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PROGRAM SYNTHESIS FROM LINEAR AND GRADED TYPES

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A Dissertation in Computer Science

November 2023 – classicthesis v4.6

ABSTRACT

A type-directed program synthesis tool can leverage the information provided by types to prune ill-resourced programs from the search space of candidate programs. Graded type systems are a class of *resourceful* type system for fine-grained quantitative reasoning about data-flow in programs. Tracing their roots from Linear types, the use of resource annotations (or *grades*) on data, allows a programmer to express structural or semantic properties of their program at the type level. Such systems have become increasingly popular in recent years, mainly for the expressive power that they offer to programmers — judicious use of grades in type specifications significantly reduces the number of typeable programs. These additional constraints on types lend themselves naturally to type-directed program synthesis, where this information can be exploited to constrain the search space of programs even further than in standard type systems. We present an approach to program synthesis for linear and graded type systems, where grades form an arbitrary pre-ordered semiring. Harnessing this grade information in synthesis is non-trivial, and we explore the issues involved in designing and implementing a resource-aware program synthesis tool, culminating in an efficient and expressive program synthesis tool for the research programming language Granule, which uses a graded type system. We show that by harnessing grades in synthesis, the majority of our benchmarking synthesis problems (many of which involve recursive functions over recursive ADTs) require less exploration of the synthesis search space than a purely type-driven approach and with fewer needed input-output examples. Our type-and-graded-directed approach is demonstrated in the Granule but we also adapt it for synthesising Haskell programs using GHC's linear types extension, demonstrating the versatility of our approach to resourceful program synthesis.

PUBLICATIONS

In this thesis, the content of some chapters is formed from previously published papers:

- Hughes and Orchard [2020] Resourceful Program Synthesis from Graded Linear Types. In Logic-Based Program Synthesis and Transformation - 30th International Symposium, pp. 151-170.

  This paper adapts the resource management techniques of previous work for Linear Logic proof search to graded types, and presents two synthesis calculi based on these approaches. This paper constitutes the majority of Chapter 3, where a program synthesis tool for a linear and graded type system is introduced.

- Hughes et al. [2020] Deriving Distributive Laws for Graded Linear Types. In 6th edition of the International Workshop on Linearity and of the 4th edition of the International Workshop on Trends in Linear Logic and its Applications.

  The majority of Chapter 4, which discusses an approach for automatically deriving certain graded programs in Granule as an alternative to synthesis, is derived from this paper.

ACKNOWLEDGMENTS

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Part I

PROGRAM SYNTHESIS FROM LINEAR AND GRADED TYPES

1 INTRODUCTION

One of the most useful and well-studied tools available to modern programmers is the type system. Not only do type systems allow many kinds of errors to be caught statically, they also help inform the design of a program. Many programmers will often tend to begin writing their programs by first defining the types, from which the program code follows naturally. This phenomenon will be familiar to any who have written programs in typed functional programming languages, and results from the fact that types form a high-level abstract specification of program behaviour.

Type-directed program synthesis is a well-studied technique for automatically generating program code from a given type specification - the *goal* type. This approach has a long history, which is deeply intertwined with automated theorem proving, thanks to the Curry-Howard correspondence [Manna and Waldinger, 1980, Green, 1969].

One lens through which we can view this task is as an inversion of type checking: we start with a goal type and inductively synthesise well-typed sub-terms by breaking the goal into sub-goals, pruning the search space of programs via typing as we go. This approach follows the treatment of program synthesis as a form of proof search in logic: given a type A we want to find a program term t which inhabits A. We can express this in terms of a synthesis *judgement* which acts as a kind of inversion of typing or proof rules:

$$\Gamma \vdash A \Rightarrow t$$

meaning that the term t can be synthesised for the goal type A under a context of assumptions Γ. We may construct a calculus of synthesis *rules* for a programming language, inductively defining the above synthesis judgement for each type former. For example, we may define a rule for standard product types in the following way:

$$\frac{\Gamma \vdash A \Rightarrow t_1 \qquad \Gamma \vdash B \Rightarrow t_2}{\Gamma \vdash A \times B \Rightarrow (t_1, t_2)}\ \times_{\text{Intro}}$$

Reading 'clockwise' from the bottom-left: to synthesise a value of type $A \times B$, we synthesise a value of type A and then a value of type

B and combine them into a pair in the conclusion. The 'ingredients' for synthesising the sub-terms t_1 and t_2 come from the free-variable assumptions contained in Γ and any constructors of A and B, assuming these types are monomorphic.

Depending on the context, there could be many possible combinations of assumption choices to synthesise such a pair. Consider the following partial program containing a program *hole*, marked with ?, specifying a position at which we wish to perform synthesis:

$$\begin{array}{l} f : A \to A \to A \to A \times A \\ f\ x\ y\ z = ? \end{array}$$

The function has three parameters all of type A which can be used to synthesise an expression of the goal type $A \times A$. Expressing this synthesis problem as an instantiation of the above $\times_{\text{INTRO}}$ rule:

$$\frac{x : A, y : A, z : A \vdash A \Rightarrow t_1 \qquad x : A, y : A, z : A \vdash A \Rightarrow t_2}{x : A, y : A, z : A \vdash A \times A \Rightarrow (t_1, t_2)}\ \times_{\text{INTRO}}$$

Even in this simple setting, the number of possibilities starts to become unwieldy: there are nine (3^2) possible candidate programs based on combinations of x, y and z. Ideally, we would like some way of constraining the number of choices that are required by the synthesis algorithm. Many systems achieve this by allowing the user to specify additional information about their desired program behaviour. For example, recent work has extended type-directed synthesis to refinement types [Polikarpova et al., 2016], cost specifications [Knoth et al., 2019], differential privacy [Smith and Albarghouthi, 2019], example-guided synthesis [Feser et al., 2015, Albarghouthi et al., 2013] or examples integrated with types Frankle et al. [2016], Osera and Zdancewic [2015], and ownership information [Fiala et al., 2023]. The general idea is that, with more information, whether that be richer types, additional examples, or behavioural specifications, the proof search / program synthesis process can be pruned and refined.

This work presents a program synthesis approach that leverages the information contained in *linear* and *graded type systems* that track and enforce program properties related to data flow, statically. We refer to these systems as *resourceful* type systems, since they treat data as though it is a physical resource, constraining how data can be used by a program and thus reducing the number of possible synthesis choices that need to be made. Our hypothesis is that resource-and-type-directed synthesis speeds up type-directed synthesis, reducing the number of paths that need to be explored and the amount of additional specification (e.g. input-output examples) required.

Graded type systems trace their roots to linear logic. In linear logic, data is treated as though it were a finite resource which must be consumed exactly once with arbitrary copying and discarding disallowed [Girard, 1987]. The identify function $\lambda x.x$ is the ideal linearly typed program: it binds a variable and then uses it exactly once. The K combinator $\lambda x.\lambda y.x$ from SKI combinatory logic, however, would not be linearly typed as the second variable y is never used inside the body. Non-linear use of data is expressed through the ! modal operator (the *exponential modality*). This gives a binary view—a value may either be used exactly once (i.e. as a resource) or in a completely unconstrained way. Bounded Linear Logic (BLL) refines this view, replacing ! with a family of indexed modal operators where the index provides an upper bound on usage [Girard et al., 1992], e.g., $!_{\leq 4}A$ represents a value A which may be used up to 4 times. In recent years, various works have generalised BLL, resulting in *graded* type systems in which these indices are drawn from an arbitrary pre-ordered semiring [Brunel et al., 2014, Ghica and Smith, 2014, Petricek et al., 2014, Abel and Bernardy, 2020, Choudhury et al., 2021, Atkey, 2018, McBride, 2016]. This allows numerous program properties to be tracked and enforced statically, including various kinds of reuse, privacy and confidentiality, and capabilities. Such systems are increasingly popular and form the basis of Linear Haskell [Bernardy et al., 2018], Idris 2 [Brady, 2021], as well as the experimental language Granule [Orchard et al., 2019].

Semantically, these semiring indexed !-modalities are modelled by *graded exponential comonads*[Gaboardi et al., 2016a]. The terminology of *graded modal types* was proposed by Orchard et al. [2019] encompassing both semiring indexed !-modalities and the notion of *graded monads* [Orchard et al., 2014, Katsumata, 2014, Smirnov, 2008], which generalise monads. In this work we do not consider the latter, dealing only with graded comonads, which are closely related to linearity. In general, we will refer to systems which make use of graded modal types as *graded* type systems.

Returning to our example in a graded setting, the arguments of the function now have *grades* (annotations) that, in this context, are natural numbers describing the exact number of times the parameters must be used (the choice here was ours):

$$\begin{aligned}&f : A^2 \to A^0 \to A^0 \to A \times A\\&f\ x\ y\ z = ?\end{aligned}$$

The first A is annotated with a grade 2, which in this context indicates that it *must* be used twice. Likewise, the types of y and z are graded with 0, enforcing zero usage, i.e., we are not allowed to use them in the body of f and must discard them.

The result is that there is only one (normal form) inhabitant for this type: (x, x); the other assumptions will not even be considered in synthesis, allowing us to effectively prune out branches which use resources in a way which violates the grades. In this example, these annotations take the form of natural numbers explaining how many times a value can be used, but we may instead wish to represent different kinds of program properties, such as sensitivity, strictness, or security levels for tracking non-interference, all of which are well-known instances of graded type systems [Orchard et al., 2019, Gaboardi et al., 2016a, Abel and Bernardy, 2020]. Note that all of these examples are technically graded presentations of *coeffects*, tracking how a programs uses its context, in contrast with graded types for side *effects* [Orchard et al., 2014, Katsumata, 2014], which we do not consider here.

As graded type systems develop into fully fledged programming languages such as Idris 2 [Brady, 2021], and as traditionally non graded languages, such as Haskell, incorporate features from graded types into their existing type systems [Bernardy et al., 2018], the proposition of harnessing the information conveyed by these richer types becomes increasingly attractive. As of this moment, program synthesis in the context of graded types has remained relatively unexplored despite the growing prevalence of such systems. Past works have tackled the problem of proof search in Linear Logic [Hodas and Miller, 1994, Cervesato et al., 2000], which are analogous, through the lens of Curry-Howard, to program synthesis solutions for linear types. These works lay the foundations for much of our approach here, however, graded types pose an entirely new set of challenges and integrating them into these existing approaches is complex.

For this work, we consider Granule Orchard et al. [2019] to be an ideal candidate for the target language of a program synthesis tool for graded type systems. Granule is a functional programming language which combines linear a type system at its core with indexed data types. On top of this, graded modalities are integrated both as graded comonads and graded monads. For this work, we forgo the treatment of Granule's indexed types in synthesis, leaving this as future work. Instead we focus on building a synthesis tool for Granule, which unlocks the expressivity held by graded comonads for use inside the synthesis algorithm itself.

Granule also provides a language extension, known as *GradedBase*, which does away with this linear type system as its foundation. Instead, grades permeate the type system à la Abel and Bernardy [2020]. This reflects a common approach to graded type systems, giving us a programming language which is capable of giving a broad representation of graded type systems. This will factor heavily into the design of our calculi, and we structure the rest of the thesis with this in mind.

1.1 CONTRIBUTIONS

The primary aim of this work is to demonstrate the feasibility and power of using resourceful types as the basis of a type-directed program synthesis tool, culminating in the development and implementation of an expressive, efficient, and feature-rich program synthesis tool for the Granule programming language.

We aim to show that using not only is program synthesis feasible in the context of graded type systems, but that the information conveyed by the grades can be used to prune the search space of programs in synthesis. This results in our synthesis algorithm needing to consider far less potential programs when constructing a program, as grade-violating candidate programs may be pruned out.

Specifically, this work makes the following contributions:

- We identify an approach which makes type-directed program synthesis in a resourceful setting feasible. Drawing inspiration from the work of Hodas and Miller on theorem proving [Hodas and Miller, 1994], we adapt their work to graded types, and propose a dual to their own method, yielding two schemes for managing resources in the synthesis of a program term.

- We use these schemes to construct two simple synthesis calculi for a simplified core of Granule, which demonstrate their effectiveness as tools for resourceful program synthesis. We implement both approaches as part of the Granule toolchain.

- We showcase an alternative and complementary approach to generating a subset of Granule programs, making use of a system inspired by Haskell's deriving mechanism Magalhães et al. [2010] adapted to graded types.

- We then define a synthesis calculus for a fully graded type system. This type system is a feature-rich language based on Granule's *graded base* language extension, which includes recursion, recursive types, and user-defined ADTs. Furthermore, we again implement this calculus as part of the Granule toolchain.

- We evaluate our tool on a benchmark suite of recursive functional programs leveraging standard data types like lists, streams, and trees. We compare against non-graded synthesis provided by MYTH [Osera and Zdancewic, 2015]. We also compare against our own tool *modulo grades*, i.e. we compare the number of synthesis paths explored by our tool when taking grades into account versus traditional un-graded synthesis.

- We prove that each of these systems is sound, i.e., synthesised programs are typed by the goal type. A ket property here is that synthesised programs are not well-typed, but also *well-resourced*, meaning that all values inside the program are used according to their resource constraints. We show that this property holds for each of our synthesis calculi as part of their soundness proofs.

- We demonstrate how our approach to resourceful program synthesis can be readily applied to other graded systems. Leveraging our calculus and implementation, we provide a prototype tool for synthesising Haskell programs written using GHC 9's Linear Types language extension.

1.2 STRUCTURE

This dissertation is structured into six chapters. In the next chapter, Chapter 2, the theoretical background of linear and graded types is laid out. In doing so, we introduce two core calculi with simple types and grades, which demonstrate the two major lineages of resourceful type systems. The first is a language based on an underlying non-graded type system (in this case a linear type system), with graded modal types introduced and eliminated explicitly. This system is the default basis of Granule [Orchard et al., 2019], the target language of our implementation. The second calculus does away with this linear basis, embedding graded modalities into the function types a la Idris 2 and Linear Haskell. McBride's QTT [McBride, 2016, Atkey, 2018], the core of Linear Haskell [Bernardy et al., 2018], and the unified graded modal calculus of [Abel and Bernardy, 2020]. This system is also present in Granule in the form of an optional language extension.

The rest of the dissertation is structured such that synthesis calculi for both of these systems are defined and presented, minimising any redundancy in the presentation. Despite the variances between the core calculi of 2, there is a substantial degree of overlap between the two. Thus, we adopt the following structure:

1. Chapter 3 introduces the core concepts of type-directed program synthesis from resourceful type systems using an extension of the typing calculus of Section 2.3.1. Specifically this chapter introduces the *resource management problem* as it relates to program synthesis: how do we ensure that a synthesised program is not only well-typed but also well-resourced? To address this question, we define two calculi of synthesis rules based on the graded linear λ-calculus which tackle the problem in different ways. To better illustrate and test the practicality of the synthesis calculi, we extend the language with multiplicative conjunction (product types $\otimes$ and unit 1) and additive disjunction (sum types $\otimes$). These calculi are then implemented targeting default Granule. We produce a comparative evaluation of the implemented synthesis tool, contrasting the efficiency of the two resource management approaches against each other, before selecting the most performant to use going forward.

2. We then consider an alternative approach to type-directed synthesis, exploring a mechanism for automatically deriving programs from their type à la Haskell's generic deriving mechanism [Magalhães et al., 2010]. Again for this we base the approach on the graded linear λ-calculus, extending it further with data constructors, pattern matching, and recursive data types.

3. Finally, in chapter 5, we present a synthesis calculus for a target language based instead on the core graded λ-calculus of 2.3.2. This calculus incorporates all of the language features that have been introduced in the previous chapters, for a rich synthesis tool implementation targeting Granule's *graded base* language extension. Furthermore, we outline several other useful extensions to the synthesis tool, such as the inclusion of example-based synthesis, and a post-synthesis refactoring process which re-writes synthesised programs in a more idiomatic style.

   We then evaluate the implementation on a set of 46 benchmarks, including several non-trivial programs which make use of these new features. In this evaluation, we compare to From our evaluation we find that using grades in synthesis outperforms purely type-driven program synthesis in terms of both speed, number of input-output examples required or number of retries to get the desired program.

   Finally, to demonstrate the practicality and versatility of our approach, we apply our synthesis algorithm of Chapter 5 to synthesising programs in Haskell from type signatures using GHC's *linear types* extension (which is implemented underneath by a graded type system).

This approach strikes a balance between maximising coverage of different approaches to resourceful type systems, and avoiding unnecessary repetition, whilst gradually increasing the complexity of the target language. By the end, we will then have two synthesis tool implementations for Granule, targeting both styles of graded type systems.

2

BACKGROUND

Before diving straight into designing program synthesis calculi, we first need to formally define our target languages. We take this chapter as an opportunity to do so, and to examine more closely some of the properties of these linear and graded systems.

Since Girard [1987]'s original work on Linear Logic, the development of type systems which convey additional information about the program's structure has evolved into a distinct paradigm, culminating in recent years with the notion of *graded types*. Approaches to graded type systems run the gamut, incorporating a wide range of effect and coeffect systems, however, they can typically be distilled into two categories, with distinct lineages:

- Systems where a graded modal type operator introduces and eliminates graded modalities above some existing type system. This is the default approach of Granule, where the underlying type system is linear, and grade modalities are introduced and eliminated via the $\Box$ modal type operator.
- Systems where grades permeate the program, and are introduced via annotations on function arrows. This is the approach taken by Linear Haskell [Bernardy et al., 2018], where grades (or "multiplicities") are specified using the `%` operator.

These two different styles to graded types mirror the dual development of effect systems and graded monadic systems in the literature. In the latter case, the two were eventually found to be equivalent, while the same treatment for the former remains ongoing work.

2.1 TERMINOLOGY

Throughout this thesis we will tend towards using a *types-and-programs* terminology rather than *propositions-and-proofs*. Via the Curry-Howard correspondence, one can switch smoothly to viewing our approach to program synthesis as proof search in logic.

The functional programming languages we discuss are presented as typed calculi given by sets of *types*, *terms* (programs), and *typing rules* that relate a term to its type. The most well-known typed calculus is the simply-typed λ-calculus (STLC), which corresponds to intuitionistic logic. We assume familiarity with STLC, and couch our explanation of linear and graded types in this context.

A *judgment* defines the typing relation between a type and a term based on a *context*. In STLC, judgments have the form: $\Gamma \vdash t : A$, stating that under some context of *assumptions* Γ the program term t can be assigned the type A. An assumption is a name with an associated type, written $x : A$ and corresponds to an in-scope variable in a program.

A term can be related to a type if we can derive a valid judgment through the application of typing rules. The application of these rules forms a tree structure known as a *typing derivation*.

2.2 LINEAR AND SUBSTRUCTURAL LOGICS

Linear logic was introduced as a way of being more descriptive about the properties of a derivation in intuitionistic logic. In type systems such as STLC, the properties of *weakening*, *contraction*, and *exchange* are assumed implicitly. These are typing rules which are *structural* as they determine how the context may be used rather than being directed by the syntax. Weakening is a rule which allows terms that are not needed in a typing derivation to be discarded. Contraction works as a dual to weakening, allowing an assumption in the context to be used more than once. Finally, exchange allows assumptions in a context to arbitrarily re-ordered. The rules themselves are provided by Figure 2.1.

$$\frac{\Gamma \vdash t : B}{\Gamma, x : A \vdash t : B}\ \textsc{Weakening} \qquad \frac{\Gamma, x : A, y : A \vdash t : B}{\Gamma, x : A \vdash t : B}\ \textsc{Contraction}$$

$$\frac{\Gamma_1, y : B, x : A, \Gamma_2 \vdash t : C}{\Gamma_1, x : A, y : B, \Gamma_2 \vdash t : C}\ \textsc{Exchange}$$

Figure 2.1: Substructural rules for weakening, contraction, and exchange

Linear logic is thus known as a *substructural* logic because it lacks the weakening and contraction rules, while permitting exchange. The disallowance of these rules means that in order to construct a typing derivation, each assumption must be used exactly once: arbitrarily copying or discarding values is disallowed, excluding a vast number of programs from being typeable.

As one can imagine, the different combinations of permitted structural rules yields various other substructural logics. Affine logic permits exchange and weakening, but disallows contraction, resulting in system where values may be used *at most* once. Relevant logic only disallows weakening (*at least* once), while Ordered logic disallows all three (values must be used exactly once and in order).

2.2.1 *The Linear λ-Calculus*

By only permitting the exchange structural rule, we can integrate linear logic into STLC, yielding a *linear* $\lambda-$calculus. This provides us with a foundation for programming with substructural logics, which we will later refine with graded types.

The types of the linear λ-calculus are given by the grammar:

$$A, B ::= A \multimap B \qquad \text{(types)}$$

Like STLC, we have one type: the function type. Here, however, our function type is a *linear* function arrow (denoted by $\multimap$).[1].

The syntax of terms is the same as STLC, given by:

$$t ::= x \mid \lambda x.t \mid t_1\, t_2 \qquad \text{(terms)}$$

Typing judgments are of the form $\Gamma \vdash t : A$, where Γ ranges over contexts of assumptions:

$$\Gamma ::= \emptyset \mid \Gamma, x : A \qquad \text{(contexts)}$$

Thus, a context may be empty $\emptyset$, extended with a linear assumption $x : A$. Throughout, comma denotes disjoint context concatenation.

$$\frac{}{x : A \vdash x : A}\ \textsc{Var}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \textsc{Abs} \qquad \frac{\Gamma_1 \vdash t_1 : A \multimap B \quad \Gamma_2 \vdash t_2 : A}{\Gamma_1 + \Gamma_2 \vdash t_1\, t_2 : B}\ \textsc{App}$$

Figure 2.2: Typing rules for the linear λ-calculus

Figure 2.2 defines the typing rules. Linear variables are typed in a singleton context (VAR). Abstraction (ABS) binds a fresh linear variable which is used to type the premise. Application (APP) combines the usages across both premises through the use of *context addition* (2.3.1). Context addition provides an analogue to contraction, combining contexts that have come from typing multiple sub-terms in a rule.

