[13.1] Let **G** be a set of elements with one element called "1".

Part 1:

Given (1) $1a = a \ \forall a$, (2) $\forall a \ \exists \ a^{-1}$ such that $a^{-1} a = 1$, and (3) Associative Law: $a(bc) = (ab)c \ \forall a, b,$ and c.

Show (A) $a1 = a \ \forall a$ and (B) $a a^{-1} = 1 \ \forall a$.

Part 2:

Replace (1) with (1') : $a1 = a \ \forall a$

Show this is not sufficient to imply (A) and (B)

Solution Part 1: From (2) $\exists \left(a^{-1}\right)^{-1}$ such that

(4) $\left(a^{-1}\right)^{-1} a^{-1} = 1$. So

(B) $a a^{-1} \overset{(1)}{=} (1a) a^{-1} \overset{(3)}{=} 1\left(a a^{-1}\right) \overset{(4)}{=} \left[\left(a^{-1}\right)^{-1} a^{-1}\right] \left(a a^{-1}\right) \overset{(3)}{=} \left(a^{-1}\right)^{-1} \left[\left(a^{-1} a\right) a^{-1}\right]$

$\overset{(2)}{=} \left(a^{-1}\right)^{-1} \left[1 a^{-1}\right] \overset{(1)}{=} \left(a^{-1}\right)^{-1} a^{-1} \overset{(4)}{=} 1$ ✔

(A) $a1 \overset{(2)}{=} a\left(a^{-1} a\right) \overset{(3)}{=} \left(a a^{-1}\right) a \overset{(B)}{=} 1a \overset{(1)}{=} 1$ ✔

Solution Part 2: Let **G** = {1, x} with multiplication defined by

$1^2 = 1x = 1$ and $x^2 = x1 = x$.

(1') : $a1 = a \quad \forall a$ (i.e., $11 = 1$ and $x1 = x$) ✔

(2) : Define $1^{-1} = x^{-1} = 1$.

Then $a^{-1} a = 1 \ \forall a$ (i.e., $1^{-1} 1 = 1^2 = 1$ and $x^{-1} x = 1x = 1$) ✔

(3) : $1(ab) = 1$ and $(1a)b = 1b = 1$ for any a and b in ***G***, and

$x(ab) = x$ and $(xa)b = xb = x$ for any a and b in ***G*** ✔

But (B) fails: $x x^{-1} = x1 = x \neq 1$ ✔

Note: Part 2 solution is a simplification of deant's simplification of Beckmann's idea.