Meeting 6:
Utility, part 1…

Utility

What do we mean by *utility* ?

Example
Consider the following lottery:
The price for buying a lottery ticket is SEK 50. If you win on this lottery you win SEK 10000. The probability of winning is 0.05.

Hence, the expected payoff for buying a ticket is SEK $(-50)\cdot 0.95 + 9950\cdot 0.05 = 450$.
The expected payoff for not buying a ticket is SEK 0.

According to the *ER* – criterion (maximize the expected payoff) you should buy a ticket.

Now, the following three persons are all considering buying a ticket:

- Martin, who has a total asset of SEK 150 000 once he has paid his monthly bills
- Zoran, who has a total asset of SEK 2100 once he has paid his monthly bills
- Sarah, who has a total asset of SEK 1 450 000 once she has paid her monthly bills

Do you think all these three would make their decisions according to the *ER*-criterion?

Temporarily we can work with *consequences* of making a decision (taking an action). Such consequences depend also on the state of the world and a consequence can thus be written

$$c = c\left(\delta\left(\boldsymbol{x}\right), \boldsymbol{\theta}\right)$$

In the rest of the course we will tune down the necessity of defining a decision procedure, and separating between procedure and action. When working with non-probabilistic criteria this separation is necessary, but with probabilistic criteria like max(ER) and min(EL) it suffices to use actions.

To be consistent with the notation used in Winkler we will from now on use a to denote a specific action (and not $\delta(\boldsymbol{x})$).

Hence, we will write $c = c\left(a, \boldsymbol{\theta}\right)$

Consequences can often be expressed in monetary terms, i.e. using payoff functions we may have $c(a, \boldsymbol{\theta}) = R(a, \boldsymbol{\theta})$.

Cash equivalents may be used to transform non-monetary consequences into monetary ones.

The following notation is used for comparison of consequences, when the consequences are such that it is not possible to use a simple numerical ordering

$c_i \prec c_j$ means that consequence c_j is preferred to consequence c_i

$c_i \sim c_j$ means that c_i and c_j are equally preferred

$c_i \precsim c_j$ means that c_i is not preferred to c_j

Back to the example with the lottery

- Martin, has a total asset of SEK 150 000 once he has paid his monthly bills
- Zoran, has a total asset of SEK 2100 once he has paid his monthly bills
- Sarah, has a total asset of SEK 1 450 000 once she has paid her monthly bills

Could the preferences be like the following?

Martin: not buying a ticket $\prec$ buying a ticket
Zoran: buying a ticket $\prec$ not buying a ticket
Sarah: buying a ticket $\sim$ not buying a ticket

Clearly, the *ER*-criterion (maximising the expected payoff) is not always the obvious probabilistic criterion.

A consequence of an action can be preferred to another consequence by one decision-maker, while the opposite can hold for another decision-maker.

Another example

Assume that when the temperature is above 25 °C and you have decided to wear long trousers and a long sleeves shirt, you will as a consequence feel unusually hot

$$c_1 = c\,(a = \text{“longs”},\ \theta > 25\ ^\circ\text{C}\ )$$

Moreover, assume that when the temperature is below 15 °C and you have decided to wear shorts and a t-shirt you will as a consequence feel unusually cold

$$c_2 = c\,(a = \text{“shorts”},\ \theta < 15\ ^\circ\text{C}\ )$$

<u>Your</u> preference order would be one of $c_1 \prec c_2$, $c_2 \prec c_1$ and $c_1 \sim c_2$

$c_1 = c\,(a = \text{“longs”},\ \theta > 25\ °\mathrm{C})$
$c_2 = c\,(a = \text{“shorts”},\ \theta < 15\ °\mathrm{C})$

