

Chapter 13. Symmetry Groups

Groups

Definitions:

A **group** is a set G with an operation \circ that is closed and associative, has an identity e , and every element g has an inverse g^{-1} such that $g \circ g^{-1} = e = g^{-1} \circ g$.

A group G is **Abelian** if it is commutative: $g \circ h = h \circ g$ for all g, h in G .

A **subgroup** is a subset of G that is a group under \circ .

Let H be a subgroup of G . A **coset of H** is a set $H \circ g = \{h \circ g : h \in H\}$, where $g \in G$. The only coset of H that is a group is the set H itself: $H = H \circ e$ where e is the identity element. The cosets of H form a partition of G .

A **normal subgroup** is a subgroup H that satisfies $g \circ H = H \circ g$ for all g in G , or equivalently $H = g^{-1} \circ H \circ g$.

A group is **simple** if it contains no non-trivial normal subgroup. The simple groups are the fundamental “building blocks” of more complex groups.

Theorem. There are precisely 4 **classical** and 5 **exceptional** simple Lie groups.

- Classical Families: A_m, B_m, C_m, D_m having dimensions $m(m+2)$, $m(2m+1)$, $m(2m+1)$, and $m(2m-1)$, respectively where $m \in \mathbb{Z}^+$.
- Exceptional Groups: E_6, E_7, E_8, F_4, G_2 of dimension 78, 133, 248, 52, and 14 respectively

Theorem. The simple finite groups have been classified into classical and exceptional groups. The largest exceptional group has $\approx 10^{60}$ elements and is known as **the monster**.

Definition. The **Product Group** of groups G and H is $\mathbf{G \times H} = \{(g, h) : g \in G, h \in H\}$

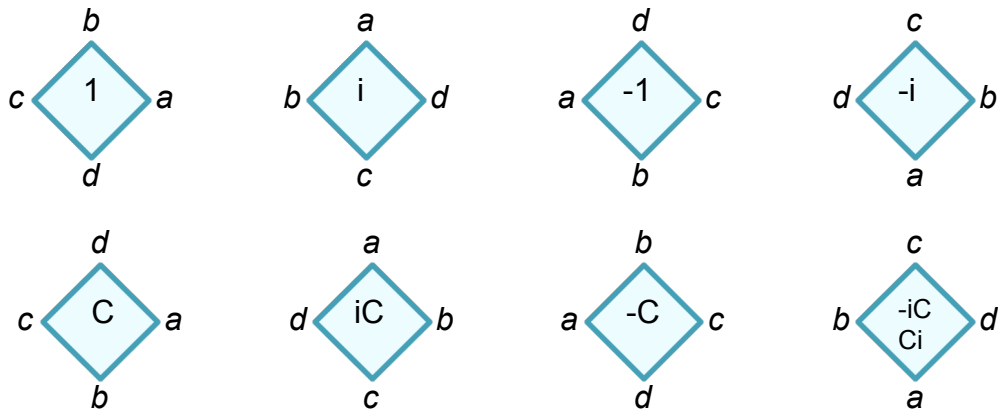
with group operation $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \circ h_2)$.

Definition. Let N be a subgroup of G . The **Factor Space G/N** is the collection of cosets $N \circ g$ along with the operation $(N \circ g_1) \circ (N \circ g_2) = N \circ (g_1 \circ g_2)$.

Theorem. If N is normal then G/N is a group, called the **Factor Group**.

Theorem. [13.10] $H \cong (G \times H) / G$.

Symmetries of a Square



Definitions:

Non-reflecting Group: $\langle i \rangle = \{1, i, -1, -i\}$

Reflecting Group: $\langle i, C \rangle = \{1, i, -1, -i, C, iC, -C, -iC = Ci\}$

C is complex conjugation: $a + bi \mapsto a - bi$. 1 is the null rotation, which is the group identity element. i is the 90° counter-clockwise rotation of the square

Convention: ab means b acts first.

A subgroup of a symmetry group is called a **reduced symmetry group**.

