[8.6]

$z = x + iy \quad r^2 = x^2 + y^2 \text{ or } r = \sqrt{x^2+y^2}$

$\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}} \quad \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

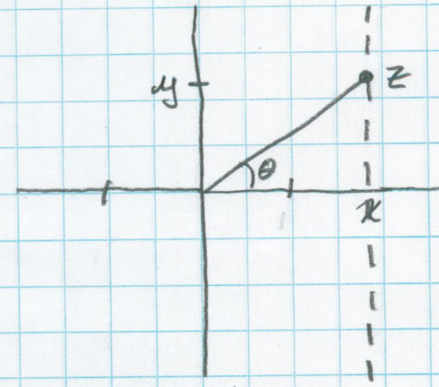


Fig 1

$z = re^{i\theta} = r[\cos\theta + i\sin\theta]$

$w \equiv \dfrac{1}{z} = \dfrac{1}{r}e^{-i\theta} = \dfrac{1}{r}[\cos\theta - i\sin\theta]$

$= u + iv \qquad u = \dfrac{\cos\theta}{r}, \quad v = -\dfrac{\sin\theta}{r}, \quad \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}, \ \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

$u = \dfrac{x}{x^2+y^2} \quad v = -\dfrac{y}{x^2+y^2} \quad$ ck: $|w|^2 = u^2+v^2 = \dfrac{1}{x^2+y^2} = \dfrac{1}{r^2} = \left|\dfrac{1}{r}e^{-i\theta}\right|^2$ ✓

$u - \dfrac{1}{2x} = \dfrac{x}{x^2+y^2} - \dfrac{1}{2x} = \dfrac{2x^2 - x^2 - y^2}{2x(x^2+y^2)} = \dfrac{x^2-y^2}{2x(x^2+y^2)}$

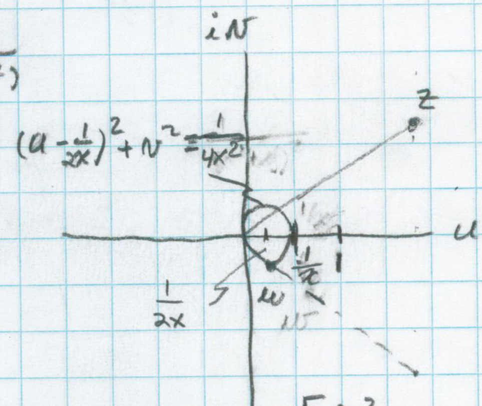


Fig 2

$\therefore \left(u - \dfrac{1}{2x}\right)^2 + v^2 = \left[\dfrac{x^2-y^2}{2x(x^2+y^2)}\right]^2 + \left[\dfrac{y}{x^2+y^2}\right]^2$

$= \dfrac{x^4 - 2x^2y^2 + y^4 + 4x^2y^2}{4x^2(x^2+y^2)^2}$

$= \dfrac{x^4 + 2x^2y^2 + y^4}{4x^2(x^2+y^2)^2} = \dfrac{\cancel{(x^2+y^2)^2}}{4x^2\cancel{(x^2+y^2)^2}} = \dfrac{1}{4x^2}$ ✓

$\therefore$ if $z \in$ line $x = x_0$, then $w \in$ circle $\left(u - \dfrac{1}{2x_0}\right)^2 + v^2 = \dfrac{1}{4x_0^2}$ ✓

Note: Figures 1 & 2 show case $x_0 > 1$. Similar for line $y = y_0$ where $y_0 > 1$.

For $x_0 < 1$, equations still hold. The circle is shown in Fig 3.

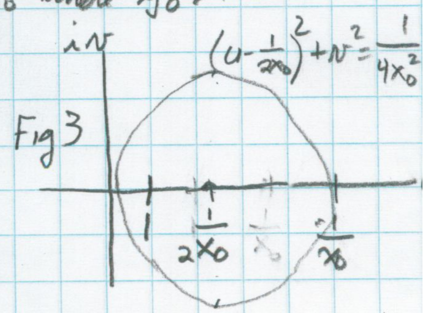
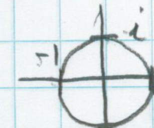


Fig 3

Finally $w = \dfrac{1}{z}$ sends top half of unit circle (Fg 4) to bottom half and vice-versa. In particular, $1 \to 1$, $-1 \to -1$, and $i \leftrightarrow -i$



Fg 4