

Chapter 6: Functions of Random Variables

6.1 The distribution function of Y is $F_Y(y) = \int_0^y 2(1-t)dt = 2y - y^2, 0 \leq y \leq 1$.

- a. $F_{U_1}(u) = P(U_1 \leq u) = P(2Y - 1 \leq u) = P(Y \leq \frac{u+1}{2}) = F_Y(\frac{u+1}{2}) = 2(\frac{u+1}{2}) - (\frac{u+1}{2})^2$. Thus,
 $f_{U_1}(u) = F'_{U_1}(u) = \frac{1-u}{2}, -1 \leq u \leq 1$.
- b. $F_{U_2}(u) = P(U_2 \leq u) = P(1 - 2Y \leq u) = P(Y \leq \frac{1-u}{2}) = F_Y(\frac{1-u}{2}) = 1 - 2(\frac{1-u}{2}) = (\frac{u+1}{2})^2$. Thus,
 $f_{U_2}(u) = F'_{U_2}(u) = \frac{u+1}{2}, -1 \leq u \leq 1$.
- c. $F_{U_3}(u) = P(U_3 \leq u) = P(Y^2 \leq u) = P(Y \leq \sqrt{u}) = F_Y(\sqrt{u}) = 2\sqrt{u} - u$. Thus,
 $f_{U_3}(u) = F'_{U_3}(u) = \frac{1}{\sqrt{u}} - 1, 0 \leq u \leq 1$.
- d. $E(U_1) = -1/3, E(U_2) = 1/3, E(U_3) = 1/6$.
- e. $E(2Y - 1) = -1/3, E(1 - 2Y) = 1/3, E(Y^2) = 1/6$.

6.2 The distribution function of Y is $F_Y(y) = \int_{-1}^y (3/2)t^2 dt = (1/2)(y^3 - 1), -1 \leq y \leq 1$.

- a. $F_{U_1}(u) = P(U_1 \leq u) = P(3Y \leq u) = P(Y \leq u/3) = F_Y(u/3) = \frac{1}{2}(u^3/18 - 1)$. Thus,
 $f_{U_1}(u) = F'_{U_1}(u) = u^2/18, -3 \leq u \leq 3$.
- b. $F_{U_2}(u) = P(U_2 \leq u) = P(3 - Y \leq u) = P(Y \geq 3 - u) = 1 - F_Y(3 - u) = \frac{1}{2}[1 - (3 - u)^3]$.
Thus, $f_{U_2}(u) = F'_{U_2}(u) = \frac{3}{2}(3 - u)^2, 2 \leq u \leq 4$.
- c. $F_{U_3}(u) = P(U_3 \leq u) = P(Y^2 \leq u) = P(-\sqrt{u} \leq Y \leq \sqrt{u}) = F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) = u^{3/2}$.
Thus, $f_{U_3}(u) = F'_{U_3}(u) = \frac{3}{2}\sqrt{u}, 0 \leq u \leq 1$.

6.3 The distribution function for Y is $F_Y(y) = \begin{cases} y^2/2 & 0 \leq y \leq 1 \\ y - 1/2 & 1 < y \leq 1.5 \\ 1 & y > 1.5 \end{cases}$.

- a. $F_U(u) = P(U \leq u) = P(10Y - 4 \leq u) = P(Y \leq \frac{u+4}{10}) = F_Y(\frac{u+4}{10})$. So,

$$F_U(u) = \begin{cases} \frac{(u+4)^2}{200} & -4 \leq u \leq 6 \\ \frac{u-1}{10} & 6 < u \leq 11 \\ 1 & u > 11 \end{cases}, \text{ and } f_U(u) = F'_U(u) = \begin{cases} \frac{u+4}{100} & -4 \leq u \leq 6 \\ \frac{1}{10} & 6 < u \leq 11 \\ 0 & \text{elsewhere} \end{cases}$$
- b. $E(U) = 5.583$.
- c. $E(10Y - 4) = 10(23/24) - 4 = 5.583$.

6.4 The distribution function of Y is $F_Y(y) = 1 - e^{-y/4}, 0 \leq y$.

- a. $F_U(u) = P(U \leq u) = P(3Y + 1 \leq u) = P(Y \leq \frac{u-1}{3}) = F_Y(\frac{u-1}{3}) = 1 - e^{-(u-1)/12}$. Thus,
 $f_U(u) = F'_U(u) = \frac{1}{12}e^{-(u-1)/12}, u \geq 1$.
- b. $E(U) = 13$.

6.5 The distribution function of Y is $F_Y(y) = y/4, 1 \leq y \leq 5$.

$$F_U(u) = P(U \leq u) = P(2Y^2 + 3 \leq u) = P(Y \leq \sqrt{\frac{u-3}{2}}) = F_Y(\sqrt{\frac{u-3}{2}}) = \frac{1}{4} \sqrt{\frac{u-3}{2}}. \text{ Differentiating,}$$

$$f_U(u) = F'_U(u) = \frac{1}{16} \left(\frac{u-3}{2} \right)^{-1/2}, 5 \leq u \leq 53.$$

6.6 Refer to Ex. 5.10 ad 5.78. Define $F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = P(Y_1 \leq Y_2 + u)$.

a. For $u \leq 0$, $F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = 0$.

$$\text{For } 0 \leq u < 1, F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = \int_0^u \int_{2y_2}^{y_2+u} 1 dy_1 dy_2 = u^2 / 2.$$

$$\text{For } 1 \leq u \leq 2, F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = 1 - \int_0^{2-u} \int_{y_2+u}^2 1 dy_1 dy_2 = 1 - (2-u)^2 / 2.$$

$$\text{Thus, } f_U(u) = F'_U(u) = \begin{cases} u & 0 \leq u < 1 \\ 2-u & 1 \leq u \leq 2 \\ 0 & \text{elsewhere} \end{cases}.$$

b. $E(U) = 1$.

6.7 Let $F_Z(z)$ and $f_Z(z)$ denote the standard normal distribution and density functions respectively.

a. $F_U(u) = P(U \leq u) = P(Z^2 \leq u) = P(-\sqrt{u} \leq Z \leq \sqrt{u}) = F_Z(\sqrt{u}) - F_Z(-\sqrt{u})$. The density function for U is then

$$f_U(u) = F'_U(u) = \frac{1}{2\sqrt{u}} f_Z(\sqrt{u}) + \frac{1}{2\sqrt{u}} f_Z(-\sqrt{u}) = \frac{1}{\sqrt{u}} f_Z(\sqrt{u}), u \geq 0.$$

$$\text{Evaluating, we find } f_U(u) = \frac{1}{\sqrt{\pi}\sqrt{2}} u^{-1/2} e^{-u/2} \quad u \geq 0.$$

b. U has a gamma distribution with $\alpha = 1/2$ and $\beta = 2$ (recall that $\Gamma(1/2) = \sqrt{\pi}$).

c. This is the chi-square distribution with one degree of freedom.

6.8 Let $F_Y(y)$ and $f_Y(y)$ denote the beta distribution and density functions respectively.

a. $F_U(u) = P(U \leq u) = P(1 - Y \leq u) = P(Y \geq 1 - u) = 1 - F_Y(1 - u)$. The density function for U is then $f_U(u) = F'_U(u) = f_Y(1 - u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\beta-1} (1-u)^{\alpha-1}, 0 \leq u \leq 1$.

b. $E(U) = 1 - E(Y) = \frac{\beta}{\alpha+\beta}$.

c. $V(U) = V(Y)$.

6.9 Note that this is the same density from Ex. 5.12: $f(y_1, y_2) = 2, 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1$.

a. $F_U(u) = P(U \leq u) = P(Y_1 + Y_2 \leq u) = P(Y_1 \leq u - Y_2) = \int_0^u \int_0^{u-y_2} 2 dy_1 dy_2 = u^2$. Thus,

$$f_U(u) = F'_U(u) = 2u, 0 \leq u \leq 1.$$

b. $E(U) = 2/3$.

c. (found in an earlier exercise in Chapter 5) $E(Y_1 + Y_2) = 2/3$.

6.10 Refer to Ex. 5.15 and Ex. 5.108.

a. $F_U(u) = P(U \leq u) = P(Y_1 - Y_2 \leq u) = P(Y_1 \leq u + Y_2) = \int_0^\infty \int_{y_2}^{u+y_2} e^{-y_1} dy_1 dy_2 = 1 - e^{-u}$, so that

$$f_U(u) = F'_U(u) = e^{-u}, u \geq 0, \text{ so that } U \text{ has an exponential distribution with } \beta = 1.$$

b. From part a above, $E(U) = 1$.

6.11 It is given that $f_i(y_i) = e^{-y_i}$, $y_i \geq 0$ for $i = 1, 2$. Let $U = (Y_1 + Y_2)/2$.

a. $F_U(u) = P(U \leq u) = P\left(\frac{Y_1 + Y_2}{2} \leq u\right) = P(Y_1 \leq 2u - Y_2) = \int_0^{2u} \int_{y_2}^{2u-y_2} e^{-y_1-y_2} dy_1 dy_2 = 1 - e^{-2u} - 2ue^{-2u}$,

$$\text{so that } f_U(u) = F'_U(u) = 4ue^{-2u}, u \geq 0, \text{ a gamma density with } \alpha = 2 \text{ and } \beta = 1/2.$$

b. From part (a), $E(U) = 1$, $V(U) = 1/2$.

6.12 Let $F_Y(y)$ and $f_Y(y)$ denote the gamma distribution and density functions respectively.

a. $F_U(u) = P(U \leq u) = P(cY \leq u) = P(Y \leq u/c)$. The density function for U is then

$$f_U(u) = F'_U(u) = \frac{1}{c} f_Y(u/c) = \frac{1}{\Gamma(\alpha)(c\beta)^\alpha} u^{\alpha-1} e^{-u/c\beta}, u \geq 0. \text{ Note that this is another gamma distribution.}$$

b. The shape parameter is the same (α), but the scale parameter is $c\beta$.

