Meeting 9:
More on the value of information…

The value of perfect information

Perfect information means that there is no uncertainty left for the decision maker. Hence the loss of the decision must be zero.

The *value of perfect information*, VPI is equal in value to the loss of taking an action, a. Since there is more than one action to be taken, VPI can vary with the action.

Assuming the decision maker takes the optimal action a^* with respect to expected loss the *expected value of perfect information*, EVPI is

$$\text{EVPI} = EL(a^*) - 0 = EL(a^*)$$

Expected loss of the optimal action.

The value of perfect information will for a specific action vary with the state of the world. For a fix state of the world, θ, the value of perfect information is

$$\text{VPI}(\theta) = R(a_\theta, \theta) - R(a^*, \theta)$$

a_θ is the optimal action when the state of the world is θ
a^* is the optimal action with respect to expected loss.

$$\Rightarrow$$

$$\text{EVPI} = \int \text{VPI}(\theta) \cdot f(\theta) d\theta \quad \text{or} \quad \sum \text{VPI}(\theta) \cdot P(\theta)$$

The value of sample information

Let

$\theta =$ State of the world

$f'(\theta) =$ Prior probability density (or mass) function of θ

$y =$ (Potential) sample result

$f(y|\theta) =$ Likelihood of θ (from sample result y)

$f''(\theta|y) =$ Posterior probability density (or mass) function of θ

$U(a) = U(a,\theta) =$ Utility of taking action a (with state of world θ)

$R(a) = R(a,\theta) =$ Payoff from taking action a (with state of world θ)

$L(a) = L(a,\theta) =$ (Opportunity) Loss of taking action a
(with state of world θ)

Then, the optimal action a' with respect to the prior distribution of θ satisfies

$$E'U(a') = \int U(a',\theta)\cdot f'(\theta)d\theta \geq \int U(a,\theta)\cdot f'(\theta)d\theta = E'U(a) \quad \forall a$$
$$E'U(a') \geq E'U(a) \quad \forall a$$

The expected utility with respect to the prior distribution of θ of the optimal action under that prior distribution is larger than or equal to the expected utility with respect to that prior distribution of any action.

and the optimal action a'' with respect to the posterior distribution of θ satisfies

$$E''U(a''|y) = \int U(a'',\theta|y)\cdot f''(\theta|y)d\theta \geq \int U(a,\theta|y)\cdot f''(\theta|y)d\theta = E''U(a|y) \quad \forall a$$
$$E''U(a''|y) \geq E''U(a|y) \quad \forall a$$

The expected utility with respect to the posterior distribution of θ of the optimal action under that posterior distribution is larger than or equal to the expected utility with respect to that posterior distribution of any action.

If we now assume that utility is linear in money it also holds that

$$E'R(a') \geq E'R(a) \quad \forall a$$
$$E''R(a''|y) \geq E''R(a|y) \quad \forall a$$

and that

$$E'L(a') \leq E'L(a) \quad \forall a$$
$$E''L(a''|y) \leq E''L(a|y) \quad \forall a$$

The *value of sample information* VSI(y) is then defined as

$$\text{VSI}(y) = E''R(a''|y) - E''R(a'|y) = E''L(a'|y) - E''L(a''|y)$$

The posterior expected payoff from taking the posterior optimal action minus the posterior expected payoff from taking the prior optimal action,

or the posterior expected loss from taking the prior optimal action minus the posterior expected loss from taking the posterior optimal action

The *expected* value of sample information

Using the definition of VSI(y):

$$\text{EVSI} = \int \text{VSI}(y) \cdot f(y)dy$$

However,

$$\int \text{VSI}(y) \cdot f(y)dy = \int \left(E''R(a''|y) - E''R(a'|y)\right) \cdot f(y)dy =$$

$$= \int E''R(a''|y) \cdot f(y)dy - \int E''R(a'|y) \cdot f(y)dy = \underline{\underline{E''R(a'') - E''R(a')}}$$

"Overall" posterior expected payoff from taking action a'' minus "overall" posterior expected payoff from taking action a' .

Note also that

$$\int E''R(a''|y) \cdot f(y)dy = \int \left(\int R(a'',\theta) \cdot f''(\theta|y)d\theta\right) \cdot f(y)dy = \int \int R(a'',\theta) \cdot f''(\theta|y) \cdot f(y)dy\,d\theta =$$

$$\int R(a'',\theta) \int f(y|\theta) \cdot f'(\theta)dy\,d\theta = \int R(a'',\theta) \cdot f'(\theta) \cdot \underbrace{\left(\int f(y|\theta)dy\right)}_{=1} d\theta = E'R(a'')$$

Analogously $\int E''R(a'|y) \cdot f(y)dy = \ldots = E'R(a')$

$$\Rightarrow \text{EVSI} = E'R(a'') - E'R(a')$$

Net gain of sampling

Since taking samples (obtain a sample result) comes with a cost the value of sample information is of interest when it is compared against the cost of sampling.

