cdot 1 \neq 0$ for all $m \neq 0$ so the kernel of ϕ is just $\{0\}$ implying the image of $\phi[\mathbb{Z}] \leq R$ is isomorphic to $\mathbb{Z}$. □

Theorem 75. A field F is either of prime characteristic p and contains a subfield isomorphic to $\mathbb{Z}_p$ or of characteristic 0 and contains a subfield isomorphic to $\mathbb{Q}$.

Proof. If the characteristic is F is not 0, then the above corollary shows F contains a subring isomorphic to $\mathbb{Z}_n$. hus, n must be a prime p or else F must contain a subring isomorphic to $\mathbb{Z}$ in which case F must contain a field of quotients which must be isomorphic to $\mathbb{Q}$. □

Definition 78 (Prime Fields)**.** The fields $\mathbb{Z}_p, \mathbb{Q}$ are *prime fields*.

Definition 79 (Principal Ideal)**.** If R is a commutative unital ring and $a \in R$, the ideal $\{ra \mid r \in R\}$ of all multiples of a is the *principal ideal generated by a* denoted by $\langle a \rangle$. An ideal N of R is a *principal ideal* if $N = \langle a \rangle$ for some $a \in R$.

Example. Every ideal of the ring $\mathbb{Z}$ is of the form $n\mathbb{Z}$ generated by N so every ideal of $\mathbb{Z}$ is a principal ideal.

Example. The ideal $\langle x \rangle$ in $F[x]$ consists of all polynomials in $F[x]$ with zero constant terms.

Theorem 76. If F is a field, then every ideal in $F[x]$ is *principal*.

Proof. For ideal N of $F[x]$, if $N = \{0\}$, then $N = \langle 0 \rangle$. Otherwise, say $g(x)$ is a nonzero element of N of minimal degree. If the degree of $g(x)$ is 0, then $g(x) \in F$ and is a unit so $N = F[x] = \langle 1 \rangle$ so N is principal. If the degree of $g(x) \geq 1$, say $f(x) \in N$ such that $f(x) = g(x)q(x) + r(x)$ where the degree of $r(x)$ is either 0 or less than that of $g(x)$. Thus, $f(x), g(x) \in N$ imply $f(x) - g(x)q(x) = r(x) \in N$ by definition of an ideal such that $g(x)$ is a nonzero element of minimal degree in N so $r(x) = 0$ and finally $f(x) = g(x)q(x)$ so $N = \langle g(x) \rangle$. □

Theorem 77. An ideal $\langle p(x)\rangle \neq \{0\}$ of $F[x]$ is maximal iff $p(x)$ is irreducible over F.

Proof. $\Rightarrow$: Suppose $\langle p(x)\rangle \neq \{0\}$ is a maximal ideal of $F[x]$. Then $\langle p(x)\rangle \neq F[x]$ so $p(x) \notin F$. Thus, if $p(x) = f(x)g(x)$, because $\langle p(x)\rangle$ is a maximal ideal and hence also a prime ideal, $(f(x)g(x)) \in \langle p(x)\rangle$ implies either $f(x)$ or $g(x) \in \langle p(x)\rangle$ so either $f(x)$ or $g(x)$ have $p(x)$ as a factor. But, the degrees of both $f(x), g(x)$ cannot be less than the degree of $p(x)$ implying $p(x)$ is irreducible over F.

$\Leftarrow$: Conversely, if $p(x)$ is irreducible over F, suppose N is an ideal such that $\langle p(x)\rangle \subseteq N \subseteq F[x]$. If N is a principal ideal, then $N = \langle g(x)\rangle$ for some $g(x) \in N$. Therefore, $p(x) \in N$ implies $p(x) = g(x)q(x)$ for some $q(x) \in F[x]$. But, $p(x)$ is irreducible so either $g(x), q(x)$ are of degree 0. If $g(x)$ is of degree 0, then it's a nonzero constant and consequently a unit in $F[x]$ so $\langle g(x)\rangle = N = F[x]$. If $q(x)$ is of degree 0, then $q(x) = c \in F$ so $g(x) = (1/c)p(x)$ is in $\langle p(x)\rangle$ meaning $N = \langle p(x)\rangle$ is maximal. □

Example. x^3+3x^2+2 is irreducible in $\mathbb{Z}_5[x]$ and therefore $\mathbb{Z}_5[x]/\langle x^3+3x+2\rangle$ is a field. Similarly, x^2-2 irreducible in $\mathbb{Q}[x]$ so $\mathbb{Q}[x]/\langle x^2-2\rangle$ is a field.

