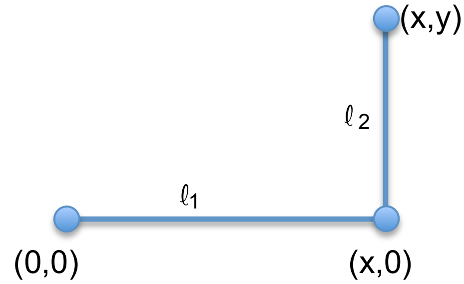
[12.13] Poincare's Lemma for p = 1 in $\mathbb{R}^2$. Let $\beta = A(x,y)\,dx + B(x,y)\,dy$ be a 1-form such that $d\beta = 0$. Show that there is a scalar field $\Phi : \mathbb{R}^2 \to \mathbb{R}$ such that locally $\beta = d\Phi$.



From problem [12.11],

$$0 = d\beta = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy$$

$$\Rightarrow (i) \quad \frac{\partial B(x,y)}{\partial x} = \frac{\partial A(x,y)}{\partial y}$$

Without loss of generality, let's choose our local point to be (0,0), and assume the point (x,y) is in an open connected neighborhood of (0,0) so that we can joint them with the lines ℓ_1 and ℓ_2 as shown.

Define $\Phi(x,y) \equiv \int_0^x A(t,0)\,dt + \int_0^y B(x,t)\,dt.$ That is, we integrate from (0,0) to (x,y) using A along ℓ_1 and B along ℓ_2.

Restricted to ℓ_2, $B_x(y) \equiv B(x,y)$ is a function of just y. Let $b(y)$ be the antiderivative of $B_x(y).$ That is, $\int_0^y B_x(t)\,dt = b(y) - b(0).$ So,

$$\begin{aligned}(ii) \quad \frac{\partial \Phi}{\partial y} &= \frac{\partial}{\partial y}\int_0^x A(t,0)\,dt + \frac{\partial}{\partial y}\int_0^y B(x,t)\,dt = \frac{\partial}{\partial y}\int_0^y B_x(t)\,dt = \frac{\partial}{\partial y}\big[b(y) - b(0)\big] = B_x(y) \\ &= B(x,y).\end{aligned}$$

Restricted to ℓ_1, $A_0(x) \equiv A(x,0)$ is a function of just x. Let $a(x)$ be the antiderivative of $A_0(x).$ That is, $\int_0^x A(t,0)\,dt = \int_0^x A_0(t)\,dt = a(x) - a(0).$ Similarly, restricted to ℓ_2, $A_x(t) \equiv A(x,t)$ is a function of just t. So,

$$\begin{aligned}(iii) \quad \frac{\partial \Phi}{\partial x} &= \frac{\partial}{\partial x}\int_0^x A(t,0)\,dt + \frac{\partial}{\partial x}\int_0^y B(x,t)\,dt = \frac{\partial}{\partial x}\big[a(x) - a(0)\big] + \frac{\partial}{\partial x}\int_0^y B(x,t)\,dt \\ &\underset{\text{of Calculus}}{\overset{\text{Fund Th}}{=}} A_0(x) + \int_0^y \frac{\partial}{\partial x} B(x,t)\,dt \overset{(i)}{=} A(x,0) + \int_0^y \frac{\partial}{\partial t} A(x,t)\,dt = A(x,0) + \int_0^y \frac{\partial}{\partial t} A_x(t)\,dt \\ &\underset{\text{of Calculus}}{\overset{\text{Fund Th}}{=}} A(x,0) + \big[A_x(y) - A_x(0)\big] = A(x,0) + \big[A(x,y) - A(x,0)\big] = A(x,y).\end{aligned}$$

Finally, we have

$$\begin{aligned}d\Phi &= \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy \overset{\text{(ii \& iii)}}{=} A(x,y)dx + B(x,y)dy \\ &= \beta.\end{aligned}$$