[13.7] SO(3) is the group of rotations of the unit sphere in 3-space. O(3) extends SO(3) by including reflections. **(A)** Show that SO(3) is a normal subgroup of O(3) and **(B)** show that it is the only proper normal subgroup.

Note. (B) is actually not true. There is one other proper normal subgroup of O(3). If 1 is the identity element (null rotation) of SO(3) and R is the reflection operator then $\mathcal{I}$ ={ 1, R(1) } is a normal subgroup of O(3). This is because if g is a (reflective or non-reflective) rotation, then $g^{-1}\, 1\, g = 1$ and $g^{-1}\, R(1)\, g = R(1)$. So we revise (B) to be that O(3) has only two proper normal subgroups.

Note: In this proof we adopt the convention that $f\, g$ represents rotating by f followed by g. So f R means to rotate and then reflect while R f means to reflect then rotate.

Proof: Penrose gives the hint: "What are the only sets in O(3) that are rotation invariant?". The answer is simple. There are only 2 such sets, which are described shortly. But first, some preliminaries…

Definitions:

1. Let **S** be the unit sphere of $\mathbb{R}^3$
2. Let **R** be the reflection operation S
3. Let **T** = R[SO(3)] be the coset of reflective rotations in O(3)
4. Let **1** be the identity of O(3), the null rotation

R is defined as an operation that reverses *xyz* orientation. It can be a reflection through the *xy*-plane, the *yz*-plane, or the *xz*-plane, or (preferred) a reflection through the origin, which is the composition of reflecting first through the *xy*-plane, then the *yz*-plane, and then the xz-plane.

While SO(3) is a group, T is not. (It is only a coset.) For example, $1 \notin T$. Also, if t_1 and t_2 belong to T, their composition $t_1\, t_2 \notin T$. Rather, $t_1\, t_2 \in SO(3)$. This is because R is applied twice during $t_1\, t_2$. In fact, any expression with an even number of reflections belongs to SO(3), and it belongs to T if the number of reflections is odd.

SO(3) and T are disjoint, and $O(3) = SO(3) \cup T$.

If $t \in T$, there are elements $s_1, s_2 \in SO(3)$ such that $t = R\, s_1$ and $t = s_2\, R$.

We need the following theorem to answer Penrose's invariant question.

Theorem 1. Let $s_1, s_2 \in SO(3)$. Then $\exists\, s_3 \in SO(3)$ such that $s_2 = s_3\, s_1$.

Proof: $s_3 = s_2\, s_3^{-1}$. ■

Theorem 2 (Answer to Penrose's question): SO(3) and T are the only proper subsets of O(3) that are rotation invariant.

Proof: SO(3) is the first rotation-invariant set because applying a rotation to any rotation in SO(3) yields another rotation, which belongs to SO(3). SO(3) has no proper subset A that is rotation-invariant because, by Theorem 1, given any $s_1 \in$ A and $s_3 \in$ SO(3) – one can find a rotation s_2 such that $s_1\, s_2 = s_3$.

Similarly, T is rotation invariant because applying a rotation to any element of T remains in T because only 1 reflection has occurred. Like SO(3), T cannot have a proper subset that is rotation invariant because, like in Theorem 1, from any $t_1 \in$ T one can obtain any other $t_2 \in$ T by applying a rotation. ■

Theorem A: SO(3) is a normal subgroup of O(3)

Proof. First, SO(3) is clearly a group because it contains the identity; inverses of are just reverse rotations which are still rotations; and the product (composition) of 2 rotations is a rotation (closed). To see that it is normal, let $s \in$ SO(3). If $g \in$ SO(3), then $g^{-1}\, s\, g \in$ SO(3) because it is the composition of 3 rotations. If $g \in$ T, then $g^{-1}\, s\, g \in$ SO(3) because it involves 2 reflections. So g^{-1} SO(3) g = SO(3). Multiplying both side by g yields SO(3) g = g SO(3). So SO(3) is normal.

That proof didn't use Penrose's hint, so here is another proof. Since SO(3) is rotation invariant, g SO(3) $\subseteq$ SO(3). By Theorem 1, g SO(3) $\supseteq$ SO(3). Therefore g SO(3) = SO(3). Similarly SO(3) g = SO(3). $\therefore$ g SO(3) = SO(3) g which proves SO(3) is normal. ■

We need the following Lemma a few times so it is worth documenting here.

Lemma: If g and h are non-trivial rotations (i.e., not the zero rotation) whose axes of rotation are perpendicular, and g is a 90° rotation, then $f = g^{-1}\, h\, g$ is a rotation whose rotation axis is perpendicular to the other two.

Proof: This is not difficult to visualize. WLOG g uses the z-axis to rotate the $x$axis into the y-axis, and h rotates by some angle θ about the x-axis. Imagine first a 90° clockwise rotation about the z-axis. Then a counterclockwise θ rotation about the x-axis followed by a 90° counterclockwise rotation about the z-axis. The result is a clockwise θ rotation about the y-axis. (I provide a rigorous, equation-based geometric proof of this claim in my version 2 solution to this problem.) ■

It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3-ball $\mathcal{R}$ of radius π in which antipodal points on the surface of $\mathcal{R}$ are identified.

Definition: For $0 \le \theta \le \pi$ let $\mathbf{S}_\theta$ be the sphere of radius θ in $\mathcal{R}$**.**

$\mathcal{R}$ consists of rotations of angle θ about all axes of rotation.

Theorem 3: If a point h of SO(3) with rotation angle θ belongs to a normal subgroup H of SO(3), then $S_\theta \subseteq H$.

