DIFFERENTIAL EQUATIONS

A differential equation is a mathematical equation that relates functions and their derivatives.

For example; $\frac{dy}{dx}=5$, $\frac{d^2y}{dx^2}-5\frac{dy}{dx}=6y$, $\left(\frac{d^2y}{dx^2}\right)^3-\frac{dy}{dx}=0$ etc

Differential equations fall into two categories, which are; **Partial differential equations** and **Ordinary differential equations.** In ordinary differential equations there is only one independent variable while in partial differential equations there is more than one independent variable. This course will focus on the ordinary differential equations

Order of differential equations

The order of differential equation is the highest derivative present in the differential equation.

For example; $\frac{dy}{dx}-5y=0$, $\left(\frac{dy}{dx}\right)^5+2y=0$ are first order differential equations while $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+3y=0$ and $\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^3=y$ are second order differential equations.

Linear and non - linear differential equation

Linear differential equation

A differential equation is said to be linear if the dependant variable and all its derivatives in the equation occur only in first degree and are not multiplied.

Non – linear differential equation

A differential equation is said to be non – linear if

a) Any of the differential coefficient has exponent more than one
b) Exponent of the dependant variable of more than one
c) Products containing dependant variable and its differential coefficient are present.

$\frac{dy}{dx}+y=e^x$, $\frac{d^2y}{dx^2}+4y=\sqrt{x}$, $\frac{d^3y}{dx^3}+y=\sin x^2$ and $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=x^3$ are the examples of linear differential equations while $\left(\frac{d^2y}{dx^2}\right)^3-7\left(\frac{dy}{dx}\right)-7y=0$, $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=e^x$,

$\left(\frac{d^3y}{dx^3}\right)^2 + y = \sin x$, $\frac{d^2y}{dx^2} + 4y^2 = x$ and $y\frac{dy}{dx} + x = 0$

Degree of differential equation

Degree of a differential equation is the highest power of the highest differential which the equation contains.

For example; $\left(\frac{d^2y}{dx^2}\right)^3 + 2\left(\frac{dy}{dx}\right)^5 = 7$ and $\left(\frac{dy}{dx}\right)^3 - y = 0$ are ordinary differential equations of degree 3 while $\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^4 + 2y = 0$ and $\frac{d^3y}{dx^3} - \left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^5 + y = 0$ are ordinary differential equations of degree 1.

Solution of differential equation

A solution to a differential equation is any function which satisfies the given differential equation.

A solution to a differential equation falls into to types; which are **general solutio**n and **particular solution.** A general solution is the one which contains one or more arbitrary constants while a particular solution is a solution in which additional information is given to a general solution to obtain the values of the constants. The additional information that gives the values of constants to the general solution is called **Boundary conditions.**

Examples

1. Show that $x = t^3$ is a solution to the differential equation $\frac{dx}{dt} = 3t^2$.

   Solution

   Replace x by t^3 in the differential equation, this gives $\frac{d(t^3)}{dx} = 3t^2 \Rightarrow 3t^2 = 3t^2$

   Hence shown.

2. Show that $y = t^2 - 3t + \frac{7}{2}$ is a solution to $y'' + 3y' + 2y = 2t^2$

   Solution

   Remember $y'' + 3y' + 2y = 2t^2$ is the same as $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3t^2$

Replacing y by $t^2-3t+\frac{7}{2}$ give $\frac{d^2(t^2-3t+\frac{7}{2})}{dx^2}+3\frac{d(t^2-3t+\frac{7}{2})}{dx}+2(t^2-3t+\frac{7}{2})=2t^2$

$\Rightarrow 2+3(2t-3)+2(t^2-3t+\frac{7}{2})=2t^2$

$\Rightarrow 2+6t-9+2t^2-6t+7=2t^2$

$\Rightarrow 2t^2=2t^2$

Shown

Exercise

1. Determine the order and degree of the following differential equations. State also , whether they are linear or non – linear

   (i) $\frac{d^2y}{dx^2}+4y=o$ (ii) $\left(\frac{dy}{dx}\right)^2+\frac{1}{\left(\frac{dy}{dx}\right)}=2$ (iii) $\sqrt[3]{\frac{d^2y}{dx^2}}=\sqrt{\frac{dy}{dx}}$

   (iv) $y\frac{d^2y}{dx^2}=y^2+1$ (v) $\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}+4y=\sin x$

   (vi) $9\frac{d^2y}{dx^2}=\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}$ (vii) $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=x\sin\left(\frac{d^2y}{dx^2}\right)$

2. Show that the function $y=e^x+1$ is a solution of the differential equation $\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$
3. Verify that the function $y=x\sin x$ is a solution of the diffferentila equation $xy'=y+x\sqrt{x^2-y^2}$, ($x\neq 0$ and $x>y$ or $x<-y$)

Formulation of differential equations

Let $f(x,y,c_1,c_2,c_3,...c_n)=0$ be a solution of a differential equation, where $c_1,c_2,c_3,...c_n$ are $n-$ arbitrary constants. If we eliminate all the $n-$ constants we obtain the differential equation of nth order. These constants are eliminated through differentiation of the given function; in the process we observe the following guidelines.