If you think it is always better to feel warm than cold your preference order will be

$$c_2 \prec c_1$$

Another person <u>feeling the same as you</u> may really dislike feeling too warm and hence has the preference order

$$c_1 \prec c_2$$

A third person <u>also feeling the same as you</u> may be someone who would always complain as soon as weather condition and choice of garments do not “fit” well probably has the preference order

$$c_1 \sim c_2$$

To be able to allow for <u>a relative desirability</u> that deviates from the linear comparability of monetary consequences we introduce a so-called ***utility function***:

$$U(c) = U(c(a, \boldsymbol{\theta})) = U(a, \boldsymbol{\theta})$$

If the difference in payoff between two pairs of action and state of world is d_R, *i.e.*

$$d_R = R(a_1, \boldsymbol{\theta}_1) - R(a_2, \boldsymbol{\theta}_2)$$

the following three differences in utility may hold

$$\begin{aligned} U(a_1, \boldsymbol{\theta}_1) - U(a_2, \boldsymbol{\theta}_2) &< k \cdot d_R \\ U(a_1, \boldsymbol{\theta}_1) - U(a_2, \boldsymbol{\theta}_2) &= k \cdot d_R \\ U(a_1, \boldsymbol{\theta}_1) - U(a_2, \boldsymbol{\theta}_2) &> k \cdot d_R \end{aligned}$$

where k is any constant > 0 that can take care of that utility and payoff may be given on different scales.

Two axioms of utility:

1. If $c_1 \prec c_2$ then $U(c_1) < U(c_2)$ and if $c_1 \sim c_2$ then $U(c_1) = U(c_2)$

2. If
   - O_1 = Obtaining consequence c_1 for certain
   - O_2 = Obtaining consequence c_2 with probability p and obtaining consequence c_3 with probability $1-p$
   - $O_1 \sim O_2$

   then $U(c_1) = p \cdot U(c_2) + (1-p) \cdot U(c_3)$

$\boxed{\text{“}p\text{-mixture”}}$

Hence, it is not necessary to work with preferences and their notations ($\prec$, $\sim$, $\precsim$).

All preferences can be expressed in terms of the utility function:

$$c_1 \prec c_2 \quad \Leftrightarrow \quad U(c_1) < U(c_2)$$
$$c_1 \sim c_2 \quad \Leftrightarrow \quad U(c_1) = U(c_2)$$
$$c_1 \precsim c_2 \quad \Leftrightarrow \quad U(c_1) \leq U(c_2)$$

Now, assume $U(a,\boldsymbol{\theta})$ is a utility function and let $W(a,\boldsymbol{\theta}) = c + d\cdot U(a,\boldsymbol{\theta})$ where c and d are constants with $d > 0$.

If $U(a_i,\boldsymbol{\theta}_k) < U(a_j,\boldsymbol{\theta}_l)$ [where $i \neq j$ or $k \neq l$ or both ; $c(a_i,\boldsymbol{\theta}_k) \prec c(a_j,\boldsymbol{\theta}_l)$]

$\Rightarrow$

$$W(a_i,\boldsymbol{\theta}_k) = c + d\cdot U(a_i,\boldsymbol{\theta}_k)$$
$$W(a_j,\boldsymbol{\theta}_l) = c + d\cdot U(a_j,\boldsymbol{\theta}_l)$$

$$\Rightarrow W(a_j,\boldsymbol{\theta}_l) - W(a_i,\boldsymbol{\theta}_k) = c + d\cdot U(a_j,\boldsymbol{\theta}_l) - (c + d\cdot U(a_i,\boldsymbol{\theta}_k)) = \underbrace{d}_{>0}\cdot\left(\underbrace{U(a_j,\boldsymbol{\theta}_l) - U(a_i,\boldsymbol{\theta}_k)}_{>0}\right) > 0$$

If $U(a_i,\boldsymbol{\theta}_k) = U(a_j,\boldsymbol{\theta}_l)$ [where $i \neq j$ or $k \neq l$ or both]

$$\Rightarrow W(a_j,\boldsymbol{\theta}_l) - W(a_i,\boldsymbol{\theta}_k) = c + d\cdot U(a_j,\boldsymbol{\theta}_l) - (c + d\cdot U(a_i,\boldsymbol{\theta}_k)) = \underbrace{d}_{>0}\cdot\left(\underbrace{U(a_j,\boldsymbol{\theta}_l) - U(a_i,\boldsymbol{\theta}_k)}_{=0}\right) = 0$$

