[13.1] Let **G** be a set of elements with one element called "1".

Part 1:

Given (1) $1a = a \ \forall a$, (2) $\forall a \ \exists \ a^{-1}$ such that $a^{-1} a = 1$, and (3) Associative Law: $a\,(b\,c) = (a\,b)\,c \ \ \forall \ a, b$, and c.

Show (A) $a\,1 = a \ \ \forall a$ and (B) $a\,a^{-1} = 1 \ \forall a$.

Part 2:

Replace (1) with (1'): $a\,1 = a \ \forall a$

Show this is not sufficient to imply (A) and (B)

Solution Part 1: From (2) $\exists \left(a^{-1}\right)^{-1}$ such that

(4) $\left(a^{-1}\right)^{-1} a^{-1} = 1$. So

$$\text{(B)}\ a\,a^{-1} \overset{(1)}{=} (1a)\,a^{-1} \overset{(3)}{=} 1\left(a\,a^{-1}\right) \overset{(4)}{=} \left[\left(a^{-1}\right)^{-1} a^{-1}\right] \left(a\,a^{-1}\right) \overset{(3)}{=} \left(a^{-1}\right)^{-1} \left[\left(a^{-1}\,a\right) a^{-1}\right]$$

$$\overset{(2)}{=} \left(a^{-1}\right)^{-1} \left[1\,a^{-1}\right] \overset{(1)}{=} \left(a^{-1}\right)^{-1} a^{-1} \overset{(4)}{=} 1 \qquad ✔$$

$$\text{(A)}\ a\,1 \overset{(2)}{=} a\left(a^{-1}\,a\right) \overset{(3)}{=} \left(a\,a^{-1}\right) a \overset{(B)}{=} 1\,a \overset{(1)}{=} 1 \qquad ✔$$

Solution Part 2: Let **G** = {1, x} with multiplication defined by

$1^2 = 1\,x = 1$ and $x^2 = x\,1 = x$.

(1'): $a\,1 = a \quad \forall a \quad$ (i.e., $1\,1 = 1$ and $x\,1 = x$) ✔

(2): Define $1^{-1} = x^{-1} = 1$.

Then $a^{-1}\,a = 1 \ \forall a$ (i.e., $1^{-1}\,1 = 1^2 = 1$ and $x^{-1}\,x = 1\,x = 1$) ✔

(3): $1\left(a\,b\right) = 1$ and $\left(1\,a\right) b = 1\,b = 1$ for any a and b in **G**, and

$x\left(a\,b\right) = x$ and $\left(x\,a\right) b = x\,b = x$ for any a and b in **G** ✔

But (B) fails: $x\,x^{-1} = x\,1 = x \neq 1$ ✔

Note: Part 2 solution is a simplification of deant's simplification of Beckmann's idea.