[12.4] Problem Statement: Let Φ be a scalar field on an n-manifold $\mathcal{M}$. Show that the (n–1) -dimensional plane elements determined by dΦ are tangential to the family of (n–1)-dimensional surfaces of constant Φ.

There were no posted solutions for this problem so I'll make an attempt.

Solution:

It suffices to show this for every point of the manifold.

Let P be a point of $\mathcal{M}$.
Let X_P be the (n–1)-dimensional plane element at P determined by dΦ.
Let ξ be a vector field.

By Penrose's definition of the (n–1)-dimensional plane element at P determined by dΦ, the direction of ξ belongs to X_P iff $0 = \mathrm{d}\Phi \bullet \xi$. But $0 = \mathrm{d}\Phi \bullet \xi = \xi(\Phi)$, the rate of change of Φ in the direction of ξ.

This implies that Φ is constant on X_P

Thus X_P is tangential to the family of (n–1)-dimensional surfaces of constant Φ.✔