Chapter 13. Symmetry Groups

Groups

Definitions:
A **group** is a set G with an operation $\circ$ that is closed and associative, has an identity e, and every element g has an inverse g^{-1} such that $g \circ g^{-1} = e = g^{-1} \circ g$.

A group G is **Abelian** if it is commutative: $g \circ h = h \circ g$ for all g, h in G.

A **subgroup** is a subset of G that is a group under $\circ$.

Let H be a subgroup of G. A **coset of H** is a set $H \circ g = \{h \circ g : h \in H\}$, where $g \in G$. The only coset of H that is a group is the set H itself: $H = H \circ e$ where e is the identity element. The cosets of H form a partition of G.

A **normal subgroup** is a subgroup H that satisfies $g \circ H = H \circ g$ for all g in G, or equivalently $H = g^{-1} \circ H \circ g$.

A group is **simple** if it contains no non-trivial normal subgroup. The simple groups are the fundamental "building blocks" of more complex groups.

Theorem. There are precisely 4 **classical** and 5 **exceptional** simple Lie groups.

- Classical Families: A_m, B_m, C_m, D_m having dimensions $m(m+2)$, $m(2m+1)$, $m(2m+1)$, and $m(2m-1)$, respectively where $m \in \mathbb{Z}^+$.
- Exceptional Groups: E_6, E_7, E_8, F_4, G_2 of dimension 78, 133, 248, 52, and 14 respectively

Theorem. The simple finite groups have been classified into classical and exceptional groups. The largest exceptional group has $\approx 10^{60}$ elements and is known as **the monster.**

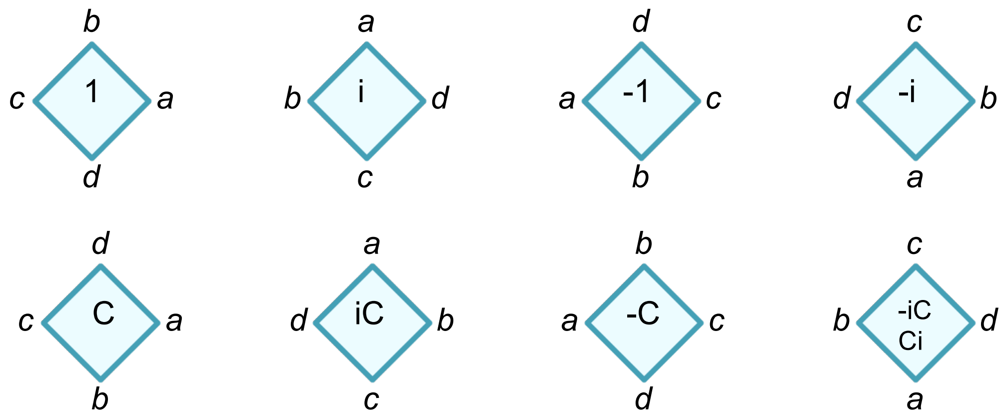
Definition. The **Product Group** of groups G and H is **GxH** = $\{(g,h) : g \in G, h \in H\}$ with group operation $(g_1,h_1) \circ (g_2,h_2) = (g_1 \circ g_2, h_1 \circ h_2)$.

Definition. Let N be a subgroup of G. The **Factor Space G/N** is the collection of cosets $N \circ g$ along with the operation $(N \circ g_1) \circ (N \circ g_2) = N \circ (g_1 \circ g_2)$.

Theorem. If N is normal then G/N is a group, called the **Factor Group.**

Theorem. [13.10] $H \cong (G \times H) / G$.

Symmetries of a Square



Definitions**:**
Non-reflecting Group: < i > = {1, i, -1, -i}
Reflecting Group: < i, C > = {1, i, -1, -i, C, iC, -C, -iC = Ci}
C is complex conjugation: $a+bi \mapsto a-bi$. **1** is the null rotation, which is the group identity element. ***i*** is the 90° counter-clockwise rotation of the square

Convention: ab means b acts first.

A subgroup of a symmetry group is called a **reduced symmetry group**.

Examples:

Normal subgroups of < i, C > :