Theorem 78. Let $p(x)$ be an irreducible polynomial in $F[x]$. If $p(x)$ divides $r(x)s(x)$ for $r(x), s(x) \in F[x]$, then either $p(x)$ divides $r(x)$ or $s(x)$.

Chapter 6

Extension Fields

6.1 Introduction to Extension Fields

Definition 80 (Extension Field)**.** A field E is an *extension field of a field* F if $F \leq E$. For instance, we can write a *tower of fields* as $\mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$ and $F \leq F(x), F(y) \leq F(x,y)$.

Theorem 79 (Kronecker's Theorem)**.** Let F be a field and $f(x)$ be some nonconstant polynomial in $F[x]$. Then, there exists some extension field E of F and an $\alpha \in E$ where $f(\alpha) = 0$.

Proof. By a prior theorem, $f(x)$ has some factorization in $F[x]$ into irreducible polynomials over F. Say $p(x)$ is one such irreducible polynomial. It is sufficient to find an extension field E of F containing an element α so $p(\alpha) = 0$. By an earlier theorem, $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ implying $F[x]/\langle p(x) \rangle$ is a field. We can naturally define $\psi \colon F \to F[x]/\langle p(x) \rangle$ where $\psi(a) = a + \langle p(x) \rangle$ for $a \in F$. This is injective as $a + \langle p(x) \rangle = b + \langle p(x) \rangle, a, b \in F$ implies $(a - b) \in \langle p(x) \rangle$ so $a - b$ is a multiple of $p(x)$ of degree ≥ 1 so $a - b = 0$ so $a = b$. ψ is easily a homomorphism which maps onto a subfield of $F[x]/\langle p(x) \rangle$. We can thus identitfy F with $\{a + \langle p(x) \rangle \mid a \in F\}$ so $E = F[x]/\langle p(x) \rangle$ is an extension field of F.

We're left to show E has some zero of $p(x)$ which we can do via $\alpha = x + \langle p(x) \rangle, \alpha \in E$ so $\phi_\alpha \colon F[x] \to E$ by a previous theorem gives $p(x) = a_0 + a_1 x + \cdots + a_n x^n, a_i \in F$ so $\phi_\alpha(p(x)) = a_0 + a_1(x + \langle p(x) \rangle) + \cdots + a_n(x + \langle p(x) \rangle)^n$ in E. But, we can compute via representatives and x is a representative so $p(\alpha) = p(x) + \langle p(x) \rangle = \langle p(x) \rangle = 0$ so there exists some $\alpha \in E$ such that $p(\alpha) = 0$ and therefore $f(\alpha) = 0$. ☐

Example. Let $F = \mathbb{R}$ and $f(x) = x^2 + 1$ which is clearly irreducible over $\mathbb{R}$ such that $\langle x^2 + 1 \rangle$ is a maximal ideal in $\mathbb{R}[x]$ so $\mathbb{R}[x]/\langle x^2 + 1 \rangle$ is a field. Identifying $r \in \mathbb{R}$ with $r + \langle x^2 + 1 \rangle$ lets us view $\mathbb{R}$ as a subfield of

$\mathbb{R}[x]/\langle x^2+1\rangle$. Now, $\alpha = x + \langle x^2+1\rangle$ so $\alpha^2+1 = (x+\langle x^2+1\rangle)^2 + (1+\langle x^2+1\rangle) = (x^2+1) + \langle x^2+1\rangle = 0$ so α is a zero of x^2+1.

Definition 81 (Algebraic + Transcendental)**.** An element α of an extension field E of a field F is *algebraic over* F if $f(\alpha) = 0$ for some nonzero $f(x) \in F[x]$. If α isn't, then it is *transcendental over* F.

