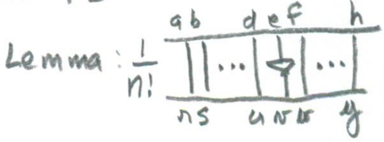
[13.24]

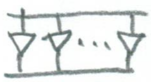
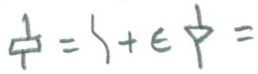
Lemma:  $= \epsilon^{\pi(a)\cdots\pi(h)}\varepsilon_{\pi(a)\cdots\pi(h)}T^{\pi(e)}{}_{\pi(e)} = \epsilon^{a\cdots h}\varepsilon_{a\cdots h}T^e{}_e$

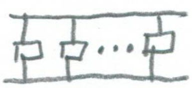
Pf:  $= \epsilon^{rs\cdots uvw\cdots y}\delta^a_r\delta^b_s\cdots\delta^d_u T^e{}_v\delta^f_w\cdots\delta^h_y\varepsilon_{ab\cdots def\cdots h}$

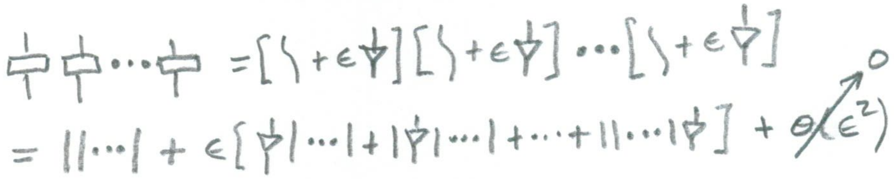
There are $n!$ permutations of $(r,\dots,y)$ times $n!$ permutations of $(a,\dots,h)$
Fix one permutation $(a,\dots,h)$. among permutations of $(r,\dots,y)$, the only non-zero term is $\epsilon^{ab\cdots def\cdots h}\delta^a_a\delta^b_b\cdots\delta^d_d T^e{}_e\delta^f_f\cdots\delta^h_h\varepsilon_{ab\cdots def\cdots h}$
$= \epsilon^{a\cdots h}\varepsilon_{a\cdots h}T^e{}_e$. When we allow all the permutations of $(a,\dots,h)$, using the summation convention we have

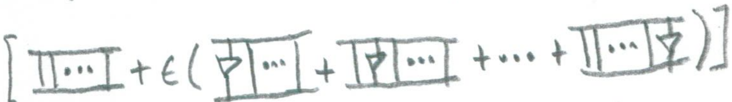
$$\epsilon^{r\cdots y}\delta^a_r\delta^b_s\cdots\delta^d_u T^e{}_v\delta^f_w\cdots\delta^h_y\varepsilon_{a\cdots h}$$
$$= \sum_\pi \epsilon^{\pi(a)\cdots\pi(h)}\varepsilon_{\pi(a)\cdots\pi(h)}T^{\pi(e)}{}_{\pi(e)}$$ ☐

[13.24] Show $\det(I+\epsilon A) = 1+\epsilon\,\mathrm{Tr}(A)$ for infinitesimal ϵ

Pf: $\det A = \frac{1}{n!}$ . Set  $= I+\epsilon A$

$\det(I+\epsilon A) = \frac{1}{n!}$ . Now



$\therefore \det(I+\epsilon A) = \frac{1}{n!}$ 

$\overset{\text{Lemma}}{=} \frac{1}{n!}\left[n! + \epsilon\sum_\pi \epsilon^{\pi(a)\cdots\pi(h)}\varepsilon_{\pi(a)\cdots\pi(h)}\left[T^{\pi(a)}{}_{\pi(a)}+\cdots+T^{\pi(h)}{}_{\pi(h)}\right]\right]$

$= 1 + \frac{\epsilon}{n!}\sum_\pi \epsilon^{a\cdots h}\varepsilon_{a\cdots h}\left[T^a{}_a+\cdots+T^h{}_h\right] = 1+\frac{\epsilon}{n!}(n!)\,\mathrm{Tr}(A) = 1+\epsilon\,\mathrm{Tr}(A)$ ☐

Note: Doing the proof using tensors:

$\det(I+\epsilon A) = \frac{1}{n!}\epsilon^{r\cdots y}\left[(\delta^a_r+\epsilon T^a{}_r)\cdots(\delta^h_y+\epsilon T^h{}_y)\right]\varepsilon_{a\cdots h}$

$= \frac{1}{n!}\epsilon^{r\cdots y}\left[\delta^a_r\cdots\delta^h_y + \epsilon(T^a{}_r\delta^b_s\cdots\delta^h_y + \delta^a_r T^b{}_s\delta^c_t\cdots\delta^h_y + \cdots + \delta^a_r\cdots\delta^g_x T^h{}_y)\right]\varepsilon_{a\cdots h}$

$= \frac{1}{n!}(n!)\left[1+\epsilon(T^a{}_a+T^b{}_b+\cdots+T^h{}_h)\right]$

$= 1+\epsilon\,\mathrm{Tr}(A)$ [This uses the lemma to set $\epsilon^{r\cdots y}\varepsilon_{a\cdots h} = n!$ considering all permutations of $(a,\dots,h)$ and $(r,\dots,y)$, and also to select $T^a{}_a, T^b{}_b$, etc.]