- If the given function has one arbitrary constant, we differentiate once and then eliminating the constant. In this case the resulting differential equation is a first order.
- If the given function has two arbitrary constants, we differentiate twice and then eliminating the constants. The resulting differential equation here is a second order,
- If the function has n – arbitrary constants, we differentiate the function n – times and then eliminating the constants. This gives the differential equation of nth order.

Examples

1. Find the differential equation of the family of curves given by $x^2 + y^2 = 2ax$.
   Solution
   Since it has only one arbitrary constant, we differentiate once and eliminating the constants.
   $2x + 2y\frac{dy}{dx} = 2a$, making a the subject we get $a = x + y\frac{dy}{dx}$
   Substituting in $x^2 + y^2 = 2ax$ gives $x^2 + y^2 = 2x(x + y\frac{dy}{dx}) \Rightarrow 2xy\frac{dy}{dx} + x^2 - y^2 = 0$
   $\therefore 2xy\frac{dy}{dx} + x^2 - y^2 = 0$ is the required differential equation
2. Form the differential equation representing the family of curves given by $(x-a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.
   Solution
   This function has only one constant, we differentiate once and eliminating the constant.
   $(x-a)^2 + 2y^2 = a^2 \Rightarrow 2(x-a) + 2.2y\frac{dy}{dx} = 0$, making a the subject we get $x + 2y\frac{dy}{dx} = a$
   substituting in $(x-a)^2 + 2y^2 = a^2$ gives $(x - (x + 2y\frac{dy}{dx}))^2 + 2y^2 = (x + 2y\frac{dy}{dx})^2$
   $\Rightarrow 4xy\frac{dy}{dx} = x^2 - 2y^2$
   $\therefore 4xy\frac{dy}{dx} = x^2 - 2y^2$ is the required differential equation.
3. Form the differential equation corresponding to $y^2 = a(b^2 - x^2)$, where a and b are arbitrary constants.
   Solution
   This function has two arbitrary constants, therefore we differentiate two times.
   Differentiating once and simplifying gives, $y\frac{dy}{dx} = -xa$ .... (i)
   Differentiating for the second time gives, $y\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = -a$
   $\Rightarrow y\frac{d^2y}{d^2x} + \left(\frac{dy}{dx}\right)^2 = -a$, substituting in (i) we get $y\frac{dy}{dx} = x\left(y\frac{d^2y}{d^2x} + \left(\frac{dy}{dx}\right)^2\right)$
   Simplifying the resulting equation gives, $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$, which is the required differential equation.
4. Form the differential equation of the family of curve $y = e^x(a\cos xa = b\sin x)$.
   Solution
   The function has two constants, we differentiate two times,

Differentiating for the first time we obtain, $\frac{dy}{dx} = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x)$,

but $y = e^x(a\cos xa = b\sin x)$ this means

$\frac{dy}{dx} = y + e^x(-a\sin x + b\cos x)$ which simplifies to $\frac{dy}{dx} - y = e^x(-a\sin x + b\cos x)$ ....(i)

Differentiating (i) gives, $\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-a\sin x + b\cos x) + e^x(-a\cos x - b\sin x)$ ....(ii)

$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-a\sin x + b\cos x) - e^x(a\cos x + b\sin x)$, but $y = e^x(a\cos xa = b\sin x)$ and

$\frac{dy}{dx} - y = e^x(-a\sin x + b\cos x)$, substituting in (ii) gives; $\frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - y$

This simplifies to, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ which is the required differential equation.

5. Form the differential equation of the following family of curves $xy = Ae^x + Be^{-x} + x^2$.

Solution

From $xy = Ae^x + Be^{-x} + x^2$ .... (i)

Differentiating (i) gives. $x\frac{dy}{dx} + y = Ae^x - Be^{-x} + 2x$ .... (ii)

Differentiating (ii) gives; $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = Ae^x + Be^{-x} + 2$, which reduces to,

$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = Ae^x + Be^{-x} + 2$ .... (iii)

Rearranging (i) gives, $xy - x^2 = Ae^x + Be^{-x}$ .... (iv)

Substituting (iv) in (iii) we get, $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$ which is the required differential equation.