If $U(a_i,\boldsymbol{\theta}_k) = p\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1}) + (1-p)\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2})$ [utilities for 3 different consequences]

$$\Rightarrow W(a_i,\boldsymbol{\theta}_k) = c + d\cdot U(a_i,\boldsymbol{\theta}_k) = c + d\cdot\left(p\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1}) + (1-p)\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2})\right) =$$
$$= c\cdot p + c\cdot(1-p) + d\cdot p\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1}) + d\cdot(1-p)\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2}) =$$
$$= p\cdot\left(c + d\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1})\right) + (1-p)\cdot\left(c + d\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2})\right) = p\cdot W(a_{j_1},\boldsymbol{\theta}_{l_1}) + (1-p)\cdot W(a_{j_2},\boldsymbol{\theta}_{l_{21}})$$

Now, assume $U(a,\boldsymbol{\theta})$ is a utility function and let $W(a,\boldsymbol{\theta}) = c + d\cdot U(a,\boldsymbol{\theta})$ where c and d are constants with $d > 0$.

If $U(a_i,\boldsymbol{\theta}_k) < U(a_j,\boldsymbol{\theta}_l)$ [where $i \neq j$ or $k \neq l$ or both ; $c(a_i,\boldsymbol{\theta}_k) \prec c(a_j,\boldsymbol{\theta}_l)$]

$\Rightarrow$

$$W(a_i,\boldsymbol{\theta}_k) = c + d\cdot U(a_i,\boldsymbol{\theta}_k)$$
$$W(a_j,\boldsymbol{\theta}_l) = c + d\cdot U(a_j,\boldsymbol{\theta}_l)$$

$\Rightarrow W(a_j,\boldsymbol{\theta}_l) - W$ … $U(a_i,\boldsymbol{\theta}_k)\Big) > 0$

A utility function is only unique up to a linear transformation

If $U(a_i,\boldsymbol{\theta}_k) = U(a_j,\boldsymbol{\theta}_l)$ [where $i \neq j$ or $k \neq l$ or both]

$$\Rightarrow W(a_j,\boldsymbol{\theta}_l) - W(a_i,\boldsymbol{\theta}_k) = c + d\cdot U(a_j,\boldsymbol{\theta}_l) - (c + d\cdot U(a_i,\boldsymbol{\theta}_k)) = \underbrace{d}_{>0}\cdot\left(\underbrace{U(a_j,\boldsymbol{\theta}_l) - U(a_i,\boldsymbol{\theta}_k)}_{=0}\right) = 0$$

If $U(a_i,\boldsymbol{\theta}_k) = p\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1}) + (1-p)\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2})$ [utilities for 3 different consequences]

$$\Rightarrow W(a_i,\boldsymbol{\theta}_k) = c + d\cdot U(a_i,\boldsymbol{\theta}_k) = c + d\cdot\left(p\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1}) + (1-p)\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2})\right) =$$
$$= c\cdot p + c\cdot(1-p) + d\cdot p\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1}) + d\cdot(1-p)\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2}) =$$
$$= p\cdot\left(c + d\cdot U(a_{j_1},\boldsymbol{\theta}_{l_1})\right) + (1-p)\cdot\left(c + d\cdot U(a_{j_2},\boldsymbol{\theta}_{l_2})\right) = p\cdot W(a_{j_1},\boldsymbol{\theta}_{l_1}) + (1-p)\cdot W(a_{j_2},\boldsymbol{\theta}_{l_{21}})$$

The *expected utility* of an action a with respect to a probability distribution of the states of the world is obtained – analogously to how expected payoff and expected loss are obtained – by integrating the utility function with the probability distribution of θ using its probability density (or mass) function $g(\theta)$:

$$EU = E_g\left(U\left(a,\tilde{\boldsymbol{\theta}}\right)\right) = \int_{\boldsymbol{\theta}} U\left(a,\boldsymbol{\theta}\right)\cdot g\left(\boldsymbol{\theta}\right)d\boldsymbol{\theta}$$