6.13 Refer to Ex. 5.8;

$$F_U(u) = P(U \leq u) = P(Y_1 + Y_2 \leq u) = P(Y_1 \leq u - Y_2) = \int_0^u \int_0^{u-y_2} e^{-y_1-y_2} dy_1 dy_2 = 1 - e^{-u} - ue^{-u}.$$

$$\text{Thus, } f_U(u) = F'_U(u) = ue^{-u}, u \geq 0.$$

6.14 Since Y_1 and Y_2 are independent, so $f(y_1, y_2) = 18(y_1 - y_1^2)y_2^2$, for $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$. Let $U = Y_1 Y_2$. Then,

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(Y_1 Y_2 \leq u) = P(Y_1 \leq u/Y_2) = P(Y_1 > u/Y_2) = 1 - \int_{u/Y_2}^1 \int_u^1 18(y_1 - y_1^2)y_2^2 dy_1 dy_2 \\ &= 9u^2 - 8u^3 + 6u^3 \ln u. \end{aligned}$$

$$f_U(u) = F'_U(u) = 18u(1 - u + u \ln u), 0 \leq u \leq 1.$$

6.15 Let U have a uniform distribution on $(0, 1)$. The distribution function for U is $F_U(u) = P(U \leq u) = u$, $0 \leq u \leq 1$. For a function G , we require $G(U) = Y$ where Y has distribution function $F_Y(y) = 1 - e^{-y^2}$, $y \geq 0$. Note that

$$F_Y(y) = P(Y \leq y) = P(G(U) \leq y) = P[U \leq G^{-1}(y)] = F_U[G^{-1}(y)] = u.$$

So it must be true that $G^{-1}(y) = 1 - e^{-y^2} = u$ so that $G(u) = [-\ln(1 - u)]^{1/2}$. Therefore, the random variable $Y = [-\ln(U - 1)]^{1/2}$ has distribution function $F_Y(y)$.

6.16 Similar to Ex. 6.15. The distribution function for Y is $F_Y(y) = b \int_b^y t^{-2} dt = 1 - \frac{b}{y}$, $y \geq b$.

$$F_Y(y) = P(Y \leq y) = P(G(U) \leq y) = P[U \leq G^{-1}(y)] = F_U[G^{-1}(y)] = u.$$

So it must be true that $G^{-1}(y) = 1 - \frac{b}{y} = u$ so that $G(u) = \frac{b}{1-u}$. Therefore, the random variable $Y = b/(1 - U)$ has distribution function $F_Y(y)$.

6.17 a. Taking the derivative of $F(y)$, $f(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha}$, $0 \leq y \leq \theta$.

b. Following Ex. 6.15 and 6.16, let $u = \left(\frac{y}{\theta}\right)^\alpha$ so that $y = \theta u^{1/\alpha}$. Thus, the random variable $Y = \theta U^{1/\alpha}$ has distribution function $F_Y(y)$.

c. From part (b), the transformation is $y = 4\sqrt{u}$. The values are 2.0785, 3.229, 1.5036, 1.5610, 2.403.

6.18 a. Taking the derivative of the distribution function yields $f(y) = \alpha\beta^\alpha y^{-\alpha-1}$, $y \geq \beta$.

b. Following Ex. 6.15, let $u = 1 - \left(\frac{\beta}{y}\right)^\alpha$ so that $y = \frac{\beta}{(1-u)^{1/\alpha}}$. Thus, $Y = \beta(1 - U)^{-1/\alpha}$.

c. From part (b), $y = 3 / \sqrt{1-u}$. The values are 3.0087, 3.3642, 6.2446, 3.4583, 4.7904.

6.19 The distribution function for X is:

$$\begin{aligned} F_X(x) &= P(X \leq x) = P(1/Y \leq x) = P(Y \geq 1/x) = 1 - F_Y(1/x) \\ &= 1 - [1 - (\beta x)^\alpha] = (\beta x)^\alpha, \quad 0 < x < \beta^{-1}, \text{ which is a power distribution with } \theta = \beta^{-1}. \end{aligned}$$

6.20 a. $F_W(w) = P(W \leq w) + P(Y^2 \leq w) = P(Y \leq \sqrt{w}) = F_Y(\sqrt{w}) = \sqrt{w}$, $0 \leq w \leq 1$.

b. $F_W(w) = P(W \leq w) + P(\sqrt{Y} \leq w) = P(Y \leq w^2) = F_Y(w^2) = w^2$, $0 \leq w \leq 1$.

6.21 By definition, $P(X = i) = P[F(i-1) < U \leq F(i)] = F(i) - F(i-1)$, for $i = 1, 2, \dots$, since for any $0 \leq a \leq 1$, $P(U \leq a) = a$ for any $0 \leq a \leq 1$. From Ex. 4.5, $P(Y = i) = F(i) - F(i-1)$, for $i = 1, 2, \dots$. Thus, X and Y have the same distribution.

6.22 Let U have a uniform distribution on the interval $(0, 1)$. For a geometric distribution with parameter p and distribution function F , define the random variable X as:

$$X = k \text{ if and only if } F(k-1) < U \leq F(k), \quad k = 1, 2, \dots$$

Or since $F(k) = 1 - q^k$, we have that:

$$X = k \text{ if and only if } 1 - q^{k-1} < U \leq 1 - q^k, \text{ OR}$$

$$X = k \text{ if and only if } q^k < 1 - U \leq q^{k-1}, \text{ OR}$$

$$X = k \text{ if and only if } k \ln q \leq \ln(1-U) \leq (k-1) \ln q, \text{ OR}$$

$$X = k \text{ if and only if } k-1 < [\ln(1-U)]/\ln q \leq k.$$

6.23 a. If $U = 2Y - 1$, then $Y = \frac{U+1}{2}$. Thus, $\frac{dy}{du} = \frac{1}{2}$ and $f_U(u) = \frac{1}{2} 2(1 - \frac{u+1}{2}) = \frac{1-u}{2}$, $-1 \leq u \leq 1$.

b. If $U = 1 - 2Y$, then $Y = \frac{1-U}{2}$. Thus, $\frac{dy}{du} = \frac{1}{2}$ and $f_U(u) = \frac{1}{2} 2(1 - \frac{1-u}{2}) = \frac{1+u}{2}$, $-1 \leq u \leq 1$.

c. If $U = Y^2$, then $Y = \sqrt{U}$. Thus, $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$ and $f_U(u) = \frac{1}{2\sqrt{u}} 2(1 - \sqrt{u}) = \frac{1-\sqrt{u}}{\sqrt{u}}$, $0 \leq u \leq 1$.

- 6.24** If $U = 3Y + 1$, then $Y = \frac{U-1}{3}$. Thus, $\frac{dy}{du} = \frac{1}{3}$. With $f_Y(y) = \frac{1}{4}e^{-y/4}$, we have that $f_U(u) = \frac{1}{3} \left[\frac{1}{4} e^{-(u-1)/12} \right] = \frac{1}{12} e^{-(u-1)/12}$, $1 \leq u$.
- 6.25** Refer to Ex. 6.11. The variable of interest is $U = \frac{Y_1+Y_2}{2}$. Fix $Y_2 = y_2$. Then, $Y_1 = 2u - y_2$ and $\frac{dy_1}{du} = 2$. The joint density of U and Y_2 is $g(u, y_2) = 2e^{-2u}$, $u \geq 0$, $y_2 \geq 0$, and $y_2 < 2u$. Thus, $f_U(u) = \int_0^{2u} 2e^{-2u} dy_2 = 4ue^{-2u}$ for $u \geq 0$.
- 6.26** a. Using the transformation approach, $Y = U^{1/m}$ so that $\frac{dy}{du} = \frac{1}{m} u^{-(m-1)/m}$ so that the density function for U is $f_U(u) = \frac{1}{\alpha} e^{-u/\alpha}$, $u \geq 0$. Note that this is the exponential distribution with mean α .
b. $E(Y^k) = E(U^{k/m}) = \int_0^\infty u^{k/m} \frac{1}{\alpha} e^{-u/\alpha} du = \Gamma\left(\frac{k}{m} + 1\right) \alpha^{k/m}$, using the result from Ex. 4.111.
- 6.27** a. Let $W = \sqrt{Y}$. The random variable Y is exponential so $f_Y(y) = \frac{1}{\beta} e^{-y/\beta}$. Then, $Y = W^2$ and $\frac{dy}{dw} = 2w$. Then, $f_Y(y) = \frac{2}{\beta} w e^{-w^2/\beta}$, $w \geq 0$, which is Weibull with $m = 2$.
b. It follows from Ex. 6.26 that $E(Y^{k/2}) = \Gamma\left(\frac{k}{2} + 1\right) \beta^{k/2}$.
- 6.28** If Y is uniform on the interval $(0, 1)$, $f_U(u) = 1$. Then, $Y = e^{-U/2}$ and $\frac{dy}{du} = -\frac{1}{2} e^{-u/2}$. Then, $f_Y(y) = 1 \mid -\frac{1}{2} e^{-u/2} \mid = \frac{1}{2} e^{-u/2}$, $u \geq 0$ which is exponential with mean 2.
- 6.29** a. With $W = \frac{mV^2}{2}$, $V = \sqrt{\frac{2W}{m}}$ and $\left| \frac{dv}{dw} \right| = \frac{1}{\sqrt{2mw}}$. Then,

$$f_W(w) = \frac{a(2w/m)}{\sqrt{2mw}} e^{-2bw/m} = \frac{a\sqrt{2}}{m^{3/2}} w^{1/2} e^{-w/kT}, w \geq 0.$$
The above expression is in the form of a gamma density, so the constant a must be chosen so that the density integrate to 1, or simply

$$\frac{a\sqrt{2}}{m^{3/2}} = \frac{1}{\Gamma(\frac{3}{2})(kT)^{3/2}}.$$
So, the density function for W is

$$f_W(w) = \frac{1}{\Gamma(\frac{3}{2})(kT)^{3/2}} w^{1/2} e^{-w/kT}.$$
b. For a gamma random variable, $E(W) = \frac{3}{2} kT$.
- 6.30** The density function for I is $f_I(i) = 1/2$, $9 \leq i \leq 11$. For $P = 2I^2$, $I = \sqrt{P/2}$ and $\frac{di}{dp} = (1/2)^{3/2} p^{-1/2}$. Then, $f_P(p) = \frac{1}{4\sqrt{2p}}$, $162 \leq p \leq 242$.

6.31 Similar to Ex. 6.25. Fix $Y_1 = y_1$. Then, $U = Y_2/y_1$, $Y_2 = y_1 U$ and $|\frac{dy_2}{du}| = y_1$. The joint density of Y_1 and U is $f(y_1, u) = \frac{1}{8} y_1^2 e^{-y_1(1+u)/2}$, $y_1 \geq 0$, $u \geq 0$. So, the marginal density for U is $f_U(u) = \int_0^\infty \frac{1}{8} y_1^2 e^{-y_1(1+u)/2} dy_1 = \frac{2}{(1+u)^3}$, $u \geq 0$.

