[13.7] This problem has 2 parts, (A) and (B). My proof subdivides (B) further. Here is an outline.

SO(3) is the group of rotations of the unit sphere in 3-space. O(3) extends SO(3) by including reflections.

(A) SO(3) is a normal subgroup of O(3)

(B) It is the only proper normal subgroup

(1) SO(3) contains no nontrivial normal subgroup

Let H be a normal subgroup of SO(3). Let $1 \neq h_1 \in$ H be a rotation of the 3-sphere by θ where $0 < \theta \leq \pi$. WLOG h_1 rotates about the *x*-axis.

(a) The rotations of amount θ about the *y*- and *z*-axes and about the negative *x*-, *y*-, and *z*-axes also belong to H

(b) $S_\theta \subseteq$ H where S_θ is the set of all rotations by θ about all axes. S_θ is a sphere in SO(3).

(c) There is a rotation angle $\psi \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ such that $S_\psi \subseteq$ H

(d) If $0 < \omega \leq \pi$, then there are a pair of elements in S_ψ whose product belongs to S_ω. Also, if the product of one pair of elements from the sphere S_ψ is in S_ω, then each element of the sphere S_ω can be generated from the product of a pair of elements in S_ψ. Therefore, H = SO(3).

(2) SO(3) is the only proper normal subgroup of O(3)

The interesting part of this solution is (B1). It is accomplished via a constructive proof that begins with any non-identity element h_1 of SO(3) and builds up to the smallest normal subgroup that contains it, eventually obtaining SO(3).

This construction exposes relationships that provide insights into the structure of SO(3). Occasional "Asides" are inserted to point out additional relationships that are not required for the proof.

(A) SO(3) is a normal subgroup of O(3)

Lemma 1.1: A subgroup H of a group G is normal iff g^{-1}Hg = H $\forall\, g \in$ G

Proof. By Penrose's definition, H is normal iff gH = Hg $\forall\, g \in$ G. Left multiplying by g^{-1} yields the lemma. ■

Theorem 1: SO(3) is a normal subgroup of O(3)

Proof: Let $g \in$ O(3) and $h \in$ SO(3). Claim $g^{-1}hg \in$ SO(3):

If $g \in$ SO(3):

Then g^{-1}, h, $g \in$ SO(3). Thus $g^{-1}hg \in$ SO(3) since groups are closed under multiplication.

If $g \notin$ SO(3):

Let **R** be the reflection operation of O(3), which swaps "handedness". Note R^{-1} = R. Set $f = Rg \in SO(3)$. Then, $g = Rf$ and $g^{-1} = Rf^{-1}$. So $g^{-1}hg = (Rf^{-1})h(Rf) = f^{-1}hf \in$ SO(3) since f^{-1}, h, and $f \in$ SO(3).

Therefore, in either case, $g^{-1}hg \in$ SO(3) $\forall h \in$ SO(3),

$$\Leftrightarrow \qquad g^{-1}\ \mathrm{SO(3)}\ g \subseteq \mathrm{SO(3)}, \qquad \text{(i)}$$

Replacing g by g^{-1} in (i) yields

$$g\ \mathrm{SO(3)}\ g^{-1} \subseteq \mathrm{SO(3)} \qquad \text{(ii)}$$

Hence,

$$\mathrm{SO(3)} = (g^{-1}g)\ \mathrm{SO(3)}\ (g^{-1}g)$$

$$= g^{-1}\,[g\mathrm{SO(3)}\,g^{-1}]\,g \overset{\text{(ii)}}{\subseteq} g^{-1}\ \mathrm{SO(3)}\ g \qquad \text{(iii)}$$

Combining (i) and (iii) yields g^{-1} SO(3) g = SO(3), proving SO(3) is normal by Lemma 1.1. ■

(B) SO(3) is the only proper normal subgroup of O(3)

It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3-ball $\mathcal{R}$ of radius π in which antipodal points on the surface of $\mathcal{R}$ are identified. In this representation, elements of $\mathcal{R}$ can be written $f = (\alpha, \beta, \gamma)$ where $\{ t\,(\alpha, \beta, \gamma) : t \geq 0 \}$ is the axis of rotation, $\theta = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$ is the (counter-clockwise) angle of rotation, and $0 \leq \theta \leq \pi$. We will refer to the axis of rotation under discussion as the **positive axis of rotation** and its opposite as the **negative axis of rotation**. We will write **| f |** = θ for the magnitude of f.

The representations for $f = (\alpha, \beta, \gamma)$ are unique except when $\theta = \pi$. In that case there are 2 representations for each point: $f = (\alpha, \beta, \gamma) = (-\alpha, -\beta, -\gamma)$ because antipodal points are identified.

The product of 2 rotations represents their composition and is written fg. The convention used here is that fg denotes the g-rotation followed by the f-rotation.

The definition of $\mathcal{R}$ only mentions counter-clockwise rotation angles $0 \le \theta \le \pi$. Multiplication of 2 rotations sometime results in larger rotations, angles θ in the half-open interval $(\pi, 2\pi]$. Such angles represent counter-clockwise rotations of amount θ but can also be interpreted as either clockwise rotations of magnitude $(2\pi - \theta)$ about the positive axis or as counter-clockwise rotations of angle $(2\pi - \theta)$ about the negative axis.

Part (B1) will be solved with the aid of $\mathcal{S}$, a second group isomorphism of SO(3). In Geometric Algebra (GA), aka Clifford Algebra, every rotation corresponds to a rotor (described shortly), and the geometric product of 2 rotors computes the equation of their composition. $\mathcal{S}$ is used in this proof to denote the subset of rotors having rotation angle $0 \le \theta \le \pi$ under the operation of geometric product. It will be shown that with certain provisions $\mathcal{S}$ is group isomorphic to $\mathcal{R}$.

GA concepts and notation used in this proof

Let **$\{ e_1, e_2, e_3 \}$** be an orthonormal basis for 3-space.

Rotors are multivectors, elements of $\mathcal{S}$ of the form

$$r = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\,e_1\,e_2 - b\,e_1\,e_3 + c\,e_2\,e_3\right]$$

where $a^2 + b^2 + c^2 = 1$ and $-\infty \le \theta \le \infty$.

A rotor r represents a rotation of angle θ about the axis $\{ t\,(a, b, c) : t \ge 0 \}$. $\frac{\theta}{2}$ is called the **rotor angle** and θ is the **rotation angle**. As this can be confusing, in this proof only "rotation angle" will be used. In this proof **$| r |$** will always mean θ and not $\frac{\theta}{2}$ and also not 1. (See "Aside", below.)

$\mathbf{Cos}\,\frac{\theta}{2}$ is referred to as the "**constant term**" of r because it does not contain any basis elements. The constant term determines the rotation angle θ. That is, if the constant term of a rotor r is known to be A, then for some θ, $A = \mathbf{Cos}\,\frac{\theta}{2}$, and so $\theta = 2\,\text{Arc}\mathbf{Cos}\,(A)$.

While e_1 is a **vector**, $e_1\, e_2$ is a **bivector**. r is also a bivector. Vectors, bivectors, trivectors, etc., belong to the class of **multivectors**.

