Estimating Risk of Dynamic Trading Strategies from High Frequency Data Flow

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Abstract. We consider the problem of risk management in the framework of low latency trading. We suggest an efficient method of real-time analysis of massive data flow from the market. The result of the analysis is a new risk measure Dynamic VaR (DVaR) for risk management of low latency trading robots. The work of DVaR is illustrated on a test example and compared with Traditional VaR and ex-post measure commonly used in high frequency trading.

1 Introduction

Financial markets have always presented researchers with Big Data type problems due to several reasons: data collection from multiple sources in different formats and of different qualities; requirement of the most advanced technology at the time; large number of data points as well as complex relationships between price indices and essentially non-linear and non-Gaussian nature of observations [7,8,13,14].

But the new era of really big data started with the spread of high frequency trading (HFT) when in 1998 SEC authorized electronic trading.

HFT gained significance in early 2000s as a market making approach. In 2005 SEC consolidated National Market System (NMS), authorized by Congress in 1975, into Regulation NMS which was implemented in 2007, a year when HFT started growing really fast.

The real key for HFT is not as much the frequency, not even latency of trading, but algorithmic trading between computers with speed that leaves no chance for human to interfere.

Appearance of trading robots gave rise to completely new types of financial data analyses: personal instincts of a floor trader need to be formalized and implemented utilizing high computing power; terabytes of high frequency price records with millions of events per day need to be analyzed to extract insights for making trading decisions; the problem of market risk management gets a new dimension with significant emphasis on timing and frequency of market events in addition to traditional emphasis on the price impact.

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Such significant paradigm shift in financial data analysis requires reevaluation of some core concepts in financial mathematics including the basic model for market price.

In this article we use suitable approach to modeling market price that goes back to [3,4] and further developed in [11] to extend the traditional market risk measure called value at risk (VaR) to account for frequency of events affecting the value of the portfolio.

The obtained results show in particular how the data flow of market prices needs to be organized into the flow of "sufficient statistics", i.e. the frequencies of events of certain types in order to process the risk measurement efficiently in real-time with very low latency.

The rest of the article is organized in the following way. In the introduction we explain the evolution of models for market price that leads to the model that is used in this study and describe the general problem of market risk measurement.

In Sect. 2 we give mathematical summary of the results including the description of the new market risk measure Dynamic VaR (DVaR).

In Sect. 3 we illustrate the work of DVaR and discuss the organization of the market risk measurement process.

1.1 Evolution of Market Price Models

Bachelier Model. The first modern description of the process of price was, probably, given by Bachelier [1]. He used Brownian motion process (later studied by Einstein [5] and rigorously defined by Wiener in early 1920s):

$$S(t) = S(0) + \mu t + \sigma W_t,$$

where W is standard Wiener process i.e. the process with independent increments starting from zero with $\mathbb{P}\{W_t \text{ is continuous }\} = 1$; $(W_{t+h} - W_t) \sim \mathbb{N}(0, h)$.

The model can be justified by dividing any time interval $[t, T]$ into n equal subintervals $[t_0 = t, t_0 + \tau, t_0 + 2\tau, \ldots, t_0 + n\tau = T]$. Then

$$S(T) - S(t) = \sum_{k=1}^{n} S(t + k\tau) - S(t + (k-1)\tau) = \sum_{k=1}^{n} \Delta S_k. \tag{1}$$

If price increments on subintervals are independent [8] and identically distributed, then according to the central limit theorem (CLT), $S(T) - S(t)$ is asymptotically normal, as $n \to \infty$, as long as $\mathbb{V}[S(t + k\tau) - S(t + (k-1)\tau)] < \infty$ (Lindeberg's condition, 1922).

Non-Gaussian Reality. It became clear soon that observed market price behavior contradicts the Gaussian distribution hypothesis.

In 1960s Fama [7,8] and Mandelbrot [13,14] suggested explanations for non-Gaussian behavior of market prices.

Initially attention was focused on the Lindeberg condition which in more broad sense means that (1) is not dominated by any finite number increments

ΔS_k and in particular, when increments are identically distributed, it means $\mathbb{V}[\Delta S_k] < \infty$.

