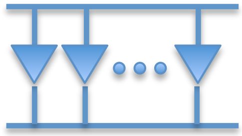
[13.21] Given Det T = $\frac{1}{n!}$  , show that

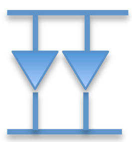
(a) $\begin{vmatrix} a & b \\ c & c \end{vmatrix} = ad - bc$,

(b) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = adj - afh + bfg - bdj + cdh - ceg$, and

(c) Det T = $\sum_{\pi \in \mathcal{P}_{12\cdots n}} \mathrm{Sign}(\pi)\, \mathrm{T}^1{}_{\pi(1)} \cdots \mathrm{T}^n{}_{\pi(n)}$ where $\mathcal{P}_{1\ldots n}$ is the set of permutations of (1, …, n).

Proof. Penrose did not ask for part (c). But, my proof is short and (c) is the standard definition of determinant given in calculus.

This proof uses the fact that if $\boldsymbol{\varepsilon}\bullet \in = \varepsilon_{a\ldots c} \in^{a\cdots c}$ is normalized to n!, then each of the n! terms satisfies $\varepsilon_{a\ldots c} \in^{a\cdots c} = 1$.

(a) $\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2!}$  = $\frac{1}{2!}\varepsilon_{rs} \in^{tu} \mathrm{T}^r{}_t \mathrm{T}^s{}_u$

$$= \frac{1}{2}\begin{pmatrix} \varepsilon_{12} \in^{12} \mathrm{T}^1{}_1 \mathrm{T}^2{}_2 + \varepsilon_{12} \in^{21} \mathrm{T}^1{}_2 \mathrm{T}^2{}_1 \\ + \varepsilon_{21} \in^{21} \mathrm{T}^2{}_2 \mathrm{T}^1{}_1 + \varepsilon_{21} \in^{12} \mathrm{T}^2{}_1 \mathrm{T}^1{}_2 \end{pmatrix}$$

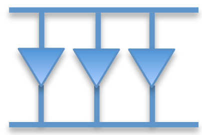
$$= \frac{1}{2}\begin{pmatrix} \cancel{\varepsilon_{12} \in^{12}} \mathrm{T}^1{}_1 \mathrm{T}^2{}_2 - \cancel{\varepsilon_{12} \in^{12}} \mathrm{T}^1{}_2 \mathrm{T}^2{}_1 \\ + \cancel{\varepsilon_{21} \in^{21}} \mathrm{T}^2{}_2 \mathrm{T}^1{}_1 - \cancel{\varepsilon_{12} \in^{12}} \mathrm{T}^2{}_1 \mathrm{T}^1{}_2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \mathrm{T}^1{}_1 \mathrm{T}^2{}_2 - \mathrm{T}^1{}_2 \mathrm{T}^2{}_1 \\ + \mathrm{T}^1{}_1 \mathrm{T}^2{}_2 - \mathrm{T}^1{}_2 \mathrm{T}^2{}_1 \end{pmatrix}$$

$$= \frac{1}{2}\left(2\mathrm{T}^1{}_1 \mathrm{T}^2{}_2 - 2\mathrm{T}^1{}_2 \mathrm{T}^2{}_1\right)$$

$$= \mathrm{T}^1{}_1 \mathrm{T}^2{}_2 - \mathrm{T}^1{}_2 \mathrm{T}^2{}_1$$

= $ad - bc$ ✔

(b) $\mathrm{Det}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = \frac{1}{3!}$  $= \frac{1}{6}\varepsilon_{rst} \in^{uvw} \mathrm{T}^r{}_u \mathrm{T}^s{}_v T^t{}_w$

$$`= \frac{1}{6}\begin{bmatrix} \varepsilon_{123}\in^{123} \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 + \varepsilon_{123}\in^{132} \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots + \varepsilon_{123}\in^{321} \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ + \varepsilon_{132}\in^{132} \mathrm{T}^1{}_1\mathrm{T}^3{}_3\mathrm{T}^2{}_2 + \varepsilon_{132}\in^{123} \mathrm{T}^1{}_1\mathrm{T}^3{}_2\mathrm{T}^2{}_3 + \cdots + \varepsilon_{132}\in^{312} \mathrm{T}^1{}_3\mathrm{T}^3{}_1\mathrm{T}^2{}_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + \varepsilon_{321}\in^{321} \mathrm{T}^3{}_3\mathrm{T}^2{}_2\mathrm{T}^1{}_1 + \varepsilon_{321}\in^{231} \mathrm{T}^3{}_2\mathrm{T}^2{}_3\mathrm{T}^1{}_1 + \cdots + \varepsilon_{321}\in^{123} \mathrm{T}^3{}_1\mathrm{T}^2{}_2\mathrm{T}^1{}_3 \end{bmatrix} \text{(36 terms)}$$