Examples:

Normal subgroups of $\langle i, C \rangle$:

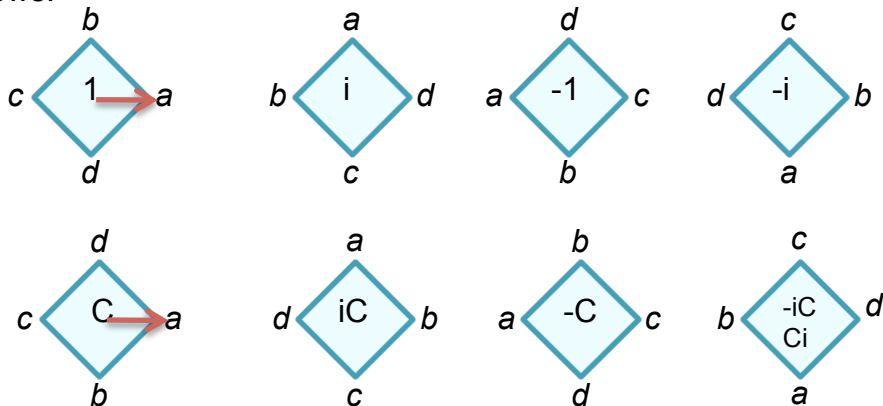
$\{1, -1, C, -C\}$, $\{1, -1\}$, $\{1, -i\}$

Non-normal subgroups of $\langle i, C \rangle$:

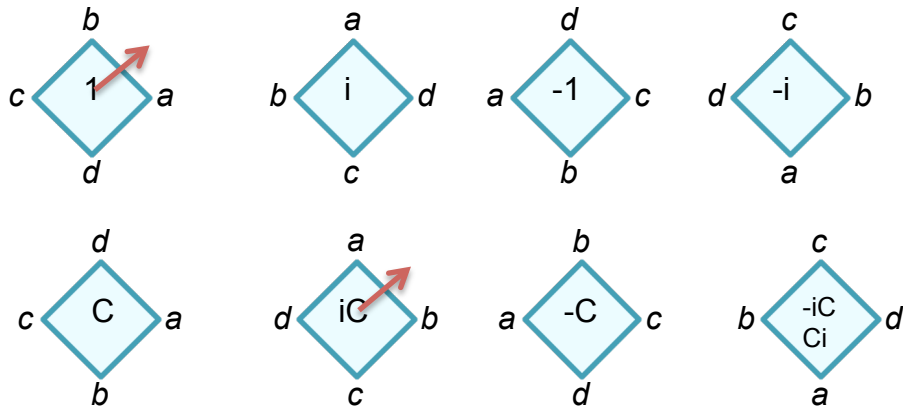
$\{1, -C\}$, $\{1, iC\}$, $\{1, C\}$

For example, $\{1, C\}i = \{i, Ci\} \neq \{i, -Ci\} = i\{1, C\}$

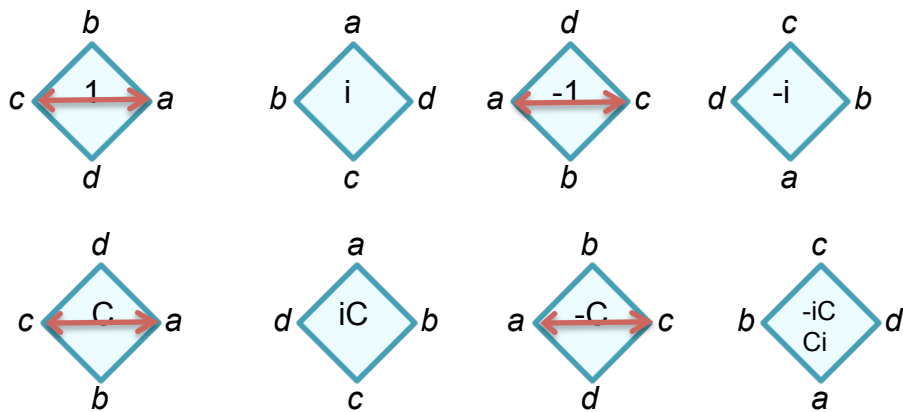
Example [13.6]: Reduced symmetry groups can be generated using one or more arrows.



