Pushforward of Symplectic Volume under the Moment Map

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§1. Abstract

We present two proofs of the Duistermaat-Heckman formula. This result expresses the pushforward of the canonical measure of a symplectic manifold under the moment map as a piecewise polynomial measure. The first proof was given by J.J. Duistermaat and G.J. Heckman using standard differential techniques [DH82]. The second proof by M. F. Atiyah and R. Bott relies on a localization theorem for equivariant cohomology [AB84].

§2. Hamiltonian Action

Let a torus T^ℓ have Hamiltonian action on symplectic manifold (M^{2n}, σ). Specifically, say $T \to \mathrm{Symp}(M, \sigma)$ with given (co)moment maps $J : M \to \mathfrak{t}^*$ and $J^* : \mathfrak{t} \to C^\infty(M)$ such that:

$$\text{(i) } \iota_{\tilde{X}}\sigma = dJ_X, \quad \text{(ii) } J \circ t(m) = \mathrm{Ad}^*_{t^{-1}} \circ J(m) = t \circ J(m) \circ t^{-1}, \quad \text{(iii) } \langle J(m), X\rangle = J_X(m)$$

where $J_X \coloneqq J^*(X)$ and $\tilde{X}|_m = \frac{d}{dt}|_0(\exp(tX)\cdot m)$ is the Hamiltonian vector field of $J_X : M \to \mathbb{R}$ for any $X \in \mathfrak{t}$. Condition (ii) effectively limits the T-action to the fibers of J.

We assume J is proper (e.g. if M is compact). And because we only intend to calculate the pushforward, which can be done component-wise, we also assume M is connected.

§3. Introduction

That the pushforward measure is *piecewise* polynomial comes from how we partition M into chunks of regular points. Say $\xi \in \mathfrak{t}^*$ is a regular value for J. For any $m \in Y_\xi \coloneqq J^{-1}(\xi)$, dJ_m is surjective so Y_ξ will be a compact submanifold of codimension $\dim \mathfrak{t}^* = \dim T = \ell$.

The orbit space $Y_\xi \overset{p_\xi}{\to} M_\xi \coloneqq Y_\xi/T$ is called the reduced phase space [MW74]. We claim M_ξ is nice enough to carry a unique symplectic form which depends linearly on ξ; in particular, we start by showing it is an orbifold [Sat56]. This is apparent via first factoring p_ξ through $r_\xi : Y_\xi \to Z_\xi \coloneqq Y_\xi/\Gamma_\xi$, where $\Gamma_\xi < T$ is the finite subgroup generated by all $T_m, m \in Y_\xi$, and then $q_\xi : Z_\xi \to M_\xi = Y_\xi/T$. Because the action of T is locally free,[1] r_ξ is a finite branched covering, i.e. a covering except on a finite subset and q_ξ is a T/Γ_ξ-principal bundle. So M_ξ has palatable singularities. The rest of this section endows M_ξ with a symplectic structure σ_ξ.

For now, fix a regular value ξ_0 of J and pick a convex open neighborhood U of ξ_0 containing just regular values. Given the fibration $J : J^{-1}U \to U$, we can pick an arbitrary connection $\nabla : \mathfrak{X}(U) \times \Gamma(J^{-1}U) \to \Gamma(J^{-1}U)$. We can make this connection T-invariant by averaging its action under T for the normalized Haar measure dt, e.g. $\widetilde{\nabla} = \int_T t^*\nabla dt$.

Such a connection lets us take unique geodesics in $J^{-1}U$ passing through Y_{ξ_0} for any straight line/tangent vector through ξ_0 (in Euclidean space $\mathfrak{t}^*$). This creates a natural projection map $\eta : J^{-1}U \to Y_{\xi_0}$. We need $\widetilde{\nabla}$ to be T-invariant as this ensures $\eta|_{Y_\xi}$ will be a T-equivariant diffeomorphism. Thus, η descends to $J^{-1}U/T \to M_{\xi_0}$, giving a smooth family of diffeomorphisms $M_\xi \to M_{\xi_0}$ for any $\xi \in U$, thus canonically identifying (by homotopy invariance of choice of local trivialization)

$$H^*(M_\xi; -) \cong H^*(M_{\xi_0}; -).$$

This will let us compare cohomology classes of the unique symplectic form $[\sigma_\xi] \in H^2(M_\xi; \mathbb{R})$ given by $p_\xi^*\sigma_\xi = i_\xi^*\sigma$ where $i_\xi : Y_\xi \hookrightarrow M$. Uniqueness holds since $\sigma_\xi([v],[w]) = \sigma(v,w)$ with $v, w \in T_mY_\xi$ $(m \in Y_\xi)$ will overdetermine σ_ξ. Accordingly, σ_ξ is non-degenerate. Smoothness is