Mandelbrot noticed extreme variability of second empirical moments of financial data, which could be interpreted as nonexistence of the theoretical second moments, i.e. Lindeberg's condition does not hold. With (1) still in place and i.i.d. ΔS_k, Mandelbrot suggested to model market prices using limit theorems different from CLT, where limit distributions are from the family of stable distributions. Fama showed agreement of the observed market prices with stable distributions.

Stable Distributions. Distribution Function $G(x)$ is Stable if for any a_1, $a_2 \in \mathbb{R}$, $b_1, b_2 > 0$ there exist $a \in \mathbb{R}$ and $b > 0$, such that

$$G(b_1 x + a_1) * G(b_2 x + a_2) = G(bx + a), x \in \mathbb{R}$$

P. Levy proved that distribution function $\mathbb{F}(x)$ can be a limit distribution for the sum $S_n = (X_1 + \ldots + X_n - a_n)/b_n$ of i.i.d. random variables X_i and some $a \in \mathbb{R}$, $b > 0$ iff $\mathbb{F}(x)$ is stable.

Various functional limit theorems establish convergence of discrete time random walks with unlimited variance to stable Levy processes.

Stable processes are fractals: trajectories observed with different frequencies belong to the same stable law, can differ only by the scale parameter.

Unfortunately, there are only 4 known stable distributions with probability densities expressed in terms of elementary mathematical functions (Gaussian, Cauchy, Levy and symmetrical Levy distributions).

Both Bachelier and Mandelbrot approaches still require structure (1) with the possibility of $n \longrightarrow \infty$.

Processes with Stochastic Time. It has been long known that intensity of trading does not remain constant.

Structural assumption (1) requires that on any time interval there is an arbitrary large number n of equidistant moments when price can be observed.

Such assumption might not always work because of random time intervals between the registered trades when the price cannot be measured, or because intensity of trading changes too fast.

Clark [3,4] suggested this as an explanation why CLT does not work with market prices. He described the process of market price as a subordinated Wiener process $W(X(t))$ where W is Wiener process and subordinator $X(t)$ is a stochastic process of "operational time" with non-decreasing trajectories starting from zero. Subordinator turns the Wiener processes into a non-Gaussian one even if increments ΔS_k satisfy CLT.

The condition of CLT that is violated in this case is the determinism of number of terms in the sum (1).

Following this approach, we replace (1) with

$$S(T) - S(t) = \sum_{k=N(t)+1}^{N(T)} S(k) - S(k-1) = \sum_{k=N(t)+1}^{N(T)} \Delta S_k \tag{2}$$

where ΔS_k satisfy CLT, the process $N(t) \geq 0$ starting from zero is the number of trades registered up until t.

If the trading conditions are such that intensity of trading is high enough and/or trading pace is stable during long enough periods of time, then there is a possibility that CLT holds at least locally in time.

Under such conditions eliminating the issue of random summation by looking at blocks of data with large enough sizes seems the most reasonable approach.

This approach in effect leads to using operational time instead of calendar time and naturally results in various stochastic volatility models of price process.

Following this direction there has been a significant theoretical work done that allows approximating the price process with continuous Gaussian martingales including some powerful methods for analyzing implied and realized volatility (see, for example, [15,16]).

However, in this article we adopt the approach based directly on (2).

Such approach originating in [3,4] was further developed in [11].

1.2 Risk Measurement

Risk management of a portfolio of financial assets usually is based on one of key risk measures, e.g. VaR or Expected Shortfall, which one can derive from the distribution of loss. Methods of estimation of such distribution can be classified into two major types: (i) application of extreme value theory to simulated sample of portfolio returns [6]; (ii) worst loss analysis for a set of stress scenarios [2].

Both types assume that until the time horizon the portfolio remains unchanged while losses result from adverse changes of market variables. Unfortunately, such assumption is not realistic in low latency case when portfolio immediately adjusts to market changes and the object of risk management is not portfolio itself, but its dynamic response to the market events, i.e. the strategy.

Most importantly, traditional methods do not account for intensity of portfolio changes.

2 Mathematical Summary

In this section using limit theorems for compound Cox processes [11] we derive a real-time risk signal which we call Dynamic VaR (DVaR). In the following section we apply DVaR to the example of a trading system breakdown.

2.1 Definitions

Cox Process. Let $\mathcal{N}$ be a set of all right-continuous non-decreasing integer-valued functions $\nu(t)$, such that $\nu(0) = 0$.