Example. $\sqrt{2}$ is an algebraic number over $\mathbb{Q}$ because it is a zero of x^2-2 while i is also an algebraic element over $\mathbb{Q}$ because it is a zero of x^2+1 inside extension field $\mathbb{C}$.

Example. The real number π is transcendental over $\mathbb{Q}$ however π is algebraic over $\mathbb{R}$ as it a zero of $(x-\pi) \in \mathbb{R}[x]$.

Theorem 80. Given extension field E of field F and $\alpha \in E$, let $\phi_\alpha\colon F[x] \to E$ be the evaluation homomorphism so $\phi_\alpha(a) = a$ for $a \in F$ and $\phi_\alpha(x) = \alpha$. Thus, α is transecendental over F iff ϕ_α gives an isomorphism of $F[x]$ with a subdomain of E, that is iff ϕ_α injective.

Proof. The element α is transcendental over F if and only if $f(\alpha) \neq 0$ for all nonzero $f(x) \in F[x]$ which is true iff (by definition), $\phi_\alpha(f(x)) \neq 0$ for all nonzero $f(x)$ which is true iff $\ker \phi_\alpha = \{0\}$ iff ϕ_α is injective. $\square$

Theorem 81. Let E be an extenson field of F with $\alpha \in E$ algebraic over F. Then, there is an irreducible polynomial $p(x) \in F[x]$ so $p(\alpha) = 0$. This polynomial is uniquely determined up to a constant factor and is a polynomial of minimal degree ≥ 1 having α as a zero. If $f(\alpha) = 0$ for some $f(x) \in F[x]$ for $f(x) \neq 0$, then $p(x) \mid f(x)$.

Proof. Given evaluation homomorphism ϕ_α of $F[x]$ into E, its kernel is an ideal and by a previous theorem, must be a principal ideal generated by some $p(x) \in F[x]$ implying $\langle p(x)\rangle$ consists precisely of those elements of $F[x]$ having α as a zero. So, if some $f(x) \neq 0$ and $f(\alpha) = 0$, then $f(x) \in \langle p(x)\rangle$ so $p(x) \mid f(x)$ making $p(x)$ a polynomial of minimal degree ≥ 1 with zero α and any other polynomial of the same degree of form $(a)p(x), a \in F$. Now, to show $p(x)$ is irreducible, if $p(x) = r(x)s(x)$ were a possible factorization into polynomials of lower degree, then $p(\alpha)$ implies either $r(\alpha)$ or $s(\alpha)$ is 0 contradicting the fact $p(x)$ is of minimal degree ≥ 1 with $p(\alpha) = 0$. So $p(x)$ is irreducible. $\square$

Definition 82 (Monic Polynomial)**.** A *monic polynomial* is one with leading coefficient 1.

Definition 83 (Irreducible Polynomial for α over F)**.** Given extension field

E of F with $\alpha \in E$ algebraic over F, the unique monic polynomial $p(x)$ is the *irreducible polynomial for* α *over* F, denoted $\operatorname{irr}(\alpha, F)$ with degree of α over F denoted $\deg(\alpha, F)$.

Example. $\operatorname{irr}(\sqrt{2}, \mathbb{Q}) = x^2 - 2$ is degree 2 of α over $\mathbb{Q}$.

Remark. With extension field E of a field F and $\alpha \in E$ and evaluation homomorphism $\phi_\alpha(a) = a$ for $a \in F$ and $\phi_\alpha(x) = \alpha$, there are two possible cases:

Case I If α is *algebraic over* F, then the kernel of ϕ_α is $\langle \operatorname{irr}(\alpha, F) \rangle$ and therefore a maximal ideal of $F[x]$. Also, $F[x]/\langle \operatorname{irr}(\alpha, F) \rangle$ is a field and isomorphic to the image $\phi_\alpha[F[x]]$ in E, making $\phi_\alpha[F[x]]$ of E the smallest subfield of E containg F and α, denoted $F(\alpha)$.

Case II If α is *algebraic over* F, then ϕ_α gives an isomorphism of $F[x]$ with a subdomain of E. Thus, $\phi_\alpha[F[x]]$ is not a field, but instead an integral domain denoted by $F[\alpha]$. Consequently, E contains a field of quotients of $F[\alpha]$ which is the smallest subfield of E containing F and α which we denote $F(\alpha)$.

