Chapter 13. Symmetry Groups

Groups

Definitions:
A **group** is a set G with an operation $\circ$ that is closed and associative, has an identity e, and every element g has an inverse g^{-1} such that $g \circ g^{-1}$ = e = $g^{-1} \circ g$.$\circ$

A group G is **Abelian** if it is commutative: $g \circ h$ = $h \circ g$ for all g, h in G.

A **coset** of G is a set Gh = {gh: g$\in$G}, where $h \in$G.

A **subgroup** is a subset of G that is a group under $\circ$.

A **normal subgroup** is a subgroup H that satisfies $g \circ$H = H$\circ g$ for all g in G, or equivalently H = $g^{-1} \circ$H$\circ g$.

A group is **simple** if it contains no non-trivial normal subgroup.

The simple groups are the fundamental "building blocks" of more complex groups.

Theorem. There are precisely 4 **classical** and 5 **exceptional** simple Lie groups.

- Classical Families: A_m, B_m, C_m, D_m having dimensions $m(m+2)$, $m(2m+1)$, $m(2m+1)$, and $m(2m-1)$, respectively where $m \in \mathbb{Z}^+$.
- Exceptional Groups: E_6, E_7, E_8, F_4, G_2 of dimension 78, 133, 248, 52, and 14 respectively

Theorem. The simple finite groups have been classified into classical and exceptional groups. The largest exceptional group has ≈ 10^{60} elements and is known as **the monster.**

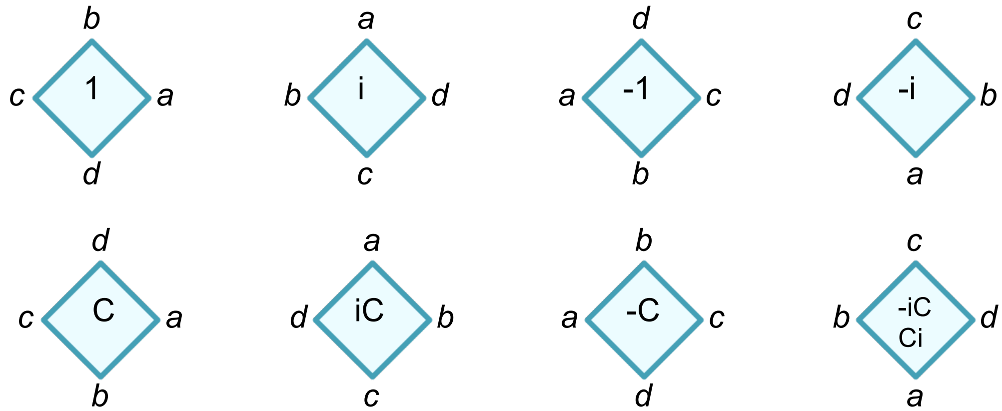
Definition. The **Product Group** of groups G and H is **GxH** = $\{(g,h): g \in G, h \in H\}$ with group operation $(g_1,h_1)\circ(g_2,h_2) = (g_1 \circ g_2, h_1 \circ h_2)$.

Definition. Let N be a subgroup of G. The **Factor Space G/N** is the collection of cosets G$\circ n$ along with the operation (G$\circ n_1$)$\circ$(G$\circ n_2$) = G$\circ$($n_1 \circ n_2$).

Theorem. If N is normal then G/N is a group, called the **Factor Group.**

Theorem. (Problem [13.10]) H$\cong G \times H$/G.

Symmetries of a Square



Definitions**:**
Non-reflecting Group: < i > = {1, i, -1, -i}
Reflecting Group: < i, C > = {1, i, -1, -i, C, iC, -C, -iC = Ci}
C is complex conjugation: $a+bi \mapsto a-bi$. **1** is the null rotation, which is the group identity element. ***i*** is the 90° counter-clockwise rotation of the square