Note! This comparison requires that the utility is assumed linear in money

Net gain of sampling given sample result y:

$$\mathrm{NGS}(y) = \mathrm{VSI}(y) - \mathrm{CS}$$

where CS is the cost of sampling (not dependent on the sample result)

Expected net gain of sampling, ENGS:

$$\mathrm{ENGS} = \mathrm{EVSI} - \mathrm{CS}$$

…as function of the sample size n :

$$\mathrm{ENGS}(n) = \mathrm{EVSI}(n) - \mathrm{CS}(n)$$

Exercise 6.20

20. In the automobile-salesman example discussed in Section 3.4, suppose that the owner of the dealership must decide whether or not to hire a new salesman. The payoff table (in terms of dollars) is as follows.

		STATE OF THE WORLD		
		Great salesman	*Good salesman*	*Poor salesman*
ACTION	*Hire*	60,000	15,000	-30,000
	Do not hire	0	0	0

The prior probabilities for the three states of the world are $P(\text{great}) = 0.10$, $P(\text{good}) = 0.50$, and $P(\text{poor}) = 0.40$. The process of selling cars is assumed to behave according to a Poisson process with $\tilde{\lambda} = 1/2$ per day for a great salesman, $\tilde{\lambda} = 1/4$ per day for a good salesman, and $\tilde{\lambda} = 1/8$ per day for a poor salesman.

(a) Find VPI(great salesman), VPI(good salesman), and VPI(poor salesman).

(b) Find the expected value of perfect information.

(c) Suppose that the owner of the dealership can purchase sample information at the rate of \$10 per day by hiring the salesman on a temporary basis. He must sample in units of four days, however, for a salesman's work week consists of four days. He considers hiring the salesman for one week (four days). Find EVSI and ENGS for this proposed sample.

(a) Since we are given a payoff table we compute VPI using the result

$$\text{VPI}(\theta) = R(a_\theta, \theta) - R(a_*, \theta)$$

where a_θ is the optimal action when θ is the state of the world and a_* is the optimal action with respect to maximised expected payoff.

$\tilde{\theta}$ can here assume the states "GREAT", "GOOD" and "POOR". The (prior) expected payoffs are:

$$\begin{aligned} ER(a = \text{"Hire"}) &= 60000 \cdot P(\tilde{\theta} = \text{GREAT}) + 15000 \cdot P(\tilde{\theta} = \text{GOOD}) \\ &\quad - 30000 \cdot P(\tilde{\theta} = \text{POOR}) = \\ &= 60000 \cdot 0.10 + 15000 \cdot 0.50 - 30000 \cdot 0.40 = 1500 \\ ER(a = \text{"Do not hire"}) &= 0 \cdot 0.10 + 0 \cdot 0.50 - 0 \cdot 0.40 = 0 \end{aligned}$$

and hence a_* is "Hire".

If $\tilde{\theta}$ = GREAT or $\tilde{\theta}$ = GOOD action "Hire" is optimal, if $\tilde{\theta}$ = POOR the action "Do not hire" is optimal.

$$\Rightarrow$$

$$\mathrm{VPI}(\mathrm{GREAT}) = \mathrm{VPI}(\mathrm{GOOD}) = 1500 - 1500 = 0$$

$$\mathrm{VPI}(\mathrm{POOR}) = 0 - (-30000) = 30000$$

(b) The expected value of perfect information is

$$\begin{aligned} EVPI &= \mathrm{VPI}(\mathrm{GREAT}) \cdot P(\tilde{\theta} = \mathrm{GREAT}) + \mathrm{VPI}(\mathrm{GOOD}) \cdot P(\tilde{\theta} = \mathrm{GOOD}) \\ &\quad + \mathrm{VPI}(\mathrm{POOR}) \cdot P(\tilde{\theta} = \mathrm{POOR}) \\ &= 0 \cdot 0.10 + 0 \cdot 0.50 + 30000 \cdot 0.40 = 12000 \end{aligned}$$