6. Show that the differential equation of which $y = 2(x^2 - 1) + ce^{-x^2}$ is a solution is $\frac{dy}{dx} + 2xy = 4x^3$.

Solution

From $y = 2(x^2 - 1) + ce^{-x^2}$ .... (i)

Differentiating (i) yield, $\frac{dy}{dx} = 4x - 2xce^{-x^2} \Rightarrow \frac{dy}{dx} - 4x = -2xce^{-x^2}$ .... (ii)

Rearranging (i) and multiplying by $-2x$ throughout gives

$-2x\left(y - 2(x^2 - 1)\right) = -2xce^{-x^2}$ .... (iii)

Substituting (iii) in (ii) we get, $\frac{dy}{dx} - 4x = -2x\left(y - 2(x^2 - 1)\right)$ which simplifies to

$\frac{dy}{dx}+2xy=4x^3$. This is the required differential equation. Shown.

Exercise

1. Form the required differential equations from the family of curves given below

   (a) $\frac{x}{a}+\frac{y}{b}=1$ (b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (c) $y=A\cos 2x+B\sin 2x$

   (d) $(y-b)^2=4(x-a)$ (e) $x^2=4ay$ (f) $y=\tan^{-1}(x)+ce^{\tan^{-1}(x)}$

   (g) $y=ae^x+be^{2x}+ce^{-3x}$

2. Shows that the differential equation represents all parabola having their axes of symmetry coincident with the axis of x is $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$.

FIRST ORDER DIFFERENTIAL EQUATIONS

First order differential equations are the differential equations appear in the form $\frac{dy}{dx}=f(y,x)$.

For example; $\frac{dy}{dx}=5y$, $5\frac{dy}{dx}+2x=3$, $x\frac{dy}{dx}=2-4x^3$ etc.

Solution of the first order ordinary differential equations.

Solution of differential equations by direct integration

A differential equation of the form of $\frac{dy}{dx}=f(x)$ is solved by direct integration.

Consider the equation above; $\frac{dy}{dx}=f(x)$

$\Rightarrow dy=f(x)dx$

$\Rightarrow \int dy=\int f(x)dx$

$\Rightarrow y=\int f(x)dx$ This is the solution to the above differential equation

Examples

1. Determine the solution of $\frac{dy}{dx}=x$

   Solution

$$\frac{dy}{dx} = x \Rightarrow dy = xdx \Rightarrow y = \int xdx \Rightarrow y = \frac{x^2}{2} + c$$

2. Find the solution of the differential equation $5\frac{dy}{dx} + 2x = 3$

   Solution

   Recall $5\frac{dy}{dx} + 2x = 3$

$$\Rightarrow \frac{dy}{dx} = \frac{3-2x}{5} \Rightarrow dy = \left(\frac{3-2x}{5}\right)dx \Rightarrow \int dy = \int\left(\frac{3-2x}{5}\right)dx \Rightarrow y = \int \frac{3}{5}dx - \int \frac{2}{5}xdx$$

$$\Rightarrow y = \frac{3}{5}x - \frac{1}{5}x^2 + c$$

Exercise

1. Find the solution of each of the following differential equations
   a) $\frac{dy}{dx} = \cos 4x - 2x$
   b) $2x\frac{dy}{dx} = 3 - x^3$
   c) $\frac{dy}{dx} + x = 3$
   d) $3\frac{dy}{d\theta} + \sin\theta = 0$
   e) $\frac{1}{e^x} + 2 = x - 3\frac{dy}{dx}$

Separable differential equations

Ordinary differential equations of the form $\frac{dy}{dx} = f(x).f(y)$ where $f(x)$ can be a constant function are called separable equations. For example; $\frac{dy}{dx} = xy$ $\frac{dy}{dx} = e^{2x+y}$ etc. Separable equations can be solved as follows

Consider the equation $\frac{dy}{dx} = f(x).f(y)$

$$\Rightarrow \frac{dy}{f(y)} = f(x)dx \Rightarrow \int \frac{dy}{f(y)} = \int f(x)dx \Rightarrow G(x) = H(x) + c$$

Examples

1. Find the solution of the following differential equations

   (i) $\frac{dy}{dx} = xy$ (ii) $\frac{dy}{dx} = e^{2x+y}$ (iii) $4xy\frac{dy}{dx} = y^2 - 1$

   Solution

   (i) $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = xdx \Rightarrow \int \frac{dy}{y} = \int xdx \Rightarrow \ln y = \frac{1}{2}x^2 + c \Rightarrow y = e^{\frac{1}{2}x^2+c} \Rightarrow y = e^{\frac{1}{2}x^2}.e^c$

   But let $e^c = A \Rightarrow y = Ae^{\frac{1}{2}x^2}$.

