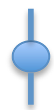
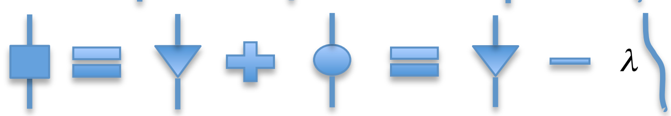
[13.26] Express the coefficients of the polynomial

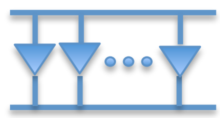
$$\det(\mathbf{T}-\lambda\mathbf{I}) = (\lambda_1-\lambda)\cdots(\lambda_n-\lambda)=0$$

in diagrammatic form. Work them out for n = 2 and n = 3.

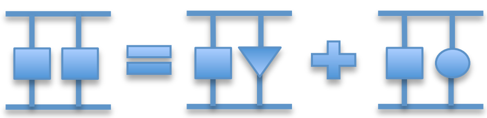
Proof. Beckmann produced a very nice proof that was then further simplified by an elegant enhancement provided by Dean. However, as far as I can tell, neither of them actually "worked out the equations for n = 2 or n = 3" as Penrose requested to generate the polynomial $(\lambda_1-\lambda)\cdots(\lambda_n-\lambda)$.

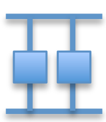
Let **R** = , **T** = , and **S** =  = -λ  , and set

 = $T^a{}_b - \lambda\delta^a_b$

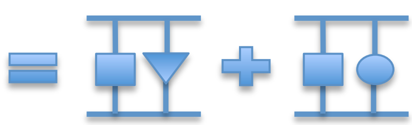
Recall that $\mathrm{Det}(\mathbf{T}) = \dfrac{1}{n!}$  .

We use the following fact repeatedly:

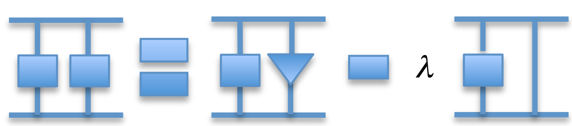
 :

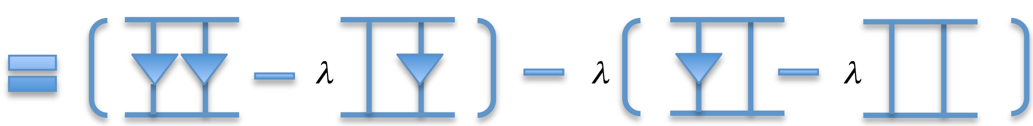
Proof:  $= \in^{ab} R^c{}_a R^d{}_b \varepsilon_{cd} = \in^{ab} R^c{}_a \left(S^d{}_b + T^d{}_b\right)\varepsilon_{cd}$

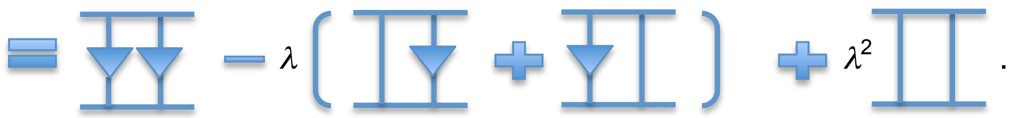
$= \in^{ab} R^c{}_a S^d{}_b \varepsilon_{cd} + \in^{ab} R^c{}_a T^d{}_b \varepsilon_{cd}$

 ✔

***n* = 2**:

2! Det (**T** – λ **I**) = 





There is a basis such that the matrix of T is triangular, so that

$$\mathbf{T} = \begin{pmatrix} T^1{}_1 & T^1{}_2 \\ T^2{}_1 & T^2{}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix}.$$

(This is the Jordan Canonical Form. Penrose mentions it in Footnote 13.12.)