$\{1, C\}$ is a reduced symmetry group



$\{1, iC\}$ is a reduced symmetry group



$\{1, -1, C, -C\}$ is a reduced symmetry group

Symmetries of a Sphere

Definitions:

A group G whose underlying set is continuous is called a **Lie Group**.

SO(3) is the group of non-reflective symmetries of a 3-sphere

O(3) is the **Orthogonal Group**. It consists of both the reflective and non-reflective symmetries of a sphere.

$O(3) = SO(3) \cup T$, the disjoint union of $O(3)$ with the coset of reflective symmetries

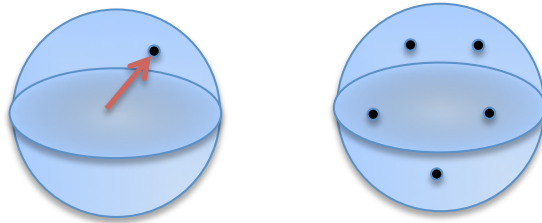
$T = R SO(3) = \{Rg : g \in SO(3)\}$ where R is the reflection operator on the sphere.

Recall problem [12.7]: $SO(3)$ is group isomorphic to the solid sphere \mathcal{R} of radius π with antipodal points identified.

Theorem. (Problem [13.7]) $SO(3)$ and $\{1, R\}$ are the only normal subgroups of $O(3)$, where **1** is the null rotation. (Penrose overlooked that the latter group is normal.)

Examples. Reduced Symmetry Groups

The set of rotations that fix a point on the sphere forms a non-normal subgroup. It is the set of rotations having the arrow as its axis.



Marking the sphere with vertices of a regular polyhedra reduces to the finite group of rotations of the sphere that take each vertex to one of the others. Such reduced symmetry groups are non-normal.

Linear Transformations and Matrices

Definition. Let V and W be vector spaces.

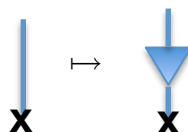
- $f: V \rightarrow W$ is a **homomorphism** if it preserves the vector space structure:
 - $f(au + bv) = af(u) + bf(v)$ for all vectors u and v and scalars a and b .
- $\text{Hom}(V, W)$ is the set of homomorphisms from V to W .
- $\mathcal{A}(V) = \text{Hom}(V, V)$.
- A **linear transformation** is a member $T \in \mathcal{A}(V)$.
 - That is, a linear transformation is a function $T: V \rightarrow V$ such that $T(au + bv) = aTu + bTv$.

Theorem. [13.12, 13.13] Let $V = \mathbb{R}^3$, using (x^1, x^2, x^3) instead of (x, y, z) . Then a linear transformation T takes the form $T: x^r \mapsto T^r_s x^s = ax^1 + bx^2 + cx^3$.

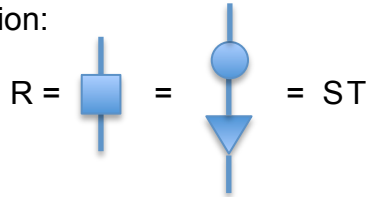
Note. Linear transformations are represented by matrices:

$$T: \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} T^1_1 & T^1_2 & T^1_3 \\ T^2_1 & T^2_2 & T^2_3 \\ T^3_1 & T^3_2 & T^3_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{or} \quad x \mapsto Tx$$

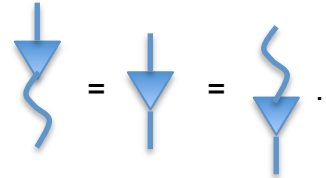
In diagrammatic form this is



Theorem. If $R = ST$ then $R^a_c = S^a_b T^b_c$. That is, the composition, R , of 2 linear transformations is the result of matrix multiplication of S and T . In diagrammatic notation:



Example. $TI = T = IT$ is written in diagrammatic form as



and, in \mathbb{R}^3 , $I = \delta^a_b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where a, b range over $\{1, 2, 3\}$.

Definitions. A linear transformation T is **singular** if $\text{Dim}(TV) < \text{Dim } W$; that is, T is not *onto*.

Theorem. [13.17] T is singular iff $\exists v \neq 0$ such that $Tv = 0$.