Definition 1. *Random process $N(t)$ with trajectories from $\mathcal{N}$ is called point process.*

Let $\mathcal{A}$ be a set of right-continuous non-decreasing functions A_t, such as $A_0 = 0$, $A_t < \infty$ for each $t < \infty$.

Definition 2. *A point process $N_A(t)$ is called a non-homogeneous Poisson process with intensity measure $A_t \in \mathcal{A}$ if $N_A(t)$ has independent increments and $N_A(t) - N_A(s)$ has Poisson distribution with mean $A_t - A_s$.*

Let $\Lambda_t, t \geq 0$, be a random process with trajectories from $\mathcal{A}$.

Cox process is a generalization of non-homogeneous Poisson process in which intensity measure can be stochastic in a certain way.

Definition 3. *A point process $N_\Lambda(t)$ is called Cox process with random intensity measure Λ_t if for any realization A_t of Λ_t the process $N_\Lambda(t)$ is a non-homogeneous Poisson process with intensity measure A_t.*

Definition of Cox process means that we can generate Cox process by first generating a trajectory of intensity measure A_t and then generating trajectory of $N_\Lambda(t)$ as a trajectory of non-homogeneous Poisson process with intensity measure A_t.

If $N_1(t)$ is a homogeneous Poisson process with unit intensity independent of random intensity measure Λ_t then Cox process $N_\Lambda(t)$ is a superposition of $N_1(t)$ and $\Lambda(t)$: $N_\Lambda(t) = N_1(\Lambda(t)), t \geq 0$.

Compound Cox Processes. Cox processes allow us to describe randomness of intensity of Poisson process.

With compound Cox processes we can also allow for randomness in the jump sizes of Poisson process or impacts.

Let $X_1, X_2, \ldots$ be i.i.d. random variables with at least two moments $\mathbb{E}[X_j] = a$, $\mathbb{V}[X_j] = \sigma^2, 0 < \sigma^2 < \infty$. Let $N_t = N(\Lambda_t)$ be a Cox process independent of $X_1, X_2, \ldots$ for any $t \geq 0$.

Then the process

$$S_t = \sum_{j=1}^{N(\Lambda_t)} X_j, t \geq 0$$

is called **Compound Cox** process ($\sum\limits_{j=1}^{0} = 0$).

If $\Lambda_t \equiv \lambda t, \lambda - const > 0$, then $S(t)$ is classical Compound Poisson process.

If $\mathbb{E}[\Lambda_t] = \mu_\Lambda < \infty$, then $\mathbb{E}[S_t] = a\mu_\Lambda$.

If $\mathbb{V}[\Lambda_t] = \sigma_\Lambda^2 < \infty$, then $\mathbb{V}[S_t] = \left(a^2 + \sigma^2\right)\mu_\Lambda + a^2\sigma_\Lambda^2$.

For more information on Cox and compound Cox processes see [10,11].

2.2 References, Descriptions of the Main Mathematical Results

Going back to the introductory part recall the main representation of the process of market price with stochastic time (2)

$$S(t) = \sum_{i=1}^{N_t} \Delta S(t_i) = \sum_{i=1}^{N_t} X_i,$$

where the counting process is a Cox process N_t, such that

$$\sum_{i=1}^{N_t} t_i \leq t < \sum_{i=1}^{N_t+1} t_i,$$

and $t_1, t_2, \ldots, t_{N_t}$ are the times of registered trades. The variables X_i are the price impacts of the corresponding trades.

We assume that $\{X_{t_i}, i = 0, 1, \ldots\}$ are i.i.d., $\mathbb{E}[X_i] = 0$, $\mathbb{V}[X_i] = \sigma_X^2 < \infty$.

Limit Theorem for Compound Cox Processes. In order to compare compound Cox process with other models of market prices we need to understand limit behavior of Cox processes when either intensity or time increases to infinity.

Let $d(t) > 0$ be a function growing unlimited when $t \longrightarrow \infty$.

