[11.3] Show defn $q^{-1} = \bar{q}(q\bar{q})^{-1}$ works.

Let $q = t + ui + vj + wk$. So $\bar{q} = t - ui - vj - wk$

and $q\bar{q} = t^2 + u^2 + v^2 + w^2 \in \mathbb{R}$. Thus $(q\bar{q})^{-1}$ exists since all real numbers have inverses. Moreover, since $q\bar{q} \in \mathbb{R}$, we can write $(q\bar{q})^{-1} = \frac{1}{q\bar{q}}$. (We cannot yet meaningfully write $\frac{1}{q}$.) To show q^{-1} is indeed the inverse for q, we must show $qq^{-1} = 1 = q^{-1}q$

(a) $qq^{-1} = q\bar{q}(q\bar{q})^{-1} = (t^2+u^2+v^2+w^2) \cdot \frac{1}{t^2+u^2+v^2+w^2}$

$= 1$

(b) $q^{-1}q = \bar{q}(q\bar{q})^{-1}q = (q\bar{q})^{-1}\bar{q}q$ (Can switch because $(q\bar{q})^{-1} \in \mathbb{R}$)

So $q^{-1}q = \frac{1}{t^2+u^2+v^2+w^2}(t^2+u^2+v^2+w^2) = 1$

So, the defn "works"

Note: Now that we have proven that q^{-1} exists, we can meaningfully write $\frac{1}{q}$. However, we are not free to switch it with another quaternion. i.e., $qp \neq pq$ in general. However $q^{-1}q = qq^{-1}$ since both equal 1, and $p(q\bar{q}) = (q\bar{q})p$ since $q\bar{q} \in \mathbb{R}$.

Note: Shaun Culver was premature to write $\frac{1}{t+ui+vj+wk}$ because at that point he had not shown that a quaternion inverse exists and has meaning.