Meeting 8:
More of utility theory…

P-mixtures of utilities

A p-mixture of two utilities U_1 and U_2 (or two payoffs or two losses) is a bet in which you obtain U_1 with probability p and U_2 with probability $1–p$.

This is essentially the utility of lottery II in the decision problem of choosing between

Lottery I: Obtaining U_0 with certainty

Lottery II: Obtaining U_1 with probability p and U_2 with probability $1–p$

Example

Assume that exchanging 1000 SEK into US$ will today give you 112 US$. Tomorrow you will receive 110 US$ with probability 0.3 and 113 US$ with probability 0.7. Then, the utility (payoff) of exchanging tomorrow is a 0.3-mixture of 110 US$ and 113 US$.

Axioms of coherence for a utility function (von Neumann & Morgenstern, 1947, Theory of Games and Economic Behaviour, 2nd ed., Princeton University Press)

1. Ordering of consequences: It is possible for the decision-maker to order the possible outcomes from best to worst (or to explicitly state their indifference between two or several of them)
2. Transitivity of preferences: If the relative preferences of three possible outcomes, expressed as utilities U_1, U_2 and U_3, are such that $U_2 > U_1$ and $U_3 > U_2$, then U_3 must be greater than U_1, i.e. $U_3 > U_1$
3. Continuity of preferences: If $U_3 > U_2 > U_1$ then it is possible to find a *p-mixture* of U_1 and U_3 that is preferable to (>) U_2 and another p-mixture of U_1 and U_3 such that U_2 is preferred to (>) that p-mixture
4. Independence: If $U_2 > U_1$ then for any another utility U_3 it holds that a p-mixture of U_2 and U_3 is preferred to the "same" p-mixture of U_1 and U_3, i.e. $pU_2 + (1-p)U_3 > . pU_1 + (1-p)U_3$

Winkler gives two more axioms that would actually follow from the axioms above.

Exercise 5.36

For each of the following utility functions for changes in assets (monetary payoffs), graph the function and comment on the attitude toward risk that is implied by the function. All of the functions are defined for $-1000 < R < 1000$.

(a) $U(R) = (R + 1000)^2$.

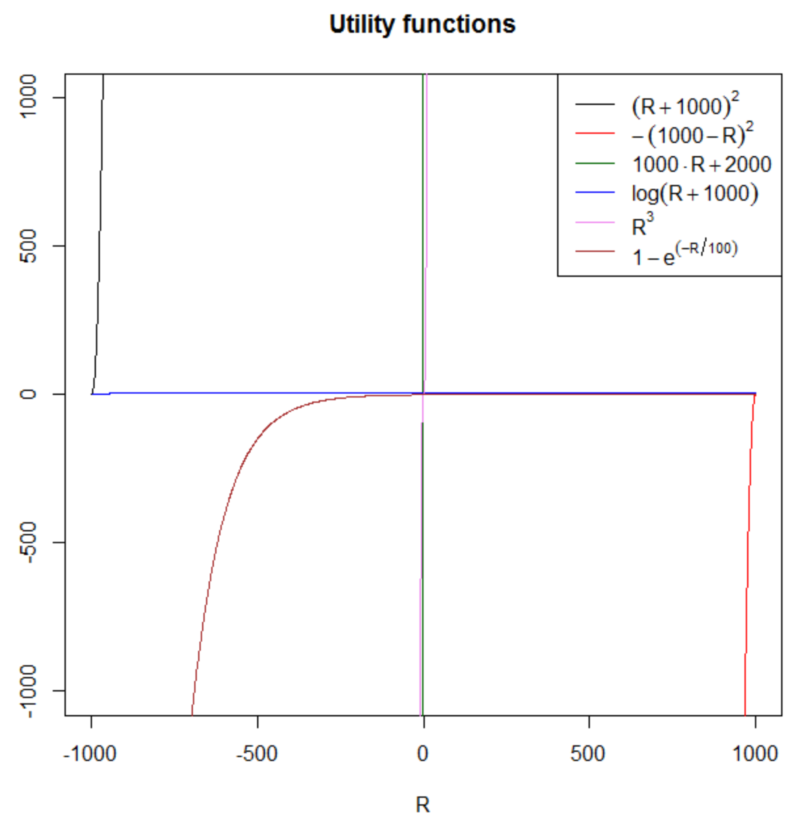
(b) $U(R) = -(1000 - R)^2$.

