

[13.18] Let V be an n -dimensional vector space. The linear transformation $T: V \rightarrow V$ is nonsingular $\Rightarrow T$ has an inverse

Proof: Let $\{a_1, \dots, a_n\}$ be a basis for V . Let $b_k = T a_k$. Then $\{b_k\}$ is also a basis for V :

Suppose $\{\beta^k\}$ is collection such that $\beta^k b_k = 0$. (Einstein Summation

Convention.) Then $0 = \beta^k b_k = \beta^k T a_k = T(\beta^k a_k) \stackrel{[13.17]}{\Leftrightarrow} \beta^k = 0 \forall k$. Thus $\{b_k\}$ is a collection of linearly independent vectors of size n , and hence $\{b_k\}$ is a basis for V .

Define $S: V \rightarrow V: S b_k = a_k$. We show $S = T^{-1}$ by showing $STv = v = TSv \forall v \in V$. (This is equivalent to $ST = I = TS$.)

Let $v = \alpha^k a_k \in V$. Then $STv = S\alpha^k b_k = \alpha^k a_k = v$ ✓

Since $\{b_n\}$ is a basis, $v = \beta^k b_k$. So $TSv = TS\beta^k b_k = T\beta^k a_k = \beta^k b_k = v$ ✓

Note: Penrose said not to use explicit expressions. By that, he may have meant to do a coordinate-free construction, which I didn't because I assumed a basis $\{a_k\}$. Beckmann avoided using a basis but otherwise his proof seems as "explicit" as mine. He proved that nonsingular \Rightarrow 1-1 and onto. Therefore $\forall w \in V \exists$ unique $v_w \in V$ such that $T(v_w) = w$. He defined $S: V \rightarrow V: S(w) = v_w$.

Thus $TS(w) = T(v_w) = w$ ✓

Next, he fixed v so that there is a w such that $w = T(v)$. So $ST(v) = S(w) = v_w$.

Since $T(v_w) = w$ and T is 1-1, then $v_w = v$; i.e., $ST(v) = v$ ✓

Therefore he concluded that $S = T^{-1}$.