[13.7] SO(3) is the group of rotations of the unit sphere in 3-space. O(3) extends SO(3) by including reflections. **(A)** Show that SO(3) is a normal subgroup of O(3) and **(B)** show that it is the only proper normal subgroup.

Note. (B) is actually not true. There is one other proper normal subgroup of O(3). If 1 is the identity element (null rotation) of SO(3) and R is the reflection operator then $\mathcal{I}$ ={ 1, R } is a normal subgroup of O(3). This is because if g is a (reflective or non-reflective) rotation, then $g^{-1}\,1\,g = 1$ and $g^{-1}\,\mathrm{R}\,g = \mathrm{R}$ (see Lemma 3). So we revise (B): show that O(3) has only two proper normal subgroups.

Note: In this proof we adopt the convention that $f\,g$ represents rotating by f followed by g. So $f\,\mathrm{R}$ means to rotate and then reflect while $\mathrm{R}\,f$ means to reflect then rotate.

Proof: Penrose gives the hint: "What are the only sets in O(3) that are rotation invariant?". The answer is simple. In Theorem 2 we show there are only 2 such sets: SO(3) and T. We begin with some preliminaries.

Definitions:

1. Let **S** be the unit sphere of $\mathbb{R}^3$
2. Let **R** be the reflection operation on S
3. Let **T** = R[SO(3)] = {Rg: $g \in$ SO(3)} be the coset of reflective rotations in O(3)
   a. SO(3) and T are disjoint, and O(3) = SO(3) $\cup$ T
4. Let **1** be the identity of O(3), the null rotation

R is defined as an operation that reverses xyz orientation. It can be a reflection through the xy-plane, the yz-plane, or the xz-plane, or (preferred) a reflection through the origin, which is the composition of reflecting first through the xy-plane, then the yz-plane, and then the xz-plane. Also, $\mathrm{R}^{-1} = \mathrm{R}$. If P is a point of S then PR = -P.

While SO(3) is a group, T is not. (It is only a coset.) For example, $1 \notin$T. Also, if t_1 and t_2 belong to T, their composition $t_1\,t_2 \notin$ T. Rather, $t_1\,t_2 \in$ SO(3). This is because R is applied twice in the expression $t_1\,t_2$. In fact, any expression with an even number of reflections belongs to SO(3), and it belongs to T if the number of reflections is odd.

If $t \in$ T, there are elements $s_1, s_2 \in$ SO(3) such that $t = \mathrm{R}\,s_1$ and $t = s_2\,\mathrm{R}$. The former is true by definition of T. The latter is seen to be true by setting $s_2 = \mathrm{R}\,s_1\,\mathrm{R}$.

We need the following theorem to answer Penrose's invariance question.

Theorem 1.

(a) Let s_1, $s_2 \in$ SO(3). Then $\exists\ s_3 \in$ SO(3) such that $s_2 = s_3\ s_1$.

(b) Let t_1, $t_2 \in$ T. Then $\exists\ s \in$ SO(3) such that $t_2 = s\ t_1$.

Proof: (a) $s_3 = s_2\ s_1^{-1}$. (b) $s = t_2\ t_1^{-1}$. $s \in$ SO(3) because this expression has 2 reflections, an even number. ■

Theorem 2 (Answer to Penrose's question): SO(3) and T are the only proper subsets of O(3) that are rotation invariant.

Proof: SO(3) is rotation-invariant because applying a rotation to any rotation in SO(3) yields another rotation, an element of SO(3). SO(3) has no proper subset A that is rotation-invariant because, by Theorem 1a, given any $s_1 \in$ A and $s_3 \notin$ A, one can find a rotation s_2 such that $s_1\ s_2 = s_3$; i.e., s_1 is rotated out of A.

