[13.18] Let V be an n-dimensional vector space. The linear transformation $T: V \to V$ is nonsingular $\Rightarrow$ T has an inverse

Proof: Let $\{a_1, \cdots, a_n\}$ be a basis for V. Let $b_k = T a_k$. Then $\{ b_k \}$ is also a basis for V:

> Suppose $\{\beta^k\}$ is collection such that $\beta^k b_k = 0$. (Einstein Summation Convention.) Then $0 = \beta^k b_k = \beta^k T a_k = T(\beta^k a_k) \overset{[13.17]}{\Leftrightarrow} \beta^k = 0 \; \forall k$. Thus $\{ b_k \}$ is a collection of linearly independent vectors of size n, and hence $\{ b_k \}$ is a basis for V.

Define $S: V \to V : S b_k = a_k$. We show $S = T^{-1}$ by showing $STv = v = TSv \; \forall v \in V$. (This is equivalent to $ST = I = TS$.)

Let $v = \alpha^k a_k \in V$. Then $STv = S\alpha^k b_k = \alpha^k a_k = v$ ✔
Since $\{b_n\}$ is a basis, $v = \beta^k b_k$. So $TSv = TS\beta^k b_k = T\beta^k a_k = \beta^k b_k = v$ ✔

Note: Penrose said not to use explicit expressions. By that, he may have meant to do a coordinate-free construction, which I didn't because I assumed a basis $\{ a_k \}$. Beckmann avoided using a basis but otherwise his proof seems as "explicit" as mine. He proved that nonsingular $\Rightarrow$ 1-1 and onto. Therefore $\forall w \in V \; \exists$ unique $v_w \in V$ such that $T(v_w) = w$. He defined $S: V \to V : S(w) = v_w$. Thus $TS(w) = T(v_w) = w$ ✔

Next, he fixed v so that there is a w such that $w = T(v)$. So $ST(v) = S(w) = v_w$. Since $T(v_w) = w$ and T is 1-1, then $v_w = v$; i.e., $ST(v) = v$ ✔

Therefore he concluded that $S = T^{-1}$.