[13.31] Let ***V*** be a vector space and ***T*** be a linear transformation on ***V*** with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, where $m \le n$. We furthermore assume

(a) For each $\lambda_j$ of multiplicity $r_j \ge 2$ (if any), there are $r_j$ independent eigenvectors.

Prove there is a basis for ***V*** composed of eigenvectors.

Note: I have reworked Beckmann's proof to help my understanding of it. I have filled in some details and changed the way he did a few things.

**Solution**. Let $r_j$ be the multiplicity of eigenvalue $\lambda_j$. Since there are $n$ eigenvalues, we have that $\sum_{j=1}^{m} r_j = n$.

If $m = 1$, all the eigenvectors arise from a single eigenvalue and are thus independent by condition (a). From the definition of "basis", those eigenvectors constitute a basis for ***V***, and we are done. So, we assume $m > 1$. Without loss of generality, if there is a zero eigenvalue, we label it $\lambda_1$.

Let $\mathscr{B}_j = \left\{ v_{j1}, v_{j2}, \cdots, v_{jr_j} \right\}$ be the set of $r_j$ independent eigenvectors corresponding to $\lambda_j$. We wish to prove

$$\mathscr{B} = \bigcup_{j=1}^{m} \mathscr{B}_j = \left\{ v_{ji} : i = 1, \ldots, r_j,\ \ j = 1, \ldots, m \right\}$$

comprises a basis for ***V***. Since $\mathscr{B}$ contains $n$ vectors, it suffices to show that these vectors are linearly independent. So, assume

$$(*) \quad \sum_{j=1}^{m} \sum_{i=1}^{r_j} \alpha_{ji} v_{ji} = 0.$$

We will be done if we can show that $\alpha_{ji} = 0 \ \ \forall\, j, i$. Set $E_j = \sum_{i=1}^{r_j} \alpha_{ji} v_{ji}$. Since

$T v_{ji} = \lambda_j v_{ji} \ \ \forall i$, then $\ T E_j = \sum_{i=1}^{r_j} \alpha_{ji} T v_{ji} = \sum_{i=1}^{r_j} \alpha_{ji} \lambda_j v_{ji} = \lambda_j \sum_{i=1}^{r_j} \alpha_{ji} v_{ji} = \lambda_j E_j$. So,

$$(1) \quad \sum_{j=1}^{m} E_j = \sum_{j=1}^{m} \sum_{i=1}^{r_j} \alpha_{ji} v_{ji} \overset{(*)}{=} 0, \quad \text{and thus}$$

$$(2) \quad \sum_{j=1}^{m} \lambda_j E_j = \sum_{j=1}^{m} T E_j = T\left( \sum_{j=1}^{m} E_j \right) = 0.$$

Since $m > 1$, $\lambda_m \ne 0$, and so we can solve (2) for $E_m$:

$(3)\quad E_m = -\sum_{j=1}^{m-1} \frac{\lambda_j}{\lambda_m} E_j.$ Plugging (3) into (1) yields

$$(4)\qquad \sum_{j=1}^{m-1}\left(1-\frac{\lambda_j}{\lambda_m}\right)E_j = 0.$$

Set $a_j = 1-\dfrac{\lambda_j}{\lambda_m}$ for $1 \le j \le m-1$. $a_j \ne 0$ since $\lambda_j \ne \lambda_m$ (because we are given that the $\lambda_j$'s are distinct). We rewrite (4) as

$(1')\qquad \sum_{j=1}^{m-1} a_j E_j = 0.$ So

$$(2')\qquad \sum_{j=1}^{m-1} a_j \lambda_j E_j = \sum_{j=1}^{m-1} a_j T E_j = T\left(\sum_{j=1}^{m-1} a_j E_j\right) = T(0) = 0.$$

If $m$ = 2, then $E_1 = 0$. Plugging this in to (1) yields $E_2 = 0$. Condition (a) implies that $\alpha_{1i} = 0$ and $\alpha_{2i} = 0$ for all $i$, which is what we are trying to prove.

If $m$ >2, we continue this process. Since $\lambda_{m-1} \ne 0,$ we can solve (2') for $a_{m-1} E_{m-1}$:

$(3')\quad a_{m-1} E_{m-1} = -\sum_{j=1}^{m-2} \frac{a_j \lambda_j}{\lambda_{m-1}} E_j.$ Plugging (3') into (1') yields

$0 = \left(\sum_{j=1}^{m-2} a_j E_j\right) + a_{m-1} E_{m-1} = \sum_{j=1}^{m-2} a_j \left(1-\frac{\lambda_j}{\lambda_{m-1}}\right) E_j,$ or

$$(4')\qquad \sum_{j=1}^{m-2} a_j \left(1-\frac{\lambda_j}{\lambda_{m-1}}\right) E_j = 0.$$

Set $b_j = a_j\left(1-\dfrac{\lambda_j}{\lambda_{m-1}}\right) \ne 0$ for $1 \le j \le m-2$.

$b_j \ne 0$ (since $a_j \ne 0$ and $\lambda_j \ne \lambda_m$), so we next rewrite (4') as

$(1'')\qquad \sum_{j=1}^{m-2} b_j E_j = 0.$ Thus

$$(2'')\qquad \sum_{j=1}^{m-2} b_j \lambda_j E_j = \sum_{j=1}^{m-2} b_j T E_j = T\left(\sum_{j=1}^{m-2} b_j E_j\right) = T(0) = 0.$$

Continuing …

$\left(1^{m-2}\right)$ $\quad d_1 E_1 + d_2 E_2 = \sum_{j=1}^{2} d_j E_j = 0$, where $d_1, d_2 \neq 0$.

…

$\left(1^{m-1}\right)$ $\quad e_1 E_1 = 0$, where $e_1 \neq 0$.

Thus $E_1 = 0$.

Plugging $E_1 = 0$ into $(1^{m-2})$ yields $E_2 = 0$.

Continuing, we get $E_j = 0 \;\; \forall \, j$.

From Condition (a), $\alpha_{j\,i} = 0 \;\; \forall \, j, i$, which is what we are trying to prove. ■