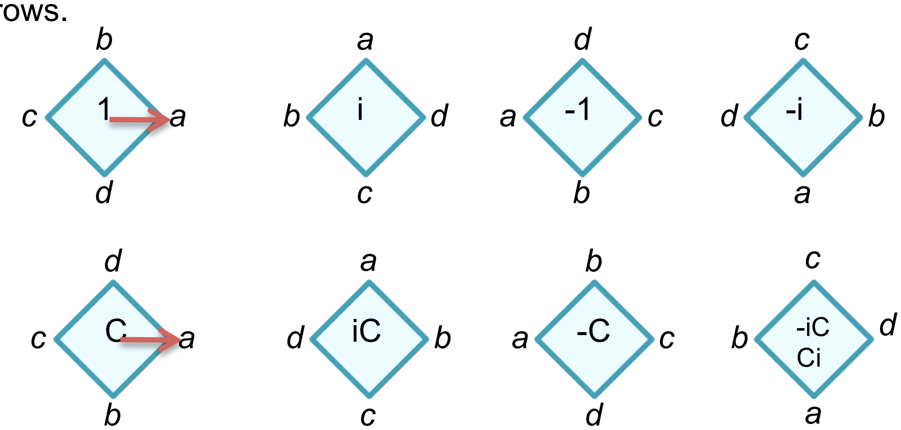
{1, -1, C, -C} , {1, -1}, {1, -1}

Non-normal subgroups of < i, C > :

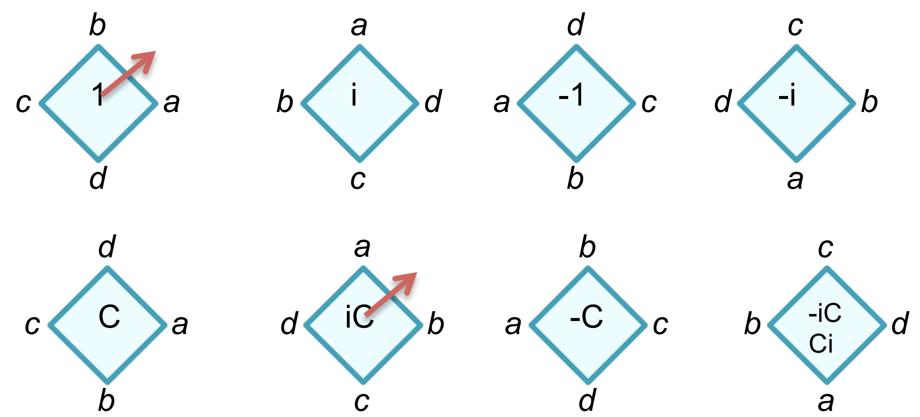
{1, -C}, {1, iC}, {1, C}

For example, {1, C} i = {i, Ci} ≠ {i, -Ci} = i {1, C}

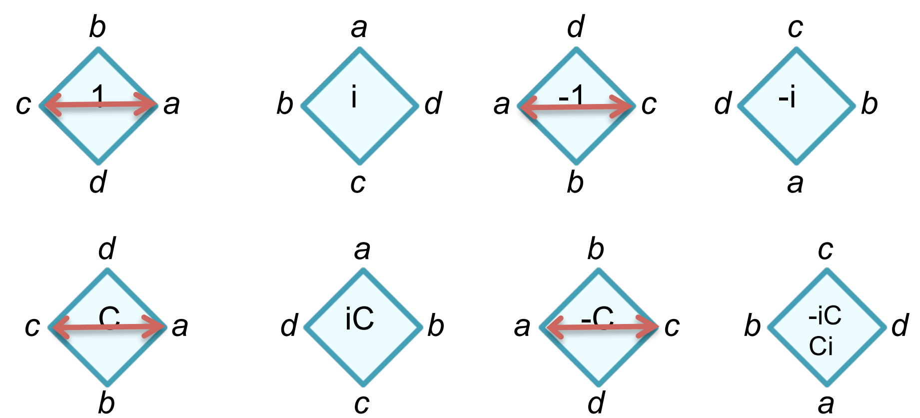
Example [13.6]: Reduced symmetry groups can be generated using one or more arrows.



{1, C} is a reduced symmetry group



{1, iC} is a reduced symmetry group



{1, -1, C, -C} is a reduced symmetry group

Symmetries of a Sphere

Definitions:
A group G whose underlying set is continuous is called a **Lie Group.**
SO(3) is the group of non-reflective symmetries of a 3-sphere
O(3) is the **Orthogonal Group**. It consists of both the reflective and non-reflective symmetries of a sphere.

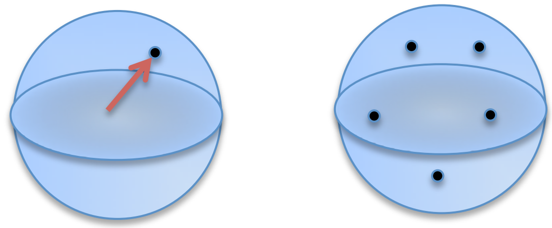
O(3) = SO(3) $\cup$ T, the disjoint union of O(3) with the coset of reflective symmetries

T $= \mathrm{R}\,\mathrm{SO}(3) = \{\mathrm{R}g : g \in \mathrm{SO}(3)\}$ where **R** is the reflection operator on the sphere.

Recall problem [12.7]: SO(3) is group isomorphic to the solid sphere $\mathcal{R}$ of radius π with antipodal points identified.