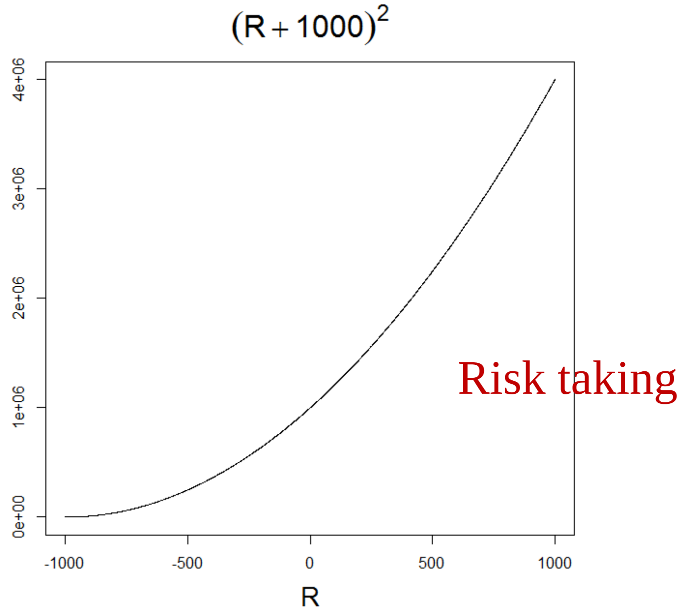
(c) $U(R) = 1000R + 2000$.

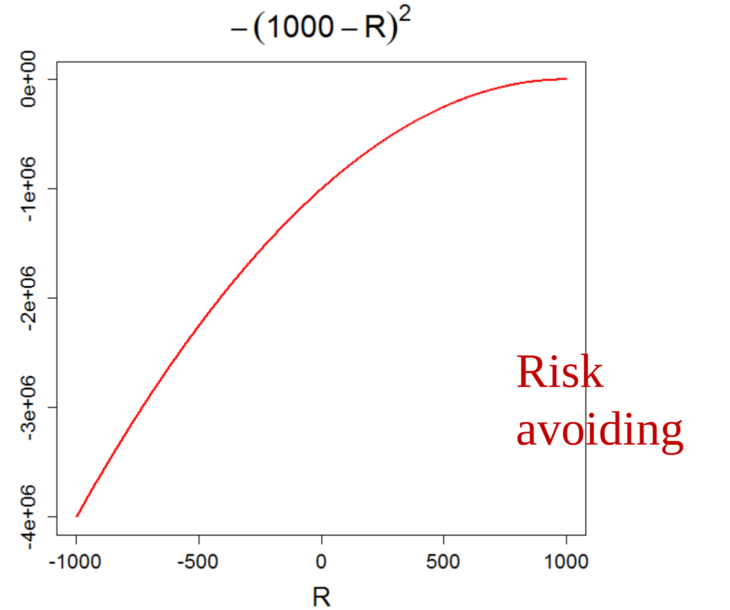
(d) $U(R) = \log(R + 1000)$.

