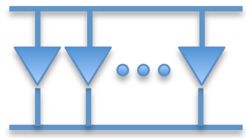
[13.21] Given Det T = $\frac{1}{n!}$  , show that

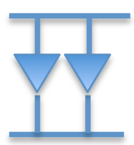
(a) $\begin{vmatrix} a & b \\ c & c \end{vmatrix} = ad - bc$,

(b) $\begin{vmatrix} q & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = adj - afh + bfg - bdj + cdh - ceg$, and

(c) Det T = $\sum_{\pi \in \mathcal{P}_{12\cdots n}} \text{Sign}(\pi)\, T^1{}_{\pi(1)} \cdots T^n{}_{\pi(n)}$.

Proof. Penrose did not ask for part (c). But, my proof is short and (c) is the standard definition of determinant given in calculus.

This proof uses the fact that if $\boldsymbol{\varepsilon} \bullet \in = \varepsilon_{a\ldots c} \in^{a\cdots c}$ is normalized to $n!$, then each of the $n!$ terms satisfies $\varepsilon_{a\ldots c} \in^{a\cdots c} = 1$.

(a) Det $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2!}$  = $\frac{1}{2!}\varepsilon_{rs} \in^{tu} T^r{}_t T^s{}_u$

$$= \frac{1}{2}\begin{pmatrix} \varepsilon_{12} \in^{12} T^1{}_1 T^2{}_2 + \varepsilon_{12} \in^{21} T^1{}_2 T^2{}_1 \\ + \varepsilon_{21} \in^{21} T^2{}_2 T^1{}_1 + \varepsilon_{21} \in^{12} T^2{}_1 T^1{}_2 \end{pmatrix}$$

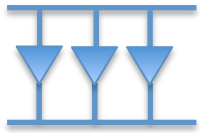
$$= \frac{1}{2}\begin{pmatrix} \cancel{\varepsilon_{12} \in^{12}} T^1{}_1 T^2{}_2 - \cancel{\varepsilon_{12} \in^{12}} T^1{}_2 T^2{}_1 \\ + \cancel{\varepsilon_{21} \in^{21}} T^2{}_2 T^1{}_1 - \cancel{\varepsilon_{12} \in^{12}} T^2{}_1 T^1{}_2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} T^1{}_1 T^2{}_2 - T^1{}_2 T^2{}_1 \\ + T^1{}_1 T^2{}_2 - T^1{}_2 T^2{}_1 \end{pmatrix}$$

$$= \frac{1}{2}\left(2T^1{}_1 T^2{}_2 - 2T^1{}_2 T^2{}_1\right)$$

$= T^1{}_1 T^2{}_2 - T^1{}_2 T^2{}_1$

$= ad - bc$ ✔

(b) $\mathrm{Det}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = \dfrac{1}{3!}$  $= \dfrac{1}{6}\varepsilon_{rst} \in^{uvw} \mathrm{T}^r{}_u \mathrm{T}^s{}_v T^t{}_w$

$$`= \frac{1}{6}\begin{bmatrix} \varepsilon_{123}\in^{123}\mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 + \varepsilon_{123}\in^{132}\mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots + \varepsilon_{123}\in^{321}\mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ + \varepsilon_{132}\in^{132}\mathrm{T}^1{}_1\mathrm{T}^3{}_3\mathrm{T}^2{}_2 + \varepsilon_{132}\in^{123}\mathrm{T}^1{}_1\mathrm{T}^3{}_2\mathrm{T}^2{}_3 + \cdots + \varepsilon_{132}\in^{312}\mathrm{T}^1{}_3\mathrm{T}^3{}_1\mathrm{T}^2{}_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + \varepsilon_{321}\in^{321}\mathrm{T}^3{}_3\mathrm{T}^2{}_2\mathrm{T}^1{}_1 + \varepsilon_{321}\in^{231}\mathrm{T}^3{}_2\mathrm{T}^2{}_3\mathrm{T}^1{}_1 + \cdots + \varepsilon_{321}\in^{123}\mathrm{T}^3{}_1\mathrm{T}^2{}_2\mathrm{T}^1{}_3 \end{bmatrix} \text{(36 terms)}$$