Note that “Rotation angle” has the same meaning whether applied to an element of $\mathcal{R}$ or $\mathcal{S}$. However, the (α, β, γ) used to describe the axis of rotation in $\mathcal{R}$ differs from the (a, b, c) used to describe the same axis of rotation in $\mathcal{S}$:

$\alpha^2 + \beta^2 + \gamma^2 = \theta^2$, while $a^2 + b^2 + c^2 = 1$.

This is because the magnitude of an element (α, β, γ) of $\mathcal{R}$ is its angle of rotation θ, while the magnitude of a rotor in $\mathcal{S}$ is $\textbf{Cos}^2\left(\frac{\theta}{2}\right) + \textbf{Sin}^2\left(\frac{\theta}{2}\right) = 1$, requiring $a^2 + b^2 + c^2 = 1$. (a, b, c) is the unit point on the axis of rotation and (α, β, γ) is the point on the axis of rotation of magnitude θ. They are related by

$$(\alpha, \beta, \gamma) = \theta\,(a, b, c) = (\theta a, \theta b, \theta c)$$

The **geometric product** of 2 multivectors is just the regular polynomial product with a couple of modifications. First, it is non-commutative, so the order of multiplication matters. Second, the basis elements are combined using the rules

$$e_i^{\,2} = 1, \quad i = 1, 2, 3 \quad \text{and}$$
$$e_i\, e_j = -e_j\, e_i \ \text{ if } \ i \neq j \quad \text{(antisymmetry)}$$

The symbol $\circ$ will be used in this proof to denote the geometric product, as in $f \circ g$.

To compute the sometimes algebra-intensive geometric products I wrote a software package in Mathematica that calculates geometric products and also performs other GA operations such as wedge product, multivector inverse, pseudoscalar, etc. The package can be downloaded for free at https://github.com/matrixbud/Geometric-Algebra. I have also saved the Mathematica file having the GA calculations used in this proof in the same directory as this file.

Note: The reason for the negative “b” term in the definition of the rotor r (above) is because $e_3\, e_1$ rather than $e_1\, e_3$ is the preferred bivector, but Mathematica always outputs $e_1\, e_3$. Since $e_3\, e_1 = -\, e_1\, e_3$, the b term is negative in the representation of r, above.

Since the normality operation $g^{-1} \circ f \circ g$ involves inverses, this is a good time to provide the formula for the inverse of a rotor. The GA **inverse of the rotor *r*** has the formula below, what is called the reverse of the rotor:

$$\boldsymbol{r}^{-1} = \ \textbf{Cos}\,\frac{\theta}{2} - \textbf{Sin}\,\frac{\theta}{2}\left[a\, e_1\, e_2 - b\, e_1\, e_3 + c\, e_2\, e_3\right]$$

Theorem 2: r^{-1} is the inverse of *r*, and $r^{-1} \in \mathcal{S}$.

Proof. By computing the geometric product we find that
$r^{-1} \circ r = \mathbf{Cos}^2 \frac{\theta}{2} + \mathbf{Sin}^2 \frac{\theta}{2}\left(a^2 + b^2 + c^2\right) = 1$ and similarly $r \circ r^{-1} = 1$. Thus, r^{-1} is the inverse of *r.* ✔

Another way of seeing that $r^{-1} \circ r = 1$ is that r^{-1} can be written
$r^{-1} = \mathbf{Cos}\left(-\frac{\theta}{2}\right) + \mathbf{Sin}\left(-\frac{\theta}{2}\right)\left[a\, e_1\, e_2 - b\, e_1\, e_3 + c\, e_2\, e_3\right]$, representing a clockwise rotation about the positive axis that cancels the counter-clockwise rotation of *r*, yielding the identity rotation, 1. ✔

Additionally, since $r^{-1} = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2}\left[-a\, e_1\, e_2 + b\, e_1\, e_3 - c\, e_2\, e_3\right]$, r^{-1} can also be regarded as a θ rotation about the negative axis, showing that $r^{-1} \in \mathcal{S}$ (since $\mathcal{S}$ is the set of all rotations $0 \le \theta \le \pi$ about all axes). ■

We are now in a position to provide the formula for a rotor *r* to rotate a point *w* = (*x*, *y*, *z*) in 3-space:

$$v = r^{-1} \circ w \circ r.$$

Effectively, r^{-1} performs half the θ rotation (i.e., $\frac{\theta}{2}$) and *r* performs the rest. In the next-to-last line of the proof of Theorem 3, it is shown algebraically, using a rotor *s* about the *z*-axis, that *w* is rotated by θ in the *xy*-plane and unchanged in *z*.

Every rotation can be represented by a rotor in $\mathcal{S}$; i.e., by a rotor
$r = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2}\left(a e_1 e_2 - b e_1 e_3 + c\, e_2 e_3\right)$ where $0 \le \theta \le \pi$. Every rotation $0 < \theta < \pi$ has a 2nd representation by a rotor with $\pi < \theta < 2\pi$ that is not in $\mathcal{S}$. To compute the geometric product of rotors in $\mathcal{S}$, we need to be able to recognize the formulas of rotors not in $\mathcal{S}$ and be able to convert them to the equivalent rotor in $\mathcal{S}$.

Definition: We say that 2 rotors r_1 and r_2 are **equivalent** if they generate the same rotation.

While the geometric product of 2 rotors (rotations) is another rotor (rotation), the product of 2 rotors from $\mathcal{S}$ may not remain in $\mathcal{S}$. For example, the geometric product of two $\frac{2}{3}\pi$ rotations about the *x*-axis is a $\frac{4}{3}\pi$ rotation about the *x*-axis, not in $\mathcal{S}$. If we identify each rotor not in $\mathcal{S}$ with its equivalent rotor in $\mathcal{S}$, then the set $\mathcal{S}$ is closed under the geometric product operation and it becomes a group.

We will show in Theorem 4 that $\mathcal{S}$ is (group) isomorphic to $\mathcal{R}$. First we wish to show how to identify and convert rotors not in $\mathcal{S}$.

Theorem 3: Let $\pi < \theta < 2\pi$. Then

$$s = r(\theta) \equiv \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2}\left(a e_1 e_2 - b e_1 e_3 + c e_2 e_3\right) \notin \mathcal{S},$$

and

$$r = r(2\pi - \theta) = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) + \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right)\left(a e_1 e_2 - b e_1 e_3 + c e_2 e_3\right) \in \mathcal{S}.$$

Also, the constant term of s is negative; the constant term of r is positive, and s is equivalent to $-s = r^{-1} \in \mathcal{S}$.