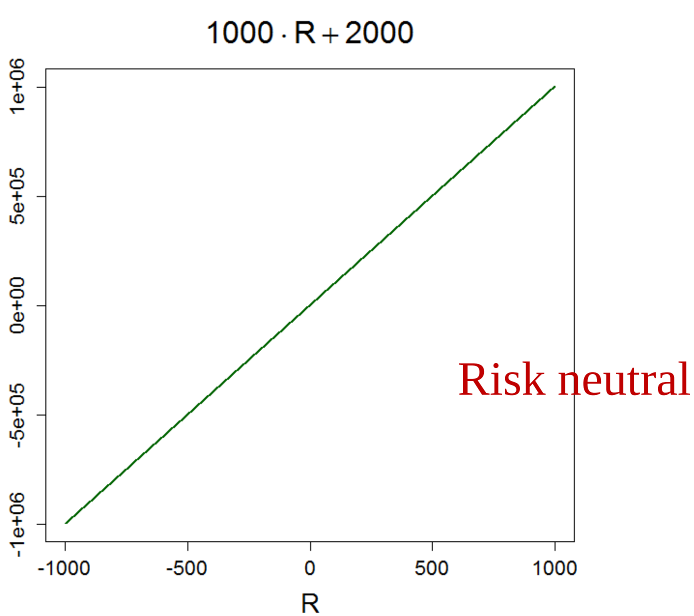
(e) $U(R) = R^3$.

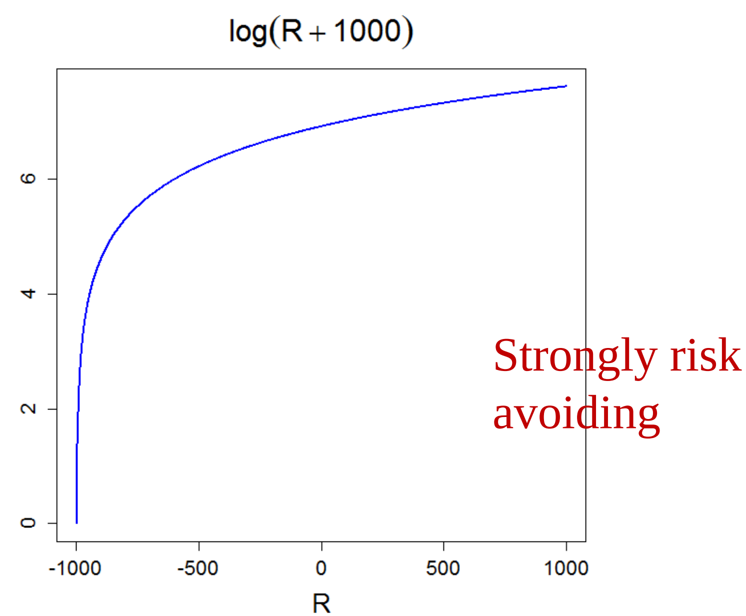
(f) $U(R) = 1 - e^{-R/100}$.