Theorem. (Problem [13.7]) SO(3) and {1, R} are the only normal subgroups of O(3), where **1** is the null rotation. (Penrose overlooked that the latter group is normal.)

Examples. Reduced Symmetry Groups



The set of rotations that fix a point on the sphere forms a non-normal subgroup. It is the set of rotations having the arrow as its axis.

Marking the sphere with vertices of a regular polyhedra reduces to the finite group of rotations of the sphere that take each vertex to one of the others. Such reduced symmetry groups are non-normal.

Linear Transformations and Matrices

Definition. Let V and W be vector spaces.

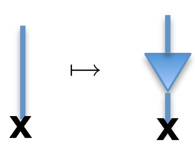
- $f: V \to W$ is a **homomorphism** if it preserves the vector space structure:
  - $f(au + bv) = af(u) + bf(v)$ for all vectors u and v and scalars a and b.
- **Hom(V,W)** is the set of homomorphisms from V to W.
- **$\mathcal{A}$(V)** = Hom(V,V).
- A **linear transformation** is a member $T \in \mathcal{A}(V)$.
  - That is, a linear transformation is a function $T: V \to V$ such that $T(au + bv) = aTu + bTv$.

Theorem. [13.12, 13.13] Let V = $\mathbb{R}^3$, using (x^1, x^2, x^2) instead of (x, y, z). Then a linear transformation T takes the form $T: x^r \mapsto T^r{}_s x^s = ax^1 + bx^2 + cx^3$.

Note. Linear transformations are represented by matrices:

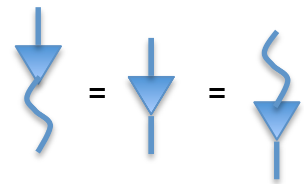
$$T: \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{or} \quad x \mapsto Tx$$

In diagrammatic form this is



Theorem. If R = ST then $R^a_{\ c} = S^a_{\ b} T^b_{\ c}$. That is, the composition, R, of 2 linear transformations is the result of matrix multiplication of S and T. In diagrammatic notation:

R =  =  = ST

Example. TI = T = IT is written in diagrammatic form as .

and, in $\mathbb{R}^3$, I = $\delta^a_b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where a, b range over {1, 2, 3}.

Definitions. A linear transformation T is **singular** if Dim(TV) < Dim W; that is, T is not *onto*.

Theorem. [13.17] T is singular iff $\exists$ $v \neq 0$ such that Tv = 0.

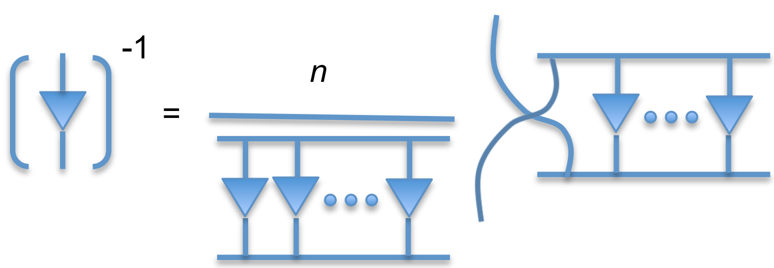
Corollary. [Bud] T is 1-1 iff T is non-singular iff T is onto.

Proof: T is 1-1 $\Leftrightarrow \forall v \neq w$ T($v - w$) = T(v) – T(w) ≠ 0 $\overset{(*)}{\Leftrightarrow}$ $\forall u \neq 0$ $\mathrm{T}(u) \neq 0$

$\overset{[13.17]}{\Leftrightarrow}$ T is non-singular $\Leftrightarrow$ T is onto.

(*) Set v = 3u and w = 2u. ■

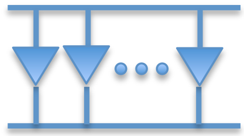
Theorem. [13.18] If T is nonsingular, then it has an inverse T^{-1}.

Theorem. [13.19] T^{-1} = 

Definition. The **transpose** of the matrix T = ($\mathrm{T}^i_{\ j}$) is the matrix $\mathbf{T}^{\mathbf{T}}$ = ($\mathrm{T}^j_{\ i}$).

Definition. A matrix T is **orthogonal** if $\mathrm{T}^{-1} = \mathrm{T}^{\mathrm{T}}$.

Determinants and Traces

Definition. Det T = $\frac{1}{n!}$  $= \frac{1}{n!} \in^{ab\cdots d} \mathrm{T}^e{}_a \mathrm{T}^f{}_b \cdots \mathrm{T}^h{}_d \, \varepsilon_{ef\cdots h}$.

