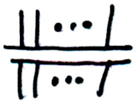
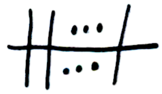
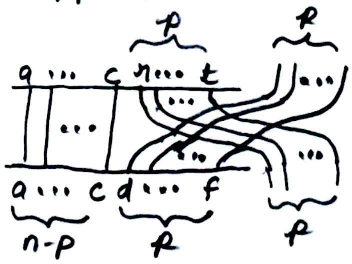
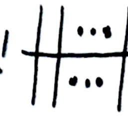
[13.22] Show why the ff diagram equalities in Fig 12.18 hold

(a) $\epsilon^{a\cdots c}\varepsilon_{f\cdots h} =$  $=$  $= n!\,\delta_{[f}^{a}\cdots\delta_{h]}^{c}$

(b) $\epsilon^{a\cdots c d\cdots f}\varepsilon_{a\cdots c\, n\cdots t} =$  $= (n-p)!$  $= (n-p)!\,p!\,\delta_{n}^{[d}\cdots\delta_{t}^{f]}$

Solution: To avoid confusion, we do not use Einstein summation convention

Th: $\epsilon^{1\cdots n}\varepsilon_{1\cdots n} = 1$

Pf: Let π be a permutation of $(1,\cdots,n)$. Observe that $\epsilon^{\pi(1)\cdots\pi(n)}\varepsilon_{\pi(1)\cdots\pi(n)}$ $= \epsilon^{1\cdots n}\varepsilon_{1\cdots n}$. For example $\epsilon^{213\cdots n}\varepsilon_{213\cdots n} = \epsilon^{123\cdots n}\varepsilon_{123\cdots n}$ because both ϵ and ε are antisymmetric. There are $n!$ permutations, π, of $(1,\cdots,n)$

$$n! = \sum_{\pi}\epsilon^{\pi(1)\cdots\pi(n)}\varepsilon_{\pi(1)\cdots\pi(n)} = n!\,\epsilon^{1\cdots n}\varepsilon_{1\cdots n} \Rightarrow \epsilon^{1\cdots n}\varepsilon_{1\cdots n} = 1 \qquad \square$$

Cor1. $\boxed{\epsilon^{a\cdots c}\varepsilon_{a\cdots c} = 1}$ for any permutation $(a,\cdots,c)$ of $(1,\cdots,n)$

Cor2. $\boxed{\epsilon^{a\cdots c}\varepsilon_{d\cdots f} = \mathrm{sign}(\pi_0) = \pm 1, \text{ where } \pi_0(d,\cdots,f) = (a,\cdots,c)}$

Pf: Let $\pi_0(d,\cdots,f) = (a,\cdots,c)$. $\mathrm{sign}(\pi_0) = \pm 1$ depending on whether π_0 is even or odd. So $\epsilon^{a\cdots c}\varepsilon_{d\cdots f} = \mathrm{sign}(\pi_0)\,\epsilon^{a\cdots c}{}_{\pi_0(d,\cdots,f)} = \mathrm{sign}(\pi_0)\,\epsilon^{a\cdots c}\varepsilon_{a\cdots c}$

$$= \mathrm{sign}(\pi_0) = \pm 1 \qquad \square$$

Proof of (a): Let π_0 be the permutation $(\pi_0(d),\cdots,\pi_0(f)) = (a,\cdots,c)$.

$LHS \overset{(\mathrm{Cor}\,2)}{=} \mathrm{sign}(\pi_0)$. The terms in the summation of RHS are zero except for $\delta_a^a\cdots\delta_c^c$. So,

$RHS = n!\,\frac{1}{n!}\sum_{\pi}\mathrm{sign}(\pi)\,\delta_{\pi(f)}^{a}\cdots\delta_{\pi(h)}^{c} = \mathrm{sign}(\pi_0)\,\delta_{\pi_0(h)}^{a}\cdots\delta_{\pi_0(h)}^{c} = \mathrm{sign}(\pi_0) = LHS \quad \square$

Note: The corolaries make ϵ and ε much less cryptic and easier to work with.

[13.22]

Proof of (b): Let π_0 be the permutation $(\pi_0(d), \dots, \pi_0(f)) = (r, \dots, t)$.

(1) $\epsilon^{a\dots c d \dots f} \varepsilon_{a \dots c r \dots t} = \text{sign}(\pi_0)\, \epsilon^{a \dots c\, \pi_0(d) \dots \pi_0(f)} \varepsilon_{a \dots c r \dots t}$

$= \text{sign}(\pi_0)\, \epsilon^{a \dots c r \dots t} \varepsilon_{a \dots c r \dots t} \overset{(\text{co}\,i)}{=} \text{sign}(\pi_0)$

Let $P = \{\pi : \pi \text{ is a permutation of } (a, \dots, c)\}$

$$\text{LHS} = \sum_{\pi \in P} \epsilon^{\pi(a) \dots \pi(c) d \dots f} \varepsilon_{\pi(a) \dots \pi(c) r \dots t}$$

$$= \sum_{\pi \in P} \epsilon^{a \dots c d \dots f} \varepsilon_{a \dots c r \dots t}$$ (same number of permutations for ϵ and ε)

$\overset{(1)}{=} (n-p)!\, \text{sign}(\pi_0)$ (P has $(n-p)!$ terms)

$$\text{RHS} = (n-p)!\, p! \left(\frac{1}{p!}\right) \sum_{\pi} \delta^{\pi(d)}_{r} \cdots \delta^{\pi(f)}_{t}$$

$= (n-p)!\, \text{sign}(\pi_0)\, \delta^{\pi_0(d)}_{r} \cdots \delta^{\pi_0(f)}_{t}$

$= (n-p)!\, \text{sign}(\pi_0)\, \delta^{r}_{r} \cdots \delta^{t}_{t}$

$= (n-p)!\, \text{sign}(\pi_0)$

$= \text{LHS}$ $\square$