Proof: $s \notin \mathcal{S}$ because $\theta \notin [0, \pi]$. ✔

Since $(2\pi - \theta) \in [0, \pi]$, $r \in \mathcal{S}$. ✔

The constant term of s is negative: $\mathbf{Cos}\frac{\theta}{2} < 0$ because $\frac{\pi}{2} < \frac{\theta}{2} < \pi$. ✔

The constant term of r is positive: $\mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) > 0$ since $0 < \pi - \frac{\theta}{2} < \frac{\pi}{2}$. ✔

$r^{-1} \in \mathcal{S}$ from Theorem 2. ✔

Claim $r^{-1} = -s$:

Without loss of generality, by a suitable rotation of 3-space, we can assume s, and hence r, are rotations about the z-axis.

$$s = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 \quad \text{and} \quad r = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) + \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2 .$$

These formulas for s and r are rotors because they satisfy the rotor definition with $a = 1$ and $b = c = 0$. They rotate about the z-axis because $e_1 e_2$ rotates the xy-plane.

$$r^{-1} = \mathbf{Cos}\left(\pi - \frac{\theta}{2}\right) - \mathbf{Sin}\left(\pi - \frac{\theta}{2}\right) e_1 e_2 = -\mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) = -s \quad ✔$$

To show s is equivalent to r^{-1}, we must show that if $w = (x, y, z)$ then $s^{-1} \circ w \circ s = r \circ w \circ r^{-1}$, i.e., that the rotations of w generated by s and r^{-1} are the same.

$$w = x e_1 + y e_2 + z e_3$$

$$
\begin{aligned}
s^{-1} \circ w \circ s &= \left[x\,\mathbf{Cos}^2\left(\frac{\theta}{2}\right) - 2y\,\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\theta}{2}\right) - x\,\mathbf{Sin}^2\left(\frac{\theta}{2}\right)\right] e_1 \\
&+ \left[y\,\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + 2x\,\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\theta}{2}\right) - y\,\mathbf{Sin}^2\left(\frac{\theta}{2}\right)\right] e_2 + \\
&+ z\left[\mathbf{Cos}^2\left(\frac{\theta}{2}\right) + \mathbf{Sin}^2\left(\frac{\theta}{2}\right)\right] e_3 \\
&= \left[x\,\mathbf{Cos}(\theta) - y\,\mathbf{Sin}(\theta)\right] e_1 + \left[y\,\mathbf{Cos}(\theta) + x\,\mathbf{Sin}(\theta)\right] e_2 + z e_3 \\
&= r \circ w \circ r^{-1} \quad ✔
\end{aligned}
$$

■

Aside: Note the change from $\frac{\theta}{2}$ to θ in the equation above, evidence that θ is indeed the rotation angle generated by the expression $s^{-1} \circ w \circ s$.

Example: Find a rotor in $\mathcal{S}$ that is equivalent to the rotor $s = -\frac{\sqrt{3}}{2} + \frac{1}{2} e_1 e_2$

Solution: First, by Theorem 3, $s \notin \mathcal{S}$ because the constant term is negative. Also by Theorem 3, s is equivalent to $r^{-1} = -s = \frac{\sqrt{3}}{2} - \frac{1}{2} e_1 e_2$ and $r^{-1} \in \mathcal{S}$.

Aside:

- If we consider $-\infty < \theta < \infty$, then there are infinitely many rotors equivalent to r since sine and cosine have period 2π. But, for $0 < \theta < 2\pi$ there are only two representations for each rotor.

Theorem 4: $\mathcal{S}$ is group isomorphic to $\mathcal{R}$ [which is group isomorphic to SO(3)]

Proof: Let $\boldsymbol{r} = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\, e_1\, e_2 - b\, e_1\, e_3 + c\, e_2\, e_3\right] \in \mathcal{S}$.

So, $0 \le \theta \le \pi$ and $a^2 + b^2 + c^2 = 1$.

r can be represented as $r = (\theta, a, b, c) \in \mathcal{S}$. Unlike $\mathcal{R}$, this representation in $\mathcal{S}$ is unique for $0 \le \theta \le \pi$ except when $\theta = 0$.

- When $\theta = 0$, there are infinitely many unit points (a, b, c) that can be used to denote the identity rotation.
- When $\theta = \pi$, there is only 1 representation because the antipodal point is not in $\mathcal{S}$. That is, if $(\theta, a, b, c) \in \mathcal{S}$ where $0 \le \theta \le \pi$, the antipodal point is $(-\theta, -a, -b, -c) = (2\pi - \theta, -a, -b, -c)$, and neither expression has a rotation angle between 0 and π.

Define

$T: \mathcal{S} \to \mathcal{R}$: $T(r) = f$ where

$r = (\theta, a, b, c)$ and

$f = (\alpha, \beta, \gamma) = \theta\,(a, b, c) = (\theta a, \theta b, \theta c)$.

That is, $T: (\theta, a, b, c) \mapsto \theta\,(a, b, c)$

T is well-defined:

To show that T is well-defined we must show that (a) $T(r) \in \mathcal{R}$ and (b) if r has 2 representations, then T assigns the same element of $\mathcal{S}$ in both cases.

(a) Since $0 \le \theta \le \pi$ and $\alpha^2 + \beta^2 + \gamma^2 = \theta^2$, $f \in \mathcal{R}$.

(b) If $\theta = 0$, then

$T(0, a, b, c) = 0\,(a, b, c) = 0 = 0\,(a', b', c') = T(0, a', b', c')$.

T is obviously 1–1 and onto since $T: (\theta, a, b, c) \mapsto \theta\,(a, b, c)$.

T is a homomorphism (i.e., $T(r_1 \circ r_2) = T(r_1)\,T(r_2)$):

$r_1 \circ r_2$ is the composition of the 2 rotations in $\mathcal{S}$, and $T(r_1)\,T(r_2)$ is the composition of the 2 rotations in $\mathcal{R}$. So,

if $r_1 \circ r_2$ = Composition of $(\theta_1, a_1, b_1, c_1)$ & $(\theta_2, a_2, b_2, c_2) = (\theta_3, a_3, b_3, c_3)$

then $T(r_1)\,T(r_2)$ = Composition of $\theta_1(a_1, b_1, c_1)$ & $\theta_2(a_2, b_2, c_2)$

$= \theta_3(a_3, b_3, c_3)$

So, $T(r_1 \circ r_2) = T(\theta_3, a_3, b_3, c_3) = \theta_3(a_3, b_3, c_3) = T(r_1)\,T(r_2)$ ✔ ■

This concludes the GA introduction and overview.

(1) SO(3) contains no nontrivial normal subgroup:

Let $1 \neq h_1 \in$ H. WLOG, the axes of the unit 3-sphere can be rotated so that h_1 is a rotation about the x-axis. For the remainder of (1), unless otherwise specified, $\theta \in (0, \pi]$, H is a normal subgroup of SO(3), and $1 \neq h_1 \in$ H is the point of SO(3) that rotates the unit 3-sphere by θ about the positive x-axis. The rotor formulas for h_1 and h_1^{-1} are:

$\boldsymbol{h_1} = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3$ (θ rotation about positive x-axis)

$\boldsymbol{h_1^{-1}} = \mathbf{Cos}\left(\frac{\theta}{2}\right) - \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3$ (θ rotation about negative x-axis or $-\theta$ rotation about positive x-axis)

(a) The rotations of amount θ about the y- and z-axes and about the negative x-, y-, and z-axes also belong to H:

Lemma 5.1: Let

$$h_2 = \text{Cos}\left(\frac{\theta}{2}\right) - \text{Sin}\left(\frac{\theta}{2}\right) e_1 e_3 \quad (\theta \text{ rotation about positive y-axis}), \text{ and}$$

$$h_3 = \text{Cos}\left(\frac{\theta}{2}\right) + \text{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 \quad (\theta \text{ rotation about positive z-axis})$$

If h_1 belongs to a normal group H, then h_2, h_3, h_1^{-1}, h_2^{-1}, $h_3^{-1} \in$ H.