$$= \frac{1}{6}\begin{bmatrix} \cancel{\varepsilon_{123}\in^{123}} \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \cancel{\varepsilon_{123}\in^{123}} \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \cancel{\varepsilon_{123}\in^{123}} \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ + \varepsilon_{132}\in^{132} \mathrm{T}^1{}_1\mathrm{T}^3{}_3\mathrm{T}^2{}_2 - \cancel{\varepsilon_{132}\in^{132}} \mathrm{T}^1{}_1\mathrm{T}^3{}_2\mathrm{T}^2{}_3 + \cdots - \cancel{\varepsilon_{132}\in^{132}} \mathrm{T}^1{}_3\mathrm{T}^3{}_1\mathrm{T}^2{}_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + \cancel{\varepsilon_{321}\in^{321}} \mathrm{T}^3{}_3\mathrm{T}^2{}_2\mathrm{T}^1{}_1 - \cancel{\varepsilon_{321}\in^{321}} \mathrm{T}^3{}_2\mathrm{T}^2{}_3\mathrm{T}^1{}_1 + \cdots - \cancel{\varepsilon_{321}\in^{321}} \mathrm{T}^3{}_1\mathrm{T}^2{}_2\mathrm{T}^1{}_3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ + \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + \mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - \mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - \mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1 \end{bmatrix}$$

$$= \frac{1}{6}\left[6\mathrm{T}^1{}_1\mathrm{T}^2{}_2\mathrm{T}^3{}_3 - 6\mathrm{T}^1{}_1\mathrm{T}^2{}_3\mathrm{T}^3{}_2 + \cdots - 6\mathrm{T}^1{}_3\mathrm{T}^2{}_2\mathrm{T}^3{}_1\right]$$

$$= aej - afh + bfg - bdj + cdh - ceg \quad ✔$$

(c) We proceed by generalizing the steps used in part (b).

$$\text{Det T} = \frac{1}{n!}\varepsilon_{r\cdots s}\in^{t\cdots u}\mathrm{T}^{r}{}_{t}\cdots\mathrm{T}^{s}{}_{u}$$

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\varepsilon_{\pi^*(1)\cdots\pi^*(n)}\in^{\pi(1)\cdots\pi(n)}\mathrm{T}^{\pi^*(1)}{}_{\pi(1)}\cdots\mathrm{T}^{\pi^*(n)}{}_{\pi(n)}$$

(Replace Einstein notation.)

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\varepsilon_{\pi^*(1)\cdots\pi^*(n)}\in^{\pi(\pi^*(1))\cdots\pi(\pi^*(n))}\mathrm{T}^{\pi^*(1)}{}_{\pi(\pi^*(1))}\cdots\mathrm{T}^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Replace π by $\pi \circ \pi^*$ in $\in$ and T. The double sum over π and π^* is unchanged, in both expressions stepping over all permutations of (1, …, n), and the exponents of $\in$ continue to match the subscripts of T. This expression generalizes the first block of 36 items in part (b).

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\mathbf{Sign}(\boldsymbol{\pi})\,\cancel{\varepsilon_{\pi^*(1)\cdots\pi^*(n)}\in^{\pi^*(1)\cdots\pi^*(n)}}\,\mathrm{T}^{\pi^*(1)}{}_{\pi(\pi^*(1))}\cdots\mathrm{T}^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Re-order superscripts of $\in$ by applying an inverse π permutation. This corresponds to the 2nd block of 36 items.)

$$= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\text{Sign}(\pi)\sum_{\pi^*\in\mathcal{P}_{1\cdots n}}\mathrm{T}^{\mathbf{1}}{}_{\pi(\mathbf{1})}\cdots\mathrm{T}^{\boldsymbol{n}}{}_{\pi(\boldsymbol{n})}$$

(This is just a simpler way to label the subscripts and superscripts of T. For example, if $\pi^*(3) = 1$ then $\mathrm{T}^{\pi^*(3))}{}_{\pi(\pi^*(3))} = \mathrm{T}^{1}{}_{\pi(1)}$. This is the 3rd block of 36 items.)

$$= \frac{n!}{n!}\sum_{\pi\in\mathcal{P}_{1\cdots n}}\text{Sign}(\pi)\mathrm{T}^{1}{}_{\pi(1)}\cdots\mathrm{T}^{n}{}_{\pi(n)}$$

$$= \sum_{\pi\in\mathcal{P}_{1\cdots n}}\text{Sign}(\pi)\mathrm{T}^{1}{}_{\pi(1)}\cdots\mathrm{T}^{n}{}_{\pi(n)} \quad ✔$$