Similarly, T is rotation invariant because applying a rotation to any element of T remains in T because only 1 reflection has occurred. Like SO(3), T cannot have a proper subset A that is rotation invariant because, by Theorem 1b, from any $t_1 \in$ A one can obtain any $t_2 \notin$ A by applying a rotation. ■

Part A

Theorem A: SO(3) is a normal subgroup of O(3)

Proof. First, SO(3) is clearly a group because it contains the identity; inverses of are just reverse rotations (which are still rotations); and the product (composition) of 2 rotations is a rotation (closed). To see that it is normal, let $s \in$ SO(3). If $g \in$ SO(3), then $g^{-1}\ s\ g \in$ SO(3) because it is the composition of 3 rotations. If $g \in$ T, then $g^{-1}\ s\ g \in$ SO(3) because the expression involves 2 reflections. So g^{-1} SO(3) g = SO(3). Left multiplying both side by g yields SO(3) g = g SO(3). So SO(3) is normal by Penrose's definition of normal.

The above proof doesn't use Penrose's hint, so here is a proof that does. Since SO(3) is rotation invariant, g SO(3) $\subseteq$ SO(3). By Theorem 1, g SO(3) $\supseteq$ SO(3). Therefore g SO(3) = SO(3). Similarly, because SO(3) is rotation invariant, SO(3) g = SO(3). Thus g SO(3) = SO(3) g which proves SO(3) is normal. ■

Part B

We need the following lemma a few times so it is worth introducing here.

Lemma 1: If g is a 90° rotation and h is a non-zero rotation having an axis of rotation perpendicular to the axis of rotation of g, then $f = g^{-1}\ h\ g$ is a rotation having an axis of rotation perpendicular to both g and h.

Proof: WLOG let

- g be a 90° counter-clockwise rotation about the z-axis and
- h be a rotation of angle θ about the x-axis.

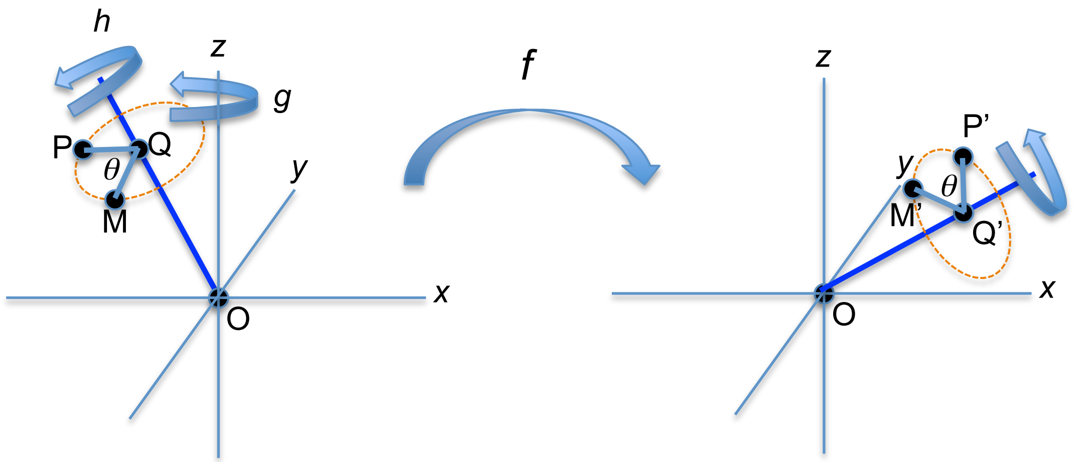
To show f is a rotation about the y-axis, it suffices to show that all points on the y-axis are fixed during rotation f.

Consider the point $(0,y,0)$ on the y-axis. Since g^{-1} spins the xy-plane 90° clockwise, $(0,y,0)\,g^{-1} = (y,0,0)$. Since points on the x-axis are fixed during rotation h, $(y,0,0)\,h = (y,0,0)$. Finally, since g spins the xy-plane 90° counter-clockwise, $(y,0,0)\,g^{-1} =(0,y,0)$. That is $(0,y,0)$ is a fixed point of the rotation f. ■

Aside: I did not see an explanation in Road To Reality of why O(3) is non-Abelian. However, using the descriptions of g^{-1} and h given in Lemma 1, it is easy to show that $g^{-1}h \neq h\,g^{-1}$:

$$(0,1,0)\,g^{-1}h = (1,0,0)\,h = (1,0,0)$$
$$(0,1,0)\,h\,g^{-1}= (0, \cos\theta, \sin\theta)\,g^{-1} = (\cos\theta, 0, \sin\theta) \quad ■$$

Lemma 2: Let $g, h \in SO(3)$ and let h have rotation angle θ. Then $f = g^{-1}\,h\,g$ has the same angle of rotation θ as h (although a possibly different axis of rotation).



