[12.8] Let $\alpha = \sum_{r=1}^{n} \alpha_r \mathrm{d}x^r, \cdots, \gamma = \sum_{u=1}^{n} \gamma_u \mathrm{d}x^u, \lambda = \sum_{j=1}^{n} \lambda_j \mathrm{d}x^j, \cdots, \upsilon = \sum_{m=1}^{n} \upsilon_m \mathrm{d}x^m$ be independent 1-forms in $\mathbb{R}^n$. Let $\phi = \alpha \wedge \cdots \wedge \gamma$ be a simple *p*-form and $\chi = \lambda \wedge \cdots \wedge \upsilon$ be a simple *q*-form. Show that $\phi \wedge \chi = \alpha \wedge \cdots \wedge \gamma \wedge \lambda \wedge \cdots \wedge \upsilon$.

**Proof**: Juergen Beckmann's proof uses the antisymmetrized-coefficients expression for the wedge product and is well presented and very insightful. This problem can also be solved using the un-antisymmetrized-coefficients expression.

Let M = {1, 2, … , *n*}, $\mathcal{P}_{r\ldots u}$ be the set of permutations of the *p*-tuple (*r*, … , u) and $\mathcal{P}_{j\ldots m}$ the set of permutations of the *q*-tuple (*j*, … , *m*). As Juergen reminds us in his proof, the un-antisymmetrized expressions for $\phi$ and $\chi$ are

$$\phi = \alpha \wedge \cdots \wedge \gamma = \sum_{(r,\cdots,u)\in \mathrm{M}^p} \alpha_{[r} \cdots \gamma_{u]} \, \mathrm{d}x^r \wedge \cdots \wedge \mathrm{d}x^u \qquad (1)$$

where

$$\alpha_{[r} \cdots \gamma_{u]} = \frac{1}{p!} \sum_{\pi \in \mathcal{P}_{r\cdots u}} sign(\pi)\, \alpha_{\pi(r)} \cdots \gamma_{\pi(u)}$$

and

$$\chi = \lambda \wedge \cdots \wedge \upsilon = \sum_{(j,\cdots,m)\in \mathrm{M}^q} \lambda_{[j} \cdots \upsilon_{m]} \, \mathrm{d}x^j \wedge \cdots \wedge \mathrm{d}x^m \qquad (2)$$

where

$$\lambda_{[j} \cdots \upsilon_{m]} = \frac{1}{q!} \sum_{\pi \in \mathcal{P}_{j\cdots m}} sign(\pi)\, \lambda_{\pi(j)} \cdots \upsilon_{\pi(m)}.$$

The un-antisymmetrized expression for $\phi \wedge \chi$ is

$$\begin{aligned}\phi \wedge \chi &= \sum_{(r,\cdots,u)\in \mathrm{M}^p} \sum_{(j,\cdots,m)\in \mathrm{M}^q} \left(\alpha_r \cdots \gamma_u\right)\left(\lambda_j \cdots \upsilon_m\right)\left(\mathrm{d}x^r \wedge \cdots \wedge \mathrm{d}x^u\right) \wedge \left(\mathrm{d}x^j \wedge \cdots \wedge \mathrm{d}x^m\right) \\ &= \sum_{(r,\cdots,u)\in \mathrm{M}^p} \sum_{(j,\cdots,m)\in \mathrm{M}^q} \alpha_r \cdots \gamma_u \, \lambda_j \cdots \upsilon_m \, \mathrm{d}x^r \wedge \cdots \wedge \mathrm{d}x^u \wedge \mathrm{d}x^j \wedge \cdots \wedge \mathrm{d}x^m.\end{aligned} \qquad (3)$$

The un-antisymmetrized expression for $\alpha \wedge \cdots \wedge \gamma \wedge \lambda \wedge \cdots \wedge \upsilon$ is

$$\alpha \wedge \cdots \wedge \gamma \wedge \lambda \wedge \cdots \wedge \upsilon = \sum_{(r,\cdots,u,j,\cdots,m)\in \mathrm{M}^{p+q}} \alpha_r \cdots \gamma_u \, \lambda_j \cdots \upsilon_m \, \mathrm{d}x^r \wedge \cdots \wedge \mathrm{d}x^u \wedge \mathrm{d}x^j \wedge \cdots \wedge \mathrm{d}x^m. \quad (4)$$

