[12.16] (Observe the use of the Einstein Summation Convention in this problem). Let $\mathcal{M}$ be an *n*-manifold, $q \leq n$, and M = {1, 2,…, *n*}. Let $\boldsymbol{\alpha}$ be a *p*-form

$\boldsymbol{\alpha} = \alpha_{r \cdots t}\, dx^r \wedge \cdots \wedge dx^t$ and let $\boldsymbol{\psi}$ be its *q*-vector Hodge-like dual $\psi = \psi^{uv \cdots w} \dfrac{\partial}{\partial x^u} \cdots \dfrac{\partial}{\partial x^w}$.

(a) Confirm the equivalency of the 3 conditions for simplicity:

1. $\boldsymbol{\alpha}$ is simple iff $\alpha_{[r \cdots t}\, \alpha_{r']s' \cdots t'} = 0$ for all *p*-tuples (*r*, …, *t*) and (*r*', …, *t*') in M^p
2. $\boldsymbol{\psi}$ is simple iff $\psi^{[u \cdots w}\, \psi^{u']v' \cdots w'} = 0$ for all *q*-tuples (*u*, …, *w*) and (*u*', …, *w*') in M^q
3. $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$ are both simple iff $\psi^{u \cdots w}\, \alpha_{ws \cdots t} = 0$ for all *p*-tuples (*w*, *s* …, *t*) and *q*-tuples (*u*, …, *w*).

(b) Show that a 2-form $\boldsymbol{\alpha} = \alpha_{rs}\, dx^r \wedge dx^s$ in $\mathcal{M}$ is simple iff $\alpha_{[rs}\, \alpha_{r']s'} = 0$ for all pairs (*r*, *s*) and (*r*', *s*') in M^2

Proof: Einstein summation convention is used throughout this proof.

This is a rework for myself of Juergen Beckmann's proof with additional details and different wording so that I can more easily follow it. It also isimplifies his part (b) proof by skipping the vectors that Penrose mentions in his hint.

(a)

Preface: I find it easier to use equations rather than proportions for the Hodge-like duals $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$. The most logical choice is to make both proportionality constants the same, $\dfrac{1}{\sqrt{n!}}$, as in

(i) $\alpha_{r \cdots t} = \dfrac{1}{\sqrt{n!}} \varepsilon_{r \cdots t u \cdots w} \psi^{u \cdots w}$ where ε is an *n*-form, and

(ii) $\psi^{u \cdots w} = \dfrac{1}{\sqrt{n!}} \alpha_{r \cdots t} \in^{r \cdots t u \cdots w}$ where $\in$ is an *n*-vector such that

(iii) $\varepsilon \bullet \in = \varepsilon_{r \cdots t u \cdots w} \in^{r \cdots t u \cdots w} = n!$:

To see that this works, set the constant to $\dfrac{1}{\sqrt{n!}}$ in (ii). We then see that (i) must have the same constant [due to equation (iii)] since

$$\alpha_{r\cdots t} \overset{\text{(iii)}}{=} \alpha_{r\cdots t}\frac{\boldsymbol{\varepsilon}\bullet\in}{n!} \overset{\text{(iii)}}{=} \frac{1}{n!}\alpha_{r\cdots t}\varepsilon_{r\cdots tu\cdots w}\in^{r\cdots tu\cdots w} = \left(\frac{1}{\sqrt{n!}}\alpha_{r\cdots t}\in^{r\cdots tu\cdots w}\right)\left(\frac{1}{\sqrt{n!}}\varepsilon_{r\cdots tu\cdots w}\right)$$

$$\overset{\text{(ii)}}{=} \frac{1}{\sqrt{n!}}\varepsilon_{r\cdots tu\cdots w}\psi^{u\cdots w} \qquad ✔$$

Notice that the indices used in the *p*-tuple in (3) are slightly different than in (1) and (2). Because in this proof we will be switching often between the expressions in (1), (2), and (3) we require a set of non-conflicting indices. To do this we re-write (i) and (ii):

(iv) $\alpha_{ws\cdots t} = \frac{1}{\sqrt{n!}}\varepsilon_{ws\cdots tx\cdots z}\psi^{x\cdots z}$ where *r…t* are given distinct members of M and *x…z* sums over the remaining members of M, and

(v) $\psi^{u\cdots w} = \frac{1}{\sqrt{n!}}\alpha_{a\cdots c}\in^{a\cdots cu\cdots w}$ where *u…w* are given distinct members of M and *a…c* sums over the remaining members of M.

