[13.12] Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Unlike Beckmann, I interpret this problem using the definition of linear transformation given on p. 255: $T\left[\alpha\left(x_1,y_1\right)+\beta\left(x_2,y_2\right)\right]=\alpha T\left(x_1,y_1\right)+\beta T\left(x_2,y_2\right)$. This makes for a much easier problem.

Show $\exists\, \alpha, \beta \in \mathbb{R} \ni T\left(x,y\right)=\left(\alpha_{1x}+\beta_{1y},\alpha_{2x}+\beta_{2y}\right)$. That is $\left\{\begin{array}{l} x \to \alpha_1 x+\beta_1 y \\ y \to \alpha_2 x+\beta_2 y \end{array}\right.$.

Solution: Define $\left(\alpha_1,\alpha_2\right)=T\left(1,0\right)$ and $\left(\beta_1,\beta_2\right)=T\left(0,1\right)$. Then

$$
\begin{aligned}
T\left(x,y\right) &= xT\left(1,0\right)+yT\left(0,1\right)=x\left(\alpha_1,\alpha_2\right)+y\left(\beta_1,\beta_2\right) \\
&=\left(\alpha_1 x+\beta_1 y,\alpha_2 x+\beta_2 y\right). \qquad \blacksquare
\end{aligned}
$$