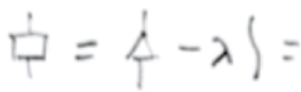
[13.26] Express the coefficients of the polynomial

$$\det(\mathbf{T} - \lambda \mathbf{I}) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda) = 0$$

in diagrammatic form. Work them out for n = 2 and n = 3.

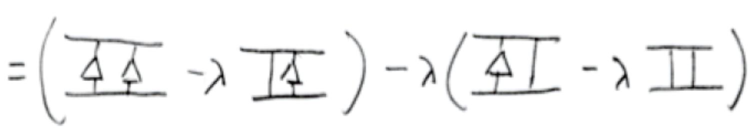
Proof. Beckmann produced a very nice proof that was then further simplified by an elegant enhancement provided by Dean. However, neither of them actually "worked out the equation for $n = 2$ or $n = 3$" as Penrose requested to generate the polynomial $(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$. When I tried, I believe I found an error in Beckmann's formula, and I have adjusted it here. I also tried to do the 2 cases using Dean's adjustment (now applied to my version) and it also didn't work. Their diagrams generated n-forms, not the real numbers needed for a determinant. My solution here is a continuation of the correct portion of Beckmann's solution.

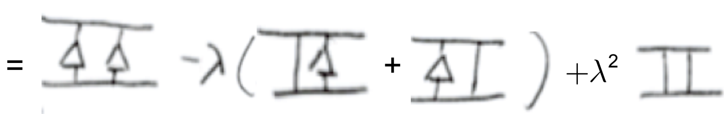
To begin, we recall that $\mathrm{Det}\,(\mathbf{T}) = \frac{1}{n!}$ 

Let  $= T^a_b - \lambda\delta^a_b$

<u>$n = 2$</u>:

2! Det ($\mathbf{T}$ – λ $\mathbf{I}$) = 





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* Great care must be taken when mimicking Penrose's diagrammatic rules. This step by Beckmann is correct but is not explicitly provided by Penrose. It must be proven. Beckmann's next step (top of next page) got him into trouble. I prove this current step in an appendix to this problem.

I will show shortly that the middle 2 terms cannot be combined into 2 

as Beckmann did.

Note that the above "expresses the coefficients of the polynomial in diagrammatic form." We now "work this out" to see that this indeed is the specified polynomial.

We use this fact. There is a basis such that the matrix of T is triangular, so that $\mathbf{T} = \begin{pmatrix} T^1{}_1 & T^1{}_2 \\ T^2{}_1 & T^2{}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}$. (This is the Jordan Canonical Form. Penrose mentions this in footnote 13.12.)

$$= \epsilon^{ab} T^c{}_a T^d{}_b \varepsilon_{cd} = \epsilon^{12} T^1{}_1 T^2{}_2 \varepsilon_{12} + \epsilon^{12} T^2{}_1 T^1{}_2 \varepsilon_{21} + \epsilon^{21} T^1{}_2 T^2{}_1 \varepsilon_{12} + \epsilon^{21} T^2{}_2 T^1{}_1 \varepsilon_{21}$$

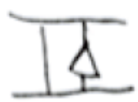
$$= \epsilon^{12} \lambda_1 \lambda_2 \varepsilon_{12} - \epsilon^{12} cb\, \varepsilon_{12} - \epsilon^{21} bc\, \varepsilon_{21} + \epsilon^{21} \lambda_2 \lambda_1 \varepsilon_{21}$$

$$= (\lambda_1 \lambda_2 - \cancelto{0}{bc})(\epsilon^{12}\varepsilon_{12} + \epsilon^{21}\varepsilon_{21}) = (\lambda_1 \lambda_2 \quad) \epsilon^{ab} \varepsilon_{ab}$$