1 Although Granule code uses $\rightarrow$ syntax rather than $\multimap$ for linear functions for the sake of familiarity with standard functional languages

Context addition, written $\Gamma_1 + \Gamma_2$, is undefined if Γ_1 and Γ_2 overlap in their assumptions, i.e. a linear assumption may not appear in both sides of the addition.

Definition 2.2.1 (Linear context addition)**.**

$$\begin{aligned}
\Gamma + \emptyset &= \Gamma \\
\emptyset + \Gamma &= \Gamma \\
(\Gamma, x : A) + \Gamma' &= (\Gamma + \Gamma'), x : A \quad \text{iff } x \notin |\Gamma'| \\
\Gamma + (\Gamma', x : A) &= (\Gamma + \Gamma'), x : A \quad \text{iff } x \notin |\Gamma|
\end{aligned}$$

This leaves us with a very simple resourceful typing calculus, which serves as the building block for further refinement.

2.3 FROM LINEARITY TO GRADES

Now that we have seen the linear λ-calculus itself, we are ready to generalise the notion of data as a resource inside a program, and formally define a graded type system. We now present two calculi, each based on the differing view of graded types mentioned at the top of this chapter. The first directly follows from the calculus in 2.2.1, merely extending it with a *graded modal type*. The second reflects the other approach to graded type systems, where we do away with the underlying linear structure.

2.3.1 *The Graded Linear λ-calculus*

We now define a core type system, based on the linear λ-calculus of section 2.2.1, extended with a graded modal type. This calculus is equivalent to the core calculus of Granule, GrMini [Orchard et al., 2019]. Granule's full type system extends this graded linear core with polymorphism, algebraic data types, indexed types, pattern matching, and recursion. We refer to the system in this section as the *graded linear λ-calculus*, reflecting the underlying linear structure of the system.

This system forms the basis of the target language for our synthesis tool in Chapter 3, although we extend it with some basic types for increased expressivity.

The types of the graded linear λ-calculus are given by:

$$A, B ::= A \multimap B \mid \Box_r A \qquad \text{(types)}$$

where the type $\Box_r A$ is an indexed family of type operators where r is a *grade* ranging over the elements of a pre-ordered semiring $(\mathcal{R}, *, 1, +, 0, \sqsubseteq)$ parametrising the calculus (where $*$ and $+$ are monotonic with respect to the pre-order $\sqsubseteq$). The syntax of terms provides the elimination and introduction forms:

$$t ::= x \mid \lambda x.t \mid t_1\, t_2 \mid [t] \mid \textbf{let}\, [x] = t_1\, \textbf{in}\, t_2 \qquad \text{(terms)}$$

In addition the the terms of the linear λ-calculus, we also have the construct $[t]$ which introduces a graded modal type $\Box_r A$ by 'promoting' a term t to the graded modality, and it's dual $\textbf{let}\,[x] = t_1 \,\textbf{in}\, t_2$ eliminates a graded modal value t_1, binding a graded variable x in scope of t_2. The typing rules relate these terms to types.

As before, typing judgments are of the form $\Gamma \vdash t : A$, where Γ ranges over contexts:

$$\Gamma ::= \emptyset \mid \Gamma, x : A \mid \Gamma, x :_r A \qquad \text{(contexts)}$$

Thus, a context may also be extended with a graded assumption $x :_r A$. For linear assumptions, structural rules of weakening and contraction are disallowed. Graded assumptions may be used non-linearly according to the constraints provided by their grade, the semiring element r.

Various operations on contexts are used to capture non-linear data flow via grading. As with the linear λ-calculus, context addition (2.3.1) combines the contexts used to provide multiple sub-terms. However, our new definition includes an extra case for dealing with graded assumptions appearing in both contexts, which are combined via the semiring $+$ of their grades.

Definition 2.3.1 (Graded linear context addition).

$$\begin{aligned}
\Gamma + \emptyset &= \Gamma \\
\emptyset + \Gamma &= \Gamma \\
(\Gamma, x : A) + \Gamma' &= (\Gamma + \Gamma'), x : A \quad \text{iff } x \notin |\Gamma'| \\
\Gamma + (\Gamma', x : A) &= (\Gamma + \Gamma'), x : A \quad \text{iff } x \notin |\Gamma| \\
(\Gamma, x :_r A) + (\Gamma', x :_s A) &= (\Gamma + \Gamma'), x :_{(r+s)} A
\end{aligned}$$

Note that this is a declarative specification of context addition. Graded assumptions may appear in any position in Γ and Γ' as witnessed by the algorithmic specification where for all Γ_1, Γ_2 *context addition* is defined as follows by ordered cases matching inductively on the structure of Γ_2, where $\Gamma_1 + \Gamma_2 =$

$$\begin{cases}
\Gamma_1 & \Gamma_2 = \emptyset \\
((\Gamma_1', \Gamma_1'') + \Gamma_2'), x :_{(r+s)} A & \Gamma_2 = \Gamma_2', x :_s A \wedge \Gamma_1 = \Gamma_1', x :_r A, \Gamma_1'' \\
(\Gamma_1 + \Gamma_2'), x : A & \Gamma_2 = \Gamma_2', x : A \ \wedge \ x : A \notin \Gamma_1
\end{cases}$$

$$\frac{}{x : A \vdash x : A}\ \textsc{Var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \textsc{Abs}$$

$$\frac{\Gamma_1 \vdash t_1 : A \multimap B \quad \Gamma_2 \vdash t_2 : A}{\Gamma_1 + \Gamma_2 \vdash t_1\, t_2 : B}\ \textsc{App}$$

$$\frac{\Gamma \vdash t : A}{\Gamma, [\Delta]_0 \vdash t : A}\ \textsc{Weak} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma, x :_1 A \vdash t : B}\ \textsc{Der}$$

$$\frac{\Gamma, x :_r A, \Gamma' \vdash t : A \quad r \sqsubseteq s}{\Gamma, x :_s A, \Gamma' \vdash t : A}\ \textsc{Approx}$$

$$\frac{[\Gamma] \vdash t : A}{r \cdot [\Gamma] \vdash [t] : \Box_r A}\ \textsc{Pr} \qquad \frac{\Gamma_1 \vdash t_1 : \Box_r A \quad \Gamma_2, x :_r A \vdash t_2 : B}{\Gamma_1 + \Gamma_2 \vdash \mathbf{let}\,[x] = t_1\,\mathbf{in}\,t_2 : B}\ \textsc{Let}\Box$$

Figure 2.3: Typing rules of the graded linear λ-calculus

Figure 2.3 gives the typing rules. The WEAK rule captures weakening of assumptions graded by 0 (where $[\Delta]_0$ denotes a context containing only graded assumptions graded by 0). Context addition and WEAK together therefore provide the rules of substructural rules of contraction and weakening. Dereliction (DER), allows a linear assumption to be converted to a graded assumption with grade 1. Grade approximation is captured by the APPROX rule, which allows a grade s to be converted to another grade r, providing that r *approximates* s, where the relation $\sqsubseteq$ is the pre-order provided with the semiring. This relation is occasionally lifted pointwise to contexts: we write $\Gamma \sqsubseteq \Gamma'$ to mean that Γ' overapproximates Γ meaning that for all $(x :_r A) \in \Gamma$ then $(x :_{r'} A) \in \Gamma'$ and $r \sqsubseteq r'$.

Introduction and elimination of the graded modality is provided by the PR and LET rules respectively. The PR rule propagates the grade r to the assumptions through *scalar multiplication* of Γ by r where every assumption in Γ must already be graded (written $[\Gamma]$ in the rule[2]), given by Definition 2.3.2.

Definition 2.3.2 (Scalar context multiplication)**.** A context which consists solely of graded assumptions can be multiplied by a semiring grade $r \in \mathcal{R}$

$$r \cdot \emptyset = \emptyset \qquad\qquad r \cdot (\Gamma, x :_s A) = (r \cdot \Gamma), x :_{(r \cdot s)} A$$

The LET rule eliminates a graded modal value $\Box_r A$ into a graded assumption $x :_r A$ with a matching grade in the scope of the **let** body. This is also referred to as “unboxing”.

2 $[\Gamma]$ here can be thought of as a partial operation on contexts, equivalent to the identity when Γ consists solely of graded assumptions, and undefined if it does not.

We give an example of graded modalities using a graded modality indexed by the semiring of natural numbers.

Example 2.3.1. The natural number semiring with discrete ordering $(\mathbb{N}, *, 1, +, 0, \equiv)$ provides a graded modality that counts exactly how many times non-linear values are used. As an example, the S combinator from the SKI system of combinatory logic is typed and defined:

$$\begin{aligned} &s : (A \multimap (B \multimap C)) \multimap (A \multimap B) \multimap (\Box_2 A \multimap C) \\ &s = \lambda x.\lambda y.\lambda z'.\mathbf{let}\,[z] = z'\,\mathbf{in}\,(x\,z)\,(y\,z) \end{aligned}$$

The graded modal value z' captures the 'capability' for a value of type A to be used twice. This capability is made available by eliminating $\Box$ (via **let**) to the variable z, which has grade 2 in the scope of the body.

2.3.2 *The Fully Graded λ-calculus*

We now define a core calculus for a fully graded type system, where grades permeate the entire program, drawing from the coeffect calculus of Petricek et al. [2014], Quantitative Type Theory (QTT) by McBride [2016] and refined further by Atkey [2018] (although we omit dependent types from our language), the calculus of Abel and Bernardy [2020], and other graded dependent type theories [Atkey, 2018, Moon et al., 2021]. Similar systems also form the basis of the core of the linear types extension to Haskell [Bernardy et al., 2018]. We refer to this system as the *fully graded λ-calculus*.

The syntax of types is given by:

$$A, B ::= A^r \to B \mid \Box_r A \qquad (types)$$

where the function arrow $A^r \to B$ annotates the input type with a *grade* r which is again drawn from a pre-ordered semiring $(\mathcal{R}, *, 1, +, 0, \sqsubseteq)$ parametrising the calculus. The graded necessity modality $\Box_r A$ is similarly annotated by the grade r being an element of the semiring.

The syntax of terms is given as:

$$t ::= x \mid \lambda x.t \mid t_1\,t_2 \mid [t] \mid \mathbf{let}\,[x] = t_1\,\mathbf{in}\,t_2 \qquad (terms)$$

Terms comprise the λ-calculus, extended with the *promotion* construct [t] as seen in Section 2.3.1. Typing judgements have the same form as Section 2.3.1, however, variable contexts are instead given by:

$$\Delta, \Gamma ::= \emptyset \mid \Gamma, x :_r A \qquad (contexts)$$

That is, a context may be empty $\emptyset$ or extended with a *graded* assumption $x :_r A$, which must be used in a way which adheres to the constraints of the grade r. As before, structural exchange is permitted, allowing a context to be arbitrarily reordered.

$$\frac{}{0 \cdot \Gamma, x :_1 A \vdash x : A}\ \textsc{Var} \qquad \frac{\Gamma, x :_r A \vdash t : B}{\Gamma \vdash \lambda x.t : A^r \to B}\ \textsc{Abs}$$

$$\frac{\Gamma_1 \vdash t_1 : A^r \to B \qquad \Gamma_2 \vdash t_2 : A}{\Gamma_1 + r \cdot \Gamma_2 \vdash t_1\, t_2 : B}\ \textsc{App}$$

$$\frac{\Gamma, x :_r A, \Gamma' \vdash t : B \qquad r \sqsubseteq s}{\Gamma, x :_s A, \Gamma' \vdash t : B}\ \textsc{Approx} \qquad \frac{\Gamma \vdash t : A}{r \cdot \Gamma \vdash [t] : \Box_r A}\ \textsc{Pr}$$

$$\frac{\Gamma_1 \vdash t_1 : \Box_r A \quad \Gamma_2, x :_r A \vdash t_2 : B}{\Gamma_1 + \Gamma_2 \vdash \mathbf{let}\, [x] = t_1\, \mathbf{in}\, t_2 : B}\ \textsc{Let}\Box$$

Figure 2.4: Typing rules for the fully graded λ-calculus

Figure 2.4 gives the full typing rules, which explains the meaning of the syntax with reference to their static semantics.

Variables (rule VAR) are typed in a context where the variable x has grade 1 denoting its single usage here. All other variable assumptions are given the grade of the 0 semiring element (providing *weakening*), using *scalar multiplication* of contexts by a grade, re-using Definition 2.3.2 from Section 2.3.1.

Abstraction (ABS) captures the assumption's grade r onto the function arrow in the conclusion, that is, abstraction binds a variable x which may be used in the body t according to grade r. Application again (APP) makes use of context addition to combine the contexts used to type the two sub-terms in the premises of the application rule:

Definition 2.3.3 (Graded context addition)**.** For all Γ_1, Γ_2 *graded context addition* is defined as follows by ordered cases matching inductively on the structure of Γ_2, where $\Gamma_1 + \Gamma_2 =$

$$\begin{cases} \Gamma_1 & \Gamma_2 = \emptyset \\ ((\Gamma_1', \Gamma_1'') + \Gamma_2'), x :_{(r+s)} A & \Gamma_2 = \Gamma_2', x :_s A \wedge \Gamma_1 = \Gamma_1', x :_r A, \Gamma_1'' \\ (\Gamma_1 + \Gamma_2'), x :_s A & \Gamma_2 = \Gamma_2', x :_s A \wedge x \notin \mathsf{dom}(\Gamma_1) \end{cases}$$

Note that 2.3.3 differs only from 2.3.1, in that the former need not consider linear assumptions.

Explicit introduction of graded modalities is achieved via the rule for promotion (PR). This rule is almost identical to that of 2.3.1 with the only difference being here Γ is known to always contain only graded assumptions. Explicit unboxing (LET$\Box$), and approximation (APPROX) are likewise identical to the calculus of 2.3.1.

2.4 TWO TYPING CALCULI

Having outlined the two lineages of graded type systems, we are left with the question: what approach should we use as the basis of a target language for a program synthesis tool? Both systems embed properties for reasoning about program structure into the language, however, they differ in how this information is expressed, as shown by the variance in typing and syntax between Sections 2.3.1 and 2.3.2.

Rather than focus entirely on one approach, we opt to instead build synthesis tools which target both systems. As we have seen, systems based on both approaches are in frequent use today, and both pose their unique challenges in designing a synthesis tool, which makes favouring a particular approach difficult to justify. Furthermore, the target programming language Granule of our implementations includes both approaches. [3]

By this point we are in a strong position to tackle the problem of program synthesis, beginning with a simple introduction to the core issues at the heart of resourceful program synthesis in the next chapter, using our graded linear λ-calculus.

3 As of Granule v0.9.3.0

3 A CORE SYNTHESIS CALCULUS

We begin our first exploration into program synthesis with a system for the graded linear λ-calculus of Section 2.3.1. The primary aim of this chapter is to introduce the core concepts of type-directed program synthesis in a resourceful setting, in particular, the problem of *resource management*. We therefore prioritise simplicity over expressivity for our target language, with the core typing calculus of Section 2.3.1 forming an ideal candidate.

As mentioned in Chapter 1, type-directed program synthesis can be framed as an inversion of type checking. In type checking, we have a judgement of the form:

$$\Gamma \vdash t : A \qquad \text{(type checking)}$$

which states that under some context of assumptions Γ we can assign the program term t the type A. Here, Γ and t constitute the "inputs" to the judgement, while the type A forms the "output". Synthesis inverts this judgement form, leaving us with a *synthesis judgement* form:

$$\Gamma \vdash A \Rightarrow t \qquad \text{(synthesis)}$$

which states that we can construct a program term t from the type A, using the assumptions in Γ. As in type checking, Γ forms an input to the judgement. However, A and t exchange roles: the former is now also an input, while the latter is the judgement's output. Program synthesis then becomes a task of inductively enumerating programs in a "bottom-up" starting from the goal type A: A is broken into sub-goals, from which sub-terms are synthesised until the goal can not be broken into further sub-goals. At this point, we either synthesise a usage of a variable from Γ if possible, or synthesis fails. This is the essence of type-directed program synthesis.

Resourceful types introduce another dimension to synthesis: how do we ensure that the assumptions in Γ are used according to their resource constraints in the synthesised term t? I.e. if $x : A$ is a linear assumption in Γ that is used in some way to construct t, then the synthesis algorithm must synthesise a t which uses x exactly once. Likewise, if $x :_r A$ is a graded assumption, then it must be used in t in a way which satisfies its grade r.

This problem has been explored before in the context of automated theorem proving for linear logic, and has been termed the *resource management problem*. We describe this problem in detail in Section 3.2 and propose two candidate solutions, basing our approach on the *input-output context management* model described by Hodas and Miller [1994], and further developed by Cervesato et al. [2000].

The challenges posed by ensuring the well-resourcedness of synthesised programs are most exemplified by the inclusion in our target language of multiplicative conjunction, and additive disjunction. Therefore, prior to fully describing the problem of resource management and our proposed solutions, we first expand our target language with multiplicative product ($\otimes$), and unit types (1), as well as disjunctive sum types ($\oplus$). These extensions are detailed in Section 3.1, which will be the target language of the synthesis calculi of this chapter. As well as helping to conceptualise the challenges posed by program synthesis in a resourceful setting, these have the added benefit of allowing the synthesis of more expressive programs, without introducing unnecessary complexity at this stage.

Having outlined both a suitable target language and two approaches to dealing with the issue of resource management, we then present two synthesis calculi in Section 3.3 as augmented inversions of the typing rules. Each calculus is based on a one of our proposed solutions to the resource management problem, which we then evaluate and contrast against each other in Section 3.6.

Both calculi are implemented as part of a synthesis tool for Granule [1]. The implementation is a fairly direct translation of the synthesis calculi into Haskell. We thus elide the details of the implementation, focusing only on an important optimisation technique in Section 3.5: focusing. Focusing removes much of the unnecessary non-determinism present in our synthesis rules by fixing an ordering on the application of rules. We present the two *focused* forms of our original synthesis calculi which comprise the basis of our Granule implementation.

3.1 A CORE TARGET LANGUAGE

The syntax for the full language is given by the following grammar:

$$\begin{aligned} t ::= {} & x \mid \lambda x.t \mid t_1\, t_2 \\ & \mid [t] \mid \mathbf{let}\,[x] = t_1\,\mathbf{in}\,t_2 \\ & \mid (t_1, t_2) \mid \mathbf{let}\,(x_1, x_2) = t_1\,\mathbf{in}\,t_2 \\ & \mid () \mid \mathbf{let}\,() = t_1\,\mathbf{in}\,t_2 \\ & \mid \mathbf{inl}\,t \mid \mathbf{inr}\,t \mid \mathbf{case}\,t_1\,\mathbf{of}\,\mathbf{inl}\,x_1 \rightarrow t_2;\ \mathbf{inr}\,x_2 \rightarrow t_3 \end{aligned} \qquad \text{(terms)}$$

1 The exact implementation of the rules as they stand is deprecated, but may be found in Granule release v0.7.8.0: https://github.com/granule-project/granule/releases/tag/v0.7.8.0

We use the syntax $()$ for the inhabitant of the multiplicative unit 1. Pattern matching via a **let** is used to eliminate products and unit types; for sum types, **case** is used to distinguish the constructors.

$$\frac{\Gamma_1 \vdash t_1 : A \qquad \Gamma_2 \vdash t_2 : B}{\Gamma_1 + \Gamma_2 \vdash (t_1, t_2) : A \otimes B}\ \textsc{Pair}$$

$$\frac{\Gamma_1 \vdash t_1 : A \otimes B \quad \Gamma_2, x_1 : A, x_2 : B \vdash t_2 : C}{\Gamma_1 + \Gamma_2 \vdash \mathbf{let}\,(x_1, x_2) = t_1\,\mathbf{in}\,t_2 : C}\ \textsc{LetPair}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathbf{inl}\,t : A \oplus B}\ \textsc{Inl} \qquad \frac{\Gamma \vdash t : B}{\Gamma \vdash \mathbf{inr}\,t : A \oplus B}\ \textsc{Inr}$$

$$\frac{\Gamma_1 \vdash t_1 : A \oplus B \qquad \Gamma_2, x_1 : A \vdash t_2 : C \qquad \Gamma_3, x_2 : B \vdash t_3 : C}{\Gamma + (\Gamma_2 \sqcup \Gamma_3) \vdash\ \mathbf{case}\ t_1\ \mathbf{of\ inl}\ x_1 \to t_2;\ \mathbf{inr}\ x_2 \to t_3 : C}\ \textsc{Case}$$

$$\frac{}{\emptyset \vdash () : 1}\ 1 \qquad \frac{\Gamma_1 \vdash t_1 : 1 \quad \Gamma_2 \vdash t_2 : A}{\Gamma_1 + \Gamma_2 \vdash \mathbf{let}\,() = t_1\,\mathbf{in}\,t_2 : A}\ \textsc{Let1}$$

Figure 3.1: Typing rules of for $\otimes$, $\oplus$, and 1

Figure 3.1 gives the typing rules. Rules for multiplicative products (pairs) and additive coproducts (sums) are routine, where pair introduction (Pair) adds the contexts used to type the pair's constituent sub-terms. Pair elimination (LetPair) binds a pair's components to two linear variables in the scope of the body t_2. The Inl and Inr rules handle the typing of constructors for the sum type $A \oplus B$. Elimination of sums (Case) takes the least upper bound (defined above) of the contexts used to type the two branches of the case.