Proof: Let $g \in SO(3)$. First, the following fact is hard to visualize, but $k = g^{-1}\, h\, g$ has the same angle of rotation θ as h, albeit about a possibly different axis of rotation. One might make a (weak) argument that based upon symmetry g and g^{-1} have the same rotation magnitude but opposite directions, so neither should dominate to make the rotation angle of k either larger or smaller than that of h. (See my version 2 solution to this problem for the equations that verify this.) Thus $\{ k = g^{-1}\, h\, g : g \in SO(3) \} \subseteq S_\theta$. That is, using just the normality operation on h, we cannot generate all of SO(3). We can only generate points in S_θ.

Since k has rotation angle θ, we can write k as $k = \theta(a, b, c)$ where $a^2 + b^2 + c^2 = 1$. The point (a, b, c) identifies the axis of rotation. Thus to generate S_θ, it suffices to generate a set of rotations $\{ k \}$ whose axes of rotation are associated each point of the unit sphere.

WLOG we can let $h = \theta(0, 0, 1)$. Consider the set $\{ g \in SO(3) \}$ having 90° rotation angles. From the Lemma if h and g have perpendicular axes of rotation, then $k = g^{-1}\, h\, g$ has a rotation axis perpendicular to both h and g. At the other extreme, if g has the same axis of rotation as h, then k has the same axis of rotation as both h and g. Let g move along a great circle from $h = \frac{\pi}{2}\ \theta(0, 0, 1)$ to $f = \frac{\pi}{2}(1, 0, 0)$. The points (a, b, c) associated with $k = g^{-1}\, h\, g$ sweep out the great circle arc from $(1, 0, 0)$ to $(0, 0, 1)$.

As f moves around the circle of radius θ about the origin in the xy-plane, the points (a, b, c) associated with the great circle arcs sweep out the northern hemisphere of the unit sphere. Note that $h^{-1} = \theta(0, 0, -1) \in H$ since H is a group, and arcs from h^{-1} similarly sweep out the southern hemisphere. Thus every point on the unit sphere is associated with either $g^{-1}\, h\, g$ or $g^{-1}\, h^{-1}\, g$ for some $g \in SO(3)$, concluding the proof. (This proof is performed rigorously using equations with graphical illustrations in my ver 2 proof.) ■

Theorem 4: SO(3) has no proper normal subgroup.

Proof: Let H be a proper normal subgroup of SO(3). $\exists\ 1 \neq h \in H$ having some rotation angle θ. By Theorem 3, $S_\theta \subseteq H$.

We take products of elements in S_θ to expand H beyond S_θ. Let f, $g \in S_\theta$ and $h = f\,g$. The maximum possible angle for h is 2θ and the minimum is 0. The maximum is obtained when $g = f$ and the minimum is obtained when $g = f^{-1}$. By letting g take a path from f to f^{-1} it is possible to generate a set of points $h = f\,g$ in H having every possible angle from 0 to 2θ. By Theorem 3, $S_\phi \subseteq$ H for $0 \le \phi \le 2\theta$. Thus every point of $\mathcal{R}$ in the closed disk of radius 2θ belongs to H. If $\theta \ge \pi$, then we are done. If not, starting from circle $S_{2\theta}$ we generate the closed disk of radius 4θ in H, then 8θ, … Since $\mathcal{R}$ is a ball of radius π, eventually we obtain that all of $\mathcal{R}$ = SO(3) $\subseteq$ H. ■

Theorem B: SO(3) and $\mathcal{I}$ ={ 1,R(1) } are the only proper normal subgroups of O(3).

Proof: Let H ≠ SO(3) and H ≠ $\mathcal{I}$ be a nontrivial normal subgroup of O(3). We need to show that H = O(3).

By Theorem 4, $\exists\, t \in$ H$\cap$T. Suppose for the moment that $t^2 \neq 1$. Then $t^2 \in$ SO(3) ∩ H ⇒ SO(3) ⊂ H by Theorem 4. Also, $\exists\, s \in$ SO(3) such that $t = s$ R. To see that T ⊆ H, let $t_1 \in$ T. $\exists\, s_1 \in$ SO(3) such that $t_1 = s_1$ R. Let $s_2 = s_1\, s^{-1} \in$ SO(3). Then $s_1 = s_2\, s$. So, $t_1 = s_1$ R = $s_2\, s$ R = $s_2\, t \in$ H since $s_2 \in$ SO(3) ⊂ H. Thus T ⊆ H. Since SO(3) ⊆ H, H = O(3).

Unfortunately if t has a 180° rotation angle, then $t^2 = 1$ and this argument doesn't quite hold. Actually, everything will still be true except that we need a new argument that SO(3) ⊂ H. Once we prove this, we again get that $t_1 \in$ H and H = SO(3).

Let g be a 90° rotation about an axis perpendicular to the axis of s. From the Lemma we know that $g^{-1}\, s\, g$ has an axis of rotation perpendicular to that of s. Consider $t_2 = g^{-1}\, t\, g$. By normality, $t_2 \in$ H. We can regard t as a rotation of the reflected sphere, and g as a 90° rotation of the reflected sphere about an axis perpendicular to t. So by the Lemma, since $g \neq 1$ and $t \neq$ R(1), then t_2 rotates the reflected sphere about an axis perpendicular to t. Let $s_3 = t\, t_2$. Because the reflection operator appears twice, $s_3 \in$ SO(3). Suppose $s_3 = 1$. That would mean $t_2 = t^{-1}$. But, inverses rotate about the same axis. Since the axes of t and t_2 are perpendicular, this is false. Therefore $s_3 \neq 1$. Because H is a group, $s_3 \in$ H. Thus, by Theorem 4, SO(3) ⊂ H, completing the proof. ■