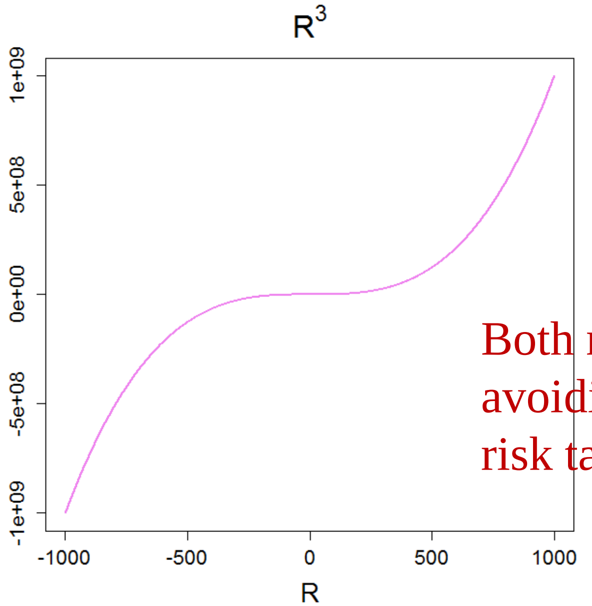



(R + 1000)^2
0e+00
1e+06
2e+06
3e+06
4e+06
-1000
-500
0
500
1000
R
Risk taking

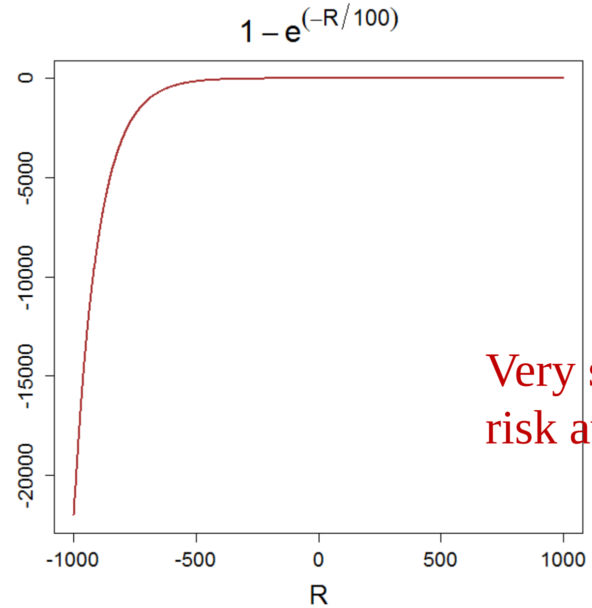

– (1000 – R)^2
0e+00
-1e+06
-2e+06
-3e+06
-4e+06
-1000
-500
0
500
1000
R
Risk
avoiding


1000 · R + 2000
1e+06
5e+05
0e+00
-5e+05
-1e+06
-1000
-500
0
500
1000
R
Risk neutral


log(R + 1000)
6
4
2
0
-1000
-500
0
500
1000
R
Strongly risk
avoiding



Both risk avoiding and risk taking



Very strongly risk avoiding

Exercise 5.37

For each of the utility functions in Exercise 36, find out if the decision maker should take a bet in which he will win \$100 with probability p and lose \$50 with probability $1-p$,

(a) if $p = 1/2$,

(b) if $p = 1/3$,

(c) if $p = 1/4$.

$$EU = U(100)\cdot p + U(-50)\cdot(1-p)$$

a)

$$EU = U(100)\cdot 0.5 + U(-50)\cdot 0.5$$

$$ER = 100\cdot 0.5 + (-50)\cdot 0.5 = 25$$

$$U(R) = (R+1000)^2 \Rightarrow EU = 1100^2\cdot 0.5 + 950^2\cdot 0.5 = 1056250$$

$$CE = U^{-1}(EU) = \sqrt{EU} - 1000 = \sqrt{1056250} - 1000 = 27.74 > 25 = ER$$

$$\Rightarrow \text{Take the bet}$$

$$U(R) = -(1000-R)^2 \Rightarrow EU = (-900)^2\cdot 0.5 + (-1050^2)\cdot 0.5 = -956250$$

$$CE = U^{-1}(EU) = 1000 - \sqrt{-EU} = 1000 - \sqrt{956250} = 22.12 < 25 = ER$$

$$\Rightarrow \text{Do not take the bet}$$

$$U(R) = 1000 \cdot R + 2000 \Rightarrow EU = 102000 \cdot 0.5 + (-48000) \cdot 0.5 = 27000$$

$$CE = U^{-1}(EU) = (EU - 2000)/1000 = 25 = ER$$

$\Rightarrow$ Indifferent

$$U(R) = \log(R + 1000) \Rightarrow EU = \log(1100) \cdot 0.5 + \log(950) \cdot 0.5 = 6.93$$

$$CE = U^{-1}(EU) = \exp(EU) - 1000 = \exp(\log(1100) \cdot 0.5 + \log(950) \cdot 0.5) - 1000 =$$

$$= 1100^{0.5} + 950^{0.5} - 1000 = 22.25 < 25 = ER$$

$\Rightarrow$ Do not take the bet

$$U(R) = R^3 \Rightarrow EU = 100^3 \cdot 0.5 + (-50)^3 \cdot 0.5 = 437500$$

$$CE = U^{-1}(EU) = EU^{1/3} = 437500^{1/3} = 75.91 > 25 = ER$$

$\Rightarrow$ Take the bet

$$U(R) = 1 - \exp(-R/100) \Rightarrow EU = (1 - \exp(-1)) \cdot 0.5 + (1 - \exp(0.5)) \cdot 0.5 = -0{,}0083$$

$$CE = U^{-1}(EU) = -100 \cdot \log(1 - EU) =$$

$$-100 \cdot \log(1 - ((1 - \exp(-1)) \cdot 0.5 + (1 - \exp(0.5)) \cdot 0.5)) == -0.83 < 25 = ER$$

$\Rightarrow$ Do not take the bet

Exercise 45

For each of the utility functions in Exercise 36, find the risk premiums for the following gambles.