Remark. Since π is transcendental over $\mathbb{Q}$, the field $\mathbb{Q}(\pi)$ is isomorphic to the field $\mathbb{Q}(x)$ of rational functions over $\mathbb{Q}[x]$.

Definition 84 (Simple Extension)**.** An extension field E of a field F is a *simple extension* of F if $E = F(\alpha)$ for some $\alpha \in E$.

Theorem 82. With simple extension $E = F(\alpha)$ of a field F and α algebraic over F, if the degree of $\operatorname{irr}(\alpha, F) \geq 1$, then every element β of E can be uniquely expressed as $\beta = \Sigma_{i=0}^{n-1} b_i \alpha^{n-1}$ for $b_i \in F$.

Proof. For the usual evaluation homomorphism ϕ_α, each element $F(\alpha) = \phi_\alpha[F[x]]$ is of the form $\phi_\alpha(f(x)) = f(\alpha)$, a formal polynomial in α with coefficients in F so $\operatorname{irr}(\alpha, F) = p(x) = x^n + \cdots + a_0$ so $p(\alpha) = 0$ means $\alpha^n = -a_{n-1}\alpha^{n-1} - \cdots - a_0$. This allows us to express any monomial $\alpha^m, m \geq n$ in terms of powers of α less than n, i.e. $\alpha^{n+1} = \alpha\alpha^n$. Thus, if $\beta \in F(\alpha)$, $\beta = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}$. For uniqueness, $b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1} = b_0' + b_1'\alpha + \cdots + b_{n-1}'\alpha^{n-1}$ for $b_i, b_i' \in F$ implies $g(x) = (b_0 - b_0') + (b_1 - b_1')x + \cdots + (b_{n-1} - b_{n-1}')x^{n-1} \in F[x]$ and $g(\alpha) = 0$. Also, the degree is less than the degree of $\operatorname{irr}(\alpha, F)$ so because $\operatorname{irr}(\alpha, F)$ is a nonzero polynomial of minimal degree with α as a zero, we must have $g(x) = 0$ so $b_i = b_i'$ proving uniqueness. $\square$

Example. The polynomial $p(x) = x^2 + x + 1$ in $\mathbb{Z}_2[x]$ is irreducible over

$\mathbb{Z}_2$ since neither 0 nor 1 is a zero however we know there is an extension field E containg a zero α of x^2+x+1. Specifically, $\mathbb{Z}_2(\alpha)$ has elements $0+0\alpha, 1+0\alpha, 0+1\alpha, 1+1\alpha$ giving us a new finite field of 4 elements. This gives us $(1+\alpha)^2 = \alpha$ because $\alpha^2 = \alpha+1$. Thus, $\mathbb{R}[x]/\langle x^2+1\rangle$ is isomorphic to the field $\mathbb{C}$ because we can view $\mathbb{R}[x]/\langle x^2+1\rangle$ as an extension field of $\mathbb{R} = \mathbb{R}(\alpha)$. Because $\alpha^2+1=0$ for some, we see α plays the role if $i \in \mathbb{C}$ and $a+b\alpha$ plays the role of $a+bi$ making $\mathbb{R}(\alpha) \sim \mathbb{C}$.

6.2 Vector Spaces

Definition 85 (Vector Space Over F)**.** Given a field F, a *vector space over* F ior F*-vector space* is an abelian group V under addition together with scalar multiplication of each element of F on the left such that, for all $a, b \in F$ and $\alpha, \beta \in V$, the following are satisfied:

($\mathscr{V}_1$) $a\alpha \in V$.

($\mathscr{V}_2$) $a(b\alpha) = (ab)\alpha$.

($\mathscr{V}_3$) $(a+b)\alpha = (a\alpha) + (b\alpha)$.

($\mathscr{V}_4$) $a(\alpha+\beta) = (a\alpha + a\beta)$.

($\mathscr{V}_5$) $1\alpha = \alpha$.

We call the elements of V *vectors* and the elements of F *scalars*. When only discussing one field F, we simply say *vector space*.

Theorem 83. If V is a vector space over F, then $0\alpha = 0, a0 = 0$, and $(-a)\alpha = a(-\alpha) = -(a\alpha)$ for all $a \in F, \alpha \in V$.