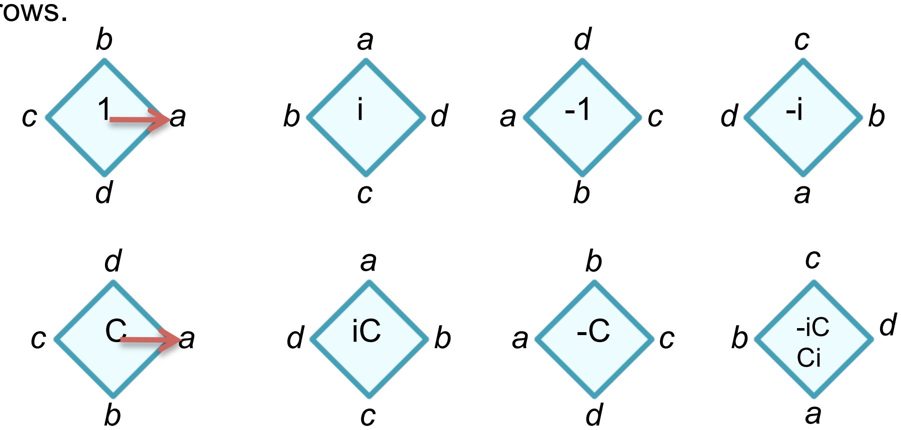
Convention: ab means b acts first.

A subgroup of a symmetry group is called a **reduced symmetry group**.

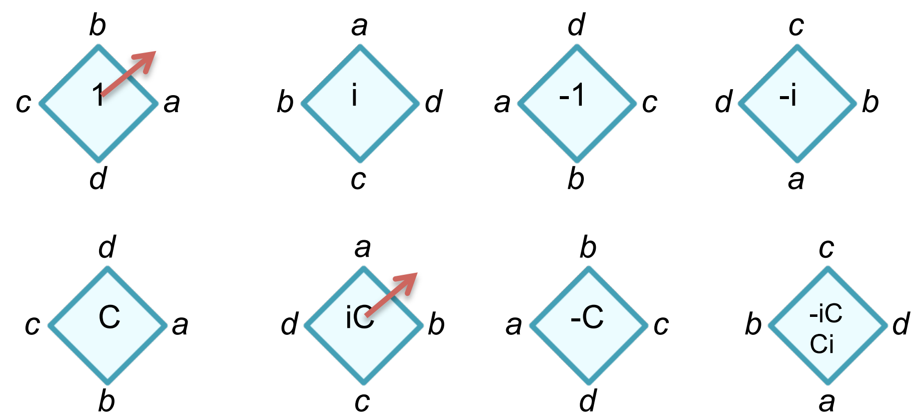
Examples:

Normal subgroups of < i, C > :
{1, -1, C, -C} , {1, -1}, {1, -1}
Non-normal subgroups of < i, C > :
{1, -C}, {1, iC}, {1, C}
For example, {1, C} i = {i, Ci} ≠ {i, -Ci} = i {1, C}

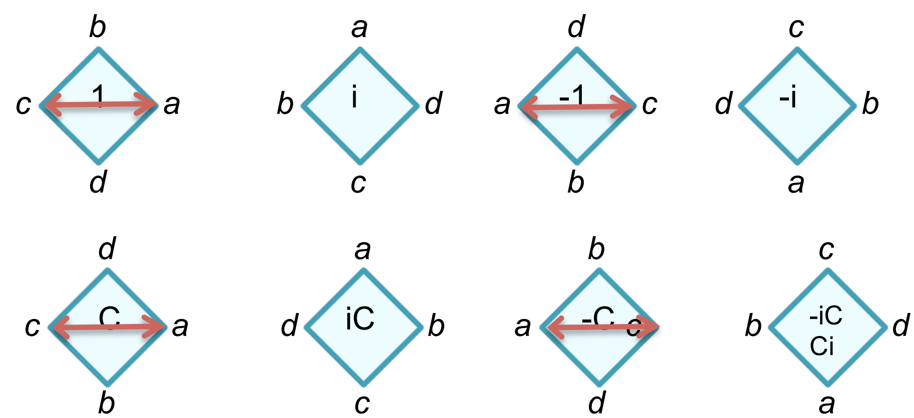
Example [13.6]: Reduced symmetry groups can be generated using one or more arrows.



{1, C} is a reduced symmetry group



{1, iC} is a reduced symmetry group



{1, -1, C, -C} is a reduced symmetry group

Symmetries of a Sphere

Definitions:
A group G whose underlying set is continuous is called a **Lie Group.**
SO(3) is the group of non-reflective symmetries of a 3-sphere
O(3) is the **Orthogonal Group**. It consists of both the reflective and non-reflective symmetries of a sphere.

