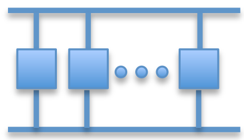
[13.24] Show $\mathrm{Det}\left(\mathrm{I}+\in \mathrm{A}\right)=1+\in \mathrm{Tr}\left(\mathrm{A}\right)$ for infinitessimal $\in$

Let A =  and $\mathrm{I}+\in \mathrm{A}$ =   $\in$   

Det($\mathrm{I}+\in \mathrm{A}$) = $\dfrac{1}{n!}$ 

$$= \frac{1}{n!}\in^{r\cdots z}\left[\left(\delta_r^a+\in \mathrm{A}^a{}_r\right)\left(\delta_s^b+\in \mathrm{A}^b{}_s\right)\cdots\left(\delta_z^j+\in \mathrm{A}^j{}_z\right)\right]\varepsilon_{ab\cdots j}$$

I provide 2 proofs, one using tensor symbols and one using diagrammatic notation.

Tensor Proof: Det($\mathrm{I}+\in \mathrm{A}$) =

$$\frac{1}{n!}\in^{r\cdots z}\left\{\left(\delta_r^a\delta_s^b\cdots\delta_z^j\right)+\in\left[\left(\mathrm{A}^a{}_r\delta_s^b\cdots\delta_z^j\right)+\left(\delta_r^a\mathrm{A}^b{}_s\cdots\delta_z^j\right)+\cdots+\left(\delta_r^a\delta s_r^b\cdots\delta_y^h\mathrm{A}^j{}_z\right)\right]+\cancel{o\left(\in^2\right)}\right\}\varepsilon_{ab\cdots j}$$

This is a double sum over permutations of (a,b,…,j) and (r,s,…,z). Fix a permutation of (a,b,…,j). The only non-zero term is

$$\frac{1}{n!}\left\{\left(\delta_a^a\delta_b^b\cdots\delta_j^j\right)+\in\left[\left(\mathrm{A}^a{}_a\delta_b^b\cdots\delta_j^j\right)+\left(\delta_a^a\mathrm{A}^b{}_b\cdots\delta_j^j\right)+\cdots+\left(\delta_a^a\delta s_b^b\cdots\delta_h^h\mathrm{A}^j{}_j\right)\right]\right\}$$

$$=\frac{1}{n!}\left\{1+\in\left[\left(\mathrm{A}^a{}_a\right)+\left(\mathrm{A}^b{}_b\right)+\cdots+\left(\mathrm{A}^j{}_j\right)\right]\right\}$$

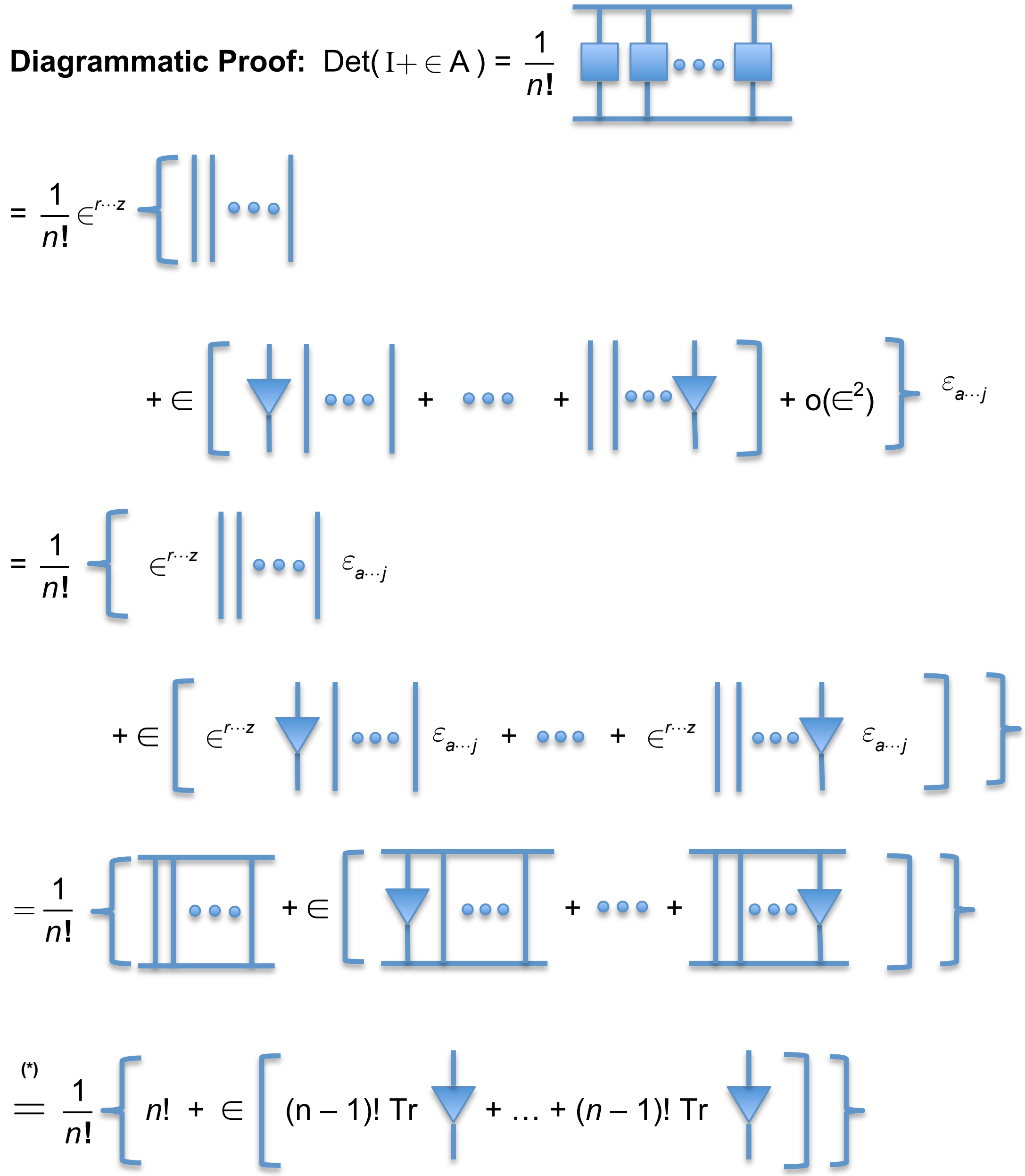
$$=\frac{1}{n!}\left[1+\in \mathrm{Tr}\left(A\right)\right].$$

