

SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

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(EXERCISES BY PER SIDÉN)

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6. CONVERGENCE

Exercise 6.1

Assume that Y, X_1, X_2, \dots are independent random variables with $Y \sim U(0, 1)$ and $X_n \sim U(\frac{1}{2n}, 1)$ for $n = 1, 2, \dots$

- (a) Does $X_n \xrightarrow{a.s.} Y$ as $n \rightarrow \infty$?
- (b) Does $X_n \xrightarrow{p} Y$ as $n \rightarrow \infty$?
- (c) Does $X_n \xrightarrow{d} Y$ as $n \rightarrow \infty$?

Exercise 6.2 (6.1 in Gut's book)

Let X_1, X_2, \dots be $U(0, 1)$ -distributed independent random variables. Show that

- (a) $\max_{1 \leq k \leq n} X_k \xrightarrow{p} 1$ as $n \rightarrow \infty$,
- (b) $\min_{1 \leq k \leq n} X_k \xrightarrow{p} 0$ as $n \rightarrow \infty$.

Exercise 6.3 (6.6 in Gut's book)

Suppose that X_1, X_2, \dots are independent, $Pa(1, 2)$ -distributed random variables, and set $Y_n = \min\{X_1, X_2, \dots, X_n\}$.

- (a) Show that $Y_n \xrightarrow{p} 1$ as $n \rightarrow \infty$.
It thus follows that $Y_n \approx 1$ with a probability close to 1 when n is large. One might therefore suspect that there exists a limit theorem to the effect that $Y_n - 1$, suitable rescaled, converges in distribution as $n \rightarrow \infty$ (note that $Y_n > 1$ always).
- (b) Show that $n(Y_n - 1)$ converges in distribution as $n \rightarrow \infty$, and determine the limit distribution.

Exercise 6.4 (6.15 in Gut's book)

Let X and Y be random variables such that

$$Y|X = x \sim N(0, x) \text{ with } X \sim Po(\lambda).$$

- (a) Find the characteristic function of Y .
- (b) Show that

$$\frac{Y}{\sqrt{\lambda}} \xrightarrow{d} N(0, 1) \text{ as } \lambda \rightarrow \infty.$$

Exercise 6.5* (6.19 in Gut's book)

Suppose that the random variables N_n, X_1, X_2, \dots are independent, that $N_n \sim Ge(p_n)$, $0 < p_n < 1$, and that X_1, X_2, \dots are equidistributed with finite mean μ . Show that if $p_n \rightarrow 0$ as $n \rightarrow \infty$ then $p_n(X_1 + X_2 + \dots + X_{N_n})$ converges in distribution as $n \rightarrow \infty$, and determine the limit distribution.

Exercise 6.6* (6.25 in Gut's book)

Let $X_n \sim \Gamma(n, 1)$, and set

$$Y_n = \frac{X_n - n}{\sqrt{X_n}}.$$

Show that $Y_n \xrightarrow{d} N(0, 1)$ as $n \rightarrow \infty$.