Proof: We use the property that rotations in $\mathbb{R}^3$ preserve rigid bodies. WLOG assume g is a counter-clockwise rotation about the z-axis. Let $P \in S$. Let $\overrightarrow{OQ}$ represent the axis of revolution of h and let h rotate point P to a point M. PQM represents the rotation angle θ of h.

Set P' = Pf, Q'= Q f, and M'= Mf.

Define the rigid body B to be the union of $\overrightarrow{OQ}$ with the angle PQM. (It looks like line segment $\overrightarrow{OQ}$ with 2 spikes.) Rotation f moves B as a rigid body so that it becomes the union of $\overrightarrow{OQ'}$ with angle P'Q'M'. The angle remains θ.

We are now in a position to find the angle and axis of rotation of f. Start with a point P' of S. We know that P' = Pf, that h rotates P an amount θ about OQ to point M, that M'= Mf and Q'= Q f. So we know that f rotates point P' to M' by an angle θ about axis OQ'.

(See my version 2 solution to [13.7] for a more rigorous, equation-based proof of this lemma.) ■

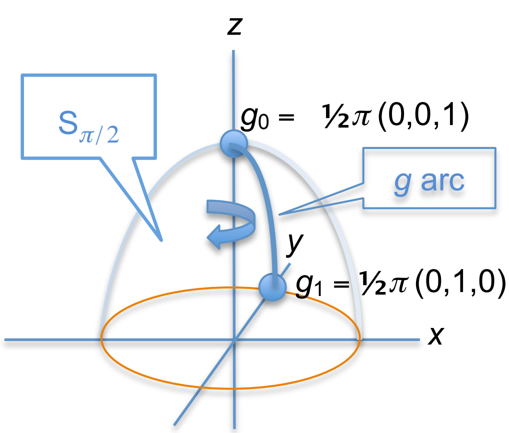
It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3-ball $\mathcal{R}$ of radius π in which antipodal points on the surface of $\mathcal{R}$ are identified. Points of $\mathcal{R}$ can be represented as $\theta(a, b, c) = (\theta a, \theta b, \theta c)$ where θ is the angle of rotation and (a, b, c) is a unit vector in the direction of the axis of rotation.

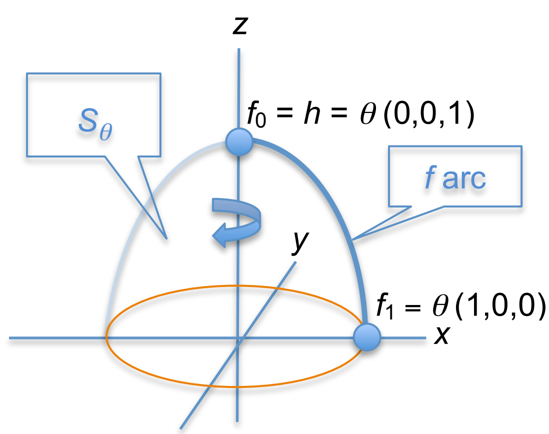
Definition: For $0 \le \theta \le \pi$ let $\mathbf{S}_\theta$ be the sphere of radius θ in $\mathcal{R}$. S_θ consists of angle θ rotations about each axis of rotation.

Theorem 3: In SO(3), if a rotation h with rotation angle θ belongs to a normal subgroup H, then $S_\theta \subseteq H$.

Proof: Let $g \in SO(3)$. Let $F = \{ f = g^{-1} h g : g \in SO(3) \} \subseteq H$. From Lemma 2, $F \subseteq S_\theta$. Thus if $f \in F$, it has rotation angle θ and can be expressed as $f = \theta(a, b, c)$ for some (a, b, c) where $a^2 + b^2 + c^2 = 1$. To prove $F = S_\theta$, we must show for every point (a, b, c) of the unit sphere S in $\mathbb{R}^3$ that $\theta(a, b, c) \in F$. We do this by building up subgroup F starting from the single element h.

