[8.4] $f(z) = \frac{1}{z}$ transforms circles to circles

Lemma 1. Let $z, w \in \mathbb{C}$. Then $\overline{z \pm w} = \bar{z} \pm \bar{w}$, $\overline{zw} = \bar{z}\,\bar{w}$,

and $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

Pf: Let $\begin{cases} z = x + iy = re^{i\theta} \\ w = u + iv = se^{i\omega} \end{cases}$ $\quad \begin{cases} \bar{z} = x - iy = re^{-i\theta} \\ \bar{w} = u - iv = se^{-i\omega} \end{cases}$

$z + w = (x+u) + i(y+v) \qquad \overline{z+w} = (x+u) - i(y+v)$

$\bar{z} + \bar{w} = (x+u) - i(y+v)$ ✓

$zw = rs\,e^{i(\theta+\omega)} \qquad \overline{zw} = rs\,e^{-i(\theta+\omega)} = \bar{z}\,\bar{w}$ ✓

$\frac{z}{w} = \frac{r}{s}e^{i(\theta-\omega)} \qquad \overline{\left(\frac{z}{w}\right)} = \frac{r}{s}e^{-i(\theta-\omega)} = \frac{\bar{z}}{\bar{w}}$ ✓ $\square$

Lemma 2. Let C be a circle in $\mathbb{C}$ with center $h \in \mathbb{C}$ and radius $r \geq 0$.

Then $C = \{z : (z-h)(\bar{z}-\bar{h}) = r^2\}$. Note: This is complex analogy of $x^2+y^2=r^2$

Pf: $C = \{z : |z-h| = r\}$. Let $w = z - h$. Note that

$|w| = r \iff |w| = r$ and $|-w| = r \iff |w|^2 = r^2$ (i.e., no new solutions are introduced by squaring).

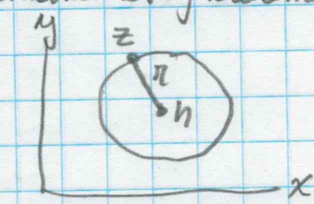
But $|w|^2 = w\bar{w}$ [if $w = a+bi$, $|w|^2 = a^2+b^2 = (a+bi)(a-bi) = w\bar{w}$]

So $C = \{z : |z-h| = r\} = \{z : |w| = r\} = \{z : w\bar{w} = r^2\} = \{z : (z-h)\overline{(z-h)} = r^2\}$

$= \{z : (z-h)(\bar{z}-\bar{h}) = r^2\}$. $\square$

Lemma 3. Conversely, $\{z : (z-h)(\bar{z}-\bar{h}) = r^2\}$ is the eq of a circle with center h and radius $r \geq 0$.

Pf: Follows from proof of Lemma 2 since every statement is iff $\square$



[8.4 con't] Pf of Theorem.

Let $h \in \mathbb{C}$ and C be the circle $\{z : (z-h)(\bar{z}-\bar{h}) = r^2\}$.

Set $\alpha \equiv r^2 - h\bar{h}$. Let $w = f(z) = \frac{1}{z}$. Then f transforms C into

$D = \{w : \alpha w\bar{w} + h\bar{w} + \bar{h}\bar{w} - 1 = 0\}$:

$$D = \{w : (\tfrac{1}{w} - h)(\tfrac{1}{\bar{w}} - \bar{h}) = r^2\} = \{w : (1-hw)(1-\bar{h}\bar{w}) = r^2 w\bar{w}\}$$

$$= \{w : 1 - hw - \bar{h}\bar{w} - h\bar{h}w\bar{w} = r^2 w\bar{w}$$

$$= \{w : (r^2 - h\bar{h})w\bar{w} + hw - \bar{h}\bar{w} - 1 = 0\}$$

$$= \{w : \alpha w\bar{w} + hw - \bar{h}\bar{w} - 1 = 0\}$$

Case 1 : $\alpha = 0$. Note : This is equivalent to origin being a pt of the circle

So $D = \{w : hw + \bar{h}\bar{w} - 1 = 0\}$ is a line since it is a linear combination in w & $\bar{w}$. i.e., f transforms C into a line :

