[3.1] Start with the number (e.g. π, $\sqrt{2}$, $7-\sqrt{3}$), then express it in decimal form. Subtract the integer part (i.e. subtract 3, in the case of π, to leave us with 0.14159265...). Take the reciprocal (i.e. $(0.14159\ldots)^{-1} = 7.0625133\ldots$). Subtract the integer part (now the number 7, in the case of π). Take reciprocal, etc. etc. The sequence of integer parts (here 3, 7, ...) give required numbers

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[3.2] Set $x = 1 + (2 + (2 + (2 + (2 + \ldots)^{-1})^{-1})^{-1})^{-1}$, then

$x - 1 = (2 + (2 + (2 + \ldots)^{-1})^{-1})^{-1}$, so $(x-1)^{-1} = 2 + (2 + (2 + \ldots)^{-1})^{-1}$
$= 1 + x$.

Thus $(x-1)(x+1) = 1$

i.e. $x^2 = 2$, so $x = \sqrt{2}$ (not $-\sqrt{2}$, since positive), convergence assumed

Set $y = 5 + (3 + (1 + (2 + (1 + (2 + \ldots)^{-1})^{-1})^{-1})^{-1})^{-1}$, so $\frac{1}{y-5} = 3 + (1 + \ldots)^{-1}$, $\therefore \left(\frac{1}{y-5} - 3\right)^{-1} = 1 + (\ldots)^{-1}$

$\therefore (((y-5)^{-1} - 3)^{-1} - 1)^{-1} - 2 = ((y-5)^{-1} - 3)^{-1}$ Put $Y = ((y-5)^{-1} - 3)^{-1} = 1 + (2 + (1 + (2 + (\ldots)^{-1})^{-1})^{-1})^{-1}$ > 0

Then $\frac{1}{Y-1} - 2 = \frac{1}{Y}$ $\therefore Y - 2Y(Y-1) = Y - 1$, i.e. $2Y^2 - 2Y - 1 = 0$ $\therefore Y = \frac{1+\sqrt{3}}{2}$

$\therefore (y-5)^{-1} - 3 = \frac{1}{Y} = \frac{2}{1+\sqrt{3}} = \sqrt{3} - 1$ $\therefore y - 5 = \frac{1}{\sqrt{3}+2} = 2 - \sqrt{3}$ $\therefore y = 7 - \sqrt{3}$

[3.3] $\frac{a}{b} > \frac{c}{d}$ (with a, b, c, d positive reals) is equivalent to saying $\frac{a}{b} > \frac{N}{M} > \frac{c}{d}$, for some positive integers N, M because the fractions are <u>dense</u> in the reals. (There's always a fraction between any two distinct reals.) This is equivalent to $Ma > Nb$, $Nd > Mc$.

[3.4] Take (positive) lengths a, b, p, q.
What is $\frac{a}{b} + \frac{p}{q}$? We could take $\frac{aq + bp}{bq}$.
What is $\frac{a}{b} \times \frac{p}{q}$? We could take $\frac{ap}{bq}$.
<u>BUT</u> the tops and bottoms aren't lengths!

So, more in accordance with the Eudoxian notion of length ratios of [3.3], we say that the <u>sum</u> of $\frac{a}{b}$ and $\frac{p}{q}$ is a ratio $\frac{x}{y}$ of (positive) lengths x, y such that neither $\frac{a}{b} + \frac{p}{q} > \frac{x}{y}$ nor $\frac{a}{b} + \frac{p}{q} < \frac{x}{y}$ which, in Eudoxian language (with no products of lengths), we can write as the non-existence of positive integers A, B, P, Q such that <u>either</u> all of the following $Ba > Ab$, $Qp > Pq$, $(AQ + BP)y > BQx$ (integer × length is OK) (expressing $\frac{a}{b} > \frac{A}{B}$, $\frac{p}{q} > \frac{P}{Q}$, $\frac{AQ+BP}{BQ} > \frac{x}{y}$) <u>or</u> all of $Ba < Ab$, $Qp < Pq$, $(AQ+BP)y < BQx$ (expressing $\frac{a}{b} < \frac{A}{B}$, $\frac{p}{q} < \frac{P}{Q}$, $\frac{AQ+BP}{BQ} < \frac{x}{y}$) hold. Likewise, we say that the <u>product</u> of $\frac{a}{b}$ and $\frac{p}{q}$ is a ratio $\frac{u}{v}$ of (positive) lengths u, v such that neither $\frac{a}{b} \times \frac{p}{q} > \frac{u}{v}$ nor $\frac{a}{b} \times \frac{p}{q} < \frac{u}{v}$, expressed as the non-existence of positive integers A, B, P, Q such that <u>either</u> all of the following $Ba > Ab$, $Qp > Pq$, $APv > BQu$, <u>or</u> all of the following $Ba < Ab$, $Qp < Pq$, $APv < BQu$ hold.