6.32 Now $f_Y(y) = 1/4$, $1 \leq y \leq 5$. If $U = 2Y^2 + 3$, then $Y = (\frac{U-3}{2})^{1/2}$ and $|\frac{dy}{du}| = \frac{1}{4}(\frac{\sqrt{2}}{\sqrt{u-3}})$. Thus, $f_U(u) = \frac{1}{8\sqrt{2(u-3)}}$, $5 \leq u \leq 53$.

6.33 If $U = 5 - (Y/2)$, $Y = 2(5 - U)$. Thus, $|\frac{dy}{du}| = 2$ and $f_U(u) = 4(80 - 31u + 3u^2)$, $4.5 \leq u \leq 5$.

6.34 a. If $U = Y^2$, $Y = \sqrt{U}$. Thus, $|\frac{dy}{du}| = \frac{1}{2\sqrt{u}}$ and $f_U(u) = \frac{1}{\theta} e^{-u/\theta}$, $u \geq 0$. This is the exponential density with mean θ .

b. From part a, $E(Y) = E(U^{1/2}) = \frac{\sqrt{\pi\theta}}{2}$. Also, $E(Y^2) = E(U) = \theta$, so $V(Y) = \theta[1 - \frac{\pi}{4}]$.

6.35 By independence, $f(y_1, y_2) = 1$, $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$. Let $U = Y_1 Y_2$. For a fixed value of Y_1 at y_1 , then $y_2 = u/y_1$. So that $\frac{dy_2}{du} = \frac{1}{y_1}$. So, the joint density of Y_1 and U is

$$g(y_1, u) = 1/y_1, \quad 0 \leq y_1 \leq 1, \quad 0 \leq u \leq y_1.$$

Thus, $f_U(u) = \int_u^1 (1/y_1) dy_1 = -\ln(u)$, $0 \leq u \leq 1$.

6.36 By independence, $f(y_1, y_2) = \frac{4y_1 y_2}{\theta^2} e^{-(y_1^2 + y_2^2)}$, $y_1 > 0$, $y_2 > 0$. Let $U = Y_1^2 + Y_2^2$. For a fixed value of Y_1 at y_1 , then $U = y_1^2 + Y_2^2$ so we can write $y_2 = \sqrt{u - y_1^2}$. Then, $\frac{dy_2}{du} = \frac{1}{2\sqrt{u - y_1^2}}$ so that the joint density of Y_1 and U is

$$g(y_1, u) = \frac{4y_1 \sqrt{u - y_1^2}}{\theta^2} e^{-u/\theta} \frac{1}{2\sqrt{u - y_1^2}} = \frac{2}{\theta^2} y_1 e^{-u/\theta}, \quad \text{for } 0 < y_1 < \sqrt{u}.$$

Then, $f_U(u) = \int_0^{\sqrt{u}} \frac{2}{\theta^2} y_1 e^{-u/\theta} dy_1 = \frac{1}{\theta^2} u e^{-u/\theta}$. Thus, U has a gamma distribution with $\alpha = 2$.

6.37 The mass function for the Bernoulli distribution is $p(y) = p^y (1-p)^{1-y}$, $y = 0, 1$.

a. $m_{Y_1}(t) = E(e^{tY_1}) = \sum_{x=0}^1 e^{tx} p(x) = 1 - p + pe^t.$

b. $m_W(t) = E(e^{tW}) = \prod_{i=1}^n m_{Y_i}(t) = [1 - p + pe^t]^n$

c. Since the mgf for W is in the form of a binomial mgf with n trials and success probability p , this is the distribution for W .

6.38 Let Y_1 and Y_2 have mgfs as given, and let $U = a_1Y_1 + a_2Y_2$. The mdg for U is

$$m_U(t) = E(e^{Ut}) = E(e^{(a_1Y_1 + a_2Y_2)t}) = E(e^{(a_1t)Y_1})E(e^{(a_2t)Y_2}) = m_{Y_1}(a_1t)m_{Y_2}(a_2t).$$

6.39 The mgf for the exponential distribution with $\beta = 1$ is $m(t) = (1 - t)^{-1}$, $t < 1$. Thus, with Y_1 and Y_2 each having this distribution and $U = (Y_1 + Y_2)/2$. Using the result from Ex. 6.38, let $a_1 = a_2 = 1/2$ so the mgf for U is $m_U(t) = m(t/2)m(t/2) = (1 - t/2)^{-2}$. Note that this is the mgf for a gamma random variable with $\alpha = 2$, $\beta = 1/2$, so the density function for U is $f_U(u) = 4ue^{-2u}$, $u \geq 0$.

6.40 It has been shown that the distribution of both Y_1^2 and Y_2^2 is chi-square with $v = 1$. Thus, both have mgf $m(t) = (1 - 2t)^{-1/2}$, $t < 1/2$. With $U = Y_1^2 + Y_2^2$, use the result from Ex. 6.38 with $a_1 = a_2 = 1$ so that $m_U(t) = m(t)m(t) = (1 - 2t)^{-1}$. Note that this is the mgf for an exponential random variable with $\beta = 2$, so the density function for U is $f_U(u) = \frac{1}{2}e^{-u/2}$, $u \geq 0$ (this is also the chi-square distribution with $v = 2$.)

6.41 (Special case of Theorem 6.3) The mgf for the normal distribution with parameters μ and σ is $m(t) = e^{\mu t + \sigma^2 t^2 / 2}$. Since the Y_i 's are independent, the mgf for U is given by

$$m_U(t) = E(e^{Ut}) = \prod_{i=1}^n E(e^{a_i t Y_i}) = \prod_{i=1}^n m(a_i t) = \exp\left[\mu t \sum_i a_i + (t^2 \sigma^2 / 2) \sum_i a_i^2\right].$$

This is the mgf for a normal variable with mean $\mu \sum_i a_i$ and variance $\sigma^2 \sum_i a_i^2$.

6.42 The probability of interest is $P(Y_2 > Y_1) = P(Y_2 - Y_1 > 0)$. By Theorem 6.3, the distribution of $Y_2 - Y_1$ is normal with $\mu = 4000 - 5000 = -1000$ and $\sigma^2 = 400^2 + 300^2 = 250,000$. Thus, $P(Y_2 - Y_1 > 0) = P(Z > \frac{0 - (-1000)}{\sqrt{250,000}}) = P(Z > 2) = .0228$.

6.43 a. From Ex. 6.41, \bar{Y} has a normal distribution with mean μ and variance σ^2/n .

b. For the given values, \bar{Y} has a normal distribution with variance $\sigma^2/n = 16/25$. Thus, the standard deviation is $4/5$ so that

$$P(|\bar{Y} - \mu| \leq 1) = P(-1 \leq \bar{Y} - \mu \leq 1) = P(-1.25 \leq Z \leq 1.25) = .7888.$$

c. Similar to the above, the probabilities are .8664, .9544, .9756. So, as the sample size increases, so does the probability that $P(|\bar{Y} - \mu| \leq 1)$.

6.44 The total weight of the watermelons in the packing container is given by $U = \sum_{i=1}^n Y_i$, so by Theorem 6.3 U has a normal distribution with mean $15n$ and variance $4n$. We require that $.05 = P(U > 140) = P(Z > \frac{140 - 15n}{\sqrt{4n}})$. Thus, $\frac{140 - 15n}{\sqrt{4n}} = z_{.05} = 1.645$. Solving this nonlinear expression for n , we see that $n \approx 8.687$. Therefore, the maximum number of watermelons that should be put in the container is 8 (note that with this value n , we have $P(U > 140) = .0002$).

- 6.45** By Theorem 6.3 we have that $U = 100 + 7Y_1 + 3Y_2$ is a normal random variable with mean $\mu = 100 + 7(10) + 3(4) = 182$ and variance $\sigma^2 = 49(.5)^2 + 9(.2)^2 = 12.61$. We require a value c such that $P(U > c) = P(Z > \frac{c-182}{\sqrt{12.61}})$. So, $\frac{c-182}{\sqrt{12.61}} = 2.33$ and $c = \$190.27$.
- 6.46** The mgf for W is $m_W(t) = E(e^{Wt}) = E(e^{(2Y/\beta)t}) = m_Y(2t/\beta) = (1 - 2t)^{-n/2}$. This is the mgf for a chi-square variable with n degrees of freedom.
- 6.47** By Ex. 6.46, $U = 2Y/4.2$ has a chi-square distribution with $v = 7$. So, by Table III, $P(Y > 33.627) = P(U > 2(33.627)/4.2) = P(U > 16.0128) = .025$.
- 6.48** From Ex. 6.40, we know that $V = Y_1^2 + Y_2^2$ has a chi-square distribution with $v = 2$. The density function for V is $f_V(v) = \frac{1}{2}e^{-v/2}$, $v \geq 0$. The distribution function of $U = \sqrt{V}$ is $F_U(u) = P(U \leq u) = P(V \leq u^2) = F_V(u^2)$, so that $f_U(u) = F'_U(u) = ue^{-u^2/2}$, $u \geq 0$. A sharp observer would note that this is a Weibull density with shape parameter 2 and scale 2.
- 6.49** The mgfs for Y_1 and Y_2 are, respectively, $m_{Y_1}(t) = [1 - p + pe^t]^{n_1}$, $m_{Y_2}(t) = [1 - p + pe^t]^{n_2}$. Since Y_1 and Y_2 are independent, the mgf for $Y_1 + Y_2$ is $m_{Y_1}(t) \times m_{Y_2}(t) = [1 - p + pe^t]^{n_1+n_2}$. This is the mgf of a binomial with $n_1 + n_2$ trials and success probability p .
- 6.50** The mgf for Y is $m_Y(t) = [1 - p + pe^t]^n$. Now, define $X = n - Y$. The mgf for X is $m_X(t) = E(e^{tX}) = E(e^{t(n-Y)}) = e^{tn}m_Y(-t) = [p + (1-p)e^t]^n$. This is an mgf for a binomial with n trials and "success" probability $(1-p)$. Note that the random variable $X = \#$ of failures observed in the experiment.
- 6.51** From Ex. 6.50, the distribution of $n_2 - Y_2$ is binomial with n_2 trials and "success" probability $1 - .8 = .2$. Thus, by Ex. 6.49, the distribution of $Y_1 + (n_2 - Y_2)$ is binomial with $n_1 + n_2$ trials and success probability $p = .2$.
- 6.52** The mgfs for Y_1 and Y_2 are, respectively, $m_{Y_1}(t) = e^{\lambda_1(e^t-1)}$, $m_{Y_2}(t) = e^{\lambda_2(e^t-1)}$.
- a. Since Y_1 and Y_2 are independent, the mgf for $Y_1 + Y_2$ is $m_{Y_1}(t) \times m_{Y_2}(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$. This is the mgf of a Poisson with mean $\lambda_1 + \lambda_2$.
- b. From Ex. 5.39, the distribution is binomial with m trials and $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$.
- 6.53** The mgf for a binomial variable Y_i with n_i trials and success probability p_i is given by $m_{Y_i}(t) = [1 - p_i + p_i e^t]^{n_i}$. Thus, the mgf for $U = \sum_{i=1}^n Y_i$ is $m_U(t) = \prod_i [1 - p_i + p_i e^t]^{n_i}$.
- a. Let $p_i = p$ and $n_i = m$ for all i . Here, U is binomial with $m(n)$ trials and success probability p .
- b. Let $p_i = p$. Here, U is binomial with $\sum_{i=1}^n n_i$ trials and success probability p .
- c. (Similar to Ex. 5.40) The cond. distribution is hypergeometric w/ $r = n_i$, $N = \sum n_i$.
- d. By definition,