(a) You win \$100 with probability 0.5 and you lose \$100 with probability 0.5.
(b) You win \$100 with probability 0.4 and you lose \$50 with probability 0.6.
(c) You win \$70 with probability 0.3 and you lose \$30 with probability 0.7.
(d) You win \$200 with probability 0.5 and you win \$50 with probability 0.5.

a)
$$EU = U(100)\cdot 0.5 + U(-100)\cdot 0.5$$
$$ER = 100\cdot 0.5 + (-100)\cdot 0.5 = 0$$
$$RP = ER - CE = -CE$$

$$U(R) = (R+1000)^2 \Rightarrow EU = 1100^2\cdot 0.5 + 900^2\cdot 0.5 = 1010000$$
$$RP = -CE = -U^{-1}(EU) = -\left(\sqrt{EU} - 1000\right) = -\sqrt{1010000} + 1000 \approx -5$$

$$U(R) = -(1000-R)^2 \Rightarrow EU = (-900)^2\cdot 0.5 + \left(-1100^2\right)\cdot 0.5 = -1010000$$
$$RP = -CE = -U^{-1}(EU) = -\left(1000 - \sqrt{-EU}\right) = -1000 + \sqrt{1010000} \approx 5$$

The value of perfect information

Perfect information means that there is no uncertainty left for the decision maker. Hence the loss of the decision must be zero.

The *value of perfect information*, VPI is what this information is worth to the decision maker. This value would be equal in value to the loss of taking an action, a. Since there is more than one action to be taken, VPI can vary with the action.

If we now assume that the decision maker would take the optimal action a^* with respect to expected (opportunity) loss the *expected value of perfect information*, EVPI is

$$\text{EVPI} = EL(a^*) - 0 = EL(a^*)$$

Hence, the expected value of perfect information is the expected loss of the optimal action.

Alternatively, we can reason with the payoff instead of the loss.

The value of perfect information will for a specific action vary with the state of the world. For a fix state of the world, θ, the value of perfect information is

$$\mathrm{VPI}(\theta) = R(a_\theta, \theta) - R(a^*, \theta)$$

where a_θ is the optimal action when the state of the world is θ while a^* is the optimal action with respect to expected loss.

Then,

$$\mathrm{EVPI} = \int \mathrm{VPI}(\theta) \cdot f(\theta) d\theta \quad \text{or} \quad \sum \mathrm{VPI}(\theta) \cdot P(\theta)$$

Exercise 6.5

5. In Exercise 22, Chapter 5, give a general expression for the expected value of perfect information regarding the weather and find the EVPI if
   (a) P(adverse weather) = 0.4, C = 3.5, and L = 10,
   (b) P(adverse weather) = 0.3, C = 3.5, and L = 10,
   (c) P(adverse weather) = 0.4, C = 10, and L = 10,
   (d) P(adverse weather) = 0.3, C = 2, and L = 8.

22. A special type of decision-making problem frequently encountered in meteorology is called the "cost-loss" decision problem. The states of the world are "adverse weather" and "no adverse weather," and the actions are "protect against adverse weather" and "do not protect against adverse weather." C represents the cost of protecting against adverse weather, while L represents the cost that is incurred if you fail to protect against adverse weather and it turns out that the adverse weather occurs. (L is usually referred to as a "loss," but it is not a loss in an opportunity-loss sense.)
    (a) Construct a payoff table and a decision tree for this decision-making problem.
    (b) For what values of P(adverse weather) should you protect against adverse weather?
    (c) Given the result in (b), is it necessary to know the absolute *magnitudes* of C and L?