   (ii) $\frac{dy}{dx} = e^{2x+y} \Rightarrow \frac{dy}{dx} = e^{2x}.e^y \Rightarrow \frac{dy}{e^y} = e^{2x}dx \Rightarrow e^{-y}dy = e^{2x}dx \Rightarrow e^{-y}dy = e^{2x}dx$

   $\Rightarrow \int e^{-y}dy = \int e^{2x}dx \Rightarrow -e^{-y} = \frac{e^{2x}}{2} + c$.

   (iii) $4xy\frac{dy}{dx} = y^2 - 1 \Rightarrow \left(\frac{4y}{y^2-1}\right)dy = \frac{dx}{x} \Rightarrow \int\left(\frac{4y}{y^2-1}\right)dy = \int\frac{dx}{x} \Rightarrow 2\int\frac{2y}{y^2-1}dy = \int\frac{dx}{x}$

Let $u = y^2 - 1$ this means $du = 2ydy$ and therefore the above equation now becomes $2\int\frac{du}{u} = \int\frac{dx}{x} \Rightarrow 2\ln u = \ln x + c \Rightarrow 2\ln u = \ln x + \ln A$, using the laws of logarithms the right hand side becomes $\ln u^2 = \ln Ax \Rightarrow u^2 = Ax$ but $u = y^2 - 1 \Rightarrow (y^2-1)^2 = Ax$

Exercise

1. Solve the following differential equations

   (a) $\frac{dy}{dx} = \frac{x^2}{y}$ (b) $\frac{dy}{dx} + y^2 \sin x = 0$ (c) $y' = (\cos^2 x)(\cos^2 2y)$

   (d) $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$ (e) $y' = \frac{x^2}{1+y^2}$ (f) $xy' = \sqrt{1-y^2}$

   (g) $y' = (3x^2 - 1)(3 + 2y)$ (h) $\frac{dy}{dx} = \frac{x^2}{y(1+x^2)}$

2. Find the solutions of the following initial value problems.

   (i) $y' = (1-2x)y^2, y(0) = -\frac{1}{6}$ (ii) $xdx + ye^{-x}dy = 0, y(0) = 1$

(iii) $y' = \dfrac{2x}{y + x^2 y}, y(2) = 0$ (iv) $y' = \dfrac{e^{-x} - e^{x}}{3 + 4y}, y(0) = 1$

(v) $y^2\sqrt{1 - x^2}\,dy = \sin^{-1}(x)dx, y(0) = 0$ (vi) $\sin(2x)dx + \cos(3y)dy = 0, y(\frac{\pi}{2}) = \frac{\pi}{3}$

First order homogeneous equations

An equation of the form $P\dfrac{dy}{dx} = Q$, where P and Q are both functions of x and y with the same degree throughout is called first order homogeneous equation. $\dfrac{dy}{dx} = \dfrac{x + 3y}{x - y}$ and $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$ are the homogeneous differential equations but $\dfrac{dy}{dx} = \dfrac{x - 3y}{x^2}$ and $\dfrac{dy}{dx} = \dfrac{y - 4x^2}{x^2 - y^2}$ are not homogeneous differential equations. These equations are not variable separable equations, but they can be made so by changing the variable. These types of equations can be solved through the following steps.

Steps

(i). Rearrange $P\dfrac{dy}{dx} = Q$ into the form $\dfrac{dy}{dx} = \dfrac{Q}{P}$ and then simplify.

(ii). Make the substitution $u = \dfrac{y}{x}$, this means $y = ux$, by using product rule this become $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$.

(iii). Substitute for both y and $\dfrac{dy}{dx}$ in $\dfrac{dy}{dx} = \dfrac{Q}{P}$, then simplify to get the separable equation.

(iv). Separate the variable and solve the resulting equation.

(v). Substitute $u = \dfrac{y}{x}$ to solve in terms of the original variables.