$$\begin{aligned}
 P(Y_1 + Y_2 = k \mid \sum_{i=1}^n Y_i) &= \frac{P(Y_1 + Y_2 = k, \sum_{i=1}^n Y_i = m)}{P(\sum_{i=1}^n Y_i = m)} = \frac{P(Y_1 + Y_2 = k, \sum_{i=3}^n Y_i = m - k)}{P(\sum_{i=1}^n Y_i = m)} = \frac{P(Y_1 + Y_2 = k)P(\sum_{i=3}^n Y_i = m - k)}{P(\sum_{i=1}^n Y_i = m)} \\
 &= \frac{\binom{n_1 + n_2}{k} \binom{\sum_{i=3}^n n_i}{m - k}}{\binom{\sum_{i=1}^n n_i}{m}}, \text{ which is hypergeometric with } r = n_1 + n_2.
 \end{aligned}$$

e. No, the mgf for U does not simplify into a recognizable form.

6.54 a. The mgf for $U = \sum_{i=1}^n Y_i$ is $m_U(t) = e^{(e^t - 1)\sum_{i=1}^n \lambda_i}$, which is recognized as the mgf for a Poisson w/ mean $\sum_{i=1}^n \lambda_i$.

b. This is similar to 6.52. The distribution is binomial with m trials and $p = \frac{\lambda_1}{\sum_{i=1}^n \lambda_i}$.

c. Following the same steps as in part d of Ex. 6.53, it is easily shown that the conditional distribution is binomial with m trials and success probability $\frac{\lambda_1 + \lambda_2}{\sum_{i=1}^n \lambda_i}$.

6.55 Let $Y = Y_1 + Y_2$. Then, by Ex. 6.52, Y is Poisson with mean $7 + 7 = 14$. Thus, $P(Y \geq 20) = 1 - P(Y \leq 19) = .077$.

6.56 Let U = total service time for two cars. Similar to Ex. 6.13, U has a gamma distribution with $\alpha = 2$, $\beta = 1/2$. Then, $P(U > 1.5) = \int_{1.5}^{\infty} 4ue^{-2u} du = .1991$.

6.57 For each Y_i , the mgf is $m_{Y_i}(t) = (1 - \beta t)^{-\alpha_i}$, $t < 1/\beta$. Since the Y_i are independent, the mgf for $U = \sum_{i=1}^n Y_i$ is $m_U(t) = \prod (1 - \beta t)^{-\alpha_i} = (1 - \beta t)^{-\sum_{i=1}^n \alpha_i}$. This is the mgf for the gamma with shape parameter $\sum_{i=1}^n \alpha_i$ and scale parameter β .

6.58 a. The mgf for each W_i is $m(t) = \frac{pe^t}{(1 - qe^t)}$. The mgf for Y is $[m(t)]^r = \left(\frac{pe^t}{1 - qe^t}\right)^r$, which is the mgf for the negative binomial distribution.

b. Differentiating with respect to t , we have

$$m'(t)\big|_{t=0} = r \left(\frac{pe^t}{1 - qe^t}\right)^{r-1} \times \frac{pe^t}{(1 - qe^t)^2} \bigg|_{t=0} = \frac{r}{p} = E(Y).$$

Taking another derivative with respect to t yields

$$m''(t)\big|_{t=0} = \frac{(1 - qe^t)^{r+1} r^2 pe^t (pe^t)^{r-1} - r(pe^t)^r (r+1)(-qe^t)(1 - qe^t)^r}{(1 - qe^t)^{2(r+1)}} \bigg|_{t=0} = \frac{pr^2 + r(r+1)q}{p^2} = E(Y^2).$$

Thus, $V(Y) = E(Y^2) - [E(Y)]^2 = rq/p^2$.

c. This is similar to Ex. 6.53. By definition,

$$P(W_1 = k | \sum W_i = m) = \frac{P(W_1 = k, \sum W_i = m)}{P(\sum W_i = m)} = \frac{P(W_1 = k, \sum_{i=2}^n W_i = m-k)}{P(\sum W_i = m)} = \frac{P(W_1 = k)P(\sum_{i=2}^n W_i = m-k)}{P(\sum W_i = m)} = \frac{\binom{m-k-1}{r-2}}{\binom{m-1}{r-1}}.$$

6.59 The mgfs for Y_1 and Y_2 are, respectively, $m_{Y_1}(t) = (1-2t)^{-v_1/2}$, $m_{Y_2}(t) = (1-2t)^{-v_2/2}$. Thus the mgf for $U = Y_1 + Y_2 = m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) = (1-2t)^{-(v_1+v_2)/2}$, which is the mgf for a chi-square variable with $v_1 + v_2$ degrees of freedom.

6.60 Note that since Y_1 and Y_2 are independent, $m_W(t) = m_{Y_1}(t) \times m_{Y_2}(t)$. Therefore, it must be so that $m_W(t)/m_{Y_1}(t) = m_{Y_2}(t)$. Given the mgfs for W and Y_1 , we can solve for $m_{Y_2}(t)$:

$$m_{Y_2}(t) = \frac{(1-2t)^{-v}}{(1-2t)^{-v_1}} = (1-2t)^{-(v-v_1)/2}.$$

This is the mgf for a chi-squared variable with $v - v_1$ degrees of freedom.

6.61 Similar to Ex. 6.60. Since Y_1 and Y_2 are independent, $m_W(t) = m_{Y_1}(t) \times m_{Y_2}(t)$. Therefore, it must be so that $m_W(t)/m_{Y_1}(t) = m_{Y_2}(t)$. Given the mgfs for W and Y_1 ,

$$m_{Y_2}(t) = \frac{e^{\lambda(e^t-1)}}{e^{\lambda_1(e^t-1)}} = e^{(\lambda-\lambda_1)(e^t-1)}.$$

This is the mgf for a Poisson variable with mean $\lambda - \lambda_1$.

6.62 $E\{\exp[t_1(Y_1 + Y_2) + t_2(Y_1 - Y_2)]\} = E\{\exp[(t_1 + t_2)Y_1 + (t_1 - t_2)Y_2]\} = m_{Y_1}(t_1 + t_2)m_{Y_2}(t_1 - t_2)$
 $= \exp[\frac{\sigma^2}{2}(t_1 + t_2)^2] \exp[\frac{\sigma^2}{2}(t_1 - t_2)^2] = \exp[\frac{\sigma^2}{2}t_1^2] \exp[\frac{\sigma^2}{2}t_2^2]$
 $= m_{U_1}(t_1)m_{U_2}(t_2).$

Since the joint mgf factors, U_1 and U_2 are independent.

6.63 a. The marginal distribution for U_1 is $f_{U_1}(u_1) = \int_0^\infty \frac{1}{\beta^2} u_2 e^{-u_2/\beta} du_2 = 1$, $0 < u_1 < 1$.

b. The marginal distribution for U_2 is $f_{U_2}(u_2) = \int_0^1 \frac{1}{\beta^2} u_2 e^{-u_2/\beta} du_1 = \frac{1}{\beta^2} u_2 e^{-u_2/\beta}$, $u_2 > 0$. This is a gamma density with $\alpha = 2$ and scale parameter β .

c. Since the joint distribution factors into the product of the two marginal densities, they are independent.

6.64 a. By independence, the joint distribution of Y_1 and Y_2 is the product of the two marginal densities:

$$f(y_1, y_2) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} y_1^{\alpha_1-1} y_2^{\alpha_2-1} e^{-(y_1+y_2)/\beta}, y_1 \geq 0, y_2 \geq 0.$$

With U and V as defined, we have that $y_1 = u_1 u_2$ and $y_2 = u_2(1-u_1)$. Thus, the Jacobian of transformation $J = u_2$ (see Example 6.14). Thus, the joint density of U_1 and U_2 is

$$\begin{aligned}
 f(u_1, u_2) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)\beta^{\alpha_1+\alpha_2}} (u_1 u_2)^{\alpha_1-1} [u_2(1-u_1)]^{\alpha_2-1} e^{-u_2/\beta} u_2 \\
 &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)\beta^{\alpha_1+\alpha_2}} u_1^{\alpha_1-1} (1-u_1)^{\alpha_2-1} u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta}, \text{ with } 0 < u_1 < 1, \text{ and } u_2 > 0.
 \end{aligned}$$

b. $f_{U_1}(u_1) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)} u_1^{\alpha_1-1} (1-u_1)^{\alpha_2-1} \int_0^\infty \frac{1}{\beta^{\alpha_1+\alpha_2}} v^{\alpha_1+\alpha_2-1} e^{-v/\beta} dv = \frac{\Gamma(\alpha_1+\alpha_a)}{\Gamma(\alpha_1)\Gamma(\alpha_a)} u_1^{\alpha_1-1} (1-u_1)^{\alpha_2-1}$, with $0 < u_1 < 1$. This is the beta density as defined.

c. $f_{U_2}(u_2) = \frac{1}{\beta^{\alpha_1+\alpha_2}} u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta} \int_0^1 \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_a)} u_1^{\alpha_1-1} (1-u_1)^{\alpha_2-1} du_1 = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)} u_2^{\alpha_1+\alpha_2-1} e^{-u_2/\beta}$, with $u_2 > 0$. This is the gamma density as defined.

d. Since the joint distribution factors into the product of the two marginal densities, they are independent.

