[12.16] Let $\mathcal{M}$ be an n-manifold, $n = p + q$, and M = {1, 2,…, n}. Let $\boldsymbol{\alpha}$ be a p-form $\boldsymbol{\alpha} = \alpha_{r\cdots t}\, dx^r \wedge \cdots \wedge dx^t$ and let $\boldsymbol{\psi}$ be its q-vector Hodge-like dual $\boldsymbol{\psi} = \psi^{uv\cdots w} \dfrac{\partial}{\partial x^u} \cdots \dfrac{\partial}{\partial x^w}$.

(a) Confirm the equivalency of these 3 conditions for simplicity:

(1) $\boldsymbol{\alpha}$ is simple iff $\alpha_{[r\cdots t}\, \alpha_{r']s'\cdots t'} = 0$ for all p-tuples (r, …, t) and (r', …, t') in M^p

(2) $\boldsymbol{\psi}$ is simple iff $\psi^{[u\cdots w}\, \psi^{u']v'\cdots w'} = 0$ for all q-tuples (u,…,w) and (u',…,w') in M^q

(3) $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$ are both simple iff $\psi^{u\cdots w}\, \alpha_{ws\cdots t} = 0$ for all p-tuples (w, s …, t) and q-tuples (u, …, w).

(b) Show that a 2-form $\boldsymbol{\alpha} = \alpha_{rs}\, dx^r \wedge dx^s$ in $\mathcal{M}$ is simple iff $\alpha_{[rs}\, \alpha_{r']s'} = 0$ for all pairs (r, s) and (r', s') in M^2

Proof: Einstein summation convention is used throughout this proof. This is a re-write of Juergen Beckmann's proof so that I can better understand it. I identify a hole in his proof of part (a) and I simplify his proof of part (b).

$\boldsymbol{\psi}$ is the Hodge-like dual of $\boldsymbol{\alpha}$ means $\psi^{u\cdots w} \propto \alpha_{r\cdots t} \in^{r\cdots tu\cdots w}$ or, equivalently, $\alpha_{r\cdots t} \propto \varepsilon_{r\cdots tu\cdots w} \psi^{u\cdots w}$. Recall

(i) $\boldsymbol{\varepsilon} \bullet \in = \varepsilon_{r\cdots tu\cdots w} \in^{r\cdots tu\cdots w} = n!$

Let K and L be the proportionality constants. Then

$$\alpha_{r\cdots t} = K \varepsilon_{r\cdots tu\cdots w} \psi^{u\cdots w} = K \varepsilon_{r\cdots tu\cdots w} L \alpha_{r\cdots t} \in^{r\cdots w} = n! K L \alpha_{r\cdots t}$$

$$\Rightarrow \quad KL = \frac{1}{n!}$$

A reasonable choice is to set $K = L = \dfrac{1}{\sqrt{n!}}$. So the Hodge-like dual definitions become

(ii) $\alpha_{r\cdots t} = \dfrac{1}{\sqrt{n!}} \varepsilon_{r\cdots tu\cdots w} \psi^{u\cdots w}$

(iii) $\psi^{u\cdots w} = \dfrac{1}{\sqrt{n!}} \alpha_{r\cdots t} \in^{r\cdots tu\cdots w}$

The definitions of Simple are:

$\boldsymbol{\alpha}$ is simple means $\exists$ 1-forms $\boldsymbol{\gamma} = \gamma_r\, dx^r, \cdots, \boldsymbol{\delta} = \delta_t\, dx^t$ such that $\boldsymbol{\alpha} = \boldsymbol{\gamma} \wedge \cdots \wedge \boldsymbol{\delta}$. That is,

(iv) $\alpha_{r\cdots t} = \gamma_{[r} \cdots \delta_{t]}$

$\boldsymbol{\psi}$ is simple means $\exists$ vector fields $\boldsymbol{\xi} = \xi^u \dfrac{\partial}{\partial x^u}, \cdots, \boldsymbol{\eta} = \eta^w \dfrac{\partial}{\partial x^w}$ such that $\boldsymbol{\psi} = \boldsymbol{\xi} \wedge \cdots \wedge \boldsymbol{\eta}$. That is,

(v) $\psi^{u\cdots w} = \xi^{[u}\cdots\eta^{w]}$

(a)

To prove $(A \Leftrightarrow B) \Leftrightarrow (C \Leftrightarrow D)$ one must show $A \Leftrightarrow C$ and $B \Leftrightarrow D$. Thus, to prove (a) we must prove the following two assertions.