Proof: Define rotors

$$g_2 = \text{Cos}\left(\frac{\pi}{4}\right) + \text{Sin}\left(\frac{\pi}{4}\right) e_1 e_2 = \frac{1+e_1 e_2}{\sqrt{2}} \text{ and}$$

$$g_3 = \text{Cos}\left(\frac{\pi}{4}\right) + \text{Sin}\left(\frac{\pi}{4}\right) e_1 e_3 = \frac{1+e_1 e_3}{\sqrt{2}}.$$

Then

$$g_2^{-1} = \frac{1-e_1 e_2}{\sqrt{2}},$$

$$g_2^{-1} \circ h_1 = \frac{\text{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} - \frac{\text{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_1 e_2 - \frac{\text{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_1 e_3 + \frac{\text{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_2 e_3$$

$$g_2^{-1} \circ h_1 \circ g_2 = \text{Cos}\left(\frac{\theta}{2}\right) - \text{Sin}\left(\frac{\theta}{2}\right) e_1 e_3 = h_2$$

and

$$g_3^{-1} = \frac{1-e_1 e_3}{\sqrt{2}},$$

$$g_3^{-1} \circ h_1 = \frac{\text{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} + \frac{\text{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_1 e_2 - \frac{\text{Cos}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_1 e_3 + \frac{\text{Sin}\left(\frac{\theta}{2}\right)}{\sqrt{2}} e_2 e_3$$

$$g_3^{-1} \circ h_1 \circ g_3 = \text{Cos}\left(\frac{\theta}{2}\right) + \text{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 = h_3$$

Since H is normal, h_2, $h_3 \in$ H. Since groups are closed under inverses, h_1^{-1}, h_2^{-1}, $h_3^{-1} \in$ H. ■

Aside: The rotors g_2 and g_3 are 90° rotations about the z and y axes, respectively. That is apparently what is required in order for the normality operation on h_1 to generate θ rotations about the y and z axes, respectively.

(b) $S_\theta \subseteq H$:

Definition. In $\mathcal{R}$, let S_θ denote the sphere of radius θ. It consists of all rotations of amount θ about all axes in 3-space. The rotor formula for S_θ is

$$S_\theta = \left\{\mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)\left(a e_1 e_2 - b e_1 e_3 + c e_2 e_3\right) : a^2 + b^2 + c^2 = 1\right\}.$$

The next theorem shows that any element r of S_θ can be obtained from h_1 by using a normality operation.

Lemma 5.2: Let

$h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3$ and

$r = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\, e_1\, e_2 - b\, e_1\, e_3 + c\, e_2\, e_3\right] \in S_\theta$,

where

$a^2 + b^2 + c^2 = 1$ and $0 < \theta \leq \pi$.

(a) If $c \neq 0$ then

$\boxed{r = g_{23}{}^{-1} \circ h_1 \circ g_{23}}$ where

$g_2 = \mathbf{Cos}\left(\frac{\beta}{2}\right) + \mathbf{Sin}\left(\frac{\beta}{2}\right) e_1 e_3$,

$g_3 = \mathbf{Cos}\left(\frac{\gamma}{2}\right) + \mathbf{Sin}\left(\frac{\gamma}{2}\right) e_1 e_2$,

$g_{23} = g_2 \circ g_3$, and

$\beta = Arc\,\mathbf{Cos}\,[\sqrt{b^2 + c^2}\,]$ and $\gamma = Arc\,\mathbf{Tan}\,[\,c, b\,]$.

(b) If $c = 0$ then

$\boxed{r = g^{-1} \circ h_3 \circ g}$ where

$g = \mathbf{Cos}\left(\frac{\alpha}{2}\right) + \mathbf{Sin}\left(\frac{\alpha}{2}\right) e_2 e_3$, and

$\alpha = Arc\,\mathbf{Sin}\,[-\,b]$.

Proof: ArcTan[c, b] denotes the arc tangent of b/c taking into account which quadrant the point (c, b) is in.

(a) g_{23} = Cos (½ *Arc***Cos** [$\sqrt{b^2+c^2}$]) Cos (½ *Arc* **Tan** [*c*, *b*])
+ Cos (½ *Arc* **Cos** [$\sqrt{b^2+c^2}$]) Sin (½ *Arc* **Tan** [*c*, *b*]) $e_1\, e_2$
+ Cos (½ *Arc* **Tan** [*c*, *b*]) Sin (½ *Arc* **Cos** [$\sqrt{b^2+c^2}$]) $e_1\, e_3$
+ Sin (½ *Arc* **Cos** [$\sqrt{b^2+c^2}$]) Sin (½ *Arc* **Tan** [*c*, *b*]) $e_2\, e_3$

g_{23}^{-1} = Cos (½ *Arc***Cos** [$\sqrt{b^2+c^2}$]) Cos (½ *Arc* **Tan** [*c*, *b*])
– Cos (½ *Arc* **Cos** [$\sqrt{b^2+c^2}$]) Sin (½ *Arc* **Tan** [*c*, *b*]) $e_1\, e_2$
– Cos (½ *Arc* **Tan** [*c*, *b*]) Sin (½ *Arc* **Cos** [$\sqrt{b^2+c^2}$]) $e_1\, e_3$
– Sin (½ *Arc* **Cos** [$\sqrt{b^2+c^2}$]) Sin (½ *Arc* **Tan** [*c*, *b*]) $e_2\, e_3$

$$
\begin{aligned}
\mathrm{g23}^{-1} \circ \mathrm{h1} = {} & \mathrm{Cos}\left[\tfrac{\theta}{2}\right] \mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] \mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right] + \\
& \mathrm{Sin}\left[\tfrac{\theta}{2}\right] \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right] + \\
& \left(\mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right] \mathrm{Sin}\left[\tfrac{\theta}{2}\right] \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] - \right. \\
& \left. \mathrm{Cos}\left[\tfrac{\theta}{2}\right] \mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right]\right) e_1\, e_2 + \\
& \left(-\mathrm{Cos}\left[\tfrac{\theta}{2}\right] \mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right] \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] - \mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] \mathrm{Sin}\left[\tfrac{\theta}{2}\right] \right. \\
& \left. \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right]\right) e_1\, e_3 + \left(\mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] \mathrm{Cos}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right] \mathrm{Sin}\left[\tfrac{\theta}{2}\right] - \right. \\
& \left. \mathrm{Cos}\left[\tfrac{\theta}{2}\right] \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcCos}\left[\sqrt{b^2+c^2}\right]\right] \mathrm{Sin}\left[\tfrac{1}{2}\mathrm{ArcTan}[c,b]\right]\right) e_2\, e_3
\end{aligned}
$$

$$
\begin{aligned}
g_{23}^{-1} \circ h_1 \circ g_{23} &= \mathbf{Cos}\left(\frac{\theta}{2}\right) + \sqrt{1-b^2-c^2}\,\mathbf{Sin}\left(\frac{\theta}{2}\right) e_1\, e_2 - b\,\mathbf{Sin}\left(\frac{\theta}{2}\right) e_1\, e_3 + c\,\mathbf{Sin}\left(\frac{\theta}{2}\right) e_3\, e_3 \\
&= r \quad \text{since } a = \sqrt{1-b^2-c^2}.
\end{aligned}
$$

Note: Mathematica was used for this and other calculations. An example of actual Mathematica output is shown in light code in the next-to-last step above. I will be suppressing most intermediate steps in the remaining calculations. The interested reader can obtain all the steps by using the attached Mathematica file.