6.65 a. By independence, the joint distribution of Z_1 and Z_2 is the product of the two marginal densities:

$$f(z_1, z_2) = \frac{1}{2\pi} e^{-(z_1^2 + z_2^2)/2}.$$

With $U_1 = Z_1$ and $U_2 = Z_1 + Z_2$, we have that $z_1 = u_1$ and $z_2 = u_2 - u_1$. Thus, the Jacobian of transformation is

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

Thus, the joint density of U_1 and U_2 is

$$f(u_1, u_2) = \frac{1}{2\pi} e^{-[u_1^2 + (u_2 - u_1)^2]/2} = \frac{1}{2\pi} e^{-(2u_1^2 - 2u_1 u_2 + u_2^2)/2}.$$

b. $E(U_1) = E(Z_1) = 0$, $E(U_2) = E(Z_1 + Z_2) = 0$, $V(U_1) = V(Z_1) = 1$,
 $V(U_2) = V(Z_1 + Z_2) = V(Z_1) + V(Z_2) = 2$, $Cov(U_1, U_2) = E(Z_1^2) = 1$

c. Not independent since $\rho \neq 0$.

d. This is the bivariate normal distribution with $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = 1$, $\sigma_2^2 = 2$, and $\rho = \frac{1}{\sqrt{2}}$.

6.66 a. Similar to Ex. 6.65, we have that $y_1 = u_1 - u_2$ and $y_2 = u_2$. So, the Jacobian of transformation is

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

Thus, by definition the joint density is as given.

b. By definition of a marginal density, the marginal density for U_1 is as given.

c. If Y_1 and Y_2 are independent, their joint density factors into the product of the marginal densities, so we have the given form.

6.67 a. We have that $y_1 = u_1 u_2$ and $y_2 = u_2$. So, the Jacobian of transformation is

$$J = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = |u_2|.$$

Thus, by definition the joint density is as given.

b. By definition of a marginal density, the marginal density for U_1 is as given.

c. If Y_1 and Y_2 are independent, their joint density factors into the product of the marginal densities, so we have the given form.

6.68 a. Using the result from Ex. 6.67,

$$f(u_1, u_2) = 8(u_1 u_2) u_2 u_2 = 8u_1 u_2^3, \quad 0 \leq u_1 \leq 1, \quad 0 \leq u_2 \leq 1.$$

b. The marginal density for U_1 is

$$f_{U_1}(u_1) = \int_0^1 8u_1 u_2^3 du_2 = 2u_1, \quad 0 \leq u_1 \leq 1.$$

The marginal density for U_1 is

$$f_{U_2}(u_2) = \int_0^1 8u_1 u_2^3 du_1 = 4u_2^3, \quad 0 \leq u_2 \leq 1.$$

The joint density factors into the product of the marginal densities, thus independence.

6.69 a. The joint density is $f(y_1, y_2) = \frac{1}{y_1^2 y_2^2}$, $y_1 > 1$, $y_2 > 1$.

b. We have that $y_1 = u_1 u_2$ and $y_2 = u_2(1 - u_1)$. The Jacobian of transformation is u_2 . So,

$$f(u_1, u_2) = \frac{1}{u_1^2 u_2^3 (1 - u_1)^2},$$

with limits as specified in the problem.

c. The limits may be simplified to: $1/u_1 < u_2$, $0 < u_1 < 1/2$, or $1/(1 - u_1) < u_2$, $1/2 \leq u_1 \leq 1$.

d. If $0 < u_1 < 1/2$, then $f_{U_1}(u_1) = \int_{1/u_1}^{\infty} \frac{1}{u_1^2 u_2^3 (1 - u_1)^2} du_2 = \frac{1}{2(1 - u_1)^2}.$

If $1/2 \leq u_1 \leq 1$, then $f_{U_1}(u_1) = \int_{1/(1 - u_1)}^{\infty} \frac{1}{u_1^2 u_2^3 (1 - u_1)^2} du_2 = \frac{1}{2u_1^2}.$

e. Not independent since the joint density does not factor. Also note that the support is not rectangular.

- 6.70 a.** Since Y_1 and Y_2 are independent, their joint density is $f(y_1, y_2) = 1$. The inverse transformations are $y_1 = \frac{u_1 + u_2}{2}$ and $y_2 = \frac{u_1 - u_2}{2}$. Thus the Jacobian is

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}, \text{ so that}$$

$$f(u_1, u_2) = \frac{1}{2}, \text{ with limits as specified in the problem.}$$

- b.** The support is in the shape of a square with corners located $(0, 0)$, $(1, 1)$, $(2, 0)$, $(1, -1)$.

c. If $0 < u_1 < 1$, then $f_{U_1}(u_1) = \int_{-u_1}^{u_1} \frac{1}{2} du_2 = u_1$.

If $1 \leq u_1 < 2$, then $f_{U_1}(u_1) = \int_{u_1-2}^{2-u_1} \frac{1}{2} du_2 = 2 - u_1$.

d. If $-1 < u_2 < 0$, then $f_{U_2}(u_2) = \int_{-u_2}^{2+u_2} \frac{1}{2} du_2 = 1 + u_2$.

If $0 \leq u_2 < 1$, then $f_{U_2}(u_2) = \int_{u_2}^{2-u_2} \frac{1}{2} du_2 = 1 - u_2$.

- 6.71 a.** The joint density of Y_1 and Y_2 is $f(y_1, y_2) = \frac{1}{\beta^2} e^{-(y_1+y_2)/\beta}$. The inverse transformations are $y_1 = \frac{u_1 u_2}{1+u_2}$ and $y_2 = \frac{u_1}{1+u_2}$ and the Jacobian is

$$J = \begin{vmatrix} \frac{u_2}{1+u_2} & \frac{u_1}{(1+u_2)^2} \\ \frac{1}{1+u_2} & \frac{-u_1}{(1+u_2)^2} \end{vmatrix} = \left| \frac{-u_1}{(1+u_2)^2} \right|$$

So, the joint density of U_1 and U_2 is

$$f(u_1, u_2) = \frac{1}{\beta^2} e^{-u_1/\beta} \frac{u_1}{(1+u_2)^2}, \quad u_1 > 0, u_2 > 0.$$

- b.** Yes, U_1 and U_2 are independent since the joint density factors and the support is rectangular (Theorem 5.5).

- 6.72** Since the distribution function is $F(y) = y$ for $0 \leq y \leq 1$,

a. $g_{(1)}(u) = 2(1-u)$, $0 \leq u \leq 1$.

- b.** Since the above is a beta density with $\alpha = 1$ and $\beta = 2$, $E(U_1) = 1/3$, $V(U_1) = 1/18$.

- 6.73** Following Ex. 6.72,

a. $g_{(2)}(u) = 2u$, $0 \leq u \leq 1$.

- b.** Since the above is a beta density with $\alpha = 2$ and $\beta = 1$, $E(U_2) = 2/3$, $V(U_2) = 1/18$.

- 6.74** Since the distribution function is $F(y) = y/\theta$ for $0 \leq y \leq \theta$,

a. $G_{(n)}(y) = (y/\theta)^n$, $0 \leq y \leq \theta$.

b. $g_{(n)}(y) = G'_{(n)}(y) = ny^{n-1}/\theta^n$, $0 \leq y \leq \theta$.

c. It is easily shown that $E(Y_{(n)}) = \frac{n}{n+1}\theta$, $V(Y_{(n)}) = \frac{n\theta^2}{(n+1)^2(n+2)}$.

6.75 Following Ex. 6.74, the required probability is $P(Y_{(n)} < 10) = (10/15)^5 = .1317$.

6.76 Following Ex. 6.74 with $f(y) = 1/\theta$ for $0 \leq y \leq \theta$,

a. By Theorem 6.5, $g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!} \left(\frac{y}{\theta}\right)^{k-1} \left(\frac{\theta-y}{\theta}\right)^{n-k} \frac{1}{\theta} = \frac{n!}{(k-1)!(n-k)!} \frac{y^{k-1}(\theta-y)^{n-k}}{\theta^n}$, $0 \leq y \leq \theta$.

b. $E(Y_{(k)}) = \frac{n!}{(k-1)!(n-k)!} \int_0^\theta \frac{y^k(\theta-y)^{n-k}}{\theta^n} dy = \frac{k}{n+1} \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)} \int_0^\theta \left(\frac{y}{\theta}\right)^k \left(1 - \frac{y}{\theta}\right)^{n-k} dy$. To evaluate this

integral, apply the transformation $z = \frac{y}{\theta}$ and relate the resulting integral to that of a beta density with $\alpha = k + 1$ and $\beta = n - k + 1$. Thus, $E(Y_{(k)}) = \frac{k}{n+1} \theta$.

c. Using the same techniques in part b above, it can be shown that $E(Y_{(k)}^2) = \frac{k(k+1)}{(n+1)(n+2)} \theta^2$ so that $V(Y_{(k)}) = \frac{(n-k+1)k}{(n+1)^2(n+2)} \theta^2$.

d. $E(Y_{(k)} - Y_{(k-1)}) = E(Y_{(k)}) - E(Y_{(k-1)}) = \frac{k}{n+1} \theta - \frac{k-1}{n+1} \theta = \frac{1}{n+1} \theta$. Note that this is constant for all k , so that the expected order statistics are equally spaced.

6.77 a. Using Theorem 6.5, the joint density of $Y_{(j)}$ and $Y_{(k)}$ is given by

$$g_{(j)(k)}(y_j, y_k) = \frac{n!}{(j-1)!(k-1-j)!(n-k)!} \left(\frac{y_j}{\theta}\right)^{j-1} \left(\frac{y_k}{\theta} - \frac{y_j}{\theta}\right)^{k-1-j} \left(1 - \frac{y_k}{\theta}\right)^{n-k} \left(\frac{1}{\theta}\right)^2, 0 \leq y_j \leq y_k \leq \theta.$$

b. $\text{Cov}(Y_{(j)}, Y_{(k)}) = E(Y_{(j)}Y_{(k)}) - E(Y_{(j)})E(Y_{(k)})$. The expectations $E(Y_{(j)})$ and $E(Y_{(k)})$ were derived in Ex. 6.76. To find $E(Y_{(j)}Y_{(k)})$, let $u = y_j/\theta$ and $v = y_k/\theta$ and write

$$E(Y_{(j)}Y_{(k)}) = c\theta \int_0^1 \int_0^v u^j (v-u)^{k-1-j} v(1-v)^{n-k} dudv,$$

where $c = \frac{n!}{(j-1)!(k-1-j)!(n-k)!}$. Now, let $w = u/v$ so $u = wv$ and $du = vdw$. Then, the integral is

$$c\theta^2 \left[\int_0^1 u^{k+1} (1-u)^{n-k} du \right] \left[\int_0^1 w^j (1-w)^{k-1-j} dw \right] = c\theta^2 [B(k+2, n-k+1)] [B(j+1, k-j)].$$

Simplifying, this is $\frac{(k+1)j}{(n+1)(n+2)} \theta^2$. Thus, $\text{Cov}(Y_{(j)}, Y_{(k)}) = \frac{(k+1)j}{(n+1)(n+2)} \theta^2 - \frac{jk}{(n+1)^2} \theta^2 = \frac{n-k+1}{(n+1)^2(n+2)} \theta^2$.

c. $V(Y_{(k)} - Y_{(j)}) = V(Y_{(k)}) + V(Y_{(j)}) - 2\text{Cov}(Y_{(j)}, Y_{(k)})$
 $= \frac{(n-k+1)k}{(n+1)^2(n+2)} \theta^2 + \frac{(n-j+1)j}{(n+1)^2(n+2)} \theta^2 - \frac{2(n-k+1)}{(n+1)^2(n+2)} \theta^2 = \frac{(k-j)(n-k+1)}{(n+1)^2(n+2)} \theta^2$.