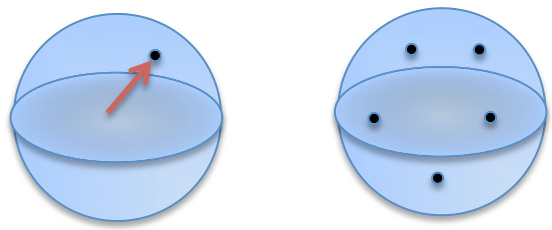
O(3) = SO(3) $\cup$ T, the disjoint union of O(3) with the coset of reflective symmetries

$\mathbf{T} = \mathrm{R\,SO}(3) = \{\mathrm{R}g : g \in \mathrm{SO}(3)\}$ where **R** is the reflection operator on the sphere.

Recall problem [12.7]: SO(3) is group isomorphic to the solid sphere $\mathcal{R}$ of radius π with antipodal points identified.

Theorem. (Problem [13.7]) SO(3) and {1, R} are the only normal subgroups of O(3), where **1** is the null rotation. (Penrose overlooked that the latter group is normal.)

Examples. Reduced Symmetry Groups



The set of rotations that fix a point on the sphere forms a non-normal subgroup. It is the set of rotations having the arrow as its axis.

Marking the sphere with vertices of a regular polyhedra reduces to the finite group of rotations of the sphere that take each vertex to one of the others. Such reduced symmetry groups are non-normal.

Linear Transformations and Matrices

Definition. Let V and W be vector spaces.

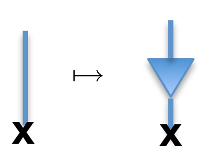
- $f: V \to W$ is a **homomorphism** if it preserves the vector space structure:
    - $f(au + bv) = af(u) + bf(v)$ for all vectors u and v and scalars a and b.
- **Hom(V,W)** is the set of homomorphisms from V to W.
- **$\mathcal{A}$(V)** = Hom(V,V).
- A **linear transformation** is a member $T \in \mathcal{A}(V)$.
    - That is, a linear transformation is a function $T: V \to V$ such that $T(au + bv) = aTu + bTv$.

Theorem. [13.12, 13.13] Let V = $\mathbb{R}^3$, using (x^1, x^2, x^2) instead of (x, y, z). Then a linear transformation T takes the form $T: x^r \mapsto T^r{}_s x^s = ax^1 + bx^2 + cx^3$.

Note. Linear transformations are represented by matrices:

$$T: \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{or} \quad x \mapsto Tx$$

In diagrammatic form this is



Theorem. If R = ST then $R^a{}_c = S^a{}_b T^b{}_c$. That is, the composition, R, of 2 linear transformations is the result of matrix multiplication of S and T. In diagrammatic notation:

R =  =  = ST

Example. TI = T = IT is written in diagrammatic form as  =  =  .

and, in $\mathbb{R}^3$, I = $\delta^a_b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where a, b range over {1, 2, 3}.

Definitions. A linear transformation T is **singular** if Dim(TV) < Dim W; that is, T is not *onto*.

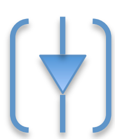
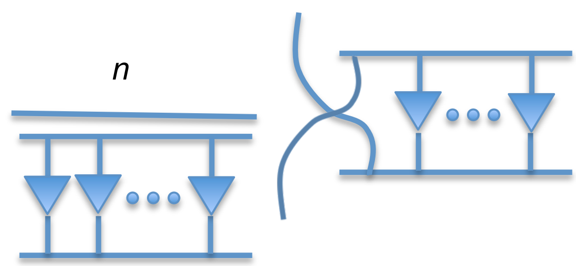
Theorem. [13.17] T is singular iff $\exists\ v \neq 0$ such that Tv = 0.

Corollary. [Bud] T is 1-1 iff T is non-singular iff T is onto.

Proof: T is 1-1 $\Leftrightarrow \forall v \neq w\ \ \mathrm{T}(v - w) = \mathrm{T}(v) - \mathrm{T}(w) \neq 0 \ \overset{(*)}{\Leftrightarrow}\ \forall u \neq 0\ \mathrm{T}(u) \neq 0$

$\overset{[13.17]}{\Leftrightarrow}$ T is non-singular $\Leftrightarrow$ T is onto.
(*) Set v = 3u and w = 2u. ■

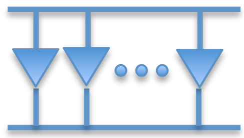
Theorem. [13.18] If T is nonsingular, then it has an inverse T^{-1}.

