[12.8] Let $\alpha = \sum_{r=1}^{n} \alpha_r dx^r, \cdots, \gamma = \sum_{u=1}^{n} \gamma_u dx^u, \lambda = \sum_{j=1}^{n} \lambda_j dx^j, \cdots, \upsilon = \sum_{m=1}^{n} \upsilon_m dx^m$ be independent 1-forms in $\Re^n$. Let $\phi = \alpha \wedge \cdots \wedge \gamma$ be a p-form and $\chi = \lambda \wedge \cdots \wedge \upsilon$ be a q-form. Show that $\phi \wedge \chi = \alpha \wedge \cdots \wedge \gamma \wedge \lambda \wedge \cdots \wedge \upsilon$.

Proof: I adopt Juergen Beckmann's excellent mathematical notation for the antisymmetrization concepts per his solution to this same problem.

Notation. Let M = {1, 2, … , n}, $\mathcal{P}_{r \dots u}$ = set of permutations of the p-tuple (r, … , u) and $\mathcal{P}_{j \dots m}$ = set of permutations of the q-tuple (j, … , m). Then

$$\phi = \alpha \wedge \cdots \wedge \gamma = \sum_{(r,\cdots,u) \in M^p} \alpha_{[r} \cdots \gamma_{u]} \, dx^r \wedge \cdots \wedge dx^u \tag{1}$$

where

$$\alpha_{[r} \cdots \gamma_{u]} = \frac{1}{p!} \sum_{\pi \in \mathcal{P}_{r \cdots u}} sign(\pi) \, \alpha_{\pi(r)} \wedge \cdots \wedge \gamma_{\pi(u)}$$

and

$$\chi = \lambda \wedge \cdots \wedge \upsilon = \sum_{(j,\cdots,m) \in M^q} \lambda_{[j} \cdots \upsilon_{m]} \, dx^j \wedge \cdots \wedge dx^m \tag{2}$$

where

$$\lambda_{[j} \cdots \upsilon_{m]} = \frac{1}{q!} \sum_{\pi \in \mathcal{P}_{j \cdots m}} sign(\pi) \, \alpha_{\pi(j)} \cdots \gamma_{\pi(m)}.$$

Thus

$$\phi \wedge \chi = \sum_{(r,\cdots,u) \in M^p} \sum_{(j,\cdots,m) \in M^q} \alpha_{[[r} \cdots \gamma_{u]} \lambda_{[j} \cdots \upsilon_{m]]} \, dx^r \wedge \cdots \wedge dx^u \wedge dx^j \wedge \cdots \wedge dx^m, \tag{3}$$

where the brackets inside a bracket [[r … u] [j … m]] means to antisymmetrize the two antisymmetrizations or, as Juergen states in his proof of this problem, to take the "average" of the two "averages".

On the other hand,

$$\alpha \wedge \cdots \wedge \gamma \wedge \lambda \wedge \cdots \wedge \upsilon$$

$$= \sum_{(r,\cdots,u,j,\cdots,m) \in M^{p+q}} \alpha_{[r} \cdots \gamma_u \lambda_j \cdots \upsilon_{m]} \, dx^r \wedge \cdots \wedge dx^u \wedge dx^j \wedge \cdots \wedge dx^m. \tag{4}$$

So the gist of this problem is to show that

$$\sum_{(r,\cdots,u)\in M^p}\ \sum_{(j,\cdots,m)\in M^q} \alpha_{[[r}\cdots\gamma_{u]}\,\lambda_{[j}\cdots\upsilon_{m]]} = \sum_{(r,\cdots,u,j,\cdots,m)\in M^{p+q}} \alpha_{[r}\cdots\gamma_u\,\lambda_j\cdots\upsilon_{m]},$$

that the average of all the terms equals the average of the two sub-averages.

Lemma. The sum of the antisymmetrized quantities $\alpha_{[r}\cdots\delta_{u]}$ equals the sum of the original quantities $\alpha_r\cdots\delta_u$. That is,

$$\sum_{(r,\cdots,u)\in M^p} \alpha_{[r}\cdots\gamma_{u]} = \sum_{(r,\cdots,u)\in M^p} \alpha_r\cdots\gamma_u \tag{5}$$

Proof: Fix a p-tuple $(r_0, \ldots, u_0)$ and consider the RHS term $\alpha_{r_0}\cdots\gamma_{u_0}$. Where does it appear in the LHS? One place is in the term

$\alpha_{[r_0}\cdots\gamma_{u_0]} = \dfrac{1}{p!}\sum_{\pi\in\mathcal{P}_{r_0\cdots u_0}} sign(\pi)\,\alpha_{\pi(r_0)}\wedge\cdots\wedge\gamma_{\pi(u_0)}$ where it appears precisely once as $\frac{1}{p!}\alpha_{r_0}\cdots\gamma_{u_0}$. In fact, there are p! permutations of $\alpha_{[r_0}\cdots\gamma_{u_0]}$ and it appears as $\frac{1}{p!}\alpha_{r_0}\cdots\gamma_{u_0}$ precisely once in each such permutation, and nowhere else. (Careful examination shows that it always appears as $+\frac{1}{p!}\alpha_{r_0}\cdots\gamma_{u_0}$, whether π is even or odd.) Thus the term $\alpha_{r_0}\cdots\gamma_{u_0}$ appears precisely once on both sides of the equation. This proves that the terms on the RHS are precisely matched by the terms on the LHS, which proves the lemma. ✔

An example is helpful to clarify the notation in (1 – 4) and the lemma (5).