Since $c = 0$,

$b = \sqrt{1-a^2}$,

g = $\mathbf{Cos}\left(\frac{1}{2}\mathit{Arc}\,\mathbf{Sin}(b)\right) - \mathbf{Sin}\left(\frac{1}{2}\mathit{Arc}\,\mathbf{Sin}(b)\right) e_2\, e_3$, and

$g^{-1} \circ h_3 \circ g$ = $\mathbf{Cos}\left(\frac{\theta}{2}\right) + a\,\mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_2 - b\,\mathbf{Sin}\left(\frac{\theta}{2}\right) e_1 e_3$

= r ■

Theorem 5: If $1 \neq h \in$ H and $\theta = |h|$, then H $\supseteq S_\theta$.

Proof: WLOG we can assume h is a θ rotation about the x-axis. That is,

$$h = h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3 .$$

Let $r \in S_\theta$. Then for some a, b, c such that $a^2 + b^2 + c^2 = 1$,

$$r = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2}\left[a\, e_1 e_2 - b\, e_1 e_3 + c\, e_2 e_3\right] .$$

If $c \neq 0$, by Lemma 5.2a, $r \in$ H since $r = g_{23}{}^{-1} \circ h_1 \circ g_{23}$ and H is normal.
If $c = 0$, by Lemma 5.2b, $r \in$ H since $r = g^{-1} \circ h_3 \circ g$, H is normal, and $h_3 \in$ H (by Lemma 5.1). ■

(c) If $0 < \theta \leq \pi$, there is a rotation angle $\psi \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ such that $S_\psi \subseteq$ H:

Observation. For any $g \in$ SO(3), the normality operation $g^{-1} \circ h \circ g$ results in a rotation having the same rotation angle as h. Thus, for a given $h \in$ SO(3), $\left\{g^{-1} \circ h \circ g : g \in \mathrm{SO(3)}\right\}$ cannot generate all of SO(3). Theorem 5 tells us that it at least generates *all* of S_θ. From there we switch tactics, taking the geometric product of pairs of elements in S_θ to generate S_ψ for some $\frac{\pi}{2} \leq \psi \leq \frac{2\pi}{3}$. Then we take the geometric product of pairs of elements in S_ψ to generate the rest of SO(3).

In Theorem 6, we will need the claim in the first sentence of this observation, so we prove it now.

Lemma 6.1: Let $0 \leq \theta \leq \pi$, $h \in S_\theta$, and $g \in$ SO(3). Then $g^{-1} \circ h \circ g \in S_\theta$.

Proof: WLOG $h = h_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3$. Since 2 vectors determine a plane, WLOG we can assume that g lies in the xy-plane. That is, if the magnitude of g is $0 \leq \phi \leq \pi$ then $g = \mathbf{Cos}\frac{\phi}{2} + \mathbf{Sin}\frac{\phi}{2}\left[a\, e_2 e_3 - b\, e_1 e_3\right]$ where $a^2 + b^2 = 1$. Then

$$
\begin{aligned}
g^{-1}hg = &\ \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2\,\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right) + b^2\,\mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\
&-2b\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}\left(\frac{\phi}{2}\right)\mathbf{Cos}\left(\frac{\phi}{2}\right)e_1e_2 - 2ab\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)e_1e_3 \\
&+\left[\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2\,\mathbf{Sin}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right]e_2e_3 \\
= &\ \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + \mathbf{Cos}\left(\frac{\theta}{2}\right)\mathbf{Sin}^2\left(\frac{\phi}{2}\right) \\
&-\mathbf{Sin}\left(\frac{\theta}{2}\right)\left\{\begin{array}{l} 2b\,\mathbf{Sin}\left(\frac{\phi}{2}\right)\mathbf{Cos}\left(\frac{\phi}{2}\right)e_1e_2 - 2ab\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)e_1e_3 \\ + \left[\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right]e_2e_3 \end{array}\right\} \\
= &\ \mathbf{Cos}\left(\frac{\theta}{2}\right) \\
&+\mathbf{Sin}\left(\frac{\theta}{2}\right)\left\{\begin{array}{l} -2b\,\mathbf{Sin}\left(\frac{\phi}{2}\right)\mathbf{Cos}\left(\frac{\phi}{2}\right)e_1e_2 + 2ab\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)e_1e_3 \\ - \left[\mathbf{Cos}^2\left(\frac{\phi}{2}\right) + a^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right) - b^2\,\mathbf{Sin}^2\left(\frac{\phi}{2}\right)\right]e_2e_3 \end{array}\right\}
\end{aligned}
$$

Since the constant term of $g^{-1}\circ h\circ g$ is $\mathbf{Cos}\frac{\theta}{2}$, $g^{-1}\circ h\circ g \in S_\theta$.■

The second and third parts of the previous Observation reference the ability to generate one sphere in $\mathcal{S}$ from another by taking products of pairs of elements in the first sphere. We will need this result in Theorem 9 so we prove it now.

Theorem 6: Let k_1, $k_2 \in S_\phi$ for some $0<\phi\le\pi$, and let $\theta = |k_1\circ k_2|$. Then S_θ can be generated by taking geometric products of pairs of elements from S_ϕ.

Proof. The theorem is trivially true if $\theta = 0$. So assume $0 < \theta \le \pi$.

Let $r \in S_\theta$. The claim is that we can find f_1, $f_2 \in S_\phi$ such that $r = f_1 \circ f_2$. Since $k_1\circ k_2$ is a random element of S_θ, WLOG we can let $k_1\circ k_2$ be the θ rotation about the *x*-axis. That is, $h_1 = k_1\circ k_2 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right)e_2e_3$.

Since $r \in S_\theta$, $\exists$ a, b, c such that $r = \mathbf{Cos}\,\frac{\theta}{2} + \mathbf{Sin}\,\frac{\theta}{2}\left[a\, e_1\, e_2 - b\, e_1\, e_3 + c\, e_2\, e_3\right]$, where $a^2 + b^2 + c^2 = 1$.

Lemma 5.2 yields that either (a) $r = g_{23}^{-1} \circ h_1 \circ g_{23}$ or (b) $r = g^{-1} \circ h_3 \circ g$, where $h_3 \in S_\theta$ is the θ rotation about the z-axis.

