

Maximum sum for exam: 20points.

Final grade calculated from exam, assignments and presentation.

Permitted aids: One double sided A4 sheet with handwritten notes, pocket calculator.

I will be available on the phone **KTORE GODZINY?**

1. (5p) Let $\mathbb{R}^k \ni \vec{X} \sim \mathcal{N}(\mu \vec{1}_k, \mathbf{I}_k)$, where \mathbf{I}_k is the identity $k \times k$ matrix and $\vec{1}_k$ is a vector of k ones. Consider the random variable $\vec{a}^T \vec{X}$ where $\vec{a}^T = (a_1, \dots, a_k)$ is a vector of known constants such that $\sum_{i=1}^k a_i = \vec{a}^T \vec{1}_k = 0$. Show that the sample mean

$$\bar{X} = \sum_{i=1}^k \vec{X}_i$$

and $\vec{a}^T \vec{X}$ are independent of each other.

2. A very important in statistics situation is when k random variables are *equicorrelated*. This means that their correlation matrix $\mathbf{R} \in \mathbb{R}^k$ is of the form :

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \dots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \dots & 1 \end{bmatrix}$$

where $0 \leq \rho < 1$.

- a) (2p) Write \mathbf{R} as a transformation of \mathbf{I}_k and $\vec{1}_k$.

- b) (4p) Verify that

$$\mathbf{R}^{-1} = \frac{1}{1-\rho} \left(\mathbf{I}_k - \frac{\rho}{(k-1)\rho} \vec{1}_k \vec{1}_k^T \right).$$

- c) (3p) Verify that $\vec{1}_k$ and $(1, -1, 0, \dots, 0)$ are two eigenvectors. Calculate the corresponding eigenvalues. Find the remaining eigenvectors and eigenvalues.

- d) (2p) Provide and interpretation of the principal components, i.e. eigenvectors.

3. (2p) Is the following valid distance functions on the set of positive real numbers? Is it a metric? Justify your answers.

$$d(x, y) = |(x - y)^{23}((x - y)^2 - 9)^8((x - y)^{18} - 56)^{100}|$$

4. (2p) Formulate the Central Limit Theorem and explain why it is important for statistical inference and applications.

Good luck!