Analysis of the Focusing NLS

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1 Introduction

The focusing cubic nonlinear Schrödinger equation (NLS) describes wave propagation in nonlinear media:

$$iu_t + \Delta u + |u|^2 u = 0 \tag{1}$$

where $u(x,t)$ is a complex-valued field. This equation models phenomena in nonlinear optics, Bose-Einstein condensates, and hydrodynamics.

2 Key Properties

- **Solitons**: Exact solutions exist of the form:

  $$u(x,t) = A\,\mathrm{sech}(B(x-x_0))e^{iv(x-x_0)}$$

  These are called solitons. A **soliton** is a nonlinear, self-reinforcing, localized wave packet that is strongly stable.

- **Elastic Collisions**: Solitons interact cleanly, preserving their form with only phase shifts.

3 Conservation Laws

3.1 Mass Conservation

Define the mass functional:

$$M(t) = \int_{-\infty}^{\infty} |u(x,t)|^2\,dx = \int_{-\infty}^{\infty} u(x,t)\,\overline{u}(x,t)\,dx.$$

Take the time derivative:

$$\begin{aligned}\frac{dM}{dt} &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\big[u\,\overline{u}\big]\,dx \\ &= \int_{-\infty}^{\infty} \big(u_t\,\overline{u} + u\,\overline{u}_t\big)\,dx.\end{aligned}$$

Substitute u_t from the NLS ($i\,u_t = -u_{xx} - |u|^2 u$) and its conjugate:

$$\begin{aligned} u_t &= i\,u_{xx} + i\,|u|^2 u, \\ \overline{u}_t &= -i\,\overline{u}_{xx} - i\,|u|^2 \overline{u}. \end{aligned}$$

Insert into the integral:

$$\begin{aligned} \frac{dM}{dt} &= \int_{-\infty}^{\infty} \big[(i\,u_{xx} + i\,|u|^2 u)\,\overline{u} + u\,(-i\,\overline{u}_{xx} - i\,|u|^2\overline{u})\big]\,dx \\ &= i\int_{-\infty}^{\infty} \big[u_{xx}\,\overline{u} - u\,\overline{u}_{xx}\big]\,dx \quad \text{(the } |u|^2 \text{ terms cancel).} \end{aligned}$$

Integrate by parts on the first term:

$$\int_{-\infty}^{\infty} u_{xx}\,\overline{u}\,dx = \big[u_x\,\overline{u}\big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x\,\overline{u}_x\,dx = -\int_{-\infty}^{\infty} u_x\,\overline{u}_x\,dx,$$

and similarly

$$\int_{-\infty}^{\infty} u\,\overline{u}_{xx}\,dx = \big[u\,\overline{u}_x\big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \overline{u}_x\,u_x\,dx = -\int_{-\infty}^{\infty} |u_x|^2\,dx.$$

Thus

$$\begin{aligned} \frac{dM}{dt} &= i\Big[-\int_{-\infty}^{\infty} |u_x|^2\,dx - \big(-\int_{-\infty}^{\infty} |u_x|^2\,dx\big)\Big] \\ &= i * 0 \\ &= 0. \end{aligned}$$

Therefore

$$\frac{dM}{dt} = 0 \quad \text{for all } t > 0.$$

3.2 Energy Conservation

Energy Functional Definition

The Hamiltonian (total energy) for the focusing NLS is

$$H[u] := \underbrace{\int_{-\infty}^{\infty} |u_x|^2\,dx}_{\text{Kinetic Energy}} - \underbrace{\tfrac{1}{2}\int_{-\infty}^{\infty} |u|^4\,dx}_{\text{Nonlinear Potential Energy}}.$$

Time Derivative Calculation

Differentiate H in time:

$$\begin{aligned} \frac{dH}{dt} &= \frac{d}{dt}\Big(\int |u_x|^2\,dx - \tfrac{1}{2}\int |u|^4\,dx\Big) \\ &= \int \frac{\partial}{\partial t}|u_x|^2\,dx - \tfrac{1}{2}\int \frac{\partial}{\partial t}|u|^4\,dx. \end{aligned}$$