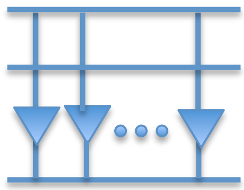
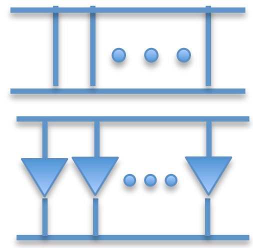
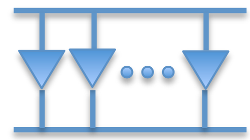
Theorem. [13.19] T^{-1} =  $^{-1}$ = 

Definition. The **transpose** of the matrix T = $(\mathrm{T}^i{}_j)$ is the matrix $\mathbf{T}^{\mathbf{T}}$ = $(\mathrm{T}^j{}_i)$.

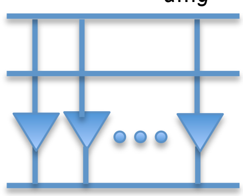
Definition. A matrix T is **orthogonal** if $\mathrm{T}^{-1} = \mathrm{T}^{\mathrm{T}}$.

Determinants and Traces

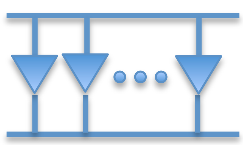
Definition. Det T = $\frac{1}{n!}$  $= \frac{1}{n!}\in^{ab\cdots d} T^e{}_a T^f{}_b \cdots T^h{}_d \,\varepsilon_{ef\cdots h}$.

Theorem. [Bud]  $=$  $=$ $n!$ 

Proof: Let $P_{a\ldots g}$ be the set of permutations of (a, …, g). Then

 $= \quad n!\,\varepsilon_{a\cdots g}\in^{r\cdots x} T^{[a}{}_r\cdots T^{g]}{}_x$

$$= \frac{n!}{n!}\varepsilon_{a\cdots g}\in^{r\cdots x}\sum_{\pi\in\mathcal{P}_{ab\cdots g}} \mathrm{Sign}(\pi)\, T^{\pi(a)}{}_r\cdots T^{\pi(g)}{}_x \overset{(*)}{=} n!\,\varepsilon_{a\cdots g} T^a{}_r\cdots T^g{}_x \in^{r\cdots x}$$

= $n!$ 

(*) π is the composition of transmutations (i.e., of pairwise permutations).

Let π^*: $\begin{matrix} c\mapsto e \\ e\mapsto c\end{matrix}$ be a transmutation. Then

$$\varepsilon_{a\cdots c\cdots e\cdots g}\in^{r\cdots t\cdots v\cdots x}\mathrm{Sign}(\pi^*)\,T^{\pi(a)}{}_r\cdots T^{\pi(c)}{}_t\cdots T^{\pi(e)}{}_v\cdots T^{\pi(g)}{}_x$$

$$=\varepsilon_{a\cdots c\cdots e\cdots g}\in^{r\cdots t\cdots v\cdots x}\mathrm{Sign}(\pi^*)\,T^{a}{}_r\cdots T^{e}{}_t\cdots T^{c}{}_v\cdots T^{g}{}_x$$

$$=\varepsilon_{a\cdots e\cdots c\cdots g}\in^{r\cdots t\cdots v\cdots x}\mathrm{Sign}(\pi^*)\,T^{a}{}_r\cdots T^{c}{}_t\cdots T^{e}{}_v\cdots T^{g}{}_x \quad \text{(Rename } c\mapsto e \text{ \& } e\mapsto c\text{)}$$

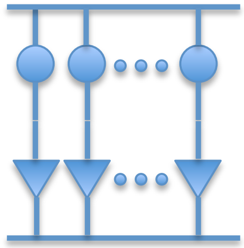
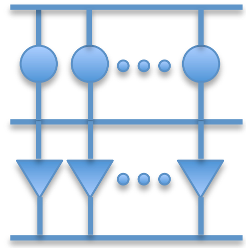
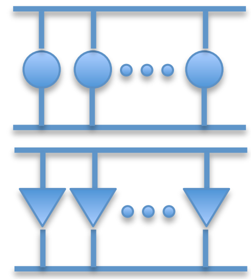
$$=\mathrm{Sign}(\pi^*)\,\varepsilon_{a\cdots c\cdots e\cdots g}\in^{r\cdots t\cdots v\cdots x}\mathrm{Sign}(\pi^*)\,T^{a}{}_r\cdots T^{c}{}_t\cdots T^{e}{}_v\cdots T^{g}{}_x$$

$$=\varepsilon_{a\cdots g}\,T^a{}_r\cdots T^g{}_x\in^{r\cdots x}.$$

So, for any permutation π, we have

$$\varepsilon_{a\cdots g}\in^{r\cdots x}\mathrm{Sign}(\pi)\,T^{\pi(a)}{}_r\cdots T^{\pi(g)}{}_x = \varepsilon_{a\cdots g}\,T^a{}_r\cdots T^g{}_x\in^{r\cdots x} \quad ■$$