Theorem 1. *Let $\Lambda_t \longrightarrow \infty$ in probability as $t \longrightarrow \infty$. For weak convergence to some random variable Z*

$$\frac{S(t)}{\sigma_X \sqrt{d(t)}} \Longrightarrow Z, t \longrightarrow \infty$$

it is necessary and sufficient that

1. $\mathbb{P}\{Z < z\} = \int\limits_0^\infty \Phi\left(zy^{-\frac{1}{2}}\right) d\mathbb{P}\{U < u\} = \mathbb{E}\left[\Phi\left(zU^{-\frac{1}{2}}\right)\right], z \in \mathbb{R};$
2. $\frac{\Lambda_t}{d(t)} \Longrightarrow U, t \longrightarrow \infty.$

Condition 2 of the theorem means that intensity process Λ_t needs to grow in some regular way and, after normalization by $d(t)$, converge to some random variable U.

But it can remain stochastic.

It is also interesting that asymptotically distribution of $\frac{\Lambda_t}{d(t)}$ does not depend on t.

Condition 1 of the theorem generally speaking means that one-dimensional distribution of Cox process does not converge to Gaussian distribution.

Instead it converges to a mix of Gaussian distributions

$$\mathbb{E}\left[\Phi\left(zU^{-\frac{1}{2}}\right)\right]$$

where U is the mixing random variable.

Such mix can have a very heavy tailed distribution.

This explains why CLT does not hold in general case when time becomes random.

2.3 Dynamic VaR

Let $L(t)$ be the loss of a low latency strategy at time moment t, $\{\tau_1, \tau_2, \ldots\}$ – times of changes of $L(t)$ (events), $N(t)$ – number of events on $[0, t]$. As noted in [16], an important feature of actual transaction prices is the existence of noise (called microstructure noise in [16]). The process L can be decomposed within that framework into two parts

$$L(t) = \widetilde{L}(t) + S(t),$$

where $\widetilde{L}$ is semimartingale, S - noise. Consider the easiest case $\widetilde{L}(t) = \mathbb{E}L(t) = at$ $(t \geqslant 0)$. The linear trend term is predictable and sometimes can be compensated. We'll focus on the behaviour of noise.

Assume that $X_j = S(\tau_j) - S(\tau_{j-i})$ are i.i.d. normal random variables with zero mean (as a consequence of condition $\mathbb{E}S(t) = 0$) and $\mathbb{V}[X_j] = \sigma^2 < \infty$; $\{N(t), X_1, X_2, \ldots\}$ are independent; $N(t) = N_1(\Lambda(t))$, where $N_1(t)$ is homogeneous Poisson process with unit intensity; $\Lambda(t) = \int\limits_0^t \lambda(s)\, ds$, instantaneous intensity $\lambda(s)$ is positive stochastic process with integrable trajectories; $\mathbb{E}[\Lambda(t)] = \mu t$.

Consider Cox process $S(t) = \sum\limits_{j=1}^{N(t)} X_j$, $\bar{S}(T) = \max\limits_{0 \leq t \leq T} S(t)$. We'll need the modification of the above theorem with fixed time interval and increasing flow intensity.

Theorem 2. *[11, Theorem 2.2.1] Assume that $\Lambda(T) = \Lambda(T; \mu) \longrightarrow \infty$ in probability as $\mu \longrightarrow \infty$. Then*

$$\mathbb{P}\left\{\frac{\bar{S}(T)}{\sigma\sqrt{\mu T}} < x\right\} \Longrightarrow F(x), (\mu \longrightarrow \infty)$$

iff there exists non-negative random variable U: $\frac{\Lambda(T;\mu)}{\mu T} \Longrightarrow U$, $\mu \longrightarrow \infty$. In addition $F(x) = 2\mathbb{E}\left[\Phi\left(\max(0, x)/\sqrt{U}\right)\right] - 1$.

The natural example of cumulated intensity process Λ (actually the random time change for the process N_1) is Gamma process. It is used in the popular Variance-Gamma model [12] as a subordinator for Brownian motion with drift.

Definition 4. *The process $\{\gamma(t; \alpha, \beta) = \gamma(t), t \geqslant 0\}$ is called Gamma process with a shape parameter α and a scale parameter β iff*

(1) $\gamma(0) = 0$ *a.s.*
(2) *for any* $0 \leqslant t_1 < t_2 < \ldots < t_n < \infty$ $\gamma(t_2) - \gamma(t_1), \gamma(t_3) - \gamma(t_2), \ldots, \gamma(t_n) - \gamma(t_{n-1})$ *are independent*
(3) *for any* $t \geqslant 0, h > 0$ $\gamma(t+h) - \gamma(t)$ *has gamma distribution with a shape parameter* αh *and a scale parameter* β.