Let $w = u + vi$ and $h = h_1 + h_2 i$. So,

$$hw + \bar{h}\bar{w} = (h_1 + h_2 i)(u + vi) + (h_1 - h_2 i)(u - vi)$$
$$= u(h_1 + h_2 i + h_1 - h_2 i) + v(h_1 i - h_2 - h_1 i - h_2)$$
$$= 2h_1 u - 2h_2 v = 1,$$

a straight line in the $w = uv$ plane

Note 1 : If h lies on x-axis of z-plane, then $h_2 = 0$

$\Rightarrow u = \frac{1}{2h_1}$ is eq of a vertical line in w-plane [See Fig 8.6 in book]

Note 2 : If h lies on y-axis of z-plane, then $h_1 = 0$

$\Rightarrow v = -\frac{1}{2h_2}$ is eq of a horizontal line in w-plane [See Fig 8.6]

Note 3 : $h \neq 0$ since $\alpha = 0$ or else $D = \{w : -1 = 0\}$ (circle at origin of radius 0)

[8.4 cont]

Case 2: $\alpha \neq 0$

Notice that $\alpha \in \mathbb{R}$, so $\bar{\alpha} = \alpha$.

If $h = 0$, then $D = \{w : |w|^2 = \frac{1}{\alpha}\}$. $D = \phi$ (empty set) if $\alpha < 0$.
If $\alpha > 0$, D is a circle at origin in w-plane with radius $\sqrt{\frac{1}{\alpha}} = \frac{1}{r}$.

So, assume $h \neq 0$. If D is a circle in w-plane, by Lemma 2

$\exists\, k \in \mathbb{C}$ and $s \in \mathbb{R} \ni D = \{w : (w-k)(\bar{w}-\bar{k}) = s^2\}$.
Since $D = \{w : \alpha w\bar{w} + hw - \bar{h}\bar{w} - 1 = 0\}$, we need to find k & s s.t.

$$\begin{aligned} \alpha w\bar{w} + hw - \bar{h}\bar{w} - 1 &= (w-k)(\bar{w}-\bar{k}) - s^2 \\ &= w\bar{w} - \bar{k}w - k\bar{w} + k\bar{k} - s^2 \\ &= \alpha w\bar{w} - \alpha\bar{k}w - \alpha k\bar{w} + \alpha(k\bar{k} - s^2) \end{aligned}$$

Thus we need $\alpha k = -\bar{h}$, $\alpha\bar{k} = -h$, and $\alpha(k\bar{k} - s^2) = -1$.

Set $k \equiv -\frac{\bar{h}}{\alpha}$. Then $\alpha k = -\bar{h}$ ✓ Also, $\bar{k} = -\left(\overline{\frac{\bar{h}}{\alpha}}\right) = -\frac{h}{\bar{\alpha}} = -\frac{h}{\alpha}$

$\Rightarrow \alpha\bar{k} = -h$ ✓ Finally, we need

$$-1 = \alpha(k\bar{k} - s^2) = \alpha(|k|^2 - s^2) = \alpha\left(\frac{|h|^2}{\alpha^2} - s^2\right) = \frac{|h|^2}{\alpha} - \alpha s^2$$

$$\Rightarrow \alpha s^2 = \frac{|h|^2}{\alpha} + 1 = \frac{|h|^2 + \alpha}{\alpha} \Rightarrow s^2 = \frac{|h|^2 + \alpha}{\alpha^2}.$$ So,

define $s = \frac{\sqrt{|h|^2 + \alpha}}{\alpha}$ and that satisfies the condition that D is a circle. □

Note 4: $s = \frac{r}{|\alpha|}$, so D is a circle centered at $-\frac{\bar{h}}{\alpha}$ with radius $\frac{r}{|\alpha|}$,

or (in original terms) centered at $-\frac{\bar{h}}{r^2 - |h|^2}$ with radius $\frac{r}{r^2 - |h|^2}$

Note 5: Center k of D

$\neq \frac{1}{h}$ except if $r = 0$ or

special case of circle about origin.

If not centered at origin, then $h \neq 0$

and $k = -\frac{\bar{h}}{\alpha} = -\frac{\bar{h}}{r^2 - |h|^2}$. If also

$k = \frac{1}{h}$, then $r^2 - |h|^2 = -|h|^2 \Rightarrow r = 0$ ✓