$$= 2(\lambda_1 \lambda_2 \quad) \; [= 2 \det(T)] \checkmark$$

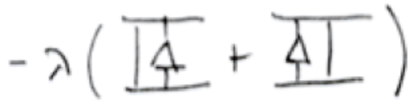
$$= \epsilon^{ab} \delta^c{}_a T^d{}_b \varepsilon_{cd} = \epsilon^{12} \delta^1{}_1 T^2{}_2 \varepsilon_{12} + \epsilon^{12} \cancelto{0}{\delta^2{}_1} T^1{}_2 \varepsilon_{21} + \epsilon^{21} \cancelto{0}{\delta^1{}_2} T^2{}_1 \varepsilon_{12} + \epsilon^{21} \delta^2{}_2 T^1{}_1 \varepsilon_{21}$$

$$= \epsilon^{12} \varepsilon_{12} \lambda_2 + \epsilon^{21} \varepsilon_{21} \lambda_1$$

This is an n-form (volume element), not a real number as required. So when Beckmann combined the middle terms into 2  he created a term that is not a real number.

Similar to above we find that  $= \epsilon^{12} \varepsilon_{12} \lambda_1 + \epsilon^{21} \varepsilon_{21} \lambda_2$

Thus,



$$= -\lambda\left(\epsilon^{12}\varepsilon_{12} + \epsilon^{21}\varepsilon_{21}\right)(\lambda_1 + \lambda_2)$$
$$= -\lambda\,\epsilon^{ab}\varepsilon_{ab}(\lambda_1 + \lambda_2)$$
$$= -2\lambda(\lambda_1 + \lambda_2)$$

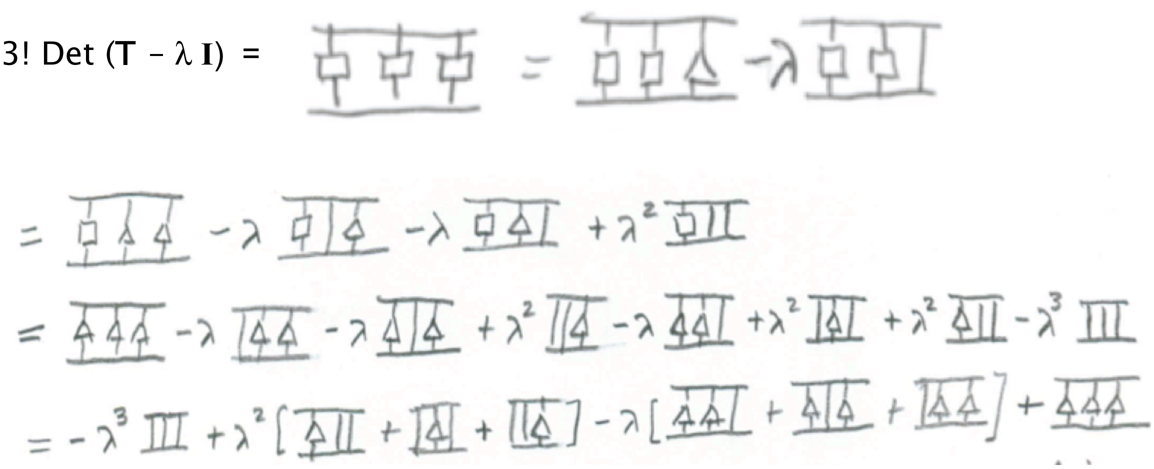
This is a real number.

The last symbol is  = 2! = 2. So

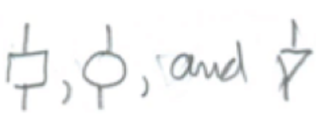
$$\det(\mathbf{T} - \lambda \mathbf{I}) = \frac{1}{2}\left[2(\lambda_1\lambda_2) - 2\lambda(\lambda_1 + \lambda_2) + 2\lambda^2\right] = \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1\lambda_2$$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \quad ✔$$

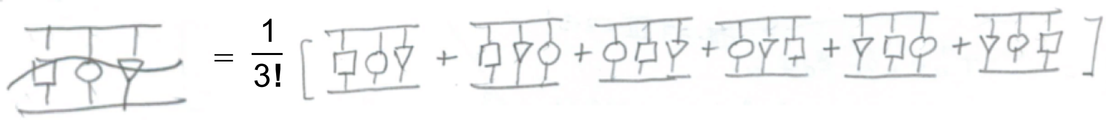
<u>$n = 3$</u>:

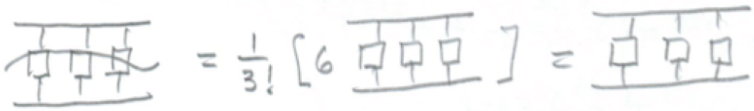
3! Det ($\mathbf{T}$ − λ $\mathbf{I}$) =

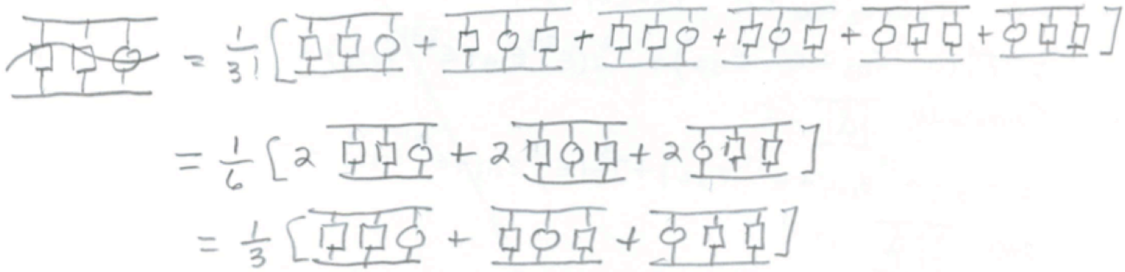


In order to write this as a single summation, we need a notation to combine the 3 factors of λ and the 3 factors of λ^2. There is likely a slicker way to do this but what I came up with is a new application of the symmetrization (wavy line) symbol.

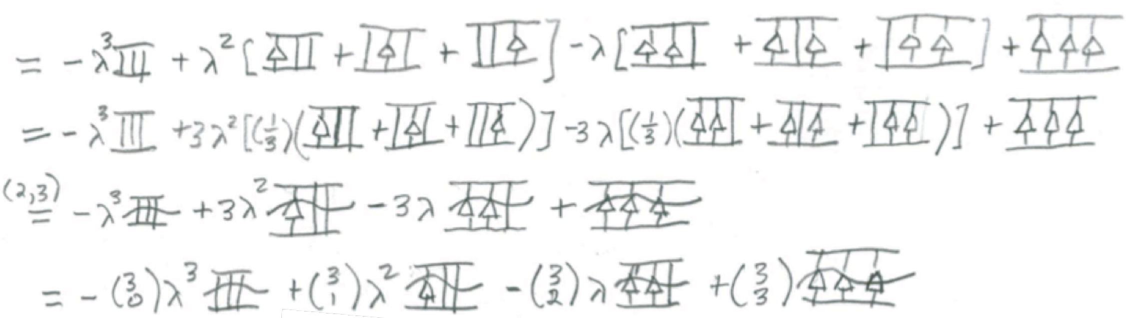
Let  be linear transformation in $\mathbb{R}^3$. Define

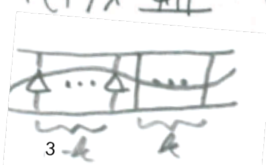
(1) 

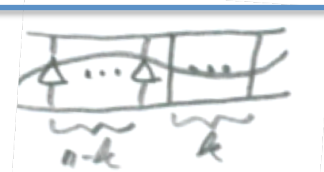
So (2)  and

(3) 