Theorem. [13.22]

Det AB = $\frac{1}{n!}$  $= \left(\frac{1}{n!}\right)^2$  $= \left(\frac{1}{n!}\right)^2$ 

= DetA DetB

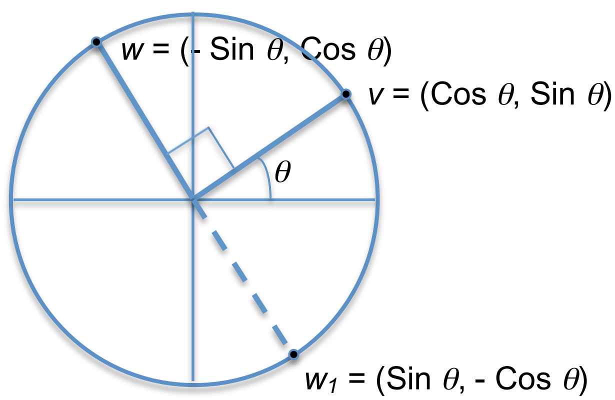
Theorem. (p.260 – no proof given) Matrix A is singular iff Det A = 0.

Proof: From [13.19], A is non-singular iff Det A ≠ 0. ■

Definition. Vectors v and w are **orthogonal** if $v \bullet w = 0$. That is, the angle between them is 90°.

Theorem. A matrix is orthogonal (i.e., $T^T = T^{-1}$) iff its column vectors are mutually orthogonal.

Example. Orthogonal 2 x 2 Matrices: A and B



Let $A = \begin{pmatrix} v & w \end{pmatrix}$

$= \begin{pmatrix} \text{Cos}\,\theta & -\text{Sin}\,\theta \\ \text{Sin}\,\theta & \text{Cos}\,\theta \end{pmatrix}.$

$A^T = \begin{pmatrix} \text{Cos}\,\theta & \text{Sin}\,\theta \\ -\text{Sin}\,\theta & \text{Cos}\,\theta \end{pmatrix}.$

$A^T = A^{-1}$:

$$A\,A^T = \begin{pmatrix} \text{Sin}^2\theta + \text{Cos}^2\theta & 0 \\ 0 & \text{Sin}^2\theta + \text{Cos}^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I} \quad ✔$$

Similarly $A^T A = \mathrm{I}$ ✔

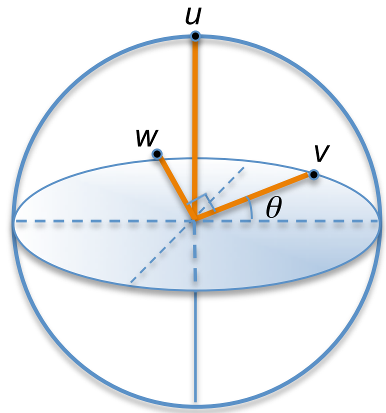
So A is an orthogonal matrix ✔

Det A = Det A^T = $\text{Cos}^2\,\theta + \text{Sin}^2\,\theta = 1$ ✔

The column vectors of A are orthogonal: $v \perp w$ ✔

Let $\mathrm{B} = \begin{pmatrix} \mathrm{Cos}\,\theta & \mathrm{Sin}\,\theta \\ \mathrm{Sin}\,\theta & -\mathrm{Cos}\,\theta \end{pmatrix} = \mathrm{B}^{\mathrm{T}}$. Then B B^{T} = I, Det B = Det B^{T} = -1, and its column vectors v and w_1 are orthogonal.