Proof. Proofs identical to before, i.e. $(0\alpha) = (0+0)\alpha = 0\alpha + 0\alpha$ and $a0 = a(0+0) = a0 + a0$ and $0 = a0 = a(\alpha + (-\alpha)) = a\alpha + a(-\alpha)$. □

Definition 86 (Span)**.** Given V vector space over F, the vectors in a subset $S = \{\alpha_i \mid i \in I\}$ of V *span* V if for every $\beta \in V$, $\beta = a_1\alpha_{i_1} + \cdots + a_n\alpha_{i_n}$ for $a_j \in F$, $a_{i_j} \in S$. This is called a *linear combination of* ai_j.

Definition 87 (Finite Dimensional)**.** A vector space V over a field F is *finite dimensional* if there is a *finite* subset of V whose vectors span V.

Example. $F[x]$ over F is not finite dimensional because polynomials of arbitrarily large degree cannot be linear combinations of elements of any finite set of polynomials.

Definition 88 (Linearly Independent)**.** We say a set of vectors in a subset $S = \{a_i \mid i \in I\}$ of a vector space V over a field F are *linearly independent over* F if any linear combination of them is 0 iff each coefficient is 0.

Otherwise, they are *linearly dependent over F*.

Definition 89 (Basis)**.** For a vector space V over a field F, the vectors in a subset $B = \{\beta_i \mid i \in I\}$ of V form a *basis for V over F* if they span V and are linearly independent.

Theorem 84. In a finite-dimensional vector space, every finite set of vectors spanning the space contains a subset that is a basis.

Corollary. A finite-dimensional vector space has a finite basis.

Theorem 85. For a finite set $S = \{\alpha_1, \cdots, \alpha_r\}$ of linearly independent vectors of a finite-dimensional vector space V over a field F, S can be extended to a basis for V over F.

Corollary. Any two bases of a finite-dimensional vector space V over F have the same number of elements.

Definition 90 (Dimension of V Over F)**.** If V is a finite-dimensional vector space over a field F, the number of elements in a basis is the *dimension of V over F*.

Example. Given an extension field E of a field F and $\alpha \in E$, if α is algebraic over F and $\deg(\alpha, F) = n$, then the dimension of $F(\alpha)$ as a vector space over F is n.

Theorem 86. For an extension field E of F with $\alpha \in E$ algebraic over F, if $\deg(\alpha, F) = n$, then $F(\alpha)$ is an n-dimensional vector space over F with basis $\{1, \alpha, \cdots, \alpha^{n-1}\}$. Furthermore, every element β of $F(\alpha)$ is algebraic over F and $\deg(\beta, F) \leq \deg(\alpha, F)$.

6.3 Algebraic Extensions

Definition 91 (Algebraic Extension of F)**.** An extension field E of a field F is an *algebraic extension of F* if every element in E is algebraic over F.

Definition 92 (Finite Extension of Degree n Over F)**.** If an extension field E of a field F is of finite dimension n as a vector space over F, then E is a *finite extension of degree n over F* denoted $[E : F]$.

Theorem 87. A finite extension field E of a field F is an algebrqaic extension of F.

Proof. If $[E : F] = n$, then $1, \alpha, \ldots, \alpha^n$ cannot be linearly independent elements for $\alpha \in E$. Thus, there exists some nonzero solution to their linear combination implying α is algebraic over F for any such α. $\square$

Theorem 88. If E is a finite extension field of a field F and K is a finite extension field of E, then K is a finite extension field of F. And, in fact,

$$[K : F] = [K : E][E : F].$$

Corollary. If F_i is a field and F_{i+1} is a finite extension of F_i, then F_r is a finite extension of F_1 for $i = 1, \ldots, r$ and $[F_r : F_1] = [F_r : F_{r-1}] \cdots [F_2 : F_1]$.

Corollary. If E is an extension field of F and $\alpha \in E$ is algebraic over F, and $\beta \in F(\alpha)$ then $\deg(\beta, F)$ divides $\deg(\alpha, F)$.

