[13.10] Let G and H be groups.

(a) Verify that G×H is a group where G×H is the set { (g, h) } with the operation $(g_1,h_1)(g_2,h_2)=(g_1g_2,h_1h_2)$

(b) Show that we can identify H with (G×H) /G

(a) (1,1) is the identity since (1,1) (g,h) = (g,h) = (g, h) (1,1) ✔

The inverse of (g,h) is (g^{-1},h^{-1}) since $(g,h)(g^{-1},h^{-1})=(1,1)=(g^{-1},h^{-1})(g,h)$ ✔

The associative law holds:

$$\left[(g_1,h_1)(g_2,h_2)\right](g_3,h_3)=\left((g_1g_2,h_1h_2)\right)(g_3,h_3)=\left((g_1g_2)g_3,(h_1h_2)h_3\right)$$
$$=\left(g_1(g_2g_3),h_1(h_2h_3)\right)=(g_1,h_1)(g_2g_3,h_2h_3)=(g_1,h_1)\left[(g_2,h_2)(g_3,h_3)\right] ✔$$

Therefore G×H is a group. ✔

(b) Recall that (G×H) /G = {G(g,h): (g,h)∈G×H }.

There are 2 preliminaries to cover.

(i) G must be a subgroup of G×H. It isn't, but it can be identified with $G^*=\left\{(g,1): g\in \mathrm{G}, 1\in \mathrm{H}\right\}$, which *is* a subgroup.

(ii) (G×H) /G* is a group only if G* is normal in G×H. In order to show that G* is normal we must show that $(g,h)\,\mathrm{G}^*\,(g^{-1},h^{-1})=\mathrm{G}^*\ \ \forall (g,h)\in \mathrm{G}\times\mathrm{H}$.

Fix (g,h) ∈ G×H. For any g_1 ∈ G,

$(g,h)(g_1,1)(g^{-1},h^{-1})=(g\,g_1\,g^{-1},1)\in \mathrm{G}^* \ \Rightarrow\ (g,h)\,\mathrm{G}^*\,(g^{-1},h^{-1})\subseteq \mathrm{G}^*$.

$(g,1) = (1,1)(g,1)(1^{-1},1^{-1}) \ \Rightarrow\ \mathrm{G}^* \subseteq (g,h)\mathrm{G}^*(g^{-1},h^{-1})$.

Therefore $\mathrm{G}^* = (g,h)\mathrm{G}^*(g^{-1},h^{-1})$ ✔

We now show that H can be identified with (G×H) /G* by providing an isomorphism. Define

p : H → (G×H) /G* : $p(h)$ = G*(1,h).

We show that p is a group homomorphism that is 1-1 and onto.

Homomorphism:

Since GG = G, then G* G* = G*, so

$p(h_1h_2)$ = G*(1, h_1h_2) = G* G* (1, h_1) (1, h_2) = G*(1, h_1) G*(1, h_2)

= $p(h_1)\,p(h_2)$. ✔

Onto

Let G*(1,h) ∈ (G×H) /G*. Then $p(h)$ = G*(1,h). ✔

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If $h_1 \neq h_2$ then G*(1, h_1) = {(g, h_1) : g∈G} ≠ {(g, h_2) : g∈G} = G*(1, h_2) ■