[1] It follows that $T \curvearrowright Y_\xi$ is locally free because if $X \in \mathrm{Lie}(\{t \in T : tm = m\})$, then $\tilde{X}|_m = 0$ so for any $v \in T_mM$, $0 = \sigma(\tilde{X}|_m, v) = dJ_X(v) = \langle dJ_m(v), X\rangle$ but J_m is a submersion so $X = 0$ implying T_m is discrete, so finite, for regular points m. The converse also holds trivially.

clear because the pullback, a restriction of a smooth form, is smooth. σ_ξ is also closed because the exterior derivative passes through pullbacks and p_ξ is surjective.

Proving σ_ξ is well-defined amounts to showing σ kills $T_m(T \cdot m) \cap T_m Y_\xi$. But these spaces are σ-orthogonal: $\sigma(\tilde{X}_m, v) = \langle dJ_m(v), X\rangle = 0$ for all $X \in \mathfrak{t}$ if and only if $dJ_m(v) = 0$, i.e. $v \in T_m Y_\xi$ so their intersection is 0. (Note we're using the fact $\tilde{X}_m$ is the infinitesimal action of T at m.)

Putting that aside, consider the short exact sequence where $\Lambda_\xi = \{X \in \mathfrak{t}, \exp X \in \Gamma_\xi\}$:

$$0 \longrightarrow \Lambda_\xi \longrightarrow \mathfrak{t} \xrightarrow{\exp} T/\Gamma_\xi \longrightarrow 0.$$

This induces the long exact sheaf cohomological sequence

$$\cdots \longrightarrow H^1(M_\xi, \mathfrak{t}) \longrightarrow H^1(M_\xi, T/\Gamma_\xi) \xrightarrow{\delta} H^2(M_\xi, \Lambda_\xi) \longrightarrow H^2(M_\xi, \mathfrak{t}) \longrightarrow \cdots$$

Blackboxing 'fine' sheaves, $\mathfrak{t}$ allegedly is one, so $H^i(M_\xi, \mathfrak{t}) = 0$ making δ an isomorphism [DH82]. If we characterize the T/Γ_ξ-bundle $q_\xi : Y_\xi/\Gamma_\xi \to M_\xi$ as an element $v_\xi \in H^1(M_\xi, T/\Gamma_\xi)$, its Chern class $c = \delta(v_\xi)$ will be constant on U. And using η, we know Λ_ξ is independent of choice of $\xi \in U$ so $\xi \mapsto \delta(v_\xi) \in H^2(M_{\xi_0}, \Lambda_{\xi_0})$ will be continuous. We can finally state our keystone theorem.

§4. Proof

Theorem 4.1. *For $\xi, \xi_0 \in \mathfrak{t}^*$ in the same connected component of J-regular values,*

$$[\sigma_\xi] = [\sigma_0] + \langle c, \xi - \xi_0\rangle \text{ where } c \text{ is the Chern class of the fibration } q_\xi.$$

We use the equivalence between cohomological class of the curvature of a connection form and first Chern class.

For simplicity, let $Y = Y_{\xi_0}$ and $M = U \times Y$ using the trivialization $J \times \eta : J^{-1}U \to U \times Y$. Because η is T-equivariant, T only acts in the second component independent of the first so we can identify M_ξ with Y/T. $\Gamma = \Gamma_\xi$ is similarly ξ-independent so Z_ξ identifies with $Z = Y/\Gamma$.

Our goal now is to reformulate σ_ξ as a $\mathfrak{t}$-valued 2-form on Y/T. We'll use the obvious projection $p : Y \twoheadrightarrow Y/T$ and embedding $i_\xi : Y \hookrightarrow U \times Y$ at level ξ. In this notation, $p^*\sigma_\xi = i_\xi^*\sigma$ fixes σ_ξ.