Examples

1. Solve the differential equation: $y - x = x\dfrac{dy}{dx}$

   Solution

   Using the steps above

   (i) Rearranging the equation gives; $\dfrac{dy}{dx} = \dfrac{y - x}{x}$ which is homogeneous in x and y

(ii) Let $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

(iii) Substituting for y and $\frac{dy}{dx}$ gives, $u + x\frac{du}{dx} = \frac{ux - x}{x} \Rightarrow u + x\frac{du}{dx} = \frac{x(u-1)}{x}$

$\Rightarrow u + x\frac{du}{dx} = u - 1 \Rightarrow x\frac{du}{dx} = u - 1 - u \Rightarrow x\frac{du}{dx} = -1$

(iv) Separating the variable gives; $du = -\frac{dx}{x} \Rightarrow \int du = -\int \frac{dx}{x} \Rightarrow u = -\ln x + c$

(v) Substituting $u = \frac{y}{x}$ gives, $\frac{y}{x} = -\ln x + c$

2. Find the general solution of the equation $x\frac{dy}{dx} = \frac{x^2 + y^2}{y}$

Solution

Using the steps above we get the following

(i) Rearranging the equation yield, $\frac{dy}{dy} = \frac{x^2 + y^2}{xy}$

(ii) Let $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

(iii) Substituting for y and $\frac{dy}{dx}$ gives, $u + x\frac{du}{dx} = \frac{x^2 + (ux)^2}{x(ux)} \Rightarrow u + x\frac{du}{dx} = \frac{x^2 + u^2x^2}{ux^2}$

$\Rightarrow u + x\frac{du}{dx} = \frac{x^2(1+u^2)}{ux^2} \Rightarrow u + x\frac{du}{dx} = \frac{1+u^2}{u} \Rightarrow x\frac{du}{dx} = \frac{1+u^2}{u} - u \Rightarrow x\frac{du}{dx} = \frac{1}{u}$

(iv) Separating the variable gives; $udu = \frac{dx}{x}$, integrating both sides we have

$\int udu = \int \frac{dx}{x} \Rightarrow \frac{1}{2}u^2 = \ln x + c$

(v) Substituting $u = \frac{y}{x}$ gives, $\frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln x + c \Rightarrow \frac{y^2}{2x^2} = \ln x + c$

Exercise

1. Solve for each of the following equations

(a) $x^2 = y^2\frac{dy}{dx}$

(b) $\frac{x+y}{y-x} = \frac{dy}{dx}$

(c) $(9xy - 11xy)\frac{dy}{dx} = 11y^2 + 16xy + 3x^2$

(d) $x - y + x\frac{dy}{dx} = 0$

(e) $\left(\frac{2y - x}{y + 2x}\right)\frac{dy}{dx} = 1$

(f) $2x\frac{dy}{dx} = x + 3y$

(g) $\left(x^2 + y^2\right)dy = xydx$

(h) $xy^3dy = \left(x^4 + y^4\right)dx$

2. Show that the solution of the differential equation $2xy\frac{dy}{dx}=x^2+y^2$ can be expressed as $x=k(x^2-y^2)$, where k is a constant.

Equations reducible to homogeneous form

The equations of the form $\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}$ are not homogeneous differential equations but they can be reduced to the homogeneous form by the substitutions; $x=X+h$ and $y=Y+k$, this means $\frac{dy}{dx}=\frac{dY}{dX}$. Using these substitutions the differential equation becomes.

$$\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C}=\frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C}$$

Choose h,k so that $ah+bk+c=0$ and $Ah+Bk+C=0$, then the equation becomes homogeneous i.e. $\frac{dY}{dX}=\frac{aX+bY}{AX+BY}$ with $\frac{a}{A}\neq\frac{b}{B}$.

If $\frac{a}{A}=\frac{b}{B}$ then, the values of h,k will not be finite, in this case $\frac{a}{A}=\frac{b}{B}=\frac{1}{m}\Rightarrow A=am$ and $B=bm$

Therefore the differential equation becomes $\frac{dy}{dx}=\frac{ax+by+c}{m(ax+by)+c}$, Now , put $ax+by=t$ and apply the method of variable separable.

Examples.

1. Solve $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

   Solution.

   We have, $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$ .... (i)

   Put $x=X+h$ and $y=Y+k$ $\Rightarrow\frac{dy}{dx}=\frac{dY}{dX}$

   This implies that (i) reduces to $\frac{dY}{dX}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$ .... (ii)

   Choose h and k such that $h+2k-3=0$ and $2h+k-3=0$, solving simultaneously gives; $h=1$ and $k=1$

   Substituting the values of h and k, equation (ii) becomes $\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$, using the steps for solving homogeneous differential equations the solution become

   $$\frac{x+y-2}{(x-y)^3}=c^2$$

2. Solve $\frac{dy}{dx}=\frac{y-x+1}{y+x-5}$

Solution

We have, $\frac{dy}{dx}=\frac{y-x+1}{y+x-5}$ ………………………………… (i)

