# Risk Neutral Portfolio Hedged via Geometric Browning Motion

**By Jack Lu, Aug, 2021**

**Abstract:** Every transaction is buying or selling risks. **The main goal of the strategy is to build a portfolio where future values have zero chance being negative, and non-zero chance of being positive or higher**.

I developed a **C++** option pricing / hedging system, a sample risk neutral portfolio with arbitrage theory. The pros / cons of each pricing algorithms are analyzed. The algorithms below in the **C++** pricing system are all based on arbitrage theory and Geometric Brownian Motion (**GBM**) theory:

- *Black-Scholes model -- analytical solution / Partial Differential Equation(PDE)*
- *Monto Carlo model -- numerical solution, Finite Differential Equation(FDE)*
- *Asian Exotic option model -- arithmetic / geometric averaging*
- *Binomial lattice model (tree) – numerical solution with discrete-time*
- *Jump Diffusion model -- discrete time / price jump*

## 1. Definitions

**Modern financial mathematics are mostly based on, not limited to, Trees, Partial Differential Equation (PDE), and martingales**. Let's define some financial terms and math, then discuss how they work.

### 1.1 Riskless and Riskiness

Fed / government bonds are considered riskless, in that future values can be determined today with certainty. A risky investment is one which returns less than government bonds in the same period. Institutions must avoid excessive riskiness of any assets whose worst possible future values are greater than today's.

### 1.2. Zero-coupon Riskless Bond

It's government bonds which pay no interest but sold at a rate discounted by risk-free rate.

### 1.3 Market Transparency and Efficiency

In a free market, prices should have been discounted with all available information. There is no such thing as good buy. Per Markov theory, there is no point looking back historic graph to predict future price. Today's market value is its best, it's pointless to think otherwise. An asset's price reflects its potential future value, discounted by today's estimated risk. Market finds its optimal equilibrium quickly and efficiently.

For example, a five dollar bill lays on street, it represents an opportunity. It will be picked up instantly by spectator, which immediately eliminates this opportunity.

However, opportunities do exist, which shall materialize in the future. As said: "**Market efficiency works only if someone does not believe in it".**

A quant's job is more about relating price movement of one asset to that of another (for risk hedging), much less of predicting future prices.

## 1.4. Arbitrage

Arbitrage is the opportunity to make money out of nothing. It proves that risk exists, there is no free lunch. Markets can only compensate investors for taking risks that are not diversifiable. If it does, diversifiable risks and compensations are cancelled out each other.

Market inefficiency / arbitrage do exist, it usually takes months or years to be realized / materialized.

## 1.5 Geometric Brownian Motion (GBM)

In nature, GBM, first used by A. Einstein in Physics, implies that things move in random walk (stochastic process) with some constant drifting, until they are hit by external force (kicked). Trend is followed until it doesn't. It's formulated as Partial Differential Equation (PDE) (derivatives which depend on more than one independent parameters are PDE):

$$dS = \mu S\, dt + \sigma S\, dW,$$

where $\mu$ and $\sigma$ are constant, $t$ stands for time, and $W$ is a Wiener process. A Wiener process is a type of stochastic process with a mean change of zero and a variance equal to $\Delta t$—this is simply another way to describe Brownian motion. If the value of a variable following a Wiener process is $x_o$ at time zero, at time $T$, it will be log-normally distributed with mean = $x_o$ and a standard deviation equal to the square root of $T$.

In practice, $\mu$ is the mean of the return over time T (it's normal distribution), σ is the volatility or Beta value of stocks, or implied volatility for option pricing.

GBM implies that asset prices follow a random walk, and the returns are distributed normally.

## 1.6 Black-Scholes Mode

This Nobel Prize winning formula is based on GBM and arbitrage theory, its prices follow log-normal distribution, the price return follows normal bell curve. Here is its formula:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Here is its solution. $d_1$ ($d_+$) and $d_2$ ($d_-$) are probability density function.

$$C(F,\tau) = D\left[N(d_+)F - N(d_-)K\right]$$
$$d_+ = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2\tau\right]$$
$$d_- = d_+ - \sigma\sqrt{\tau}$$

The auxiliary variables are:

- $D = e^{-r\tau}$ is the discount factor
- $F = e^{r\tau}S = \frac{S}{D}$ is the forward price of the underlying asset, and $S = DF$

with $d_+ = d_1$ and $d_- = d_2$ to clarify notation.

