SEMINAR EXERCISES IN PROBABILITY THEORY 732A63

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(EXERCISES BY PER SIDÉN)
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5. Multivariate Normal Distribution

Exercise 5.1 (5.4 in Gut's book)
The random vector $(X, Y)'$ has a two-dimensional normal distribution with $Var\,(X) = Var\,(Y)$. Show that $X + Y$ and $X - Y$ are independent random variables.

Exercise 5.2 (5.12 in Gut's book)
Let X_1 and X_2 be independent, $N\,(0, 1)$-distributed random variables. Set $Y_1 = X_1 - 3X_2 + 2$ and $Y_2 = 2X_1 - X_2 - 1$. Determine the distribution of

(a) $\mathbf{Y}$, and

(b) $Y_1 | Y_2 = y$.

Exercise 5.3 (5.18 in Gut's book)
The random vector $\mathbf{X}$ has a three-dimensional normal distribution with expectation $\mathbf{0}$ and covariance matrix $\mathbf{\Lambda}$ given by

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ -1 & 0 & 7 \end{pmatrix}.$$

Find the distribution of X_3 given that $X_1 = 1$.

Exercise 5.4 (5.20 in Gut's book)
The random vector $\mathbf{X}$ has a three-dimensional normal distribution with mean vector $\boldsymbol{\mu} = \mathbf{0}$ and covariance matrix

$$\mathbf{\Lambda} = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the distribution of $X_1 + X_3$ given that

(a) $X_2 = 0$,

(b) $X_2 = 2$.

Exercise 5.5* (5.33 in Gut's book)
Let X and Y be random variables, such that

$$Y|X = x \sim N\left(x, \tau^2\right) \text{ with } X \sim N\left(\mu, \sigma^2\right).$$

(a) Compute $E(Y)$, $Var(Y)$ and $Cov(X, Y)$.
(b) Determine the distribution of the vector $(X, Y)'$.
(c) Determine the (posterior) distribution of $X|Y = y$.

Exercise 5.6* (5.34 in Gut's book)
Let X and Y be jointly normal with means 0, variances 1, and correlation coefficient ρ. Compute the moment generating function of $X \cdot Y$ for

(a) $\rho = 0$, and
(b) general ρ.