[9.6] Show $g(p) = \frac{1}{\sqrt{2\pi}} \int_{p=-\infty}^{\infty} f(x) e^{-pxi} dx$

Givens: $z = e^{\frac{x}{N}i}$

$\alpha_n = \frac{1}{2\pi i} \oint_C \frac{F(z)}{z^{n+1}} dz$ where C is unit complex circle

$F(z) = \sqrt{2\pi}\, f(x)$, where $\sqrt{2\pi}$ is scaling factor introduced in middle of p. 166 (§9.4)

as period $l = 2\pi N \to \infty$, n is replaced by $\frac{n}{N} \to p$ and $\alpha_n \to g(p)$.

Proof: as z moves CCW on unit circle from $z=-1$:

z	-1	$-i$	0	i	-1
x	$-N\pi$	$-\frac{1}{2}N\pi$	0	$\frac{1}{2}N\pi$	$N\pi$

Also $dz = \frac{i}{N} e^{\frac{x}{N}i} dx$. So,

$$\alpha_n = \frac{1}{2\pi i} \int_{x=-N\pi}^{N\pi} \frac{f(x)\sqrt{2\pi}}{e^{(n+1)\frac{x}{N}i}} \frac{i}{N} e^{\frac{x}{N}i} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x=-N\pi}^{N\pi} \frac{f(x)}{e^{\frac{n}{N}xi}} dx$$

So $g(p) = \lim\limits_{N\to\infty} \alpha_n = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{\infty} e^{-pxi} f(x) dx$ □

Overview of derivation of Fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(p) e^{ipx} dp$ on p. 166

Fix N and set $\sqrt{2\pi}\, f_N(x) = F(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^n = \sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{n}{N}xi}$

$f(x) = \lim\limits_{N\to\infty} f_N(x) = \frac{1}{\sqrt{2\pi}} \int_{p=-\infty}^{\infty} g(p) e^{pxi} dp$ (uses "Givens" from above)

□