(Note: The above is actually multiplied by the permutation of $\in^{ab\cdots j}\varepsilon_{ab\cdots j}$. However, this factor can be omitted because all such products are equal to $\in^{12\cdots n}\varepsilon_{12\cdots n}=1$ as I showed in the proof of Problem [13.22].)

There are n! permutations of (a,b,…,j), so

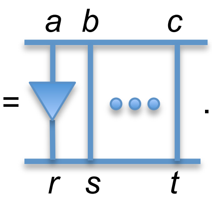
Det($\mathrm{I}+\in \mathrm{A}$) = $\dfrac{n!}{n!}\left[1+\in \mathrm{Tr}\left(A\right)\right]$ = $1+\in \mathrm{Tr}\left(A\right)$ ✔

Diagrammatic Proof: Det(I+ ∈ A) = $\frac{1}{n!}$



= $\frac{1}{n!}$ [$n!$ + ∈ (n) (n – 1)! Tr(A)]

= 1 + ∈ Tr(A)

(*) Let B = 

.

Let $\mathcal{P}_{ab\ldots c}$ and $\mathcal{P}_{rs\ldots t}$ be the sets of permutations of (a,b,…,c) and (r,s,…,t), respectively.

$$\mathrm{B} = \in^{rs\cdots t}\varepsilon_{ab\cdots c}\mathrm{T}^{a}{}_{r}\delta^{b}_{s}\cdots\delta^{c}_{t} = \sum_{\pi\in\mathcal{P}_{ab\cdots c}}\sum_{\pi'\in\mathcal{P}_{rs\cdots t}}\in^{\pi'(r)\pi'(s)\cdots\pi'(t)}\varepsilon_{\pi(a)\pi(b)\cdots\pi(c)}\mathrm{T}^{\pi(a)}{}_{\pi'(r)}\delta^{\pi(b)}_{\pi'(s)}\cdots\delta^{\pi(c)}_{\pi'(t)}.$$

Fix π. The only non-zero term in the sum is

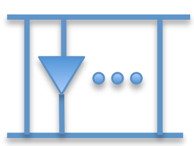
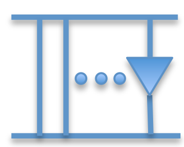
$$\in^{\pi(a)\pi(b)\cdots\pi(c)}\varepsilon_{\pi(a)\pi(b)\cdots\pi(c)}\mathrm{T}^{\pi(a)}{}_{\pi(a)}\delta^{\pi(b)}_{\pi(b)}\cdots\delta^{\pi(c)}_{\pi(c)} = \mathrm{T}^{\pi(a)}{}_{\pi(a)}.$$

(Note: I showed in Problem [13.22] that $\in^{xy\cdots z}\varepsilon_{xy\cdots z} = 1$ for any fixed (x,y,…,z).)

Thus, B = $\sum_{\pi\in\mathcal{P}_{ab\cdots c}}\mathrm{T}^{\pi(a)}{}_{\pi(a)}$. This sum has n! terms composed of (n – 1)! terms equal to $\mathrm{T}^{a}{}_{a}$, (n – 1)! terms equal to $\mathrm{T}^{b}{}_{b}$, …, and (n – 1)! terms equal to $\mathrm{T}^{c}{}_{c}$. So,

B = (n – 1)! ($\mathrm{T}^{a}{}_{a}$ + $\mathrm{T}^{b}{}_{b}$ + … + $\mathrm{T}^{c}{}_{c}$) = (n – 1)! Tr (A) = (n – 1)! Tr 

Similarly,

 = (n – 1)! Tr  , … ,  = (n – 1)! Tr  .

■

I believe that (*) merits being a standalone problem all on its own.