

[13.7] $SO(3)$ is the group of rotations of the unit sphere in 3-space. $O(3)$ extends $SO(3)$ by including reflections. (A) Show that $SO(3)$ is a normal subgroup of $O(3)$ and (B) show that it is the only proper normal subgroup.

Note. (B) is actually not true. There is one other proper normal subgroup of $O(3)$. If 1 is the identity element (null rotation) of $SO(3)$ and R is the reflection operator then $\mathcal{I} = \{1, R\}$ is a normal subgroup of $O(3)$. This is because if g is a (reflective or non-reflective) rotation, then $g^{-1} 1 g = 1$ and $g^{-1} R g = R$ (see Lemma 3). So we revise (B): show that $O(3)$ has only two proper normal subgroups.

Note: In this proof we adopt the convention that $f g$ represents rotating by f followed by g . So $f R$ means to rotate and then reflect while $R f$ means to reflect then rotate.

Proof: Penrose gives the hint: “What are the only sets in $O(3)$ that are rotation invariant?”. The answer is simple. In Theorem 2 we show there are only 2 such sets: $SO(3)$ and T . We begin with some preliminaries.

Definitions:

1. Let \mathbf{S} be the unit sphere of \mathbb{R}^3
2. Let \mathbf{R} be the reflection operation on S
3. Let $\mathbf{T} = R[SO(3)] = \{Rg: g \in SO(3)\}$ be the coset of reflective rotations in $O(3)$
 - a. $SO(3)$ and T are disjoint, and $O(3) = SO(3) \cup T$
4. Let $\mathbf{1}$ be the identity of $O(3)$, the null rotation

\mathbf{R} is defined as an operation that reverses xyz orientation. It can be a reflection through the xy -plane, the yz -plane, or the xz -plane, or (preferred) a reflection through the origin, which is the composition of reflecting first through the xy -plane, then the yz -plane, and then the xz -plane. Also, $\mathbf{R}^{-1} = \mathbf{R}$. If P is a point of S then $\mathbf{PR} = -P$.

While $SO(3)$ is a group, T is not. (It is only a coset.) For example, $1 \notin T$. Also, if t_1 and t_2 belong to T , their composition $t_1 t_2 \notin T$. Rather, $t_1 t_2 \in SO(3)$. This is because R is applied twice in the expression $t_1 t_2$. In fact, any expression with an even number of reflections belongs to $SO(3)$, and it belongs to T if the number of reflections is odd.

If $t \in T$, there are elements $s_1, s_2 \in SO(3)$ such that $t = R s_1$ and $t = s_2 R$. The former is true by definition of T . The latter is seen to be true by setting $s_2 = R s_1 R$.

We need the following theorem to answer Penrose's invariance question.

Theorem 1.

- (a) Let $s_1, s_2 \in \text{SO}(3)$. Then $\exists s_3 \in \text{SO}(3)$ such that $s_2 = s_3 s_1$.
 (b) Let $t_1, t_2 \in T$. Then $\exists s \in \text{SO}(3)$ such that $t_2 = s t_1$.

Proof: (a) $s_3 = s_2 s_1^{-1}$. (b) $s = t_2 t_1^{-1}$. $s \in \text{SO}(3)$ because this expression has 2 reflections, an even number. ■

Theorem 2 (Answer to Penrose's question): $\text{SO}(3)$ and T are the only proper subsets of $O(3)$ that are rotation invariant.

Proof: $\text{SO}(3)$ is rotation-invariant because applying a rotation to any rotation in $\text{SO}(3)$ yields another rotation, an element of $\text{SO}(3)$. $\text{SO}(3)$ has no proper subset A that is rotation-invariant because, by Theorem 1a, given any $s_1 \in A$ and $s_3 \notin A$, one can find a rotation s_2 such that $s_1 s_2 = s_3$; i.e., s_1 is rotated out of A .