For $\lambda \in \mathfrak{t}^*$, take the constant vector field $\tilde{\lambda} = (\lambda, 0)$ over M and corresponding differentiation ∂_λ. This defines a $\mathfrak{t}$-valued 1-form $\alpha_\xi : \lambda \mapsto i_\xi^*(\iota_{\tilde{\lambda}}\sigma)$ on Y. Cartan's formula and $d\sigma = 0$ imply

$$p^*(\partial_\lambda \sigma_\xi) = \partial_\lambda(i_\xi^*\sigma) = i_\xi^*(\mathscr{L}_{\tilde{\lambda}}\sigma) = i_\xi^*(d(\iota_{\tilde{\lambda}}\sigma)) = d\alpha_\xi.$$

Leaving that for a moment, the T-invariance of α_ξ fixes a unique $\mathfrak{t}$-valued 1-form β_ξ on $Z = Y/\Gamma$ under $r^*\beta_\xi = \alpha_\xi$ from $r : Y \twoheadrightarrow Y/\Gamma$. We claim β_ξ is a connection form for the T/Γ-fibration $q : Z \twoheadrightarrow Y/T$. Γ-invariance is clear so we are left to show $\beta_\xi(\tilde{X}) = X$. This follows from

$$\langle X, \lambda\rangle = (dJ_X)_m(\tilde{\lambda}_m) = \sigma(\tilde{\lambda}, \tilde{X})_m = \langle \alpha_\xi(\tilde{X}), \lambda\rangle_m, \quad \text{for any } X \in \mathfrak{t} \text{ and } \lambda \in \mathfrak{t}^*.$$

Thus there is a unique closed $\mathfrak{t}$-valued curvature 2-form Ω_ξ on Y/T for which $d\beta_\xi = q^*\Omega_\xi$. Hence,

$$p^*(\partial_\lambda \sigma_\xi) = d\alpha_\xi = r^*q^*\Omega_\xi = p^*\Omega_\xi.$$

And because p_* is surjective, we get an equivalence

$$\{\lambda \to \partial_\lambda \sigma_\xi\} = \{\lambda \to \langle \Omega_\xi, \lambda\rangle\}.$$

By our initial comment, this curvature corresponds with the Chern class so in cohomology,

$$\partial_\lambda[\sigma_\xi] = \langle c, \lambda\rangle \quad \text{independent of } \xi.$$

Now integrating any $\xi_0 \to \xi$-path in a connected component gives $[\sigma_\xi - \sigma_{\xi_0}] = \langle c, \xi - \xi_0\rangle$. $\square$

§5. Measure under Free T-Action

To carry on, we have to assume T's action is effective. For instance, replace T by $T/\bigcap_{m\in M} T_m$. This is a small demand as we know each T_m is finite from earlier.[2] Let M' be the subset of M on which T acts freely. We now prove the below result due to Atiyah - Guillemin - Sternberg:

Theorem 5.1. *$M - M'$ is a measure 0 locally finite union of closed symplectic submanifolds of codimension ≥ 2. In particular, M' is open, connected, dense, and consists of regular points.*

The theorem is trivial as long as we can characterize M's complement as this union. The proof overlaps slightly with content presented in class as well as sections 5.5 of [MS17] and 27 of [Sil06]. We can analyze the subset $M - M'$ on which T does not act effectively by focusing on subgroups of the torus with a nonempty fixed point set. In particular, we first show that for any subgroup $S < T$,

$$\mathrm{Fix}(S) = \bigcap_{s\in S} \mathrm{Fix}(s) \text{ is a symplectic submanifold of } M.$$

For an arbitrary Riemannian metric g, let $\tilde{g} = \int_{t\in T} t^* g dt$ so it becomes T-invariant. Per section 12 of [Sil06], this gives back a σ-compatible, T-invariant ($t^*J = J$) almost complex structure J. Now, for $p \in \mathrm{Fix}(S)$ and $s \in S$, $ds_p : T_pM \to T_pM$ is a unitary transformation:

$$g(ds_p(v), ds_p(w)) = \sigma(ds_p(v), Jw) = g(ds_p(v), w) = g(w, ds_p(v)) = \sigma(w, Jv) = g(v, w).$$

Each s is a local isometry under $\tilde{g}$ so by $ds_p(v) = (s \circ \exp(v))'(0)$ and the uniqueness of geodesics,

$$\exp_p(ds_p(v)) = t \circ \exp_p(v).$$

Therefore, the fixed points of s near p are in bijection with the fixed points of ds_p, i.e.

$$T_p\mathrm{Fix}(S) = \bigcap_{s\in S} \ker(\mathbb{1} - ds_p) = \bigcap_{s\in S} E_1(ds_p).$$