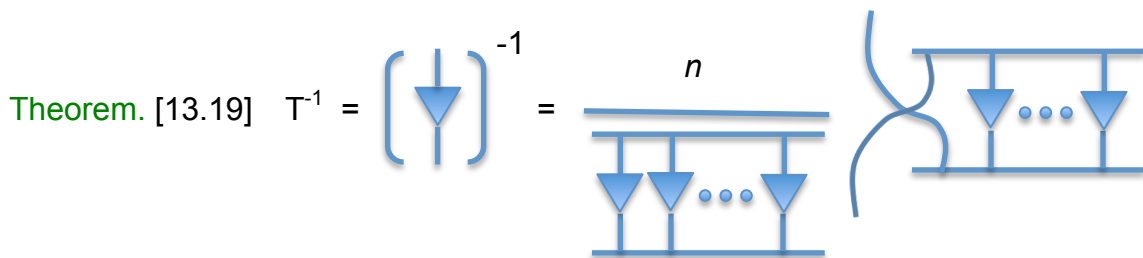
Corollary. [Bud] T is 1-1 iff T is non-singular iff T is onto.

Proof: T is 1-1 $\Leftrightarrow \forall v \neq w \quad T(v - w) = T(v) - T(w) \neq 0 \stackrel{(*)}{\Leftrightarrow} \forall u \neq 0 \quad T(u) \neq 0$

$\stackrel{[13.17]}{\Leftrightarrow} T$ is non-singular $\Leftrightarrow T$ is onto.

(*) Set $v = 3u$ and $w = 2u$. ■

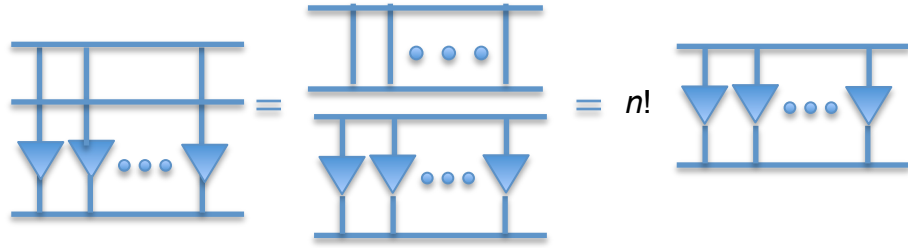
Theorem. [13.18] If T is nonsingular, then it has an inverse T^{-1} .



Definition. The **transpose** of the matrix $T = (T^i_j)$ is the matrix $T^T = (T^j_i)$.

Definition. A matrix T is **orthogonal** if $T^{-1} = T^T$.

Theorem. [Bud]



Proof: Let $P_{a\dots g}$ be the set of permutations of (a, \dots, g) . Then

$$\begin{aligned}
 & \text{Diagram} = n! \varepsilon_{a\dots g} \in^{r\dots x} T_r^a \dots T_x^g \\
 &= \frac{n!}{n!} \varepsilon_{a\dots g} \in^{r\dots x} \sum_{\pi \in P_{ab\dots g}} \text{Sign}(\pi) T_r^{\pi(a)} \dots T_x^{\pi(g)} = n! \varepsilon_{a\dots g} T_r^a \dots T_x^g \in^{r\dots x} \\
 &= n! \text{Diagram}
 \end{aligned}$$

(*) π is the composition of transmutations (i.e., of pairwise permutations).

Let $\pi^*: \begin{matrix} c \mapsto e \\ e \mapsto c \end{matrix}$ be a transmutation. Then