6.78 From Ex. 6.76 with $\theta = 1$, $g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!} y^{k-1} (1-y)^{n-k} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} y^{k-1} (1-y)^{n-k}$. Since $0 \leq y \leq 1$, this is the beta density as described.

6.79 The joint density of $Y_{(1)}$ and $Y_{(n)}$ is given by (see Ex. 6.77 with $j = 1, k = n$),

$$g_{(1)(n)}(y_1, y_n) = n(n-1) \left(\frac{y_n}{\theta} - \frac{y_1}{\theta}\right)^n \left(\frac{1}{\theta}\right)^2 = n(n-1) \left(\frac{1}{\theta}\right)^n (y_n - y_1)^{n-2}, 0 \leq y_1 \leq y_n \leq \theta.$$

Applying the transformation $U = Y_{(1)}/Y_{(n)}$ and $V = Y_{(n)}$, we have that $y_1 = uv$, $y_n = v$ and the Jacobian of transformation is v . Thus,

$$f(u, v) = n(n-1) \left(\frac{1}{\theta}\right)^n (v - uv)^{n-2} v = n(n-1) \left(\frac{1}{\theta}\right)^n (1-u)^{n-2} v^{n-1}, 0 \leq u \leq 1, 0 \leq v \leq \theta.$$

Since this joint density factors into separate functions of u and v and the support is rectangular, thus $Y_{(1)}/Y_{(n)}$ and $V = Y_{(n)}$ are independent.

6.80 The density and distribution function for Y are $f(y) = 6y(1-y)$ and $F(y) = 3y^2 - 2y^3$, respectively, for $0 \leq y \leq 1$.

a. $G_{(n)}(y) = (3y^2 - 2y^3)^n, 0 \leq y \leq 1$.

b. $g_{(n)}(y) = G'_{(n)}(y) = n(3y^2 - 2y^3)^{n-1}(6y - 6y^2) = 6ny(1-y)(3y^2 - 2y^3)^{n-1}, 0 \leq y \leq 1$.

c. Using the above density with $n = 2$, it is found that $E(Y_{(2)}) = .6286$.

6.81 a. With $f(y) = \frac{1}{\beta} e^{-y/\beta}$ and $F(y) = 1 - e^{-y/\beta}, y \geq 0$:

$$g_{(1)}(y) = n \left[e^{-y/\beta} \right]^{n-1} \frac{1}{\beta} e^{-y/\beta} = \frac{n}{\beta} e^{-ny/\beta}, y \geq 0.$$

This is the exponential density with mean β/n .

b. With $n = 5, \beta = 2, Y_{(1)}$ has an exponential distribution with mean .4. Thus

$$P(Y_{(1)} \leq 3.6) = 1 - e^{-9} = .99988.$$

6.82 Note that the distribution function for the largest order statistic is

$$G_{(n)}(y) = [F(y)]^n = [1 - e^{-y/\beta}]^n, y \geq 0.$$

It is easily shown that the median m is given by $m = \phi_{.5} = \beta \ln 2$. Now,

$$P(Y_{(m)} > m) = 1 - P(Y_{(m)} \leq m) = 1 - [F(\beta \ln 2)]^n = 1 - (.5)^n.$$

6.83 Since $F(m) = P(Y \leq m) = .5, P(Y_{(m)} > m) = 1 - P(Y_{(n)} \leq m) = 1 - G_{(n)}(m) = 1 - (.5)^n$. So, the answer holds regardless of the continuous distribution.

6.84 The distribution function for the Weibull is $F(y) = 1 - e^{-y^m/\alpha}, y > 0$. Thus, the distribution function for $Y_{(1)}$, the smallest order statistic, is given by

$$G_{(1)}(y) = 1 - [1 - F(y)]^n = 1 - [e^{-y^m/\alpha}]^n = 1 - e^{-ny^m/\alpha}, y > 0.$$

This is the Weibull distribution function with shape parameter m and scale parameter α/n .

6.85 Using Theorem 6.5, the joint density of $Y_{(1)}$ and $Y_{(2)}$ is given by

$$g_{(1)(2)}(y_1, y_2) = 2, 0 \leq y_1 \leq y_2 \leq 1.$$

$$\text{Thus, } P(2Y_{(1)} < Y_{(2)}) = \int_0^{1/2} \int_{2y_1}^1 2dy_2 dy_1 = .5.$$

6.86 Using Theorem 6.5 with $f(y) = \frac{1}{\beta} e^{-y/\beta}$ and $F(y) = 1 - e^{-y/\beta}, y \geq 0$:

a. $g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!} (1 - e^{-y/\beta})^{k-1} (e^{-y/\beta})^{n-k} \frac{e^{-y/\beta}}{\beta} = \frac{n!}{(k-1)!(n-k)!} (1 - e^{-y/\beta})^{k-1} (e^{-y/\beta})^{n-k+1} \frac{1}{\beta}, y \geq 0$.

b. $g_{(j)(k)}(y_j, y_k) = \frac{n!}{(j-1)!(k-1-j)!(n-k)!} (1 - e^{-y_j/\beta})^{j-1} (e^{-y_j/\beta} - e^{-y_k/\beta})^{k-1-j} (e^{-y_k/\beta})^{n-k+1} \frac{1}{\beta^2} e^{-y_j/\beta},$
 $0 \leq y_j \leq y_k < \infty.$

- 6.87** For this problem, we need the distribution of $Y_{(1)}$ (similar to Ex. 6.72). The distribution function of Y is

$$F(y) = P(Y \leq y) = \int_4^y (1/2)e^{-(1/2)(t-4)} dy = 1 - e^{-(1/2)(y-4)}, y \geq 4.$$

a. $g_{(1)}(y) = 2[e^{-(1/2)(y-4)}] \cdot \frac{1}{2} e^{-(1/2)(y-4)} = e^{-(y-4)}, y \geq 4.$

b. $E(Y_{(1)}) = 5.$

- 6.88** This is somewhat of a generalization of Ex. 6.87. The distribution function of Y is

$$F(y) = P(Y \leq y) = \int_{\theta}^y e^{-(t-\theta)} dy = 1 - e^{-(y-\theta)}, y > \theta.$$

a. $g_{(1)}(y) = n[e^{-(y-\theta)}]^{n-1} e^{-(y-\theta)} = ne^{-n(y-\theta)}, y > \theta.$

b. $E(Y_{(1)}) = \frac{1}{n} + \theta.$

- 6.89** Theorem 6.5 gives the joint density of $Y_{(1)}$ and $Y_{(n)}$ is given by (also see Ex. 6.79)

$$g_{(1)(n)}(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2}, 0 \leq y_1 \leq y_n \leq 1.$$

Using the method of transformations, let $R = Y_{(n)} - Y_{(1)}$ and $S = Y_{(1)}$. The inverse transformations are $y_1 = s$ and $y_n = r + s$ and Jacobian of transformation is 1. Thus, the joint density of R and S is given by

$$f(r, s) = n(n-1)(r + s - s)^{n-2} = n(n-1)r^{n-2}, 0 \leq s \leq 1 - r \leq 1.$$

(Note that since $r = y_n - y_1$, $r \leq 1 - y_1$ or equivalently $r \leq 1 - s$ and then $s \leq 1 - r$).

The marginal density of R is then

$$f_R(r) = \int_0^{1-r} n(n-1)r^{n-2} ds = n(n-1)r^{n-2}(1-r), 0 \leq r \leq 1.$$

FYI, this is a beta density with $\alpha = n - 1$ and $\beta = 2$.

- 6.90** Since the points on the interval $(0, t)$ at which the calls occur are uniformly distributed, we have that $F(w) = w/t, 0 \leq w \leq t$.

a. The distribution of $W_{(4)}$ is $G_{(4)}(w) = [F(w)]^4 = w^4 / t^4, 0 \leq w \leq t$. Thus $P(W_{(4)} \leq 1) = G_{(4)}(1) = 1/16$.

b. With $t = 2$, $E(W_{(4)}) = \int_0^2 4w^4 / 2^4 dw = \int_0^2 w^4 / 4 dw = 1.6$.

- 6.91** With the exponential distribution with mean θ , we have $f(y) = \frac{1}{\theta} e^{-y/\theta}$, $F(y) = 1 - e^{-y/\theta}$, for $y \geq 0$.

a. Using Theorem 6.5, the joint distribution of order statistics $W_{(j)}$ and $W_{(j-1)}$ is given by

$$g_{(j-1)(j)}(w_{j-1}, w_j) = \frac{n!}{(j-2)!(n-j)!} \left(1 - e^{-w_{j-1}/\theta}\right)^{j-2} \left(e^{-w_j/\theta}\right)^{n-j} \frac{1}{\theta^2} \left(e^{-(w_{j-1}+w_j)/\theta}\right), 0 \leq w_{j-1} \leq w_j < \infty.$$

Define the random variables $S = W_{(j-1)}$, $T_j = W_{(j)} - W_{(j-1)}$. The inverse transformations are $w_{j-1} = s$ and $w_j = t_j + s$ and Jacobian of transformation is 1. Thus, the joint density of S and T_j is given by

$$\begin{aligned}
 f(s, t_j) &= \frac{n!}{(j-2)!(n-j)!} (1 - e^{-s/\theta})^{j-2} (e^{-(t_j+s)/\theta})^{n-j} \frac{1}{\theta^2} (e^{-(2s+t_j)/\theta}) \\
 &= \frac{n!}{(j-2)!(n-j)!} e^{-(n-j+1)t_j/\theta} \frac{1}{\theta^2} (1 - e^{-s/\theta})^{j-2} (e^{-(n-j+2)s/\theta}), \quad s \geq 0, t_j \geq 0.
 \end{aligned}$$

The marginal density of T_j is then

$$f_{T_j}(t_j) = \frac{n!}{(j-2)!(n-j)!} e^{-(n-j+1)t_j/\theta} \frac{1}{\theta^2} \int_0^\infty (1 - e^{-s/\theta})^{j-2} (e^{-(n-j+2)s/\theta}) ds.$$

Employ the change of variables $u = e^{-s/\theta}$ and the above integral becomes the integral of a scaled beta density. Evaluating this, the marginal density becomes

$$f_{T_j}(t_j) = \frac{n-j+1}{\theta} e^{-(n-j+1)t_j/\theta}, \quad t_j \geq 0.$$

This is the density of an exponential distribution with mean $\theta/(n-j+1)$.

b. Observe that

$$\begin{aligned}
 \sum_{j=1}^r (n-j+1)T_j &= nW_1 + (n-1)(W_2 - W_1) + (n-2)(W_3 - W_2) + \dots + (n-r+1)(W_r - W_{r-1}) \\
 &= W_1 + W_2 + \dots + W_{r-1} + (n-r+1)W_r = \sum_{j=1}^r W_j + (n-r)W_r = U_r.
 \end{aligned}$$

$$\text{Hence, } E(U_r) = \sum_{j=1}^r (n-j+1)E(T_j) = r\theta.$$

6.92 By Theorem 6.3, U will have a normal distribution with mean $(1/2)(\mu - 3\mu) = -\mu$ and variance $(1/4)(\sigma^2 + 9\sigma^2) = 2.5\sigma^2$.

