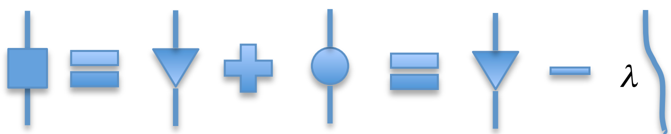
[13.26] Express the coefficients of the polynomial

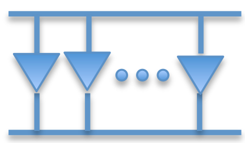
$$\det(\mathbf{T} - \lambda \mathbf{I}) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda) = 0$$

in diagrammatic form. Work them out for n = 2 and n = 3.

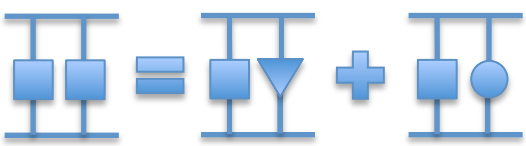
Proof. Beckmann produced a very nice proof that was then further simplified by an elegant enhancement provided by Dean. However, as far as I can tell, neither of them actually "worked out the equations for n = 2 or n = 3" as Penrose requested to generate the polynomial $(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$.

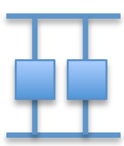
Let **R** = , **T** = , and **S** =  = $-\lambda$ , and set

 = $T^a{}_b - \lambda\delta^a_b$

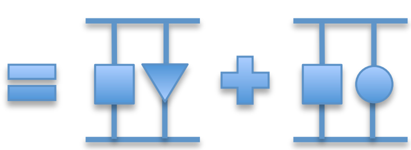
Recall that $\mathrm{Det}(\mathbf{T}) = \dfrac{1}{n!}$  .

We use the following fact repeatedly:

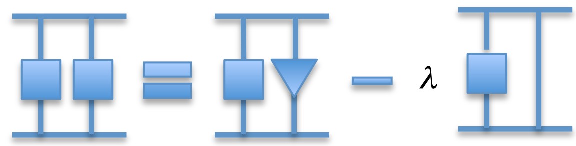
 :

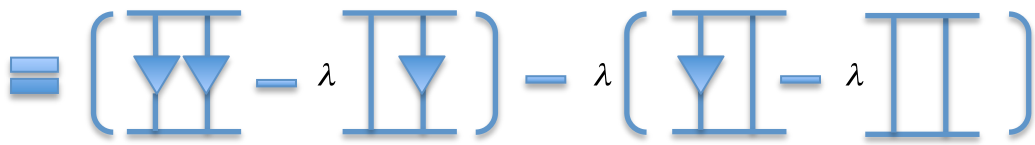
Proof:  $= \in^{ab} R^c{}_a R^d{}_b \varepsilon_{cd} = \in^{ab} R^c{}_a \left(S^d{}_b + T^d{}_b\right)\varepsilon_{cd}$

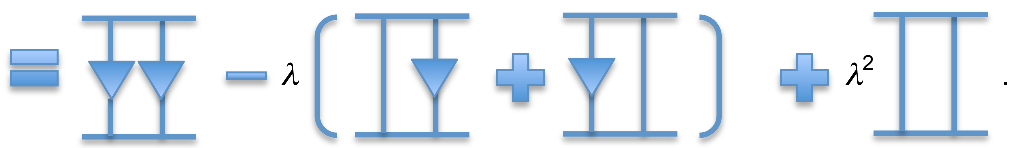
$= \in^{ab} R^c{}_a S^d{}_b \varepsilon_{cd} + \in^{ab} R^c{}_a T^d{}_b \varepsilon_{cd}$

 ✔

***n* = 2**:

2! Det ($\mathbf{T} - \lambda\,\mathbf{I}$) = 



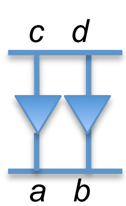


There is a basis such that the matrix of T is triangular, so that

$$\mathbf{T} = \begin{pmatrix} T^1{}_1 & T^1{}_2 \\ T^2{}_1 & T^2{}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}.$$

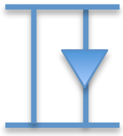
(This is the Jordan Canonical Form. Penrose mentions it in Footnote 13.12.)