Theorem. [Bud] Det T = $\sum_{\pi \in \mathcal{P}_{12\cdots n}} \mathrm{Sign}(\pi)\, \mathrm{T}^1{}_{\pi(1)} \cdots \mathrm{T}^n{}_{\pi(n)}$ (the normal definition of Det)

Proof. Let $\mathcal{P}_{1\ldots n}$ be the set of permutations of (1, …, n).

$$\mathrm{Det\ T} = \frac{1}{n!} \varepsilon_{r\cdots s} \in^{t\cdots u} \mathrm{T}^r{}_t \cdots \mathrm{T}^s{}_u$$

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi(1)\cdots\pi(n)} \mathrm{T}^{\pi^*(1)}{}_{\pi(1)} \cdots \mathrm{T}^{\pi^*(n)}{}_{\pi(n)}$$

(Replace Einstein notation.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi(\pi^*(1))\cdots\pi(\pi^*(n))} \mathrm{T}^{\pi^*(1)}{}_{\pi(\pi^*(1))} \cdots \mathrm{T}^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Replace π by $\pi \circ \pi^*$ in $\in$ and T. The double sum over π and π^* is unchanged, in both expressions stepping over all permutations of (1, …, n), and the exponents of $\in$ continue to match the subscripts of T.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \mathbf{Sign}(\pi) \cancel{\varepsilon_{\pi^*(1)\cdots\pi^*(n)} \in^{\pi^*(1)\cdots\pi^*(n)}} \mathrm{T}^{\pi^*(1)}{}_{\pi(\pi^*(1))} \cdots \mathrm{T}^{\pi^*(n)}{}_{\pi(\pi^*(n))}$$

(Re-order superscripts of $\in$ by applying an inverse π permutation.)

$$= \frac{1}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \mathrm{Sign}(\pi) \sum_{\pi^* \in \mathcal{P}_{1\cdots n}} \mathrm{T}^{\mathbf{1}}{}_{\pi(\mathbf{1})} \cdots \mathrm{T}^{\mathbf{n}}{}_{\pi(\mathbf{n})}$$

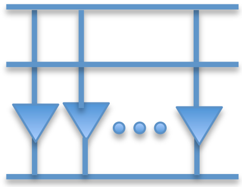
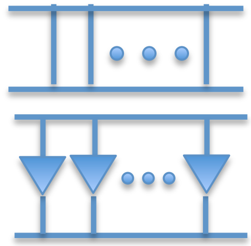
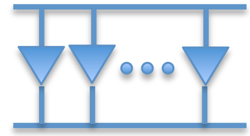
(This is just a simpler way to label the subscripts and superscripts of T. For example, if $\pi^*(3) = 1$ then $\mathrm{T}^{\pi^*(3))}{}_{\pi(\pi^*(3))} = \mathrm{T}^1{}_{\pi(1)}$.)

$$= \frac{n!}{n!} \sum_{\pi \in \mathcal{P}_{1\cdots n}} \mathrm{Sign}(\pi) \mathrm{T}^1{}_{\pi(1)} \cdots \mathrm{T}^n{}_{\pi(n)}$$

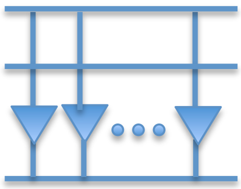
$$= \sum_{\pi \in \mathcal{P}_{1\cdots n}} \mathrm{Sign}(\pi) \mathrm{T}^1{}_{\pi(1)} \cdots \mathrm{T}^n{}_{\pi(n)}$$

■

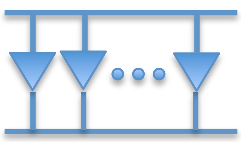
(See my solution to [13.21] for examples of this for n = 2 and 3.)

Theorem. [Bud]  =  = $n!$ 

Proof: Let $P_{a\ldots g}$ be the set of permutations of (a, …, g). Then

 $= \; n!\,\varepsilon_{a\cdots g} \in^{r\cdots x} T^{[a}{}_{r} \cdots T^{g]}{}_{x}$

$$= \frac{n!}{n!}\varepsilon_{a\cdots g} \in^{r\cdots x} \sum_{\pi\in\mathcal{P}_{ab\cdots g}} \mathrm{Sign}(\pi)\, T^{\pi(a)}{}_{r}\cdots T^{\pi(g)}{}_{x} \overset{(*)}{=} n!\,\varepsilon_{a\cdots g}\, T^{a}{}_{r}\cdots T^{g}{}_{x} \in^{r\cdots x}$$

= $n!$ 

(*) π is the composition of transmutations (i.e., of pairwise permutations).