Definition 93 (Adjoining to F). We obtain the field $F(\alpha_1, \ldots, \alpha_n)$ from the field F by adjoining to F the elements $\alpha_i \in E$ for extension field E of field F. This is the smallest field containing all α_i for $i = 1, \ldots, n$.

Example. $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. Note that $x^4 - 10x^2 + 1$ has root $\sqrt{2} + \sqrt{3}$. This is irreducible in $\mathbb{Q}[x]$ so $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$ but $(\sqrt{2} + \sqrt{3}) \notin \mathbb{Q}(\sqrt{2})$ so $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ so $\{1, \sqrt{3}\}$ is a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{2})$. Thus, by the previous theoreom, $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.

Example. $2^{1/2} \notin \mathbb{Q}(2^{1/3})$ because $\deg(2^{1/2}, \mathbb{Q}) = 2$ and 2 is not a divisor of $3 = \deg(2^{1/3}, \mathbb{Q})$.

Theorem 89. If E is an algebraic extension of a field F, then there exists a finite number of elements $\alpha_1, \ldots \alpha_n \in E$ so $E = F(\alpha_1, \ldots, \alpha_n)$ if and only if E is a finite-dimensional vector space over F if and only if E is a finite extension of F.

Proof. Suppose $E = F(\alpha_1, \ldots \alpha_n)$. Thus, E is an algebraic extension of F and each α_i is algebraic over F and thus over any extension field of F in E implying $F(\alpha_1)$ is algebraic over F and in general $F(\alpha_1, \ldots, \alpha_j)$ is algebraic over $F(\alpha_1, \ldots, \alpha_{j-1})$. $F(\alpha_1, \ldots, F_n)$ shows E is a finite extension of F. Conversely, if E is a finite algebraic extension of F, then if $[E : F] = 1$, $E = F(1) = F$. If $E \neq F$, then $\alpha_1 \in E - F$ so $[F(\alpha_1) : F] > 1$. If $F(\alpha_1) = E$ we are done; otherwise, $\alpha_2 \in E - F(\alpha_1)$. Continuiing this process, because $[E : F]$ is finite, we must arrive at some α_n such that $F(\alpha_1, \ldots, \alpha_n) = E$. $\square$

Remark. We have not yet shown that if E is an extension field of a field F and $\alpha, \beta \in E$ are algebraic over F, then so are $\alpha+\beta, \alpha\beta, \alpha-\beta, \alpha/\beta(\beta \neq 0)$.

Definition 94 (Algebraic Closure of F in E)**.** For extension field E of F, $\bar{F_E} = \{\alpha \in E \mid \alpha \text{ is algebraic over } F\}$ is a subfield of E and called the *algebraic closure of F in E.*

Proof. Take $\alpha, \beta \in \bar{F_E}$. By the last theorem, $F(\alpha, \beta)$ is a finite extension of F so every element $F(\alpha, \beta)$ is algebraic over F and therefore $F(\alpha, \beta) \subseteq \bar{F_E}$ so $\bar{F_E}$ is a subfield of E containing $\alpha + \beta$, etc. $\square$

Corollary. The set of all algebraic numbers forms a field.

Definition 95 (Algebraically Closed)**.** A field F is *algebraically closed* if every nonconstant polynomial in $F[x]$ has a zero in F.

Theorem 90. A field F is algebraically closed if and only if every nonconstant polynomial in $F[x]$ factors into linear factors in $F[x]$

Proof. If F is algebraically closed, let $f(x)$ be a nonstnt polynomial in $F[x]$ with a zero $a \in F$. Therefore, $x - a$ is a factor so $f(x) = (x - a)g(x)$. If $g(x)$ is nonconstant, it also must have a zero $b \in F$ and we have $f(x) = (x - a)(x - b)h(x)$. We can continue this to get linear factors of $f(x)$.

Conversely, if every nonconstant polynomial of $F[x]$ has a factorization into linear factors, then if $ax - b$ is a linear factor of $f(x)$ then b/a is a zer of $f(x)$ so F is algebraically closed. $\square$

Corollary. An algebraically closed field F has no proper algebraic extensions, i.e. no algebraic extensions E with $F < E$.

Theorem 91. Every field F has an algrebaic closure.