Given put–call parity, which is expressed in these terms as:

$$C - P = D(F - K) = S - DK$$

the price of a put option is:

$$P(F,\tau) = D\left[N(-d_-)K - N(-d_+)F\right]$$

## 2 Portfolio Construction and Evaluation

Here are the steps to construct a sample risk neutral portfolio with zero probability of negative value and greater than zero probability of positive value in the future:

a) Given the conditions below, the bank sells call option:

   1) *S = current stock price*
   2) *K = strike price (S < K)*
   3) *r = interest rate*
   4) *T = time to expire*
   5) *σ = volatility*
   6) *C = call option premium calculated using the Black-Scholes model*

b) Calculate the call premium and delta derivative
c) If stock price stays below K, we do nothing
d) Whenever stock price go above S, we buy shares (maybe factional) to cover the portfolio. Whenever stock price goes below K, we sell some shares, until we reach S where we sell all stock shares.
e) At expiry, stock price is S1. Below is the bank's profit:

   1) *If S1 <= S, profit = C*
   2) *If S < S1 <= K, profit = C + (K – S1)*
   3) *If S1 > K, we sell all shares to cover the call option, profit = C*

The key of this sample risk neutral portfolio is to calculate the reasonable call option price, and the delta ratio of stock shares, <u>with assumptions of no transaction cost, constant interest rate, constant volatility, no dividend, no tax</u>.

## 3 C++ Implementation of GBM/PDE

The table below compares option prices from various quant formulas. Please refer to Github for the C++ implementation. The same parameters below are applied to all quant strategies for the purpose of comparison:

| | |
|---|---|
| *K = 100* | *Strike* |
| *S = 100* | *Spot Price* |
| *r = 0.05* | *Risk-Free Interest Rate / Year* |
| *T = 1* | *Time to Expiry (years)* |
| *σ = 0.25* | *volatility* |
| *N = 50* | *Terms in the finite sum approximation* |
| *m = 1.083278* | *Scale Factor* |
| *lambda = 1.0* | *Intensity of Jump* |
| *mu = 0.4* | *Standard Deviation of Log-normal Jump Process* |

**Here are the noticeable from the results:**

a) *Black-Scholes, Monto Carlo, Binomial Lattice produce similar Put / Call prices. Binomial Lattice model can be adjusted to take into dividend;*

b) *Asian Exotic produces much lower Put / Call prices due to its averaging feature, which reduces volatility substantially. Suitable for Fixed Income / Commodity;*

c) *Jump Diffusion Black-Scholes produces much higher Put / Call prices. Jumps happen far more frequent in real market, which leads to higher volatility, and higher option prices.*

d) *Monte Carlo simulations are also less susceptible to sampling errors when comparing to binomial techniques which uses discrete time units.*

| ID | Quant Formula | Call | Put | Delta<br>Call/Put | Gamma<br>Call/Put | Applicable Assets |
|---|---|---|---|---|---|---|
| 1 | *Black-Scholes Analytic / PDE* | 12.336 | 7.45893 | 0.627409/<br>0.372591 | 0.0151368/<br>0.0151368 | EUR Vanilla Options |
| 2 | *Monto Carlo Finite Differential* | 12.342 | 7.45892 | 0.627411/ -<br>0.372589 | 0.0151388/<br>0.0151388 | EUR Vanilla Options |
| 3 | *Asian Exotic – Arithmetical* | 6.82947 | 4.40931 | N/A | N/A | Fixed Income, Capital Mkt, Commodity |
| 4 | *Asian Exotic – Geometric* | 6.4881 | 4.63759 | N/A | N/A | Fixed Income, Capital Mkt, Commodity |
| 5 | *Binomial Lattice Discrete-Time* | 12.3261 | 7.97021 | N/A | N/A | American Options w/o dividend, Fixed Income, Interest Rate Derivative |
| 6 | *Jump Diffusion* | 19.9209 | 15.0438 | N/A | N/A | EUR / American Options with discrete time / price jump, closer to reality |

## 4 Conclusion

Geometric Brownian Motion / Black-Scholas model provide a reasonable good model to estimate option prices, with some limited assumptions, such as constant interest rate, volatility etc. These issues can be overcome through iterative pricing as rate / volatility change.

Binomial Lattice takes a different approach, yet with limitations, such as bigger sampling errors compared to Monto Carlo method, and handling fewer complex variables.

Not losing money is the main goal, maximize profit is another. A risk neutral profolio is a good way to achieve these two goals.

*Performance*:

| | |
|---|---|
| *Black-Scholes analytic* | *O(1) fastest* |
| *Monto Carlo* | *$O(n^k)$ polynomial time* |
| *Asian Exotic* | *$O(n^k)$ polynomial time* |
| *Binomial Lattice* | *$O(2^n)$ slowest* |
| *Jump Diffusion BSM* | *$O(n^k)$ polynomial time* |

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