Payoff table

Action	State of the world	
	adverse weather	no adverse weather
protect	$-C$	$-C$
do not protect	$-L$	0

$$\mathrm{VPI}(\text{adverse weather}) =$$

$$= \begin{cases} R(\text{protect}, \text{adverse weather}) - R(a^*, \text{adverse weather}) & \text{if } C < L \\ R(\text{do not protect}, \text{adverse weather}) - R(a^*, \text{adverse weather}) & \text{if } C \geq L \end{cases}$$

$$\mathrm{VPI}(\text{no adverse weather}) =$$

$$= R(\text{do not protect}, \text{no adverse weather}) - R(a^*, \text{no adverse weather})$$

$$C < L \Rightarrow$$

$$\left.\begin{aligned} &\mathrm{VPI}(\text{adverse weather}) = -C - (-C) = 0 \\ &\mathrm{VPI}(\text{no adverse weather}) = 0 - 0 = 0 \end{aligned}\right\} \Rightarrow EVPI = 0$$

$$C \geq L \Rightarrow$$

$$\left.\begin{aligned} &\mathrm{VPI}(\text{adverse weather}) = -C - (-L) = L - C \\ &\mathrm{VPI}(\text{no adverse weather}) = 0 - 0 = 0 \end{aligned}\right\} \Rightarrow EVPI = (L - C) \cdot P(\text{adverse weather})$$

(a)

$$EVPI = (L - C) \cdot P(\text{adverse weather}) = (10 - 3.5) \cdot 0.4 = 2.6$$

(b)

$$EVPI = (L - C) \cdot P(\text{adverse weather}) = (10 - 3.5) \cdot 0.3 = 1.95$$

(c)

$$EVPI = (L - C) \cdot P(\text{adverse weather}) = (10 - 10) \cdot 0.4 = 0$$

(d)

$$EVPI = (L - C) \cdot P(\text{adverse weather}) = (8 - 2) \cdot 0.3 = 1.8$$

Winkler's answer seems to be wrong here!

Exercise 6.8

8. In decision-making problems for which the uncertain quantity of primary interest can be viewed as a future sample outcome $\tilde{y}$, the relevant distribution of interest to the decision maker is the predictive distribution of $\tilde{y}$.
   (a) Given a prior distribution $f(\theta)$ and a likelihood function $f(y|\theta)$, how would you find the expected value of perfect information about $\tilde{y}$?
   (b) Given a prior distribution $f(\theta)$ and a likelihood function $f(y|\theta)$, how would you find the expected value of perfect information about $\tilde{\theta}$?
   (c) Explain the difference between your answers to (a) and (b).

(a)

Prior predictive distribution:

$$f(y) = \int f(y|\theta) \cdot f(\theta) d\theta$$

$$\text{EVPI} = EL_{\tilde{y}}(a^*) = \int L(a^*, y) \cdot f(y)\, dy =$$

$$= \int_y L(a^*, y) \cdot \left(\int_\theta f(y|\theta) \cdot f(\theta) d\theta \right) dy =$$

$$= \int_y \int_\theta L(a^*, y) \cdot f(y|\theta) \cdot f(\theta) d\theta\, dy = \int_\theta \int_y L(a^*, y) \cdot f(y|\theta) \cdot f(\theta) dy\, d\theta$$

(b)

$$\text{EVPI} = EL_{\theta|y}\left(a^*\right) = \int L\left(a^*, \theta\right) \cdot f\left(\theta|y\right) d\theta =$$

$$= \int_y L\left(a^*, \theta\right) \cdot \left(\frac{f\left(y|\theta\right) \cdot f\left(\theta\right)}{\int_\lambda f\left(y|\theta\right) \cdot f\left(\lambda\right) d\lambda} \right) d\theta =$$

$$= \int_\theta L\left(a^*, \theta\right) \cdot \left(\frac{f\left(y|\theta\right) \cdot f\left(\theta\right)}{f\left(y\right)} \right) d\theta =$$