(c) Let $\tilde{y}$ = Number of automobiles sold during one week. Then

$$P(\tilde{y} = y | \tilde{\theta} = \mathrm{GREAT}) = \frac{(4 \cdot (1/2))^y}{y!} \cdot e^{-4 \cdot (1/2)} = \frac{2^y}{y!} \cdot e^{-2} \quad ; P(\tilde{y} = y | \tilde{\theta} = \mathrm{GOOD}) = \frac{(4 \cdot (1/4))^y}{y!} \cdot e^{-4 \cdot (1/4)} = \frac{1^y}{y!} \cdot e^{-1}$$

$$P(\tilde{y} = y | \tilde{\theta} = \mathrm{POOR}) = \frac{(4 \cdot (1/8))^y}{y!} \cdot e^{-4 \cdot (1/8)} = \frac{0.5^y}{y!} \cdot e^{-0.5}$$

Now,

$$EVSI = \sum_y \mathrm{VSI}(y) \cdot P(y)$$

$$ENGS = EVSI - CS$$

where $\quad \mathrm{VSI}(y) = E''\left(R\left(a''\middle|y\right)\right) - E''\left(R\left(a'\middle|y\right)\right)$

In subtask (a) we obtained a_* =“Hire” and a' is thus equal to “Hire”.

The expected posterior payoffs are

$$\begin{aligned}
E''\left(R\left(\text{"Hire"}\middle|y\right)\right) = {} & R\left(\text{"Hire"}, \tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\middle|\tilde{y} = y\right) \\
& + R\left(\text{"Hire"}, \tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\middle|\tilde{y} = y\right) \\
& + R\left(\text{"Hire"}, \tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\middle|\tilde{y} = y\right) \\
= {} & 60000 \cdot P\left(\tilde{\theta} = \text{GREAT}\middle|\tilde{y} = y\right) + 15000 \cdot P\left(\tilde{\theta} = \text{GOOD}\middle|\tilde{y} = y\right) \\
& - 12000 \cdot P\left(\tilde{\theta} = \text{POOR}\middle|\tilde{y} = y\right)
\end{aligned}$$

$$\begin{aligned}
E''\left(R\left(\text{"Do not hire"}\middle|y\right)\right) = \ldots = {} & 0 \cdot P\left(\tilde{\theta} = \text{GREAT}\middle|\tilde{y} = y\right) + 0 \cdot P\left(\tilde{\theta} = \text{GOOD}\middle|\tilde{y} = y\right) \\
& - 0 \cdot P\left(\tilde{\theta} = \text{POOR}\middle|\tilde{y} = y\right) = 0
\end{aligned}$$

Now,

$$P\left(\tilde{\theta} = \text{GREAT} \middle| \tilde{y} = y\right) = \frac{P\left(Y = y \middle| \tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\right)}{P\left(\tilde{y} = y\right)}$$

$$P\left(\tilde{\theta} = \text{GOOD} \middle| \tilde{y} = y\right) = \frac{P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\right)}{P\left(\tilde{y} = y\right)}$$

$$P\left(\tilde{\theta} = \text{POOR} \middle| \tilde{y} = y\right) = \frac{P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\right)}{P\left(\tilde{y} = y\right)}$$

$$\text{where } P\left(\tilde{y} = y\right) = P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GREAT}\right) \cdot P\left(\tilde{\theta} = \text{GREAT}\right) + P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{GOOD}\right) \cdot P\left(\tilde{\theta} = \text{GOOD}\right)$$
$$+ P\left(\tilde{y} = y \middle| \tilde{\theta} = \text{POOR}\right) \cdot P\left(\tilde{\theta} = \text{POOR}\right)$$

This gives

$$E''\left(R\left(\text{"Hire"} \middle| y\right)\right) =$$
$$= 60000 \cdot \frac{\left(2^y / y!\right) e^{-2} \cdot 0.10}{P\left(\tilde{y} = y\right)} + 15000 \cdot \frac{\left(1^y / y!\right) e^{-1} \cdot 0.50}{P\left(\tilde{y} = y\right)} - 30000 \cdot \frac{\left(0.5^y / y!\right) e^{-0.5} \cdot 0.40}{P\left(\tilde{y} = y\right)}$$
$$= \frac{6000 \cdot \left(2^y / y!\right) e^{-2} + 7500 \cdot \left(1^y / y!\right) e^{-1} - 12000 \cdot \left(0.5^y / y!\right) e^{-0.5}}{P\left(\tilde{y} = y\right)}$$

We now must investigate for which values of y the optimal action a" is equal to "Hire" and for which y this action is equal to "Do not hire".