(4) $\boldsymbol{\alpha}$ is simple iff its Hodge-like dual $\boldsymbol{\psi}$ is simple

(5) $\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'} = 0$ iff $\psi^{[u\cdots w}\,\psi^{u']v'\cdots w'} = 0$ iff $\psi^{u\cdots w}\,\alpha_{ws\cdots t} = 0$

Note: In his proof, Juergen Beckmann did not include (4). Rather he claimed that (5) plus Part (b) above (his Part a) is sufficient. But Part (b) is merely an example of proving (1) for a special case. I don't see that (1) + (5) can result in $\boldsymbol{\psi}$ being simple. I believe (4) + (5) is required.

Proof of (4)

$\boldsymbol{\alpha}$ is simple $\overset{\text{(iv)}}{\Leftrightarrow}$ $\exists p$ 1-forms $\boldsymbol{\gamma} = \gamma_r\, dx^r, \cdots, \boldsymbol{\delta} = \delta_t\, dx^t$ such that

$$\boldsymbol{\alpha} = \boldsymbol{\gamma} \wedge \cdots \wedge \boldsymbol{\delta} = \sum_{(r,\cdots,t)\in M^p} \gamma_{[r}\cdots\delta_{t]}\, dx^r \wedge \cdots dx^t .$$

That is, such that

$$\alpha_{r\cdots t} = \gamma_{[r}\cdots\delta_{t]} .$$

Note that if any subscript repeats then that term is zero. Thus we only consider permutations (r, …, t) of (1, …,p) in the sum.

Its Hodge-like dual $\boldsymbol{\psi}$ has components

$$\psi^{u\cdots w} \overset{\text{(iii)}}{=} \frac{1}{\sqrt{n!}}\alpha_{r\cdots t}\in^{r\cdots tu\cdots w} = \frac{1}{\sqrt{n!}}\gamma_{[r}\cdots\delta_{t]}\in^{r\cdots tu\cdots w} = \frac{\in^{1\cdots n}\, sign(\pi^+)}{\sqrt{n!}}\gamma_{[r}\cdots\delta_{t]}$$

where π^+ is the permutation $(r,\cdots,t) \mapsto (1,2,\cdots,p)$ and sign(π^+) is +1 if the permutation is even, -1 if it is odd.

We need to define q 1-vector fields $\boldsymbol{\xi} = \xi^u \frac{\partial}{\partial x^u}, \boldsymbol{\zeta} = \zeta^v \frac{\partial}{\partial x^v}, \cdots, \boldsymbol{\eta} = \eta^w \frac{\partial}{\partial x^w}$ such that

$$\boldsymbol{\psi} = \boldsymbol{\xi} \wedge \boldsymbol{\zeta} \wedge \cdots \wedge \boldsymbol{\eta} = \sum_{(u,\cdots,w)\in M^q} \xi^{[u}\zeta^v\cdots\eta^{w]} \frac{\partial}{\partial x^u} \wedge \frac{\partial}{\partial x^v} \wedge \cdots \wedge \frac{\partial}{\partial x^w} .$$

That is, such that

$$\psi^{uv\cdots w} = \xi^{[u}\zeta^v\cdots\eta^{w]} .$$

Again, we only need consider permutations of (u, v, …, w) of (p+1, p+2, …, n) in the sum. Thus, (r, …, t, u, v, …, w) is a permutation of (1, 2, …, n).

So, we wish to define the q 1-vector fields so that

$$\xi^{[u}\zeta^{v}\cdots\eta^{w]}=\frac{\in^{1\cdots n}\,sign\left(\pi^{+}\right)}{\sqrt{n!}}\gamma_{[r}\cdots\delta_{t]}\,.$$

Because there is not a 1-1 matchup of vectors and 1-forms, we cannot make a simple definition like $\xi^{u}=\gamma_{r},\cdots,\eta^{w}=\delta_{t}$. Rather, to define the *q* components of $\boldsymbol{\psi}$ in terms of the *p* components of $\boldsymbol{\alpha}$ we define x^u to be the entire RHS (and we have to toss in a *q*! factor) and we set the other components to either 0 or 1 as required:

$$\xi^{u}=\frac{q!\in^{1\cdots n}\,sign\left(\pi^{+}\right)}{\sqrt{n!}}\gamma_{[r}\cdots\delta_{t]},\ \xi^{v}=\cdots=\xi^{w}=0\,,$$

$$\zeta^{v}=1,\zeta^{u}=\cdots\left(\text{except }\zeta^{v}\right)=\zeta^{w}=0,\cdots\eta^{u}=\eta^{v}=\cdots=0,\eta^{w}=1.$$

Note: Since the term $\xi^{[u}\zeta^{v}\cdots\eta^{w]}$ includes $\xi^{u},\xi^{v},\cdots,\xi^{w},\zeta^{u},\zeta^{v},\cdots,\zeta^{w},\cdots,\eta^{u},\eta^{v},\cdots,\eta^{w}$, we must define them all.