When data is not taken into account $g(\boldsymbol{\theta})$ is the prior pdf/pmf $f'(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$:

$$EU = \int_{\boldsymbol{\theta}} U\left(a,\boldsymbol{\theta}\right)\cdot f'\left(\boldsymbol{\theta}\right)d\boldsymbol{\theta} = \int_{\boldsymbol{\theta}} U\left(a,\boldsymbol{\theta}\right)\cdot p\left(\boldsymbol{\theta}\right)d\boldsymbol{\theta}$$

When data ($\boldsymbol{x}$) is taken into account, $g(\boldsymbol{\theta})$ is the posterior pdf/pmf $f''(\boldsymbol{\theta}|\,\boldsymbol{x}) = q(\boldsymbol{\theta}|\,\boldsymbol{x})$:

$$EU = \int_{\boldsymbol{\theta}} U\left(a,\boldsymbol{\theta}\right)\cdot f''\left(\boldsymbol{\theta}\,\middle|\,\boldsymbol{x}\right)d\boldsymbol{\theta} = \int_{\boldsymbol{\theta}} U\left(a,\boldsymbol{\theta}\right)\cdot q\left(\boldsymbol{\theta}\,\middle|\,\boldsymbol{x}\right)d\boldsymbol{\theta}$$

Assessing/finding a utility function

For a particular state of nature $\boldsymbol{\theta}$ let

c_1 be the worst consequence and c_2 be the best consequence

Normalise – without loss of generality – the utility function $U(c(a, \boldsymbol{\theta}))$ such that $U(c_1) = 0$ and $U(c_2) = 1$

For a particular action with consequence c it must always hold that $0 \leq U(c(a, \boldsymbol{\theta})) \leq 1$

Now, assume a gamble in which you should choose between

1. Obtaining consequence c for certain
2. Obtain consequence c_1 with probability $1-p$ and consequence c_2 with probability p

With the first choice the expected utility is $U(c(a, \boldsymbol{\theta}))$.
With the second choice the expected utility is

$$U(c_1)\cdot[1-p]+U(c_2)\cdot p = 0\cdot(1-p)+1\cdot p = p$$

1. Obtaining consequence c for certain
2. Obtain consequence c_1 with probability $1–p$ and consequence c_2 with probability p

For a certain value of p, p_0 say, you will be indifferent between 1 and 2

Hence $$U(a,\theta) = (1 - p_0)\cdot\underbrace{U(c_1)}_{=0} + p_0 \cdot\underbrace{U(c_2)}_{=1} = p_0$$

This means that $U(a, \theta)$ can be seen as proportional to the probability of obtaining the best consequence.

$$\Pr\left(\text{Best consequence} \mid a,\theta\right) \propto U\left(a,\theta\right)$$

$$\Rightarrow$$

$$\Pr\left(\text{Best consequence} \mid a\right) \propto \int_{\theta} U\left(a,\theta\right) \cdot g\left(\theta\right) d\theta = \overline{U\left(a,g\right)}$$

Hence, the optimal action is the action that ***maximises the expected utility*** under the probability distribution that rules the state of nature

$$a_g^{(\text{optimal})} = \arg\max_{a \in \mathcal{A}} \left(\overline{U\left(a,g\right)}\right)$$

$\mathcal{A}$ is the set of possible actions

$\Rightarrow$ The *Bayes action (decision)* is

$$a^{(B)} = \begin{cases} \arg\max_{a \in \mathcal{A}} \left(\overline{U\left(a,p\right)}\right) & \text{when no data are used} \\ \arg\max_{a \in \mathcal{A}} \left(\overline{U\left(a,q,\boldsymbol{x}\right)}\right) & \text{when data, } \boldsymbol{x} \text{ are used} \end{cases}$$

Example

Assume you are choosing between fixing the interest rate of your mortgage loan for one year or keeping the floating interest rate for this period.

Let us say that the floating rate for the moment is 4 % and the fixed rate is 5 %.

The floating rate may however increase during the period and we may approximately assume that with probability g_1 = 0.10 the average floating rate will be 7 %, with probability g_2 = 0.20 the average floating rate will be 6 % and with probability g_3 = 0.70 the floating rate will stay at 4 %.