Both expressions (3) and (4) have $n^{p+q}$ terms, and all terms are identical. Therefore

$\phi\wedge\chi=\alpha\wedge\cdots\wedge\gamma\wedge\lambda\wedge\cdots\wedge\upsilon$ , which finishes the proof. ■

**Note:** Juergen's proof has several nice features and he clarifies issues only hinted at by Penrose.

First, he introduces an elegant notation to represent the antisymmetrization of two antisymmetrizations:

$$\phi\wedge\chi=\sum_{(r,\cdots,u,j,\cdots,m)\in\mathrm{M}^{p+q}}\alpha_{[[r}\cdots\gamma_{u]}\,\lambda_{[j}\cdots\upsilon_{m]}\,\mathrm{d}x^r\wedge\cdots\wedge\mathrm{d}x^u\wedge\mathrm{d}x^j\wedge\cdots\wedge\mathrm{d}x^m\ . \quad (5)$$

I, for one, had a hard time figuring out where to put the outside brackets in order to express the outer antisymmetrization.

He also clarified the meaning of the RHS of (5). I considered expanding the RHS using a 2! permutation because I thought that the outer antisymmetrization was operating on the 2 quantities $\alpha_{[r}\cdots\delta_{u]}$ and $\lambda_{[r}\cdots\upsilon_{u]}$, but that was incorrect. Juergen's explanation uses a permutation expression involving ($p$ + $q$)! and I believe it translates to

$$\phi\wedge\chi$$
$$=\sum_{(r,\cdots,u,j,\cdots,m)\in\mathrm{M}^{p+q}}\frac{1}{(p+q)!}\sum_{\pi\in\mathcal{P}_{r\cdots u\,j\cdots m}}\mathrm{sign}(\pi)\alpha_{\pi[r}\cdots\gamma_{u]}\,\lambda_{\pi[j}\cdots\upsilon_{m]}\,\mathrm{d}x^r\wedge\cdots\wedge\mathrm{d}x^u\wedge\mathrm{d}x^j\wedge\cdots\wedge\mathrm{d}x^m.$$

Next he expresses the inner antisymmetrizations in terms of permutations:

$$\begin{aligned}\phi\wedge\chi=&\sum_{(r,\cdots,u,j,\cdots,m)\in\mathrm{M}^{p+q}}\frac{1}{(p+q)!}\sum_{\pi\in\mathcal{P}_{r\cdots u\,j\cdots m}}\mathrm{sign}(\pi)\\ &\frac{1}{p!}\sum_{\pi_1\in\mathcal{P}_{r\cdots u}}\mathrm{sign}(\pi_1)\alpha_{\pi(\pi_1(r)}\cdots\gamma_{\pi_1(u))}\,\mathrm{d}x^r\wedge\cdots\wedge\mathrm{d}x^u\\ &\frac{1}{q!}\sum_{\pi_2\in\mathcal{P}_{j\cdots m}}\mathrm{sign}(\pi_2)\lambda_{\pi(\pi_2(j)}\cdots\upsilon_{\pi_2(m))}\wedge\mathrm{d}x^j\wedge\cdots\wedge\mathrm{d}x^m\end{aligned}$$

$$\begin{aligned}=&\sum_{(r,\cdots,u,j,\cdots,m)\in\mathrm{M}^{p+q}}\frac{1}{(p+q)!}\frac{1}{p!}\frac{1}{q!}\sum_{\pi\in\mathcal{P}_{r\cdots u\,j\cdots m}}\sum_{\pi_1\in\mathcal{P}_{r\cdots u}}\sum_{\pi_2\in\mathcal{P}_{j\cdots m}}\mathrm{sign}(\pi)\mathrm{sign}(\pi_1)\mathrm{sign}(\pi_2)\\ &\alpha_{\pi\circ\pi_1(r)}\cdots\gamma_{\pi\circ\pi_1(u))}\,\lambda_{\pi\circ\pi_2(j)}\cdots\upsilon_{\pi\circ\pi_2(m))}\,\mathrm{d}x^r\wedge\cdots\wedge\mathrm{d}x^u\wedge\mathrm{d}x^j\wedge\cdots\wedge\mathrm{d}x^m.\end{aligned}$$

Juergen then finished his proof with some clever re-writing of the permutations to eliminate the two inner antisymmetrizations (i.e., the 2nd and 3rd summation signs), leading to the (desired) expression for $\alpha\wedge\cdots\wedge\gamma\wedge\lambda\wedge\cdots\wedge\upsilon$. My only purpose here was to attempt to translate his notation without his reference to a generic antisymmetric operator.