Since ds_p is unitary, each eigenspace is preserved by J_p so this subspace is almost complex and thus symplectic so of even, and minimally 2, codimension. Noting also that $\mathrm{Fix}(s) = (s \times \mathrm{id}_M)^{-1}(\Delta)$ is closed, each $\mathrm{Fix}(S)$ is a closed symplectic submanifold of M. Local finiteness of the submanifolds follows from the locally finite T_m. The above proof is rather geometric and offers some intuition for Theorem 5.1. However, Duistermaat-Heckman achieved the same result less candidly in their paper using the equivariant Darboux-Weinstein Theorem [Wei71]:

Lemma 5.2. *There exist S-linear Darboux coordinates around an S-fixed submanifold.*

This theorem's proof is identical to the application of Moser's trick to a normal bundle (as done in class) because everything here is S-equivariant. To be clear though, our case automatically satisfies the omitted assumptions that S is compact and M is finite-dimensional.

Under such coordinates, it's evident these fixed sets have the same description as before. If they are also open and nonempty though, then by our assumptions that M is connected and T's action is effective, $\mathrm{Fix}(S) = M$ so $S = 1$ which is not a submanifold under consideration so we're done. □

§6. Measure Pushforward

We now calculate the pushforward Liouville, or symplectic, measure. Specifically, we want to measure the density of $J_*(dm)$ in terms of $d\xi$ where dm is the Liouville measure $\frac{\sigma^n}{n!}$ and $d\xi$ is the dual Lebesgue measure on $\mathfrak{t}^*$.

Here, we regard measures as continuous linear forms from compactly supported continuous functions to $[0, \infty)$. In this sense, our assumption that J is proper implies the pushforward $J_*(dm)(-) = dm(J^{-1} \circ -)$ acts on functions with continuous support. At regular values, it's

[2] In fact, the moment map is constant on a finite union of proper symplectic submanifolds which make up these fixed points so our final volume is not changed. See: [Ati81] or 5.5.1 in [MS17].

clear $J_*(dm)$ will be continuous linear. By the previous section, this suffices to get an a.e. smooth, locally integrable function f for which:

$$J_*(dm) = f d\xi.$$

If we can calculate the density of f, i.e. the Radon-Nikodym derivative of J_*dm with respect to $d\xi$, at the J-regular values ξ, we're done. Recall the trivialization $J \times \eta : J^{-1}U \to U \times Y$.

To make this volume-preserving, we need determine a symplectic structure ω on $U \times Y$ that is a restriction of σ. In particular, it must satisfy the Hamiltonian condition

$$\iota_{\tilde{X}}\omega = d\langle \mathrm{pr}_{\mathfrak{t}^*}, X\rangle \quad \text{for all } X \in \mathfrak{t}^*$$

where the moment map has been factored via $J^{-1}\eta \overset{J\times\eta}{\to} U \times Y \overset{\mathrm{pr}_{\mathfrak{t}^*}}{\to} U$ into $\mathrm{pr}_{\mathfrak{t}^*}$ exactly. Seeing this interior derivative will kill any form purely on Y, we make the Ansatz that

$$\omega = \mathrm{pr}_Y^* i_{\xi_0}^* \sigma - d\langle \mathrm{pr}_{t^*}, \mathrm{pr}_Y^*\alpha\rangle.$$

It holds that ω is closed because d passes through pullbacks with $d\sigma = 0$ and $d^2 = 0$. For non-degeneracy, note that $\omega = \mathrm{pr}_Y^* i_{\xi_0}^*\sigma - \langle d\mathrm{pr}_{t^*}, \mathrm{pr}_Y^*\alpha\rangle - \langle \mathrm{pr}_{t^*}, d\mathrm{pr}_Y^*\alpha\rangle$'s last term will die on $\xi_0 \times Y$. Split T_mY into vertical and horizontal spaces $V_m \oplus H_m$ according to α so $T_{(\xi_0,m)}U \times Y = \mathfrak{t}^* \oplus V_m \oplus H_m$. Given any vector $w = (X, v, h)$ for which

$$\iota_{\tilde{w}}\omega(-) = i_{\xi_0}^*\sigma(h, -) - \langle v, \mathrm{pr}_Y^*\alpha(-)\rangle + \langle d\mathrm{pr}_{t^*}(-), X\rangle = 0,$$

the components must all vanish seeing that they are independent. The non-degeneracy of $i_{\xi_0}^*\sigma$ implies $h = 0$, the third term killing all $\mathfrak{t}^*$ confirms $X = 0$, and similarly for $v = 0$. Hence, ω is non-degenerate, so symplectic, on a neighborhood of $\xi_0 \times Y$ in $U \times Y$. Restricting U to a convex open subset of this neighborhood gives us a measure.