$$\begin{aligned}
 & \varepsilon_{a\dots c\dots e\dots g} \in^{r\dots t\dots v\dots x} \text{Sign}(\pi^*) T_r^{\pi(a)} \dots T_t^{\pi(c)} \dots T_v^{\pi(e)} \dots T_x^{\pi(g)} \\
 &= \varepsilon_{a\dots c\dots e\dots g} \in^{r\dots t\dots v\dots x} \text{Sign}(\pi^*) T_r^a \dots T_t^e \dots T_v^c \dots T_x^g \\
 &= \varepsilon_{a\dots e\dots c\dots g} \in^{r\dots t\dots v\dots x} \text{Sign}(\pi^*) T_r^a \dots T_t^c \dots T_v^e \dots T_x^g \text{ (Rename } c \mapsto e \text{ \& } e \mapsto c) \\
 &= \text{Sign}(\pi^*) \varepsilon_{a\dots c\dots e\dots g} \in^{r\dots t\dots v\dots x} \text{Sign}(\pi^*) T_r^a \dots T_t^c \dots T_v^e \dots T_x^g \\
 &= \varepsilon_{a\dots g} T_r^a \dots T_x^g \in^{r\dots x}.
 \end{aligned}$$

So, for any permutation π , we have

$$\varepsilon_{a\dots g} \in^{r\dots x} \text{Sign}(\pi) T_r^{\pi(a)} \dots T_x^{\pi(g)} = \varepsilon_{a\dots g} T_r^a \dots T_x^g \in^{r\dots x} \quad \blacksquare$$

Theorem. [13.22]

$$\text{Det } AB = \frac{1}{n!} = \left(\frac{1}{n!}\right)^2 = \left(\frac{1}{n!}\right)^2$$

= DetA DetB

Theorem. (p.260 – no proof given) Matrix A is singular iff Det A = 0.

Proof: From [13.19], A is non-singular iff Det A \neq 0. ■

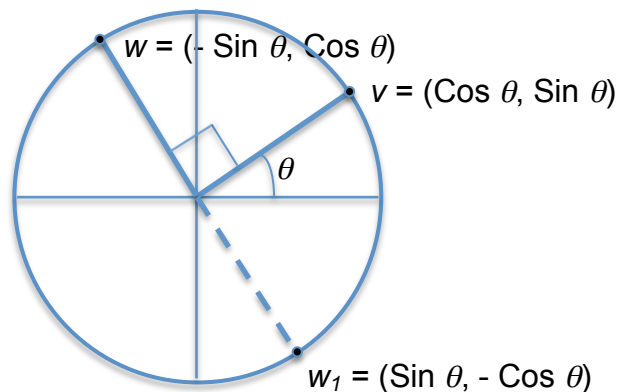
Definition. Vectors v and w are **orthogonal** if $v \cdot w = 0$. That is, the angle between them is 90° .

Theorem. A matrix is orthogonal (i.e., $T^T = T^{-1}$) iff its column vectors are mutually orthogonal.

Example. Orthogonal 2 x 2 Matrices: A and B

$$\text{Let } A = \begin{pmatrix} v & w \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$A^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$



$$A^T = A^{-1} :$$

$$A A^T = \begin{pmatrix} \sin^2 \theta + \cos^2 \theta & 0 \\ 0 & \sin^2 \theta + \cos^2 \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \checkmark$$

$$\text{Similarly } A^T A = I \quad \checkmark$$

So A is an orthogonal matrix \checkmark

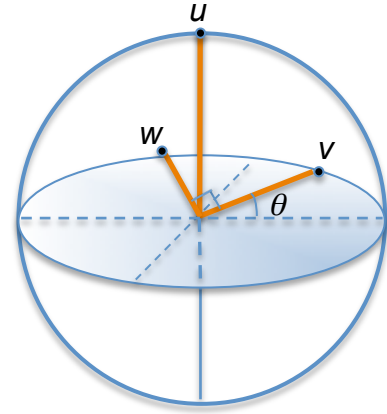
$$\text{Det } A = \text{Det } A^T = \cos^2 \theta + \sin^2 \theta = 1 \quad \checkmark$$

The column vectors of A are orthogonal: $v \perp w$ \checkmark

Let $B = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} = B^T$. Then $B B^T = I$, $\text{Det } B = \text{Det } B^T = -1$, and its column vectors v and w_1 are orthogonal.

