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TENTAMEN I MATEMATISK STATISTIK
Multivariate Methods 1MS003
2015 VIII 21

Maximum sum for exam: 20points.
Final grade calculated from exam, assignments and presentation.
Permitted aids: One double sided A4 sheet with handwritten notes, pocket calculator.
I will visit the exam room about 1 hour after the start of the exam.

1. (5p) Let $\mathbb{R}^k \ni \vec{X} \sim \mathcal{N}(\mu\vec{1}_k, \mathbf{I}_k)$, where $\mathbf{I}_k$ is the identity $k \times k$ matrix and $\vec{1}_k$ is a vector of k ones. Consider the random variable $\vec{a}^T\vec{X}$ where, $\vec{a}^T = (a_1, \ldots, a_k)$ is a vector of known constants such that $\sum\limits_{i=1}^{k} a_i = \vec{a}^T\vec{1}_k = 0$. Show that the sample mean
$$\bar{X} = \sum_{i=1}^{k} \vec{X}_i$$
and $\vec{a}^T\vec{X}$ are independent of each other.

2. A very important in statistics situation is when k random variables are *equicorrelated.* This means that their correlation matrix $\mathbf{R} \in \mathbb{R}^k$ is of the form :
$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \ldots & \rho \\ \rho & 1 & \ldots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \ldots & 1 \end{bmatrix}$$
where $0 \leq \rho < 1$.
   a) (2p) Write $\mathbf{R}$ as a transformation of $\mathbf{I}_k$ and $\vec{1}_k$.
   b) (3p) Verify that
   $$\mathbf{R}^{-1} = \frac{1}{1-\rho}\left(\mathbf{I}_k - \frac{\rho}{(k-1)\rho}\vec{1}_k\vec{1}_k^T\right).$$
   c) (3p) Verify that $\vec{1}_k$ and $(1, -1, 0, \ldots, 0)$ are two eigenvectors. Calculate the corresponding eigenvalues. Find the remaining eigenvectors and eigenvalues.
   d) (2p) Provide an interpretation of the principal components, i.e. eigenvectors.

3. (2p) Is the following a valid distance function on the set of positive real numbers? Is it a metric? Justify your answers.
$$d(x, y) = |(x-y)^{23}((x-y)^2 - 9)^8((x-y)^{18} - 56)^{100}|$$

4. (3p) Formulate the Central Limit Theorem and explain why it is important for statistical inference and applications.

Good luck!