4.18 Find the maximum likelihood estimates of the 2×1 mean vector $\boldsymbol{\mu}$ and the 2×2 covariance matrix $\boldsymbol{\Sigma}$ based on the random sample

$$\mathbf{X} = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

from a bivariate normal population.

Sln: Since the random samples $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ and $\mathbf{X}_4$ are from normal population, the MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are $\bar{X}$ and $\frac{1}{n}\sum_{j=1}^{n}\left(\bar{\mathbf{X}}_j - \bar{\mathbf{X}}\right)\left(\bar{\mathbf{X}}_j - \bar{\mathbf{X}}\right)^T$. Therefore,

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}} = \frac{1}{n}\underset{p\times n}{\mathbf{X}^T}\cdot \underset{n\times 1}{\mathbb{1}} = \frac{1}{4}\underset{2\times 4}{\mathbf{X}^T}\cdot \underset{4\times 1}{\mathbb{1}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{3+4+5+4}{3} \\ \frac{6+4+7+7}{3} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

$$\begin{aligned}
\hat{\boldsymbol{\Sigma}} &= \frac{1}{n}\sum_{j=1}^{n}\left(\mathbf{X}_j - \bar{\mathbf{X}}\right)\left(\mathbf{X}_j - \bar{\mathbf{X}}\right)^T \\
&= \frac{1}{4}\left\{\begin{bmatrix} -1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\begin{bmatrix} 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\right\} \\
&= \frac{1}{4}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} \\
&= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}.
\end{aligned}$$

4.19 Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{20}$ be a random sample of size $n = 20$ from an $N_6(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ population. Specify each of the following completely.

(a) the distribution of $\left(\mathbf{X}_1 - \boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1}\left(\mathbf{X}_1 - \boldsymbol{\mu}\right)$

(b) the distributions of $\bar{\mathbf{X}}$ and $\sqrt{n}\left(\bar{\mathbf{X}} - \boldsymbol{\mu}\right)$

(c) the distribution of $(n-1)\mathbf{S}$

Sln: (a) $\left(\mathbf{X}_1 - \boldsymbol{\mu}\right)^T \boldsymbol{\Sigma}^{-1}\left(\mathbf{X}_1 - \boldsymbol{\mu}\right)$ is distributed as χ_6^2.

(b) $\bar{\mathbf{X}}$ is distributed as $N_6\left(\boldsymbol{\mu}, \frac{1}{20}\boldsymbol{\Sigma}\right)$ and $\sqrt{n}\left(\bar{\mathbf{X}} - \boldsymbol{\mu}\right)$ is distributed as $N_6\left(\mathbf{0}, \boldsymbol{\Sigma}\right)$.

(c) $(n-1)\mathbf{S}$ is distributed as Wishart distribution $\sum_{i=1}^{20-1} \underbrace{\mathbf{Z}_i \mathbf{Z}_i^T}_{6\times 6}$, where $\mathbf{Z}_i \sim N_6(\mathbf{0}, \mathbf{\Sigma})$.

4.21 Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{60}$ be a random sample of size 60 from a four-variate normal distribution having mean $\boldsymbol{\mu}$ and covariance $\mathbf{\Sigma}$. Specify each of the following completely.

(a) The distribution of $\overline{\mathbf{X}}$

(b) The distribution of $(\mathbf{X}_1 - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{X}_1 - \boldsymbol{\mu})$

(c) The distribution of $n(\overline{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu})$

(d) The approximate distribution of $n(\overline{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu})$

Sln: (a) $\overline{\mathbf{X}}$ is distributed as $N_4\left(\boldsymbol{\mu}, \frac{1}{60}\mathbf{\Sigma}\right)$

(b) $(\mathbf{X}_1 - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{X}_1 - \boldsymbol{\mu})$ is distributed as χ_4^2.

(c) $n(\overline{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu})$ is distributed as χ_4^2, because $\overline{\mathbf{X}}$ is distributed as $N_4\left(\boldsymbol{\mu}, \frac{1}{60}\mathbf{\Sigma}\right)$

(d) Since $60 \gg 4$, $n(\overline{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu})$ can be approximated as χ_4^2.

4.22 Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{75}$ be a random sample from population distribution with mean $\boldsymbol{\mu}$ and covariance $\mathbf{\Sigma}$. What is the approximate distribution of each of the following.

(a) $\overline{\mathbf{X}}$

(b) $n(\overline{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu})$

Sln: (a) $\overline{\mathbf{X}}$ can be approximated by $N_p\left(\boldsymbol{\mu}, \frac{1}{75}\mathbf{\Sigma}\right)$

(b) $n(\overline{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\overline{\mathbf{X}} - \boldsymbol{\mu})$ can be approximated by χ_p^2