Examples. Orthogonal 3 x 3 Matrices: A, B, and C

$$\text{Let } v = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, w = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, \text{ and } u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$



$$\text{Let } A = \begin{pmatrix} v & w & u \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$A^T = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. A \text{ is orthogonal, its columns are orthogonal vectors,}$$

and its determinant is +1. ✓

$$\text{Let } B = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}. B \text{ is orthogonal and its determinant is } -1. \checkmark$$

Let C be a θ -rotation of A about an axis $\{t(a, b, c) : 0 < t < \infty, a^2 + b^2 + c^2 = 1\}$:

$$C = \begin{pmatrix} \frac{1}{2}[1 + a^2 - b^2 - c^2 + (1 - a^2 + b^2 + c^2)\cos \theta] & 2\sin \frac{\theta}{2} \left(-c \cos \frac{\theta}{2} + ab \sin \frac{\theta}{2} \right) & 2\sin \frac{\theta}{2} \left(b \cos \frac{\theta}{2} + ac \sin \frac{\theta}{2} \right) \\ 2\sin \frac{\theta}{2} \left(c \cos \frac{\theta}{2} + ab \sin \frac{\theta}{2} \right) & \frac{1}{2}[1 - a^2 + b^2 - c^2 + (1 + a^2 - b^2 + c^2)\cos \theta] & 2\sin \frac{\theta}{2} \left(-a \cos \frac{\theta}{2} + bc \sin \frac{\theta}{2} \right) \\ 2\sin \frac{\theta}{2} \left(-b \cos \frac{\theta}{2} + ac \sin \frac{\theta}{2} \right) & 2\sin \frac{\theta}{2} \left(a \cos \frac{\theta}{2} + bc \sin \frac{\theta}{2} \right) & \frac{1}{2}[1 - a^2 - b^2 + c^2 + (1 + a^2 + b^2 - c^2)\cos \theta] \end{pmatrix}$$

It can be directly verified that C is an orthogonal matrix with mutually orthogonal column vectors and determinant +1. ✓

Definition. A **symmetry** of a vector space $(V, +)$ is a transformation $T : V \mapsto V$ that is 1-1 and onto that preserves the vector space structure:

$$T(a v + b w) = a T v + b T w$$

Definition. The **General Linear Group** $GL(n)$ is the group of symmetries of an n -dimensional vector space.

Theorem. $GL(n)$ is the group of non-singular $(n \times n)$ matrices.

Proof. Let $T \in GL(n)$. Since $T(a v + b w) = a T v + b T w$, T is a linear transformation. Were T singular, then by [13.17] $\dim TV < n \Rightarrow T$ is not onto, a

contradiction. Therefore T is a non-singular linear transformation. Thus in any basis, T is represented by a non-singular matrix. ■

Definition. The **Special Linear Group $SL(n)$** is the subset of $GL(n)$ having determinant = 1.

Theorem. $SL(n)$ is a normal subgroup of $GL(n)$.

Proof. First, $SL(n)$ is a **group**:

Closed: If $S_1, S_2 \in SL(n)$, then $\text{Det}(S_1 S_2) = \text{Det}(S_1) \text{Det}(S_2) = 1$
 $\Rightarrow S_1 S_2 \in SL(n)$.

Identity: $\text{Det}(I) = 1 \Rightarrow I \in SL(n)$

Inverse: $1 = \text{Det}(I) = \text{Det}(S_1 S_1^{-1}) = \text{Det}(S_1) \text{Det}(S_1^{-1}) = \text{Det}(S_1^{-1})$
 $\Rightarrow S_1^{-1} \in SL(n)$

Also, $SL(n)$ is **normal**:

Let $S \in SL(n)$ and $G \in GL(n)$. Then

$$\begin{aligned} \text{Det}(G^{-1} S G) &= \text{Det}(G^{-1}) \text{Det}(S) \text{Det}(G) = \text{Det}(G^{-1}) \text{Det}(G) \\ &= \text{Det}(G G^{-1}) = \text{Det}(I) = 1 \end{aligned}$$

$$\Rightarrow G^{-1} S G \in SL(n) \Rightarrow G^{-1} SL(n) G = SL(n) \quad \blacksquare$$

The groundwork has now been laid to introduce the table, below, that shows the relationships between $GL(3)$, $O(3)$, $SL(3)$, general linear transformations, orthogonality, determinants, and symmetries. The table shows that $SL(3) \subseteq O(3) \subseteq GL(3) \subseteq \mathcal{A}(\mathbb{R}^3)$, and $GL(3)$ is both the set of symmetries of \mathbb{R}^3 and the set of non-singular matrices. It also shows that the orthogonal group $O(3)$ is a proper subset of the set of orthogonal matrices (shaded blue).