$$= \frac{1}{6}\begin{bmatrix} \cancel{\varepsilon_{123}\in^{123}}\mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \cancel{\varepsilon_{123}\in^{123}}\mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \cancel{\varepsilon_{123}\in^{123}}\mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ + \varepsilon_{132}\in^{132}\mathrm{T}^1{}_1\mathrm{T}^3{}_3\mathrm{T}^2{}_2 - \cancel{\varepsilon_{132}\in^{132}}\mathrm{T}^1{}_1\mathrm{T}^3{}_2\mathrm{T}^2{}_3 + \cdots - \cancel{\varepsilon_{132}\in^{132}}\mathrm{T}^1{}_3\mathrm{T}^3{}_1\mathrm{T}^2{}_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + \cancel{\varepsilon_{321}\in^{321}}\mathrm{T}^3{}_3\mathrm{T}^2{}_2\mathrm{T}^1{}_1 - \cancel{\varepsilon_{321}\in^{321}}\mathrm{T}^3{}_2\mathrm{T}^2{}_3\mathrm{T}^1{}_1 + \cdots - \cancel{\varepsilon_{321}\in^{321}}\mathrm{T}^3{}_1\mathrm{T}^2{}_2\mathrm{T}^1{}_3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ + \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \end{bmatrix}$$

$$= \frac{1}{6}\left[6\mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - 6\mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - 6\mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1\right]$$

$$= aej - afh + bfg - bdj + cdh - ceg \quad ✔$$

(c) Let $\mathcal{P}_{1\ldots n}$ be the set of permutations of (1, …, n). We proceed by generalizing the steps used in part (b).

$$\text{Det T} = \frac{1}{n!}\varepsilon_{r\cdots s}\in^{t\cdots u}\mathrm{T}^{r}{}_{t}\cdots\mathrm{T}^{s}{}_{u}$$

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\varepsilon_{\pi^*(1)\cdots\pi^*(n)}\in^{\pi(1)\cdots\pi(n)}\mathrm{T}^{\pi^*(1)}{}_{\pi(1)}\cdots\mathrm{T}^{\pi^*(n)}{}_{\pi(n)}$$

(Replace Einstein notation.)

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\varepsilon_{\pi^*(1)\cdots\pi^*(n)}\in^{\pi(\pi^*(1))\cdots\pi(\pi^*(n))}\mathrm{T}^{\pi^*(1)}{}_{\pi(\pi^*(1))}\cdots\mathrm{T}^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Replace π by $\pi \circ \pi^*$ in $\in$ and T. The double sum over π and π^* is unchanged, stepping over all permutations of (1, …, n), and the exponents of $\in$ continue to match the subscripts of T. This expression generalizes the first block of 36 items in part (b).

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\mathbf{Sign}(\boldsymbol{\pi})\,\cancel{\varepsilon_{\pi^*(1)\cdots\pi^*(n)}\in^{\pi^*(1)\cdots\pi^*(n)}}\,\mathrm{T}^{\pi^*(1)}{}_{\pi(\pi^*(1))}\cdots\mathrm{T}^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Re-order superscripts of $\in$ by applying an inverse π permutation. This corresponds to the 2nd block of 36 items.)

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\mathrm{Sign}(\pi)\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\mathrm{T}^{\mathbf{1}}{}_{\boldsymbol{\pi(1)}}\cdots\mathrm{T}^{\boldsymbol{n}}{}_{\boldsymbol{\pi(n)}}$$

(This is just a simpler way to label the subscripts and superscripts of T. For example, if $\pi^*(3) = 1$ then $\mathrm{T}^{\pi^*(3))}{}_{\pi(\pi^*(3))} = \mathrm{T}^{1}{}_{\pi(1)}$. This is the 3rd block of 36 items.)

$$= \frac{n!}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\mathrm{Sign}(\pi)\mathrm{T}^{1}{}_{\pi(1)}\cdots\mathrm{T}^{n}{}_{\pi(n)}$$

$$= \sum_{\pi\in\mathcal{P}_{1\cdots n}}\mathrm{Sign}(\pi)\mathrm{T}^{1}{}_{\pi(1)}\cdots\mathrm{T}^{n}{}_{\pi(n)} \qquad ✔$$