The optimal action is "Hire" when $E''\left(R\left(\text{"Hire"}\middle| y\right)\right) > E''\left(R\left(\text{"Do not hire"}\middle| y\right)\right) = 0$

$$\Leftrightarrow$$

$$\frac{6000 \cdot \left(2^y / y!\right) e^{-2} + 7500 \cdot \left(1^y / y!\right) e^{-1} - 12000 \cdot \left(0.5^y / y!\right) e^{-0.5}}{P(\tilde{y} = y)} > 0 \quad \Leftrightarrow \left\langle \text{since } P(\tilde{y} = y) > 0 \right\rangle$$

$$6000 \cdot \left(2^y / y!\right) e^{-2} + 7500 \cdot \left(1^y / y!\right) e^{-1} - 12000 \cdot \left(0.5^y / y!\right) e^{-0.5} > 0 \quad \Leftrightarrow \left\langle \text{since } y! > 0 \right\rangle$$

$$6000 \cdot 2^y \cdot e^{-2} + 7500 \cdot e^{-1} - 12000 \cdot 0.5^y \cdot e^{-0.5} > 0 \Leftrightarrow 6000 \cdot 2^y \cdot e^{-2} + 7500 \cdot e^{-1} - 12000 \cdot \frac{1}{2^y} \cdot e^{-0.5} > 0$$

$$\Leftrightarrow \left\langle \text{since } 2^y > 0 \right\rangle$$

$$6000 \cdot e^{-2} \cdot 2^{2y} + 7500 \cdot e^{-1} \cdot 2^y - 12000 \cdot e^{-0.5} > 0 \Leftrightarrow 2^{2y} + \frac{6000}{7500} e^{1} \cdot 2^y - \frac{12000}{7500} e^{1.5} > 0$$

$$\Rightarrow \left\langle \text{since } 2^y > 0 \right\rangle$$

$$2^y > -\frac{6000}{2 \cdot 7500} e^{1} + \sqrt{\left(-\frac{6000}{2 \cdot 7500} e^{1}\right)^2 + \frac{12000}{7500} e^{1.5}}$$

$$\Leftrightarrow 2^y > 1.802835 \Leftrightarrow y > \frac{\log(1.802835)}{\log(2)} \approx 0.85$$

Hence,

$$E''\left(R\left(a''\middle|y\right)\right)=\begin{cases} E''\left(R\left(\text{"Hire"}\middle|y\right)\right) & y\ge 1 \\ E''\left(R\left(\text{"Do not hire"}\middle|y\right)\right) & y=0 \end{cases}$$

and

$$\begin{aligned}\mathrm{VSI}(y) &= E''\left(R\left(a''\middle|y\right)\right)-E''\left(R\left(a'\middle|y\right)\right) \\ &= \begin{cases} E''\left(R\left(\text{"Hire"}\middle|y\right)\right)-E''\left(R\left(\text{"Hire"}\middle|y\right)\right)=0 & y\ge 1 \\ E''\left(R\left(\text{"Do not hire"}\middle|y=0\right)\right)-E''\left(R\left(\text{"Hire"}\middle|y=0\right)\right) & y=0 \end{cases}\end{aligned}$$

$$E''\left(R\left(\text{"Do not hire"}\middle|y=0\right)\right)=0$$

$$\begin{aligned}E''\left(R\left(\text{"Hire"}\middle|y=0\right)\right) &= \frac{6000\cdot\left(2^0/0!\right)e^{-2}+7500\cdot\left(1^0/0!\right)e^{-1}-12000\cdot\left(0.5^0/0!\right)e^{-0.5}}{P\left(\tilde{y}=0\right)} \\ &= \frac{6000\,e^{-2}+7500\,e^{-1}-12000\,e^{-0.5}}{P\left(\tilde{y}=0\right)}\end{aligned}$$

and thus

$$\mathrm{VSI}(y)=\begin{cases} 0 & y\ge 1 \\ -\dfrac{6000\,e^{-2}+7500\,e^{-1}-12000\,e^{-0.5}}{P\left(Y=0\right)} & y=0 \end{cases}$$

…and finally,

$$EVSI = \sum_{y=0}^{\infty} \mathrm{VSI}(y) \cdot P(\tilde{y} = y) = \mathrm{VSI}(0) \cdot P(\tilde{y} = 0) + \sum_{y=1}^{\infty} 0 \cdot P(\tilde{y} = y) = \mathrm{VSI}(0) \cdot P(\tilde{y} = 0)$$

$$= -\frac{6000\,e^{-2} + 7500\,e^{-1} - 12000\,e^{-0.5}}{\cancel{P(\tilde{y} = 0)}} \times \cancel{P(\tilde{y} = 0)} \approx 3707$$

$$ENGS = EVSI - CS = 3707 - 4 \cdot 10 = 3667$$