In the typing of **case** expressions, the *least-upper bound* of the two contexts used to type each branch is used, defined:

Definition 3.1.1 (Partial least-upper bounds of contexts)**.** For all Γ_1, Γ_2, $\Gamma_1 \sqcup \Gamma_2 =$

$$\begin{cases} \emptyset & \Gamma_1 = \emptyset & \wedge\ \Gamma_2 = \emptyset \\ (\emptyset \sqcup \Gamma_2'), x :_{0 \sqcup s} A & \Gamma_1 = \emptyset & \wedge\ \Gamma_2 = \Gamma_2', x :_s A \\ (\Gamma_1' \sqcup (\Gamma_2', \Gamma_2'')), x : A & \Gamma_1 = \Gamma_1', x : A & \wedge\ \Gamma_2 = \Gamma_2', x : A, , \Gamma_2'' \\ (\Gamma_1' \sqcup (\Gamma_2', \Gamma_2'')), x :_{r \sqcup s} A & \Gamma_1 = \Gamma_1', x :_r A & \wedge\ \Gamma_2 = \Gamma_2', x :_s A, \Gamma_2'' \end{cases}$$

where $r \sqcup s$ is the least-upper bound of grades r and s if it exists, derived from $\sqsubseteq$.

As an example of the partiality of $\sqcup$, if one branch of a **case** uses a linear variable, then the other branch must also use it to maintain linearity overall, otherwise the upper-bound of the two contexts for these branches is not defined.

With these extensions in place, we now have the capacity to write more idiomatic functional programs in our target language. As a demonstration of this, and to showcase how graded modalities interact with these new type extensions, we provide two further examples of different graded modalities which complement these new types.

Example 3.1.1. Exact usage analysis is less useful when control-flow is involved, e.g., eliminating sum types where each control-flow branch uses variables differently. The above $\mathbb{N}$-semiring can be imbued with a notion of *approximation* via less-than-equal ordering, providing upper bounds. A more expressive semiring is that of natural number intervals Orchard et al. [2019], given by pairs $\mathbb{N} \times \mathbb{N}$ written $r...s$ here for the lower-bound $r \in \mathbb{N}$ and upper-bound usage $s \in \mathbb{N}$ with $0 = 0...0$ and $1 = 1...1$, addition and multiplication defined pointwise, and ordering $r...s \sqsubseteq r'...s' = r' \leq r \wedge s \leq s'$. Thus a coproduct elimination function can be written and typed:

$$\begin{aligned}
&\oplus_e : \Box_{0...1}(A \multimap C) \multimap \Box_{0...1}(B \multimap C) \multimap (A \oplus B) \multimap C \\
&\oplus_e = \lambda x'.\lambda y'.\lambda z.\textbf{let } [x] = x' \textbf{ in} \\
&\qquad\qquad\qquad\qquad \textbf{let } [y] = y' \textbf{ in} \\
&\qquad\qquad\qquad\qquad\quad \textbf{case } z \textbf{ of inl } u \to x\ u \mid \textbf{inr } v \ \to y\ v
\end{aligned}$$

Example 3.1.2. Graded modalities can capture a form of information-flow security, tracking the flow of labelled data through a program Orchard et al. [2019], with a lattice-based semiring on $\mathcal{R} = \{\mathsf{Unused} \sqsubseteq \mathsf{Hi} \sqsubseteq \mathsf{Lo}\}$ where $0 = \mathsf{Unused}$, $1 = \mathsf{Hi}$, $+ = \sqcup$ and if $r = \mathsf{Unused}$ or $s = \mathsf{Unused}$ then $r \cdot s = \mathsf{Unused}$ otherwise $r \cdot s = \sqcup$. This allows the following well-typed program, eliminating a pair of Lo and Hi security values, picking the left one to pass to a continuation expecting a Lo input:

$$\begin{aligned}
&\mathit{noLeak} : (\Box_{\mathsf{Lo}} A \otimes \Box_{\mathsf{Hi}} A) \multimap (\Box_{\mathsf{Lo}}(A \otimes 1) \multimap B) \multimap B \\
&\mathit{noLeak} = \lambda z.\lambda u.\textbf{let } <x',\ y'> \ = \ z \textbf{ in} \\
&\qquad\qquad\qquad \textbf{let } [x] \ = \ x' \textbf{ in} \\
&\qquad\qquad\qquad\quad \textbf{let } [y] \ = \ y' \textbf{ in } u\ [(x,\ ())]
\end{aligned}$$

3.1.1 *Metatheory*

Finally, the admissibility of substitution is a key result that holds for this language [Orchard et al., 2019], which is leveraged in soundness of the synthesis calculi.

Lemma 3.1.1 (Admissibility of substitution). *Let $\Delta \vdash t' : A$, then:*

- *(Linear) If $\Gamma, x : A, , \Gamma' \vdash t : B$ then $\Gamma + \Delta + \Gamma' \vdash [t'/x]t : B$*

- *(Graded) If* $\Gamma, x :_r A, , \Gamma' \vdash t : B$ *then* $\Gamma + (r \cdot \Delta) + \Gamma' \vdash [t'/x]t : B$

3.2 THE RESOURCE MANAGEMENT PROBLEM

In Chapter 1 we considered a synthesis rule for pairs and highlighted how graded types could be use to control the number of times assumptions are used in the synthesised term.

Chapter 1 considered (Cartesian) product types $\times$, but in our target language we use the multiplicative product of linear types, given in Figure 3.1Each sub-term is typed by a different context Γ_1 and Γ_2 which are then combined via *disjoint* union: the pair cannot be formed if variables are shared between Γ_1 and Γ_2. This prevents the structural behaviour of *contraction* (where a variable appears in multiple sub-terms). Naïvely inverting this typing rule into a synthesis rule yields:

$$\frac{\Gamma_1 \vdash A \Rightarrow t_1 \qquad \Gamma_2 \vdash B \Rightarrow t_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B \Rightarrow (t_1,\ t_2)}\ \otimes\textsc{Intro}$$

As a declarative specification, the $\otimes\textsc{Intro}$ synthesis rule is sufficient. However, this rule embeds a considerable amount of non-determinism when considered from an algorithmic perspective. Reading 'clockwise' starting from the bottom-left, given some context Γ and a goal $A \otimes B$, we have to split the context into disjoint subparts Γ_1 and Γ_2 such that $\Gamma = \Gamma_1, \Gamma_2$ in order to pass the Γ_1 and Γ_2 to the sub-goals for A and B. For a context of size n there are 2^n possible such partitions! This quickly becomes intractable. Instead, Hodas and Miller [1994] developed a technique for linear logic programming, refined by Cervesato et al. [2000], where proof search for linear logic has both an *input context* of available resources and an *output context* of the remaining resources, which we write as judgements of the form $\Gamma \vdash A \Rightarrow^- t \mid \Gamma'$ for input context Γ and output context Γ'. Synthesis for multiplicative products then becomes:

$$\frac{\Gamma_1 \vdash A \Rightarrow^- t_1 \mid \Gamma_2 \qquad \Gamma_2 \vdash B \Rightarrow^- t_2 \mid \Gamma_3}{\Gamma_1 \vdash A \otimes B \Rightarrow^- (t_1, t_2) \mid \Gamma_3}\ \otimes^-_{\textsc{Intro}}$$

where the remaining resources after synthesising for A the first term t_1 are Γ_2 which are then passed as the resources for synthesising the second term B. There is an ordering implicit here in 'threading through' the contexts between the premises. For example, starting with a context $x : A, y : B$, then this rule can be instantiated as:

$$\frac{x : A, y : B \vdash A \Rightarrow^- x \mid y : B \qquad y : B \vdash B \Rightarrow^- y \mid \emptyset}{x : A, y : B \vdash A \otimes B \Rightarrow^- (x, y) \mid \emptyset}\ \otimes^-_{\textsc{Intro}}$$

(example)

Thus this approach neatly avoids the problem of having to split the input context, and facilitates efficient proof search for linear types. We extend this input-output context management model to graded types to graded types to facilitate the synthesis of programs in Granule. We term the above approach *subtractive* resource management (in a style similar to *left-over* type checking for linear type systems [Allais, 2018, Zalakain and Dardha, 2020]).

Graded type systems, as we consider them here, have typing contexts in which free-variables are assigned a type, and a grade. In a graded setting, the subtractive approach is problematic as there is not necessarily a notion of actual subtraction for grades. Consider a version of the above example for subtractively synthesising a pair, but now for a context with some grades r and s on the input variables. Using a variable to synthesise a sub-term now does not result in that variable being left out of the output context. Instead a new grade must be assigned in the output context that relates to the first by means of an additional constraint describing that some usage took place:

$$\dfrac{\exists s'.s' + 1 = s \qquad \begin{array}{c}\exists r'.r' + 1 = r \\ x :_r A, y :_s B \vdash A \Rightarrow^- x \mid x :_{r'} A, y :_s B\end{array} \quad \\ x :_{r'} A, y :_s B \vdash B \Rightarrow^- y \mid x :_{r'} A, y :_{s'} B}{x :_r A, y :_s B \vdash A \otimes B \Rightarrow^- (x, y) \mid x :_{r'} A, y :_{s'} B}\;\otimes^-_{\text{INTRO}} \qquad \text{(example)}$$

In the first synthesis premise, x has grade r in the input context, x is synthesised for the goal, and thus the output context has some grade r' where $r' + 1 = r$, denoting that some usage of x occurred (which is represented by the 1 element of the semiring in graded systems).

For the natural numbers semiring, with $r = 1$ and $s = 1$ then the constraints above are satisfied with $r' = 0$ and $s' = 0$. In a general setting, this subtractive approach to synthesis for graded types requires solving many such existential equations over semirings, which also introduces a new source of non-determinism is there is more than one solution. These constraints can be discharged via an off-the-shelf SMT solver, such as Z3 [de Moura and Bjørner, 2008]. Such calls to an external solver are costly, however, and thus efficiency of resource management is a key concern.

We propose a dual approach to the subtractive: the *additive* resource management scheme. In the additive approach, output contexts describe what was *used* not what was is *left*. In the case of synthesising a term with multiple sub-terms (like pairs), the output context from each premise is then added together using the semiring addition operation applied pointwise on contexts to produce the final output in the conclusion. For pairs this looks like:

$$\dfrac{\Gamma \vdash A \Rightarrow^+ t_1 \mid \Delta_1 \qquad \Gamma \vdash B \Rightarrow^+ t_2 \mid \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^+ (t_1, t_2) \mid \Delta_1 + \Delta_2}\;\otimes^+_{\text{INTRO}}$$

The entirety of Γ is used to synthesise both premises. For example, for a goal of $A \otimes A$:

$$\dfrac{\begin{array}{c} x :_r A, y :_s B \vdash A \Rightarrow^{+} x \mid x :_1 A, y :_0 B \\ x :_r A, y :_s B \vdash A \Rightarrow^{+} x \mid x :_1 A, y :_0 B \end{array}}{x :_r A, y :_s B \vdash A \otimes A \Rightarrow^{+} (x, x) \mid x :_{1+1} A, y :_0 B} \otimes^{+}_{\textsc{Intro}} \tag{example}$$

Later checks in synthesis then determine whether the output context describes usage that is within the grades given by Γ, i.e., that the synthesised terms are *well-resourced*.

Both the subtractive and additive approaches avoid having to split the incoming context Γ into two prior to synthesising sub-terms.

We adapt the input-output context management model of linear logic synthesis to graded types, pruning the search space via the quantitative constraints of grades. We implement synthesis calculi based on both the additive and subtractive approaches, evaluating their performance on a set of benchmarking synthesis problems.

3.2.1 *Related Work*

Before Hodas and Miller, the problem of resource non-determinism was first identified by Harland and Pym [2000]. Their solution delays splitting of contexts at a multiplicative connective. They later explored the implementation details of this approach, proposing a solution where proof search is formulated in terms of constraints on propositions. Propositions which occur in the conclusion of a multiplicative connective are assigned a Boolean expression whose solution Constraints generated during the proof search, with a solution to these constituting a valid proof. The logic programming language Lygon [lyg] implements this approach.

Our approach to synthesis implements a *backward* style of proof search: starting from the goal, recursively search for solutions to sub-goals. In contrast to this, *forward* reasoning approaches attempt to reach the goal by building sub-goals from previously proved sub-goals until the overall goal is proved. Chaudhuri and Pfenning [2005a,b] consider forward approaches to proof search in linear logic using the *inverse method* [Degtyarev and Voronkov, 2001] where the issue of resource non-determinism that is typical to backward approaches is absent.

3.3 THE SYNTHESIS CALCULI

We now present two synthesis calculi based on the subtractive and additive resource management schemes, respectively. The structure of the synthesis calculi mirrors a cut-free sequent calculus, with *left* and *right* rules for each type constructor. Right rules synthesise an introduction form for the goal type. Left rules eliminate (deconstruct) assumptions so that they may be used inductively to synthesise sub-terms. Each type in the core language has right and left rules corresponding to its constructors and destructors respectively.

$$\frac{}{\Gamma, x : A \vdash A \Rightarrow^{-} x \mid \Gamma}\,\textsc{LinVar}^{-} \qquad \frac{\exists s.\, r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash A \Rightarrow^{-} x \mid \Gamma, x :_s A}\,\textsc{GrVar}^{-}$$

$$\frac{\Gamma, x : A \vdash B \Rightarrow^{-} t \mid \Delta \quad x \notin |\Delta|}{\Gamma \vdash A \multimap B \Rightarrow^{-} \lambda x.t \mid \Delta}\,\multimap_R^{-}$$

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^{-} t_1 \mid \Delta_1 \qquad x_2 \notin |\Delta_1| \qquad \Delta_1 \vdash A \Rightarrow^{-} t_2 \mid \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^{-} [(x_1\, t_2)/x_2]t_1 \mid \Delta_2}\,\multimap_L^{-}$$

$$\frac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^{-} t \mid \Delta, x :_{s'} A \qquad y \notin |\Delta| \qquad \exists s.\, r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash B \Rightarrow^{-} [x/y]t \mid \Delta, x :_{s'} A}\,\textsc{der}^{-}$$

$$\frac{\Gamma \vdash A \Rightarrow^{-} t \mid \Delta}{\Gamma \vdash \Box_r A \Rightarrow^{-} [t] \mid \Gamma - r \cdot (\Gamma - \Delta)}\,\Box_R^{-}$$

$$\frac{\Gamma, x_2 :_r A \vdash B \Rightarrow^{-} t \mid \Delta, x_2 :_s A \qquad 0 \sqsubseteq s}{\Gamma, x_1 : \Box_r A \vdash B \Rightarrow^{-} \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t \mid \Delta}\,\Box_L^{-}$$

$$\frac{\Gamma \vdash A \Rightarrow^{-} t_1 \mid \Delta_1 \qquad \Delta_1 \vdash B \Rightarrow^{-} t_2 \mid \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^{-} (t_1, t_2) \mid \Delta_2}\,\otimes_R^{-}$$

$$\frac{\Gamma, x_1 : A, x_2 : B \vdash C \Rightarrow^{-} t_2 \mid \Delta \qquad x_1 \notin |\Delta| \qquad x_2 \notin |\Delta|}{\Gamma, x_3 : A \otimes B \vdash C \Rightarrow^{-} \textbf{let}\,(x_1, x_2) = x_3\,\textbf{in}\,t_2 \mid \Delta}\,\otimes_L^{-}$$

$$\frac{\Gamma \vdash A \Rightarrow^{-} t \mid \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{-} \textbf{inl}\,t \mid \Delta}\,\oplus 1_R^{-} \qquad \frac{\Gamma \vdash B \Rightarrow^{-} t \mid \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{-} \textbf{inr}\,t \mid \Delta}\,\oplus 2_R^{-}$$

$$\frac{\Gamma, x_2 : A \vdash C \Rightarrow^{-} t_1 \mid \Delta_1 \quad \Gamma, x_3 : B \vdash C \Rightarrow^{-} t_2 \mid \Delta_2 \quad x_2 \notin |\Delta_1| \quad x_3 \notin |\Delta_2|}{\Gamma, x_1 : A \oplus B \vdash C \Rightarrow^{-} \textbf{case}\; x_1\; \textbf{of inl}\; x_2 \to t_1;\; \textbf{inr}\; x_3 \to t_2 \mid \Delta_1 \sqcap \Delta_2}\,\oplus_L^{-}$$

$$\frac{}{\Gamma \vdash 1 \Rightarrow^{-} () \mid \Gamma}\,1_R^{-} \qquad \frac{\Gamma \vdash C \Rightarrow^{-} t \mid \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^{-} \textbf{let}\,() = x\,\textbf{in}\,t \mid \Delta}\,1_L^{-}$$

Figure 3.2: Collected rules of the subtractive synthesis calculus

3.3.1 *Subtractive Resource Management*

Our subtractive approach follows the philosophy of earlier work on linear logic proof search [Hodas and Miller, 1994, Cervesato et al., 2000], structuring synthesis rules around an input context of the available resources and an output context of the remaining resources that can be used to synthesise subsequent sub-terms. Synthesis rules are read bottom-up, with judgments $\Gamma \vdash A \Rightarrow^{-} t \mid \Delta$ meaning from the *goal type* A we can synthesise a term t using assumptions in Γ, with output context Δ. We describe the rules in turn to aid understanding. Figure 3.2 collects the rules for reference.

3.3.1.1 *Variables*

Variable terms can be synthesised from assumptions in Γ by rules:

$$\frac{}{\Gamma, x : A \vdash A \Rightarrow^{-} x \mid \Gamma}\ \textsc{LinVar}^{-} \qquad \frac{\exists s.\, r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash A \Rightarrow^{-} x \mid \Gamma, x :_s A}\ \textsc{GrVar}^{-}$$

On the left, a variable x may be synthesised for the goal A if a linear assumption $x : A$ is present in the input context. The input context without x is then returned as the output context, since x has been used. On the right, we can synthesise a variable x for A we have a graded assumption of x matching the type. However, the grading r must permit x to be used once here. Therefore, the premise states that there exists some grade s such that grade r approximates $s + 1$. The grade s represents the use of x in the rest of the synthesised term, and thus $x :_s A$ is in the output context. For the natural numbers semiring, this constraint is satisfied by $s = r - 1$ whenever $r \neq 0$, e.g., if $r = 3$ then $s = 2$. For intervals, the role of approximation is more apparent: if $r = 0...3$ then this rule is satisfied by $s = 0...2$ where $s + 1 = 0...2 + 1...1 = 1...3 \sqsubseteq 0...3$. This is captured by the instantiation of a new existential variable representing the new grade for x in the output context of the rule. In the natural numbers semiring, this could be done by simply subtracting 1 from the assumption's existing grade r. However, as not all semirings have an additive inverse, this is instead handled via a constraint on the new grade s, requiring that $r \sqsupseteq s + 1$. In the implementation, the constraint is discharged via an SMT solver, where an unsatisfiable result terminates this branch of synthesis.

3.3.1.2 *Functions*

In typing, λ-abstraction binds linear variables to introduce linear functions. Synthesis from a linear function type therefore mirrors typing:

$$\frac{\Gamma, x : A \vdash B \Rightarrow^{-} t \mid \Delta \qquad x \notin |\Delta|}{\Gamma \vdash A \multimap B \Rightarrow^{-} \lambda x.t \mid \Delta}\ {\multimap}_{R}^{-}$$

Thus, $\lambda x.t$ can be synthesised given that t can be synthesised from B in the context of Γ extended with a fresh linear assumption $x : A$. To ensure that x is used linearly by t we must therefore check that it is not present in Δ.

The left-rule for linear function types then synthesises applications (as in Hodas and Miller [1994]):

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^- t_1 \mid \Delta_1 \qquad x_2 \notin |\Delta_1| \qquad \Delta_1 \vdash A \Rightarrow^- t_2 \mid \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^- [(x_1\, t_2)/x_2]t_1 \mid \Delta_2} \multimap_L^-$$

The rule synthesises a term for type C in a context that contains an assumption $x_1 : A \multimap B$. The first premise synthesises a term t_1 for C under the context extended with a fresh linear assumption $x_2 : B$, i.e., assuming the result of x_1. This produces an output context Δ_1 that must not contain x_2, i.e., x_2 is used by t_1. The remaining assumptions Δ_1 provide the input context to synthesise t_2 of type A: the argument to the function x_1. In the conclusion, the application $x_1\, t_2$ is substituted for x_2 inside t_1, and Δ_2 is the output context.

3.3.1.3 *Dereliction*

Note that the above rule synthesises the application of a function given by a linear assumption. What if we have a graded assumption of function type? Rather than duplicating every left rule for both linear and graded assumptions, we mirror the dereliction typing rule (converting a linear assumption to graded) as:

$$\frac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^- t \mid \Delta, x :_{s'} A \qquad y \notin |\Delta| \qquad \exists s.\, r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash B \Rightarrow^- [x/y]t \mid \Delta, x :_{s'} A} \textsc{der}^-$$

Dereliction captures the ability to reuse a graded assumption being considered in a left rule. A fresh linear assumption y is generated that represents the graded assumption's use in a left rule, and must be used linearly in the subsequent synthesis of t. The output context of this premise then contains x graded by s', which reflects how x was used in the synthesis of t, i.e. if x was not used then $s' = s$. The premise $\exists s.\, r \sqsupseteq s + 1$ constrains the number of times dereliction can be applied so that it does not exceed x's original grade r.

One may observe that the DER$^-$ rule makes the presence of the GRVAR$^-$ rule admissible. Synthesising the usage of a graded variable can instead be achieved through the use of dereliction on the graded assumption, followed by the LINVAR$^-$. Nonetheless, we find the inclusion of GRVAR$^-$ useful as an explanatory tool and optimisation in the implementation of the calculus.