6.93 By independence, the joint distribution of I and R is $f(i, r) = 2r$, $0 \leq i \leq 1$ and $0 \leq r \leq 1$. To find the density for W , fix $R=r$. Then, $W = I^2 r$ so $I = \sqrt{W/r}$ and $\left|\frac{di}{dw}\right| = \frac{1}{2r} \left(\frac{w}{r}\right)^{-1/2}$ for the range $0 \leq w \leq r \leq 1$. Thus, $f(w, r) = \sqrt{r/w}$ and

$$f(w) = \int_w^1 \sqrt{r/w} dr = \frac{2}{3} \left(\frac{1}{\sqrt{w}} - w \right), \quad 0 \leq w \leq 1.$$

6.94 Note that Y_1 and Y_2 have identical gamma distributions with $\alpha = 2$, $\beta = 2$. The mgf is

$$m(t) = (1 - 2t)^{-2}, \quad t < 1/2.$$

The mgf for $U = (Y_1 + Y_2)/2$ is

$$m_U(t) = E(e^{tU}) = E(e^{t(Y_1+Y_2)/2}) = m(t/2)m(t/2) = (1-t)^{-4}.$$

This is the mgf for a gamma distribution with $\alpha = 4$ and $\beta = 1$, so that is the distribution of U .

6.95 By independence, $f(y_1, y_2) = 1$, $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$.

a. Consider the joint distribution of $U_1 = Y_1/Y_2$ and $V = Y_2$. Fixing V at v , we can write $U_1 = Y_1/v$. Then, $Y_1 = vU_1$ and $\frac{dy_1}{du} = v$. The joint density of U_1 and V is $g(u, v) = v$. The ranges of u and v are as follows:

- if $y_1 \leq y_2$, then $0 \leq u \leq 1$ and $0 \leq v \leq 1$
- if $y_1 > y_2$, then u has a minimum value of 1 and a maximum at $1/y_2 = 1/v$.
Similarly, $0 \leq v \leq 1$

So, the marginal distribution of U_1 is given by

$$f_{U_1}(u) = \begin{cases} \int_0^1 v dv = \frac{1}{2} & 0 \leq u \leq 1 \\ \int_0^{1/u} v dv = \frac{1}{2u^2} & u > 1 \end{cases}.$$

- b.** Consider the joint distribution of $U_2 = -\ln(Y_1 Y_2)$ and $V = Y_1$. Fixing V at v , we can write $U_2 = -\ln(v Y_2)$. Then, $Y_2 = e^{-U_2} / v$ and $\frac{dy_2}{du} = -e^{-u} / v$. The joint density of U_2 and V is $g(u, v) = -e^{-u} / v$, with $-\ln v \leq u < \infty$ and $0 \leq v \leq 1$. Or, written another way, $e^{-u} \leq v \leq 1$.

So, the marginal distribution of U_2 is given by

$$f_{U_2}(u) = \int_{e^{-u}}^1 -e^{-u} / v dv = u e^{-u}, 0 \leq u.$$

- c.** Same as Ex. 6.35.

- 6.96** Note that $P(Y_1 > Y_2) = P(Y_1 - Y_2 > 0)$. By Theorem 6.3, $Y_1 - Y_2$ has a normal distribution with mean $5 - 4 = 1$ and variance $1 + 3 = 4$. Thus,
 $P(Y_1 - Y_2 > 0) = P(Z > -1/2) = .6915$.

- 6.97** The probability mass functions for Y_1 and Y_2 are:

y_1	0	1	2	3	4	y_2	0	1	2	3
$p_1(y_1)$.4096	.4096	.1536	.0256	.0016	$p_2(y_2)$.125	.375	.375	.125

Note that $W = Y_1 + Y_2$ is a random variable with support $(0, 1, 2, 3, 4, 5, 6, 7)$. Using the hint given in the problem, the mass function for W is given by

w	$p(w)$
0	$p_1(0)p_2(0) = .4096(.125) = \mathbf{.0512}$
1	$p_1(0)p_2(1) + p_1(1)p_2(0) = .4096(.375) + .4096(.125) = \mathbf{.2048}$
2	$p_1(0)p_2(2) + p_1(2)p_2(0) + p_1(1)p_2(1) = .4096(.375) + .1536(.125) + .4096(.375) = \mathbf{.3264}$
3	$p_1(0)p_2(3) + p_1(3)p_2(0) + p_1(1)p_2(2) + p_1(2)p_2(1) = .4096(.125) + .0256(.125) + .4096(.375) + .1536(.375) = \mathbf{.2656}$
4	$p_1(1)p_2(3) + p_1(3)p_2(1) + p_1(2)p_2(2) + p_1(4)p_2(0) = .4096(.125) + .0256(.375) + .1536(.375) + .0016(.125) = \mathbf{.1186}$
5	$p_1(2)p_2(3) + p_1(3)p_2(2) + p_1(4)p_2(1) = .1536(.125) + .0256(.375) + .0016(.375) = \mathbf{.0294}$
6	$p_1(4)p_2(2) + p_1(3)p_2(3) = .0016(.375) + .0256(.125) = \mathbf{.0038}$
7	$p_1(4)p_2(3) = .0016(.125) = \mathbf{.0002}$

Check: $.0512 + .2048 + .3264 + .2656 + .1186 + .0294 + .0038 + .0002 = 1$.

- 6.98** The joint distribution of Y_1 and Y_2 is $f(y_1, y_2) = e^{-(y_1+y_2)}$, $y_1 > 0$, $y_2 > 0$. Let $U_1 = \frac{Y_1}{Y_1+Y_2}$, $U_2 = Y_2$. The inverse transformations are $y_1 = u_1 u_2 / (1 - u_1)$ and $y_2 = u_2$ so the Jacobian of transformation is

$$J = \begin{vmatrix} \frac{u_2}{(1-u_1)^2} & \frac{u_1}{1-u_1} \\ 0 & 1 \end{vmatrix} = \frac{u_2}{(1-u_1)^2}.$$

Thus, the joint distribution of U_1 and U_2 is

$$f(u_1, u_2) = e^{-[u_1 u_2 / (1-u_1) + u_2]} \frac{u_2}{(1-u_1)^2} = e^{-[u_2 / (1-u_1)]} \frac{u_2}{(1-u_1)^2}, \quad 0 \leq u_1 \leq 1, u_2 > 0.$$

Therefore, the marginal distribution for U_1 is

$$f_{U_1}(u_1) = \int_0^{\infty} e^{-[u_2 / (1-u_1)]} \frac{u_2}{(1-u_1)^2} du_2 = 1, \quad 0 \leq u_1 \leq 1.$$

Note that the integrand is a gamma density function with $\alpha = 1$, $\beta = 1 - u_1$.

- 6.99** This is a special case of Example 6.14 and Ex. 6.63.

- 6.100** Recall that by Ex. 6.81, $Y_{(1)}$ is exponential with mean $15/5 = 3$.

- $P(Y_{(1)} > 9) = e^{-3}$.
- $P(Y_{(1)} < 12) = 1 - e^{-4}$.

- 6.101** If we let $(A, B) = (-1, 1)$ and $T = 0$, the density function for X , the landing point is

$$f(x) = 1/2, \quad -1 < x < 1.$$

We must find the distribution of $U = |X|$. Therefore,

$$F_U(u) = P(U \leq u) = P(|X| \leq u) = P(-u \leq X \leq u) = [u - (-u)]/2 = u.$$

So, $f_U(u) = F'_U(u) = 1$, $0 \leq u \leq 1$. Therefore, U has a uniform distribution on $(0, 1)$.

- 6.102** Define Y_1 = point chosen for sentry 1 and Y_2 = point chosen for sentry 2. Both points are chosen along a one-mile stretch of highway, so assuming independent uniform distributions on $(0, 1)$, the joint distribution for Y_1 and Y_2 is

$$f(y_1, y_2) = 1, \quad 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1.$$

The probability of interest is $P(|Y_1 - Y_2| < \frac{1}{2})$. This is most easily solved using geometric considerations (similar to material in Chapter 5): $P(|Y_1 - Y_2| < \frac{1}{2}) = .75$ (this can easily be found by considering the complement of the event).

- 6.103** The joint distribution of Y_1 and Y_2 is $f(y_1, y_2) = \frac{1}{2\pi} e^{-(y_1^2 + y_2^2)/2}$. Considering the transformations $U_1 = Y_1/Y_2$ and $U_2 = Y_2$. With $y_1 = u_1 u_2$ and $y_2 = |u_2|$, the Jacobian of transformation is u_2 so that the joint density of U_1 and U_2 is

$$f(u_1, u_2) = \frac{1}{2\pi} |u_2| e^{-[(u_1 u_2)^2 + u_2^2]/2} = \frac{1}{2\pi} |u_2| e^{-[u_2^2(1+u_1^2)]/2}.$$

The marginal density of U_1 is

$$f_{U_1}(u_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi} |u_2| e^{-[u_2^2(1+u_1^2)]/2} du_2 = \int_0^{\infty} \frac{1}{\pi} u_2 e^{-[u_2^2(1+u_1^2)]/2} du_2.$$

Using the change of variables $v = u_2^2$ so that $du_2 = \frac{1}{2\sqrt{v}} dv$ gives the integral

$$f_{U_1}(u_1) = \int_0^{\infty} \frac{1}{2\pi} e^{-[v(1+u_1^2)]/2} dv = \frac{1}{\pi(1+u_1^2)}, \quad -\infty < u_1 < \infty.$$

The last expression above comes from noting the integrand is related an exponential density with mean $2/(1+u_1^2)$. The distribution of U_1 is called the Cauchy distribution.