Substituting Gamma process as the random time change into $N_1(t)$ gives Levy process S. Then $\Lambda(T)$ has gamma distribution with a shape parameter αT and a scale parameter β. We can increase Poisson process N intensity by either increasing shape parameter or decreasing scale parameter. Choose the latter $\alpha = 1/T$, $\beta = (\mu T)^{-1}$. Then for any $x \geqslant 0$

$$\mathbb{P}\{\Lambda(T;\mu)/\mu T < x\} = 1 - \exp(-\beta x \mu T) = 1 - e^{-x}$$

and $\frac{\Lambda(T;\mu)}{\mu T}$ converges weakly to the exponentially distributed U with unit mean. Calculate $\mathbb{E}\left[\Phi\left(x/\sqrt{U}\right)\right]$ for $x \geqslant 0$

$$\begin{aligned}\mathbb{E}\left[\Phi\left(x/\sqrt{U}\right)\right] &= \int_0^\infty \Phi\left(x/\sqrt{u}\right) d\left(1 - e^{-u}\right)\\ &= \int_0^\infty \left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^{x/\sqrt{u}} \exp\left(-\frac{v^2}{2}dv\right)\right] e^{-u}du.\end{aligned}$$

Changing the order of integration gives

$$\mathbb{E}\left[\Phi\left(x/\sqrt{U}\right)\right] = 1 - \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{u^2}{2} - \frac{x^2}{u^2}\right) du.$$

The last integral can be found in [9] formula 3.325. Finally

$$\mathbb{P}\left\{\frac{\overline{S}(T)}{\sigma\sqrt{\mu T}} < x\right\} \approx \begin{cases} 0, & x < 0,\\ 1 - e^{-\sqrt{2}x}, & x \geq 0. \end{cases} \tag{3}$$

From (3) we find q-level quantile of the maximum loss distribution and define Dynamic VaR as

$$D(T,q) = \frac{\sigma\sqrt{\mu T}}{\sqrt{2}} \ln\frac{1}{1-q}. \tag{4}$$

Note than the maximum loss distribution has heavier tail (exponential) than random variables X_j.

3 Example: Trading System Breakdown

3.1 Description of the Test Example

Dynamic VaR is used for real-time analysis of market risk associated with low latency strategies.

Here we reconstruct the situation similar to the incident of August 1, 2012 when reportedly a computer at Knight Capital Group (KCG Holdings Inc.) started mistakenly sending frequent orders buying shares at the ask level and immediately selling them at the bid level losing the bid-ask spread on each cycle.

As a result portfolio did not change more than 1 lot at any particular moment in time, but increased frequency of trades, 40 trades per second (or 25 ms between the trades), resulted in a steady rate of loss about \$10 million per minute.

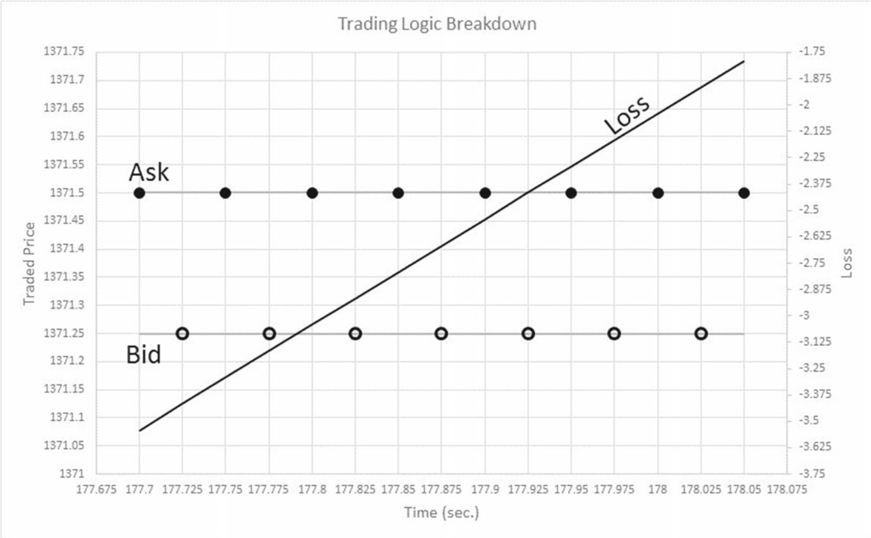

Fig. 1. Trading Logic Breakdown

The following chart explains our test case using S&P 500 e-mini futures (Fig. 1): by repeatedly buying the asset at the ask price and immediately selling it one tick lower at the bid level results in a significant loss even if only one contract is traded at a time.