3.3.1.4 *Graded modalities*

For a graded modal goal type $\Box_r A$, we synthesise a promotion $[t]$ if we can synthesise the 'unpromoted' t from A:

$$\frac{\Gamma \vdash A \Rightarrow^- t \mid \Delta}{\Gamma \vdash \Box_r A \Rightarrow^- [t] \mid \Gamma - r \cdot (\Gamma - \Delta)} \Box_R^-$$

A non-graded value t may be promoted to a graded value using the box syntactic construct. Recall that typing of a promotion $[t]$ scales all the graded assumptions used to type t by r. Therefore, to compute the output context we must "subtract" r-times the use of the variables in t. However, in the subtractive model Δ tells us what is left, rather than what is used. Thus we first compute the *context subtraction* of Γ and Δ yielding the variable usage information about t:

Definition 3.3.1 (Context subtraction)**.** For all Γ_1, Γ_2 where $\Gamma_2 \subseteq \Gamma_1$, $\Gamma_1 - \Gamma_2 =$

$$\begin{cases} \Gamma_1 & \Gamma_2 = \emptyset \\ (\Gamma_1', \Gamma_1'') - \Gamma_2' & \Gamma_2 = \Gamma_2', x : A \quad \wedge \Gamma_1 = \Gamma_1', x : A, \Gamma_1'' \\ ((\Gamma_1', \Gamma_1'') - \Gamma_2'), x :_q A & \Gamma_2 = \Gamma_2', x :_s A \quad \wedge \Gamma_1 = \Gamma_1', x :_r A, \Gamma_1'' \\ & \wedge\ \exists q. r \sqsupseteq q + s \ \wedge \forall q'. r \sqsupseteq q' + s \implies q \sqsupseteq q' \end{cases}$$

As in graded variable synthesis, context subtraction existentially quantifies a variable q to express the relationship between grades on the right being "subtracted" from those on the left. The last conjunct states q is the greatest element (wrt. to the pre-order) satisfying this constraint, i.e., for all other $q' \in \mathcal{R}$ satisfying the subtraction constraint then $q \sqsupseteq q'$ e.g., if $r = 2...3$ and $s = 0...1$ then $q = 2...2$ instead of, say, $0...1$. This *maximality* condition is important for soundness (that synthesised programs are well-typed).

Thus for $\Box_R^-$, $\Gamma - \Delta$ is multiplied by the goal type grade r to obtain how these variables are used in t after promotion. This is then subtracted from the original input context Γ giving an output context containing the left-over variables and grades. Context multiplication requires that $\Gamma - \Delta$ contains only graded variables, preventing the incorrect use of linear variables from Γ in t.

Synthesis of graded modality elimination, is handled by the $\Box_L^-$ left rule:

$$\frac{\Gamma, x_2 :_r A \vdash B \Rightarrow^- t \mid \Delta, x_2 :_s A \qquad 0 \sqsubseteq s}{\Gamma, x_1 : \Box_r A \vdash B \Rightarrow^- \mathbf{let}\, [x_2] = x_1 \,\mathbf{in}\, t \mid \Delta} \Box_L^-$$

Given an input context comprising Γ and a linear assumption x_1 of graded modal type, we can synthesise an unboxing of x_1 if we can synthesise a term t under Γ extended with a graded assumption $x_2 :_r A$. This returns an output context that must contain x_2 graded by s with the constraint that s must approximate 0. This enforces that x_2 has been used as much as stated by the grade r.

3.3.1.5 *Products*

The right rule for products $\otimes_R^-$ behaves similarly to the $\multimap_L^-$ rule, passing the entire input context Γ to the first premise. This is in then used to synthesise the first sub-term of the pair t_1, yielding an output context Δ_1, which is passed to the second premise. After synthesising the second sub-term t_2, the output context for this premise becomes the output context of the rule's conclusion.

The left rule equivalent $\otimes_L^-$ binds two assumptions $x_1 : A\ x_2 : B$ in the premise, representing the constituent sides of the pair. As with $\multimap_L^-$, we also ensure that these bound assumptions must not present in the premise's output context Δ.

$$\frac{\Gamma \vdash A \Rightarrow^- t_1 \mid \Delta_1 \qquad \Delta_1 \vdash B \Rightarrow^- t_2 \mid \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^- (t_1, t_2) \mid \Delta_2}\otimes_R^-$$

$$\frac{\Gamma, x_1 : A, x_2 : B \vdash C \Rightarrow^- t_2 \mid \Delta \qquad x_1 \notin |\Delta| \qquad x_2 \notin |\Delta|}{\Gamma, x_3 : A \otimes B \vdash C \Rightarrow^- \textbf{let}\,(x_1, x_2) = x_3\,\textbf{in}\,t_2 \mid \Delta}\otimes_L^-$$

3.3.1.6 *Sums*

The introduction rules for sum types, $\oplus1_R^-$ and $\oplus2_R^-$, are straightforward:

$$\frac{\Gamma \vdash A \Rightarrow^- t \mid \Delta}{\Gamma \vdash A \oplus B \Rightarrow^- \textbf{inl}\,t \mid \Delta}\oplus1_R^- \qquad \frac{\Gamma \vdash B \Rightarrow^- t \mid \Delta}{\Gamma \vdash A \oplus B \Rightarrow^- \textbf{inr}\,t \mid \Delta}\oplus2_R^-$$

The $\oplus_L^-$ rule synthesises the left and right branches of a case statement that may use resources differently:

$$\frac{\Gamma, x_2 : A \vdash C \Rightarrow^- t_1 \mid \Delta_1 \quad \Gamma, x_3 : B \vdash C \Rightarrow^- t_2 \mid \Delta_2 \quad x_2 \notin |\Delta_1| \quad x_3 \notin |\Delta_2|}{\Gamma, x_1 : A \oplus B \vdash C \Rightarrow^- \textbf{case}\ x_1\ \textbf{of inl}\ x_2 \to t_1;\ \textbf{inr}\ x_3 \to t_2 \mid \Delta_1 \sqcap \Delta_2}\oplus_L^-$$

The output context therefore takes the *greatest lower bound* ($\sqcap$) of Δ_1 and Δ_2, given by definition 3.3.2,

Definition 3.3.2 (Partial greatest-lower bounds of contexts). For all Γ_1, $\Gamma_2, \Gamma_1 \sqcap \Gamma_2 =$

$$\begin{cases} \emptyset & \Gamma_1 = \emptyset & \wedge\ \Gamma_2 = \emptyset \\ (\emptyset \sqcap \Gamma_2'), x :_{0 \sqcap s} A & \Gamma_1 = \emptyset & \wedge\ \Gamma_2 = \Gamma_2', x :_s A \\ (\Gamma_1' \sqcap (\Gamma_2', \Gamma_2'')), x : A & \Gamma_1 = \Gamma_1', x : A & \wedge\ \Gamma_2 = \Gamma_2', x : A, \Gamma_2'' \\ (\Gamma_1' \sqcap (\Gamma_2', \Gamma_2'')), x :_{r \sqcap s} A & \Gamma_1 = \Gamma_1', x :_r A & \wedge\ \Gamma_2 = \Gamma_2', x :_s A, \Gamma_2'' \end{cases}$$

where $r \sqcap s$ is the greatest-lower bound of grades r and s if it exists, derived from $\sqsubseteq$.

As an example of $\sqcap$, consider the semiring of intervals over natural numbers and two judgements that could be used as premises for the $(\oplus_L^-)$ rule:

$$\Gamma, y :_{0...5} A', x_2 : A \vdash C \Rightarrow^- t_1 \mid y :_{2...5} A'$$
$$\Gamma, y :_{0...5} A', x_3 : B \vdash C \Rightarrow^- t_2 \mid y :_{3...4} A'$$

where t_1 uses y such that there are 2-5 uses remaining and t_2 uses y such that there are 3-4 uses left. To synthesise **case** x_1 **of inl** $x_2 \rightarrow t_1$; **inr** $x_3 \rightarrow t_2$ the output context must be pessimistic about what resources are left, thus we take the greatest-lower bound yielding the interval $[2 \ldots 4]$ here: we know y can be used at least twice and at most 4 times in the rest of the synthesised program.

3.3.1.7 *Unit*

The right and left rules for units are then self-explanatory following the subtractive resource model:

$$\frac{}{\Gamma \vdash 1 \Rightarrow^- () \mid \Gamma} \; 1_R^- \qquad \frac{\Gamma \vdash C \Rightarrow^- t \mid \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^- \textbf{let}\, () = x \,\textbf{in}\, t \mid \Delta} \; 1_L^-$$

This completes subtractive synthesis. We conclude with a key result, that synthesised terms are well-typed at the type from which they were synthesised:

Lemma 3.3.1 (Subtractive synthesis soundness)**.** *For all* Γ *and* A *then:*

$$\Gamma \vdash A \Rightarrow^- t \mid \Delta \quad \Longrightarrow \quad \Gamma - \Delta \vdash t : A$$

i.e. t has type A under context $\Gamma - \Delta$, that contains just those linear and graded variables with grades reflecting their use in t.

The proof of soundness can be found in Section B.1.1 of Appendix B

$$\frac{}{\Gamma, x : A \vdash A \Rightarrow^{+} x;\ x : A}\ \textsc{LinVar}^{+} \qquad \frac{}{\Gamma, x :_r A \vdash A \Rightarrow^{+} x;\ x :_1 A}\ \textsc{GrVar}^{+}$$

$$\frac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^{+} t;\ \Delta, y : A}{\Gamma, x :_s A \vdash B \Rightarrow^{+} [x/y]t;\ \Delta + x :_1 A}\ \textsc{der}^{+}$$

$$\frac{\Gamma, x : A \vdash B \Rightarrow^{+} t;\ \Delta, x : A}{\Gamma \vdash A \multimap B \Rightarrow^{+} \lambda x.t;\ \Delta}\ \multimap_R^{+}$$

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1;\ \Delta_1, x_2 : B \qquad \Gamma \vdash A \Rightarrow^{+} t_2;\ \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^{+} [(x_1\, t_2)/x_2]t_1;\ (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \multimap_L^{+}$$

$$\frac{\Gamma \vdash A \Rightarrow^{+} t;\ \Delta}{\Gamma \vdash \Box_r A \Rightarrow^{+} [t];\ r \cdot \Delta}\ \Box_R^{+}$$

$$\frac{\Gamma, x_2 :_r A \vdash B \Rightarrow^{+} t;\ \Delta \qquad \textit{if } x_2 :_s A \in \Delta \textit{ then } s \sqsubseteq r \textit{ else } 0 \sqsubseteq r}{\Gamma, x_1 : \Box_r A \vdash B \Rightarrow^{+} \mathbf{let}\,[x_2] = x_1\,\mathbf{in}\,t;\ (\Delta \backslash x_2), x_1 : \Box_r A}\ \Box_L^{+}$$

$$\frac{\Gamma \vdash A \Rightarrow^{+} t_1;\ \Delta_1 \qquad \Gamma \vdash B \Rightarrow^{+} t_2;\ \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^{+} (t_1, t_2);\ \Delta_1 + \Delta_2}\ \otimes_R^{+}$$

$$\frac{\Gamma, x_1 : A, x_2 : B \vdash C \Rightarrow^{+} t_2;\ \Delta, x_1 : A, x_2 : B}{\Gamma, x_3 : A \otimes B \vdash C \Rightarrow^{+} \mathbf{let}\,(x_1, x_2) = x_3\,\mathbf{in}\,t_2;\ \Delta, x_3 : A \otimes B}\ \otimes_L^{+}$$

$$\frac{\Gamma \vdash A \Rightarrow^{+} t;\ \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{+} \mathbf{inl}\,t;\ \Delta}\ \oplus1_R^{+} \qquad \frac{\Gamma \vdash B \Rightarrow^{+} t;\ \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{+} \mathbf{inr}\,t;\ \Delta}\ \oplus2_R^{+}$$

$$\frac{\Gamma, x_2 : A \vdash C \Rightarrow^{+} t_1;\ \Delta_1, x_2 : A \qquad \Gamma, x_3 : B \vdash C \Rightarrow^{+} t_2;\ \Delta_2, x_3 : B}{\Gamma, x_1 : A \oplus B \vdash C \Rightarrow^{-} \mathbf{case}\ x_1\ \mathbf{of\ inl}\ x_2 \to t_1;\ \mathbf{inr}\ x_3 \to t_2 \mid \Delta_1 \sqcup \Delta_2, x_1 : A \oplus B}\ \oplus_L^{+}$$

$$\frac{}{\Gamma \vdash 1 \Rightarrow^{+} ();\ \emptyset}\ 1_R^{+} \qquad \frac{\Gamma \vdash C \Rightarrow^{+} t;\ \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^{+} \mathbf{let}\,() = x\,\mathbf{in}\,t;\ \Delta, x : 1}\ 1_L^{+}$$

Figure 3.3: Collected rules of the additive synthesis calculus

3.3.2 *Additive Resource Management*

We now present the dual to subtractive resource management — the *additive* approach. Additive synthesis also uses the input-output context approach, but where output contexts describe exactly which assumptions were used to synthesise a term, rather than which assumptions are still available. Additive synthesis rules are read bottom-up, with $\Gamma \vdash A \Rightarrow^{+} t;\ \Delta$ meaning that from the type A we synthesise a term t using exactly the assumptions Δ that originate from the input context Γ.

3.3.2.1 *Variables*

We unpack the rules, starting with variables:

$$\frac{}{\Gamma, x : A \vdash A \Rightarrow^{+} x;\ x : A}\ \textsc{LinVar}^{+} \qquad \frac{}{\Gamma, x :_r A \vdash A \Rightarrow^{+} x;\ x :_1 A}\ \textsc{GrVar}^{+}$$

For a linear assumption, the output context contains just the variable that was synthesised. For a graded assumption $x :_r A$, the output context contains the assumption graded by 1. To synthesise a variable from a graded assumption, we must check that the use is compatible with the grade.

3.3.2.2 *Graded modalities*

The subtractive approach handled the $\textsc{GrVar}^{-}$ by a constraint $\exists s. r \sqsupseteq s + 1$. Here however, the point at which we check that a graded assumption has been used according to the grade takes place in the $\Box^{+}_{L}$ rule, where graded assumptions are bound:

$$\frac{\Gamma, x_2 :_r A \vdash B \Rightarrow^{+} t;\ \Delta \qquad \textit{if}\ x_2 :_s A \in \Delta\ \textit{then}\ s \sqsubseteq r\ \textit{else}\ 0 \sqsubseteq r}{\Gamma, x_1 : \Box_r A \vdash B \Rightarrow^{+} \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t;\ (\Delta \backslash x_2), x_1 : \Box_r A}\ \Box^{+}_{L}$$

Here, t is synthesised under a fresh graded assumption $x_2 :_r A$. This produces an output context containing x_2 with some grade s that describes how x_2 is used in t. An additional premise requires that the original grade r approximates either s if x_2 appears in Δ or 0 if it does not, ensuring that x_2 has been used correctly. For the $\mathbb{N}$-semiring with equality as the ordering, this would ensure that a variable has been used exactly the number of times specified by the grade.

The synthesis of a promotion is considerably simpler in the additive approach. In subtractive resource management it was necessary to calculate how resources were used in the synthesis of t before then applying the scalar context multiplication by the grade r and subtracting this from the original input Γ. In additive resource management, however, we can simply apply the multiplication directly to the output context Δ to obtain how our assumptions are used in $[t]$:

$$\frac{\Gamma \vdash A \Rightarrow^{+} t;\ \Delta}{\Gamma \vdash \Box_r A \Rightarrow^{+} [t];\ r \cdot \Delta}\ \Box^{+}_{R}$$

3.3.2.3 *Functions*

Synthesis rules for $\multimap$ have a similar shape to the subtractive calculus:

$$\frac{\Gamma, x : A \vdash B \Rightarrow^{+} t;\ \Delta, x : A}{\Gamma \vdash A \multimap B \Rightarrow^{+} \lambda x.t;\ \Delta}\ \multimap^{+}_{R}$$

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1;\ \Delta_1, x_2 : B \qquad \Gamma \vdash A \Rightarrow^{+} t_2;\ \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^{+} [(x_1\,t_2)/x_2]t_1;\ (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \multimap^{+}_{L}$$

Synthesising an abstraction ($\multimap^+_R$) requires that $x : A$ is in the output context of the premise, ensuring that linearity is preserved. Likewise for application ($\multimap^+_L$), the output context of the first premise must contain the linearly bound $x_2 : B$ and the final output context must contain the assumption being used in the application $x_1 : A \multimap B$. This output context computes the *context addition* (Def. 2.3.1) of both output contexts of the premises $\Delta_1 + \Delta_2$. If Δ_1 describes how assumptions were used in t_1 and Δ_2 respectively for t_2, then the addition of these two contexts describes the usage of assumptions for the entire subprogram. Recall, context addition ensures that a linear assumption may not appear in both Δ_1 and Δ_2, preventing us from synthesising terms that violate linearity.

3.3.2.4 *Dereliction*

As in the subtractive calculus, we avoid duplicating left rules to match graded assumptions by giving a synthesising version of dereliction:

$$\frac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^+ t;\ \Delta, y : A}{\Gamma, x :_s A \vdash B \Rightarrow^+ [x/y]t;\ \Delta + x :_1 A}\ \textsc{der}^+$$

The fresh linear assumption $y : A$ must appear in the output context of the premise, ensuring it is used. The final context therefore adds to Δ an assumption of x graded by 1, accounting for this use of x (temporarily renamed to y). As with the subtractive case, $\textsc{der}^+$ makes $\textsc{GrVar}^+$ admissible.

3.3.2.5 *Products*

The right rule for products $\otimes^+_R$ follows the same structure as its subtractive equivalent, however, here Γ is passed to both premises. The conclusion's output context is then formed by taking the context addition of the Δ_1 and Δ_2. The left rule, $\otimes^+_L$ follows fairly straightforwardly from the resource scheme.

$$\frac{\Gamma \vdash A \Rightarrow^+ t_1;\ \Delta_1 \qquad \Gamma \vdash B \Rightarrow^+ t_2;\ \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^+ (t_1, t_2);\ \Delta_1 + \Delta_2}\ \otimes^+_R$$

$$\frac{\Gamma, x_1 : A, x_2 : B \vdash C \Rightarrow^+ t_2;\ \Delta, x_1 : A, x_2 : B}{\Gamma, x_3 : A \otimes B \vdash C \Rightarrow^+ \mathbf{let}\,(x_1, x_2) = x_3\,\mathbf{in}\,t_2;\ \Delta, x_3 : A \otimes B}\ \otimes^+_L$$

3.3.2.6 *Sums*

In contrast to the subtractive rule, the rule $\oplus^+_L$ takes the least-upper bound of the premise's output contexts (see definition 3.1.1). Otherwise, the right and left rules for synthesising programs from sum types are straightforward.

$$\frac{\Gamma \vdash A \Rightarrow^{+} t;\, \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{+} \mathbf{inl}\, t;\, \Delta}\ \oplus 1^{+}_{R} \qquad \frac{\Gamma \vdash B \Rightarrow^{+} t;\, \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{+} \mathbf{inr}\, t;\, \Delta}\ \oplus 2^{+}_{R}$$

$$\frac{\Gamma, x_2 : A \vdash C \Rightarrow^{+} t_1;\, \Delta_1, x_2 : A \qquad \Gamma, x_3 : B \vdash C \Rightarrow^{+} t_2;\, \Delta_2, x_3 : B}{\Gamma, x_1 : A \oplus B \vdash C \Rightarrow^{-} \mathbf{case}\ x_1\ \mathbf{of\ inl}\ x_2 \to t_1;\ \mathbf{inr}\ x_3 \to t_2 \mid \Delta_1 \sqcup \Delta_2, x_1 : A \oplus B}\ \oplus^{+}_{L}$$

3.3.2.7 *Unit*

As in the subtractive approach, the right and left rules for unit types, are as expected.

$$\frac{}{\Gamma \vdash 1 \Rightarrow^{+} ();\, \emptyset}\ 1^{+}_{R} \qquad \frac{\Gamma \vdash C \Rightarrow^{+} t;\, \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^{+} \mathbf{let}\,() = x\,\mathbf{in}\, t;\, \Delta, x : 1}\ 1^{+}_{L}$$

Thus concludes the rules for additive synthesis. As with subtractive, we have prove that this calculus is sound.

Lemma 3.3.2 (Additive synthesis soundness)**.** *For all* Γ *and* A*:*

$$\Gamma \vdash A \Rightarrow^{+} t;\, \Delta \quad \Longrightarrow \quad \Delta \vdash t : A$$

Thus, the synthesised term t is well-typed at A using only the assumptions Δ. , where Δ is a subset of Γ. i.e., synthesised terms are well typed at the type from which they were synthesised. The proof of soundness can be found in Section B.1.2 of Appendix B

3.3.2.8 *Additive pruning*

As seen above, the additive approach delays checking whether a variable is used according to its linearity/grade until it is bound. We hypothesise that this can lead additive synthesis to explore many ultimately ill-typed (or *ill-resourced*) paths for too long. Subsequently, we define a "pruning" variant of any additive rules with multiple sequenced premises. For $\otimes^{+}_{R}$ this is:

$$\frac{\Gamma \vdash A \Rightarrow^{+} t_1;\, \Delta_1 \qquad \Gamma - \Delta_1 \vdash B \Rightarrow^{+} t_2;\, \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^{+} (t_1, t_2);\, \Delta_1 + \Delta_2}\ \otimes'^{+}_{R}$$

Instead of passing Γ to both premises, Γ is the input only for the first premise. This premise outputs context Δ_1 that is subtracted from Γ to give the input context of the second premise. This provides an opportunity to terminate the current branch of synthesis early if $\Gamma - \Delta_1$ does not contain the necessary resources to attempt the second premise. The $\multimap^{+}_{L}$ rule is similarly adjusted:

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1;\, \Delta_1, x_2 : B \qquad \Gamma - \Delta_1 \vdash A \Rightarrow^{+} t_2;\, \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^{+} [(x_1\, t_2)/x_2]t_1;\, (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \multimap'^{+}_{L}$$

Lemma 3.3.3 (Additive pruning synthesis soundness)**.** *For all* Γ *and* A*:*

$$\Gamma \vdash A \Rightarrow^{+} t; \Delta \quad \Longrightarrow \quad \Delta \vdash t : A$$

The proof of soundness can be found in Section B.1.3 of Appendix B

3.4 POST-SYNTHESIS REFACTORING

A synthesised term often contains some artefacts of the fact that it was constructed automatically. The structure of our synthesis rules means aspects of our synthesised programs are unrepresentative in some stylistic ways of the kind of programs functional programmers typically write. We consider two examples of these below using Granule code, and show how we apply a refactoring procedure to any synthesised term to rewrite them in a more idiomatic style.

