[13.10] Let G and H be groups.

(a) Verify that G×H is a group where G×H is the set { (g, h) } with the operation $(g_1,h_1)(g_2,h_2)=(g_1g_2,h_1h_2)$

(b) Show that we can identify H with G×H /G

(a) (1,1) is the identity since (1,1) (g,h) = (g,h) = (g, h) (1,1) ✔

The inverse of (g,h) is (g^{-1},h^{-1}) since $(g,h)(g^{-1},h^{-1})=(1,1)=(g^{-1},h^{-1})(g,h)$ ✔

The associative law holds:

$$[(g_1,h_1)(g_2,h_2)](g_3,h_3)=((g_1g_2,h_1h_2))(g_3,h_3)=((g_1g_2)g_3,(h_1h_2)h_3)$$

$$=(g_1(g_2g_3),h_1(h_2h_3))=(g_1,h_1)(g_2g_3,h_2h_3)=(g_1,h_1)[(g_2,h_2)(g_3,h_3)]$$ ✔

Therefore G×H is a group. ✔

(b) There are 3 preliminaries to cover.

(i) G must be a subgroup of G×H. It isn't, but it can be identified with $\{(g,1): g\in \mathrm{G}, 1\in \mathrm{H}\}$, which *is* a subgroup. We henceforth use the symbol G to represent group G = { g } as well as group G×{1} = { (g,1) : $g \in$ G }, and it should always be obvious which is meant.

(ii) G×H /G Is a group only if G is normal in G×H. In order to show that G is normal we must show that $(g,h)\,\mathrm{G}\,(g^{-1},h^{-1})=\mathrm{G}\ \ \forall (g,h)\in \mathrm{G}\times\mathrm{H}$.

Fix (g,h) ∈ G×H. For any g_1 ∈ G,

$$(g,h)(g_1,1)(g^{-1},h^{-1})=(gg_1g^{-1},1)\in \mathrm{G} \quad\Rightarrow\quad (g,h)\,\mathrm{G}\,(g^{-1},h^{-1})\subseteq \mathrm{G}.$$

To show equality, let g_2 ∈ G. We need to find a g_1 ∈ G such that $(g,h)(g_1,1)(g^{-1},h^{-1})=(g_2,1)$.

Define $g_1=g^{-1}g_2g\in \mathrm{G}$. Then

$$(g,h)(g_1,1)(g^{-1},h^{-1})=(g(g_1)g^{-1},1)=(g(g^{-1}g_2g)g^{-1},1)=(g_2,1).$$ ✔

(iii) We require the fact that gG = G for any g ∈ G:

Fix g. Clearly $g\mathrm{G}\subseteq \mathrm{G}$ (since gg_1 ∈ G for any g_1 ∈ G). To show equality, let g_1 ∈ G. We want to find a g_2 such that $gg_2=g_1$. Define $g_2 = g^{-1} g_1$. Then indeed $gg_2=g(g^{-1}g_1)=g_1$. ✔

We are now ready to show the identification of H with G×H /G. By definition G×H /G = { G(g,h) }. Since gG = G, $\mathrm{G}(g_1,h) = \mathrm{G}(g_2,h)$ for any g_1, $g_2 \in$ G. So for each coset G(g,h) we can choose any element of G to be the representative element. Choose g = 1. That is, G×H /G = { G(1,h)) : $h \in$ H }, and we readily identify H = { $h \in$ H } with { G(1,h) : $h \in$ H } = G×H /G. In fact we have shown that H is group isomorphic to G×H /G. ✔