We provide two examples, one illustrating the permutation notation and the other illustrating the [12.8] problem statement.

**Example: Permutation Notation**

Let $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2$ and $\beta = \beta_1 dx^1 + \beta_2 dx^2$ be 1-forms in $\mathbb{R}^2$. Then $n = p = 2$, M = {1, 2}, and $M^p = \{1,2\}^2 = \{1,2\} x \{1,2\} = \{(1,1),(1,2),(2,1),(2,2)\}$. So $\mathcal{P}_{12} = \{\pi_1, \pi_2\}$ where

$$\pi_1 : \begin{cases} 1 \to 1 \\ 2 \to 2 \end{cases} \quad , \quad \pi_2 : \begin{cases} 1 \to 2 \\ 2 \to 1 \end{cases} \quad , \quad sign(\pi_1) = +1, \text{ and } sign(\pi_2) = -1.$$

So, for example,

$$\alpha_{[1}\beta_{2]} = \frac{1}{2}(\alpha_1\beta_2 - \alpha_2\beta_1) = \frac{1}{2!}\left(sign(\pi_1)\alpha_{\pi_1(1)}\beta_{\pi_1(2)} + sign(\pi_2)\alpha_{\pi_2(1)}\beta_{\pi_2(2)}\right)$$
$$= \frac{1}{p!}\sum_{\pi \in \mathcal{P}_{12}} sign(\pi)\alpha_{\pi(1)\pi(2)}. \quad \blacksquare$$

**Example: $\boldsymbol{\alpha} \wedge (\boldsymbol{\lambda} \wedge \boldsymbol{\upsilon})$**

Let $\alpha = \sum_{r=1}^{n} \alpha_r dx^r$, $\lambda = \sum_{s=1}^{n} \lambda_s dx^s$, and $\upsilon = \sum_{t=1}^{n} \upsilon_t dx^t$. Set $\sigma = \lambda \wedge \upsilon = \sum_{s,t} \sigma_{st} dx^s dx^t$ where

$\sigma_{st} = \frac{1}{2}\sum_{\pi_1 \in \mathcal{P}_{st}} \text{sign}(\pi_1)\lambda_{\pi_1(s)}\upsilon_{\pi_1(t)} = \frac{1}{2}(\lambda_s\upsilon_t - \lambda_t\upsilon_s)$. So $\sigma = \sum_{s,t}\frac{1}{2}(\lambda_s\upsilon_t - \lambda_t\upsilon_s) dx^s \wedge dx^t$.

Next, let $\tau = \alpha \wedge \sigma = \alpha \wedge (\lambda \wedge \upsilon) = \sum_{r,s,t} \tau_{rst}\, dx^r \wedge (dx^s \wedge dx^t) = \sum_{r,s,t} \tau_{rst}\, dx^r \wedge dx^s \wedge dx^t$ where

$$\tau_{rst} = \alpha_{[r}\sigma_{st]} = \frac{1}{3!}\sum_{\pi \in \mathcal{P}_{rst}} \text{sign}(\pi)\alpha_{\pi(r)}\sigma_{\pi(st)} = \frac{1}{3!}\sum_{\pi \in \mathcal{P}_{rst}} \text{sign}(\pi)\alpha_{\pi(r)}\lambda_{[\pi(s)}\upsilon_{\pi(t)]}$$
$$= \frac{1}{3!}\sum_{\pi \in \mathcal{P}_{rst}} \text{sign}(\pi)\alpha_{\pi(r)}\frac{1}{2!}\sum_{\pi_1 \in \mathcal{P}_{st}} \text{sign}(\pi_1)\lambda_{\pi\circ\pi_1(s)}\upsilon_{\pi\circ\pi_1(t)}$$
$$= \frac{1}{3!}\frac{1}{2!}\sum_{\pi \in \mathcal{P}_{rst}}\sum_{\pi_1 \in \mathcal{P}_{st}} \text{sign}(\pi)\text{sign}(\pi_1)\alpha_{\pi(r)}\lambda_{\pi\circ\pi_1(s)}\upsilon_{\pi\circ\pi_1(t)} \ .$$