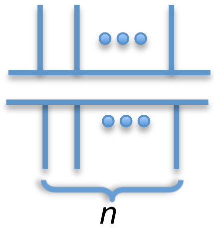
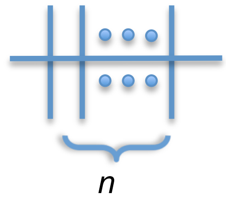
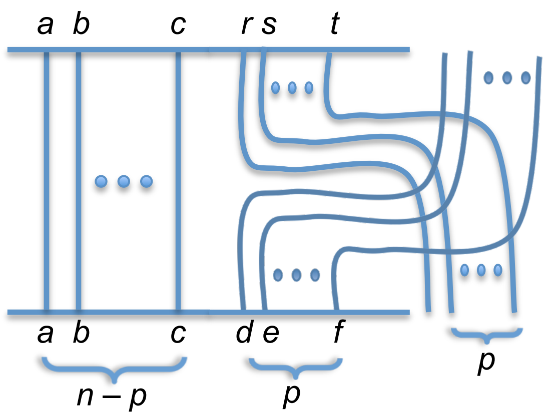
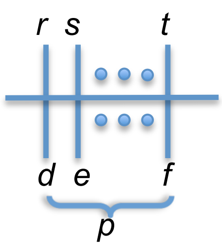
[13.22] Show why the following diagram equalities in Figure 12.18 hold:

(a) $\in^{a\cdots c}\varepsilon_{d\cdots f} =$ 

$=$ 

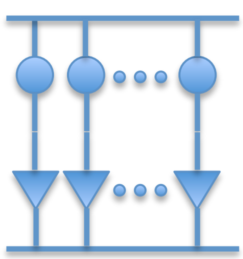
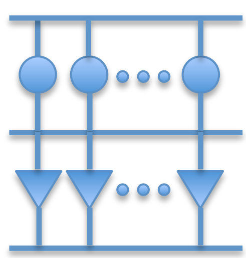
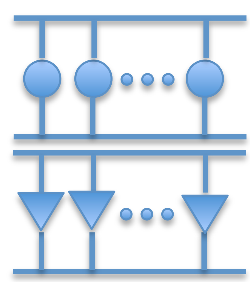
$n!\delta^{a}_{[d}\cdots\delta^{c}_{f]}$

(b) $\in^{a\cdots cd\cdots f}\varepsilon_{a\cdots cr\cdots t} =$ 

$=(n-p)!$ 

$\overset{(a)}{=}(n-p)!p!\delta^{r}_{[d}\cdots\delta^{t}_{f]}$

I have embedded below a couple of useful corollaries to Part (b). I have also included a Part (c) because I don't see that the equalities of Figure 13.8 (b) follow effortlessly from (a) and (b) above even though the diagrams have similarities.

(c)  $= \dfrac{1}{n!}$  $= \dfrac{1}{n!}$ 

Proof: The dot product $\in\bullet\varepsilon$ of the Levi-Civita antisymmetrization quantities is a sum of *n* terms. If the sum is normalized to *n*!, then each term $\in^{ab\cdots c}\varepsilon_{ab\cdots c}$ equals 1. That is the essence of the theorem below. This fact is used to simplify the proofs of (a – c). To avoid confusion, I do not use Einstein notation in the theorem and its corollaries.

Theorem: $\in^{12\cdots n}\varepsilon_{12\cdots n}=1$.

Proof: Let π be a permutation of (1, 2, …, n). Observe that $\in^{\pi(1)\pi(2)\cdots\pi(n)}\varepsilon_{\pi(1)\pi(2)\cdots\pi(n)}=\left[\text{Sign}(\pi)\in^{12\cdots n}\right]\left[\text{Sign}(\pi)\varepsilon_{12\cdots n}\right]=\in^{12\cdots n}\varepsilon_{12\cdots n}$.There are n! permutations, π, of (1, …, n). So,

$$n!=\in\bullet\varepsilon=\sum_{\pi}\in^{\pi(1)\pi(2)\cdots\pi(n)}\varepsilon_{\pi(1)\pi(2)\cdots\pi(n)}=\sum_{\pi}\in^{12\cdots n}\varepsilon_{12\cdots n}=n!\in^{12\cdots n}\varepsilon_{12\cdots n}.\quad\blacksquare$$

Corollary 1: Let (a,b,…,c) be a permutation of (1, 2, …, n). Then $\in^{ab\cdots c}\varepsilon_{ab\cdots c}=1$.

Proof: Let $\pi^*:(a,b,\cdots,c)\mapsto(1,2,\cdots,n)$. Then

$$\in^{ab\cdots c}\varepsilon_{ab\cdots c}=\text{Sign}(\pi^*)\in^{12\cdots n}\text{Sign}(\pi^*)\varepsilon_{12\cdots n}\overset{(\text{Theorem})}{=}1.$$

Corollary 2: Let (a, b, …, c) and (d, e, …, f) be permutations of (1, 2, …, n), and let $\pi_0:(d,e,\cdots,f)\mapsto(a,b,\cdots,c)$. Then $\in^{ab\cdots c}\varepsilon_{de\cdots f}=\text{Sign}(\pi_0)=\pm1$ depending on whether π_0 is an even or odd permutation.