3.4.1 *Abstractions*

A function definition synthesised from a function type using the $\multimap_R$ will take the form of a sequence of nested abstractions which bind the function's arguments, with the sub-term of the innermost abstraction containing the function body, e.g.

```
pair : ∀ { a b : Type } . a → b → (a, b)
pair = λx → λy → (x, y)
```

In most cases, a programmer would write a function definition as a series of equations with the function arguments given as patterns. Our refactoring procedure collects the outermost abstractions of a synthesised term and transforms them into equation-level patterns with the innermost abstraction body forming the equation body:

```
pair : ∀ { a b : Type } . a → b → (a, b)
pair x y = (x, y)
```

3.4.2 *Unboxing*

An unboxing term is synthesised via the $\Box_L$ rule as a case statement which pattern matches over a box pattern, yielding an assumption with the grade's usage. Such terms can also be refactored both into function equations and to avoid nested let bindings. For example, we may write the k combinator using an explicit graded modality:

```
k : ∀ { a b : Type } . a → b [0] → a
k x y = let [z] = y in x
```

which we can then refactor into

```
k : ∀ { a b : Type } . a → b [0] → a
k x [z] = z
```

This procedure included programs which perform a nested unboxing, refactoring:

```
comp : ∀ {k : Coeffect, n m : k, a b c : Type}
     . (a [m] → b) [n]
     → (b [n] → c)
     → a [n * m]
     → c
comp x y z = let [u] = x in let [v] = z in y [ u [v] ]
```

into:

```
comp : ∀ {k : Coeffect, n m : k, a b c : Type}
     . (a [m] → b) [n]
     → (b [n] → c)
     → a [n * m]
     → c
comp [u] y [v] = y [ u [v] ]
```

3.5 FOCUSING

The additive and subtractive calculi presented in Sections 3.3.1 and 3.3.2 provide the foundations for the implementations of a synthesis tool for Granule programs. Implementing the rules as they currently stand, however, would yield a highly inefficient tool. In their current form, the rules of both calculi exhibit a high degree of non-determinism with regard to order in which rules can be applied.

This leads to us exploring a large number of redundant search branches: something which can be avoided through the application of a technique from linear logic proof theory called *focusing* [Andreoli, 1992]. Focusing is based on the observation that some of the synthesis rules are invertible, i.e. whenever the conclusion of the rule is derivable, then so are its premises. In other words, the order in which we apply invertible rules doesn't matter. By fixing a particular ordering on the application of these rules, we eliminate much of the non-determinism that arises from trying branches which differ only in the order in which invertible rules are applied.

We take our both of our calculi and apply this focusing technique to them, yielding two *focusing* calculi. To do so, we augment our previous synthesis judgement with an additional input context Ω:

$$\Gamma; \Omega \vdash A \Rightarrow t \mid \Delta$$

Unlike Γ and Δ, Ω is an *ordered* context, which behaves like a stack.

Using the terminology of Andreoli [1992], we refer to rules that are invertible as *asynchronous* and rules that are not as *synchronous*. The intuition is that of asynchronous communication: asynchronous rules can be applied eagerly, while the non-invertible synchronous rules require us to *focus* on a particular part of the judgement: either on the assumption (if we are in an elimination rule) or on the goal (for an introduction rule). When focusing we apply a chain of synchronous rules, until we either reach a position where no rules may be applied (at which point the branch terminates), we have synthesised a term for our goal, or we have exposed an asynchronous connective at which point we switch back to applying asynchronous rules.

We divide our synthesis rules into four categories, each with their own judgement form, which refines the focusing judgement above with an arrow indicating which part of the judgement is currently in focus. An $\Uparrow$ indicates an asynchronous phase, while a $\Downarrow$ indicates a synchronous (focused) phase. The location of the arrow in the judgement indicates whether we are focusing on the left or right:

1. Right Async: $\multimap_R$ rule with the judgement:

$$\Gamma; \Omega \vdash A \Uparrow \Rightarrow t \mid \Delta$$

2. Left Async: $\otimes_L$, $\oplus_L$, 1_L, DER, and $\Box_L$ rules with the judgement:

$$\Gamma; \Omega \Uparrow \vdash A \Rightarrow t \mid \Delta$$

3. Right Sync: $\otimes_R$, $\oplus 1_R$, $\oplus 2_R$, 1_R, and $\Box_R$ rules with the judgement:

$$\Gamma; \Omega \vdash A \Downarrow \Rightarrow t \mid \Delta$$

4. Left Sync: $\multimap_L$ rule with the judgement:

$$\Gamma; \Omega \Downarrow \vdash A \Rightarrow t \mid \Delta$$

The complete calculi of focusing synthesis rules are given in Figures 3.4-3.8 for the subtractive calculus, and 3.9-3.13 for the additive, divided into focusing phases. The focusing rules for the additive pruning calculus are identical to the additive calculus, save for the $\otimes_R^+$ and $\multimap_L^+$ rules, which are given in Figure 3.14.

For the most part, the translation from non-focused to focused rules is straightforward. The most notable change occurs in rules in which assumptions are bound. In the cases where a fresh assumption's type falls into the Left Async category (i.e. $\otimes$, $\oplus$, etc.), then it is bound in the ordered context Ω instead of Γ. Left Async rules operate on assumptions in Ω, rather than Γ. This results in invertible elimination rules being applied as fully as possible before *focusing* on non-invertible rules when Ω is empty.

In addition to the focused forms of the original synthesis calculi, each calculus has a set of rules which determine which part of the synthesis judgement will be focused on: the FOCUS rules. These rules are given by Figures 3.6, and 3.11 for the subtractive and additive calculi, respectively.

$$\frac{\Gamma; \Omega, x : A \vdash B \Uparrow \Rightarrow^{-} t \mid \Delta \quad x \notin |\Delta|}{\Gamma; \Omega \vdash A \multimap B \Uparrow \Rightarrow^{-} \lambda x.t \mid \Delta} \multimap_R^{-}$$

$$\frac{\Gamma; \Omega \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta \qquad C \text{ not right async}}{\Gamma; \Omega \vdash C \Uparrow \Rightarrow^{-} t \mid \Delta} \Uparrow_R^{-}$$

Figure 3.4: Right Async rules of the focused subtractive synthesis calculus

$$\frac{\Gamma; \Omega, x_1 : A, x_2 : B \Uparrow \vdash C \Rightarrow^{-} t_2 \mid \Delta \qquad x_1 \notin |\Delta| \qquad x_2 \notin |\Delta|}{\Gamma; \Omega, x_3 : A \otimes B \Uparrow \vdash C \Rightarrow^{-} \textbf{let}\,(x_1, x_2) = x_3\,\textbf{in}\,t_2 \mid \Delta} \otimes_L^{-}$$

$$\frac{\Gamma; \Omega, x_2 : A \Uparrow \vdash C \Rightarrow^{-} t_1 \mid \Delta_1 \quad \Gamma; \Omega, x_3 : B \Uparrow \vdash C \Rightarrow^{-} t_2 \mid \Delta_2 \quad x_2 \notin |\Delta_1| \quad x_3 \notin |\Delta_2|}{\Gamma; \Omega, x_1 : A \oplus B \Uparrow \vdash C \Rightarrow^{-} \textbf{case}\ x_1\ \textbf{of inl}\ x_2 \to t_1;\ \textbf{inr}\ x_3 \to t_2 \mid \Delta_1 \sqcap \Delta_2} \oplus_L^{-}$$

$$\frac{\Gamma; \Omega, x_2 :_r A \Uparrow \vdash B \Rightarrow^{-} t \mid \Delta, x_2 :_s A \qquad 0 \sqsubseteq s}{\Gamma; \Omega, x_1 : \Box_r A \Uparrow \vdash B \Rightarrow^{-} \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t \mid \Delta} \Box_L^{-}$$

$$\frac{\Gamma; \emptyset \vdash C \Rightarrow^{-} t \mid \Delta}{\Gamma; x : 1 \vdash C \Rightarrow^{-} \textbf{let}\,() = x\,\textbf{in}\,t \mid \Delta} 1_L^{-}$$

$$\frac{\Gamma; x :_s A, y : A \Uparrow \vdash B \Rightarrow^{-} t \mid \Delta, x :_{s'} A \qquad y \notin |\Delta| \qquad \exists s.\, r \sqsupseteq s + 1}{\Gamma; x :_r A \Uparrow \vdash B \Rightarrow^{-} [x/y]t \mid \Delta, x :_{s'} A} \textsc{der}^{-}$$

$$\frac{\Gamma, x : A; \Omega \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta \qquad \text{A not left async}}{\Gamma; \Omega, x : A \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta} \Uparrow_L^{-}$$

Figure 3.5: Left Async rules of the focused subtractive synthesis calculus

$$\frac{\Gamma; \emptyset \vdash C \Downarrow \Rightarrow^{-} t \mid \Delta \qquad C \text{ not atomic}}{\Gamma; \emptyset \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta} \textsc{focus}_R^{-} \qquad \frac{\Gamma; x : A \Downarrow \vdash C \Rightarrow^{-} t \mid \Delta}{\Gamma, x : A; \emptyset \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta} \textsc{focus}_L^{-}$$

Figure 3.6: Focus rules of the focused subtractive synthesis calculus

$$\frac{\Gamma;\emptyset \vdash A \Downarrow \Rightarrow^{-} t_1 \mid \Delta_1 \qquad \Delta_1;\emptyset \vdash B \Downarrow \Rightarrow^{-} t_2 \mid \Delta_2}{\Gamma;\emptyset \vdash A \otimes B \Downarrow \Rightarrow^{-} (t_1, t_2) \mid \Delta_2}\otimes_R^{-}$$

$$\frac{\Gamma;\emptyset \vdash B \Downarrow \Rightarrow^{-} t \mid \Delta}{\Gamma;\emptyset \vdash A \oplus B \Downarrow \Rightarrow^{-} \mathbf{inr}\, t \mid \Delta}\oplus 2_L^{+} \quad \frac{\Gamma;\emptyset \vdash A \Downarrow \Rightarrow^{-} t \mid \Delta}{\Gamma;\emptyset \vdash A \oplus B \Downarrow \Rightarrow^{-} \mathbf{inl}\, t \mid \Delta}\oplus 1_L^{+}$$

$$\frac{\Gamma;\emptyset \vdash A \Uparrow \Rightarrow^{-} t \mid \Delta}{\Gamma;\emptyset \vdash \Box_r A \Downarrow \Rightarrow^{-} t \mid \Gamma - r \cdot (\Gamma - \Delta)}\Box_R^{-}$$

$$\frac{}{\Gamma \vdash 1 \Rightarrow^{-} () \mid \Gamma}1_R^{-} \quad \frac{\Gamma;\emptyset \vdash A \Uparrow \Rightarrow^{-} t \mid \Delta}{\Gamma;\emptyset \vdash A \Downarrow \Rightarrow^{-} t \mid \Delta}\Downarrow_R^{-}$$

Figure 3.7: Right Sync rules of the focused subtractive synthesis calculus

$$\frac{\Gamma; x_2 : B \Downarrow \vdash C \Rightarrow^{-} t_1 \mid \Delta_1 \qquad x_2 \notin |\Delta_1| \qquad \Delta_1;\emptyset \vdash A \Downarrow \Rightarrow^{-} t_2 \mid \Delta_2}{\Gamma; x_1 : A \multimap B \Downarrow \vdash C \Rightarrow^{-} [(x_1\, t_2)/x_2]t_1 \mid \Delta_2}\multimap_L^{-}$$

$$\frac{}{\Gamma; x : A \Downarrow \vdash A \Rightarrow^{-} x \mid \Gamma}\textsc{LinVar}^{-} \quad \frac{\exists s.\, r \sqsubseteq s + 1}{\Gamma; x :_r A \Downarrow \vdash A \Rightarrow^{-} x \mid \Gamma, x :_s A}\textsc{GrVar}^{-}$$

$$\frac{\Gamma; x : A \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta \qquad \text{A not atomic and not left sync}}{\Gamma; x : A \Downarrow \vdash C \Rightarrow^{-} t \mid \Delta}\Downarrow_L^{-}$$

Figure 3.8: Left Sync and Var rules of the focused subtractive synthesis calculus

$$\frac{\Gamma; \Omega, x : A \vdash B \Uparrow \Rightarrow t \mid \Delta, x : A}{\Gamma; \Omega \vdash A \multimap B \Uparrow \Rightarrow \lambda x.t \mid \Delta}\multimap_R^{+} \quad \frac{\Gamma; \Omega \Uparrow \vdash C \Rightarrow t \mid \Delta \qquad C \text{ not right async}}{\Gamma; \Omega \vdash C \Uparrow \Rightarrow t \mid \Delta}\Uparrow_R^{+}$$

Figure 3.9: Right Async rules of the focused additive synthesis calculus

$$\frac{\Gamma; \Omega, x_1 : A, x_2 : B \vdash C \Rightarrow t_2 \mid \Delta, x_1 : A, x_2 : B}{\Gamma; \Omega, x_3 : A \otimes B \vdash C \Rightarrow \textbf{let}\,(x_1, x_2) = x_3\,\textbf{in}\,t_2 \mid \Delta, x_3 : A \otimes B}\ \otimes_L^+$$

$$\frac{\Gamma; \Omega, x_2 : A \Uparrow \vdash C \Rightarrow t_1 \mid \Delta_1, x_2 : A \qquad \Gamma; \Omega, x_3 : B \Uparrow \vdash C \Rightarrow t_2 \mid \Delta_2, x_3 : B}{\Gamma; \Omega, x_1 : A \oplus B \Uparrow \vdash C \Rightarrow^- \textbf{case}\ x_1\ \textbf{of inl}\ x_2 \to t_1;\ \textbf{inr}\ x_3 \to t_2 \mid \Delta_1 \sqcup \Delta_2, x_1 : A \oplus B}\ \oplus_L^+$$

$$\frac{\Gamma; \Omega, x_2 :_r A \Uparrow \vdash B \Rightarrow t \mid \Delta \qquad \textit{if } x_2 :_s A \in \Delta \textit{ then } s \sqsubseteq r \textit{ else } 0 \sqsubseteq r}{\Gamma; \Omega, x_1 : \Box_r A \vdash B \Rightarrow \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t \mid (\Delta \backslash x_2), x_1 : \Box_r A}\ \Box_L^+$$

$$\frac{\Gamma; x :_s A, y : A \Uparrow \vdash B \Rightarrow t \mid \Delta, y : A}{\Gamma; x :_s A \Uparrow \vdash B \Rightarrow [x/y]t \mid \Delta + x :_1 A}\ \textsc{der}^+$$

$$\frac{\Gamma; \emptyset \vdash C \Rightarrow t \mid \Delta}{\Gamma; x : 1 \vdash C \Rightarrow \textbf{let}\,() = x\,\textbf{in}\,t \mid \Delta, x : 1}\ 1_L^+$$

$$\frac{\Gamma, x : A; \Omega \Uparrow \vdash C \Rightarrow t \mid \Delta \qquad \text{A not left async}}{\Gamma; \Omega, x : A \Uparrow \vdash C \Rightarrow t \mid \Delta}\ \Uparrow_L^+$$

Figure 3.10: Left Async rules of the focused additive synthesis calculus

$$\frac{\Gamma; \emptyset \vdash C \Downarrow \Rightarrow t \mid \Delta \qquad \text{C not atomic}}{\Gamma; \emptyset \Uparrow \vdash C \Rightarrow t \mid \Delta}\ \textsc{focus}_R^+$$

$$\frac{\Gamma; x : A \Downarrow \vdash C \Rightarrow t \mid \Delta}{\Gamma, x : A; \emptyset \Uparrow \vdash C \Rightarrow t \mid \Delta}\ \textsc{focus}_L^+$$

Figure 3.11: Focus rules of the focused additive synthesis calculus

$$\frac{\Gamma; \emptyset \vdash A \Downarrow \Rightarrow t_1 \mid \Delta_1 \qquad \Gamma; \emptyset \vdash B \Downarrow \Rightarrow t_2 \mid \Delta_2}{\Gamma; \emptyset \vdash A \otimes B \Downarrow \Rightarrow (t_1, t_2) \mid \Delta_1 + \Delta_2}\ \otimes_R^+$$

$$\frac{\Gamma; \emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta}{\Gamma; \emptyset \vdash A \oplus B \Downarrow \Rightarrow \textbf{inl}\,t \mid \Delta}\ \oplus 1_L^+ \qquad \frac{\Gamma; \emptyset \vdash B \Downarrow \Rightarrow t \mid \Delta}{\Gamma; \emptyset \vdash A \oplus B \Downarrow \Rightarrow \textbf{inr}\,t \mid \Delta}\ \oplus 2_L^+$$

$$\frac{\Gamma; \emptyset \vdash A \Uparrow \Rightarrow t \mid \Delta}{\Gamma; \emptyset \vdash \Box_r A \Downarrow \Rightarrow [t] \mid r \cdot \Delta}\ \Box_R^+ \qquad \frac{}{\Gamma; \emptyset \vdash 1 \Rightarrow () \mid \emptyset}\ 1_R^+$$

$$\frac{\Gamma; \emptyset \vdash A \Uparrow \Rightarrow t \mid \Delta}{\Gamma; \emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta}\ \Downarrow_R^+$$

Figure 3.12: Right Sync rules of the focused additive synthesis calculus

$$\textsc{LeftSync}\quad \dfrac{\Gamma; x_2 : B \Downarrow \vdash C \Rightarrow t_1 \mid \Delta_1, x_2 : B \qquad \Gamma; \emptyset \vdash A \Downarrow \Rightarrow t_2 \mid \Delta_2}{\Gamma; x_1 : A \multimap B \Downarrow \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + \Delta_2), x_1 : A \multimap B}\;\multimap_L^+$$

$$\dfrac{}{\Gamma; x : A \vdash A \Rightarrow x \mid x : A}\;\textsc{LinVar}^+ \qquad \dfrac{}{\Gamma; x :_r A \vdash A \Rightarrow x \mid x :_1 A}\;\textsc{GrVar}^+$$

$$\dfrac{\Gamma; x : A \Uparrow \vdash C \Rightarrow t \mid \Delta \qquad A \text{ not atomic and not left sync}}{\Gamma; x : A \Downarrow \vdash C \Rightarrow t \mid \Delta}\;\Downarrow_L^+$$

Figure 3.13: Left Sync and Var rules of the focused additive synthesis calculus

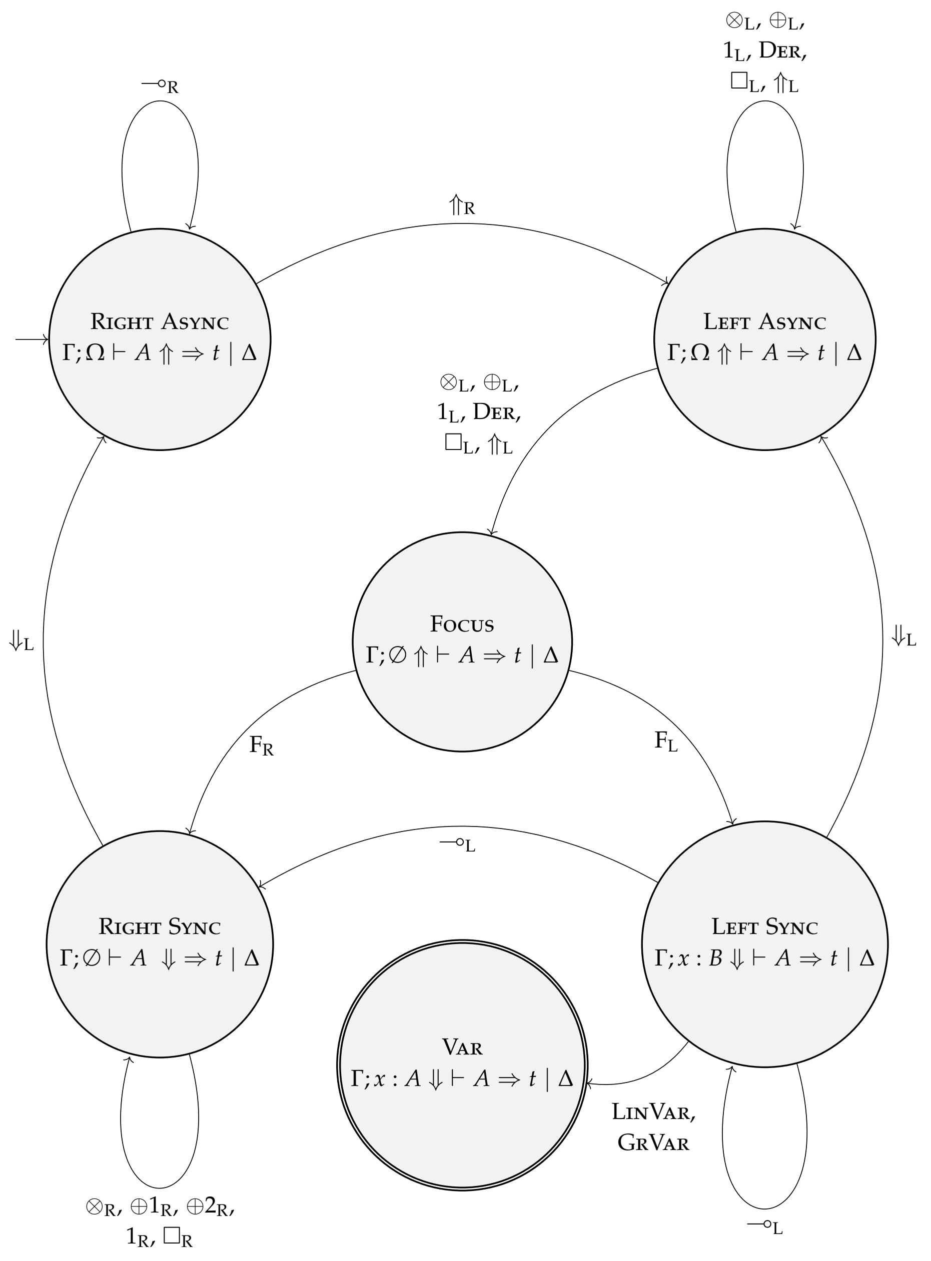
$$\dfrac{\Gamma; x_2 : B \vdash C \Rightarrow t_1 \mid \Delta_1, x_2 : B \qquad \Gamma - \Delta_1; \emptyset \vdash A \Rightarrow t_2 \mid \Delta_2}{\Gamma; x_1 : A \multimap B \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + \Delta_2), x_1 : A \multimap B}\;\multimap_L'^+$$

$$\dfrac{\Gamma; \emptyset \vdash A \Rightarrow t_1 \mid \Delta_1 \qquad \Gamma - \Delta_1; \emptyset \vdash B \Rightarrow t_2 \mid \Delta_2}{\Gamma; \emptyset \vdash A \otimes B \Rightarrow (t_1, t_2) \mid \Delta_1 + \Delta_2}\;\otimes_R'^+$$

Figure 3.14: Rules of the focused additive pruning synthesis calculus

One way to view focusing is in terms of a finite state machine, such as Figure 3.15. States comprise the four phases of focusing, plus two additional states, Focus, and Var. Edges are then the synthesis rules that direct the transition between focusing phases. The transitions between these focusing phases are handled by dedicated focusing rules for each transition. For the asynchronous phases, the $\Uparrow_R / \Uparrow_L$ handle the transition between right to left phases, and left to focusing phases, respectively. Conversely, the $\Downarrow R$ rule deals with the transition from a right synchronous phase back to a right asynchronous phase, with the $\Downarrow L$ rule likewise transitioning to a left asynchronous phase. Depending on the current phase of focusing, these rules consider the goal type, the assumption currently being focused on's type, as well as the size of Ω, to decide when to transition between focusing phases.

