[10.11] Show that $\frac{\partial \Phi}{\partial \bar{z}} = 0$ implies $\frac{\partial \Phi}{\partial x} - i\frac{\partial \Phi}{\partial y} = 0$.

Proof: Recall that $x = \frac{z+\bar{z}}{2}$ and $y = -\frac{z-\bar{z}}{2}i$.

So,

$$0 = \frac{\partial \Phi}{\partial \bar{z}} = \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{\partial \Phi}{\partial x}\frac{1}{2} + \frac{\partial \Phi}{\partial y}\frac{1}{2}i$$

$$or \quad \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}i = 0$$

We invent a Part 2, to derive the Cauchy-Riemann equations shown on page 194 from this equation.

Part 2: Derive the Cauchy-Riemann Equations in middle of p. 194 from [10.11], above.

Let $\Phi(z) = \Phi(x+yi) = \alpha + \beta i$

Then $\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial \Phi}{\partial \beta}\frac{\partial \beta}{\partial x}$ and $\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial \Phi}{\partial \beta}\frac{\partial \beta}{\partial y}$

Now $\frac{\partial \Phi}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\alpha + \beta i\right) = \frac{\partial \alpha}{\partial \alpha} + \frac{\partial \beta}{\partial \alpha}i = 1 + 0 = 1$

because we hold β constant when taking the partial with respect to α.

Similarly, $\frac{\partial \Phi}{\partial \beta} = \frac{\partial \alpha}{\partial \beta} + \frac{\partial \beta}{\partial \beta}i = 0 + i = i$.

So, $\frac{\partial \Phi}{\partial x} = \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial x}i$ and $\frac{\partial \Phi}{\partial y} = \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial y}i$.

$$\therefore \quad 0 = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}i = \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial x}i + \frac{\partial \alpha}{\partial y}i - \frac{\partial \beta}{\partial y}.$$

Thus, $\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}$ and $\frac{\partial \beta}{\partial x} = -\frac{\partial \alpha}{\partial y}$.