[13.17] Let $T \in a(V)$; i.e. $T: V \to V$ is a lin trans on vector space V

(a) T is singular (i.e. TV is a subspace of dim $< n$) $\Longleftrightarrow \exists\ 0 \neq v \in V \ni Tv = 0$

(b) $$T = \begin{bmatrix} T^1{}_1 & T^1{}_2 & \cdots & T^1{}_n \\ T^2{}_1 & T^2{}_2 & \cdots & T^2{}_n \\ \cdots & \cdots & \cdots & \cdots \\ T^n{}_1 & T^n{}_2 & \cdots & T^n{}_n \end{bmatrix}$$

if a column is comprised of zeros, or if 2 rows or 2 columns are equal, show $\exists\ v \neq 0 \ni Tv = 0$

Pf: (Beckman)

(a) Let $\{a_k\}_{k=1}^n$ be a basis for V. (We use Einstein Summation convention below.)

"$\Rightarrow$" $\ni$ T is singular. Let $w = \sum_{k=1}^n a_k$. $Tw = \sum_{k=1}^n T a_k \in TV$

and $\dim TV < n$. Hence $\{Ta_k\}$ is linearly dependent.

So $\exists\ \{\beta^k\}_{k=1}^n$ not all zero $\ni$ $^{(1)}\ \beta^k T a_k = 0$. wlog $\beta^n \neq 0$.

So $T a_n = -\sum_{k=1}^{n-1} \frac{\beta^k}{\beta^n} a_k$ Let $v = -\frac{\beta^k}{\beta^n} a_k = -\left(\sum_{k=1}^{n-1} \frac{\beta^k}{\beta^n} a_k\right) + a_n \neq 0$

But $Tv = -\frac{1}{\beta^n} \beta^k T a_k \overset{(1)}{=} 0$

$\Leftarrow$ $\ni$ $v \neq 0$ but $Tv = 0$, $v = \alpha^i a_i$ wlog $\alpha^n \neq 0$.

So, as above, $T a_n = -\sum_{k=1}^{n-1} \frac{\beta^k}{\beta^n} T a_k$. Thus $Tv \in \langle a_1, \cdots, a_{n-1} \rangle$.

$\therefore TV \subset \langle a_1, \cdots, a_{n-1} \rangle$; i.e. T is singular

[13.17]

(b) Let $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ So $Tv = \begin{bmatrix} T^1{}_1 v_1 + T^1{}_2 v_2 + \cdots + T^1{}_n v_n \\ T^2{}_1 v_1 + T^2{}_2 v_2 + \cdots + T^2{}_n v_n \\ \cdots \\ T^n{}_1 v_1 + T^n{}_2 v_2 + \cdots + T^n{}_n v_n \end{bmatrix}$

If, for example, col 2 of T is all zeros, then

$$T\begin{bmatrix}0\\1\\0\\ \vdots \\0\end{bmatrix} = \begin{bmatrix} 0+T^1{}_2(1)+0+\cdots+0 \\ 0+T^2{}_2(1)+0+\cdots+0 \\ \vdots \\ 0+T^n{}_2(1)+0+\cdots+0\end{bmatrix} = 0 \quad ✓$$

If, for example, the 1st 2 columns of T are identical, then

$$T\begin{bmatrix}1\\-1\\0\\ \vdots \\0\end{bmatrix} = \begin{bmatrix} T^1{}_1(1)+T^1{}_1(-1)+0+\cdots+0 \\ T^2{}_1(1)+T^2{}_1(-1)+0+\cdots+0 \\ \vdots \\ T^n{}_1(1)+T^n{}_1(-1)+0+\cdots+0\end{bmatrix} = 0 \quad ✓$$

& the 1st 2 rows of T are identical. Let $v = \alpha^\lambda a_\lambda \in V$

$$Tv = T\alpha^\lambda a_\lambda = \begin{bmatrix} T^1{}_1\alpha^1 a_1 + \cdots + T^1{}_n \alpha^n a_n \\ T^1{}_1\alpha^1 a_1 + \cdots + T^1{}_n \alpha^n a_n \\ T^3{}_1\alpha^1 a_1 + \cdots + T^3{}_n \alpha^n a_n \\ \cdots \\ T^n{}_1\alpha^1 a_1 + \cdots + T^n{}_n \alpha^n a_n \end{bmatrix} = \begin{bmatrix} w_1 \\ w_1 \\ w_3 \\ \vdots \\ w_n \end{bmatrix}$$

$\therefore\ Tv \in \langle w_1, w_3, w_4, \dots, w_n\rangle \ \forall v$ i.e. $TV \in \langle w_1, w_3, \dots, w_n\rangle$

$\Rightarrow \dim TV < n \Rightarrow T$ is singular $\overset{(a)}{\Rightarrow} \exists\, 0 \ne v \in V \ni Tv = 0$ ✓