Example. Let $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2$ and $\alpha = \beta_1 dx^1 + \beta_2 dx^2$ be 1-forms in $\Re^2$. Then p = 2, M = {1, 2}, so $M^p = \{1,2\}^2 = \{1,2\}\,x\,\{1,2\} = \{(1,1),(1,2),(2,1),(2,2)\}$. So,

$$\begin{aligned}
\phi = \alpha\wedge\beta &= \sum_{(r\,s)\in M^2} \alpha_{[r}\,\beta_{s]}\, dx^r\wedge dx^s \\
&= \alpha_{[1}\beta_{1]}\,dx^1\wedge dx^1 + \alpha_{[1}\beta_{2]}\,dx^1\wedge dx^2 + \alpha_{[2}\beta_{1]}\,dx^2\wedge dx^1 + \alpha_{[2}\beta_{2]}\,dx^2\wedge dx^2 \\
&= \alpha_{[1}\beta_{2]}\,dx^1\wedge dx^2 + \alpha_{[2}\beta_{1]}\,dx^2\wedge dx^1 \\
&= \tfrac{1}{2}\left(\alpha_1\beta_2 - \alpha_2\beta_1\right)dx^1\wedge dx^2 + \tfrac{1}{2}\left(\alpha_2\beta_1 - \alpha_1\beta_2\right)dx^2\wedge dx^1 \\
&= \left[\tfrac{1}{2}\left(\alpha_1\beta_2 - \alpha_2\beta_1\right) - \tfrac{1}{2}\left(\alpha_2\beta_1 - \alpha_1\beta_2\right)\right]dx^1\wedge dx^2 \\
&= \left[\tfrac{1}{2}\left(\alpha_1\beta_2 - \alpha_2\beta_1\right) + \tfrac{1}{2}\left(\alpha_1\beta_2 - \alpha_2\beta_1\right)\right]dx^1\wedge dx^2
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_1 \beta_2\, dx^1 \wedge dx^2 - \alpha_2 \beta_1\, dx^1 \wedge dx^2 \\
&= \alpha_1 \beta_2\, dx^1 \wedge dx^2 + \alpha_2 \beta_1\, dx^2 \wedge dx^1 \\
&= \alpha_1 \beta_1\, dx^1 \wedge dx^1 + \alpha_1 \beta_2\, dx^1 \wedge dx^2 + \alpha_2 \beta_1\, dx^2 \wedge dx^1 + \alpha_2 \beta_2\, dx^2 \wedge dx^2 \\
&= \sum_{(r\ s) \in M^2} \alpha_r \beta_s\, dx^r \wedge dx^s
\end{aligned}
$$

This illustrates both the summation notation and the lemma.

$\mathcal{P}_{12} = \{\pi_1, \pi_2\}$ where

$$
\pi_1 \mathbf{:} \begin{cases} 1 \to 1 \\ 2 \to 2 \end{cases} \mathbf{,} \quad \pi_2 \mathbf{:} \begin{cases} 1 \to 2 \\ 2 \to 1 \end{cases} , \quad sign(\pi_1) = +1, \text{ and } sign(\pi_2) = -1.
$$

So, for example,

$$
\begin{aligned}
\alpha_{[1} \beta_{2]} &= \frac{1}{2}\left(\alpha_1 \beta_2 - \alpha_2 \beta_1\right) = \frac{1}{2!}\left(sign(\pi_1)\, \alpha_{\pi_1(1)} \beta_{\pi_1(2)} + sign(\pi_2)\, \alpha_{\pi_2(1)} \beta_{\pi_2(2)}\right) \\
&= \frac{1}{p!} \sum_{\pi = \mathcal{P}_{12}} sign(\pi)\, \alpha_{\pi(1)\pi(2)}.
\end{aligned}
$$

This illustrates the permutation notation and concludes the example.

Continuing the proof, we can use the lemma (5) to rewrite equations (3) and (4):

$$
\phi \wedge \chi = \sum_{(r, \cdots, u) \in M^p} \sum_{(j, \cdots, m) \in M^q} \alpha_r \cdots \gamma_u\, \lambda_j \cdots \upsilon_m\, dx^r \wedge \cdots \wedge dx^u \wedge dx^j \wedge \cdots \wedge dx^m \qquad \mathbf{(3')}
$$

and

$$
\begin{aligned}
&\alpha \wedge \cdots \wedge \gamma \wedge \lambda \wedge \cdots \wedge \upsilon \\
&= \sum_{(r, \cdots, u, j, \cdots, m) \in M^{p+q}} \alpha_r \cdots \gamma_u\, \lambda_j \cdots \upsilon_m\, dx^r \wedge \cdots \wedge dx^u \wedge dx^j \wedge \cdots \wedge dx^m. \qquad \mathbf{(4')}
\end{aligned}
$$

Observe that both expressions (3') and (4') have n^{p+q} terms that are identical. Thus

$\phi \wedge \chi = \alpha \wedge \cdots \wedge \gamma \wedge \lambda \wedge \cdots \wedge \upsilon$, concluding the proof. ✔

Note. Juergen solved this problem by showing directly that antisymmetrization of the *p* and *q* antisymmetrizations equals the single *p*+*q* antisymmetrization. He argued that the inner transpositions are self-cancelling. (Transpositions between one member of { *r*, …, *u }* and one member from { *j* …, *m* } are "**inner transpositions**". Transpositions that involves terms only from { *r* , …, *u }* or only from { *j* …, *m* }, but not both, are "**outer transpositions**"). This perhaps is what Penrose looking for in this problem.

By collapsing the antisymmetrizations I side-stepped the very difficult issue of showing that the outer antisymmetrizations cancel, leaving just the inner antisymmetrizations.