Let $x=X+h$ and $y=Y+k \Rightarrow dx=dX$ and $dy=dY \Rightarrow \frac{dy}{dx}=\frac{dY}{dX}$, using these substitutions equation (i) now becomes $\frac{dY}{dX}=\frac{Y-X+(k-h+1)}{Y+X+(k+h-5)}$ ………………… (ii)

Choose h and k such that $h-k+1=0$ and $k+h-5=0$, solving simultaneously gives; $h=3$ and $k=2$. This means equation (ii) now becomes $\frac{dY}{dX}=\frac{Y-X}{Y+X}$. Using the technique for solving homogeneous equations the solution becomes

$$\frac{1}{2}\ln\left((x-3)^2+(y-2)^2\right)+\tan^{-1}\left(\frac{y-2}{x-3}\right)=c$$

Exercise

Solve the following differential equations

(a) $\frac{dy}{dx}=\frac{2x-9y-20}{6x+2y-10}$

(b) $\frac{dy}{dx}=\frac{y-x+1}{y+x+5}$

(c) $\frac{dy}{dx}=\frac{x-y-2}{y+x+6}$

(d) $\frac{dy}{dx}=\frac{y+x-2}{y-x-4}$

(e) $\frac{dy}{dx}=\frac{2x-5y+3}{2x+4x-6}$

First order exact differential equations

The expression $M(x,y)dx+N(x,y)dy$ is called an exact differential in a domain D if there exist a function F of two real variables such that this expression equals the total differential $dF(x,y)$ for all (x,y). That is $M(x,y)dx+N(x,y)dy$ is an exact differential in D if there exist a function F such that $\frac{\partial F(x,y)}{\partial x}=M(x,y)$ and $\frac{\partial F(x,y)}{\partial y}=N(x,y)$ for all $(x,y)\in D$.Thus if $M(x,y)dx+N(x,y)dy$ is an exact differential, then the differential equation $M(x,y)dx+N(x,y)dy=0$ is called an exact differential equation. The general solution of these equations are in the form $f(x,y)=c$

Theorem 1: Test for Exactness

Let *M* and *N* have continuous partial derivatives on an open disc *R*. The differential equation $M(x,y)dx+N(x,y)dy=0$ is exact if and only if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Examples

Test for exactness in the following differential equations

(a) $\left(xy^2+x\right)dx+yx^2dy=0$

(b) $\cos y dx+\left(y^2-x\sin y\right)dy=0$

Solutions

(a) Comparing $\left(xy^2+x\right)dx+yx^2dy=0$ with $M\left(x,y\right)dx+N\left(x,y\right)dy=0$, gives $M(x,y)=y^2x+x$ and $N(x,y)=yx^2$, this implies that $\frac{\partial M}{\partial y}=2xy$ and $\frac{\partial N}{\partial x}=2xy$ which satisfy the condition of exactness.

(b) Likewise in (b) $M(x,y)=\cos y$ and $N(x,y)=y^2-x\sin y \Rightarrow \frac{\partial M}{\partial y}=-\sin y$ and $\frac{\partial N}{\partial x}=-\sin y$, this shows that the equation is an exact one.

Solution of the first order exact differential equations

Examples

1. Solve the differential equation $(2xy-3x^2)dx+(x^2-2y)dy=0$

   Solution

   The differential equation is an exact because,

   $$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(2xy-3x^2)=2x=\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(x^2-2y)$$

   The general solution $f(x,y)=c$ is given by $f(x,y)=\int M(x,y)dx \Rightarrow \int\left(2xy-3x^2\right)dx$

   Which gives, $x^2y-x^3+g(y)$ .... (i)

   The function $g(y)$ is determined by integrating $N(x,y)$ with respect to y and reconciling the two expressions for $f(x,y)$. It follows that

   $\int N(x,y)dy=\int(x^2-2y)dy$, which yield $x^2y-y^2+g(x)$ ....(ii)

   Comparing equation (i) and (ii) shows that $g(y)=-y^2$, substituting this result in equation (i) we get $x^2y-x^3-y^2=c$.

2. Solve the following differential equation $(\cos x-x\sin x+y^2)dx+2xydy=0$

   Solution

   This differential equation is an exact one because,

   $\frac{\partial}{\partial y}(\cos x-x\sin x+y^2)=2y=\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2xy)$. But its solution is of the form $f(x,y)=c$

which is given by $f(x,y)=\int(\cos x - x\sin x + y^2)dx$ which reduces to

$x\cos x + xy^2 + g(y)$ ………………………………………………… (i)

Our $N(x,y)$ is $2xy$,this means $\int N(x,y)dy = \int 2xydy$, this reduces to

$xy^2 + g(x)$ ……………………………………………………… (ii)

Comparing the two equations gives, $g(x)=o$, So the solution to this differential equation is $x\cos x + xy^2 = c$.