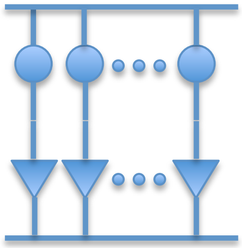
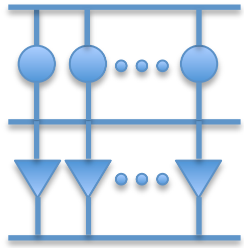
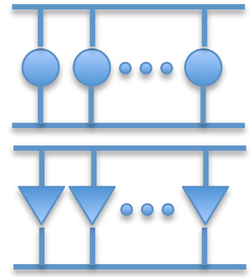
Let π^*: $\begin{matrix} c \mapsto e \\ e \mapsto c \end{matrix}$ be a transmutation. Then

$$\begin{aligned}
&\varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*)\, T^{\pi(a)}{}_{r}\cdots T^{\pi(c)}{}_{t}\cdots T^{\pi(e)}{}_{v}\cdots T^{\pi(g)}{}_{x} \\
&= \varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*)\, T^{a}{}_{r}\cdots T^{e}{}_{t}\cdots T^{c}{}_{v}\cdots T^{g}{}_{x} \\
&= \varepsilon_{a\cdots e\cdots c\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*)\, T^{a}{}_{r}\cdots T^{c}{}_{t}\cdots T^{e}{}_{v}\cdots T^{g}{}_{x} \quad \text{(Rename } c \mapsto e \text{ \& } e \mapsto c\text{)} \\
&= \mathrm{Sign}(\pi^*)\,\varepsilon_{a\cdots c\cdots e\cdots g} \in^{r\cdots t\cdots v\cdots x} \mathrm{Sign}(\pi^*)\, T^{a}{}_{r}\cdots T^{c}{}_{t}\cdots T^{e}{}_{v}\cdots T^{g}{}_{x} \\
&= \varepsilon_{a\cdots g}\, T^{a}{}_{r}\cdots T^{g}{}_{x} \in^{r\cdots x}.
\end{aligned}$$

So, for any permutation π, we have

$$\varepsilon_{a\cdots g} \in^{r\cdots x} \mathrm{Sign}(\pi)\, T^{\pi(a)}{}_{r}\cdots T^{\pi(g)}{}_{x} = \varepsilon_{a\cdots g}\, T^{a}{}_{r}\cdots T^{g}{}_{x} \in^{r\cdots x} \quad \blacksquare$$

Theorem. [13.22]

Det AB = $\frac{1}{n!}$  $= \left(\frac{1}{n!}\right)^2$  $= \left(\frac{1}{n!}\right)^2$ 

= DetA DetB

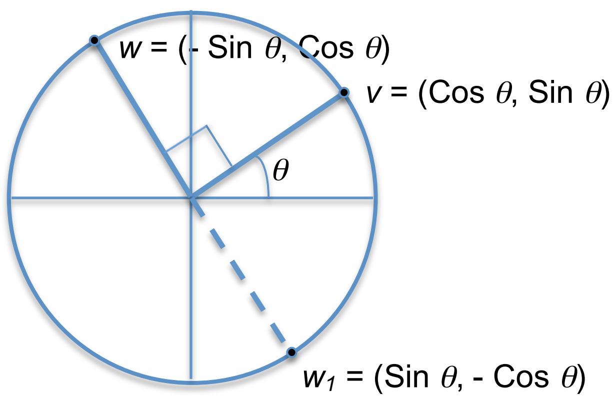
Theorem. (p.260 – no proof given) Matrix A is singular iff Det A = 0.

Proof: From [13.19], A is non-singular iff Det A ≠ 0. ■

Definition. Vectors v and w are **orthogonal** if $v \bullet w = 0$. That is, the angle between them is 90°.

Theorem. A matrix is orthogonal (i.e., $T^T = T^{-1}$) iff its column vectors are mutually orthogonal.

Example. Orthogonal 2 x 2 Matrices: A and B



Let $A = \begin{pmatrix} v & w \end{pmatrix}$

$= \begin{pmatrix} \text{Cos } \theta & \text{- Sin } \theta \\ \text{Sin } \theta & \text{Cos } \theta \end{pmatrix}.$

$A^T = \begin{pmatrix} \text{Cos } \theta & \text{Sin } \theta \\ \text{- Sin } \theta & \text{Cos } \theta \end{pmatrix}.$

$A^T = A^{-1}$:

$$A\,A^T = \begin{pmatrix} \text{Sin}^2\theta + \text{Cos}^2\theta & 0 \\ 0 & \text{Sin}^2\theta + \text{Cos}^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I} \quad ✔$$

Similarly $A^T A = \mathrm{I}$ ✔

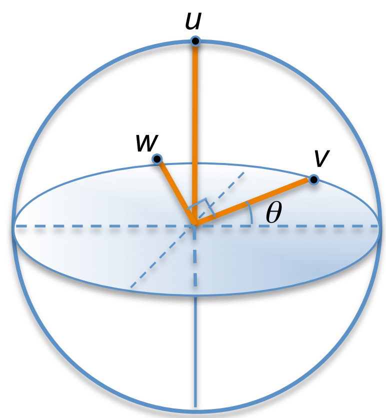
So A is an orthogonal matrix ✔

Det A = Det A^T = $\text{Cos}^2\,\theta + \text{Sin}^2\,\theta$ = 1 ✔

The column vectors of A are orthogonal: $v \perp w$ ✔

Let $B = \begin{pmatrix} \text{Cos } \theta & \text{Sin } \theta \\ \text{Sin } \theta & \text{- Cos } \theta \end{pmatrix} = B^T$. Then $B\,B^T = \mathrm{I}$, Det B = Det B^T = -1, and its column vectors v and w_1 are orthogonal.