We also must keep in mind the following fact:

(vi) Each set *r…t x…z* and *a…c u…w* is a permutation of the members of M = {1, 2,…, *n*}.

To prove part (a) it is sufficient to show that $\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'} = 0$ iff $\psi^{[u\cdots w}\,\psi^{u']v'\cdots w'} = 0$ iff $\psi^{u\cdots w}\alpha_{ws\cdots t} = 0$ since then we will have $\boldsymbol{\alpha}$ is simple iff $\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'} = 0$ iff $\psi^{[u\cdots w}\,\psi^{u']v'\cdots w'} = 0$ iff $\boldsymbol{\psi}$ is simple iff $\psi^{u\cdots w}\alpha_{ws\cdots t} = 0$ iff $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$ are both simple.

$$\psi^{u\cdots w}\alpha_{ws\cdots t} \overset{\text{(v)}}{=} \left(\frac{1}{\sqrt{n!}}\alpha_{a\cdots c}\in^{a\cdots cu\cdots w}\right)\alpha_{ws\cdots t} = \frac{1}{\sqrt{n!}}\alpha_{a\cdots c}\,\alpha_{ws\cdots t}\in^{a\cdots cu\cdots w}$$

$$\overset{\text{(vi)}}{=} \frac{\pm 1}{\sqrt{n!}}\sum_{\pi\in\mathcal{P}} sign(\pi)\,\alpha_{\pi(a)\cdots\pi(c)}\,\alpha_{\pi(w)s\cdots t}\in^{1\,2\cdots n} \qquad \text{(± because } \in \text{ is antisymmetric)}$$

$$\underset{\text{antisym}}{\overset{\text{Defn of}}{=}} \frac{\pm 1}{\sqrt{n!}}\alpha_{[a\cdots c}\,\alpha_{w]s\cdots t}\in^{1\,2\cdots n}$$

$$\underset{\text{indices}}{\overset{\text{Rename}}{=}} \frac{\pm 1}{\sqrt{n!}}\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'}\in^{1\,2\cdots n} \textit{ where } \mathcal{P} \textit{ is the set of permutations of } (a,\ldots,c,w).$$

Therefore $\psi^{u\cdots w}\alpha_{ws\cdots t} = 0$ iff $\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'} = 0$. ✔

$$\psi^{u\cdots w}\,\alpha_{ws\cdots t} \overset{\text{(iv)}}{=} \psi^{u\cdots w}\left(\frac{1}{\sqrt{n!}}\varepsilon_{ws\cdots t\,x\cdots z}\psi^{x\cdots z}\right)\alpha_{ws\cdots t} = \frac{1}{\sqrt{n!}}\psi^{x\cdots z}\;\psi^{u\cdots vw}\;\varepsilon_{ws\cdots t\,x\cdots z}$$

$$= \frac{\pm 1}{\sqrt{n!}}\psi^{x\cdots z}\;\psi^{wu\cdots v}\;\varepsilon_{x\cdots z ws\cdots t} \overset{\text{(vi)}}{=} \frac{\pm 1}{\sqrt{n!}}\sum_{\pi\in\mathcal{P}^*} sign(\pi)\;\psi^{\pi(x)\cdots\pi(z)}\;\psi^{\pi(w)u\cdots v}\;\varepsilon_{1\,2\cdots n}$$

$$\underset{\text{antisym}}{\overset{\text{Defn of}}{=}} \frac{\pm 1}{\sqrt{n!}}\psi^{[x\cdots z}\;\psi^{w]u\cdots v}\;\varepsilon_{1\,2\cdots n}$$

$$\underset{\text{indices}}{\overset{\text{Rename}}{=}} \frac{\pm 1}{\sqrt{n!}}\psi^{[u\cdots w}\;\psi^{u']v'\cdots w'}\;\varepsilon_{1\,2\cdots n} \text{ where } \mathcal{P}^* \text{ is the set of permutations of } (x,\ldots,z,w).$$

Therefore $\psi^{u\cdots w}\,\alpha_{ws\cdots t} = 0$ iff $\psi^{[u\cdots w}\;\psi^{u']v'\cdots w'} = 0.$ ✔

(b).