Examples. Orthogonal 3 x 3 Matrices: A, B, and C



Let $v = \begin{pmatrix} \mathrm{Cos}\,\theta \\ \mathrm{Sin}\,\theta \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} -\mathrm{Sin}\,\theta \\ \mathrm{Cos}\,\theta \\ 0 \end{pmatrix}$, and $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Let $\mathrm{A} = \begin{pmatrix} v & w & u \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}\,\theta & -\mathrm{Sin}\,\theta & 0 \\ \mathrm{Sin}\,\theta & \mathrm{Cos}\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} \mathrm{Cos}\,\theta & \mathrm{Sin}\,\theta & 0 \\ -\mathrm{Sin}\,\theta & \mathrm{Cos}\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. A is orthogonal, its columns are orthogonal vectors, and its determinant is +1. ✔

Let $\mathrm{B} = \begin{pmatrix} \mathrm{Cos}\,\theta & -\mathrm{Sin}\,\theta & 0 \\ \mathrm{Sin}\,\theta & \mathrm{Cos}\,\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$. B is orthogonal and its determinant is -1. ✔

Let C be a θ-rotation of A about an axis $\left\{t(a,b,c)\!:\ 0 < t < \infty,\ a^2+b^2+c^2=1\right\}$:

$$\mathrm{C} = \begin{pmatrix} \frac{1}{2}\left[1+a^2-b^2-c^2+\left(1-a^2+b^2+c^2\right)\mathrm{Cos}\,\theta\right] & 2\mathrm{Sin}\,\frac{\theta}{2}\left(-c\,\mathrm{Cos}\,\frac{\theta}{2}+ab\,\mathrm{Sin}\,\frac{\theta}{2}\right) & 2\mathrm{Sin}\,\frac{\theta}{2}\left(b\,\mathrm{Cos}\,\frac{\theta}{2}+ac\,\mathrm{Sin}\,\frac{\theta}{2}\right) \\ 2\mathrm{Sin}\,\frac{\theta}{2}\left(c\,\mathrm{Cos}\,\frac{\theta}{2}+ab\,\mathrm{Sin}\,\frac{\theta}{2}\right) & \frac{1}{2}\left[1-a^2+b^2-c^2+\left(1+a^2-b^2+c^2\right)\mathrm{Cos}\,\theta\right] & 2\mathrm{Sin}\,\frac{\theta}{2}\left(-a\,\mathrm{Cos}\,\frac{\theta}{2}+bc\,\mathrm{Sin}\,\frac{\theta}{2}\right) \\ 2\mathrm{Sin}\,\frac{\theta}{2}\left(-b\,\mathrm{Cos}\,\frac{\theta}{2}+ac\,\mathrm{Sin}\,\frac{\theta}{2}\right) & 2\mathrm{Sin}\,\frac{\theta}{2}\left(a\,\mathrm{Cos}\,\frac{\theta}{2}+bc\,\mathrm{Sin}\,\frac{\theta}{2}\right) & \frac{1}{2}\left[1-a^2-b^2+c^2+\left(1+a^2+b^2-c^2\right)\mathrm{Cos}\,\theta\right] \end{pmatrix}$$

It can be directly verified that C is an orthogonal matrix with mutually orthogonal column vectors and determinant +1. ✔

Definition. A **symmetry** of a vector space (V,+) is a transformation T : V$\mapsto$V that is 1-1 and onto that preserves the vector space structure:

T(a v + b w) = a Tv + b Tw

Definition. The **General Linear Group GL(*n*)** is the group of symmetries of an *n*-dimensional vector space.

Theorem. GL(n) is the group of non-singular (n x n) matrices.

Proof. Let T $\in$ GL(n). Since T(a v + b w) = a Tv + b Tw, T is a linear transformation. Were T singular, then by [13.17] Dim TV < n $\Rightarrow$ T is not onto, a contradiction. Therefore T is a non-singular linear transformation. Thus in any basis, T is represented by a non-singular matrix. ■

Definition. The **Special Linear Group SL(*n*)** is the subset of GL(n) having determinant = 1.

Theorem. SL(n) is a normal subgroup of GL(n).

Proof. First, SL(n) is a group:

Closed: If S_1, $S_2 \in$ SL(n), then Det($S_1 S_2$) = Det(S_1) Det(S_2) = 1
$\Rightarrow$ $S_1 S_2 \in$ SL(n).

Identity: Det(I) = 1 $\Rightarrow$ I $\in$ SL(n)

Inverse: 1 = Det(I)= Det($S_1 S_1^{-1}$) = Det(S_1) Det(S_1^{-1}) = Det(S_1^{-1})
$\Rightarrow$ $S_1^{-1} \in$ SL(n)

Also, SL(n) is normal:

Let S $\in$ SL(n) and G $\in$ GL(n). Then

Det(G^{-1} S G) = Det(G^{-1}) Det(S) Det(G) = Det(G^{-1}) Det(G)
= Det(G G^{-1}) = Det(I) = 1

$\Rightarrow G^{-1} S G \in SL(n) \Rightarrow G^{-1} SL(n) G = SL(n)$ ■

The groundwork has now been laid to introduce the table, below, that shows the relationships between GL(3), O(3), SL(3), general linear transformations, orthogonality, determinants, and symmetries. The table shows that SL(3) $\subseteq$ O(3) $\subseteq$ GL(3)$\subseteq$ $\mathcal{A}(\mathbb{R}^3)$, and GL(3) is both the set of symmetries of $\mathbb{R}^3$ and the set of non-singular matrices. It also shows that the orthogonal group O(3) is a proper subset of the set of orthogonal matrices (shaded blue).