In general, non-singular matrices squeeze and stretch the unit sphere (or the reflected sphere) into an ellipsoid. However, singular matrices are more severe. They squash the unit sphere down to a 2-dimensional circle or ellipse or even to a line or a point.

Only orthogonal matrices preserve the sphere without squeezing or stretching any portion of it. This is achieved by limiting its operation to rotations and reflections. However, if determinant $\neq \pm 1$ then orthogonal matrices also expand or contract the sphere.

Non-orthogonal matrices also squeeze, stretch, or preserve the sphere but not as rotations. Rather, the matrix columns would contain non-orthogonal vectors. In such a case the angle between the 1st and 2nd column vectors might be less than 90° , squeezing the sphere along associated plane. The angle between the 2nd and 3rd vectors would then be greater than 90° , stretching the sphere along that plane.

$A(\mathbb{R}^3) = 3 \times 3$ Matrices

Determinant	Orthogonal	Sphere maps to a ...	Matrix Type
0	Yes	Circle or line or point	Singular
	No	Ellipse or line or point	
Between -1 and 0	Yes	Contracted reflected sphere	GL(3) Non-singular Symmetries of \mathbb{R}^3
	No	Contracted reflected ellipsoid	
Between 0 and +1	Yes	Contracted sphere	
	No	Contracted ellipsoid	
-1	Yes	Reflected sphere	
	No	Reflected ellipsoid	
+1	Yes	SL(3) = sphere	
	No	Ellipsoid	
< -1	Yes	Expanded reflected sphere	
	No	Expanded reflected ellipsoid	
> 1	Yes	Expanded sphere	
	No	Expanded ellipsoid	

Matrices with positive determinant act on the sphere. Matrices with negative determinant behave exactly the same but act on the reflected sphere.

Definition. The **Trace** of A is $\text{Tr}(A) = \text{Tr} \downarrow = T^k_k = T^1_1 + \dots T^n_n$.

Theorem: [Bud]

$$\text{Tr} \downarrow = \frac{1}{(n-1)!} \begin{array}{c} a \quad b \quad c \\ \downarrow \quad \downarrow \quad \downarrow \\ r \quad s \quad t \end{array} = \frac{1}{(n-1)!} \begin{array}{c} \downarrow \quad \dots \quad \downarrow \\ \downarrow \quad \dots \quad \downarrow \end{array} = \dots$$

$$= \frac{1}{(n-1)!} \begin{array}{c} \downarrow \quad \dots \quad \downarrow \\ \downarrow \quad \dots \quad \downarrow \end{array}$$

Proof: Let $\mathcal{P}_{ab\dots c}$ and $\mathcal{P}_{rs\dots t}$ be the sets of permutations of (a,b,\dots,c) and (r,s,\dots,t) ,

respectively. Let $B =$

$$\begin{array}{c} a \quad b \quad c \\ \downarrow \quad \downarrow \quad \downarrow \\ r \quad s \quad t \end{array}$$

$$= \epsilon^{rs\dots t} \epsilon_{ab\dots c} T^a_r \delta^b_s \dots \delta^c_t = \sum_{\pi \in \mathcal{P}_{ab\dots c}} \sum_{\pi' \in \mathcal{P}_{rs\dots t}} \epsilon^{\pi'(r)\pi'(s)\dots\pi'(t)} \epsilon_{\pi(a)\pi(b)\dots\pi(c)} T^{\pi(a)}_{\pi'(r)} \delta^{\pi(b)}_{\pi'(s)} \dots \delta^{\pi(c)}_{\pi'(t)}.$$

Fix π . The only non-zero term in the sum is

$$\in^{\pi(a)\pi(b)\dots\pi(c)} \varepsilon_{\pi(a)\pi(b)\dots\pi(c)} T_{\pi(a)}^{\pi(a)} \delta_{\pi(a)}^{\pi(b)} \dots \delta_{\pi(c)}^{\pi(c)} = T_{\pi(a)}^{\pi(a)}.$$

I showed in Problem [13.22] that $\in^{xy\dots z} \varepsilon_{xy\dots z} = 1$ for any fixed (x,y,\dots,z) .

