[12.9] Given a curve $\gamma$ in a 1-manifold $\mathcal{M}$, a 1-form $\alpha$ on $\mathcal{M}$, coordinate patches ($x$) and ($X$) in $\mathcal{M}$, and transfer function $x = x(X)$ and its inverse $X = X(x)$. Let $\alpha = f(x)\, dx$ in ($x$)-coordinates. Show

$$\int_\gamma \alpha = \int_a^b f(x)\,dx = \int_A^B F(X)\,dX,$$ and find $A$, $B$, $F(X)$, and $dX$.

First, a "curve" $\gamma$ in $\mathcal{M}$ is simply a closed interval, say [$a$,$b$]. This shows that

$$\int_\gamma \alpha = \int_a^b f(x)\,dx.$$

Define $\boxed{A = X(a) \text{ and } B = X(b)}$. So $X = A$ when $x = a$ and $X = B$ when $x = b$.

Also, $x = x(X) \quad \Rightarrow \quad dx = d[x(X)]$. So $F(X)\,dX = \alpha = f(x)\,dx = f[x(X)]\, d[x(X)]$.
Thus, $\boxed{F(X) = f[x(X)] \text{ and } dX = d[x(X)].}$

Finally,

$$\int_\gamma \alpha = \int_{x=a}^{x=b} f(x)\,dx = \int_{X=A}^{X=B} F(X)\,dX.$$

Postscript 1. Dimbulb in his approach to this problem was concerned about the definition of $A$ and $B$ in the case that the relationship between $x$ and $X$ were not 1-1. But, that cannot be the case because whenever one has inverse functions, they are 1-1 (and onto). So there should be no concern about simply defining $A = X(a)$ and $B = X(b)$.

Postscript 2. This problem is insightful in many respects but in other ways its simplicity hides key insights. Let me propose and solve a simple change of variables example for a 1-form in a 2-manifold (choose $\mathcal{M} = \Re^2$) to gain a larger perspective on this process. This problem has 3 givens and a bunch of unknowns.

Given:

(1) A curve $\gamma$ (actually, a straight line segment) in $\Re^2$ defined by:

$T : [0, 1] \rightarrow \Re^2$: $T(t) = (x(t), y(t)) = (6t + 2, 3t + 1)$

Thus,

$\gamma = T([0, 1])$,

$x(t) = 6t + 2 = 2(3t + 1)$, and

$y(t) = 3t + 1$.

(2) A transfer function: $\left\{ \begin{array}{l} X = X(x,y) = 2x \\ Y = Y(x,y) = x + y \end{array} \right\}$

(3) A 1-form $\alpha = f(x,y)\,dx + g(x,y)\,dy \equiv x^2\,dx + xy\,dy.$

So $f(x,y) = x^2$ and $g(x, y) = xy$.

Find the following:

(a) Compute $\int_\gamma \alpha = \int_\gamma f\,dx + g\,dy$.

(b) Find $F(X,Y)\,dX$ and $G(X,Y)\,dY$

(c) Show directly that $\int_\gamma \alpha = \int_\gamma f\,dx + g\,dy = \int_\gamma F\,dX + G\,dY.$

Solution:

(a) $$\int_\gamma \alpha = \int_\gamma f\,dx + g\,dy = \int_\gamma \left[ f(x(t),y(t))\frac{dx(t)}{dt} + g(x(t),y(t))\frac{dy(t)}{dt} \right] dt$$

$$= \int_\gamma \left[ x(t)^2\,(6) + x(t)\,y(t))\,(3) \right] dt$$

$$= \int_\gamma \left[ [2(3t+1)]^2\,(6) + [2(3t+1)](3t+1)\,(3) \right] dt$$

$$= \int_{t=0}^{1} \left[ 24\,(3t+1)^2 + 6\,(3t+1)^2 \right] dt = \int_{t=0}^{1} 30\,(3t+1)^2\,dt = 210$$

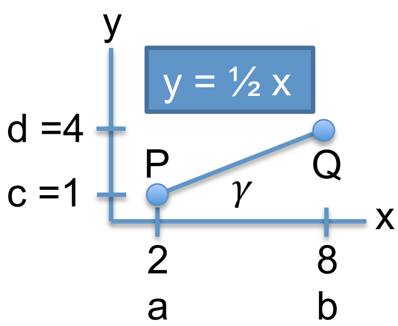
(b)

