

[13.24]

Lemma: $\frac{1}{n!} \begin{array}{|c|c|c|c|} \hline a & b & d & e & f & h \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline n & s & u & v & w & y \\ \hline \end{array} = \epsilon^{\pi(a)\dots\pi(h)} \epsilon_{\pi(a)\dots\pi(h)} T^{\pi(e)}_{\pi(e)} = \epsilon^{a\dots h} \epsilon_{a\dots h} T^e_e$

Pf: $\begin{array}{|c|c|c|c|} \hline \vdots & \vdots & \vdots & \vdots \\ \hline \end{array} = \epsilon^{n s \dots u v \dots w y} \delta^a_n \delta^b_s \dots \delta^d_u T^e_{\dots} \delta^f_w \dots \delta^h_y \epsilon_{a b \dots d e \dots f h}$

There are $n!$ permutations of (n, \dots, y) times $n!$ permutations of (a, \dots, h)

Fix one permutation (a, \dots, h) . Among permutations of (n, \dots, y) , the only non-zero term is $\epsilon^{a b \dots d e \dots f h} \delta^a_n \delta^b_s \dots \delta^d_u T^e_{\dots} \delta^f_w \dots \delta^h_y \epsilon_{a b \dots d e \dots f h} = \epsilon^{a\dots h} \epsilon_{a\dots h} T^e_e$. When we allow all the permutations of (a, \dots, h) , using the summation convention we have

$$\epsilon^{n\dots y} \delta^a_n \delta^b_s \dots \delta^d_u T^e_{\dots} \delta^f_w \dots \delta^h_y \epsilon_{a\dots h} = \sum_{\pi} \epsilon^{\pi(a)\dots\pi(h)} \epsilon_{\pi(a)\dots\pi(h)} T^{\pi(e)}_{\pi(e)} = \sum_{\pi} T^{\pi(e)}_{\pi(e)}$$

□

[13.24] Show $\det(I + \epsilon A) = 1 + \epsilon \text{Tr}(A)$ for infinitesimal ϵ

Pf: $\det A = \frac{1}{n!} \begin{array}{|c|c|c|c|} \hline \vdots & \vdots & \vdots & \vdots \\ \hline \end{array}$. Set $\phi = I + \epsilon A$

$\det(I + \epsilon A) = \frac{1}{n!} \begin{array}{|c|c|c|c|} \hline \phi & \phi & \dots & \phi \\ \hline \end{array}$. Now

$$\begin{aligned} \phi \phi \dots \phi &= [I + \epsilon A][I + \epsilon A] \dots [I + \epsilon A] \\ &= |I| + \epsilon [A|I| + |A|I| + \dots + |I|A|] + O(\epsilon^2) \end{aligned}$$

$$\therefore \det(I + \epsilon A) = \frac{1}{n!} \left[\begin{array}{|c|c|c|c|} \hline I & I & \dots & I \\ \hline \end{array} + \epsilon \left(\begin{array}{|c|c|c|c|} \hline A & I & \dots & I \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline I & A & \dots & I \\ \hline \end{array} + \dots + \begin{array}{|c|c|c|c|} \hline I & I & \dots & A \\ \hline \end{array} \right) \right]$$

$$\begin{aligned} &= \frac{1}{n!} \left[n! + \epsilon \sum_{\pi} \epsilon^{\pi(a)\dots\pi(h)} \epsilon_{\pi(a)\dots\pi(h)} [T^{\pi(a)}_{\pi(a)} + \dots + T^{\pi(h)}_{\pi(h)}] \right] \\ &= 1 + \frac{\epsilon}{n!} \left(\sum_{\pi} \epsilon^{a\dots h} \epsilon_{a\dots h} [T^a_a + \dots + T^h_h] \right) = 1 + \frac{\epsilon}{n!} (n!) \text{Tr}(A) = 1 + \epsilon \text{Tr}(A) \end{aligned}$$

□

Note: Doing the proof using tensors:

$$\begin{aligned} \det(I + \epsilon A) &= \frac{1}{n!} \epsilon^{a\dots h} [\delta^a_n + \epsilon T^a_n] \dots [\delta^h_y + \epsilon T^h_y] \epsilon_{a\dots h} \\ &= \frac{1}{n!} \epsilon^{a\dots h} [\delta^a_n \dots \delta^h_y + \epsilon (T^a_n \delta^b_s \dots \delta^h_y + \delta^a_n T^b_s \delta^c_t \dots \delta^h_y + \dots + \delta^a_n \dots \delta^g_x T^h_y)] \epsilon_{a\dots h} \\ &= \frac{1}{n!} (n!) [1 + \epsilon (T^a_a + T^b_b + \dots + T^h_h)] \\ &= 1 + \epsilon \text{Tr}(A) \quad [\text{This uses the lemma to set } \epsilon^{a\dots h} \epsilon_{a\dots h} = n! \text{ considering all permutations of } (a, \dots, h) \text{ and } (n, \dots, y), \text{ and also to select } T^a_a, T^b_b, \text{ etc.}] \end{aligned}$$