(a) Define $f_1 = g_{23}^{-1} \circ k_1 \circ g_{23}$ and $f_2 = g_{23}^{-1} \circ k_2 \circ g_{23}$. By Lemma 6.1, $f_1, f_2 \in S_\phi$. Thus

$$r = g_{23}^{-1} \circ h_1 \circ g_{23} = g_{23}^{-1} \circ k_1 \circ k_2 \circ g_{23} = \left(g_{23}^{-1} \circ k_1 \circ g_{23}\right) \circ \left(g_{23}^{-1} \circ k_2 \circ g_{23}\right)$$
$$= f_1 \circ f_2 \qquad ✔$$

(b) As was shown in Lemma 5.1, h_3 can be obtained from h_1 by the 90° rotation $g_3 = \frac{1 + e_1 e_3}{\sqrt{2}}$. The rotation formula is $h_3 = g_3^{-1} \circ h_1 \circ g_3$. Apply the same 90° rotation to k_1 and k_2 :

$$k_3 = g_3^{-1} \circ k_1 \circ g_3 \quad \text{and} \quad k_4 = g_3^{-1} \circ k_2 \circ g_3 \ .$$

$k_3, k_4 \in S_\phi$ and

$$h_3 = g_3^{-1} \circ\ h_1 \circ\ g_3 = g_3^{-1} \circ\ k_1 \circ k_2 \circ g_3$$
$$= \left(g_3^{-1} \circ\ k_1 \circ\ g_3\right) \circ \left(g_3^{-1} \circ\ k_2 \circ\ g_3\right) \ .$$
$$= k_3 \circ k_4$$

Define $f_1 = g^{-1} \circ k_3 \circ g$ and $f_2 = g^{-1} \circ k_4 \circ g$. By Lemma 6.1, $f_1, f_2 \in S_\phi$. Thus

$$r = g^{-1} \circ h_3 \circ g = g^{-1} \circ k_3 \circ k_4 \circ g = \left(g^{-1} \circ k_3 \circ g\right) \circ \left(g^{-1} \circ k_4 \circ g\right)$$
$$= f_1 \circ f_2 \qquad ✔ \qquad \blacksquare$$

The interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, below, is magical. Theorem 7 does not hold for most angles outside this interval.

Lemma 7.1: If $0 < \theta < \pi$, there is a positive integer n such that $\frac{\pi}{2} \le |\, h_1^n \,| \le \frac{2\pi}{3}$.

Proof: h_1 = $\mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2\, e_3$ rotates the 3-sphere an amount θ about the positive x-axis. So, h_1^n rotates the 3-sphere an amount $n\theta$ about the positive x-axis. A rotation angle $\theta > 2\pi$ is equivalent to a rotation amount θ mod (2π). So, the rotation angle $\psi = |h_1^n|$ satisfies the formula $\psi = n\theta$ mod (2π).

The claim is:

$$\text{for } 0 < \theta < \pi \ \exists\, n \text{ such that } [n\theta \bmod (2\pi)] \in \left[\frac{1}{2}\pi,\ \frac{2}{3}\pi\right] \qquad \text{(iv)}$$

It is convenient to take π out of the discussion. (iv) is equivalent to:

$$\text{for } 0 < x < 1 \ \exists\, n \text{ such that } [nx \bmod 2] \in \left[\frac{1}{2},\ \frac{2}{3}\right]. \qquad \text{(v)}$$

If x is an irrational number, then $\{nx \bmod 2 : n \in \mathbb{Z}^+\}$ is an infinite set that is dense in the interval [0, 2]. Therefore (v) is satisfied.

Suppose x is a rational number. x is between 0 and 1. For a denominator of 2, set $n = 1$ for $x = ½$. For a denominator of 3, set $n = 1$ for $x = ⅔$ and $n = 2$ for $x = ⅓$. Thus rational numbers with denominators of 2 and 3 satisfy (v).

For a rational number x with a larger denominator, we can assume x is written as a reduced fraction $\frac{p}{q}$ where $q > 3$, $p < q$, and p and q are relatively prime. Call the set $\{x, 2x, 3x, \cdots\}$ mod 2 the **orbit** of x. For any such rational number $x = \frac{p}{q}$, the orbit of x is the same as the orbit of $\frac{1}{q}$. Therefore it is sufficient to examine the orbit of the rational number $\frac{1}{q}$.

If q is even, then $n = \frac{q}{2}$ satisfies (v) since $nx = ½$. If q is odd, then $q = 2s + 1$ for some s. Setting $n = s + 1$ satisfies (v) since $\frac{1}{2} < \frac{s+1}{2s+1} < \frac{2}{3}$ since $s > 1$ (because q > 3). ■

Observation: Lemma 7.1 does not hold for $\theta = \pi$ because h_1^2 has angle of rotation 2π, which equals 0 mod 2π. Thus $h_1^3 = \pi \bmod 2\pi$, $h_1^4 = 0 \bmod 2\pi$, etc., and so no power of h_1 lies in the interval $\left[\frac{1}{2}\pi,\ \frac{2}{3}\pi\right]$.

The next theorem, based on Lemma 7.1, holds also for $\theta = \pi$, which is handled as a special case.

Theorem 7: If $0 < \theta \leq \pi$, there is an angle $\frac{\pi}{2} \leq \psi \leq \frac{2\pi}{3}$ such that $S_\psi \subseteq H$.

Proof: Suppose 0 < θ < π. Set $h = h_1^n$. Since $h_1 \in H$, then $h \in H$. $h \neq 1$ since $|h| > 0$. Set $\psi = |h|$. By Lemma 7.1, $\frac{\pi}{2} \leq \psi \leq \frac{2\pi}{3}$. Replacing θ by ψ in Theorem 5 yields $S_\psi \subseteq H$. This proves Theorem 7 except for the special case of $\theta = \pi$.

Suppose $\theta = \pi$. Then h_1 = $\mathbf{Cos}\left(\frac{\pi}{2}\right) + \mathbf{Sin}\left(\frac{\pi}{2}\right) e_2 e_3$ = $e_2 e_3$. By Theorem 5, $H \supseteq S_\pi$.

Define

$$h \equiv \frac{1}{\sqrt{2}}\left(e_1 e_2 + e_2 e_3\right) = \mathbf{Cos}\frac{\pi}{2} + \mathbf{Sin}\frac{\pi}{2}\left(\frac{1}{\sqrt{2}} e_1 e_2 + \frac{1}{\sqrt{2}} e_2 e_3\right) \in S_\pi \subseteq H.$$

Then

$$h \circ h_1 = \frac{1}{\sqrt{2}}\left(e_1 e_2 + e_2 e_3\right) \circ e_2 e_3 = \frac{1}{\sqrt{2}}\left(e_1 e_3 - 1\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(e_1 e_3\right)$$

By Theorem 3, $h \circ h_1 \notin S$ and is equivalent to to $-h \circ h_1 \in S$. Since $\mathbf{Sin}\frac{\pi}{4}$ = $\mathbf{Cos}\frac{\pi}{4}$ = $\frac{1}{\sqrt{2}}$, then $-h \circ h_1 = \mathbf{Cos}\frac{\pi}{4} - \mathbf{Sin}\frac{\pi}{4} e_1 e_3 \in S_{\frac{\pi}{2}}$. By Theorem 5, $H \supseteq S_{\frac{\pi}{2}}$ because $1 \neq h \circ h_1 \in H$. Set $\psi = \frac{\pi}{2}$. Then, trivially, $\frac{\pi}{2} \leq \psi \leq \frac{2\pi}{3}$ and $H \supseteq S_\psi$. ■

(d) SO(3) is generated from geometric products of pairs of elements in S_ψ if $\frac{\pi}{2} \leq \psi \leq \frac{2\pi}{3}$ or $\psi = \pi$.

The next theorem shows the "magical" property of the rotation angle interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ being able to generate any angle ω by the geometric product of a pair of rotors.