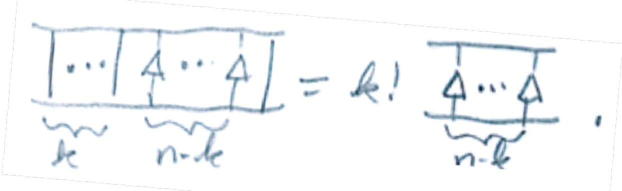
Thus, $3!\ \text{Det}\left(\mathbf{T}-\lambda\mathbf{I}\right)$



$$=\sum_{k=0}^{3}\binom{3}{k}\left(-\lambda\right)^{k}$$ 

In general, $\boxed{n!\ \text{Det}\left(\mathbf{T}-\lambda\mathbf{I}\right)=\sum_{k=0}^{n}\binom{n}{k}\left(-\lambda\right)^{k}}$ 

Note: Dean suggested that we can simplify this by using the relation

 I didn't use this because I cannot discern the meaning of  in $\mathbb{R}^n$. When I tried to understand this I

came up with an $(n-k)$-form rather than a real number. For example in $\mathbb{R}^2$,

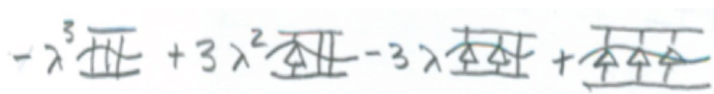
 $= \in^{ad} T^{n}{}_{a} \varepsilon_{nu} = \in^{12} T^1{}_1 \varepsilon_{12} + \in^{12} T^2{}_1 \varepsilon_{21} + \in^{21} T^1{}_2 \varepsilon_{12} + \in^{21} T^2{}_2 \varepsilon_{21}$

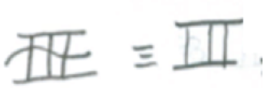
$= \in^{12} \varepsilon_{12} (T^1{}_1 - T^2{}_1) + \in^{21} \varepsilon_{21} (T^2{}_2 - T^1{}_2)$, not a real number.

Finally, we show that the formula for $n = 3$ generates the desired polynomial. We assume T is triangular:

$$T = \begin{pmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & b & c \\ 0 & \lambda_2 & f \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

$\mathrm{Det}\,(\mathbf{T}) = \lambda_1 \lambda_2 \lambda_3.$

$3!\ \mathrm{Det}(\mathbf{T} - \lambda \mathbf{I}) =$ 

 $= 3!$

$$= \in^{abc} T^d{}_a \delta^e_b \delta^f_c \varepsilon_{def}$$
$$= \in^{123} T^1{}_1 \delta^2_2 \delta^3_3 \varepsilon_{123} + \in^{132} T^1{}_1 \delta^3_3 \delta^2_2 \varepsilon_{132} + \in^{213} T^2{}_2 \delta^1_1 \delta^3_3 \varepsilon_{213} + \in^{231} T^2{}_2 \delta^3_3 \delta^1_1 \varepsilon_{231}$$
$$+ \in^{312} T^3{}_3 \delta^1_1 \delta^2_2 \varepsilon_{312} + \in^{321} T^3{}_3 \delta^2_2 \delta^1_1 \varepsilon_{321}$$
$$= T^1{}_1 \left(\in^{123} \varepsilon_{123} + \in^{132} \varepsilon_{132} \right) + T^2{}_2 \left(\in^{213} \varepsilon_{213} + \in^{231} \varepsilon_{231} \right) + T^3{}_3 \left(\in^{312} \varepsilon_{312} + \in^{321} \varepsilon_{321} \right)$$

Similarly

 $= \in^{abc}\ \delta_a^d T^e{}_b \delta_c^f \varepsilon_{def}$

$$= T^2{}_2\left(\in^{123}\ \varepsilon_{123} + \in^{132}\ \varepsilon_{132}\right) + T^3{}_3\left(\in^{213}\ \varepsilon_{213} + \in^{231}\ \varepsilon_{231}\right) + T^1{}_1\left(\in^{312}\ \varepsilon_{312} + \in^{321}\ \varepsilon_{321}\right)$$

and

 $= \in^{abc}\ \delta_a^d \delta_b^e T^f{}_c \varepsilon_{def}$

$$= T^3{}_3\left(\in^{123}\ \varepsilon_{123} + \in^{132}\ \varepsilon_{132}\right) + T^1{}_1\left(\in^{213}\ \varepsilon_{213} + \in^{231}\ \varepsilon_{231}\right) + T^2{}_2\left(\in^{312}\ \varepsilon_{312} + \in^{321}\ \varepsilon_{321}\right)$$