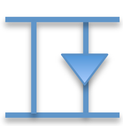
I will use the fact that $\in^{12} \varepsilon_{12} = -\in^{12} \varepsilon_{21} = -\in^{21} \varepsilon_{12} = \in^{21} \varepsilon_{21} = 1$ which I proved in problem [13.22]. So,

$$= \in^{ab} T^c{}_a T^d{}_b \varepsilon_{cd}$$

$$= \in^{12} T^1{}_1 T^2{}_2 \varepsilon_{12} + \in^{12} T^2{}_1 T^1{}_2 \varepsilon_{21} + \in^{21} T^1{}_2 T^2{}_1 \varepsilon_{12} + \in^{21} T^2{}_2 T^1{}_1 \varepsilon_{21}$$

$$= \lambda_1 \lambda_2 - (0)(b) - (b)(0) + \lambda_2 \lambda_1$$

$$= 2\lambda_1 \lambda_2$$



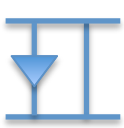
$$= \in^{ab} \delta^c_a T^d{}_b \varepsilon_{cd}$$

$$= \in^{12} \delta^1_1 T^2{}_2 \varepsilon_{12} + \in^{12} \delta^2_1 T^1{}_2 \varepsilon_{21} + \in^{21} \delta^1_2 T^2{}_1 \varepsilon_{12} + \in^{21} \delta^2_2 T^1{}_1 \varepsilon_{21}$$

$$= \lambda_2 - (0)(b) - (0)(0) + \lambda_1$$

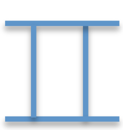
$$= \lambda_1 + \lambda_2 .$$

Similarly we find that



$$= \lambda_1 + \lambda_2 .$$

Finally,

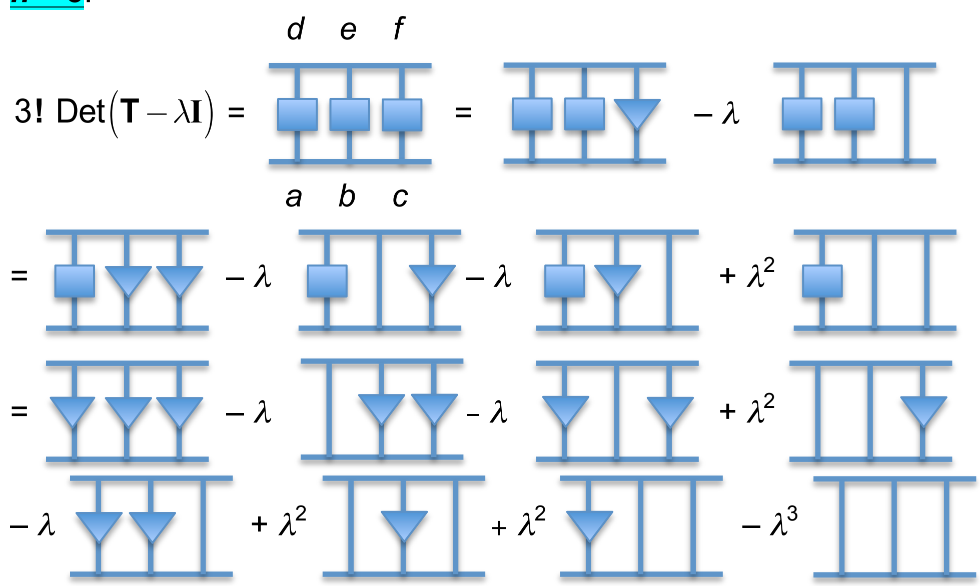


= 2! = 2.

So

$$\det(\mathbf{T}-\lambda\mathbf{I}) = \frac{1}{2}\left[2(\lambda_1\,\lambda_2) - 2\lambda(\lambda_1+\lambda_2) + 2\lambda^2\right] = \lambda^2 - \lambda(\lambda_1+\lambda_2) + \lambda_1\,\lambda_2$$
$$= (\lambda_1-\lambda)(\lambda_2-\lambda) \qquad ✔$$