In general, non-singular matrices squeeze and stretch the unit sphere (or the reflected sphere) into an ellipsoid. However, singular matrices are more severe. They squash the unit sphere down to a 2-dimensional circle or ellipse or even to a line or a point.

Only orthogonal matrices preserve the sphere without squeezing or stretching any portion of it. This is achieved by limiting its operation to rotations and reflections. However, if determinant ≠ ±1 then orthogonal matrices also expand or contract the sphere.

Non-orthogonal matrices also squeeze, stretch, or preserve the sphere but not as rotations. Rather, the matrix columns would contain non-orthogonal vectors. In such a case the angle between the 1st and 2nd column vectors might be less than

$\mathcal{A}(\mathbb{R}^3)$ = 3 x 3 Matrices

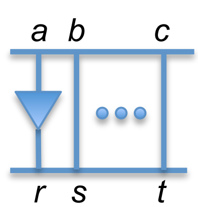
Determinant	**Orthogonal**	**Sphere maps to a …**	**Matrix Type**
0	Yes	Circle or line or point	Singular
	No	Ellipse or line or point	
Between -1 and 0	Yes	Contracted reflected sphere	**GL(3)** Non-singular Symmetries of $\mathbb{R}^3$
	No	Contracted reflected ellipsoid	
Between 0 and +1	Yes	Contracted sphere	
	No	Contracted ellipsoid	
-1	Yes	Reflected sphere	
	No	Reflected ellipsoid O(3)	
+1	Yes	SL(3) = sphere	
	No	Ellipsoid	
< -1	Yes	Expanded reflected sphere	
	No	Expanded reflected ellipsoid	
> 1	Yes	Expanded sphere	
	No	Expanded ellipsoid	

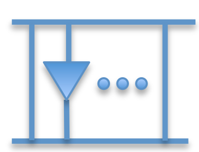
90°, squeezing the sphere along associated plane. The angle between the 2nd and 3rd vectors would then be greater than 90°, stretching the sphere along that plane.

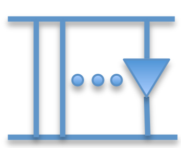
Matrices with positive determinant act on the sphere. Matrices with negative determinant behave exactly the same but act on the reflected sphere.

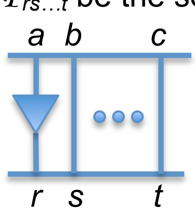
Definition. The **Trace** of A is $T_r(A)$ = Tr  $= T^k{}_k = T^1{}_1 + \cdots T^n{}_n$.

Theorem: [Bud]

Tr  $= \dfrac{1}{(n-1)!}$ 

$= \dfrac{1}{(n-1)!}$  $= \cdots$

$= \dfrac{1}{(n-1)!}$ 

Proof: Let $\mathcal{P}_{ab\ldots c}$ and $\mathcal{P}_{rs\ldots t}$ be the sets of permutations of (a,b,…,c) and (r,s,…,t), respectively. Let B = 

$$= \in^{rs\cdots t}\varepsilon_{ab\cdots c}\mathrm{T}^a{}_r\delta^b_s\cdots\delta^c_t = \sum_{\pi\in\mathcal{P}_{ab\cdots c}}\sum_{\pi'\in\mathcal{P}_{rs\cdots t}} \in^{\pi'(r)\pi'(s)\cdots\pi'(t)}\varepsilon_{\pi(a)\pi(b)\cdots\pi(c)}\mathrm{T}^{\pi(a)}{}_{\pi'(r)}\delta^{\pi(b)}_{\pi'(s)}\cdots\delta^{\pi(c)}_{\pi'(t)}.$$

Fix π. The only non-zero term in the sum is

$$\in^{\pi(a)\pi(b)\cdots\pi(c)}\varepsilon_{\pi(a)\pi(b)\cdots\pi(c)}\mathrm{T}^{\pi(a)}{}_{\pi(a)}\delta^{\pi(b)}_{\pi(b)}\cdots\delta^{\pi(c)}_{\pi(c)} = \mathrm{T}^{\pi(a)}{}_{\pi(a)}.$$

I showed in Problem [13.22] that $\in^{xy\cdots z}\varepsilon_{xy\cdots z} = 1$ for any fixed $(x,y,\ldots,z)$.