Finally, we check that these definitions give the desired answer:

$$\psi^{uv\cdots w}=\xi^{[u}\zeta^{v}\cdots\eta^{w]}=\frac{1}{q!}\sum_{\pi\in\mathcal{P}_{uv\cdots w}}\xi^{\pi(u)}\zeta^{\pi(v)}\cdots\eta^{\pi(w)}=\frac{1}{q!}\xi^{u}\zeta^{v}\cdots\eta^{w}$$

$$=\frac{1}{q!}\frac{q!\in^{1\cdots n}\,sign\left(\pi^{+}\right)}{\sqrt{n!}}\gamma_{[r}\cdots\delta_{t]}$$

$$=\frac{\in^{1\cdots n}\,sign\left(\pi^{+}\right)}{\sqrt{n!}}\gamma_{[r}\cdots\delta_{t]}$$

Proof of (5)

By renaming subscripts, (ii) and (iii) can also be expressed as

(ii') $\alpha_{ws\cdots t}=\frac{1}{\sqrt{n!}}\varepsilon_{ws\cdots t\,x\cdots z}\psi^{x\cdots z}$ where *ws*…*t* are *p* given distinct members of M and *x*…*z* are the remaining *q* members of M, and

(iii') $\psi^{u\cdots w}=\frac{1}{\sqrt{n!}}\alpha_{a\cdots c}\in^{a\cdots c\,u\cdots w}$ where *u*…*w* are *q* distinct members of M and *a*…*c* are the remaining *p* members of M.

We also must keep in mind that each set *r*…*t x*…*z* and *a*…*c u*…*w* is a permutation of the members of M = {1, 2,…, *n*}. There is no need to consider terms with duplicate indices since $dx^{s}\wedge dx^{s}=0$ and $\frac{\partial^{2}}{\partial x^{v}\,\partial x^{v}}=0$.

Let $\mathcal{P}$ be the set of permutations of (*r*, … , *t*, *w*). Then

$$\psi^{u\cdots w}\,\alpha_{ws\cdots t} \overset{\text{(iii')}}{=} \left(\frac{1}{\sqrt{n!}}\alpha_{a\cdots c} \in^{a\cdots cu\cdots w}\right)\alpha_{ws\cdots t} = \frac{1}{\sqrt{n!}}\alpha_{a\cdots c}\,\alpha_{ws\cdots t} \in^{a\cdots cu\cdots w}$$

$$= \frac{\in^{1\,2\cdots n}}{\sqrt{n!}}\sum_{\pi\in\mathcal{P}} sign(\pi)\,\alpha_{\pi(a)\cdots\pi(c)}\,\alpha_{\pi(w)s\cdots t} \qquad \left(\text{because } \in^{a\cdots cu\cdots w} = sign(\pi)\in^{12\cdots n}\right)$$

$$= \frac{(p+1)!\,\in^{1\,2\cdots n}}{\sqrt{n!}}\alpha_{[a\cdots c}\,\alpha_{w]s\cdots t} \qquad \left(\begin{aligned}&\text{because } \alpha_{[a\cdots c}\,\alpha_{w]s\cdots t}\\ &= \frac{1}{(p+1)!}\sum_{\pi\in\mathcal{P}} sign(\pi)\,\alpha_{\pi(a)\cdots\pi(c)}\,\alpha_{\pi(w)s\cdots t}\end{aligned}\right)$$

$$= \frac{(p+1)!\,\in^{1\,2\cdots n}}{\sqrt{n!}}\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'} \qquad (\text{renaming of subscripts}).$$

Therefore $\psi^{u\cdots w}\,\alpha_{ws\cdots t} = 0$ iff $\alpha_{[r\cdots t}\,\alpha_{r']s'\cdots t'} = 0$. ✔