Proof: $\in^{ab\cdots c}\varepsilon_{de\cdots f}=\text{Sign}(\pi_0)\in^{ab\cdots c}\varepsilon_{ab\cdots c}\overset{(\text{Cor 1})}{=}\text{Sign}(\pi_0)$ $\blacksquare$

We resume Einstein summation notation.

Proof of (a): From Corollary 2, LHS = $\in^{ab\cdots c}\varepsilon_{de\cdots f}=\text{Sign}(\pi_0)$. The RHS summation is $n!\delta^a_{[d}\cdots\delta^c_{f]}=\dfrac{n!}{n!}\sum_{\pi}\text{Sign}(\pi)\delta^a_{\pi(d)}\cdots\delta^c_{\pi(f)}$. All the terms in the summation are zero except for $\text{Sign}(\pi_0)\delta^a_{\pi_0(d)}\cdots\delta^c_{\pi_0(f)}=\text{Sign}(\pi_0)\delta^a_a\cdots\delta^c_c=\text{Sign}(\pi_0)$ = LHS. ✔

Proof of (b): Let $\pi_1:(d,e,\cdots,f)\mapsto(r,s,\cdots,t)$. Claim:

(i) $\in^{a\cdots cd\cdots f}\varepsilon_{a\cdots cr\cdots t}=\text{Sign}(\pi_1)$

$$\in^{a\cdots cd\cdots f}\varepsilon_{a\cdots cr\cdots t}=\text{Sign}(\pi_1)\in^{a\cdots cr\cdots t}\varepsilon_{a\cdots cr\cdots t}\overset{(\text{Cor 1})}{=}\text{Sign}(\pi_1)\quad ✔$$

Let $\mathcal{P}_{ab\ldots c}$ and $\mathcal{P}_{de\ldots f}$ be the sets of permutations of (a, b, …, c) and (d, e, …, f), respectively. (a,b,c) has (n –p) terms and (d,e,f) has p terms.

$$\text{LHS} = \in^{a\cdots cd\cdots f}\varepsilon_{a\cdots cr\cdots t}=\sum_{\pi\in\mathcal{P}_{ab\cdots c}}\in^{\pi(a)\cdots\pi(c)d\cdots f}\varepsilon_{\pi(a)\cdots\pi(c)r\cdots t}$$

$$=\sum_{\pi\in\mathcal{P}_{ab\cdots c}}\left[\text{Sign}(\pi)\in^{a\cdots cd\cdots f}\right]\left[\text{Sign}(\pi)\varepsilon_{a\cdots cr\cdots t}\right]$$

$$= \sum_{\pi\in\mathcal{P}_{ab\cdots c}} \in^{a\cdots cd\cdots f}\varepsilon_{a\cdots cr\cdots t} \overset{(i)}{=} \sum_{\pi\in\mathcal{P}_{ab\cdots c}} \text{Sign}(\pi_1) = (n-p)!\,\text{Sign}(\pi_1).$$

$$\text{RHS} = (n-p)!\,p!\,\delta^r_{[d}\cdots\delta^t_{f]} = (n-p)!\,\frac{p!}{p!}\sum_{\pi\in\mathcal{P}_{de\cdots f}} \text{Sign}(\pi)\,\delta^r_{\pi(d)}\cdots\delta^t_{\pi(f)}.$$

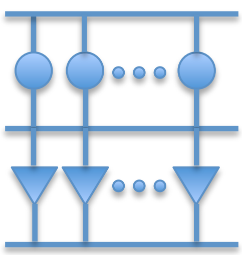
All of the terms are zero except

$$(n-p)!\,\text{Sign}(\pi_1)\,\delta^r_{\pi_1(d)}\cdots\delta^t_{\pi_1(f)} = (n-p)!\,\text{Sign}(\pi_1)\,\delta^r_r\cdots\delta^t_t = (n-p)!\,\text{Sign}(\pi_1)$$

$$= \text{LHS} \quad \blacksquare$$

Proof of (c): We first show equality of the left and center diagrams. Interpretation of the antisymmetry diagram in this problem is a little different than in other diagrams we have so far encountered. Previously we have only bracketed one set of subscripts or superscripts. However, in this instance the subscripts of S must match the superscripts of T. If one set is rearranged, the other must match the arrangement. Hence we must bracket both of them, as below. Even though there are two brackets, because the arrangements are synchronized there is only a single summation. Because the signs of the terms must match, there are two instances of Sign(π).