Recall that $\boldsymbol{\alpha}$ is antisymmetric in (r, s) :

$\boldsymbol{\alpha}$ is a sum of, say, q simple 2-forms. So there are simple 1-forms $\boldsymbol{\gamma}^k$ and $\boldsymbol{\delta}^k$ such that $\boldsymbol{\alpha} = \sum_{k=1}^{q}\boldsymbol{\gamma}^k \wedge \boldsymbol{\delta}^k = \sum_{k=1}^{q}\gamma^k_{[r}\,\delta^k_{s]}\;dx^r \wedge dx^s = \alpha_{rs}\;dx^r \wedge dx^s$

That is,

$$(*) \qquad \alpha_{rs} = \sum_{k=1}^{n}\gamma^k_{[r}\delta^k_{s]} = -\sum_{k=1}^{n}\gamma^k_{[s}\delta^k_{r]} = -\alpha_{rs}$$

Thus $\boldsymbol{\alpha}$ is antisymmetric in (r, s). ✔

$\boldsymbol{\alpha}$ is simple iff $\exists$ 1-forms $\boldsymbol{\gamma} = \gamma_r\,dx^r$ and $\boldsymbol{\delta} = \delta_s\,dx^s \ni \boldsymbol{\alpha} = \boldsymbol{\gamma}\wedge\boldsymbol{\delta} = \gamma_{[r}\,\delta_{s]}\;dx^r \wedge dx^s$. That is, $\boldsymbol{\alpha}$ is simple iff the components α_{rs} of α satisfy $\alpha_{rs} = \gamma_{[r}\delta_{s]}\;\forall r,s$.

"IF"

Suppose $\alpha_{[rs}\alpha_{r']s'} = 0$ for all pairs (r, s) and (r', s') in M^2.

If $\boldsymbol{\alpha}$ = 0, clearly $\boldsymbol{\alpha}$ is simple (e.g., $\boldsymbol{\alpha} = dx^1 \wedge dx^1$). So, suppose $\boldsymbol{\alpha} \neq 0$. Then $\exists\ a, b \in \{1, \ldots, n\}$ such that $\alpha_{ab} \neq 0$. Define

$$\text{(i)} \quad \gamma = \gamma_r\,dx^r \text{ where } \gamma_r = 2\frac{\alpha_{rb}}{\alpha_{ab}} \quad \text{and} \quad \text{(ii)} \quad \boldsymbol{\delta} = \delta_s\,dx^s \text{ where } \delta_s = \alpha_{as}.$$

Consequently (i') $\gamma_s = 2\dfrac{\alpha_{sb}}{\alpha_{ab}}$ and (ii') $\delta_r = \alpha_{ar}$.

$$\text{(iii)} \quad \gamma_{[r}\delta_{s]} = \tfrac{1}{2}\left(\gamma_r\delta_s - \gamma_s\delta_r\right) \overset{\text{(i, ii, i', ii')}}{=} \frac{\alpha_{rb}\alpha_{as} - \alpha_{sb}\alpha_{ar}}{\alpha_{ab}}.$$

Now $0=\alpha_{[rs}\alpha_{a]b}=\frac{1}{6}\left(\alpha_{rs}\alpha_{ab}+\alpha_{sa}\alpha_{rb}+\alpha_{ar}\alpha_{sb}-\alpha_{sr}\alpha_{ab}-\alpha_{as}\alpha_{rb}-\alpha_{ra}\alpha_{sb}\right)$, or

$$\text{(iv)}\quad \alpha_{rb}\alpha_{as}-\alpha_{sb}\alpha_{ar}=\alpha_{ab}\alpha_{rs}+\alpha_{rb}\alpha_{sa}-\alpha_{ab}\alpha_{sr}-\alpha_{sb}\alpha_{ra}$$