6.104 a. The event $\{Y_1 = Y_2\}$ occurs if

$$\{(Y_1 = 1, Y_2 = 1), (Y_1 = 2, Y_2 = 2), (Y_1 = 3, Y_2 = 3), \dots\}$$

So, since the probability mass function for the geometric is given by $p(y) = p(1-p)^{y-1}$, we can find the probability of this event by

$$\begin{aligned} P(Y_1 = Y_2) &= p(1)^2 + p(2)^2 + p(3)^2 \dots = p^2 + p^2(1-p)^2 + p^2(1-p)^4 + \dots \\ &= p^2 \sum_{j=0}^{\infty} (1-p)^{2j} = \frac{p^2}{1-(1-p)^2} = \frac{p}{2-p}. \end{aligned}$$

b. Similar to part a, the event $\{Y_1 - Y_2 = 1\} = \{Y_1 = Y_2 + 1\}$ occurs if

$$\{(Y_1 = 2, Y_2 = 1), (Y_1 = 3, Y_2 = 2), (Y_1 = 4, Y_2 = 3), \dots\}$$

Thus,

$$\begin{aligned} P(Y_1 - Y_2 = 1) &= p(2)p(1) + p(3)p(2) + p(4)p(3) + \dots \\ &= p^2(1-p) + p^2(1-p)^3 + p^2(1-p)^5 + \dots = \frac{p(1-p)}{2-p}. \end{aligned}$$

c. Define $U = Y_1 - Y_2$. To find $p_U(u) = P(U = u)$, assume first that $u > 0$. Thus,

$$\begin{aligned} P(U = u) &= P(Y_1 - Y_2 = u) = \sum_{y_2=1}^{\infty} P(Y_1 = u + y_2)P(Y_2 = y_2) = \sum_{y_2=1}^{\infty} p(1-p)^{u+y_2-1} p(1-p)^{y_2-1} \\ &= p^2(1-p)^u \sum_{y_2=1}^{\infty} (1-p)^{2(y_2-1)} = p^2(1-p)^u \sum_{x=1}^{\infty} (1-p)^{2x} = \frac{p(1-p)^u}{2-p}. \end{aligned}$$

If $u < 0$, proceed similarly with $y_2 = y_1 - u$ to obtain $P(U = u) = \frac{p(1-p)^{-u}}{2-u}$. These two

results can be combined to yield $p_U(u) = P(U = u) = \frac{p(1-p)^{|u|}}{2-u}$, $u = 0, \pm 1, \pm 2, \dots$.

6.105 The inverse transformation is $y = 1/u - 1$. Then,

$$f_U(u) = \frac{1}{B(\alpha, \beta)} \left(\frac{1-u}{u}\right)^{\alpha-1} u^{\alpha+\beta} \frac{1}{u^2} = \frac{1}{B(\alpha, \beta)} u^{\beta-1} (1-u)^{\alpha-1}, \quad 0 < u < 1.$$

This is the beta distribution with parameters β and α .

6.106 Recall that the distribution function for a continuous random variable is monotonic increasing and returns values on $[0, 1]$. Thus, the random variable $U = F(Y)$ has support on $(0, 1)$ and has distribution function

$$F_U(u) = P(U \leq u) = P(F(Y) \leq u) = P(Y \leq F^{-1}(u)) = F[F^{-1}(u)] = u, \quad 0 \leq u \leq 1.$$

The density function is $f_U(u) = F'_U(u) = 1$, $0 \leq u \leq 1$, which is the density for the uniform distribution on $(0, 1)$.

- 6.107** The density function for Y is $f(y) = \frac{1}{4}$, $-1 \leq y \leq 3$. For $U = Y^2$, the density function for U is given by

$$f_U(u) = \frac{1}{2\sqrt{u}} [f(\sqrt{u}) + f(-\sqrt{u})],$$

as with Example 6.4. If $-1 \leq y \leq 3$, then $0 \leq u \leq 9$. However, if $1 \leq u \leq 9$, $f(-\sqrt{u})$ is not positive. Therefore,

$$f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}} \left(\frac{1}{4} + \frac{1}{4} \right) = \frac{1}{4\sqrt{u}} & 0 \leq u < 1 \\ \frac{1}{2\sqrt{u}} \left(\frac{1}{4} + 0 \right) = \frac{1}{8\sqrt{u}} & 1 \leq u \leq 9 \end{cases}.$$

- 6.108** The system will operate provided that C_1 and C_2 function and C_3 or C_4 function. That is, defining the system as S and using set notation, we have

$$S = (C_1 \cap C_2) \cap (C_3 \cup C_4) = (C_1 \cap C_2 \cap C_3) \cup (C_1 \cap C_2 \cap C_4).$$

At some y , the probability that a component is operational is given by $1 - F(y)$. Since the components are independent, we have

$$P(S) = P(C_1 \cap C_2 \cap C_3) + P(C_1 \cap C_2 \cap C_4) - P(C_1 \cap C_2 \cap C_3 \cap C_4).$$

Therefore, the reliability of the system is given by

$$[1 - F(y)]^3 + [1 - F(y)]^3 - [1 - F(y)]^4 = [1 - F(y)]^3 [1 + F(y)].$$

- 6.109** Let C_3 be the production cost. Then U , the profit function (per gallon), is

$$U = \begin{cases} C_1 - C_3 & \frac{1}{3} < Y < \frac{2}{3} \\ C_2 - C_3 & \text{otherwise} \end{cases}.$$

So, U is a discrete random variable with probability mass function

$$P(U = C_1 - C_3) = \int_{1/3}^{2/3} 20y^3(1-y)dy = .4156.$$

$$P(U = C_2 - C_3) = 1 - .4156 = .5844.$$

- 6.110** a. Let X = next gap time. Then, $P(X \leq 60) = F_X(60) = 1 - e^{-6}$.
b. If the next four gap times are assumed to be independent, then $Y = X_1 + X_2 + X_3 + X_4$ has a gamma distribution with $\alpha = 4$ and $\beta = 10$. Thus,

$$f(y) = \frac{1}{\Gamma(4)10^4} y^3 e^{-y/10}, \quad y \geq 0.$$

- 6.111** a. Let $U = \ln Y$. So, $\frac{du}{dy} = \frac{1}{y}$ and with $f_U(u)$ denoting the normal density function,

$$f_Y(y) = \frac{1}{y} f_U(\ln y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right], \quad y > 0.$$

- b. Note that $E(Y) = E(e^U) = m_U(1) = e^{\mu + \sigma^2/2}$, where $m_U(t)$ denotes the mgf for U . Also, $E(Y^2) = E(e^{2U}) = m_U(2) = e^{2\mu + 2\sigma^2}$ so $V(Y) = e^{2\mu + 2\sigma^2} - \left(e^{\mu + \sigma^2/2}\right)^2 = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$.

6.112 a. Let $U = \ln Y$. So, $\frac{du}{dy} = \frac{1}{y}$ and with $f_U(u)$ denoting the gamma density function,

$$f_Y(y) = \frac{1}{y} f_U(\ln y) = \frac{1}{y \Gamma(\alpha) \beta^\alpha} (\ln y)^{\alpha-1} e^{-(\ln y)/\beta} = \frac{1}{\Gamma(\alpha) \beta^\alpha} (\ln y)^{\alpha-1} y^{-(1+\beta)/\beta}, y > 1.$$

b. Similar to Ex. 6.111: $E(Y) = E(e^U) = m_U(1) = (1 - \beta)^{-\alpha}$, $\beta < 1$, where $m_U(t)$ denotes the mgf for U .

c. $E(Y^2) = E(e^{2U}) = m_U(2) = (1 - 2\beta)^{-\alpha}$, $\beta < .5$, so that $V(Y) = (1 - 2\beta)^{-\alpha} - (1 - \beta)^{-2\alpha}$.

6.113 a. The inverse transformations are $y_1 = u_1/u_2$ and $y_2 = u_2$ so that the Jacobian of transformation is $1/|u_2|$. Thus, the joint density of U_1 and U_2 is given by

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(u_1/u_2, u_2) \frac{1}{|u_2|}.$$

b. The marginal density is found using standard techniques.

c. If Y_1 and Y_2 are independent, the joint density will factor into the product of the marginals, and this is applied to part b above.

6.114 The volume of the sphere is $V = \frac{4}{3} \pi R^3$, or $R = \left(\frac{3}{4\pi} V\right)^{1/3}$, so that $\frac{dr}{dv} = \frac{1}{3} \left(\frac{3}{4\pi}\right)^{1/3} v^{-2/3}$. Thus, $f_V(v) = \frac{2}{3} \left(\frac{3}{4\pi}\right)^{2/3} v^{-1/3}$, $0 \leq v \leq \frac{4}{3} \pi$.

6.115 a. Let R = distance from a randomly chosen point to the nearest particle. Therefore,

$$P(R > r) = P(\text{no particles in the sphere of radius } r) = P(Y = 0 \text{ for volume } \frac{4}{3} \pi r^3).$$

Since Y = # of particles in a volume v has a Poisson distribution with mean λv , we have

$$P(R > r) = P(Y = 0) = e^{-(4/3)\pi r^3 \lambda}, r > 0.$$

Therefore, the distribution function for R is $F(r) = 1 - P(R > r) = 1 - e^{-(4/3)\pi r^3 \lambda}$ and the density function is

$$f(r) = F'(r) = 4\lambda \pi r^2 e^{-(4/3)\pi r^3 \lambda}, r > 0.$$

b. Let $U = R^3$. Then, $R = U^{1/3}$ and $\frac{dr}{du} = \frac{1}{3} u^{-2/3}$. Thus,

$$f_U(u) = \frac{4\lambda \pi}{3} e^{-(4\lambda \pi/3)u}, u > 0.$$

This is the exponential density with mean $\frac{3}{4\lambda \pi}$.

6.116 a. The inverse transformations are $y_1 = u_1 + u_2$ and $y_2 = u_2$. The Jacobian of transformation is 1 so that the joint density of U_1 and U_2 is

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(u_1 + u_2, u_2).$$

b. The marginal density is found using standard techniques.

c. If Y_1 and Y_2 are independent, the joint density will factor into the product of the marginals, and this is applied to part b above.