WLOG we can let $h = \theta(0, 0, 1)$. Consider a great circle arc from $g_0 = \frac{\pi}{2}(0, 0, 1)$ to $g_1 = \frac{\pi}{2}(0, 1, 0)$ on the surface of sphere $S_{\pi/2}$. Since g_0 has same axis of rotation as h, then $f_0 = g_0^{-1} h g_0 = h = \theta(0, 0, 1)$. Since g_1 has an axis of rotation perpendicular to that of h, by Lemma 1 $f_1 = g_1^{-1} h g_1$ has an axis of rotation perpendicular to both h and g_1. That is, $f_1 = \theta(1, 0, 0)$. Thus as





$g = \frac{\pi}{2}$ (0, sin ϕ, cos ϕ) moves along the arc on $S_{\pi/2}$ from g_0 to g_1 (i.e., from $\phi = 0$ to $\phi = \frac{\pi}{2}$), $f = g^{-1} h g = \theta$ (sin ϕ, 0, cos ϕ) moves along the great circle arc in S_θ from f_0 to f_1.

Now rotate the entire *g* arc in a clockwise 360° circle as indicated in the figure. This sweeps out the northern hemisphere on the surface of the sphere $S_{\pi/2}$:

$$\left\{g = \frac{\pi}{2}\left(\sin\omega \sin\phi, \cos \omega \sin \phi, \cos \phi\right) : 0 \le \phi \le \frac{\pi}{2}, 0 \le \omega \le 2\pi\right\}.$$

The corresponding *f* arc sweeps out the northern hemisphere of S_θ:

$$\left\{f = \theta\left(\cos\omega \sin\phi, -\sin\omega \sin \phi, \cos \phi\right) : 0 \le \phi \le \frac{\pi}{2}, 0 \le \omega \le 2\pi\right\}.$$

That is, for every point (*a*, *b*, *c*) on the northern hemisphere of the unit sphere S there are angles $\phi \in \left[0, \frac{\pi}{2}\right]$ and $\omega \in \left[0, 2\pi\right]$ such that $a = \cos\omega \sin\phi$, $b = -\sin\omega \sin\phi$, and $c = \cos\phi$. Thus there is a θ rotation *f* on the northern hemisphere of S_θ and a 90° rotation *g* such that $\theta(a, b, c) = f = g^{-1} h g$. Thus $\theta(a, b, c) \in$ F.

For the southern hemisphere, note that $h^{-1} = \theta(0, 0, -1) \in$ H since H is a group. The *g* and *f* arcs based on h^{-1} similarly sweep out their southern hemispheres.

Thus for every point (*a*, *b*, *c*) on the unit sphere, $\theta(a, b, c)$ equals either $g^{-1} h g$ or $g^{-1} h^{-1} g$ for some 90° rotation *g*, proving that $\theta(a, b, c) \in$ F and concluding the proof. (A different proof using Clifford Algebra rotation equations in provided my version 2 proof.) ■

Theorem 4: SO(3) has no proper normal subgroup.

Proof: Let H be a non-trivial normal subgroup of SO(3). ∃ 1 ≠ *h* ∈ H having some rotation angle θ. By Theorem 3, $S_\theta \subseteq$ H.

We take products of elements in S_θ to grow H into the solid ball of radius 2θ:

Let $g, h \in S_\theta$ and $f = h g \in$ H. The maximum possible angle for *f* is 2θ, obtained when $g = h$, and the minimum angle is 0, obtained when $g = h^{-1}$. By moving *g* along a path on S_θ from *h* to h^{-1} we generate a continuous curve of points f in H having every possible angle ϕ from 0 to 2θ. From Theorem 3, $S_\phi \subseteq H$ for $0 \le \phi \le 2\theta$.

If $2\theta \ge \pi$, then we are done. If not, starting from sphere $S_{2\theta}$ we similarly grow H to include the closed ball of radius 4θ, then 8θ, … Eventually we grow H to include the ball SO(3) of radius π.