Examples. Orthogonal 3 x 3 Matrices: A, B, and C



Let $v = \begin{pmatrix} \text{Cos}\,\theta \\ \text{Sin}\,\theta \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} -\text{Sin}\,\theta \\ \text{Cos}\,\theta \\ 0 \end{pmatrix}$, and $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Let $\text{A} = \begin{pmatrix} v & w & u \end{pmatrix} = \begin{pmatrix} \text{Cos}\,\theta & -\text{Sin}\,\theta & 0 \\ \text{Sin}\,\theta & \text{Cos}\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\text{A}^{\text{T}} = \begin{pmatrix} \text{Cos}\,\theta & \text{Sin}\,\theta & 0 \\ -\text{Sin}\,\theta & \text{Cos}\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. A is orthogonal, its columns are orthogonal vectors, and its determinant is +1. ✔

Let $\text{B} = \begin{pmatrix} \text{Cos}\,\theta & -\text{Sin}\,\theta & 0 \\ \text{Sin}\,\theta & \text{Cos}\,\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$. B is orthogonal and its determinant is -1. ✔

Let C be a θ-rotation of A about an axis $\left\{t(a,b,c)\text{: } 0 < t < \infty,\ a^2 + b^2 + c^2 = 1\right\}$:

$$\text{C} = \begin{pmatrix} \frac{1}{2}\left[1+a^2-b^2-c^2+\left(1-a^2+b^2+c^2\right)\text{Cos}\,\theta\right] & 2\text{Sin}\,\frac{\theta}{2}\left(-c\,\text{Cos}\,\frac{\theta}{2}+ab\,\text{Sin}\,\frac{\theta}{2}\right) & 2\text{Sin}\,\frac{\theta}{2}\left(b\,\text{Cos}\,\frac{\theta}{2}+ac\,\text{Sin}\,\frac{\theta}{2}\right) \\ 2\text{Sin}\,\frac{\theta}{2}\left(c\,\text{Cos}\,\frac{\theta}{2}+ab\,\text{Sin}\,\frac{\theta}{2}\right) & \frac{1}{2}\left[1-a^2+b^2-c^2+\left(1+a^2-b^2+c^2\right)\text{Cos}\,\theta\right] & 2\text{Sin}\,\frac{\theta}{2}\left(-a\,\text{Cos}\,\frac{\theta}{2}+bc\,\text{Sin}\,\frac{\theta}{2}\right) \\ 2\text{Sin}\,\frac{\theta}{2}\left(-b\,\text{Cos}\,\frac{\theta}{2}+ac\,\text{Sin}\,\frac{\theta}{2}\right) & 2\text{Sin}\,\frac{\theta}{2}\left(a\,\text{Cos}\,\frac{\theta}{2}+bc\,\text{Sin}\,\frac{\theta}{2}\right) & \frac{1}{2}\left[1-a^2-b^2+c^2+\left(1+a^2+b^2-c^2\right)\text{Cos}\,\theta\right] \end{pmatrix}$$

It can be directly verified that C is an orthogonal matrix with mutually orthogonal column vectors and determinant +1. ✔

Definition. A **symmetry** of a vector space (V,+) is a transformation T : V $\mapsto$ V that is 1-1 and onto that preserves the vector space structure:

T(a v + b w) = a Tv + b Tw

Definition. The **General Linear Group GL(*n*)** is the group of symmetries of an *n*-dimensional vector space.

Theorem. GL(n) is the group of non-singular (n x n) matrices.

Proof. Let T $\in$ GL(n). Since T(a v + b w) = a Tv + b Tw, T is a linear transformation. Were T singular, then by [13.17] Dim TV < n $\Rightarrow$ T is not onto, a contradiction. Therefore T is a non-singular linear transformation. Thus in any basis, T is represented by a non-singular matrix. ■

Definition. The **Special Linear Group SL(*n*)** is the subset of GL(*n*) having determinant = 1.

Theorem. SL(*n*) is a normal subgroup of GL(*n*).

Proof. First, SL(*n*) is a group:

Closed: If $S_1, S_2 \in SL(n)$, then $Det(S_1 S_2) = Det(S_1)\, Det(S_2) = 1$
$\Rightarrow S_1 S_2 \in SL(n)$.