The main reason for the loss in such case is not the size of the position and not a significant shift in the market price: in our example the level of the price is not changing. It is the frequency of trading that should alert risk management system as early as possible.

And what about the traditional measure for market risk, VaR, calculated from moving time window of 5 s?

It actually drops because the change of the Profit and Loss (P&L) becomes predictable which means that the standard deviation of the P&L increment drops down.

Since VaR is based on he standard deviation of the increment of the P&L it drops also.

This is shown on Fig. 2: during the period of software malfunction intensity of P&L changes jumps, at the same time standard deviation of the P&L drops down.

This, of course makes the traditional VaR not applicable to analysis of the market risk in low latency environment when risk has significant frequency component.

This explains why low latency trading systems do not use any of the traditional market risk measures in their safety logic.

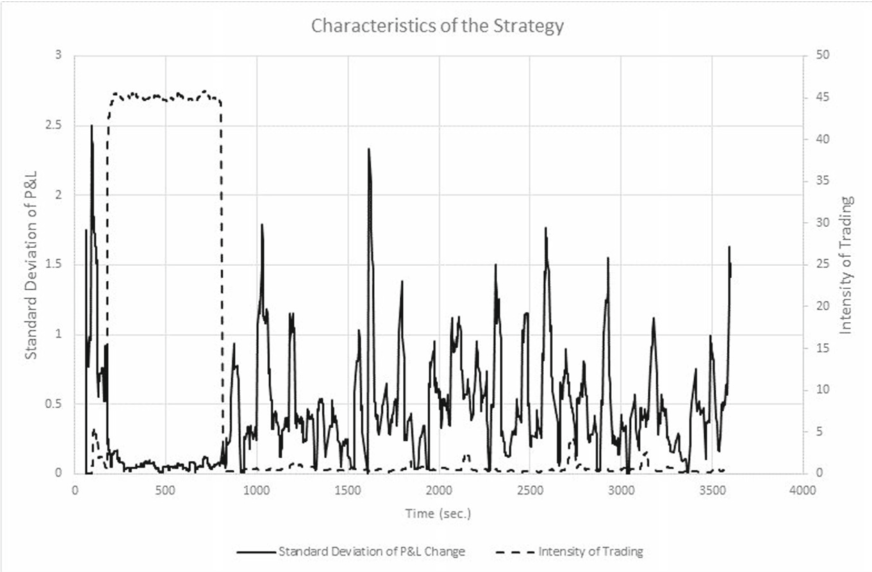

Fig. 2. Characteristics of the Strategy

Instead they use different ex-post measures such as realized P&L, i.e. the trading system would shut down when the realized P&L drops below the preset limit.

But would an appropriate predictive market risk measure give any benefit if used as part of the safety logic of trading robot?

To answer this question zoom into the previous picture to see events on the time scale at which high frequency trading systems make decisions.

On Fig. 3 we see the moment when the software accident occurs and the immediate aftermath of it.

3.2 Dynamic Var

We see on Fig. 3 the beginning of the drop of the standard deviation of P&L and simultaneous sharp increase in intensity of trading events.

Contrary to the traditional VaR that utilizes only standard deviation the new measure Dynamic VaR (DVaR) defined in the previous section (4) utilizes both characteristics: the standard deviation and the intensity.

On Fig. 4 we see that immediately after the incident DVaR (dashed line) jumps sharply to its maximum value in about 600 ms.

The realized P&L (dot-dashed line) used here as an ex-post measure of market risk has not reached the same level yet after 2 s.

In the environment when decisions are made in few microseconds advantage of the predictive measure DVaR relative to realized P&L can make a significant difference.

