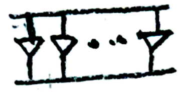
* [13.21] Show that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ and

$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = aej - afh + bfg - bdj + cdh - ceg$

follows from the diagram $\det A = \frac{1}{n!}$ 

(a) $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2!}$  $= \frac{1}{2!} \varepsilon_{rs} T^r{}_t T^s{}_u \epsilon^{tu}$

Note that $\{r, s\} = \{t, u\} = \{1, 2\}$. The last term above involves summation over permutations of $\{r, s\} = \{1, 2\}$ and $\{t, u\} = \{1, 2\}$.

So $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\left[\epsilon^{12}\varepsilon_{12} T^1{}_1 T^2{}_2 + \epsilon^{21}\varepsilon_{12} T^1{}_2 T^2{}_1 + \epsilon^{21}\varepsilon_{21} T^2{}_2 T^1{}_1 + \epsilon^{12}\varepsilon_{21} T^2{}_1 T^1{}_2\right]$

$= \frac{1}{2}\left[\epsilon^{12}\varepsilon_{12} T^1{}_1 T^2{}_2 - \epsilon^{12}\varepsilon_{12} T^1{}_2 T^2{}_1 + \epsilon^{21}\varepsilon_{21} T^2{}_2 T^1{}_1 - \epsilon^{21}\varepsilon_{21} T^2{}_1 T^1{}_2\right]$

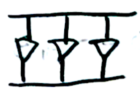
Note that $T^1{}_1 = a$, $T^1{}_2 = b$, $T^2{}_1 = c$, and $T^2{}_2 = d$. So,

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\left[ad\left(\epsilon^{12}\varepsilon_{12} + \epsilon^{21}\varepsilon_{21}\right) - bc\left(\epsilon^{12}\varepsilon_{12} + \epsilon^{21}\varepsilon_{21}\right)\right]$

$= \frac{1}{2}(ad - bc)\left(\epsilon^{12}\varepsilon_{12} + \epsilon^{21}\varepsilon_{21}\right)$

$= \frac{1}{2}(ad - bc)\,\epsilon^{rs}\varepsilon_{rs} = \frac{1}{2}(ad - bc)(2!)$

$= ad - bc$ ✓

(b) $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = \frac{1}{3!}$  $= \frac{1}{6}\varepsilon_{rst} T^r{}_u T^s{}_v T^t{}_w \epsilon^{uvw}$

$= \frac{1}{6}\left[\varepsilon_{123}\epsilon^{123} T^1{}_1 T^2{}_2 T^3{}_3 + \varepsilon_{123}\epsilon^{132} T^1{}_1 T^2{}_3 T^3{}_2 + \varepsilon_{123}\epsilon^{213} T^1{}_2 T^2{}_1 T^3{}_3\right.$

+ 33 more terms (i.e., $\exists$ 6×6 permutations of $\{1,2,3\}$ $\{1,2,3\}$

$= \frac{1}{6} aej\left(\epsilon^{123}\varepsilon_{123} + \epsilon^{132}\varepsilon_{132} + \epsilon^{213}\varepsilon_{213} + \epsilon^{231}\varepsilon_{231} + \epsilon^{312}\varepsilon_{312} + \epsilon^{321}\varepsilon_{321}\right)$

$- \frac{1}{6} afh\,($ " $)$

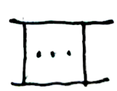
⋮

$= \frac{1}{6}(aej - afh + bfg - bdj + cdh - ceg)\left(\epsilon^{rst}\varepsilon_{rst}\right)$

$= \frac{1}{6}($ " $)(3!)$

$= aej - afh + bfg - bdj + cdh - ceg$ ✓

Cor.  $= n!$

Pf: Let $T = I = \delta$. Since $\det(I) = 1$, we have $1 = \det(I) = \frac{1}{n!}$ 

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