Theorem 8: Suppose $\psi \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ and $0 < \omega \leq \pi$. Then there is a pair of elements in S_ψ whose geometric product lies in S_ω.

Proof: Let k be the rotor in S_ψ that rotates by ψ about the x-axis:

$$k = \mathbf{Cos}\left(\frac{\psi}{2}\right) + \mathbf{Sin}\left(\frac{\psi}{2}\right) e_2 e_3.$$

k is the first desired rotor. We will find a second rotor, $h \in S_\psi$, such that $h \circ k \in S_\omega$.To define h, we need a, b, and c such that

$$a^2 + b^2 + c^2 = 1,$$

$$h = \mathbf{Cos}\left(\frac{\psi}{2}\right) + \mathbf{Sin}\left(\frac{\psi}{2}\right)\left(a e_1 e_2 - b e_1 e_3 + c e_2 e_3\right), \text{ and}$$

$$|h \circ k| = \omega.$$

Recall that $|h \circ k| = \omega \Leftrightarrow$ Constant term of $h \circ h_1$ = $\mathbf{Cos}\left(\frac{\omega}{2}\right)$.

By performing the geometric product calculation we find that

$$\begin{aligned} h \circ k = {} & \mathbf{Cos}^2\left(\frac{\psi}{2}\right) - c\,\mathbf{Sin}^2\left(\frac{\psi}{2}\right) \\ & + \left(a\,\mathbf{Cos}\left(\frac{\psi}{2}\right)\mathbf{Sin}\left(\frac{\psi}{2}\right) + b\,\mathbf{Sin}^2\left(\frac{\psi}{2}\right) \right) e_1 e_2 \\ & + \left(-b\,\mathbf{Cos}\left(\frac{\psi}{2}\right)\mathbf{Sin}\left(\frac{\psi}{2}\right) + a\,\mathbf{Sin}^2\left(\frac{\psi}{2}\right) \right) e_1 e_3 \\ & + \left(\mathbf{Cos}\left(\frac{\psi}{2}\right)\mathbf{Sin}\left(\frac{\psi}{2}\right) + c\,\mathbf{Cos}\left(\frac{\psi}{2}\right)\mathbf{Sin}\left(\frac{\psi}{2}\right) \right) e_2 e_3 \end{aligned}$$

The constant term of rotor $h \circ k$ = $\mathbf{Cos}^2\left(\frac{\psi}{2}\right) - c\,\mathbf{Sin}^2\left(\frac{\psi}{2}\right)$.

Setting $\mathbf{Cos}\left(\frac{\omega}{2}\right) = \mathbf{Cos}^2\left(\frac{\psi}{2}\right) - c\,\mathbf{Sin}^2\left(\frac{\psi}{2}\right)$ and solving for c yields

$$c = \frac{\mathbf{Cos}^2\left(\frac{\psi}{2}\right) - \mathbf{Cos}\left(\frac{\omega}{2}\right)}{\mathbf{Sin}^2\left(\frac{\psi}{2}\right)}.$$

This is well defined except when $\mathbf{Sin}^2\left(\frac{\psi}{2}\right) = 0$, which occurs only when $\psi = 0$ and when $\psi = 2\pi$. Since these values are outside the domain $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ of ψ, the definition of c is well defined. "a" and "b" can be any numbers so long as $a^2 + b^2 + c^2 = 1$.

We also require $c^2 \le 1$ (since $a^2 + b^2 + c^2 = 1$). So to complete the proof of this theorem, we must show that $|c| \le 1$. (Then we can set, for example, $b = 0$ and $a = \sqrt{1-c^2}$.)

Since $0 < \omega \le \pi$,

$$0 < \frac{\omega}{2} \le \frac{\pi}{2},$$

$$\mathbf{Cos}\frac{\pi}{2} \le \mathbf{Cos}\frac{\omega}{2} < \mathbf{Cos}(0),$$

$$0 \le \mathbf{Cos}\left(\frac{\omega}{2}\right) < 1,$$

$$-1 < -\mathbf{Cos}\left(\frac{\omega}{2}\right) \le 0,$$

$$1 - 2\,\mathbf{Cos}\left(\frac{\omega}{2}\right) \le 1, \qquad \text{(vi)}$$

and

$$-1 < \frac{1}{3} - \frac{4}{3}\mathbf{Cos}\left(\frac{\omega}{2}\right). \qquad \text{(vii)}$$

Here is where the magical interval comes into play. Since $\frac{\pi}{2} \le \psi \le \frac{2\pi}{3}$, then

$$\frac{\pi}{4} \le \frac{\psi}{2} \le \frac{\pi}{3},$$

$$\frac{1}{2} \le \mathbf{Cos}\left(\frac{\psi}{2}\right) \le \frac{1}{\sqrt{2}},$$

$$\frac{1}{4} \le \mathbf{Cos}^2\left(\frac{\psi}{2}\right) \le \frac{1}{2}, \qquad \text{(viii)}$$

$$\frac{1}{4} \le 1 - \mathbf{Sin}^2\left(\frac{\psi}{2}\right) \le \frac{1}{2},$$

$$-\frac{1}{2} \le \mathbf{Sin}^2\left(\frac{\psi}{2}\right) - 1 \le -\frac{1}{4},$$

$$\frac{1}{2} \le \mathbf{Sin}^2\left(\frac{\psi}{2}\right) \le \frac{3}{4}. \qquad \text{(ix)}$$

Thus,

$$-1 \overset{\text{(vii)}}{<} \frac{1}{3} - \frac{4}{3}\mathbf{Cos}\frac{\omega}{2} = \frac{\frac{1}{4} - \mathbf{Cos}\frac{\omega}{2}}{\frac{3}{4}} \overset{\text{(viii, ix)}}{\leq} \frac{\mathbf{Cos}^2\left(\frac{\psi}{2}\right) - \mathbf{Cos}\left(\frac{\omega}{2}\right)}{\mathbf{Sin}^2\left(\frac{\psi}{2}\right)} = c$$

$$\overset{\text{(viii, ix)}}{\leq} \frac{\frac{1}{2} - \mathbf{Cos}\frac{\omega}{2}}{\frac{1}{2}} = 1 - 2\mathbf{Cos}\frac{\omega}{2} \overset{\text{(vi)}}{\leq} 1$$

That is, $|\, c \,| \leq 1$. This completes the proof of the theorem. ■

Corollary 8.1: Suppose $\psi \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, $0 < \omega \leq \pi$, and $S_\psi \subseteq H$, where H is a group, not necessarily normal. Then S_ω is generated by geometric products of pairs of elements from S_ψ, and so $S_\omega \subseteq H$.

Proof: By Theorem 8, $\exists\, k_1, k_2 \in S_\psi \subseteq H$ such that $k_1 \circ k_2 \in S_\omega$. Replacing ϕ by ψ and θ by ω in Theorem 6 yields $S_\omega = \{h \circ k : h, k \in S_\psi\} \subseteq H$. ■

Theorem 9: SO(3) contains no proper normal subgroup.