Identity: $Det(I) = 1 \Rightarrow I \in SL(n)$

Inverse: $1 = Det(I) = Det(S_1 S_1^{-1}) = Det(S_1)\, Det(S_1^{-1}) = Det(S_1^{-1})$
$\Rightarrow S_1^{-1} \in SL(n)$

Also, SL(*n*) is normal:

Let $S \in SL(n)$ and $G \in GL(n)$. Then

$$Det(G^{-1} S G) = Det(G^{-1})\, Det(S)\, Det(G) = Det(G^{-1})\, Det(G)$$
$$= Det(G G^{-1}) = Det(I) = 1$$
$$\Rightarrow G^{-1} S G \in SL(n) \Rightarrow G^{-1} SL(n) G = SL(n)$$ ■

The groundwork has now been laid to introduce the table, below, that shows the relationships between GL(3), O(3), SL(3), general linear transformations, orthogonality, determinants, and symmetries. The table shows that $SL(3) \subseteq O(3) \subseteq GL(3) \subseteq \mathcal{A}(\mathbb{R}^3)$, and GL(3) is both the set of symmetries of $\mathbb{R}^3$ and the set of non-singular matrices. It also shows that the orthogonal group O(3) is a proper subset of the set of orthogonal matrices (shaded blue).

In general, non-singular matrices squeeze and stretch the unit sphere (or the reflected sphere) into an ellipsoid. However, singular matrices are more severe. They squash the unit sphere down to a 2-dimensional circle or ellipse or even to a line or a point.

Only orthogonal matrices preserve the sphere without squeezing or stretching any portion of it. This is achieved by limiting its operation to rotations and reflections. However, if determinant ≠ ±1 then orthogonal matrices also expand or contract the sphere.

Non-orthogonal matrices also squeeze, stretch, or preserve the sphere but not as rotations. Rather, the matrix columns would contain non-orthogonal vectors. In such a case the angle between the 1st and 2nd column vectors might be less than 90°, squeezing the sphere along associated plane. The angle between the 2nd and 3rd vectors would then be greater than 90°, stretching the sphere along that plane.

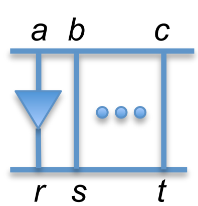
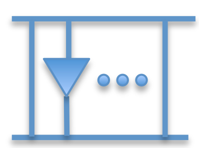
$\mathcal{A}(\mathbb{R}^3)$ = 3 x 3 Matrices

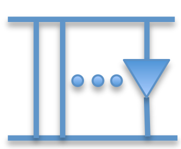
Determinant	Orthogonal	Sphere maps to a …	Matrix Type
0	Yes	Circle or line or point	Singular
	No	Ellipse or line or point	
Between -1 and 0	Yes	Contracted reflected sphere	GL(3) Non-singular Symmetries of $\mathbb{R}^3$
	No	Contracted reflected ellipsoid	
Between 0 and +1	Yes	Contracted sphere	
	No	Contracted ellipsoid	
-1	Yes	Reflected sphere	
	No	Reflected ellipsoid O(3)	
+1	Yes	SL(3) = sphere	
	No	Ellipsoid	
< -1	Yes	Expanded reflected sphere	
	No	Expanded reflected ellipsoid	
> 1	Yes	Expanded sphere	
	No	Expanded ellipsoid	

Matrices with positive determinant act on the sphere. Matrices with negative determinant behave exactly the same but act on the reflected sphere.

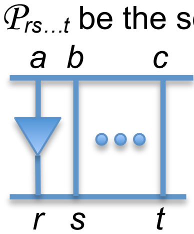
Definition. The **Trace** of A is $T_r(A)$ = Tr  $= T^k{}_k = T^1{}_1 + \cdots T^n{}_n$.

Theorem: [Bud]

Tr  $= \dfrac{1}{(n-1)!}$  $= \dfrac{1}{(n-1)!}$  $= \cdots$

$= \dfrac{1}{(n-1)!}$ 

Proof: Let $\mathcal{P}_{ab\ldots c}$ and $\mathcal{P}_{rs\ldots t}$ be the sets of permutations of (a,b,…,c) and (r,s,…,t),

respectively. Let B = 

$$= \epsilon^{rs\cdots t}\,\varepsilon_{ab\cdots c}\,T^a{}_r\,\delta^b_s\cdots\delta^c_t = \sum_{\pi\in\mathcal{P}_{ab\cdots c}}\ \sum_{\pi'\in\mathcal{P}_{rs\cdots t}} \epsilon^{\pi'(r)\pi'(s)\cdots\pi'(t)}\,\varepsilon_{\pi(a)\pi(b)\cdots\pi(c)}\,T^{\pi(a)}{}_{\pi'(r)}\,\delta^{\pi(b)}_{\pi'(s)}\cdots\delta^{\pi(c)}_{\pi'(t)}.$$