Similarly, T is rotation invariant because applying a rotation to any element of T remains in T because only 1 reflection has occurred. Like $\text{SO}(3)$, T cannot have a proper subset A that is rotation invariant because, by Theorem 1b, from any $t_1 \in A$ one can obtain any $t_2 \notin A$ by applying a rotation. ■

Part A

Theorem A: $\text{SO}(3)$ is a normal subgroup of $O(3)$

Proof. First, $\text{SO}(3)$ is clearly a group because it contains the identity; inverses of are just reverse rotations (which are still rotations); and the product (composition) of 2 rotations is a rotation (closed). To see that it is normal, let $s \in \text{SO}(3)$. If $g \in \text{SO}(3)$, then $g^{-1} s g \in \text{SO}(3)$ because it is the composition of 3 rotations. If $g \in T$, then $g^{-1} s g \in \text{SO}(3)$ because the expression involves 2 reflections. So $g^{-1} \text{SO}(3) g = \text{SO}(3)$. Left multiplying both side by g yields $\text{SO}(3) g = g \text{SO}(3)$. So $\text{SO}(3)$ is normal by Penrose's definition of normal.

The above proof doesn't use Penrose's hint, so here is a proof that does. Since $\text{SO}(3)$ is rotation invariant, $g \text{SO}(3) \subseteq \text{SO}(3)$. By Theorem 1, $g \text{SO}(3) \supseteq \text{SO}(3)$. Therefore $g \text{SO}(3) = \text{SO}(3)$. Similarly, because $\text{SO}(3)$ is rotation invariant, $\text{SO}(3) g = \text{SO}(3)$. Thus $g \text{SO}(3) = \text{SO}(3) g$ which proves $\text{SO}(3)$ is normal. ■

Part B

We need the following lemma a few times so it is worth introducing here.

Lemma 1: If g is a 90° rotation and h is a non-zero rotation having an axis of rotation perpendicular to the axis of rotation of g , then $f = g^{-1} h g$ is a rotation having an axis of rotation perpendicular to both g and h .

Proof: WLOG let

- g be a 90° counter-clockwise rotation about the z -axis and
- h be a rotation of angle θ about the x -axis.

To show f is a rotation about the y -axis, it suffices to show that all points on the y -axis are fixed during rotation f .

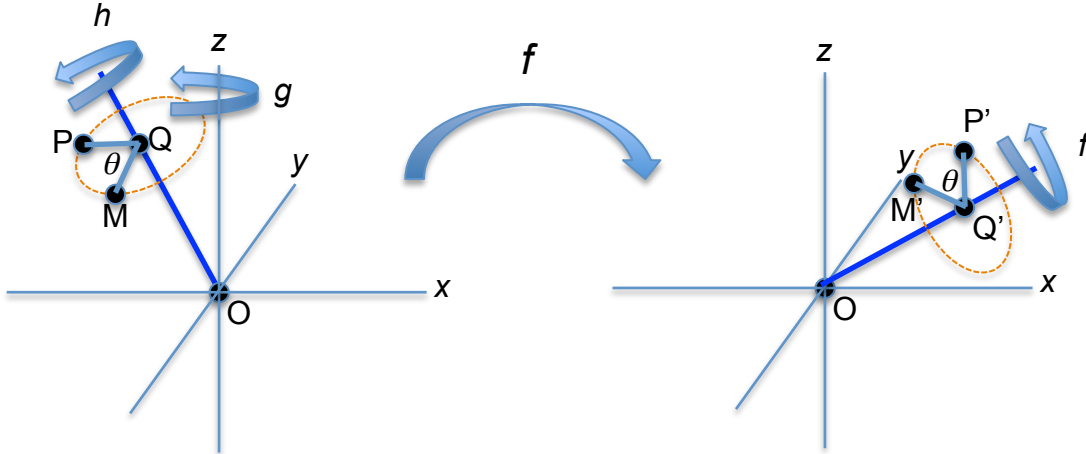
Consider the point $(0,y,0)$ on the y -axis. Since g^{-1} spins the xy -plane 90° clockwise, $(0,y,0) g^{-1} = (y,0,0)$. Since points on the x -axis are fixed during rotation h , $(y,0,0) h = (y,0,0)$. Finally, since g spins the xy -plane 90° counter-clockwise, $(y,0,0) g = (0,y,0)$. That is $(0,y,0)$ is a fixed point of the rotation f . ■

