[12.4] Problem Statement: Let $\boldsymbol{\xi}$ be a vector field, Φ a scalar field, and dΦ the covector with components $\frac{\partial\Phi}{\partial x^1}, \frac{\partial\Phi}{\partial x^2}, \ldots, \frac{\partial\Phi}{\partial x^n}$. (i) At each point of $\mathcal{M}$, a (non-zero) covector $\boldsymbol{\alpha}$ determines an (n-1) dimensional plane element. (ii) When $\boldsymbol{\alpha} = d\Phi$, these (n-1) plane elements are tangential to the family of (n-1) dimensional surfaces of constant Φ.

Solution:

(i) In general, $\xi = \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial x^k} = \langle \xi_1, \ldots, \xi_n \rangle$, $\alpha = \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x^k} = \langle \alpha_1, \ldots, \alpha_n \rangle$, and

$$\alpha(\xi) \equiv \alpha \bullet \xi = \langle \alpha_1, \cdots, \alpha_n \rangle \bullet \langle \xi_1, \cdots, \xi_n \rangle = \sum_{k=1}^{n} \alpha_k \xi_k.$$

First, consider α = dx^1.

$\alpha(\xi) = dx^1(\xi) = \langle 1, 0, \cdots, 0 \rangle \bullet \langle \xi_1, \xi_2, \cdots, \xi_n \rangle = \xi_1$. Thus $\alpha(\xi)$ = dx^1 represents the magnitude of the rate of change of ξ in the x^1 direction. That is, at a point P$\in$ $\mathcal{M}$ it represents the magnitude of the rate of change of ξ at P when x^2, x^3, …, x^n are held constant. Thus $[\alpha(\xi)]$ (Φ) represents the (n-1)-dimensional plane element $\langle x^2, x^3, \ldots, x^n \rangle$ at P.

Now consider α in general. $\alpha(\xi)$ represents the magnitude of the rate of change of ξ in the direction of ξ, namely $\sum_{k=1}^{n} \xi_k x^k$, where $\{x^k\}$ are unit vectors along the x^k-axes. That is, like $dx^1(\xi)$, $[\alpha(\xi)]$ (Φ) represents a vector at P, and thus it determines the (n-1)-dimensional plane element at P perpendicular to the vector.

(ii) Let α = $d\Phi$. We know $\xi(\Phi) = \frac{d\Phi}{d\xi} = d\Phi \bullet \xi$. So when $\frac{d\Phi}{d\xi} = d\Phi \bullet \xi = \alpha \bullet \xi = 0$, ξ points along a direction of constant Φ. Thus, the (n-1)-dimensional plane element at P generated by α in part (i) is a hyperplane where Φ is constant.