Proof: Let H be a non-trivial normal subgroup of SO(3). H contains a non-identity element. WLOG $1 \neq h_1 \in H$ where h_1 is a θ rotation about the x-axis and $0 < \theta \leq \pi$. By Theorem 7, there is an angle $\frac{\pi}{2} \leq \psi \leq \frac{2\pi}{3}$ such that $S_\psi \subseteq H$.

We wish to show that H = SO(3). It suffices to show that H contains every sphere S_ω in $\mathcal{R}$. For $\omega = 0$, $S_\omega \subseteq H$ since $1 \in H$. So let $0 < \omega \leq \pi$ be an arbitrary rotation angle. By Corollary 8.1 $S_\psi \subseteq H$. ■

(2) Theorem 10: SO(3) is the only proper normal subgroup of O(3).

Proof: Recalling that R is the reflection operator, let

T = R [SO(3)].

That is, O(3) is the disjoint union **O(3) = T ∪ SO(3)**.

To help the reader keep track of variables, we will use s, s_1, s_2 and s_3 for elements of SO(3) and t, t_1, and t_2 for elements of T.

Suppose H ≠ SO(3) is a non-trivial normal subgroup of O(3). From Theorem 9 we know that H is not a subset of SO(3). Thus,

$$\exists\, \boldsymbol{t_1} \in \mathrm{H} \cap \mathrm{T}.$$

Let

$$\boldsymbol{s_1} = \mathrm{R}\, t_1 \in \mathrm{SO}(3).$$

Then

$$t_1 = \mathrm{R}\, s_1.$$

Let

$$\boldsymbol{\theta} = |\, s_1 \,| = |\, t_1 \,|.$$

WLOG, s_1 is a rotation about the ***x*-axis**:

$$s_1 = \mathbf{Cos}\left(\frac{\theta}{2}\right) + \mathbf{Sin}\left(\frac{\theta}{2}\right) e_2 e_3.$$

Define

$$\boldsymbol{g_2} = \mathbf{Cos}\left(\frac{\pi}{4}\right) + \mathbf{Sin}\left(\frac{\pi}{4}\right) e_1 e_3 = \frac{1 + e_1 e_3}{\sqrt{2}}$$, a 90° rotation about the ***y-axis***.

$g_2 \in \mathrm{SO}(3)$ since it is a rotation. Hence,

$$\boldsymbol{s_2} \equiv g_2^{-1} \circ s_1 \circ g_2 \in \mathrm{SO}(3).$$

Computing the geometric product yields

$$s_2 = \mathbf{Cos}\frac{\theta}{2} + \mathbf{Sin}\frac{\theta}{2} e_1 e_2,$$

the rotation of magnitude θ about the positive ***z*-axis**. Then

$$\boldsymbol{t_2} \equiv \mathrm{R}(s_2) = g_2^{-1} \circ \mathrm{R}(s_1) \circ g_2 = g_2^{-1} \circ t_1 \circ g_2 \in \mathrm{H}$$

because H is normal. So,

$$s_3 \equiv t_1 \circ t_2 = \mathrm{R}(s_1) \circ \mathrm{R}(s_2) = s_1 \circ s_2.$$

s_1 and s_2 are not inverses of each other because they are rotations about perpendicular axes. (Recall that inverse pairs rotate in opposite directions about the same axis.) Because $s_3 = s_1 \circ s_2$,

$$s_3 \neq 1.$$

$s_3 \in \mathrm{H}$ since $t_1, t_2 \in \mathrm{H}$. $s_3 \in \mathrm{SO}(3)$ since $s_1, s_2 \in \mathrm{SO}(3)$. So,

$$1 \neq s_3 \in \mathrm{H} \cap \mathrm{SO}(3).$$

$\mathrm{H} \cap \mathrm{SO}(3)$ is thus a non-trivial normal subgroup of SO(3), so by Theorem 9

$$\mathrm{H} \cap \mathrm{SO}(3) = \mathrm{SO}(3).$$

Thus,

$$\mathrm{SO}(3) \subset \mathrm{H}.$$

Finally, we are in a position to show that $\mathrm{T} \subset \mathrm{H}$, implying that H = O(3). Let $t \in \mathrm{T}$ be any element of T. This proof is finished if we can show $t \in \mathrm{H}$. Let

$$s \equiv \mathrm{R}(t) \in \mathrm{SO}(3).$$

Then

$$t = \mathrm{R}(s).$$

Set

$$g \equiv s_1^{-1} \circ s \in \mathrm{SO}(3) \subset \mathrm{H}.$$

Then

$$s = (s_1 \circ s_1^{-1}) \circ s = s_1 \circ (s_1^{-1} \circ s) = s_1 \circ g.$$

Therefore

$$t = \mathrm{R}(s) = \mathrm{R}(s_1 \circ g) = \mathrm{R}(s_1) \circ g = t_1 \circ g \in \mathrm{H}$$

since $t_1, g \in \mathrm{H}$, and that completes the proof the theorem and part (B). ■

APPENDIX: The 2 insights that suggested how to define the rotor g_{23} in Lemma 5.2.

The hardest part of this proof was to invent the rotor g_{23}. I couldn't solve a certain system of 4 non-linear simultaneous equations that would have provided it, but I was able to guess a solution after examining lots of rotations.

The **1st insight**, gained during Lemma 5.1, is that a rotor, say h_3, about the z-axis, is obtained by $g_2^{-1} \circ h_1 \circ g_2$ where g_2 is a rotor about the y-axis with a 90° rotation angle and h_1 is a rotor about the x-axis having the same angle of rotation, θ, *as* h_2. Loosely speaking, I learned that a 90° rotation g_2 yields a rotation about an axis 90° away from h_1.

2nd Insight comes from discovering that reducing the g_2 angle of rotation to 45° generates a rotation about the 45° diagonal between the x-axis and z-axis (i.e., about an axis 45° away from h_1), seen below:

Let

$$g_2 = \mathbf{Cos}\left(\frac{\pi}{8}\right) - \mathbf{Sin}\left(\frac{\pi}{8}\right) e_1\, e_3.$$

Then

$$\begin{aligned} g_2^{-1} \circ h_1 \circ g_2 &= \mathbf{Cos}\left(\frac{\theta}{2}\right) - \frac{1}{\sqrt{2}}\mathbf{Sin}\left(\frac{\theta}{2}\right) e_1\, e_2 + \frac{1}{\sqrt{2}}\mathbf{Sin}\left(\frac{\theta}{2}\right) e_2\, e_3 \\ &= \text{45° diagonal axis in the } zx\text{-plane.} \end{aligned}$$

To generate a 45° diagonal axis not in the xy-, yz-, or zx-planes, I made a guess that it requires products of rotors like g_2 and g_3 where g_2 generates a skew axis between the z and x-axes, and then g_3 generates an out-of-zx-plane skew axis towards the y-axis. Armed with the guess $g_{23} = g_2 \circ g_3$, for a given angle ψ, I assigned g_2 and g_3 rotor angles β and γ, respectively, performed the $g_{23}^{-1} \circ h \circ g_{23}$ GA multiplication with Mathematica, and solved for a constant term equal to $\mathbf{Cos}\left(\frac{\psi}{2}\right)$, yielding the *Arc***Cos** and *Arc***Tan** values used in Lemma 5.2. The proof in Lemma 5.2 is basically just the confirmation that that the solution I found is correct.