Aside: I did not see an explanation in Road To Reality of why $O(3)$ is non-Abelian. However, using the descriptions of g^{-1} and h given in Lemma 1, it is easy to show that $g^{-1}h \neq hg^{-1}$:

$$(0,1,0) g^{-1}h = (1,0,0) h = (1,0,0)$$

$$(0,1,0) h g^{-1} = (0, \cos \theta, \sin \theta) g^{-1} = (\cos \theta, 0, \sin \theta) \quad \blacksquare$$

Lemma 2: Let $g, h \in SO(3)$ and let h have rotation angle θ . Then $f = g^{-1} h g$ has the same angle of rotation θ as h (although a possibly different axis of rotation).



Proof: We use the property that rotations in \mathbb{R}^3 preserve rigid bodies. WLOG assume g is a counter-clockwise rotation about the z -axis. Let $P \in S$. Let \overrightarrow{OQ} represent the axis of revolution of h and let h rotate point P to a point M . PQM represents the rotation angle θ of h .

Set $P' = P f$, $Q' = Q f$, and $M' = M f$.

Define the rigid body B to be the union of \overrightarrow{OQ} with the angle PQM . (It looks like line segment \overrightarrow{OQ} with 2 spikes.) Rotation f moves B as a rigid body so that it becomes the union of $\overrightarrow{OQ'}$ with angle $P'Q'M'$. The angle remains θ .

We are now in a position to find the angle and axis of rotation of f . Start with a point P' of S . We know that $P' = Pf$, that h rotates P an amount θ about OQ to point M , that $M' = Mf$ and $Q' = Qf$. So we know that f rotates point P' to M' by an angle θ about axis OQ' .

(See my version 2 solution to [13.7] for a more rigorous, equation-based proof of this lemma.) ■

It was shown in problem [12.17] that $SO(3)$ is group isomorphic to the (solid) 3-ball \mathcal{R} of radius π in which antipodal points on the surface of \mathcal{R} are identified. Points of \mathcal{R} can be represented as $\theta(a, b, c) = (\theta a, \theta b, \theta c)$ where θ is the angle of rotation and (a, b, c) is a unit vector in the direction of the axis of rotation.

Definition: For $0 \leq \theta \leq \pi$ let S_θ be the sphere of radius θ in \mathcal{R} . S_θ consists of angle θ rotations about each axis of rotation.

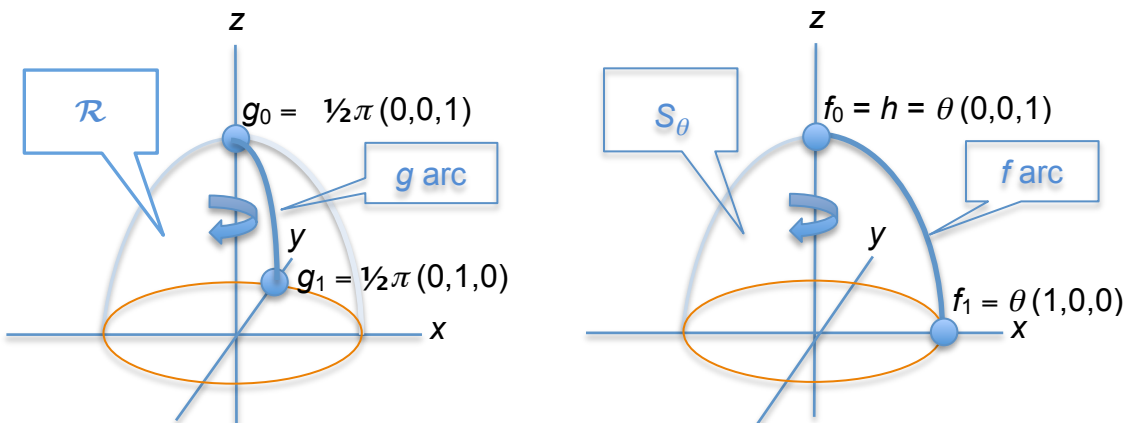
Theorem 3: In $SO(3)$, if a rotation h with rotation angle θ belongs to a normal subgroup H , then $S_\theta \subseteq H$.

