[12.7] If ϕ is antisymmetric in (r, s, … , u), show that $\phi_{[rs\ldots u]} = \phi_{rs\ldots u}$ where there are p indices r, s, … , u.

Proof: There are p! permutations of r, s, … , u, half of them even and half odd. Let $\mathcal{P}^{+}$ be the set of even permutations, $\mathcal{P}^{-}$ the set of odd permutations, and $\mathcal{P}$ the set of all permutations. Since ϕ is antisymmetric in (r, s, …, u),

$$\phi_{\pi(r)\pi(s)\ldots\pi(u)} = \left\{ \begin{array}{ll} \phi_{rs\ldots u} & \text{if } \pi \in \mathcal{P}^{+} \\ -\phi_{rs\ldots u} & \text{if } \pi \in \mathcal{P}^{-} \end{array} \right\} = sign(\pi)\,\phi_{rs\ldots u} \qquad \textbf{(1)}$$

So,

$$\begin{aligned} \phi_{[rs\ldots u]} &\overset{defn}{=} \frac{1}{p!}\sum_{\pi\in\mathcal{P}} sign(\pi)\ \phi_{\pi(r)\pi(s)\ldots\pi(u)} \overset{(1)}{=} \frac{1}{p!}\sum_{\pi\in\mathcal{P}} sign(\pi)\left[sign(\pi)\ \phi_{rs\ldots u}\right] \\ &= \frac{1}{p!}\sum_{\pi\in\mathcal{P}} \phi_{rs\ldots u} = \frac{1}{p!}\left(p!\phi_{rs\ldots u}\right) \\ &= \phi_{rs\ldots u} \end{aligned}$$

Alternatively, we could solve this as

$$\begin{aligned} \phi_{[rs\ldots u]} &\overset{defn}{=} \frac{1}{p!}\sum_{\pi\in\mathcal{P}} sign(\pi)\ \phi_{\pi(r)\pi(s)\ldots\pi(u)} \overset{(a)}{=} \frac{1}{p!}\left(\sum_{\pi\in\mathcal{P}^{+}} \phi_{\pi(r)\pi(s)\ldots\pi(u)} + \sum_{\pi\in\mathcal{P}^{-}} -\phi_{\pi(r)\pi(s)\ldots\pi(u)}\right) \\ &\overset{(1)}{=} \frac{1}{p!}\left(\frac{p!}{2}\phi_{rs\ldots u} + \frac{p!}{2}(-)\left(-\phi_{rs\ldots u}\right)\right) \\ &= \phi_{rs\ldots u} \end{aligned}$$

(a) $sign$ (π) = 1 for $\pi \in \mathcal{P}^{+}$ and $sign$ (π) = -1 for $\pi \in \mathcal{P}^{-}$