This focused approach to synthesis ensures that we are restricted to generating programs in β-normal form, which eliminates a class of redundant programs for which behaviourally equivalent β-normal forms can be synthesised in less steps.



Right Async
$\Gamma;\Omega \vdash A \Uparrow \Rightarrow t \mid \Delta$
Left Async
$\Gamma;\Omega \Uparrow \vdash A \Rightarrow t \mid \Delta$
Focus
$\Gamma;\emptyset \Uparrow \vdash A \Rightarrow t \mid \Delta$
Right Sync
$\Gamma;\emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta$
Left Sync
$\Gamma;x:B \Downarrow \vdash A \Rightarrow t \mid \Delta$
Var
$\Gamma;x:A \Downarrow \vdash A \Rightarrow t \mid \Delta$
$\multimap_R$
$\Uparrow_R$
$\otimes_L$, $\oplus_L$, 1_L, Der, $\Box_L$, $\Uparrow_L$
$\otimes_L$, $\oplus_L$, 1_L, Der, $\Box_L$, $\Uparrow_L$
$\Downarrow_L$
$\Downarrow_L$
F_R
F_L
$\multimap_L$
LinVar, GrVar
$\otimes_R$, $\oplus1_R$, $\oplus2_R$, 1_R, $\Box_R$
$\multimap_L$

Figure 3.15: Focusing State Machine

We conclude with three key results: that applying focusing is sound for the subtractive (Lemma 3.5.1) and additive (Lemma 3.5.2), and additive pruning (Lemma 3.5.3) synthesis calculi. The proofs are contained in Sections B.1.4, B.1.5, and B.1.6 of Appendix B, respectively.

Lemma 3.5.1 (Soundness of focusing for subtractive synthesis)**.** *For all contexts* Γ, Ω *and types* A *then:*

$$\begin{array}{lllll}
1. \textit{Right Async}: & \Gamma;\Omega \vdash A \Uparrow \Rightarrow^{-} t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash A \Rightarrow^{-} t \mid \Delta \\
2. \textit{Left Async}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash C \Rightarrow^{-} t \mid \Delta \\
3. \textit{Right Sync}: & \Gamma;\emptyset \vdash A \Downarrow \Rightarrow^{-} t \mid \Delta & \Longrightarrow & \Gamma \vdash A \Rightarrow^{-} t \mid \Delta \\
4. \textit{Left Sync}: & \Gamma; x : A \Downarrow \vdash C \Rightarrow^{-} t \mid \Delta & \Longrightarrow & \Gamma, x : A \vdash C \Rightarrow^{-} t \mid \Delta \\
5. \textit{Focus Right}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{-} t \mid \Delta \\
6. \textit{Focus Left}: & \Gamma, x : A;\Omega \Uparrow \vdash C \Rightarrow^{-} t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{-} t \mid \Delta
\end{array}$$

i.e. t *has type* A *under context* Δ*, which contains assumptions with grades reflecting their use in* t*.*

Lemma 3.5.2 (Soundness of focusing for additive synthesis)**.** *For all contexts* Γ, Ω *and types* A *then:*

$$\begin{array}{lllll}
1. \textit{Right Async}: & \Gamma;\Omega \vdash A \Uparrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash A \Rightarrow^{+} t; \Delta \\
2. \textit{Left Async}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash C \Rightarrow^{+} t; \Delta \\
3. \textit{Right Sync}: & \Gamma;\emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash A \Rightarrow^{+} t; \Delta \\
4. \textit{Left Sync}: & \Gamma; x : A \Downarrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma, x : A \vdash C \Rightarrow^{+} t; \Delta \\
5. \textit{Focus Right}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t; \Delta \\
6. \textit{Focus Left}: & \Gamma, x : A;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t; \Delta
\end{array}$$

i.e. t *has type* A *under context* Δ*, which contains assumptions with grades reflecting their use in* t*.*

Lemma 3.5.3 (Soundness of focusing for additive pruning synthesis)**.** *For all contexts* Γ, Ω *and types* A *then:*

$$\begin{array}{lllll}
1. \textit{Right Async}: & \Gamma;\Omega \vdash A \Uparrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash A \Rightarrow^{+} t; \Delta \\
2. \textit{Left Async}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash C \Rightarrow^{+} t; \Delta \\
3. \textit{Right Sync}: & \Gamma;\emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash A \Rightarrow^{+} t; \Delta \\
4. \textit{Left Sync}: & \Gamma; x : A \Downarrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma, x : A \vdash C \Rightarrow^{+} t; \Delta \\
5. \textit{Focus Right}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t; \Delta \\
6. \textit{Focus Left}: & \Gamma, x : A;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t; \Delta
\end{array}$$

i.e. t *has type* A *under context* Δ*, which contains assumptions with grades reflecting their use in* t*.*

3.6 EVALUATING THE SYNTHESIS CALCULI

Prior to evaluation, we made the following hypotheses about the relative performance of the additive versus subtractive approaches:

H1. (**Solving; Additive requires less**) Additive synthesis should make fewer calls to the solver, with lower complexity theorems (fewer quantifiers). Dually, subtractive synthesis makes more calls to the solver with higher complexity theorems.

H2. (**Paths; Subtractive explores fewer**) For complex problems, additive will explore more paths as it cannot tell whether a variable is not well-resourced until closing a binder; additive pruning and subtractive will explore fewer paths as they can fail sooner.

H3. (**Performance; additive faster on simpler examples**) A corollary of the above two: simple examples will likely be faster in additive mode, but more complex examples will be faster in subtractive.

3.6.1 *Methodology*

We implemented our approach as a synthesis tool for Granule, integrated with its core tool. Granule features ML-style polymorphism (rank-0 quantification) but we do not address polymorphism here. Instead, programs are synthesised from type schemes treating universal type variables as logical atoms. Multiplicative products are primitive in Granule, although additives coproducts are provided via ADTs, from which we define a core sum type to use here.

Constraints on resource usage are handled via Granule's existing symbolic engine, which compiles constraints on grades (for various semirings) to the SMT-lib format for Z3 [de Moura and Bjørner, 2008]. We use the LogicT monad for backtracking search [Kiselyov et al., 2005] and the Scrap Your Reprinter library for splicing synthesised code into syntactic "holes" (represented by ? in Granule), preserving the rest of the program text [Clarke et al., 2017].

To evaluate our synthesis tool we developed a suite of benchmarks comprising Granule type schemes for a variety of operations using linear and graded modal types. We divide our benchmarks into several classes of problem:

- **Hilbert**: the Hilbert-style axioms of intuitionistic logic (including SKI combinators), with appropriate $\mathbb{N}$ and $\mathbb{N}$-intervals grades where needed (see, e.g., S combinator in Example 2.3.1 or coproduct elimination in Example 3.1.1).
- **Comp**: various translations of function composition into linear logic: multiplicative, call-by-value and call-by-name using ! [Girard, 1987], I/O using ! [Liang and Miller, 2009], and coKleisli composition over $\mathbb{N}$ and arbitrary semirings: e.g. $\forall r, s \in \mathcal{R}$:

$$\textit{comp-coK}_{\mathcal{R}} : \Box_r(\Box_s A \multimap B) \multimap (\Box_r B \multimap C) \multimap \Box_{r \cdot s} A \multimap C$$

- **Dist**: distributive laws of various graded modalities over functions, sums, and products, e.g., $\forall r \in \mathbb{N}$, or $\forall r \in \mathcal{R}$ in any semiring, or $r = 0 ... \infty$:

$$\textit{pull}_{\oplus} : (\Box_r A \oplus \Box_r B) \multimap \Box_r (A \oplus B)$$

$$\textit{push}_{\multimap} : \Box_r (A \multimap B) \multimap \Box_r A \multimap \Box_r B$$

- **Vec**: map operations on fixed size vectors encoded as products, e.g.:

$$\begin{aligned} vmap_5 : \ & \Box_5(A \multimap B) \\ \multimap \ & ((((A \otimes A) \otimes A) \otimes A) \otimes A) \\ \multimap \ & ((((B \otimes B) \otimes B) \otimes B) \otimes B) \end{aligned}$$

- **Misc**: includes Example 3.1.2 (information-flow security) and functions which must share resources between graded modalities, e.g.:

$$\begin{aligned} share : \ & \Box_4 A \\ \multimap \ & \Box_6 A \\ \multimap \ & \Box_2(((((A \otimes A) \otimes A) \otimes A) \otimes A) \multimap B) \\ \multimap \ & (B \otimes B) \end{aligned}$$

Table 3.1 provides the complete list of Granule type schemes used for these synthesis problems (32 in total). Note that these are type schemes which quantify over type variables (a, and b), however, we simply treat each type variable as a logical atom, unifiable only with itself. Chapter 5 provides a proper treatment of synthesis from Rank-1 polymorphic type schemes. Note also that $\Box A$ is used as shorthand for $\Box_{0...\infty} A$ (graded modality with indices drawn from intervals over $\mathbb{N} \cup \infty$). The compete synthesised program code for the benchmarking problems can be found in Section A.1 of Appendix A

We found that Z3 is highly variable in its solving time, so timing measurements are computed as the mean of 20 trials. We used Z3 version 4.8.8 on a Linux laptop with an Intel i7-8665u @ 4.8 Ghz and 16 Gb of RAM.

3.6.2 *Results and Analysis*

For each synthesis problem, we recorded whether synthesis was successful or not (denoted ✓ or ×), the mean total synthesis time (μT), and the number of calls made to the SMT solver (N). Table 3.2 summarises the results with the fastest case for each benchmark highlighted. For all benchmarks that used the SMT solver, the solver accounted for 91.73% – 99.98% of synthesis time, so we report only the mean total synthesis time μT. We set a timeout of 120 seconds.

3.6.2.1 *Additive versus subtractive*

As expected, the additive approach generally synthesises programs faster than the subtractive. Our first hypothesis (that the additive approach in general makes fewer calls to the SMT solver) holds for almost all benchmarks, with the subtractive approach often far exceeding the number made by the additive. This is explained by the

Hilbert	
$\otimes$Intro	$\otimes_i : \forall a, b.\ a \multimap b \multimap (a \otimes b)$
$\otimes$Elim	$\otimes_{e1} : \forall a, b.\ (a \otimes \Box_0 b) \multimap a$
	$\otimes_{e2} : \forall a, b.\ ()(\Box_0 a \otimes b) \multimap b$
$\oplus$Intro	$\oplus_{i1} : \forall a, b.\ a \multimap a \oplus b$
	$\oplus_{i2} : \forall a, b.\ b \multimap a \oplus b$
$\oplus$Elim	$\oplus_e : \forall a, b, c.\ \Box_{0...1}(a \multimap c) \multimap \Box_{0...1}(b \multimap c) \multimap (a \oplus b) \multimap c$
SKI	$s : \forall a, b, c.\ (a \multimap (b \multimap c)) \multimap (a \multimap b) \multimap (\Box_2 a \multimap c)$
	$k : \forall a, b.\ a \multimap \Box_0 b \multimap a$
	$i : \forall a.\ a \multimap a$
Comp	
0/1	$\circ_{I/O} : \forall a, b, c.\ \Box(\Box a \multimap \Box b) \multimap \Box(\Box b \multimap \Box c) \multimap \Box(\Box a \multimap c)$
CBN	$\circ_{\text{CBN}} : \forall a, b, c.\ \Box(\Box a \multimap b) \multimap \Box(\Box b \multimap c) \multimap \Box a \multimap c$
CBV	$\circ_{\text{CBV}} : \forall a, b, c.\ \Box(\Box a \multimap \Box b) \multimap \Box(\Box b \multimap \Box c) \multimap \Box\Box a \multimap \Box c$
coK-$\mathcal{R}$	$\circ_{\mathcal{R}} : \forall \mathcal{R}, r, s \in \mathcal{R}, a, b, c.\ \Box_r(\Box_s a \multimap b) \multimap (\Box_r b \multimap c) \multimap \Box_{r \cdot s} a \multimap c$
mult	$\circ : \forall a, b, c.\ (a \multimap b) \multimap (b \multimap c) \multimap (a \multimap c)$
coK-$\mathbb{N}$	$\circ_{\mathbb{N}} : \forall r, s \in \mathbb{N}, a, b, c.\ \Box_r(\Box_s a \multimap b) \multimap (\Box_r b \multimap c) \multimap \Box_{r \cdot s} a \multimap c$
Dist	
$\oplus$-$\mathbb{N}$	$pull_{\oplus} : \forall r : \mathbb{N}, a, b.\ (\Box_r a \oplus \Box_r b) \multimap \Box_r(a \oplus b)$
$\oplus$-!	$pull_{\oplus} : \forall a, b.\ (\Box a \oplus \Box b) \multimap \Box(a \oplus b)$
$\oplus$-$\mathcal{R}$	$pull_{\oplus} : \forall \mathcal{R}, r \in \mathcal{R}, a, b.\ (\Box_r a \oplus \Box_r b) \multimap \Box_r(a \oplus b)$
$\otimes$-$\mathbb{N}$	$pull_{\otimes} : \forall r : \mathbb{N}, a, b.\ (\Box_r a \otimes \Box_r b) \multimap \Box_r(a \otimes b)$
$\otimes$-!	$pull_{\otimes} : \forall a, b.\ (\Box a \otimes \Box b) \multimap \Box(a \otimes b)$
$\otimes$-$\mathbb{R}$	$pull_{\otimes} : \forall \mathcal{R}, r, a, b.\ (\Box_r a \otimes \Box_r b) \multimap \Box_r(a \otimes b)$
$\multimap$-$\mathbb{N}$	$push_{\multimap} : \forall r : \mathbb{N}, a, b.\ \Box_r(a \multimap b) \multimap \Box_r a \multimap \Box_r b$
$\multimap$-!	$push_{\multimap} : \forall a, b.\ \Box(a \multimap b) \multimap \Box a \multimap \Box b$
$\multimap$-$\mathcal{R}$	$push_{\multimap} : \forall \mathcal{R}, r : \mathcal{R}, a, b.\ \Box_r(a \multimap b) \multimap \Box_r a \multimap \Box_r b$
Vec	
vec5	$vmap_5 : \forall a, b.\ \Box_5(a \multimap b) \multimap ((((a \otimes a) \otimes a) \otimes a) \otimes a)$
	$\multimap ((((b \otimes b) \otimes b) \otimes b) \otimes b)$
vec10	$vmap_{10} : \forall a, b.$ *as above but for 10-tuples*
vec15	$vmap_{15} : \forall a, b.$ *as above but for 15-tuples*
vec20	$vmap_{20} : \forall a, b.$ *as above but for 20-tuples*
Misc	
split$\oplus$	$split : \forall a, b, c. \Box_{2...3} b \multimap (a \oplus c) \multimap ((a \otimes \Box_{2..2} b) \oplus (c \otimes \Box_{3...3} b))$
split$\otimes$	$split : \forall a, b. \Box_{0...2}(a \multimap a \multimap a) \multimap \Box_{10...10} a \multimap (\Box_{2...2} a \otimes \Box_{6...6} a)$
share	$share : \forall a, b. \Box_4 a \multimap \Box_6 a \multimap \Box_2(((((a \otimes a) \otimes a) \otimes a) \otimes a) \multimap b) \multimap (b \otimes b)$
Exm. 3.1.2	$noLeak : \forall a, b. (\Box_{\mathsf{Lo}} a \otimes \Box_{\mathsf{Hi}} a) \multimap (\Box_{\mathsf{Lo}}(a \otimes 1) \multimap b) \multimap b$

Table 3.1: List of benchmark synthesis problems

difference in graded variable synthesis between approaches. In the additive, a constant grade 1 is given for graded assumptions in the output context, whereas in the subtractive, a fresh grade variable is created with a constraint on its usage which is checked immediately. As the total synthesis time is almost entirely spent in the SMT solver (more than 90%), solving constraints is by far the most costly part of synthesis leading to the additive approach synthesising most examples in a shorter amount of time.

Graded variable synthesis in the subtractive case also results in several examples failing to synthesise. In some cases, e.g., the first three *comp* benchmarks, the subtractive approach times-out as synthesis diverges with constraints growing in size due to the maximality condition and absorbing behaviour of $0...\infty$ interval. In the case of *coK*-$\mathcal{R}$ and *coK*-$\mathbb{N}$, the generated constraints have the form $\forall r.\exists s.r \sqsupseteq s + 1$ which is not valid $\forall r \in \mathbb{N}$ (e.g., when $r = 0$), which suggests that the subtractive approach does not work well for polymorphic grades. As further work, we are considering an alternate rule for synthesising promotion with constraints of the form $\exists s.s = s' * r$, i.e., a multiplicative inverse constraint.

In more complex examples we see evidence to support our second hypothesis. The *share* problem requires a lot of graded variable synthesis which is problematic for the additive approach, for the reasons described in the second hypothesis. In contrast, the subtractive approach performs better, with $\mu T = 193.3ms$ as opposed to additive's $292.02ms$. However, additive pruning outperforms both.

Notably, on examples which are purely linear such as *andElim* from Hilbert's axioms or *mult* for function composition, the subtractive approach generally performs better. Linear programs without graded modalities can be synthesised without the need to interface with Z3 at all, making the differences here somewhat negligible as solver time generally makes up for the vast proportion of total synthesis time.

3.6.2.2 *Additive pruning*

The pruning variant of additive synthesis (where subtraction takes place in the premises of multiplicative rules) had mixed results compared to the default. In simpler examples, the overhead of pruning (requiring SMT solving) outweighs the benefits obtained from reducing the space. However, in more complex examples which involve synthesising many graded variables (e.g. *share*), pruning is especially powerful, performing better than the subtractive approach. However, additive pruning failed to synthesis two examples which are polymorphic in their grade ($\otimes$-$\mathbb{N}$) and in the semiring ($\otimes$-$\mathcal{R}$).

Overall, the additive approach outperforms the subtractive and is successful at synthesising more examples, including ones polymorphic in grades and even the semiring itself. Given that the literature on linear logic theorem proving is typically subtractive, this is an interesting result. Going forward, we will focus on the additive scheme.

3.7 CONCLUSION

At this point we have constructed a simple program synthesis tool for Granule, paramterised by a resource management scheme, which effectively deals with the problems of treating data as a resource inside a program. Both schemes would be a reasonable choice for further development of a synthesis tool for our language based on the graded linear λ-calculus.

Going forward, however, we focus primarily on the additive resource management scheme, using this as the basis for our more feature-rich fully-graded synthesis calculus in Chapter 5. The evaluation in Section 3.6 showed that the additive approach generally yields smaller and simpler theorems than the subtractive, requiring less time to solve. Theorem proving becomes even more prevalent in synthesis for a fully graded typing calculus - potentially every rule introduces new constraints that require solving, thus the speed at which this can be carried out is especially important.

