[13.31] Let ***V*** be a vector space and ***T*** be a linear transformation on ***V*** with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, where $m \le n$. Furthermore assume

(a) For each λ_j of multiplicity $r_j \ge 2$ (if any), there are r_j independent eigenvectors.

Prove **(A)** there is a basis for ***V*** composed of eigenvectors and **(B)**

$$T = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_m \end{pmatrix}$$

where each λ_j appears r_j times.

Solution. **(A)**

Let r_j be the multiplicity of eigenvalue λ_j. Since there are n eigenvalues, we have that $\sum_{j=1}^{m} r_j = n$. Let $\mathscr{B}_j = \left\{ v_{j1}, v_{j2}, \cdots, v_{jr_j} \right\}$ be the set of r_j independent eigenvectors corresponding to λ_j. We wish to prove

$$\mathscr{B} = \bigcup_{j=1}^{m} \mathscr{B}_j = \left\{ v_{ji} : i = 1, \cdots, r_j,\ j = 1, \cdots, m \right\}$$

comprises a basis for ***V***. Since $\mathscr{B}$ contains n vectors, it suffices to show that these vectors are linearly independent. So, assume

$$(*) \quad \sum_{j=1}^{m} \sum_{i=1}^{r_j} \alpha_{ji} v_{ji} = 0.$$

We will be done if we can show that $\alpha_{ji} = 0 \ \forall\, j, i,$ so suppose some $\alpha_{ji} \ne 0.$ We show this leads to a contradiction, which will complete the proof.

Since the double sum (*) has a finite number of terms, there is some collection $\left\{ \alpha_{ji} \right\}$ of non-zero coefficients satisfying (*) having as few terms as possible. That is, $\exists p \le m,$ numbers $\left\{ s_j \le r_j \right\},$ and a set $\left\{ \alpha_{ji} \ne 0 : i = 1, \cdots, s_j,\ j = 1, \cdots, p \right\}$ such that

$$(1) \quad \sum_{j=1}^{p} \sum_{i=1}^{s_j} \alpha_{ji} v_{ji} = 0 \text{ has the minimum number of terms.}$$

If p = 1, all the eigenvectors arise from a single eigenvalue and are thus independent by condition (a). Hence $\alpha_{ji} = 0 \ \forall \ j, i$, contradicting that they are all non-zero.

So, we assume p > 1. Since for all j and i, $Tv_{ji} = \lambda_j v_{ji}$, we can apply T to equation (1) to get

$$(2) \quad \sum_{j=1}^{p} \sum_{i=1}^{s_j} \lambda_j \alpha_{ji} v_{ji} = 0.$$

Multiplying equation (1) by λ_p gives

$$(3) \quad \sum_{j=1}^{p} \sum_{i=1}^{s_j} \lambda_p \alpha_{ji} v_{ji} = 0.$$

Subtracting (3) from (2) gives

$$0 = \sum_{j=1}^{p} \sum_{i=1}^{s_j} (\lambda_j - \lambda_p) \alpha_{ji} v_{ji} = \sum_{j=1}^{p-1} \sum_{i=1}^{s_j} (\lambda_j - \lambda_p) \alpha_{ji} v_{ji} + \sum_{j=p}^{p} \sum_{i=1}^{s_j} \cancel{(\lambda_p - \lambda_p)} \alpha_{ji} v_{ji}, \text{ or}$$

$$(4) \quad \sum_{j=1}^{p-1} \sum_{i=1}^{s_j} (\lambda_j - \lambda_p) \alpha_{ji} v_{ji} = 0.$$

In equation (4), $\lambda_j - \lambda_p \neq 0$ for all j since $j < p$ and the eigenvalues are distinct. Thus we have produced a shorter relation than (1), yielding the afore-mentioned contradiction, completing the proof of (A). ✔

(B) Re-label $\{v_{ji}\}$ as $\{e_k : k = 1, 2, \cdots, n\}$ and re-label the corresponding eigenvalues as $\{\lambda_k : k = 1, 2, \cdots, n\}$. (For clarification, in this notation if, for example, λ_1 has multiplicity > 1, then we will have $\lambda_1 = \lambda_2 = \cdots = \lambda_{r_1}$. In the basis $\{e_k\}$, $\lambda_k e_k = T e_k$ is written in matrix form as

$$\left[\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right] = \begin{bmatrix} T^1{}_1 & T^1{}_2 & \cdots & T^1{}_k & \cdots & T^1{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T^{k-1}{}_1 & T^{k-1}{}_2 & \cdots & T^{k-1}{}_k & \cdots & T^{k-1}{}_n \\ T^k{}_1 & T^k{}_2 & \cdots & T^k{}_k & \cdots & T^k{}_n \\ T^{k+1}{}_1 & T^{k+1}{}_2 & \cdots & T^{k+1}{}_k & \cdots & T^{k+1}{}_n \\ \vdots & \vdots & & \vdots & & \vdots \\ T^n{}_1 & T^n{}_2 & \cdots & T^n{}_k & \cdots & T^n{}_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} T^1{}_k \\ \vdots \\ T^{k-1}{}_k \\ T^k{}_k \\ T^{k+1}{}_k \\ \vdots \\ T^n{}_k \end{bmatrix}$$

$\Rightarrow \forall k \;\; T^k{}_k = \lambda_k$ and $T^j{}_k = 0$ if $j \neq k$.

That is,

$$T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

If we revert to the original notation where each Eigenvalue appears according to its multiplicity, then this matrix becomes the required matrix. ■