***n* = 3**:



$$= \in^{abc}\,\varepsilon_{def}\begin{pmatrix} T^d{}_a T^e{}_b T^f{}_c - \lambda\delta^d_a T^e{}_b T^f{}_c - \lambda T^d{}_a \delta^e_b T^f{}_c + \lambda^2\delta^d_a\delta^e_b T^f{}_c \\ -\lambda T^d{}_a T^e{}_b \delta^f_c + \lambda^2\,\delta^d_a T^e{}_b \delta^f_c + \lambda^2\,T^d{}_a \delta^e_b \delta^f_c - \lambda^3\delta^d_a\delta^e_b\delta^f_c \end{pmatrix}$$
$$= \in^{123}\,\varepsilon_{123}\begin{pmatrix} T^1{}_1 T^2{}_2 T^3{}_3 - \lambda\delta^1_1 T^2{}_2 T^3{}_3 - \lambda T^1{}_1 \delta^2_2 T^3{}_3 + \lambda^2\delta^1_1\delta^2_2 T^3{}_3 \\ -\lambda T^1{}_1 T^2{}_2 \delta^3_3 + \lambda^2\,\delta^1_1 T^2{}_2 \delta^3_3 + \lambda^2\,T^1{}_1 \delta^2_2 \delta 3^3_3 - \lambda^3\delta^1_1\delta^2_2\delta^3_3 \end{pmatrix}$$
$$+ \in^{123}\,\varepsilon_{312}\begin{pmatrix} T^1{}_3 T^2{}_1 T^3{}_2 - \lambda\delta^1_3 T^2{}_1 T^3{}_2 - \lambda T^1{}_3 \delta^2_1 T^3{}_2 + \lambda^2\delta^1_3\delta^2_1 T^3{}_2 \\ -\lambda T^1{}_3 T^2{}_1 \delta^3_2 + \lambda^2\,\delta^1_3 T^2{}_1 \delta^3_2 + \lambda^2\,T^1{}_3 \delta^2_1 \delta^3_2 - \lambda^3\delta^1_3\delta^2_1\delta^3_2 \end{pmatrix}$$
$$+ \ldots$$
$$+ \in^{321}\,\varepsilon_{321}\begin{pmatrix} T^3{}_3 T^2{}_2 T^1{}_1 - \lambda\delta^3_3 T^2{}_2 T^1{}_1 - \lambda T^3{}_3 \delta^2_2 T^1{}_1 + \lambda^2\delta^3_3\delta^2_2 T^1{}_1 \\ -\lambda T^3{}_3 T^2{}_2 \delta^1_1 + \lambda^2\,\delta^3_3 T^2{}_2 \delta^1_1 + \lambda^2\,T^3{}_3 \delta^2_2 \delta^1_1 - \lambda^3\delta^3_3\delta^2_2\delta^1_1 \end{pmatrix}.$$

(There are 36 sets of expressions involving T and δ corresponding to 6 permutations of $\in$ times 6 permutations of ε.)

By choosing an appropriate basis we can assume T is in triangular form:

$$\mathbf{T} = \begin{pmatrix} T^1{}_1 & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & T^2{}_2 & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & T^3{}_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & b & c \\ 0 & \lambda_2 & f \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

We also use the following fact that I proved in problem [13.22]:

$$\in^{123} \varepsilon_{123} = \in^{132} \varepsilon_{132} = \cdots = \in^{321} \varepsilon_{321} = - \in^{123} \varepsilon_{132} = - \in^{132} \varepsilon_{123} = \cdots = - \in^{321} \varepsilon_{31} = 1$$

In any of the 36 sets of expressions, unless all three of the upper and lower indices match, the given set consists of the sum of eight zeros:

- Each of the 8 terms has at least one factor of δ or T from the lower left of its matrix. Those values are zero.