I will use the fact that $\in^{12}\varepsilon_{12} = -\in^{12}\varepsilon_{21} = -\in^{21}\varepsilon_{12} = \in^{21}\varepsilon_{21} = 1$ which I proved in problem [13.22]. So,



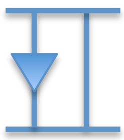
$$= \in^{ab} T^c{}_a T^d{}_b \varepsilon_{cd}$$

$$\begin{aligned} &= \in^{12} T^1{}_1 T^2{}_2 \varepsilon_{12} + \in^{12} T^2{}_1 T^1{}_2 \varepsilon_{21} + \in^{21} T^1{}_2 T^2{}_1 \varepsilon_{12} + \in^{21} T^2{}_2 T^1{}_1 \varepsilon_{21} \\ &= \lambda_1 \lambda_2 - (0)(0) - (0)(0) + \lambda_2 \lambda_1 \\ &= 2\lambda_1 \lambda_2 \end{aligned}$$



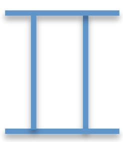
$$
\begin{aligned}
&= \in^{ab}\, \delta_a^c T^d{}_b \varepsilon_{cd} \\
&= \in^{12} T^1{}_1 T^2{}_2 \varepsilon_{12} + \in^{12} T^2{}_1 T^1{}_2 \varepsilon_{21} + \in^{21} T^1{}_2 T^2{}_1 \varepsilon_{12} + \in^{21} T^2{}_2 T^1{}_1 \varepsilon_{21} \\
&= \lambda_1 \lambda_2 - (0)(0) - (0)(0) + \lambda_2 \lambda_1 \\
&= 2\lambda_1 \lambda_2 .
\end{aligned}
$$

Similarly we find that



$$= \lambda_1 + \lambda_2 .$$

Finally,



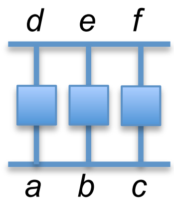
= 2! = 2.

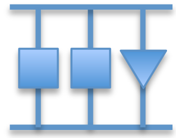
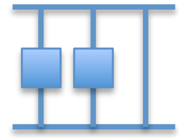
So

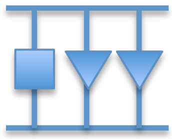
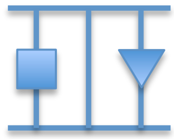
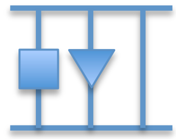
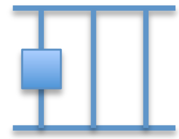
$$
\begin{aligned}
\det(\mathbf{T} - \lambda \mathbf{I}) &= \frac{1}{2}\left[2(\lambda_1 \lambda_2) - 2\lambda(\lambda_1 + \lambda_2) + 2\lambda^2\right] = \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 \\
&= (\lambda_1 - \lambda)(\lambda_2 - \lambda)
\end{aligned}
$$

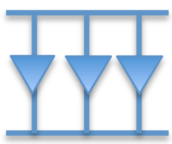
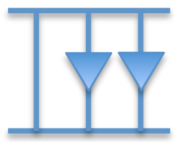
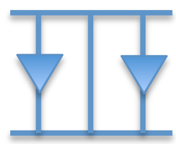
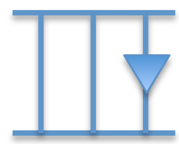
✔

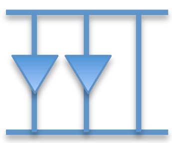
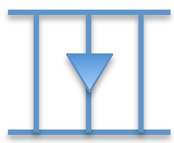
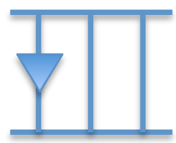
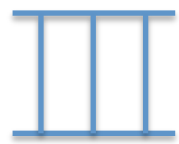
***n* = 3**:

$3!\ \mathrm{Det}\left(\mathbf{T}-\lambda\mathbf{I}\right) =$ 

$=$  $-\,\lambda$ 

$=$  $-\,\lambda$  $-\,\lambda$  $+\,\lambda^2$ 

$=$  $-\,\lambda$  $-\,\lambda$  $+\,\lambda^2$ 

$-\,\lambda$  $+\,\lambda^2$  $+\,\lambda^2$  $-\,\lambda^3$ 

$$=\in^{abc}\varepsilon_{def}\begin{pmatrix} T^d{}_a T^e{}_b T^f{}_c - \lambda\delta^d_a T^e{}_b T^f{}_c - \lambda T^d{}_a \delta^e_b T^f{}_c + \lambda^2\delta^d_a\delta^e_b T^f{}_c \\ -\lambda T^d{}_a T^e{}_b \delta^f_c + \lambda^2\,\delta^d_a T^e{}_b\delta^f_c + \lambda^2\,T^d{}_a\delta^e_b\delta^f_c - \lambda^3\delta^d_a\delta^e_b\delta^f_c \end{pmatrix}$$

$$=\in^{123}\varepsilon_{123}\begin{pmatrix} T^1{}_1 T^2{}_2 T^3{}_3 - \lambda\delta^1_1 T^2{}_2 T^3{}_3 - \lambda T^1{}_1 \delta^2_2 T^3{}_3 + \lambda^2\delta^1_1\delta^2_2 T^3{}_3 \\ -\lambda T^1{}_1 T^2{}_2 \delta^3_3 + \lambda^2\,\delta^1_1 T^2{}_2\delta^3_3 + \lambda^2\,T^1{}_1\delta^2_2\delta 3^3_3 - \lambda^3\delta^1_1\delta^2_2\delta^3_3 \end{pmatrix}$$

$$+\in^{123}\varepsilon_{312}\begin{pmatrix} T^1{}_3 T^2{}_1 T^3{}_2 - \lambda\delta^1_3 T^2{}_1 T^3{}_2 - \lambda T^1{}_3 \delta^2_1 T^3{}_2 + \lambda^2\delta^1_3\delta^2_1 T^3{}_2 \\ -\lambda T^1{}_3 T^2{}_1 \delta^3_2 + \lambda^2\,\delta^1_3 T^2{}_1\delta^3_2 + \lambda^2\,T^1{}_3\delta^2_1\delta^3_2 - \lambda^3\delta^1_3\delta^2_1\delta^3_2 \end{pmatrix}$$

$+\ \ldots$

$$+\in^{321}\varepsilon_{321}\begin{pmatrix} T^3{}_3 T^2{}_2 T^1{}_1 - \lambda\delta^3_3 T^2{}_2 T^1{}_1 - \lambda T^3{}_3 \delta^2_2 T^1{}_1 + \lambda^2\delta^3_3\delta^2_2 T^1{}_1 \\ -\lambda T^3{}_3 T^2{}_2 \delta^1_1 + \lambda^2\,\delta^3_3 T^2{}_2\delta^1_1 + \lambda^2\,T^3{}_3\delta^2_2\delta^1_1 - \lambda^3\delta^3_3\delta^2_2\delta^1_1 \end{pmatrix}.$$

(There are 36 sets of expressions involving T and δ corresponding to 6 permutations of $\in$ times 6 permutations of ε.)

By choosing an appropriate basis we can assume T is in triangular form:

$$\mathbf{T}=\begin{pmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{pmatrix}=\begin{pmatrix} \lambda_1 & b & c \\ 0 & \lambda_2 & f \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

We also use the following fact that I proved in problem [13.22]:

$$\in^{123} \varepsilon_{123} = \in^{132} \varepsilon_{132} = \cdots = \in^{321} \varepsilon_{321} = -\in^{123} \varepsilon_{132} = -\in^{132} \varepsilon_{123} = \cdots = -\in^{321} \varepsilon_{31} = 1$$

In any of the 36 sets of expressions, unless all three of the upper and lower indices match, the given set consists of the sum of eight zeros because all of the T’s and δ’s are zero. So, there are only 6 sets that are non-zero, and