Exercise

Integrating Factor

This is a function multiplied throughout the non – exact differential equation to make it exact one.

Consider the general form of the first order linear differential equation bellow.

$\frac{dy}{dx} + P(x)y = Q(x)$.

Let us take a simple case $\frac{dy}{dx} + P(x)y = 0$. Rearranging the equation gives

$dy + P(x)ydx = 0$ ………………………………………………………. (i)

Equation (i) is not exact, to make it exact we multiply by an integrating factor.

If $\mu(x)$ is our integrating factor, multiplying it throughout equation (i) we get,

$\mu(x)dy + \mu(x)P(x)ydx = 0$ The equation now is exact only if $\frac{\partial\mu(x)}{\partial x} = \frac{\partial\left(\mu(x)P(x)y\right)}{\partial y}$

Since both μ and P are functions of x, then the identity reduces to $\frac{d\mu}{dx} = \mu P$

Separating the variable we have $\frac{d\mu}{\mu} = Pdx$

$\Rightarrow \int\frac{d\mu}{\mu} = \int Pdx \Rightarrow \ln\mu = \int Pdx \Rightarrow \mu = e^{\int Pdx}$

The integrating factor(μ) is given by $\mu = e^{\int P(x)dx}$

Examples

Determine the integrating factors in each of the following differential equations

(a) $\frac{dy}{dx} + xy = x$ (b) $\frac{dy}{dx} - 4y = x$

Solution

(a) We have, $\frac{dy}{dx} + xy = x$, in this equation $P(x) = x$ and its integrating factor μ is

$\mu = e^{\int xdx} \Rightarrow \mu = e^{\frac{1}{2}x^2}$

(b) We have, $\frac{dy}{dx} - 4y = x$, in this equation $P(x) = -4$, so its integrating factor(μ) become

$$\mu = e^{\int -4dx} \Rightarrow \mu = e^{-4x}$$

First order linear differential equations

A differential equation of the form $\frac{dy}{dx} + Py = Q$ is called a linear differential equation in y, where P and Q are functions of x (but not y) or constants. For example; $\frac{dy}{dx} + x^2y = x^4$, $\frac{dy}{dx} + \frac{y}{x} = x^2$.

Solution of first order differential equations

The solution of the first order linear differential equations follows the steps below

1. Find the integrating factor of the given equation
2. Multiply the integrating factor in (i) throughout the given equation
3. Integrate both sides of the equation obtained in (ii) to get the solution, which is in the form

$$y(\mu) = \int (Q \times \mu)dx + c$$

Examples

1. Solve the differential equation $\frac{dy}{dx} + 2y = 6e^x$

Solutions

We have, $\frac{dy}{dx} + 2y = 6e^x$ .... (i)

This equation is linear with $P = 2, Q = 6e^x$

The integrating factor(μ) is $\mu = e^{2dx} \Rightarrow \mu = e^{2x}$

The solution is $y(e^{2x}) = \int 6e^x \times e^{2x}dx + c$

$\Rightarrow y(e^{2x}) = \int 6e^{3x}dx + c \Rightarrow y = 2e^x + ce^{-x}$ which is the required solution

2. Solve the differential equation $x\frac{dy}{dx} + y = x^4$

Solution

We have, $x\frac{dy}{dx} + y = x^4 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^3 \Rightarrow \frac{dy}{dx} + \left(\frac{1}{x}\right)y = x^3$

This means $P = \frac{1}{x}$ and $Q = x^3$

The integrating factor is $\mu = e^{\int \frac{1}{x}dx} \Rightarrow \mu = e^{\ln x} = x$

The solution is $y.x = \int x^3.xdx + c \Rightarrow yx = \int x^4dx + c \Rightarrow y = \frac{1}{5}x^4 + \frac{c}{x}$ which is the required solution.