While the tool presented in this chapter allows users to synthesise a considerable subset of Granule programs, our language is still limited in its expressivity. Data types comprise only product, sum, and unit types, while synthesis of recursive function defintions or functions which make use of other in-scope values such as top-level definitions is not permitted. One notable limitation of our typing calculi is the inability to express (and therefore synthesise) programs which perform a deep pattern match over a graded data type. A clear example of this can be found in the synthesis of programs which distribute a graded modality over a data type. Consider a classic example of a distributive program, *push*:

$$\mathit{push} : \Box_r(A \otimes B) \multimap \Box_r A \otimes \Box_r B$$

which takes a data type graded by r (in this case the product type $A \otimes B$), and distributes r over the constituent elements of the product A and B. Given this goal type, how would we go about synthesising a program in our tool?

We instatiate the $\multimap_R^+$ rule at this type, building a partial synthesis derivation. Although we use the additive scheme for this example, the exact same situation arises in the subtractive.

$$\dfrac{\dfrac{x_2 :_r A \otimes B \vdash \Box_r A \otimes \Box_r B \Rightarrow ? \mid ?}{x_1 : \Box_r(A \otimes B) \vdash \Box_r A \otimes \Box_r B \Rightarrow \textbf{let } [x_2] = x_1 \textbf{ in } ? \mid ?}\ \Box_L^+}{\emptyset \vdash \Box_r(A \otimes B) \multimap \Box_r A \otimes \Box_r B \Rightarrow \lambda x_1.? \mid ?}\ \multimap_R^+$$

After applying $\multimap^+_R$ followed by $\Box^+_L$, we now have the graded assumption $x_2 :_r A \otimes B$ in our context which we must use to construct a term of type $\Box_r A \otimes \Box_r B$. We might expect that the path synthesis should take now would be to break x_2 down into two graded assumptions with types A and B, promote these graded assumptions using the $\Box^+_R$ rule, before finally peforming a pair introduction to yield $\Box_r A \otimes \Box_r B$. However, in order to apply the pair elimination rule $\otimes^+_L$ and break our graded assumption into two, we must perform a dereliction on x_2, to yield a linear copy:

$$\frac{x_2 :_r A \otimes B, x_3 : A \otimes B \vdash \Box_r A \otimes \Box_r B \Rightarrow ?}{x_2 :_r A \otimes B \vdash \Box_r A \otimes \Box_r B \Rightarrow ? \mid ?}\ \textsc{der}^+$$

Clearly, this cannot lead us to the goal: the $\Box^+_R$ rule cannot promote terms using linear assumptions. Therefore, *push* and other types which exhibit this distributive behaviour are not derivable in our calculi.

In the following chapter, we present an alternative approach to generating programs which exhibit this distributive behaviour using a generic programming methodology. The approach we present in Chapter 4 is not type-directed program synthesis, per se. This approach complements the calculi presented here and in Chapter 5, providing users with a means to automatically generate programs purely from a type for a common class of graded programs. In describing this mechanism, we also begin to enhance our language with more advanced features such pattern matching, and recursion, further laying the foundations for Chapter 5.

		Additive			Additive (pruning)			Subtractive		
	Problem		μT (ms)	N		μT (ms)	N		μT (ms)	N
Hilbert	$\otimes$Intro	✓	6.69 (0.05)	2	✓	9.66 (0.23)	2	✓	10.93 (0.31)	2
	$\otimes$Elim	✓	0.22 (0.01)	0	✓	0.05 (0.00)	0	✓	0.06 (0.00)	0
	$\oplus$Intro	✓	0.08 (0.00)	0	✓	0.07 (0.00)	0	✓	0.07 (0.00)	0
	$\oplus$Elim	✓	7.26 (0.30)	2	✓	13.25 (0.58)	2	✓	204.50 (8.78)	15
	SKI	✓	8.12 (0.25)	2	✓	24.98 (1.19)	2	✓	41.92 (2.34)	4
Comp	01	✓	28.31 (3.09)	5	✓	41.86 (0.38)	5	×	Timeout	-
	cbn	✓	13.12 (0.84)	3	✓	26.24 (0.27)	3	×	Timeout	-
	cbv	✓	19.68 (0.98)	5	✓	34.15 (0.98)	5	×	Timeout	-
	$\circ coK_{\mathcal{R}}$	✓	33.37 (2.01)	2	✓	27.37 (0.78)	2	×	92.71 (2.37)	8
	$\circ coK_{\mathbb{N}}$	✓	27.59 (0.67)	2	✓	21.62 (0.59)	2	×	95.94 (2.21)	8
	mult	✓	0.29 (0.02)	0	✓	0.12 (0.00)	0	✓	0.11 (0.00)	0
Dist	$\otimes$-!	✓	12.96 (0.48)	2	✓	32.28 (1.32)	2	✓	10487.92 (4.38)	7
	$\otimes$-$\mathbb{N}$	✓	24.83 (1.01)	2	×	32.18 (0.80)	2	×	31.33 (0.65)	2
	$\otimes$-$\mathcal{R}$	✓	28.17 (1.01)	2	×	29.72 (0.90)	2	×	31.91 (1.02)	2
	$\oplus$-!	✓	7.87 (0.23)	2	✓	16.54 (0.43)	2	✓	160.65 (2.26)	4
	$\oplus$-$\mathbb{N}$	✓	22.13 (0.70)	2	✓	30.30 (1.02)	2	×	23.82 (1.13)	1
	$\oplus$-$\mathcal{R}$	✓	22.18 (0.60)	2	✓	31.24 (1.40)	2	×	16.34 (0.40)	1
	$\multimap$-!	✓	6.53 (0.16)	2	✓	10.01 (0.25)	2	✓	342.52 (2.64)	4
	$\multimap$-$\mathbb{N}$	✓	29.16 (0.82)	2	✓	28.71 (0.67)	2	×	54.00 (1.53)	4
	$\multimap$-$\mathcal{R}$	✓	29.31 (1.84)	2	✓	27.44 (0.60)	2	×	61.33 (2.28)	4
Vec	vec5	✓	4.72 (0.07)	1	✓	14.93 (0.21)	1	✓	78.90 (2.25)	6
	vec10	✓	5.51 (0.36)	1	✓	20.81 (0.77)	1	✓	142.87 (5.86)	11
	vec15	✓	9.75 (0.25)	1	✓	22.09 (0.24)	1	✓	195.24 (3.20)	16
	vec20	✓	13.40 (0.46)	1	✓	30.18 (0.20)	1	✓	269.52 (4.25)	21
Misc	split$\oplus$	✓	3.79 (0.04)	1	✓	5.10 (0.16)	1	✓	10732.65 (8.01)	6
	split$\otimes$	✓	14.07 (1.01)	3	✓	46.27 (2.04)	3	×	Timeout	-
	share	✓	292.02 (11.37)	44	✓	100.85 (2.44)	6	✓	193.33 (4.46)	17
	Exm. 3.1.2	✓	8.09 (0.46)	2	✓	26.03 (1.21)	2	✓	284.76 (0.31)	3

Table 3.2: Results. μT in *ms* to 2 d.p. with standard sample error in brackets

4 AUTOMATICALLY DERIVING GRADED COMBINATORS

Thus far we have considered program synthesis from the perspective of enumerative search, using our types to guide us and pruning the space of programs where possible. This approach yielding a synthesis tool which was highly expressive, allowing the synthesis of a program term for each syntactic form in our language. In this chapter we present an alternative approach, which targets a specific class of graded programs: graded *distributive* combinators. We view this approach as a useful complement to the more powerful type-directed synthesis.

When programming with graded modal types, we have observed there is often a need to 'distribute' a graded modality over a type, and vice versa, in order to compose programs. That is, we may find ourselves in possession of a $\Box_r(\mathsf{F}\alpha)$ value (for some parametric data type F) which needs to be passed to a pre-existing function (of our own codebase or a library) which requires a $\mathsf{F}(\Box_r\alpha)$ value, or perhaps vice versa. A *distributive law* (in the categorical sense, e.g., Street [1972]) provides a conversion from one to the other. In this chapter, we present a procedure to automatically synthesise these distributive operators, applying a generic programming methodology [Hinze, 2000] to compute these operations given the base type (e.g., $\mathsf{F}\alpha$ in the above description). This serves to ease the use of graded modal types in practice, removing boilerplate code by automatically generating these 'interfacing functions' on-demand, for user-defined data types as well as built-in types.

Throughout, we refer to distributive laws of the form $\Box_r(\mathsf{F}\alpha) \to \mathsf{F}(\Box_r\alpha)$ as *push* operations (as they 'push' the graded modality inside the type constructor F), and dually $\mathsf{F}(\Box_r\alpha) \to \Box_r(\mathsf{F}\alpha)$ as *pull* operations (as they 'pull' the graded modality outside F).

As a standalone methodology for generating a common class of graded programs, this "deriving mechanism" serves as a complement to the synthesis calculi of Chapter 3. Synthesis problems which exhibit this distributive behaviour pose issues in the existing calculi. However, in many cases the solution programs to these distributive problems are straightforwardly derivable from the type alone, making the costly enumerative search of type-directed synthesis unnecessary.

Thus, we present a tool for an automatic procedure which calculates distributive laws from graded types and present a formal analysis of their properties. This approach is realised in Granule, embedded into the compiler. In doing so, we extend our graded linear λ-calculus of Section 2.3.1 to incorporate data constructors and pattern matching, as well as recursive data types.

This extended calculus is defined in Section 4.1 providing an idealised, simply-typed subset of Granule with which we develop the core deriving mechanism. Section 4.2 gives the procedures for deriving *push* and *pull* operators for the calculus. Section 4.3 describes the details of how these procedures are realised in the Granule language. We then provide examples of how several other structural combinators in Granule may be derived using this tool in Section 4.4. Finally, Section 4.6 discusses more related and future work.

We start with a motivating example typifying the kind of software engineering impedance problem that distributive laws solve. We do so in Granule code since it is the main vehicle for the developments here.

4.0.1 *Motivating Example*

Consider the situation of projecting the first element of a pair. In Granule, this first-projection is defined and typed as the following polymorphic function (whose syntax is reminiscent of Haskell or ML):

```
fst : ∀ { a b : Type } . (a, b [0]) → a
fst (x, [y]) = x
```

Linearity is the default, so this represents a linear function applied to linear values. However, the second component of the pair has a *graded modal type*, written `b [0]`, which means that we can use the value "inside" the graded modality 0 times by first 'unboxing' this capability via the pattern match `[y]` which allows weakening to be applied in the body to discard `y` of type `b`. In calculus of Section 4.1, we denote '`b [0]`' as the type $\Box_0 b$.

The type for `fst` is however somewhat restrictive: what if we are trying to use such a function with a value (call it `myPair`) whose type is not of the form `(a, b [0])` but rather `(a, b) [r]` for some grading term `r` which permits weakening? Such a situation readily arises when we are composing functional code, say between libraries or between a library and user code. In this situation, `fst myPair` is ill-typed. Instead, we could define a different first projection function for use with `myPair : (a, b) [r]` as:

```
fst' : ∀ { a b : Type, s : Semiring, r : s }
     . {0 ⩽ r} ⇒ (a, b) [r] → a
fst' [(x, y)] = x
```

This implementation uses various language features of Granule to make it as general as possible. Firstly, the function is polymorphic in the grade r and in the semiring s of which r is an element. Next, a refinement constraint 0 ⩽ r specifies that by the pre-ordering ⩽ associated with the semiring s, that 0 is approximated by r (essentially, that r permits weakening). The rest of the type and implementation looks more familiar for computing a first projection, but now the graded modality is over the entire pair.

From a software engineering perspective, it is cumbersome to create alternate versions of generic combinators every time we are in a slightly different situation with regards the position of a graded modality. Fortunately, this is an example to which a general *distributive law* can be deployed. In this case, we could define the following distributive law of graded modalities over products, call it pushPair:

```
pushPair : ∀ { a b : Type, s : Semiring, r : s }
         . (a, b) [r] → (a [r], b [r])
pushPair [(x, y)] = ([x], [y])
```

This 'pushes' the graded modality r into the pair (via pattern matching on the modality and the pair inside it, and then reintroducing the modality on the right hand side via [x] and [y]), distributing the graded modality to each component. Given this combinator, we can now apply fst (pushPair myPair) to yield a value of type a [r], on which we can apply the Granule standard library function extract:

```
extract : ∀ { a : Type, s : Semiring, r : s }
        . {(1 : s) ⩽ r} ⇒ a [r] → a
extract [x] = x
```

to get the original a value we desired:

```
extract (fst (pushPair myPair)) : a
```

The pushPair function could be provided by the standard library, and thus we have not had to write any specialised combinators ourselves: we have applied supplied combinators to solve the problem.

Now imagine we have introduced some custom data type List on which we have a *map* function:

```
data List a = Cons a (List a) | Nil

map : ∀ { a b : Type } . (a → b) [0..∞] → List a → List b
map [f] Nil = Nil;
map [f] (Cons x xs) = Cons (f x) (map [f] xs)
```

Note that, via a graded modality, the type of map specifies that the parameter function, of type a → b is non-linear, used between 0 and ∞ times. Imagine now we have a value myPairList : (List (a, b)) [r] and we want to map first projection over it. But fst expects (a, b [0]) and even with pushPair we require (a, b) [r]. *We need another distributive law*, this time of the graded modality over the List data type. Since List was user-defined, we now have to roll our own `pushList` operation, and so we are back to having to make specialised combinators for our data types.

The crux of this chapter is that such distributive laws can be automatically calculated given the definition of a type. With our Granule implementation of this approach (Section 4.3), we can then solve this combination problem via the following composition of combinators:

```
map (extract . fst . push @(,)) (push @List myPairList) :
    List a
```

where the `push` operations are written with their base type via `@` (a type application) and whose definitions and types are automatically generated during type checking. Thus the `push` operation is a *data-type generic function* [Hinze, 2000]. This generic function is defined inductively over the structure of types, thus a programmer can introduce a new user-defined algebraic data type and have the implementation of the generic distributive law derived automatically. This reduces both the initial and future effort (e.g., if an ADT definition changes or new ADTs are introduced).

Dual to the above, there are situations where a programmer may wish to *pull* a graded modality out of a structure. This is possible with a dual distributive law, which could be written by hand as:

```
pullPair : ∀ { a b : Type, s : Semiring, m n : s }
         . (a [n], b [m]) → (a, b) [n ⊓ m]
pullPair ([x], [y]) = [(x, y)]
```

Note that the resulting grade is defined by the greatest-lower bound (meet) of `n` and `m`, if it exists as defined by a pre-order for semiring `s` (that is, ⊓ is not a total operation). This allows some flexibility in the use of the *pull* operation when grades differ in different components but have a greatest-lower bound which can be 'pulled out'. Our approach also allows such operations to be generically derived.

4.1 EXTENDING THE GRADED LINEAR-λ-CALCULUS

We define here a typing calculus which extends the graded linear λ-calculus of 2.3.1 with data constructors, pattern matching, and recursive data types. This language constitutes a simplified monomorphic subset of Granule. We include notions of data constructors and their elimination via **case** expressions as a way to unify the handling of regular type constructors.

The full syntax of terms and types is given by:

$$\begin{aligned} t ::= {} & x \mid t_1\, t_2 \mid \lambda x.t \mid [t] \mid C\, t_0 \ldots t_n \\ & \mid \mathbf{case}\ t\ \mathbf{of}\ p_1 \mapsto t_1; \ldots; p_n \mapsto t_n \mid \mathbf{letrec}\ x = t_1\ \mathbf{in}\ t_2 \qquad \text{(terms)} \end{aligned}$$

$$p ::= x \mid \_ \mid [p] \mid C\, p_0 \ldots p_n \qquad \text{(patterns)}$$

$$A, B ::= A \multimap B \mid \alpha \mid A \otimes B \mid A \oplus B \mid 1 \mid \Box_r A \mid \mu X.A \mid X \qquad \text{(types)}$$

$$C ::= () \mid \mathsf{inl} \mid \mathsf{inr} \mid (,) \qquad \text{(data constructors)}$$

For the most part, typing follows the calculus defined in section 3.1. Figure 4.1 gives the additional rules. We briefly explain the extensions introduced for this chapter.

$$\frac{\Gamma, x : A \vdash t_1 : A \qquad \Gamma', x : A \vdash t_2 : B}{\Gamma + \Gamma' \vdash \textbf{letrec}\ x\ =\ t_1\ \textbf{in}\ t_2 : B}\ \textsc{Letrec}$$

$$\frac{(C : B_1 \multimap ... \multimap B_n \multimap A) \in \mathrm{D}}{\emptyset\ \vdash\ C : B_1 \multimap ... \multimap B_n \multimap A}\ \textsc{Con}$$

$$\frac{\Gamma \vdash t : A \qquad \cdot \vdash p_i : A \rhd \Delta_i \quad \Gamma', \Delta_i \vdash t_i : B}{\Gamma + \Gamma' \vdash \textbf{case}\ t\ \textbf{of}\ p_1 \mapsto t_1; ...; p_n \mapsto t_n : B}\ \textsc{Case}$$

Figure 4.1: Typing rules for the extended graded linear λ-calculus

The LetRec rule provides recursive bindings in the standard way.

Data constructors with zero or more arguments are introduced via the Con rule. Here, the constructors that concern us are units, products, and coproducts (sums), given by D, a global set of data constructors with their types, defined:

$$\begin{aligned} D = &\{() : 1\} \\ &\cup \{(,) : A \multimap B \multimap A \otimes B \mid \forall A, B\} \\ &\cup \{\mathsf{inl} : A \multimap A \oplus B \mid \forall A, B\} \\ &\cup \{\mathsf{inr} : B \multimap A \oplus B \mid \forall A, B\} \end{aligned}$$

Constructors are eliminated by pattern matching via the CASE rule. Patterns p are typed by judgments of the form $?r \vdash p : A \rhd \Delta$ meaning that a pattern p has type A and produces a context of typed binders Δ (used, e.g., in the typing of the case branches). The information to the left of the turnstile denotes optional grade information arising from being in an unboxing pattern and is syntactically defined as either:

$$r :?R\ ::= - \mid r \qquad \text{(enclosing grade)}$$

where $-$ means the present pattern is not nested inside an unboxing pattern and r that the present pattern is nested inside an unboxing pattern for a graded modality with grade r.

The rules of pattern typing are given in Figure 4.2. The rule (PBox) provides graded modal elimination (an 'unboxing' pattern), propagating grade information into the typing of the sub-pattern. Thus

$$\frac{}{\cdot \vdash x : A \triangleright x : A}\ \textsc{Pvar} \qquad \frac{\cdot \vdash p_i : B_i \triangleright \Gamma_i}{\cdot \vdash C p_1..p_n : A \triangleright \Gamma_1, .., \Gamma_n}\ \textsc{Pcon}$$

$$\frac{r \vdash p : A \ \triangleright\ \Gamma}{\cdot \vdash [p] : \Box_r A \triangleright \Gamma}\ \textsc{Pbox}$$

$$\frac{r \vdash p_i : B_i \ \triangleright\ \Gamma_i \qquad |A| > 1 \Rightarrow 1 \sqsubseteq r}{r \vdash C p_1..p_n : A \triangleright \Gamma_1, .., \Gamma_n}\ [\textsc{Pcon}]$$

$$\frac{}{r \vdash\ x : A\ \triangleright\ x :_r A}\ [\textsc{Pvar}] \qquad \frac{0 \sqsubseteq r}{r \vdash\ \_ : A\ \triangleright\ \emptyset}\ [\textsc{Pwild}]$$

Figure 4.2: Pattern typing rules for the extended graded linear λ-calculus

case t **of** $[p] \to t'$ can be used to eliminate a graded modal value. Variable patterns are typed via two rules depending on whether the variable occurs inside an unbox pattern ([PVAR]) or not (PVAR), with the [PVAR] rule producing a binding with the grade of the enclosing box's grade r. As with variable patterns, constructor patterns are split between rules for patterns which either occur inside an unboxing pattern or not. In the former case, the grade information is propagated to the sub-pattern(s), with the additional constraint that if there is more than one data constructor for the type A (written $|A| > 1$), then the grade r must approximate 1 (written $1 \sqsubseteq r$) as pattern matching incurs a usage to inspect the constructor. The operation $|A|$ counts the number of data constructors for a type:

$$|1| = 1 \quad |A \multimap B| = 1 \quad |\Box_r A| = |A|$$
$$|A \oplus B| = 2(|A| + |B|) \quad |A \otimes B| = |A||B|$$
$$|\mu X.A| = |A[\mu X.A/X]|$$

and $|X|$ is undefined (or effectively 0) since we do not allow unguarded recursion variables in types. A type A must therefore involve a sum type for $|A| > 1$.

Since a wildcard pattern _ discards a value, this is only allowed inside an unboxing pattern where the enclosing grade permits weakening, captured via $0 \sqsubseteq r$ in rule [PWILD].