Let d_1 = Fix the interest rate and d_2 = Keep the floating interest rate
Let θ = average floating rate for the coming period

$$U(d_1,\theta)=\begin{cases} 4-5=-1 & \theta=4 \\ 6-5=1 & \theta=6 \\ 7-5=2 & \theta=7 \end{cases} \qquad U(d_2,\theta)=\begin{cases} 5-4=1 & \theta=4 \\ 5-6=-1 & \theta=6 \\ 5-7=-2 & \theta=7 \end{cases}$$

$$\Rightarrow$$

$$\overline{U(d_1,g)}=(-1)\cdot 0.7+1\cdot 0.2+2\cdot 0.1=-0.3$$

$$\overline{U(d_2,g)}=1\cdot 0.7+(-1)\cdot 0.2+(-2)\cdot 0.1=0.3 \quad \Rightarrow d^{(B)}=d_2$$

Loss function

When utilities are all non-desirable it is common to describe the decision problem in terms of losses than utilities. The loss function in Bayesian decision theory is defines as

$$L_S(a,\theta) = \max_{a' \in \mathcal{A}} \left(U(a',\theta)\right) - U(a,\theta)$$

Then, the Bayes action with the use of data can be written

$$\begin{aligned}
a^{(B)} &= \arg\max_{a \in \mathcal{A}} \left(\int_\theta U(a,\theta) q(\theta \mid \boldsymbol{x}) d\theta \right) = \\
&= \arg\max_{a \in \mathcal{A}} \left(\int_\theta \left[\max_{a' \in \mathcal{A}} \left(U(a',\theta) \right) - L_S(a,\theta) \right] q(\theta \mid \boldsymbol{x}) d\theta \right) = \\
&= \arg\min_{a \in \mathcal{A}} \left(\int_\theta L_S(a,\theta) q(\theta \mid \boldsymbol{x}) d\theta \right) = \arg\min_{a \in \mathcal{A}} \overline{L_S(a, q, \boldsymbol{x})}
\end{aligned}$$

i.e. the action that minimises the expected posterior loss.

Example

A person asking for medical care has some symptoms that may be connected with two different diseases A and B. But the symptoms could also be temporary and disappear within reasonable time.

For A there is a therapy treatment that cures the disease if it is present and hence removes the symptoms. If however the disease is not present the treatment will lead to that the symptoms remain with the same intense.

For B there is a therapy treatment that generally "reduces" the intensity of the symptoms by 10 % regardless of whether B is present or not. If B is present the reduction is 40 %.

Assume that A is present with probability 0.3, that B is present with probability 0.4. Assume further that A and B cannot be present at the same time and therefore that the probability of the symptoms being just temporary is 0.3.

What is the Bayes action in this case: Treatment for A, treatment for B or no treatment?

$\Pr(A) = 0.3$

$\Pr(B) = 0.4$

$\Pr(A \cap B) = 0 \Rightarrow \Pr(\overline{A \cup B}) = 0.3$

Use normalised utilities:

U(Decision, State of nature) = 0 $\Leftrightarrow$ Symptoms remain with same intense

U(Decision, State of nature) = 1 $\Leftrightarrow$ Symptoms disappear

Treatment for A (T_A): $U(T_A, A) = 1$

$U(T_A, B) = 0$

$U(T_A, \overline{A \cup B}) = 0$

Treatment for B (T_B): $U(T_B, A) = 0.1$

$U(T_B, B) = 0.4$

$U(T_B, \overline{A \cup B}) = 1$

No treatment (NT): $U(NT, A) = 0$

$U(NT, B) = 0$

$U(NT, \overline{A \cup B}) = 1$

$\Rightarrow$

$\overline{U(T_A)} = 1 \cdot 0.3 + 0 \cdot 0.4 + 0 \cdot 0.3 = 0.30$

$\overline{U(T_B)} = 0.1 \cdot 0.3 + 0.4 \cdot 0.4 + 1 \cdot 0.3 = 0.49$

$\overline{U(NT)} = 0 \cdot 0.3 + 0 \cdot 0.4 + 1 \cdot 0.3 = 0.30$

The Bayes' action is therapy treatment for B