Thus, B = $\sum_{\pi\in\mathcal{P}_{ab\cdots c}}\mathrm{T}^{\pi(a)}{}_{\pi(a)}$. This sum has n! terms composed of (n – 1)! terms equal to $\mathrm{T}^a{}_a$, (n – 1)! terms equal to $\mathrm{T}^b{}_b$, …, and (n – 1)! terms equal to $\mathrm{T}^c{}_c$. So,

B = (n – 1)! ($\mathrm{T}^a{}_a$ + $\mathrm{T}^b{}_b$ + … + $\mathrm{T}^c{}_c$) = (n – 1)! Tr (A) = (n – 1)! Tr 

Similarly for the other figures. ■

Theorem. [13.24] $\mathrm{Det}(\mathrm{I}+\in\mathrm{A}) = 1+\in\mathrm{T_r}(\mathrm{A})$ if we ignore 2nd order and higher $\in$ terms.

Theorem. [13.25] Det e^{A} = $e^{\mathrm{T_r}(\mathrm{A})}$.

Definition. An **Eigenvector** is a non-zero vector v for which $\exists\lambda\in\mathbb{C}$ such that $\mathrm{T}v = \lambda v \;\Leftrightarrow\; (\mathrm{T}-\lambda\mathrm{I})v = 0$. λ is called an **Eigenvalue.**

Note: $\mathrm{Det}(\mathrm{T}-\lambda\mathrm{I}) = 0$ and so $(\mathrm{T}-\lambda\mathrm{I})$ is singular

Theorem. [13.26] $\mathrm{Det}(\mathrm{T}\text{-}\lambda\mathrm{I}) = (\lambda_1-\lambda)\cdots(\lambda_n-\lambda) = 0$ is a polynomial equation of degree n.

Definition. λ **has multiplicity** $\boldsymbol{r}$ means that λ appears r times in the equation above. Eigenvalue multiplicities are called **degeneracies** in Quantum Mechanics.

Definition. The set of Eigenvectors corresponding to λ is a linear space called an **Eigenspace**.

Theorem. If d is the dimension of the Eigenspace of λ and r is the multiplicity of λ then $1 \le d \le r$.

Theorem. [13.27] Let $\{\lambda_i\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then $\sum r_i = n$.

Corollary. A linear transformation T has at least 1 Eigenvector.

Theorem. [13.30] Suppose $\{e_k\}$ and $\{f_k\}$ are bases for a vector space V, and $f_k = \mathrm{T} e_k$. Then

$$f_j = \begin{pmatrix} \mathrm{T}^1{}_j \\ \vdots \\ \mathrm{T}^n{}_j \end{pmatrix}.$$

That is, the components of f_j in basis $\{e_k\}$ are $\left(\mathrm{T}^1{}_j, \cdots, \mathrm{T}^n{}_j\right)$.

Theorem. [13.31] If the Eigenspace dimension of every multiple Eigenvector equals its multiplicity, then there is a basis for V composed of Eigenvectors, and the matrix of T in this basis is

$$\mathrm{T} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The next theorem states that even when the hypothesis of the above theorem is not satisfied, the matrix of T can at least be written in upper triangular form.

Theorem. (Note 13.12): **Jordan Canonical Form:** Let $\{\lambda_i\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then there is a basis for V such that the matrix of T in this basis is

$$\mathrm{T} = \left(\begin{array}{c|c|c}
\begin{matrix} \lambda_1 & 1 & & & \\ & \lambda_1 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_1 \end{matrix} & \begin{matrix} \\ \\ \\ \\ 0 \end{matrix} & \begin{matrix} \cdots & \mathbf{0} \\ \iddots & \vdots \end{matrix} \\
\hline
 & \begin{matrix} \lambda_2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_{n-1} \end{matrix} & \begin{matrix} \\ \\ \\ 0 \end{matrix} \\
\hline
\begin{matrix} \vdots & \iddots \\ \mathbf{0} & \cdots \end{matrix} & & \begin{matrix} \lambda_n & 1 & & \\ & \lambda_n & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_n \end{matrix}
\end{array}\right).$$