Regarding the Hamiltonian condition, recall that α is a connection and pr_{t^*} is in fact the moment map. So, for $X \in \mathfrak{t}$, $X^*\langle \mathrm{pr}_{t^*}, \alpha\rangle = \langle \mathrm{pr}_{t^*}, \alpha\rangle$. Taking the derivative of this trivial action with respect to any X will of course vanish. Hence, by Cartan's formula,

$$\iota_{\tilde{X}}(i_{\xi_0}^*\sigma - d\langle \mathrm{pr}_{t^*}, \mathrm{pr}_Y^*\alpha\rangle) = 0 - (\mathcal{L}_{\tilde{X}} - d\iota_{\tilde{X}})\langle \mathrm{pr}_{t^*}, \mathrm{pr}_Y^*\alpha\rangle = 0 - 0 + d\langle \mathrm{pr}_t^*, X\rangle \quad \text{as desired.}$$

Moreover, because ω is T-invariant, it shares the same Hamiltonian T-action and will therefore pullback to a restriction of σ under our trivialization.

The only work remaining is to calculate ω^n. We can do this locally in standard θ^i coordinates on $U \subset \mathfrak{t}^* \cong \mathbb{R}^\ell$. For ease of notation, we write $\langle \mathrm{pr}_{t^*}, \mathrm{pr}_Y^*\alpha\rangle = \langle \theta, \alpha\rangle = \theta^i\alpha_i$, drop the pr_Y^*'s, shorten pr_{t^*} to pr, and let $i_{\xi_0} = i$. Finally recalling that Y is of codimension ℓ, this form simplifies to

$$\begin{aligned}
\omega^n = (i_{\xi_0}\sigma - d\langle \mathrm{pr}, \alpha\rangle)^n &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} (i^*\sigma - \langle \mathrm{pr}, d\alpha\rangle)^{n-k} \wedge \langle d\mathrm{pr}, \alpha\rangle^k \\
&= (-1)^\ell \binom{n}{\ell} (i^*\sigma - \langle \mathrm{pr}, d\alpha\rangle)^{n-\ell} \wedge (\ell!)\alpha_1 \wedge \cdots \wedge \wedge \alpha_\ell \wedge d\theta. \\
&= (-1)^\ell \frac{n!}{(n-\ell)!}(i_\xi^*\sigma)^{n-\ell} \wedge \alpha \wedge d\theta.
\end{aligned}$$

At this point, reframe θ^i so $\alpha(\tilde{\theta^i}) = \pm 1$ over T-orbits. We simplify even further by moving to reduced phase space. Note that in this trivialization around regular values, T-action is independent of ξ so the principal T-fibration $p_\xi : M' \cap Y_\xi \to M'_\xi := (M' \cap Y_\xi)/T$ will have similar measure 0 complement of codimension ≥ 2 closed symplectic submanifolds as M'.

In particular, we can take local trivializations $\psi_i : Z_i \to T \times \tilde{Z}_i$ where $\{\tilde{Z}_i\}_i$ form an open cover of M_{ξ_0}. If we pick a partition of unity t_i subordinate to this cover, it will hold that

$$\sum_i \int_{Z_i} t_i\omega = \sum_i \int_{T \times \tilde{Z}_i} t_i p_{\xi_0} t_i^{-1*} = \sum_i \int_{\tilde{Z}_i} t_i \pi_T^* \int_T \mathrm{dvol}(T).$$

For $\mathrm{vol}(T) = 1$, observe the integral of $i_\xi^*\omega$ over phase space will therefore equal the integral of ω_ξ over reduced phase space, allowing us to utilize our previous result granting linear ξ-dependence.

Remembering that we can build any real-valued function on $\mathfrak{t}^*$ out of a partition of functions with compact support in some U, it suffices to see that (ignoring the signs of measure):

$$\int_U f d\xi = \int_{J^{-1}U} \frac{\sigma^n}{n!} = \int_{U\times Y} \frac{(i_\xi^*\sigma)^{n-\ell}}{(n-\ell)!} \wedge \alpha \wedge d\xi = \int_U \left[\int_Y \frac{(i_\xi^*\sigma)^{n-\ell}}{(n-\ell)!} \wedge \alpha\right] d\xi.$$

Now moving to reduced phase space:

$$f(\xi) = \int_Y \frac{(i_\xi^*\sigma)^{n-\ell}}{(n-\ell)!} \wedge \alpha = \int_{M_{\xi_0}} \frac{(\sigma_\xi)^{n-\ell}}{(n-\ell)!} = \mathrm{vol}(M_\xi).$$