Plugging (iv) into (iii) yields

$$\begin{aligned}\gamma_{[r}\delta_{s]}&\overset{\text{(iii)}}{=}\frac{\alpha_{rb}\alpha_{as}-\alpha_{sb}\alpha_{ar}}{\alpha_{ab}}\overset{\text{(iv)}}{=}\frac{\alpha_{ab}\alpha_{rs}+\alpha_{rb}\alpha_{sa}-\alpha_{ab}\alpha_{sr}-\alpha_{sb}\alpha_{ra}}{\alpha_{ab}}\\&\overset{(*)}{=}\alpha_{rs}-\left(\frac{\alpha_{rb}}{\alpha_{ab}}\right)\alpha_{as}-\alpha_{sr}+\left(\frac{\alpha_{sb}}{\alpha_{ab}}\right)\alpha_{ar}\\&\overset{\text{(i, ii, i', ii')}}{=}\alpha_{rs}-\tfrac{1}{2}\gamma_r\delta_s-\alpha_{sr}+\tfrac{1}{2}\gamma_s\delta_r\overset{(*)}{=}2\alpha_{rs}-\gamma_{[r}\delta_{s]}\end{aligned}$$

$\Rightarrow\quad \alpha_{rs}=\delta_{[r}\gamma_{s]}$

$\Rightarrow\quad \boldsymbol{\alpha}$ is simple ✔

“ONLY IF”

Suppose $\alpha=\alpha_{rs}\,dx^r\wedge dx^s$ is simple. Then $\exists$ 1-forms $\gamma=\gamma_r dx^r$ and $\delta=\delta_s dx^s$ such that

$$\text{(v)}\quad \alpha_{rs}=\gamma_{[r}\delta_{s]}\quad \forall r,s.$$

So

$$\begin{aligned}\alpha_{[rs}\alpha_{u]v}&=\tfrac{1}{6}\left(\alpha_{rs}\alpha_{uv}+\alpha_{su}\alpha_{rv}+\alpha_{ur}\alpha_{sv}-\alpha_{sr}\alpha_{uv}-\alpha_{us}\alpha_{rv}-\alpha_{ru}\alpha_{sv}\right)\\&=\tfrac{1}{6}\left[\left(\alpha_{rs}-\alpha_{sr}\right)\alpha_{uv}+\left(\alpha_{su}-\alpha_{us}\right)\alpha_{rv}+\left(\alpha_{ur}-\alpha_{ru}\right)\alpha_{sv}\right]\\&\overset{(*)}{=}\tfrac{1}{3}\left(\alpha_{rs}\alpha_{uv}+\alpha_{su}\alpha_{rv}+\alpha_{ur}\alpha_{sv}\right)\\&\overset{(v)}{=}\tfrac{1}{3}\left(\gamma_{[r}\delta_{s]}\gamma_{[u}\delta_{v]}+\gamma_{[s}\delta_{u]}\gamma_{[r}\delta_{v]}+\gamma_{[u}\delta_{r]}\gamma_{[s}\delta_{v]}\right)\\&=\left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\begin{Bmatrix}\left(\left[\gamma_r\delta_s-\gamma_s\delta_r\right]\left[\gamma_u\delta_v-\gamma_v\delta_u\right]\right)+\left(\left[\gamma_s\delta_u-\gamma_u\delta_s\right]\left[\gamma_r\delta_v-\gamma_v\delta_r\right]\right)\\+\left(\left[\gamma_u\delta_r-\gamma_r\delta_u\right]\left[\gamma_s\delta_v-\gamma_v\delta_s\right]\right)\end{Bmatrix}\\&=\tfrac{1}{12}\begin{bmatrix}\gamma_r\delta_s\gamma_u\delta_v-\gamma_r\delta_s\gamma_v\delta_u-\gamma_s\delta_r\gamma_u\delta_r+\gamma_s\delta_r\gamma_v\delta_u\\+\gamma_s\delta_u\gamma_r\delta_v-\gamma_s\delta_u\gamma_v\delta_r-\gamma_u\delta_s\gamma_r\delta_v+\gamma_u\delta_s\gamma_v\delta_r\\+\gamma_u\delta_r\gamma_s\delta_v-\gamma_u\delta_r\gamma_v\delta_s-\gamma_r\delta_u\gamma_s\delta_v+\gamma_r\delta_u\gamma_v\delta_s\end{bmatrix}\\&\overset{(4)}{=}0\,.\end{aligned}$$

✔