Thus 

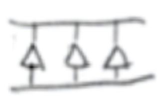
$$= \left(T^1{}_1 + T^2{}_2 + T^3{}_3\right)\left[\left(\in^{123}\ \varepsilon_{123} + \in^{132}\ \varepsilon_{132}\right) + \left(\in^{213}\ \varepsilon_{213} + \in^{231}\ \varepsilon_{231}\right) + \left(\in^{312}\ \varepsilon_{312} + \in^{321}\ \varepsilon_{321}\right)\right]$$
$$= \left(\lambda_1 + \lambda_2 + \lambda_3\right)\left[\in^{abc}\ \varepsilon_{abc}\right] = 3!\left(\lambda_1 + \lambda_2 + \lambda_3\right)$$

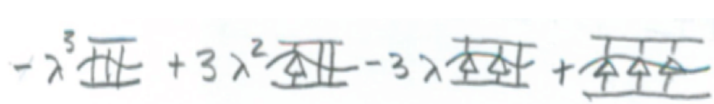
Next, 

$$= \in^{abc}\ T^d{}_a T^e{}_b \delta_c^f \varepsilon_{def}$$
$$= \in^{123}\ T^1{}_1 T^2{}_2 \delta_3^3 \varepsilon_{123} + \in^{132}\ T^1{}_1 T^3{}_3 \delta_2^2 \varepsilon_{132} + \in^{213}\ T^2{}_2 T^1{}_1 \delta_3^3 \varepsilon_{213} + \in^{231}\ T^2{}_2 T^3{}_3 \delta_1^1 \varepsilon_{231}$$
$$+ \in^{312}\ T^3{}_3 T^1{}_1 \delta_2^2 \varepsilon_{312} + \in^{321}\ T^3{}_3 T^2{}_2 \delta_1^1 \varepsilon_{321}$$
$$= \lambda_1\lambda_2\left(\in^{123}\ \varepsilon_{123} + \in^{213}\ \varepsilon_{213}\right) + \lambda_2\lambda_3\left(\in^{231}\ \varepsilon_{231} + \in^{321}\ \varepsilon_{321}\right) + \lambda_1\lambda_3\left(\in^{132}\ \varepsilon_{132} + \in^{312}\ \varepsilon_{312}\right)$$

Eventually we get

 $= \left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)\left[\in^{abc}\ \varepsilon_{abc}\right] = 3!\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)$

Finally,  =  $= 3!\ \mathrm{Det}(\mathbf{T}) = 3!\ \lambda_1\lambda_2\lambda_3$

Thus $3!\ \mathrm{Det}(\mathbf{T} - \lambda\mathbf{I}) =$ 

$= 3!\ \left[-\lambda^3 + 3\lambda^2\left(\lambda_1 + \lambda_2 + \lambda_3\right) - 3\lambda\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) + \lambda_1\lambda_2\lambda_3\right]$, and so

$\mathrm{Det}(\mathbf{T} - \lambda\mathbf{I}) = \left(\lambda_1 - \lambda\right)\left(\lambda_2 - \lambda\right)\left(\lambda_3 - \lambda\right)$ ✔

Appendix.

Theorem. Let R =  = S + T in $\mathbb{R}^2$. Then

Proof.

 $= \epsilon^{ab} R^{c}{}_{a} R^{d}{}_{b}\, \varepsilon_{cd} = \epsilon^{ab} R^{c}{}_{a} (S^{d}{}_{b} + T^{d}{}_{b})\, \varepsilon_{cd}$

$= \epsilon^{ab} R^{c}{}_{a} S^{d}{}_{b}\, \varepsilon_{cd} + \epsilon^{ab} R^{c}{}_{a} T^{d}{}_{b}\, \varepsilon_{cd}$

=