Let $\mathcal{P}^*$ be the set of permutations of (x, … , z, w). Then

$$\psi^{u\cdots w}\,\alpha_{ws\cdots t} \overset{\text{(ii')}}{=} \psi^{u\cdots w}\left(\frac{1}{\sqrt{n!}}\varepsilon_{ws\cdots t\,x\cdots z}\psi^{x\cdots z}\right)\alpha_{ws\cdots t} = \frac{1}{\sqrt{n!}}\psi^{x\cdots z}\,\psi^{u\cdots vw}\,\varepsilon_{ws\cdots t\,x\cdots z}$$

$$= \frac{(-1)^{q-1}}{\sqrt{n!}}\psi^{x\cdots z}\,\psi^{wu\cdots v}\,\varepsilon_{x\cdots zws\cdots t} = \frac{(-1)^{q-1}\,\varepsilon_{1\,2\cdots n}}{\sqrt{n!}}\sum_{\pi\in\mathcal{P}^*} sign(\pi)\,\psi^{\pi(x)\cdots\pi(z)}\,\psi^{\pi(w)u\cdots v}$$

$$= \frac{(-1)^{q-1}\,(q+1)!\,\varepsilon_{1\,2\cdots n}}{\sqrt{n!}}\psi^{[x\cdots z}\,\psi^{w]u\cdots v}$$

$$= \frac{(-1)^{q-1}\,(q+1)!\,\varepsilon_{1\,2\cdots n}}{\sqrt{n!}}\psi^{[u\cdots w}\,\psi^{u']v'\cdots w'}.$$

Therefore $\psi^{u\cdots w}\,\alpha_{ws\cdots t} = 0$ iff $\psi^{[u\cdots w}\,\psi^{u']v'\cdots w'} = 0$. ✔

(b). This is a simplification of Juergen Beckmann's proof (he calls this his part a). The simplification occurs in the "IF" part of the proof by eliminating Beckmann's use of the vectors that Penrose mentions in his hint for this problem.

Recall that $\boldsymbol{\alpha}$ is antisymmetric in (r, s) :

(*) $\qquad \alpha_{sr} = -\alpha_{rs}$

$\boldsymbol{\alpha}$ is simple iff ∃ 1-forms $\boldsymbol{\gamma} = \gamma_r\,dx^r$ and $\boldsymbol{\delta} = \delta_s\,dx^s \ni\ \boldsymbol{\alpha} = \boldsymbol{\gamma}\wedge\boldsymbol{\delta} = \gamma_{[r}\,\delta_{s]}\,dx^r\wedge dx^s$. That is,

(**) $\qquad \boldsymbol{\alpha}$ is simple iff ∃ 1-forms $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ such that the components α_{rs} of α satisfy $\alpha_{rs} = \gamma_{[r}\delta_{s]}\ \forall r,s$.

"IF"

Suppose $\alpha_{[rs}\alpha_{r']s'} = 0$ for all pairs (r, s) and (r', s') in M^2.

If $\boldsymbol{\alpha}$ = 0, clearly $\boldsymbol{\alpha}$ is simple (e.g., $\boldsymbol{\alpha} = dx^1\wedge dx^1$). So, suppose $\boldsymbol{\alpha}$ ≠ 0. Then

$\exists\, a, b \in \{1, \ldots, n\}$ such that $\alpha_{ab}\, dx^a \wedge dx^b \neq 0 \quad \Rightarrow \quad a \neq b$ and $\alpha_{ab} \neq 0$. Fix $(r, s) \in \mathrm{M}^2$. Define

$$\text{(I)} \quad \boldsymbol{\gamma} = \gamma_r\, dx^r \text{ where } \gamma_r = 2\frac{\alpha_{rb}}{\alpha_{ab}} \quad \text{and} \quad \text{(II)} \quad \boldsymbol{\delta} = \delta_s\, dx^s \text{ where } \delta_s = \alpha_{as}.$$

We will show that $\alpha_{rs} = \gamma_{[r}\delta_{s]}$. First,

$$\text{(I')} \quad \gamma_s = 2\frac{\alpha_{sb}}{\alpha_{ab}} \quad \text{and} \quad \text{(II')} \quad \delta_r = \alpha_{ar}.$$

So,

$$\text{(III)} \quad \gamma_{[r}\delta_{s]} = \tfrac{1}{2}\left(\gamma_r\delta_s - \gamma_s\delta_r\right) \overset{\text{(I, II, I', II')}}{=} \frac{\alpha_{rb}\alpha_{as} - \alpha_{sb}\alpha_{ar}}{\alpha_{ab}}.$$

Now $0 = \alpha_{[rs}\alpha_{a]b} = \tfrac{1}{6}\left(\alpha_{rs}\alpha_{ab} + \alpha_{sa}\alpha_{rb} + \alpha_{ar}\alpha_{sb} - \alpha_{sr}\alpha_{ab} - \alpha_{as}\alpha_{rb} - \alpha_{ra}\alpha_{sb}\right)$, or

$$\text{(IV)} \quad \alpha_{rb}\alpha_{as} - \alpha_{sb}\alpha_{ar} = \alpha_{ab}\alpha_{rs} + \alpha_{rb}\alpha_{sa} - \alpha_{ab}\alpha_{sr} - \alpha_{sb}\alpha_{ra}$$