Fix π. The only non-zero term in the sum is

$$\in^{\pi(a)\pi(b)\cdots\pi(c)} \varepsilon_{\pi(a)\pi(b)\cdots\pi(c)} \mathrm{T}^{\pi(a)}{}_{\pi(a)} \delta^{\pi(b)}_{\pi(b)} \cdots \delta^{\pi(c)}_{\pi(c)} = \mathrm{T}^{\pi(a)}{}_{\pi(a)}.$$

I showed in Problem [13.22] that $\in^{xy\cdots z} \varepsilon_{xy\cdots z} = 1$ for any fixed $(x,y,\ldots,z)$.

Thus, B = $\sum_{\pi \in \mathcal{P}_{ab\cdots c}} \mathrm{T}^{\pi(a)}{}_{\pi(a)}$. This sum has n! terms composed of $(n-1)!$ terms equal to $\mathrm{T}^a{}_a$, $(n-1)!$ terms equal to $\mathrm{T}^b{}_b$, …, and $(n-1)!$ terms equal to $\mathrm{T}^c{}_c$. So,

$$\mathrm{B} = (n-1)!\,(\mathrm{T}^a{}_a + \mathrm{T}^b{}_b + \ldots + \mathrm{T}^c{}_c) = (n-1)!\,\mathrm{Tr}\,(\mathrm{A}) = (n-1)!\,\mathrm{Tr}$$

Similarly for the other figures. ■

Theorem. [13.24] $\mathrm{Det}(\mathrm{I} + \in \mathrm{A}) = 1 + \in \mathrm{T_r}(\mathrm{A})$ if we ignore 2nd order and higher $\in$ terms.

Theorem. [13.25] Det e^{A} = $e^{\mathrm{T_r}(\mathrm{A})}$.

Definition. An **Eigenvector** is a non-zero vector v for which $\exists \lambda \in \mathbb{C}$ such that $\mathrm{T}v = \lambda v \iff (\mathrm{T} - \lambda \mathrm{I})v = 0$. λ is called an **Eigenvalue.**

Note: $\mathrm{Det}(\mathrm{T} - \lambda \mathrm{I}) = 0$ and so $(\mathrm{T} - \lambda \mathrm{I})$ is singular

Theorem. [13.26] $\mathrm{Det}(\mathrm{T}\text{-}\lambda \mathrm{I}) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda) = 0$ is a polynomial equation of degree n.

Definition. ***λ* has multiplicity *r*** means that λ appears r times in the equation above. Eigenvalue multiplicities are called **degeneracies** in Quantum Mechanics.

Definition. The set of Eigenvectors corresponding to λ is a linear space called an **Eigenspace**.

Theorem. If d is the dimension of the Eigenspace of λ and r is the multiplicity of λ then $1 \le d \le r$.

Theorem. [13.27] Let $\{\lambda_i\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then $\sum r_i = n$.

Corollary. A linear transformation T has at least 1 Eigenvector.

Theorem. [13.30] Suppose $\{e_k\}$ and $\{f_k\}$ are bases for a vector space V, and $f_k = \mathrm{T}e_k$. Then

$$f_j = \begin{pmatrix} \mathrm{T}^1{}_j \\ \vdots \\ \mathrm{T}^n{}_j \end{pmatrix}.$$

That is, the components of f_j in basis $\{e_k\}$ are $\left(\mathrm{T}^1{}_j, \cdots, \mathrm{T}^n{}_j\right)$.

Theorem. [13.31] If the Eigenspace dimension of every multiple Eigenvector equals its multiplicity, then there is a basis for V composed of Eigenvectors, and the matrix of T in this basis is

$$\mathrm{T} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The next theorem states that even when the hypothesis of the above theorem is not satisfied, the matrix of T can at least be written in upper triangular form.

Theorem. (Note 13.12): **Jordan Canonical Form:** Let $\{\lambda_i\}$ be the set of Eigenvalues of an n x n matrix T, and let r_i be the multiplicity of λ_i. Then there is a basis for V such that the matrix of T in this basis is

$$\mathrm{T} = \left(\begin{array}{c|c|c}
\begin{matrix} \lambda_1 & 1 & & & \\ & \lambda_1 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_1 \end{matrix} & 0 & \begin{matrix} \cdots & 0 \\ \iddots & \vdots \end{matrix} \\ \hline
 & \begin{matrix} \lambda_2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_{n-1} \end{matrix} & 0 \\ \hline
\begin{matrix} \vdots & \iddots \\ 0 & \cdots \end{matrix} & & \begin{matrix} \lambda_n & 1 & & \\ & \lambda_n & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_n \end{matrix}
\end{array}\right).$$