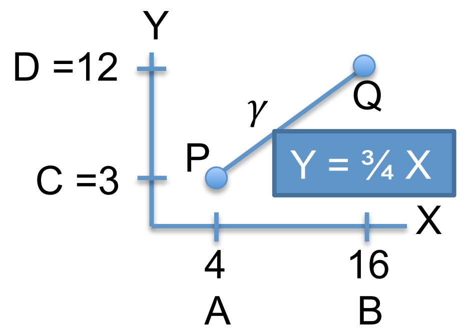
Let P and Q represent the endpoints of $\gamma$ in coordinate-free notation.

P = $(a, c)$ = $(x(0), y(0))$ = (2,1) and Q = $(b, d)$ = $(x(1), y(1))$ = (8,4) in (x,y)-coordinates.

P = $(A, C)$ = $(2a, a + c)$ = (4, 3) and
Q = $(B, D)$ = $(2b, b + d)$ = (16, 12) in (X,Y)-coordinates.

The $(x,y)$- and $(X,Y)$-plots are below, along with their equations.




Now,

$$
\begin{aligned}
F(X,Y)\,dX + G(X,Y)\,dY = \alpha \\
&= f(x,y)\,dx + g(x,y)\,dy = x^2\,dx + xy\,dy \\
&= x(X,Y)^2\,d[x(X,Y)] + x(X,Y)\,y(X,Y)\,d[y(X,Y)] \\
&= \tfrac{1}{4}X^2\left(\tfrac{1}{2}dX\right) + \left(\tfrac{1}{2}X\right)\left(-\tfrac{1}{2}X + Y\right)\left(-\tfrac{1}{2}dX + dY\right) \\
&= \tfrac{1}{8}X^2\,dX + \tfrac{1}{2}X\left[\left(\tfrac{1}{4}X - \tfrac{1}{2}Y\right)dX + \left(-\tfrac{1}{2}X + Y\right)dY\right] \\
&= \tfrac{1}{8}X^2\,dX + \left[\left(\tfrac{1}{8}X^2 - \tfrac{1}{4}XY\right)dX + \left(-\tfrac{1}{4}X^2 + \tfrac{1}{2}XY\right)dY\right] \\
&= \left(\tfrac{1}{4}X^2 - \tfrac{1}{4}XY\right)dX + \left(-\tfrac{1}{4}X^2 + \tfrac{1}{2}XY\right)dY.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
F(X,Y)\,dX &= \tfrac{1}{4}\left(X^2 - \quad XY\right)dX, \text{ and} \\
G(X,Y)\,dY &= -\tfrac{1}{4}\left(X^2 - 2XY\right)dY.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\int_\gamma F\,dX + G\,dY &= \int_{X=A}^{B} F(X,Y)\,dX + \int_{Y=C}^{D} G(X,Y)\,dY \\
&= \tfrac{1}{4}\int_{X=4}^{16}\left(X^2 - XY\right)dX - \tfrac{1}{4}\int_{Y=3}^{12}\left(X^2 - 2XY\right)dY.
\end{aligned}
$$

From the plot, $Y = \frac{3}{4}X$ and $X = \frac{4}{3}Y$. Substituting yields

$$
\begin{aligned}
\int_\gamma F\,dX + G\,dY &= \tfrac{1}{4}\int_{X=4}^{16}\left(X^2 - \tfrac{3}{4}X^2\right)dX - \tfrac{1}{4}\int_{Y=3}^{12}\left(\tfrac{16}{9}Y^2 - 2(\tfrac{4}{3})Y^2\right)dY \\
&= \tfrac{1}{16}\int_{X=4}^{16} X^2\,dX + \tfrac{2}{9}\int_{Y=3}^{12} Y^2\,dY = \tfrac{1}{48}X^3\Big|_4^{16} + \tfrac{2}{27}Y^3\Big|_3^{12} \\
&= 84 + 126 = 210. \qquad \surd
\end{aligned}
$$