Using the chain rule,

$$\frac{\partial}{\partial t}|u_x|^2 = \frac{\partial}{\partial t}(u_x\bar{u}_x) = u_{xt}\bar{u}_x + u_x\bar{u}_{xt} = 2\,\Re(\bar{u}_x\,u_{xt}),$$
$$\frac{\partial}{\partial t}|u|^4 = \frac{\partial}{\partial t}(|u|^2)^2 = 2|u|^2\,\frac{\partial}{\partial t}|u|^2 = 2|u|^2(\bar{u}\,u_t + u\,\bar{u}_t) = 4|u|^2\,\Re(\bar{u}\,u_t).$$

Hence

$$\frac{dH}{dt} = 2\,\Re\int \bar{u}_x\,u_{xt}\,dx - 2\,\Re\int |u|^2\,\bar{u}\,u_t\,dx.$$

Integration by Parts

For the first integral,

$$\int_{-\infty}^{\infty} \bar{u}_x\,u_{xt}\,dx = \big[\bar{u}_x\,u_t\big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \bar{u}_{xx}\,u_t\,dx = -\int_{-\infty}^{\infty} \bar{u}_{xx}\,u_t\,dx.$$

Thus

$$\frac{dH}{dt} = -2\,\Re\int \bar{u}_{xx}\,u_t\,dx - 2\,\Re\int |u|^2\,\bar{u}\,u_t\,dx.$$

Using the NLS Equation

From

$$i\,u_t = -u_{xx} - |u|^2u \quad\Longrightarrow\quad \bar{u}_{xx} + |u|^2\bar{u} = -i\,\bar{u}_t,$$

we substitute and get

$$\frac{dH}{dt} = -2\,\Re\int \big(\bar{u}_{xx} + |u|^2\bar{u}\big)\,u_t\,dx = -2\,\Re\int (-i\bar{u}_t)\,u_t\,dx = 2\,\Re\int i\,|u_t|^2\,dx.$$

Final Evaluation

Since $i\,|u_t|^2$ is purely imaginary,

$$\Re\big(i\,|u_t|^2\big) = 0,$$

it follows that

$$\frac{dH}{dt} = 0,$$

so energy is conserved:

$$\frac{dH}{dt} \equiv 0 \quad \text{for all } t > 0.$$

4 Fourier Representation

Transforming the NLS to Fourier space yields:

$$i\,\hat{u}_t + k^2\,\hat{u} - \mathscr{F}\big[\,|u|^2 u\big] = 0,$$

or equivalently

$$i\,\hat{u}_t = -\,k^2\,\hat{u} + \mathscr{F}[\,|u|^2 u].$$

The term $-k^2\hat{u}$ is "stiff" for large $|k|$, making explicit time-stepping restrictive. We handle this using a method found in Trefethen's *Spectral Methods in Matlab*. More precisely, this method involves using an integrating factor. So we define the integrating-factor variable

$$v(k,t) \;=\; e^{-ik^2 t}\,\hat{u}(k,t).$$

Differentiating,

$$\frac{d}{dt}v = -i\,k^2\,e^{-ik^2t}\hat{u} + e^{-ik^2t}\hat{u}_t = e^{-ik^2t}\big(-i\,k^2\,\hat{u} + \hat{u}_t\big).$$

Using the Fourier-space NLS,

$$\hat{u}_t = \;i\,k^2\,\hat{u} - i\,\mathscr{F}[\,|u|^2 u],$$

we obtain

$$\frac{d}{dt}v = e^{-ik^2t}\left(-i\,k^2\,\hat{u} + \;i\,k^2\,\hat{u} - i\,\mathscr{F}[\,|u|^2 u]\right) = -i\,e^{-ik^2t}\,\mathscr{F}\big[\,|u|^2 u\big].$$

Re-expressing $u = \mathscr{F}^{-1}(e^{ik^2t}v)$, the final form is

$$\frac{d}{dt}v(k,t) = -\,i\,e^{-ik^2t}\,\mathscr{F}\Big[\big|\mathscr{F}^{-1}[e^{ik^2t}v]\big|^2\,\mathscr{F}^{-1}[e^{ik^2t}v]\Big].$$

- *Exact linear integration:* the factor e^{ik^2t} removes the stiff k^2 term.
- *Non-stiff nonlinear evolution:* only the cubic nonlinearity remains to be stepped in time.

This yields the formulation which we implement in our simulation.