[13.32] Show that every finite group **G** has a faithful representation in GL(n) where n is the order of **G.**

Solution

A representation is a function $T: \boldsymbol{G} \to \mathrm{GL}(n)$ that preserves the group structure; i.e., for all $g_i, g_j \in \boldsymbol{G}$, $T(g_i)T(g_j) = T(g_i g_j)$. T is faithful if it is one-to-one; i.e., if $T(g_i) = T(g_j) \Rightarrow g_i = g_j$.

Part A. Show T is a representation

I do this 2 ways. I do it my way first, then I repeat it using Robin's method which is very slick. For motivation, I use Penrose's hint to label the matrix for $T(g_i)$.

$$T(g_i) = \begin{array}{c} \\ \\ \\ g_j \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} \phantom{T(g_i)^1{}_1} & \phantom{T(g_i)^1{}_2} & & g_k & & \phantom{T(g_i)^1{}_n} \end{array} \\ \begin{bmatrix} T(g_i)^1{}_1 & T(g_i)^1{}_2 & \cdots & T(g_i)^1{}_k & \cdots & T(g_i)^1{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T(g_i)^j{}_1 & T(g_i)^j{}_2 & \cdots & T(g_i)^j{}_k & \cdots & T(g_i)^j{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T(g_i)^n{}_1 & T(g_i)^n{}_2 & \cdots & T(g_i)^n{}_k & \cdots & T(g_i)^n{}_n \end{bmatrix} \end{array}$$

Given k there is a unique j such that $g_k = g_i\, g_j$ (namely, $g_j = g_i^{-1} g_k$). So I think of this matrix as g_i taking g_j to g_k. (Robin had g_i take g_k to g_j, which also works, as does right multiplication by g_i.) Thus I make the definition

$$(1) \qquad T(g_i) \equiv \begin{cases} 1 & \text{if } i, j, \text{ and } k \text{ are such that } g_k = g_i\, g_j \\ 0 & \text{Otherwise} \end{cases}$$

which in matrix notation is $$T(g_i)\begin{bmatrix} 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} T(g_i)^1{}_j \\ \vdots \\ T(g_i)^k{}_j \\ \vdots \\ T(g_i)^n{}_j \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1_k \\ \vdots \\ 0 \end{bmatrix}$$

It suffices to show that $T(g_i)\,T(g_j)$ agrees with $T(g_i\,g_j)$ on each basis element $\begin{bmatrix} 0 \\ \vdots \\ 1_r \\ \vdots \\ 0 \end{bmatrix}$.

From definition (1) we get that

$$(2) \qquad T(g_j)^s{}_r = \left\{ \begin{array}{ll} 1 & \text{if } j, r, \text{ and } s \text{ are such that } g_s = g_j\,g_r \\ 0 & \text{Otherwise} \end{array} \right\}, \text{ or } \; T(g_j)\begin{bmatrix} 0 \\ \vdots \\ 1_r \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1_s \\ \vdots \\ 0 \end{bmatrix}$$

and

$$(3) \qquad T(g_i)^t{}_s = \left\{ \begin{array}{ll} 1 & \text{if } i, s, \text{ and } t \text{ are such that } g_t = g_i\,g_s \\ 0 & \text{Otherwise} \end{array} \right\}, \text{ or } \; T(g_i)\begin{bmatrix} 0 \\ \vdots \\ 1_s \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1_t \\ \vdots \\ 0 \end{bmatrix}.$$

Thus,

$$\left[T(g_i)\,T(g_j)\right]\begin{bmatrix} 0 \\ \vdots \\ 1_r \\ \vdots \\ 0 \end{bmatrix} = T(g_i)\left(T(g_j)\begin{bmatrix} 0 \\ \vdots \\ 1_r \\ \vdots \\ 0 \end{bmatrix}\right) \overset{(2)}{=} T(g_i)\begin{bmatrix} 0 \\ \vdots \\ 1_s \\ \vdots \\ 0 \end{bmatrix} \overset{(3)}{=} \begin{bmatrix} 0 \\ \vdots \\ 1_t \\ \vdots \\ 0 \end{bmatrix}.$$

Again, from definition (1), we get that

$$(4) \qquad T(g_k)^p{}_r = \left\{ \begin{array}{ll} 1 & \text{if } k, r, \text{ and } p \text{ are such that } g_p = g_k\,g_r \\ 0 & \text{Otherwise} \end{array} \right\}, \text{ or}$$

$$T(g_k)\begin{bmatrix}0\\ \vdots\\ 1_r\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 1_p\\ \vdots\\ 0\end{bmatrix}.$$ So $g_p=g_k\,g_r=(g_i\,g_j)\,g_r=g_i\,(g_j\,g_r)=g_i\,g_s=g_t$; i.e., $p=t$.

So, $$T(g_i\,g_j)\begin{bmatrix}0\\ \vdots\\ 1_r\\ \vdots\\ 0\end{bmatrix}=T(g_k)\begin{bmatrix}0\\ \vdots\\ 1_r\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 1_p\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 1_t\\ \vdots\\ 0\end{bmatrix}=\left[T(g_i)\,T(g_j)\right]\begin{bmatrix}0\\ \vdots\\ 1_r\\ \vdots\\ 0\end{bmatrix}.$$

That is, $T(g_i)\,T(g_j)=T(g_i\,g_j)$. ✔

Alternate proof of Part A

$$\left[T(g_i)\,T(g_j)\right]^a{}_b=T(g_i)^a{}_c\,T(g_j)^c{}_b=\begin{cases}1 & \text{if } i, c, a, j, \text{ and } b \text{ satisfy } g_a=g_i\,g_c \text{ and } g_c=g_j\,g_b\\ 0 & \text{Otherwise}\end{cases}.$$

$g_a=g_i\,g_c$ and $g_c=g_j\,g_b \;\Leftrightarrow\; g_i^{-1}\,g_a=g_j\,g_b \;\Leftrightarrow\; g_a\,g_b^{-1}=g_i\,g_j$. Therefore

$$\left[T(g_i)\,T(g_j)\right]^a{}_b=\left\{\begin{matrix}1 & \text{if } a, b, i, \text{ and } j \text{ satisfy } g_a\,g_b^{-1}=g_i\,g_j\\ 0 & \text{Otherwise}\end{matrix}\right\}.$$ But,

$$T(g_i\,g_j)^a{}_b=\begin{cases}1 & \text{if a, b, i, and } j \text{ satisy } g_a=(g_i\,g_j)\,g_b \Leftrightarrow g_a\,g_b^{-1}=g_i\,g_j\\ 0 & \text{Otherwise}\end{cases}$$ ✔

Part B Show T is faithful

We suppose $T(g_i)=T(g_j)$ and we must show that $g_i=g_j$. Since $T(g_i)=T(g_j)$, $\forall\, a, b\;\; T(g_i)^a{}_b=T(g_j)^a{}_b \;\Leftrightarrow\; T(g_i)^a{}_b=1$ if and only if $T(g_j)^a{}_b=1$.
$T(g_i)^a{}_b=1$ iff $g_a=g_i\,g_b$ and $T(g_j)^a{}_b=1$ iff $g_a=g_j\,g_b$. So for all a and b we have $g_a=g_i\,g_b$ iff $g_a=g_j\,g_b$. Letting b = e (the identity of **G**) we get
$g_i=g_a$ iff $g_j=g_a \;\Leftrightarrow\; g_i=g_j$. ✔