So, there are only 6 sets that have non-zero terms, and we can write

$$\begin{aligned}
&3!\ \mathrm{Det}\left(\mathbf{T} - \lambda\mathbf{I}\right) \\
&= \in^{123} \varepsilon_{123} \begin{pmatrix} T^1{}_1 T^2{}_2 T^3{}_3 - \lambda \delta^1_1 T^2{}_2 T^3{}_3 - \lambda T^1{}_1 \delta^2_2 T^3{}_3 + \lambda^2 \delta^1_1 \delta^2_2 T^3{}_3 \\ -\lambda T^1{}_1 T^2{}_2 \delta^3_3 + \lambda^2 \delta^1_1 T^2{}_2 \delta^3_3 + \lambda^2 T^1{}_1 \delta^2_2 \delta^3_3 - \lambda^3 \delta^1_1 \delta^2_2 \delta^3_3 \end{pmatrix} \\
&+ \in^{312} \varepsilon_{312} \begin{pmatrix} T^3{}_3 T^2{}_1 T^3{}_2 - \lambda \delta^3_3 T^1{}_1 T^2{}_2 - \lambda T^3{}_3 \delta^1_1 T^2{}_2 + \lambda^2 \delta^3_3 \delta^1_1 T^2{}_2 \\ -\lambda T^3{}_3 T^1{}_1 \delta^2_2 + \lambda^2 \delta^3_3 T^1{}_1 \delta^2_2 + \lambda^2 T^3{}_3 \delta^1_1 \delta^2_2 - \lambda^3 \delta^3_3 \delta^1_1 \delta^2_2 \end{pmatrix} \\
&+ \in^{231} \varepsilon_{231} \begin{pmatrix} T^2{}_2 T^3{}_3 T^1{}_1 - \lambda \delta^2_2 T^3{}_3 T^1{}_1 - \lambda T^2{}_2 \delta^3_3 T^1{}_1 + \lambda^2 \delta^2_2 \delta^3_3 T^1{}_1 \\ -\lambda T^2{}_2 T^3{}_3 \delta^1_1 + \lambda^2 \delta^2_2 T^3{}_3 \delta^1_1 + \lambda^2 T^2{}_2 \delta^3_3 \delta^1_1 - \lambda^3 \delta^2_2 \delta^3_3 \delta^1_1 \end{pmatrix} \\
&+ \in^{132} \varepsilon_{132} \begin{pmatrix} T^1{}_1 T^3{}_3 T^2{}_2 - \lambda \delta^1_1 T^3{}_3 T^2{}_2 - \lambda T^1{}_1 \delta^3_3 T^2{}_2 + \lambda^2 \delta^1_1 \delta^3_3 T^2{}_2 \\ -\lambda T^1{}_1 T^3{}_3 \delta^2_2 + \lambda^2 \delta^1_1 T^3{}_3 \delta^2_2 + \lambda^2 T^1{}_1 \delta^3_3 \delta^2_2 - \lambda^3 \delta^1_1 \delta^3_3 \delta^2_2 \end{pmatrix} \\
&+ \in^{213} \varepsilon_{213} \begin{pmatrix} T^2{}_2 T^1{}_1 T^3{}_3 - \lambda \delta^2_2 T^1{}_1 T^3{}_3 - \lambda T^2{}_2 \delta^1_1 T^3{}_3 + \lambda^2 \delta^2_2 \delta^1_1 T^3{}_3 \\ -\lambda T^2{}_2 T^1{}_1 \delta^3_3 + \lambda^2 \delta^2_2 T^1{}_1 \delta^3_3 + \lambda^2 T^2{}_2 \delta^1_1 \delta^3_3 - \lambda^3 \delta^2_2 \delta^1_1 \delta^3_3 \end{pmatrix} \\
&+ \in^{321} \varepsilon_{321} \begin{pmatrix} T^3{}_3 T^2{}_2 T^1{}_1 - \lambda \delta^3_3 T^2{}_2 T^1{}_1 - \lambda T^3{}_3 \delta^2_2 T^1{}_1 + \lambda^2 \delta^3_3 \delta^2_2 T^1{}_1 \\ -\lambda T^3{}_3 T^2{}_2 \delta^1_1 + \lambda^2 \delta^3_3 T^2{}_2 \delta^1_1 + \lambda^2 T^3{}_3 \delta^2_2 \delta^1_1 - \lambda^3 \delta^3_3 \delta^2_2 \delta^1_1 \end{pmatrix} \\
&= 6\lambda_1\lambda_2\lambda_3 - 6\lambda\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) + 6\lambda^2\left(\lambda_1 + \lambda_2 + \lambda_3\right) - 6\lambda^3.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathrm{Det}\left(\mathbf{T} - \lambda\mathbf{I}\right) = \lambda_1\lambda_2\lambda_3 - \lambda\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda\right) + \lambda^2\left(\lambda_1 + \lambda_2 + \lambda_3\right) - \lambda^3 \\
&= \left(\lambda_1 - \lambda\right)\left(\lambda_2 - \lambda\right)\left(\lambda_3 - \lambda\right).\ ✔
\end{aligned}$$