Let $\mathcal{P}$ be the set of permutations of (a, …, c). Then



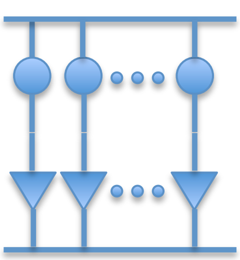
$$\overset{\text{defn}}{=} n!\,\varepsilon_{r\cdots t}\, S^r{}_{[a}\cdots S^t{}_{c]}\, T^{[a}{}_u\cdots T^{c]}{}_w \in^{u\cdots w}$$

$$= \frac{n!}{n!}\sum_{\pi\in\mathcal{P}} \varepsilon_{r\cdots t}\,\text{Sign}(\pi)\,S^r{}_{\pi(a)}\cdots S^t{}_{\pi(c)}\,\text{Sign}(\pi)\,T^{\pi(a)}{}_u\cdots T^{\pi(c)}{}_w \in^{u\cdots w}$$

$$= \sum_{\pi\in\mathcal{P}} \varepsilon_{r\cdots t}\, S^r{}_a\cdots S^t{}_c\, T^a{}_u\cdots T^c{}_w \in^{u\cdots w} \qquad \left[\text{Rename } \pi(a)\mapsto a,\cdots,\pi(c)\mapsto c\right]$$

$$= n!\,\varepsilon_{r\cdots t}\, S^r{}_a\cdots S^t{}_c\, T^a{}_u\cdots T^c{}_w \in^{u\cdots w}$$

=



✔

We now show equality of the center and right diagrams. Since $\frac{1}{n!}\in^{a\cdots c}\varepsilon_{a\cdots c}=1$, it is compelling to insert $\frac{1}{n!}\in^{a\cdots c}\varepsilon_{a\cdots c}$ between S and T in the last equation above in order to generate $\left(\varepsilon_{r\cdots t}S^r{}_a\cdots S^t{}_c\in^{a\cdots c}\right)\left(\varepsilon_{a\cdots c}T^a{}_u\cdots T^c{}_w\in^{u\cdots w}\right)$, which would conclude the proof. However, it is not as easy as that because the subscripts of S are linked to the superscripts of T.

Here is an example of the difficulty (using summation notation for clarity). Consider 2 quantities S_a and T_c.

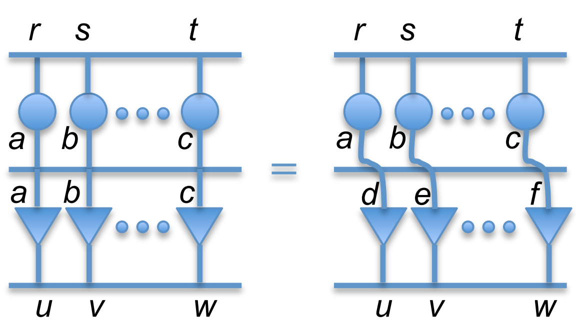
$$\sum_{a=1}^{2}S_a T_a \neq \left(\sum_{a=1}^{2}S_a\right)\left(\sum_{a=1}^{2}T_a\right)$$

because their subscripts are linked. However,

$$\sum_{a=1}^{2}\sum_{c=1}^{2}S_a T_c = \left(\sum_{a=1}^{2}S_a\right)\left(\sum_{c=1}^{2}T_c\right)$$

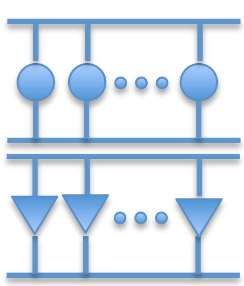
when they are not linked.

The key is to find a way to unlink $S^r{}_a\cdots S^t{}_c$ and $T^a{}_u\cdots T^c{}_w$. A solution is to interpret the middle diagram as having identity operators $I\cdots I$ = $\delta^a_d\cdots\delta^c_f$ connecting the S and T terms:

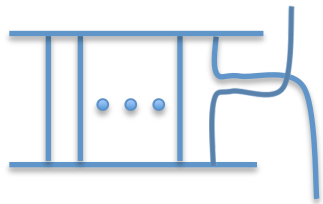


$$=n!\,\varepsilon_{r\cdots t}S^r{}_a\cdots S^t{}_c\,\delta^a_{[d}\cdots\delta^c_{f]}T^d{}_u\cdots T^f{}_w\in^{u\cdots w}$$

$$\overset{(a)}{=}\varepsilon_{r\cdots t}S^r{}_a\cdots S^t{}_c\in^{a\cdots c}\varepsilon_{d\cdots f}T^d{}_u\cdots T^f{}_w\in^{u\cdots w}=\left(\varepsilon_{r\cdots t}S^r{}_a\cdots S^t{}_c\in^{a\cdots c}\right)\left(\varepsilon_{d\cdots f}T^d{}_u\cdots T^f{}_w\in^{u\cdots w}\right)$$

$=$ 

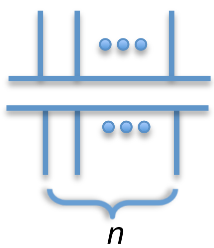
The following corollaries are useful.

Corollary b1:  = $(n-1)!$ 

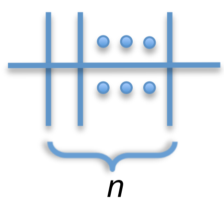
Proof:

LHS $\overset{(b)}{=}$ $(n-1)!$  $= (n-1)!\,\delta^a_{[b]} = (n-1)!\,\delta^a_b$ $= (n-1)!$  ■

Corollary b2:



$\overset{(a)}{=}$



$\overset{(b)}{=}$