Proof: Let $g \in SO(3)$. Let $F = \{ f = g^{-1} h g : g \in SO(3) \} \subseteq H$. From Lemma 2, $F \subseteq S_\theta$. Thus if $f \in F$, it has rotation angle θ and can be expressed as $f = \theta(a, b, c)$ for some (a, b, c) where $a^2 + b^2 + c^2 = 1$. To prove $F = S_\theta$, we must show for every point (a, b, c) of the unit sphere S in \mathbb{R}^3 that $\theta(a, b, c) \in F$. We do this by building up subgroup F starting from the single element h .

WLOG we can let $h = \theta(0, 0, 1)$. Consider a great circle arc from $g_0 = \frac{\pi}{2}(0, 0, 1)$

to $g_1 = \frac{\pi}{2}(0, 1, 0)$ on the surface of ball \mathcal{R} . Since g_0 has same axis of rotation

as h , then $f_0 = g_0^{-1} h g_0 = h = \theta(0, 0, 1)$. Since g_1 has an axis of rotation perpendicular to that of h , by Lemma 1 $f_1 = g_1^{-1} h g_1$ has an axis of rotation perpendicular to both h and g_1 . That is, $f_1 = \theta(1, 0, 0)$. Thus as



$g = (0, \sin \phi, \cos \phi)$ moves along the arc in \mathcal{R} from g_0 to g_1 (i.e., from $\phi = 0$ to $\phi = \frac{\pi}{2}$), $f = g^{-1} h g = (\sin \phi, 0, \cos \phi)$ moves along the great circle arc in S_θ from f_0 to f_1 .

Now sweep the g arc in a clockwise 360° circle as indicated in the figure. This sweeps out the northern hemisphere on the surface of the ball \mathcal{R} . The corresponding f arc sweeps out the northern hemisphere of S_θ . That is, for every point (a, b, c) on the northern hemisphere of the unit sphere S there is a θ rotation f on the northern hemisphere of S_θ such that $f = \theta(a, b, c) = g^{-1} h g \in F$.

For the southern hemisphere, note that $h^{-1} = \theta(0, 0, -1) \in H$ since H is a group. The g and f arcs based on h^{-1} similarly sweep out the southern hemispheres.

Thus for every point (a, b, c) on the unit sphere, $\theta(a, b, c)$ equals either $g^{-1} h g$ or $g^{-1} h^{-1} g$ for some $g \in \text{SO}(3)$, proving that $\theta(a, b, c) \in F$ and concluding the proof. (This proof is performed more rigorously using equations in my version 2 proof.) ■

Theorem 4: $\text{SO}(3)$ has no proper normal subgroup.

Proof: Let H be a non-trivial normal subgroup of $\text{SO}(3)$. $\exists 1 \neq h \in H$ having some rotation angle θ . By Theorem 3, $S_\theta \subseteq H$.