$$\begin{aligned}
&3!\ \mathrm{Det}\left(\mathbf{T} - \lambda\mathbf{I}\right) \\
&= \in^{123} \varepsilon_{123} \begin{pmatrix} T^1{}_1 T^2{}_2 T^3{}_3 - \lambda\delta^1_1 T^2{}_2 T^3{}_3 - \lambda T^1{}_1 \delta^2_2 T^3{}_3 + \lambda^2 \delta^1_1 \delta^2_2 T^3{}_3 \\ -\lambda T^1{}_1 T^2{}_2 \delta^3_3 + \lambda^2 \delta^1_1 T^2{}_2 \delta^3_3 + \lambda^2 T^1{}_1 \delta^2_2 \delta^3_3 - \lambda^3 \delta^1_1 \delta^2_2 \delta^3_3 \end{pmatrix} \\
&+ \in^{312} \varepsilon_{312} \begin{pmatrix} T^3{}_3 T^2{}_1 T^3{}_2 - \lambda\delta^3_3 T^1{}_1 T^2{}_2 - \lambda T^3{}_3 \delta^1_1 T^2{}_2 + \lambda^2 \delta^3_3 \delta^1_1 T^2{}_2 \\ -\lambda T^3{}_3 T^1{}_1 \delta^2_2 + \lambda^2 \delta^3_3 T^1{}_1 \delta^2_2 + \lambda^2 T^3{}_3 \delta^1_1 \delta^2_2 - \lambda^3 \delta^3_3 \delta^1_1 \delta^2_2 \end{pmatrix} \\
&+ \in^{231} \varepsilon_{231} \begin{pmatrix} T^2{}_2 T^3{}_3 T^1{}_1 - \lambda\delta^2_2 T^3{}_3 T^1{}_1 - \lambda T^2{}_2 \delta^3_3 T^1{}_1 + \lambda^2 \delta^2_2 \delta^3_3 T^1{}_1 \\ -\lambda T^2{}_2 T^3{}_3 \delta^1_1 + \lambda^2 \delta^2_2 T^3{}_3 \delta^1_1 + \lambda^2 T^2{}_2 \delta^3_3 \delta^1_1 - \lambda^3 \delta^2_2 \delta^3_3 \delta^1_1 \end{pmatrix} \\
&+ \in^{132} \varepsilon_{132} \begin{pmatrix} T^1{}_1 T^3{}_3 T^2{}_2 - \lambda\delta^1_1 T^3{}_3 T^2{}_2 - \lambda T^1{}_1 \delta^3_3 T^2{}_2 + \lambda^2 \delta^1_1 \delta^3_3 T^2{}_2 \\ -\lambda T^1{}_1 T^3{}_3 \delta^2_2 + \lambda^2 \delta^1_1 T^3{}_3 \delta^2_2 + \lambda^2 T^1{}_1 \delta^3_3 \delta^2_2 - \lambda^3 \delta^1_1 \delta^3_3 \delta^2_2 \end{pmatrix} \\
&+ \in^{213} \varepsilon_{213} \begin{pmatrix} T^2{}_2 T^1{}_1 T^3{}_3 - \lambda\delta^2_2 T^1{}_1 T^3{}_3 - \lambda T^2{}_2 \delta^1_1 T^3{}_3 + \lambda^2 \delta^2_2 \delta^1_1 T^3{}_3 \\ -\lambda T^2{}_2 T^1{}_1 \delta^3_3 + \lambda^2 \delta^2_2 T^1{}_1 \delta^3_3 + \lambda^2 T^2{}_2 \delta^1_1 \delta^3_3 - \lambda^3 \delta^2_2 \delta^1_1 \delta^3_3 \end{pmatrix} \\
&+ \in^{321} \varepsilon_{321} \begin{pmatrix} T^3{}_3 T^2{}_2 T^1{}_1 - \lambda\delta^3_3 T^2{}_2 T^1{}_1 - \lambda T^3{}_3 \delta^2_2 T^1{}_1 + \lambda^2 \delta^3_3 \delta^2_2 T^1{}_1 \\ -\lambda T^3{}_3 T^2{}_2 \delta^1_1 + \lambda^2 \delta^3_3 T^2{}_2 \delta^1_1 + \lambda^2 T^3{}_3 \delta^2_2 \delta^1_1 - \lambda^3 \delta^3_3 \delta^2_2 \delta^1_1 \end{pmatrix} \\
&= 6\lambda_1\lambda_2\lambda_3 - 6\lambda\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) + 6\lambda^2\left(\lambda_1 + \lambda_2 + \lambda_3\right) - 6\lambda^3.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathrm{Det}\left(\mathbf{T} - \lambda\mathbf{I}\right) = \lambda_1\lambda_2\lambda_3 - \lambda\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda\right) + \lambda^2\left(\lambda_1 + \lambda_2 + \lambda_3\right) - \lambda^3 \\
&= \left(\lambda_1 - \lambda\right)\left(\lambda_2 - \lambda\right)\left(\lambda_3 - \lambda\right).
\end{aligned}$$ ✔