And Theorem 5.1 lets us conclude that

$$f(\xi) = \int_{[M_{\xi_0}]} \frac{[\sigma_\xi]^{n-\ell}}{(n-\ell)!} = \frac{1}{(n-\ell)!}\left\langle [M_{\xi_0}], ([\sigma_{\xi_0}] - \langle c, \xi - \xi_0\rangle)^{n-\ell}\right\rangle \quad \text{for regular values } \xi \text{ of } J.$$

In other words, we can describe the volume density of $J_*(dm)$ as a polynomial on each connected component of the set of regular values of the momentum map J. Namely, the moment map pushforward Liouville measure on M is a piecewise polynomial on $\mathbb{R}^\ell$. □

§7. Equivariant Proof

We briefly survey a subsequent proof from Atiyah-Bott that achieves the same result. For brevity, to not lose scope with commutative algebra rather than geometry, and for fun, we present nearly all the results of [AB84] without explanation.

Take G to be a connected and compact Lie group. We define the equivariant cohomology of a G-space M (in complex coefficients) to be $H_G^*(M) := H^*(EG \times_G M)$ where EG is the universal G-bundle such that we identify the left G-action on EG with the right action on M.

If a torus T is acting smoothly on a compact manifold M with a set of fixed points F made up of connected components, say P's. Atiyah-Bott show that the normal bundle ν_P of P will decompose as a direct sum of non-trivial 2-dimensional representations of T. If compatibly oriented with M, P, these representations become complex characters

$$\exp(2\pi i a_j) = \alpha_j : T \to U(1).$$

The equivariant Euler class E of the normal bundle ν_P in M can be expressed as a product $\Pi_j a_j$ of these characters. Further, writing $a_j = \sum a_{jk} u_k$ as a linear form on the Lie algebra makes $E(\nu_p)$ some polynomial f_P.

Consider the $H_T^*(M)$-module homomorphism $i_* : H_T^*(F) \to H_T^*(M)$. If we localize the ring of polynomials with respect to (i.e. allow fractions with powers of) the product polynomial $f_F = \Pi_P f_p$, the map $i^* i_*$ will become invertible since $i^* i_* 1 = E(\nu_F)$. In particular,

$$Q = \sum_P \frac{i_P^*}{E(\nu_P)} \quad \text{is inverse to } i_*^F : H_T^*(F) \to H_T^*(M) \text{ in the localization.}$$

Therefore, $\phi = i_*^F Q \phi = \sum_P \frac{i_*^P i_P^*}{E(\nu_P)}$. So for any $\phi \in H_T^*(M)$, applying $\pi_*^M : H_T^*(M) \to H_T^*(pt)$ gives the integration formula

$$\pi_*^M \phi = \sum_P \pi_*^P \frac{i_P^* \phi}{E(\nu_P)}.$$

For simplicity, suppose F is a finite set of points and $T = S^1$. Now, we just substitute. This next step resembles Dusitermaat-Heckman's primary cohomological theorem and is similarly nontrivial, though we direct those interested in its proof to [AB84]. For a fixed $X \in \mathfrak{t}$, there is some u so $\sigma^\sharp = \sigma - J_X u$ extends σ to a closed equivariant class implying $e^{\sigma^\sharp} = e^\sigma e^{-J_X u}$ is a class in the formal power series of $H_T^*(M)$. Given F consists of finite points, $E(\nu_P) = e_P u^n$ for integers $e_P \in \mathbb{Z}$. π_*^M (resp. π_*^P) kills all $q \neq 2n$ (resp. $q \neq 0$)-forms. So, our integration formula simplifies to

$$\int_M \frac{\sigma^n}{n!} e^{-J_X u} = \sum_P \frac{e^{-J_X(P)u}}{e_P u^n}.$$

Invoking $u = -it$, our integral becomes

$$\int_M e^{itJ_X} \frac{\sigma^n}{n!} = \frac{1}{(it)^n} \sum_P \frac{e^{itJ_X(P)}}{e_P}.$$

Finally taking the Fourier transform implies

$$\int_M \langle J(m), X\rangle \frac{\sigma^n}{n!} = \sum_P \frac{\mathrm{vol}(M_P)}{g(P,X)}$$

for some function g dependent on the specific Hamiltonian action. Miraculously, this exact homological solution precisely matches the leading term of the stationary phase approximation of the inverse Fourier transform of our original, analytic solution [DH82].

§8. References

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