Thus, $B = \sum_{\pi \in \mathcal{P}_{ab\dots c}} T_{\pi(a)}^{\pi(a)}$. This sum has $n!$ terms composed of $(n-1)!$ terms equal to T_a^a , $(n-1)!$ terms equal to T_b^b , ..., and $(n-1)!$ terms equal to T_c^c . So,

$$B = (n-1)! (T_a^a + T_b^b + \dots + T_c^c) = (n-1)! \text{Tr}(A) = (n-1)! \text{Tr}$$

Similarly for the other figures. ■

Theorem. [13.24] $\text{Det}(I + \in A) = 1 + \in \text{Tr}(A)$ if we ignore 2^{nd} order and higher \in terms.

Theorem. [13.25] $\text{Det } e^A = e^{\text{Tr}(A)}$.

Definition. An **Eigenvector** is a non-zero vector v for which $\exists \lambda \in \mathbb{C}$ such that $Tv = \lambda v \Leftrightarrow (T - \lambda I)v = 0$. λ is called an **Eigenvalue**.

Note: $\text{Det}(T - \lambda I) = 0$ and so $(T - \lambda I)$ is singular

Theorem. [13.26] $\text{Det}(T - \lambda I) = (\lambda_1 - \lambda) \dots (\lambda_n - \lambda) = 0$ is a polynomial equation of degree n .

Definition. λ has **multiplicity r** means that λ appears r times in the equation above. Eigenvalue multiplicities are called **degeneracies** in Quantum Mechanics.

Definition. The set of Eigenvectors corresponding to λ is a linear space called an **Eigenspace**.

Theorem. If d is the dimension of the Eigenspace of λ and r is the multiplicity of λ then $1 \leq d \leq r$.

Theorem. [13.27] Let $\{\lambda_i\}$ be the set of Eigenvalues of an $n \times n$ matrix T , and let r_i be the multiplicity of λ_i . Then $\sum r_i = n$.

Corollary. A linear transformation T has at least 1 Eigenvector.

Theorem. [13.30] Suppose $\{e_k\}$ and $\{f_k\}$ are bases for a vector space V , and $f_k = T e_k$. Then

$$f_j = \begin{pmatrix} T^1_j \\ \vdots \\ T^n_j \end{pmatrix}.$$

That is, the components of f_j in basis $\{e_k\}$ are (T^1_j, \dots, T^n_j) .

Theorem. [13.31] If the Eigenspace dimension of every multiple Eigenvector equals its multiplicity, then there is a basis for V composed of Eigenvectors, and the matrix of T in this basis is

$$T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The next theorem states that even when the hypothesis of the above theorem is not satisfied, the matrix of T can at least be written in upper triangular form.

Theorem. (Note 13.12): **Jordan Canonical Form:** Let $\{\lambda_i\}$ be the set of Eigenvalues of an $n \times n$ matrix T , and let r_i be the multiplicity of λ_i . Then there is a basis for V such that the matrix of T in this basis is

$$T = \begin{pmatrix} \begin{array}{ccc|ccc|ccc} \lambda_1 & 1 & & & & & & & \dots & 0 \\ & \lambda_1 & 1 & & & & & & \ddots & \vdots \\ & & \ddots & \ddots & & & & & & \\ & & & \ddots & \ddots & & & & & 1 \\ & & & & \lambda_1 & & & & & \lambda_1 \\ & & & & & 0 & & & & \end{array} & \begin{array}{ccc} 0 & & \\ \lambda_2 & 1 & \\ & \ddots & \ddots \\ & & \ddots & 1 \\ & & & \lambda_{n-1} & 0 \end{array} & \begin{array}{ccc} & & \\ & & \\ \lambda_n & 1 & \\ & \lambda_n & \ddots \\ & & \ddots & 1 \\ & & & \lambda_n \end{array} \end{pmatrix}.$$