3. Solve $x\frac{dy}{dx}-y=x^2$

Solution

We have, $x\frac{dy}{dx}-y=x^2 \Rightarrow \frac{dy}{dx}-\frac{y}{x}=x \Rightarrow \frac{dy}{dx}-\left(\frac{1}{x}\right)y=x$, this means $P=-\frac{1}{x}$ and $Q=x$

The integrating factor is $\mu=e^{\int -\frac{dx}{x}} \Rightarrow \mu=e^{-\ln x} \Rightarrow \mu=e^{\ln(x^{-1})}=e^{\ln\left(\frac{1}{x}\right)}=\frac{1}{x}$

The solution is $y\left(\frac{1}{x}\right)=\int x.\left(\frac{1}{x}\right)dx+c \Rightarrow \frac{y}{x}=\int dx+c \Rightarrow \frac{y}{x}=x+c \Rightarrow y=x^2+cx$

Exercise

Solve for each of the following differential equations

(a) $\frac{dy}{dx}-\frac{y}{x}=2x^2$ (b) $2x\frac{dy}{dx}+y=6x^3$ (c) $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$

(d) $(x^2-1)\frac{dy}{dx}+2xy=\frac{2}{x^21}$ (e) $\frac{dy}{dx}+y=\cos x-\sin x$ (f) $\cos^2 x\frac{dy}{dx}+y=\tan x$

(g) $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$ (h) $\frac{dy}{dx}+y=\cos x$ (i) $(1+x^2)\frac{dy}{dx}+y=\tan^{-1}(x)$

Equations reducible to linear form (Bernoulli's equation)

The equation of the form $\frac{dy}{dx}+Py=Qy^n$ where P and Q are constants or functions of x can be reduced to the linear form on dividing by y^n and substituting $\frac{1}{y^{n-1}}=t$.

Consider the equation $\frac{dy}{dx}+Py=Qy^n$ ……………………………………………………… (i)

On dividing equation (i) by y^n, we get $\frac{1}{y^n}\frac{dy}{dx}+\frac{1}{y^{n-1}}P=Q$ ……………………….. (ii)

Put $\frac{1}{y^{n-1}}=t$ so that $\frac{1-n}{y^n}\frac{dy}{dx}=\frac{dt}{dx}$, equation (ii) now become $\frac{1}{1-n}\frac{dt}{dx}+Pt=Q$

$\Rightarrow \frac{dt}{dx}+P(1-n)t=Q(1-n)$ which is a linear in t and can be solved by the same technique as in linear differential equations.

Examples

1. Solve $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$

Solution

We have, $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2} \Rightarrow \frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=\frac{1}{x^2} \Rightarrow \frac{1}{y^2}\frac{dy}{dx}+\frac{1}{x}.\left(\frac{1}{y}\right)=\frac{1}{x^2}$ .... (i)

Let $t=-\frac{1}{y} \Rightarrow \frac{dt}{dx}=\frac{1}{y^2}\frac{dy}{dx}$ .... (ii)

Substituting equation (ii) in (i) we get $\frac{dt}{dx}-\frac{t}{x}=\frac{1}{x^2}$, this now is a linear equation, solving by the technique we get $t=-\frac{1}{2x}+cx$ but $t=-\frac{1}{y}$

Which implies that $\frac{1}{xy}=\frac{1}{2x^2}-c$

2. Solve $x\frac{dy}{dx}+y\log y=xye^x$

Solution

We have, $x\frac{dy}{dx}+y\log y=xye^x \Rightarrow \frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x$ .... (i)

Let $\log y=t \Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{dt}{dx}$, substituting in equation (i) gives $\frac{dt}{dx}+\frac{t}{x}=e^x$ which is a linear equation whose solution is $x\log y=xe^x-e^x+c$

3. Solve $\left(\sec x.\tan x.\tan y-e^x\right)dx+\sec x.\sec^2 y dy=0$

Solution

We have, $\left(\sec x.\tan y-e^x\right)dx+\sec x.\sec^2 y dy=0 \Rightarrow \sec^2 y\frac{dy}{dx}+\tan x\tan y=e^x\cos x$ ...... (i)

Put $\tan y=t \Rightarrow \sec^2 y\frac{dy}{dx}=\frac{dt}{dx}$, substituting in equation (i) gives $\frac{dt}{dx}+t\tan x=e^x\cos x$

This is equation is linear and its solution is $\tan y\sec x=e^x+c$.

Exercise

Solve the following differential equations

(a) $\tan y\frac{dy}{dx}+\tan x=\cos y\cos^2 x$

(b) $\sec^2 y\frac{dy}{dx}+x\tan y=x^3$

(c) $y\sin x dx-(1+y^2+\cos^2 x)dy=0$

(d) $x\frac{dy}{dx}=y(\log y-\log x-1)$

(e) $ydy-(1-x^2-y^2)xdx=0$

(f) $\frac{dy}{dx}+xy=xy^3$

(g) $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$