We take products of elements in S_θ to grow H beyond S_θ . Let $g, h \in S_\theta$ and $f = h g$. The maximum possible angle for f is 2θ and the minimum is 0. The maximum is obtained when $g = h$ and the minimum is obtained when $g = h^{-1}$. By letting g take a path in \mathcal{R} from h to h^{-1} we generate a path of points $f = h g$ in H having every possible angle ϕ from 0 to 2θ . By Theorem 3, $S_\phi \subseteq H$ for $0 \leq \phi \leq 2\theta$. Thus every point of \mathcal{R} in the closed ball of radius 2θ belongs to H . If $2\theta \geq \pi$, then we are done. If not, starting from sphere $S_{2\theta}$ we similarly grow H to include the closed ball of radius 4θ , then 8θ , ... Eventually we obtain that all of $\text{SO}(3) = \mathcal{R} \subseteq H$. ■

Lemma 3: Let $g \in \text{SO}(3)$. Then $g^{-1} R g = R$.

Proof: Let P be a point on the unit sphere S . Let $Q = P g^{-1}$. Then $-Q = Q R = P g^{-1} R$ and $P g^{-1} R g = (-Q) g = -(Q g) = -P = P R$. ■

Theorem B: $SO(3)$ and $\mathcal{I} = \{1, R\}$ are the only proper normal subgroups of $O(3)$.

Proof: Let H be a nontrivial normal subgroup of $O(3)$ such that $H \neq SO(3)$ and $H \neq \mathcal{I}$. We need to show that $H = O(3)$.

Claim: There is an element $t \in H \cap T$ such that $t \neq R$:

By Theorem 4, $\exists t_0 \in H \cap T$. If $t_0 \neq R$, the claim is true. If $t_0 = R$, since $H \neq \mathcal{I}$, H contains another element. If that element is in T , the claim is true. If the other element is $s_0 \in SO(3)$, we can assume $s_0 \neq 1$ since $H \neq \mathcal{I}$. Set $t = s_0 R$. Then $t \in H \cap T$ and $t \neq R$.

$t^2 \in SO(3)$. Suppose for the moment that $t^2 \neq 1$. Then $t^2 \in SO(3) \cap H$
 $\Rightarrow SO(3) \subseteq H$ by Theorem 4. Also, $\exists s \in SO(3)$ such that $t = s R$. Since $t \neq R$ then $s \neq 1$.

Claim: $T \subseteq H$:

Let $t_1 \in T$. $\exists s_1 \in SO(3)$ such that $t_1 = s_1 R$. Let $s_2 = s_1 s^{-1} \in SO(3)$. Then $s_1 = s_2 s$. Since $s_2 \in SO(3) \subset H$, $t_1 = s_1 R = s_2 s R = s_2 t \in H$. Thus $T \subseteq H$.

Since $SO(3) \subseteq H$, we have $O(3) = SO(3) \cup T \subseteq H$. Therefore $H = O(3)$.

Unfortunately if s has a 180° rotation angle, then $s^2 = 1$ and thus $t^2 = 1$, and the above argument doesn't quite hold. (Note: $t^2 = 1$ because if P is a point, then $P t^2 = P s R s R = [P s] R s R = [-P s] s R = -P s^2 R = -P R = P$). However, everything in the above argument remains true except that we haven't proved $SO(3) \subseteq H$. Once we prove this, we are done.

Let g be a 90° rotation about an axis perpendicular to the axis of s and let $s_3 = g^{-1} s g$. By Lemma 1, the rotation axis of s_3 is perpendicular to that of s . Let $t_3 = g^{-1} t g \in H$. We have

$$t_3 = g^{-1} s R g = g^{-1} s (g g^{-1}) R g = (g^{-1} s g) (g^{-1} R g) = s_3 R$$

by Lemma 3. Hence the axis of rotation of t is perpendicular to that of t_3 . (See footnote¹.) Let $s_4 = t t_3$. Because inverses have the same axis of rotation, $t_3 \neq t^{-1}$ and so $s_4 \neq 1$. Because H is a group, $s_4 \in H$. Thus, by Theorem 4, $SO(3) \subseteq H$, completing the proof. ■

¹ The axis of rotation of the reflective rotations t and t_3 can be considered to be located in the reflected unit sphere. They point in the opposite directions from the axes of rotation of s and s_3 , respectively. Thus, if the axes of s and s_3 are perpendicular, then so are the axes of t and t_3 .