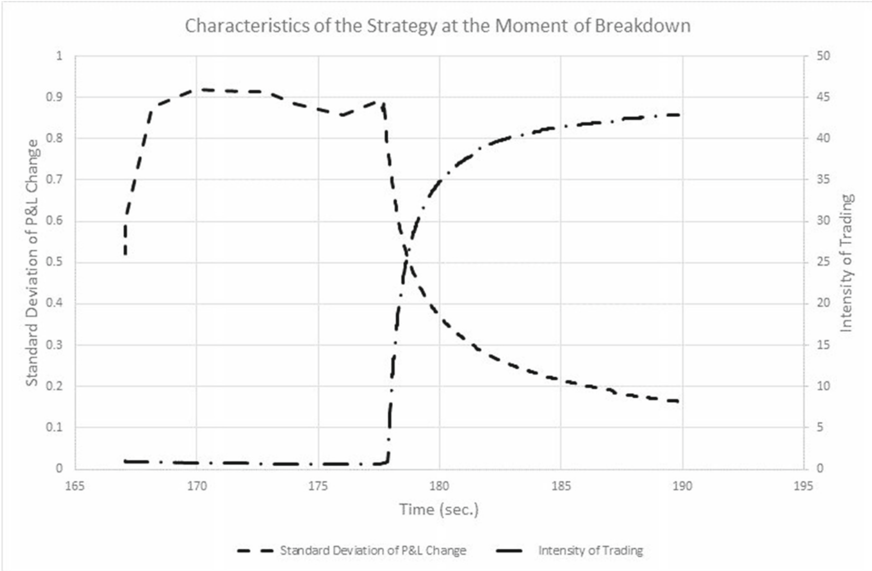

Fig. 3. Characteristics of the Strategy at the Moment of Breakdown

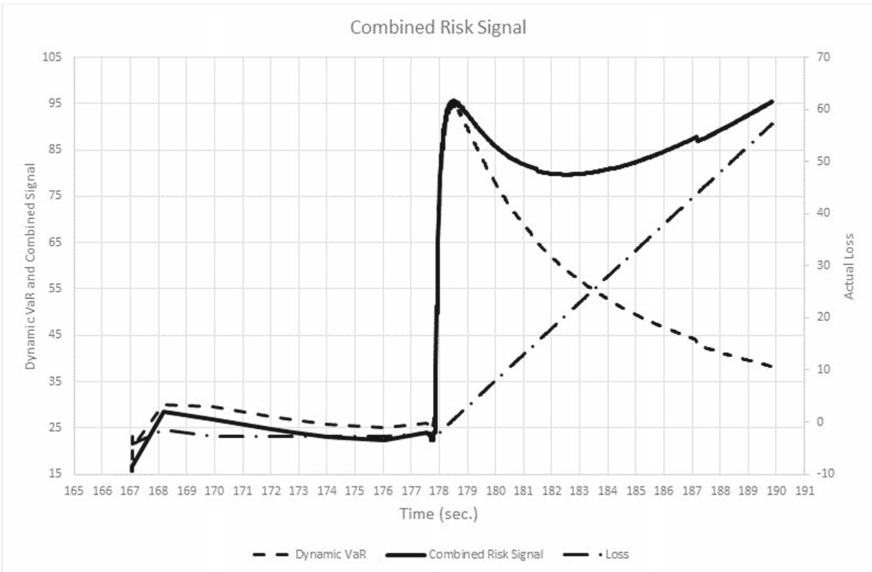

Fig. 4. Combined Risk Signal

While DVaR showed very good reaction at the very beginning of the incident it started declining in the following couple of seconds.

This is because DVaR combines the standard deviation and the frequency in a multiplicative expression: after growing in the first half second the product is dominated by falling standard deviation as seen on Fig. 3.

One way of improving this drawback is combining DVaR with the ex-post P&L in one index shown on Fig. 4 as the black line.

3.3 Organization of the Data Flow

Knowing that the low latency environment is characterized by very high intensity of the data flow and very tight restriction on computer resources it is important to have the process of computation organized as efficiently as possible.

The structure of DVaR allows very efficient calculation because it is based on accumulated intensities of the corresponding events which can be preprocessed, require limited storage in computer memory and can be easily recalculated when necessary.

This follows from the basic model for market price in the form of compounded Cox process and the results of the previous section.

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