[12.17] Represent a rotation in ordinary 3-space as a vector pointing along the rotation axis of length equal to the angle of rotation. As in Euclidean Space, every vector can be represented as a point. Let $\mathcal{R}$ = { (u, v, w) } be the Rotation Space where each point (u, v, w) represents a rotation. Show that the topology of $\mathcal{R}$ can be described (a) as a solid ball (of radius π) bounded by an ordinary sphere and (b) that each point of the sphere is identified with its antipodal point. (c) Give a direct argument to show why a closed loop representing a 2π-rotation cannot be continuously deformed to a point.

Proof: Penrose defines the Configuration Space $\mathcal{C}$ as the 6-dimensional space $\mathbb{R}^3$ x $\mathcal{R}$, where $\mathbb{R}^3 = \{ (x, y, z): x, y, z \in \mathbb{R} \}$ is Euclidean 3-space.

(a) In Figure 1 (taken from a John Denker paper) we can visualize vectors in $\mathcal{R}$ as rotations of an airplane (roll, pitch, yaw, or combinations). Each rotation vector in $\mathcal{R}$ has magnitude (representing the angle of the rotation, between $-\pi$ and π) and direction (namely, the axis of axis of rotation). Considered as points, members of $\mathcal{R}$ can have any direction in $\mathbb{R}^3$ and any magnitude from 0 to π. Thus $\mathcal{R}$ is a solid 3D ball of radius π.

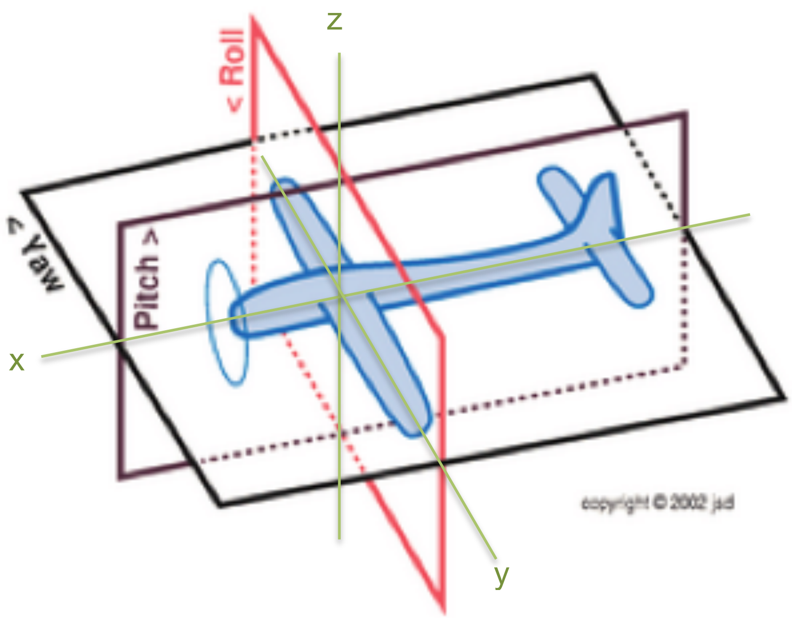

Figure 1

(b) To see that antipodal points of $\mathcal{R}$ are identified, imagine an airplane heading in the x-direction in the xy-plane. A pitch (rotation in the xz-plane) of either π or $-\pi$ results in an upside-down airplane lying in the xy-plane and heading in the $-x$ direction. Thus the points $(0, \pi, 0)$ and $(0, -\pi, 0)$ of $\mathcal{R}$ are identified. Given any other direction in $\mathbb{R}^3$, a simple rotation of axes to point the airplane in the new direction allows this argument to identify the endpoints of that direction. Thus any pair of antipodal points is identified.

(c) Figure 2 represents the *uv*-cross section of the solid 3-ball $\mathcal{R}$. The *w*-axis points up out of the page. O is the origin (0, 0, 0). Observe that the line segment $\overline{\mathrm{POQ}}$ in the ball is a loop in $\mathcal{R}$ because P is identified with Q.

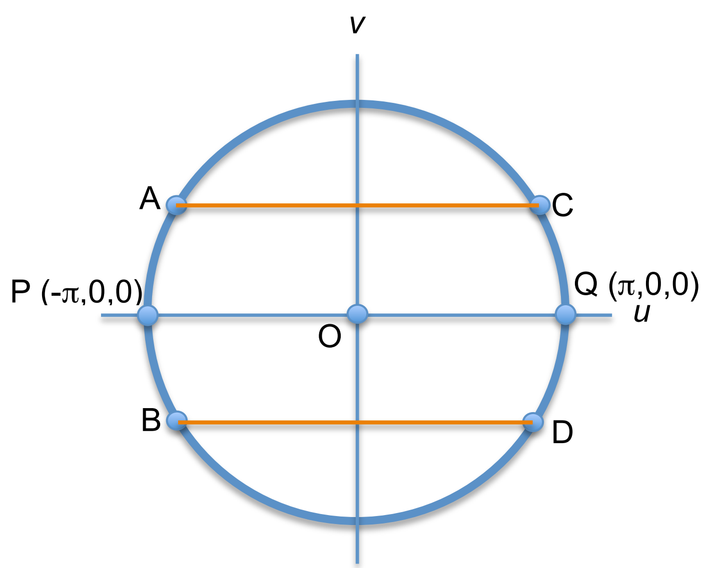

Figure 2

Penrose states in this problem that $\overline{\mathrm{POQ}}$ represents a 2π-rotation. To understand this, observe that P and Q represent rotations. $\overline{\mathrm{POQ}}$ is a loop in $\mathcal{R}$ where every point of the loop is a rotation. As the trajectory moves through the loop from P to O to Q, the rotations smoothly change from $-\pi$ at P through $-\pi/2$ to zero at O through $\pi/2$ to π at Q = P. So $\overline{\mathrm{POQ}}$ represents a 2π–rotation of the rotations.

The planar region ABCD bounded by the lines $\overline{AC}$, $\overline{BD}$, and the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$, is a Mobius strip because A = D and B = C (antipodal points). It is well-known that the center line $\overline{POQ}$ of a Mobius strip is not homotopic (i.e., cannot be continuously deformed) to a point, which completes this problem.