[9.2] Let C be unit circle in $\mathbb{C}$. $\mathfrak{F}$ $F(z) = \sum_{k=-\infty}^{\infty} \alpha_k z^k : C \to \mathbb{C}$

$= \ldots \frac{\alpha_{-2}}{z^2} + \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots$ is anal on C. Show that

$\alpha_n = (2\pi i)^{-1} \oint z^{-n-1} F(z)\, dz$.

Define $F_n : C \to \mathbb{C} : F_n(z) = \alpha_n z^n$. So $F = \sum_{n=-\infty}^{\infty} F_n = \lim_{n\to\infty} \sum_{k=-n}^{n} F_k$ on C.

Let D be the closed unit disk in $\mathbb{C}$. We extend F_n to D as follows:

Let $z \in D$ $\quad z = re^{i\theta}$, $r \le 1$, $0 \le \theta < 2\pi$. Let $w = e^{i\theta} = \frac{z}{r} = \frac{z}{|z|}$

(1) Define $F_n : D \to \mathbb{C} : F_n(z) = \alpha_n z^n = \alpha_n (rw)^n = r^n \alpha_n w^n = r^n F_n(w)$ for $z \ne 0$.

Define $F_n(0) = 0$. This extends the defn of F_n since the extended formula is the same as the original formula. We extend F to D by $F = \sum_{n=-\infty}^{\infty} F_n$.

Claim F_n is anal on D except at $z=0$ for all n:

By defn, we must show $F_n'(z)$ exists $\forall\, 0 \ne z \in D$. On C,

(2) $F_n'(w) = \lim_{v \to w} \frac{F_n(w) - F_n(v)}{w - v} = n \alpha_n w^{n-1}$ exists. Let $z \in D$, $z \ne 0$,

$F_n'(z) = \lim_{u \to z} \frac{F_n(z) - F_n(u)}{z - u}$ if $F_n'(z)$ exists. Let $z \equiv re^{i\theta} \equiv rw$ and $u \equiv se^{i\phi} \equiv sv$

Then $F_n'(z) = \lim_{u \to z} \frac{\alpha_n z^n - \alpha_n u^n}{z - u} \overset{(1)}{=} \lim_{\substack{s \to r \\ v \to w}} \frac{r^n F_n(w) - s^n F_n(v)}{rw - sv}$

$= \lim_{v \to w} r^{n-1} \frac{F_n(w) - F_n(v)}{w - v} = r^{n-1} F_n'(w) \overset{(2)}{=} r^{n-1} n \alpha_n w^{n-1}$ exists ✓

So $F'(z) = \sum_{n=-\infty}^{\infty} F_n'(z)$ exists, and so F is anal on D except possibly at $z=0$.

Consider $\frac{F(z)}{z^{n+1}} = z^{-n-1} F(z) = \sum_{k=-\infty}^{\infty} \alpha_k z^{k-n-1} = \ldots \frac{\alpha_{n-1}}{z^2} + \frac{\alpha_n}{z} + \alpha_{n+1} + \alpha_{n+2} z + \alpha_{n+3} z^2 + \ldots$

Since $\frac{F(z)}{z^{n+1}}$ is anal on D except at $z=0$, (since it is ratio of 2 anal fns), by Residue Th

$\oint \frac{F(z)}{z^{n+1}} dz = 2\pi i \operatorname{Res}\left[\frac{F(z)}{z^{n+1}}\right] = 2\pi i \sum_{k=-\infty}^{\infty} \operatorname{Res}\left[\alpha_k z^{k-n-1}\right]$. For $k \ge n+1$,

$\alpha_k z^{k-n-1}$ has no pole, thus no residue. For $k \le n-1$, $\operatorname{Res}\left[\alpha_k z^{k-n-1}\right] = \frac{d^{n-k}}{dz^{n-k}} \alpha_k = 0$

For $k = n$, $\operatorname{Res}\left(\frac{\alpha_n}{z}\right) = \alpha_n$. So $\oint \frac{F(z)}{z^{n+1}} dz = (2\pi i)\alpha_n$ $\square$

Note: Since F is analytic, by defn $\alpha_k = 0$ for $k<0$. So didn't have to consider them.