Thus, $SO(3) = \mathcal{R} \subseteq H$. ■

Lemma 3: Let $g \in SO(3)$. Then $g^{-1}Rg = R$.

Proof: Let P be a point on the unit sphere S. Let $Q = Pg^{-1}$. Then $-Q = QR = Pg^{-1}R$ and $Pg^{-1}Rg = (-Q)g = -(Qg) = -P = PR$. ■

Theorem B: SO(3) and $\mathcal{I} = \{ 1,R \}$ are the only proper normal subgroups of O(3).

Proof: Let H be a nontrivial normal subgroup of O(3) such that $H \ne SO(3)$ and $H \ne \mathcal{I}$. We need to show that H = O(3).

Claim: There is an element $t \in H \cap T$ such that $t \ne R$:

By Theorem 4, $\exists\, t_0 \in H \cap T$. If $t_0 \ne R$, the claim is true. So suppose $t_0 = R$. $1 \in H$ since H is a group. Since $H \ne \mathcal{I}$, H contains another element besides 1 and R. If that element is in T, the claim is true. Suppose the other element is $s_0 \in SO(3)$. Set $t = s_0 R$. Then $t \in H \cap T$ and $t \ne R$ since $s_0 \ne 1$.

$t^2 \in SO(3)$. Suppose for the moment that $t^2 \ne 1$. Then $t^2 \in SO(3) \cap H$ $\Rightarrow SO(3) \subseteq H$ by Theorem 4. Also, $\exists\; s \in SO(3)$ such that $t = s\,R$. Since $t \ne R$ then $s \ne 1$.

Claim: $T \subseteq H$:

Let $t_1 \in T$. $\exists\, s_1 \in SO(3)$ such that $t_1 = s_1 R$. Let $s_2 = s_1\, s^{-1} \in SO(3)$. Then $s_1 = s_2\, s$. Since $s_2 \in SO(3) \subset H$, $t_1 = s_1 R = s_2\, s\, R = s_2\, t \in H$. Thus $T \subseteq H$.

Since $SO(3) \subseteq H$, we have $O(3) = SO(3) \cup T \subseteq H$. Therefore H = O(3).

Unfortunately if s has a 180° rotation angle, then $s^2 = 1$ and thus $t^2 = 1$, and the above argument doesn't quite hold. (Note: $t^2 = 1$ because if P is a point, then $Pt^2 = PsRsR = [Ps]RsR = [-Ps]sR = -Ps^2R = -PR = P$). However, everything in the above argument remains true except that we haven't proved $SO(3) \subseteq H$. Once we prove this, we are done.

Let g be a 90° rotation about an axis perpendicular to the axis of s and let $s_3 = g^{-1}sg$. By Lemma 1, the rotation axis of s_3 is perpendicular to that of s. Let $t_3 = g^{-1}\, t\, g \in \mathrm{H}$. We have

$$t_3 = g^{-1}\, s\, \mathrm{R}\, g = g^{-1}\, s\, (g\, g^{-1})\, \mathrm{R}\, g = (g^{-1}\, s\, g)\, (g^{-1}\, \mathrm{R}\, g) = s_3\, \mathrm{R}$$

by Lemma 3. Hence the axis of rotation of t_3 is perpendicular to that of t. (See footnote[1].) Let $s_4 = t\, t_3$. Because inverses have the same axis of rotation, $t_3 \neq t^{-1}$ and so $s_4 \neq 1$. Because H is a group, $s_4 \in \mathrm{H}$. Thus, by Theorem 4, $\mathrm{SO(3)} \subseteq \mathrm{H}$, completing the proof. ■

[1] We have $s_3 = g^{-1}sg$, $t_3 = s_3\, \mathrm{R}$, and $t = s\, \mathrm{R}$. The axis of rotation of the reflective rotations t and t_3 can be considered to be located in the reflected unit sphere. They point in the opposite directions from the axes of rotation of s and s_3, respectively. Thus, since the axes of s and s_3 are perpendicular, then so are the axes of t and t_3.