[13.32] Show that every finite group **G** has a faithful representation in GL(n) where n is the order of **G.**

Solution

Part A. Show T is a representation

This proof of this part is just an elaboration of Robin's method, which is very slick.

Let **G** = {g_1, …, g_n}. A representation is a group homomorphism $T:\ \boldsymbol{G} \rightarrow \mathrm{GL}(n)$, a function that preserves the group structure:

For all $g_i, g_j \in \boldsymbol{G},\ T(g_i)T(g_j) = T(g_i\, g_j)$.

Thus, T(g_i) is an n x n matrix. I use Penrose's hint to label the rows and columns of matrix $T(g_i)$ to indicate that the matrix takes g_s to g_r:

$$T(g_i) = \begin{array}{c|cccccc|} & 1 & 2 & \cdots & s & \cdots & n \\ \hline 1 & T(g_i)^1{}_1 & T(g_i)^1{}_2 & \cdots & T(g_i)^1{}_s & \cdots & T(g_i)^1{}_n \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ r & T(g_i)^r{}_1 & T(g_i)^r{}_2 & \cdots & T(g_i)^r{}_s & \cdots & T(g_i)^r{}_n \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ n & T(g_i)^n{}_1 & T(g_i)^n{}_2 & \cdots & T(g_i)^n{}_s & \cdots & T(g_i)^n{}_n \end{array}.$$

Matrix T(g_j) can be written

$$T(g_j) = \begin{array}{c|cccccc|} & 1 & 2 & \cdots & t & \cdots & n \\ \hline 1 & T(g_j)^1{}_1 & T(g_j)^1{}_2 & \cdots & T(g_j)^1{}_t & \cdots & T(g_j)^1{}_n \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ s & T(g_j)^s{}_1 & T(g_j)^s{}_2 & \cdots & T(g_j)^s{}_t & \cdots & T(g_j)^s{}_n \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ n & T(g_j)^n{}_1 & T(g_j)^n{}_2 & \cdots & T(g_j)^n{}_t & \cdots & T(g_j)^n{}_n \end{array},$$

matrix $T(g_i\, g_j)$ can be written

$$\mathrm{T}(g_i\, g_j) = \begin{bmatrix} & \vdots & \\ \cdots & \mathrm{T}(g_i\, g_j)^r_t & \cdots \\ & \vdots & \end{bmatrix},$$

and the product matrix $T(g_i)T(g_j)$ is

$$\mathrm{T}(g_i)\mathrm{T}(g_j)=\begin{bmatrix} & \vdots & \\ \cdots & \sum_{s=1}^{n}\mathrm{T}(g_i)_s^r\,\mathrm{T}(g_j)_t^s & \cdots \\ & \vdots & \end{bmatrix}.$$

A strategy to define T such that $T(g_i g_j)= T(g_i)T(g_j)$ is to put as many zeros as possible into the matrix so that the calculation becomes simpler. To that end, define

$$T(g_i)_s^r \equiv \begin{cases} 1 & \text{if } g_r = g_i\, g_s \\ 0 & \text{Otherwise} \end{cases}.$$

This matrix has precisely one 1 in every row and every column. The element $\sum_{s=1}^{n}\mathrm{T}(g_i)_s^r\,\mathrm{T}(g_j)_t^s$ of the matrix $T(g_i)T(g_j)$ then becomes

$$\sum_{s=1}^{n}T(g_i)_s^r\,T(g_j)_t^s \equiv \begin{cases} 1 & \begin{aligned} &\text{if } T(g_i)_s^r = 1 \text{ and } T(g_j)_t^s = 1 \text{ for some } s \\ &\Leftrightarrow \text{if } g_r = g_i\, g_s \text{ and } g_s = g_j\, g_t \text{ for some } s \\ &\Leftrightarrow \text{if } g_i^{-1} g_r = g_s = g_j\, g_t \text{ for some } s \\ &\Leftrightarrow \text{if } (g_i\, g_j)\, g_t = g_r \end{aligned} \\ 0 & \text{Otherwise} \end{cases}.$$

$$\Leftrightarrow T(g_i\, g_j)_t^r$$

That is, $T(g_i)\,T(g_j) = T(g_i\, g_j)$. ✔

Part B Show T is faithful

T is faithful if it is one-to-one; i.e., if $T(g_i) = T(g_j) \Rightarrow g_i = g_j$. So, suppose

$$\begin{aligned} T(g_i) = T(g_j) &\Leftrightarrow \forall\, a, b \;\; T(g_i)^a{}_b = T(g_j)^a{}_b \\ &\Rightarrow \forall\, a,b \;\; T(g_i)^a{}_b = 1 \;\; \text{if and only if} \;\; T(g_j)^a{}_b = 1 \\ &\Leftrightarrow \forall\, a,b \;\; g_i\, g_b = g_a = g_j\, g_b \\ &\Rightarrow g_i = g_j. \end{aligned}$$

✔