Plugging (IV) into (III) yields

$$\begin{aligned}
\gamma_{[r}\delta_{s]} &\overset{\text{(III)}}{=} \frac{\alpha_{rb}\alpha_{as} - \alpha_{sb}\alpha_{ar}}{\alpha_{ab}} \overset{\text{(IV)}}{=} \frac{\alpha_{ab}\alpha_{rs} + \alpha_{rb}\alpha_{sa} - \alpha_{ab}\alpha_{sr} - \alpha_{sb}\alpha_{ra}}{\alpha_{ab}} \\
&\overset{(*)}{=} \alpha_{rs} - \left(\frac{\alpha_{rb}}{\alpha_{ab}}\right)\alpha_{as} - \alpha_{sr} + \left(\frac{\alpha_{sb}}{\alpha_{ab}}\right)\alpha_{ar} \\
&\overset{\text{(I, II, I', II')}}{=} \alpha_{rs} - \tfrac{1}{2}\gamma_r\delta_s - \alpha_{sr} + \tfrac{1}{2}\gamma_s\delta_r \overset{(*)}{=} 2\alpha_{rs} - \gamma_{[r}\delta_{s]}
\end{aligned}$$

$\Rightarrow \quad \alpha_{rs} = \gamma_{[r}\delta_{s]} \qquad \Rightarrow \qquad \boldsymbol{\alpha}$ is simple [due to (**)] ✔

“ONLY IF”

Suppose $\boldsymbol{\alpha} = \alpha_{rs}\, dx^r \wedge dx^s$ is simple. Then from (**) $\exists$ 1-forms $\boldsymbol{\gamma} = \gamma_r dx^r$ and $\boldsymbol{\delta} = \delta_s dx^s$ such that

$$\text{(V)} \quad \alpha_{rs} = \gamma_{[r}\delta_{s]} \quad \forall\, r, s.$$

So

$$
\begin{aligned}
\alpha_{[rs}\alpha_{u]v} &= \tfrac{1}{6}\left(\alpha_{rs}\alpha_{uv} + \alpha_{su}\alpha_{rv} + \alpha_{ur}\alpha_{sv} - \alpha_{sr}\alpha_{uv} - \alpha_{us}\alpha_{rv} - \alpha_{ru}\alpha_{sv}\right) \\
&= \tfrac{1}{6}\left[\left(\alpha_{rs} - \alpha_{sr}\right)\alpha_{uv} + \left(\alpha_{su} - \alpha_{us}\right)\alpha_{rv} + \left(\alpha_{ur} - \alpha_{ru}\right)\alpha_{sv}\right] \\
&\overset{(*)}{=} \tfrac{1}{3}\left(\alpha_{rs}\alpha_{uv} + \alpha_{su}\alpha_{rv} + \alpha_{ur}\alpha_{sv}\right) \\
&\overset{(V)}{=} \tfrac{1}{3}\left(\gamma_{[r}\delta_{s]}\gamma_{[u}\delta_{v]} + \gamma_{[s}\delta_{u]}\gamma_{[r}\delta_{v]} + \gamma_{[u}\delta_{r]}\gamma_{[s}\delta_{v]}\right) \\
&= \left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left\{\begin{array}{l} \left(\left[\gamma_r\delta_s - \gamma_s\delta_r\right]\left[\gamma_u\delta_v - \gamma_v\delta_u\right]\right) + \left(\left[\gamma_s\delta_u - \gamma_u\delta_s\right]\left[\gamma_r\delta_v - \gamma_v\delta_r\right]\right) \\ + \left(\left[\gamma_u\delta_r - \gamma_r\delta_u\right]\left[\gamma_s\delta_v - \gamma_v\delta_s\right]\right) \end{array}\right\} \\
&= \tfrac{1}{12}\left[\begin{array}{l} \gamma_r\delta_s\gamma_u\delta_v - \gamma_r\delta_s\gamma_v\delta_u - \gamma_s\delta_r\gamma_u\delta_v + \gamma_s\delta_r\gamma_v\delta_u \\ + \gamma_s\delta_u\gamma_r\delta_v - \gamma_s\delta_u\gamma_v\delta_r - \gamma_u\delta_s\gamma_r\delta_v + \gamma_u\delta_s\gamma_v\delta_r \\ + \gamma_u\delta_r\gamma_s\delta_v - \gamma_u\delta_r\gamma_v\delta_s - \gamma_r\delta_u\gamma_s\delta_v + \gamma_r\delta_u\gamma_v\delta_s \end{array}\right] \\
&= 0\ .
\end{aligned}
$$