Theorem 92 (Fundamental Theorem of Algebra)**.** The field $\mathbb{C}$ is an algebraically closed field.

Proof. Let the polynomial $f(z) \in \mathbb{C}[z]$ have no zero in $\mathbb{C}$ such that $1/f(z)$ gives a function that is analytic everywhere. Therefore, if $f \notin \mathbb{C}$, then $\lim_{|z|\to\infty} |f(z)| = \infty$ so $\lim_{|z|\to\infty} |1/f(z)| = 0$ implying $1/f$ must be bounded in the plane so by Liouville's theorem, $1/f$ is constant so f is constant. Thus, any nonconstant polynomial in $\mathbb{C}[z]$ must have a zero in $\mathbb{C}$ so $\mathbb{C}$ is algebraically closed. $\square$

Definition 96 (Poset)**.** A *partial ordering of a set* S is given by an equivalence relation defined for only certain ordered pairs of elements. A subset T of a poset is a *chain* if any two elements of T are comparable.

Lemma 5 (Zorn's Lemma)**.** If S is a partially ordered set such that every chain in S has an upper bound in S, then S has at least one maximal element.

Remark. This is equivalent to the axiom of choice.

Theorem 93. Every field F has an algebraic closure $\bar{F}$.

Proof. It can be shown in set theory that given any set, there exists a set with *strictly more elements*. Take a set $A = \{\omega_{f,i} \mid f \in F[x]; i = 0, \ldots, (\text{degree } f)\}$ that has an element for every possible zero of any $f(x) \in F[x]$ and Ω be a set with strictly more elements. We can assume $F \subset \Omega$ even if by $\Omega' = F \cup \Omega$. Thus, if E is any extension field of F and $\gamma \in E$ is a zero $f(x) \in F[x]$ for $\gamma \notin F$ and $\deg(\gamma, F) = n$, then renaming γ by ω for $\omega \in \Omega - F$ and renaming elements $a_0 + \ldots + a_{n-1}\gamma^{n-1}$ of $F(\gamma)$ by distinct elements of Ω and a_i over F, we can consider $F(\gamma)$ as an algebraic extension field $F(\omega)$ of F with $F(\omega) \subset \Omega$ and $f(\omega) = 0$. This set has enough elements to form $F(\omega)$ since Ω has enough elements to provide n different zeros for each element of degree n in any subset of $F[x]$.

All algebraic extension fields $E_j \subseteq \Omega$ of F form a set S that is partially ordered under subfield inclusion $\leq$. One element of which is F itself.

Given a chain T in S and W the union of all elements, we can make W into a field. The field axioms follow since the operations were defined in terms of addition and multiplication in fields. If we can show W is algebraic over F, then $W \in S$ will be an upper bound for T. But, if $\alpha \in W$, then $\alpha \in E_j$ for some $E_j \in T$ so α is algebraic over F. Hence W is an algebraic extension of F and an upper bound of T.

Now, by Zorn's lemma, there is a maximal element $\bar{F}$ of S. Let $f(x) \in \bar{F}[x]$ where $f(x) \notin \bar{F}$. If $f(x)$ has no zero in $\bar{F}$, then because Ω has many more elements than $\bar{F}$, we can take $\omega \in \Omega - \bar{F}$ and form a field $\bar{F}(\omega) \subseteq \Omega$ with ω a zero of $f(x)$ as in the first part of this proof. Thus, we can take $\beta \in \bar{F}(\omega)$ so β is a zero of the polynomial $g(x) = \alpha_0 + \cdots + \alpha_n x^n$ in $\bar{F}[x], \alpha_i \in \bar{F}$ implying α_i is algebraic over F. Thus, $F(\alpha_0, \ldots, \alpha_n)$ is a finite extension of F and since β is algebraic over $F(\alpha_0, \ldots, \alpha_n)$, $F(\alpha_0, \ldots, \alpha_n, \beta)$ must be a finite extension of it where β is algebraic over F. Thus, $\bar{F}(\omega) \in S$, and $\bar{F} < \bar{F}(\omega)$ contradicting the choice of $\bar{F}$ as maximal so $f(x)$ must have some zero in $\bar{F}$ implying $\bar{F}$ is algebraically closed. □