4.2 AUTOMATICALLY DERIVING *push* AND *pull*

Now that we have established the language, we describe the algorithmic calculation of distributive laws. Note that whilst our language is simply typed (monomorphic), it includes type variables (ranged

$$\begin{aligned}
[\![1]\!]^{\Sigma}_{\mathsf{push}}\ z &= \textbf{case}\ z\ \textbf{of}\ [()] \to () \\
[\![\alpha]\!]^{\Sigma}_{\mathsf{push}}\ z &= z \\
[\![X]\!]^{\Sigma}_{\mathsf{push}}\ z &= \Sigma(X)\ z \\
[\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}}\ z &= \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ [\![A]\!]^{\Sigma}_{\mathsf{push}}[x]; \\
&\qquad\qquad\qquad\ [\mathsf{inr}\ y] \to \mathsf{inr}\ [\![B]\!]^{\Sigma}_{\mathsf{push}}[y] \\
[\![A \otimes B]\!]^{\Sigma}_{\mathsf{push}}\ z &= \textbf{case}\ z\ \textbf{of}\ [(x, y)] \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}[x], [\![B]\!]^{\Sigma}_{\mathsf{push}}[y]) \\
[\![A \multimap B]\!]^{\Sigma}_{\mathsf{push}}\ z &= \lambda y.\textbf{case}\ z\ \textbf{of}\ [f] \to \\
&\qquad\qquad \textbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \textbf{of}\ [u] \to [\![B]\!]^{\Sigma}_{\mathsf{push}}[(f\ u)] \\
[\![\mu X.A]\!]^{\Sigma}_{\mathsf{push}}\ z &= \textbf{letrec}\ f = [\![A]\!]^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i / \alpha_i]}}_{\mathsf{push}}\ \textbf{in}\ f\ z
\end{aligned}$$

Figure 4.3: Interpretation rules for $[\![A]\!]_{\mathsf{push}}$

over by α) to enable the distributive laws to be derived on parametric types. In the implementation, these will really be polymorphic type variables, but the derivation procedure need only treat them as some additional syntactic type construct.

4.2.1 *Notation*

Let $\mathsf{F} : \mathsf{Type}^n \to \mathsf{Type}$ be an n-ary type constructor (i.e. a constructor which takes n type arguments), whose free type variables provide the n parameter types. We write $\mathsf{F}\overline{\alpha_i}$ for the application of F to type variables α_i for all $1 \leq i \leq n$.

4.2.2 *Push*

We automatically calculate *push* for F applied to n type variables $\overline{\alpha_i}$ as the operation:

$$[\![\mathsf{F}\overline{\alpha_i}]\!]_{\mathsf{push}} : \Box_r \mathsf{F}\overline{\alpha_i} \multimap \mathsf{F}(\overline{\Box_r \alpha_i})$$

where we require $1 \sqsubseteq r$ *if* $|\mathsf{F}\overline{\alpha_i}| > 1$ due to the [PCON] rule (e.g., if F contains a sum).

For types A closed with respect to recursion variables, let $[\![A]\!]_{\mathsf{push}} = \lambda z.[\![A]\!]^{\emptyset}_{\mathsf{push}}\ z$ given by an intermediate interpretation $[\![A]\!]^{\Sigma}_{\mathsf{push}}$ where Σ is a context of *push* combinators for the recursive type variables. This interpretation is defined by Figure 4.3. In the case of *push* on a value of type 1, we pattern match on the value, eliminating the graded modality via the unboxing pattern match and returning the unit value. For type variables, *push* is simply the identity of the value, while for recursion variables we lookup the X's binding in Σ and apply

$$
\begin{aligned}
[\![1]\!]^{\Sigma}_{\mathsf{pull}}\ z &= \textbf{case}\ z\ \textbf{of}\ () \to [()] \\
[\![\alpha]\!]^{\Sigma}_{\mathsf{pull}}\ z &= z \\
[\![X]\!]^{\Sigma}_{\mathsf{pull}}\ z &= \Sigma(X)\ z \\
[\![A \oplus B]\!]^{\Sigma}_{\mathsf{pull}}\ z &= \textbf{case}\ z\ \textbf{of}\ \ \mathsf{inl}\ x \to \textbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; \\
&\qquad\qquad\qquad\ \mathsf{inr}\ y \to \textbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] \\
[\![A \otimes B]\!]^{\Sigma}_{\mathsf{pull}}\ z &= \textbf{case}\ z\ \textbf{of}\ (x, y) \to \\
&\qquad\qquad \textbf{case}\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}}\ x, [\![B]\!]^{\Sigma}_{\mathsf{pull}}\ y)\ \textbf{of}\ ([u], [v]) \to [(u, v)] \\
[\![\mu X.A]\!]^{\Sigma}_{\mathsf{pull}}\ z &= \textbf{letrec}\ f = [\![A]\!]^{\Sigma, X \mapsto f : \mu X.A[\overrightarrow{\Box_{r_i}\alpha_i / \alpha_i}] \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \textbf{in}\ f\ z
\end{aligned}
$$

Figure 4.4: Interpretation rules for $[\![A]\!]_{\mathsf{pull}}$

it to the value. For sum and product types, *push* works by pattern matching on the type's constructor(s) and then inductively applying *push* to the boxed arguments, re-applying them to the constructor(s). Unlike *pull* below, the *push* operation can be derived for function types, with a contravariant use of *pull*. For recursive types, we inductively apply *push* to the value with a fresh recursion variable bound in Σ, representing a recursive application of push. There is no derivation of a distributive law for types which are themselves graded modalities.

Section B.2.3 in Appendix B gives the proof that $[\![A]\!]_{\mathsf{pull}}$ is type sound, i.e., its derivations are well-typed.

4.2.3 *Pull*

We automatically calculate *pull* for F applied to n type variables $\overline{\alpha_i}$ as the operation:

$$[\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{pull}} : \mathsf{F}\ (\overline{\Box_{r_i}\alpha_i}) \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mathsf{F}\ \overline{\alpha_i})$$

Type constructor F here is applied to n arguments each of the form $\Box_{r_i}\alpha_i$, i.e., each with a different grading of which the greatest-lower bound[1] $\bigwedge_{i=1}^{n} r_i$ is the resulting grade (see `pullPair` from Section 4.0.1).

For types A closed with respect to recursion variables, let $[\![A]\!]_{\mathsf{pull}} = \lambda z.[\![A]\!]^{\emptyset}_{\mathsf{pull}}\ z$ given by an intermediate interpretation $[\![A]\!]^{\Sigma}_{\mathsf{pull}}$ where Σ is a context of *pull* combinators for the recursive type variables. This interpretation is defined by Figure 4.4.

Just like *push*, we cannot apply *pull* to graded modalities themselves. Unlike *push*, we cannot apply *pull* to function types. That is, we cannot

1 The greatest-lower bound $\wedge$ is partial operation which can be defined in terms of the semiring's pre-order: $r \wedge s = t$ if $t \sqsubseteq r$, $t \sqsubseteq s$ and there exists no other t' where $t' \sqsubseteq r$ and $t' \sqsubseteq s$ and $t \sqsubseteq t'$.

derive a distributive law of the form $(\Box_r A \multimap \Box_r B) \multimap \Box_r(A \multimap B)$ since introducing the concluding $\Box_r$ would require the incoming function $(\Box_r A \multimap \Box_r B)$ to itself be inside $\Box_r$ due to the promotion rule (PR), which does not match the type scheme for *pull*.

The rest of the derivation above is similar but dual to that of *push*.

Section B.2.3 in Appendix B gives the proof that $[\![A]\!]_{\mathsf{pull}}$ is type sound, i.e., its derivations are well-typed.

Example 4.2.1. To illustrate the above procedures, the derivation of $\lambda z.[\![(\alpha \otimes \alpha) \multimap \beta]\!]_{\mathsf{push}}\ z : \Box_r((\alpha \otimes \alpha) \multimap \beta) \multimap ((\Box_r\alpha \otimes \Box_r\alpha) \multimap \Box_r\beta)$ is:

$$\begin{aligned}
& \lambda z.[\![(\alpha \otimes \alpha) \multimap \beta]\!]^{\varnothing}_{\mathsf{push}}\ z \\
= {} & \lambda z.\lambda y.\mathbf{case}\ z\ \mathbf{of}\ [f] \to \mathbf{case}\ [\![\alpha \otimes \alpha]\!]^{\varnothing}_{\mathsf{pull}}\ y\ \mathbf{of}\ [u] \to [\![\beta]\!]^{\varnothing}_{\mathsf{push}}[(f\ u)] \\
= {} & \lambda z.\lambda y.\mathbf{case}\ z\ \mathbf{of}\ [f] \to \\
& \qquad \mathbf{case}\ (\mathbf{case}\ y\ \mathbf{of}\ (x', y') \to \\
& \qquad\qquad \mathbf{case}\ ([\![\alpha]\!]^{\varnothing}_{\mathsf{pull}}\ x', [\![\alpha]\!]^{\varnothing}_{\mathsf{pull}}\ y')\ \mathbf{of}\ ([u], [v]) \to [(u, v)])\ \mathbf{of} \\
& \qquad\quad [u] \to [\![\beta]\!]^{\varnothing}_{\mathsf{push}}[(f\ u)] \\
= {} & \lambda z.\lambda y.\mathbf{case}\ z\ \mathbf{of}\ [f] \to \\
& \qquad \mathbf{case}\ (\mathbf{case}\ y\ \mathbf{of}\ (x', y') \to \\
& \qquad\qquad \mathbf{case}\ (x', y')\ \mathbf{of}\ ([u], [v]) \to [(u, v)])\ \mathbf{of}\ [u] \to [(f\ u)]
\end{aligned}$$

Remark 1. One might ponder whether linear logic's exponential $!A$ [Girard, 1987] is modelled by the graded necessity modality over $\mathbb{N}_\infty$ intervals, i.e., with $!A \triangleq \Box_{0..\infty}A$. This is a reasonable assumption, but $\Box_{0..\infty}A$ has a slightly different meaning to $!A$, exposed here: whilst $[\![A \otimes B]\!]_{\mathsf{push}} : \Box_{0..\infty}(A \otimes B) \multimap (\Box_{0..\infty}A \otimes \Box_{0..\infty}B)$ is derivable in our language, linear logic does not permit $!(A \otimes B) \multimap (!A \otimes !B)$. Models of ! provide only a monoidal functor structure which gives *pull* for $\otimes$, but not *push* [Benton et al., 1992]. This structure can be recovered in Granule through the introduction of a partial type-level operation which selectively disallows *push* for $\otimes$ in semirings which model the ! modality of linear logic[2].

The algorithmic definitions of 'push' and 'pull' can be leveraged in a programming context to automatically yield these combinators for practical purposes. We discuss how this is leveraged inside the Granule compiler in Section 4.3. Before that, we study the algebraic behaviour of the derived distributive laws.

4.2.4 *Properties*

We consider here the properties of these derived operations. Prima facie, the above *push* and *pull* operations are simply distributive laws

2 The work in Hughes et al. [2021] arose as a result of this work.

between two (parametric) type constructors F and $\Box_r$, the latter being the graded modality. However, both F and $\Box_r$ have additional structure. If the mathematical terminology of 'distributive laws' is warranted, then such additional structure should be preserved by *push* and *pull* (e.g., as in how a distributive law between a monad and a comonad must preserve the behaviour of the monad and comonad operations after applying the distributive law [Power and Watanabe, 2002]); we explain here the relevant additional structure and verify the distributive law properties.

We note that these distributive laws are mutually inverse:

Proposition 4.2.1 (Pull is right inverse to push)**.** *For all n-arity types* F *which do not contain function types, then for type variables* $(\alpha_i)_{i\in[1..n]}$ *and for all grades* $r \in \mathcal{R}$ *where* $1 \sqsubseteq r$ *if* $|\mathsf{F}\overline{\alpha_i}| > 1$*, then:*

$$[\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{pull}}([\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{push}}) = id \ : \Box_r\mathsf{F}\overline{\alpha_i} \multimap \Box_r\mathsf{F}\overline{\alpha_i}$$

Proposition 4.2.2 (Pull is left inverse to push)**.** *For all n-arity types* F *which do not contain function types, then for type variables* $(\alpha_i)_{i\in[1..n]}$ *and for all grades* $r \in \mathcal{R}$ *where* $1 \sqsubseteq r$ *if* $|\mathsf{F}\overline{\alpha_i}| > 1$*, then:*

$$[\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{push}}([\![\mathsf{F}\ \overline{\alpha_i}]\!]_{\mathsf{pull}}) = id \ : \mathsf{F}(\Box_r\overline{\alpha_i}) \multimap \mathsf{F}(\Box_r\overline{\alpha_i})$$

Section B.2.4 of Appendix B gives the proofs, leveraging an equational theory for our lanugage.

Additional properties of these distributive laws can be found in Hughes et al. [2020]. These include the naturality of *push* and *pull* (in a categorical sense), and their preservation of graded comonadic structure. We choose to omit this properties here both for brevity.

4.3 IMPLEMENTATION IN GRANULE

The Granule type checker implements the algorithmic derivation of *push* and *pull* distributive laws as covered in the previous section. Whilst the syntax of our language types had unit, sum, and product types as primitives, in Granule these are provided by a more general notion of type constructor which can be extended by user-defined, generalized algebraic data types (GADTs). The procedure outlined in Section 4.2 is therefore generalised slightly so that it can be applied to any data type: the case for $A \oplus B$ is generalised to types with an arbitrary number of data constructors.

Our deriving mechanism is exposed to programmers via explicit (visible) type application (akin to that provided in GHC Haskell [Eisenberg et al., 2016]) on reserved names `push` and `pull`. Written `push @T` or `pull @T`, this signals to the compiler that we wish to derive the corresponding distributive laws at the type `T`. For example, for the

`List : Type → Type` data type from the standard library, we can write the expression `push @List` which the type checker recognises as a function of type:

```
push @List : ∀ { a : Type, s : Semiring, r : s }
           . {1 ⩽ r} ⇒ (List a) [r] → List (a [r])
```

Note this function is not only polymorphic in the grade, but polymorphic in the semiring itself. Granule identifies different graded modalities by their semirings, and thus this operation is polymorphic in the graded modality. When the type checker encounters such a type application, it triggers the derivation procedure of Section 4.2, which also calculates the type. The result is then stored in the state of the frontend to be passed to the interpreter (or compiler) after type checking. The derived operations are memoized so that they need not be re-calculated if a particular distributive law is required more than once. Otherwise, the implementation largely follows Section 4.2 without surprises, apart from some additional machinery for specialising the types of data constructors coming from (generalized) ADTs.

4.3.1 *Examples*

At the start of this chapter, we motivated the crux of this work with a concrete example, which we can replay here in concrete Granule, using its type application technique for triggering the automatic derivation of the distributive laws. Previously, we defined `pushPair` by hand which can now be replaced with:

```
push @(,) : ∀ { a b : Type, s : Semiring, r : s }
          . (a, b) [r] → (a [r], b [r])
```

Note that in Granule `(,)` is an infix type constructor for products as well as terms. We could then replace the previous `fst'` with:

```
fst' : ∀ { a b : Type, r : Semiring }
     . {0 ⩽ r} ⇒ (a, b) [r] → a
fst' = let [x'] = fst (push @(,) x) in x'
```

The point however in the example is that we need not even define this intermediate combinator, but can instead write the following wherever we need to compute the first projection of `myPair : (a, b) [r]`:

```
extract (fst (push @(,) myPair)) : a
```

We already saw that we can then generalise this by applying this first projection inside of the list
`myPairList : (List (a, b)) [r]` directly, using `push @List`.

In a slightly more elaborate example, we can use the `pull` combinator for pairs to implement a function that duplicates a pair (given that both elements can be consumed twice):

```
copyPair : ∀ { a, b : Type }
         . (a [0..2], b [2..4]) → ((a, b), (a, b))
-- where, copy : a [2] → (a, a)
copyPair x = copy (pull @(,) x)
```

Note `pull` computes the greatest-lower bound of intervals `0..2` and `2..4` which is `2..2`, i.e., we can provide a pair of `a` and `b` values which can each be used exactly twice: exactly what is required for `copy`.

As another example, interacting with Granule's indexed types (GADTs), consider a simple programming task of taking the head of a sized-list (vector) and duplicating it into a pair. The `head` operation is typed:

```
head : ∀ { a : Type, n : Nat }
     . (Vec (n + 1) a) [0..1] → a
```

which has a graded modal input with grade `0..1` meaning the input vector is used 0 or 1 times: the head element is used once (linearly) for the return but the tail is discarded.

This head element can then be copied via a graded modality, e.g., a value of type `(Vec (n + 1) (a [2])) [0..1]` permits:

```
copyHead' : ∀ { a : Type, n : Nat :}
          . (Vec (n + 1) (a [2])) [0..1] → (a, a)
-- [y] unboxes (a [2]) to y:a usable twice
copyHead' xs = let [y] = head xs in (y, y)
```

Here we "unbox" the graded modal value of type `a [2]` to get a non-linear variable `y` which we can use precisely twice. However, what if we are in a programming context where we have a value `Vec (n + 1) a` with no graded modality on the type `a`? We can employ two idioms here: (i) take a value of type `(Vec (n + 1) a) [0..2]` and split its modality in two: `(Vec (n + 1) a) [2] [0..1]` (ii) then use *push* on the inner graded modality `[2]` to get `(Vec (n + 1) (a [2])) [0..1]`.

Using `push @Vec` we can thus write the following to duplicate the head element of a vector:

```
copyHead : ∀ { a : Type, n : Nat }
         . (Vec (n + 1) a) [0..2] → (a, a)
copyHead = copy . head . boxmap [push @Vec] . disject
```

which employs combinators from the standard library and the derived distributive law, of type:

```
boxmap    : ∀ { a b : Type, s : Semiring, r : s }
          . (a  → b) [r] → a [r] → b [r]
disject   : ∀ { a : Type, s : Semiring, n m : s }
          . a [m * n] → (a [n]) [m]
push @Vec : ∀ { a : Type, n : Nat, s : Semiring, r : s }
          . (Vec n a) [r] → Vec n (a [r])
```

$$
\begin{aligned}
[\![C^w]\!]^{\Sigma}_{\mathsf{drop}} z &= \mathtt{drop}\ z \\
[\![1]\!]^{\Sigma}_{\mathsf{drop}} z &= \mathbf{case}\ z\ \mathbf{of}\ () \to () \\
[\![X]\!]^{\Sigma}_{\mathsf{drop}} z &= \Sigma(X) z \\
[\![A \oplus B]\!]^{\Sigma}_{\mathsf{drop}} z &= \mathbf{case}\ z\ \mathbf{of}\ \ \mathsf{inl}\ x \to [\![A]\!]_{\mathsf{drop}}(x);\ \ \mathsf{inr}\ y \to [\![B]\!]_{\mathsf{drop}}(y) \\
[\![A \otimes B]\!]^{\Sigma}_{\mathsf{drop}} z &= \mathbf{case}\ z\ \mathbf{of}\ (x, y) \to \\
&\qquad\qquad \mathbf{case}\ [\![A]\!]_{\mathsf{drop}}(x)\ \mathbf{of}\ () \to \\
&\qquad\qquad\qquad \mathbf{case}\ [\![B]\!]_{\mathsf{drop}}(y)\ \mathbf{of}\ () \to () \\
[\![\mu X.A]\!]^{\Sigma}_{\mathsf{drop}} z &= \mathbf{letrec}\ f = [\![A]\!]^{\Sigma, X \mapsto f : A \multimap 1}_{\mathsf{drop}}\ \mathbf{in}\ f\ z
\end{aligned}
$$

Figure 4.5: Interpretation rules for $[\![A]\!]_{\mathsf{drop}}$

4.4 DERIVING OTHER USEFUL STRUCTURAL COMBINATORS

So far we have motivated the use of distributive laws, and demonstrated that they are useful in practice when programming in languages with linear and graded modal types. The same methodology we have been discussing can also be used to derive other useful generic combinators for programming with linear and graded modal types. In this section, we consider two structural combinators, `drop` and `copyShape`, in Granule as well as related type classes for dropping, copying, and moving resources in Linear Haskell.

4.4.1 *A Combinator for Weakening ("drop")*

The built-in type constants of Granule can be split into those which permit structural weakening C^{w} such as `Int`, `Char`, and those which do not C^{l} such as `Handle` (file handles) and `Chan` (concurrent channels).

Those that permit weakening contain non-abstract values that can in theory be systematically inspected in order to consume them. Granule provides a built-in implementation of `drop` for C^{w} types, which is then used by the derivation procedure of 4.5 to derive weakening on compound types.

Note we cannot use this procedure in a polymorphic context (over type variables α) since type polymorphism ranges over all types, including those which cannot be dropped like C^{l}.

4.4.2 *A Combinator for Copying "shape"*

The "shape" of values for a parametric data types F can be determined by a function *shape* : $\mathsf{F}A \to \mathsf{F}1$, usually derived when F is a functor by mapping with $A \to 1$ (dropping elements) [Jay and Cockett, 1994].

This provides a way of capturing the size, shape, and form of a data structure. Often when programming with data structures which must be used linearly, we may wish to reason about properties of the data structure (such as the length or "shape" of the structure) but we may not be able to drop the contained values. Instead, we wish to extract the shape but without consuming the original data structure itself.

This can be accomplished with a function which copies the data structure exactly, returning this duplicate along with a data structure of the same shape, but with the terminal nodes replaced with values of the unit type 1 (the 'spine'). For example, consider a pair of integers: `(1, 2)`. Then applying `copyShape` to this pair would yield `(((), ()), (1, 2))`. The original input pair is duplicated and returned on the right of the pair, while the left value contains a pair with the same structure as the input, but with values replaced with `()`. This is useful, as it allows us to use the left value of the resulting pair to reason about the structure of the input (e.g., its depth / size), while preserving the original input. This is particularly useful for deriving size and length combinators for collection-like data structures. As with "drop", we can derive such a function automatically:

$$[\![\mathsf{F}\alpha]\!]_{\mathsf{copyShape}} : \mathsf{F}\alpha \multimap \mathsf{F1} \otimes \mathsf{F}\alpha$$

defined by $[\![A]\!]_{\mathsf{copyShape}} = \lambda z.[\![A]\!]^{\emptyset}_{\mathsf{copyShape}} z$ by an intermediate interpretation $[\![A]\!]^{\Sigma}_{\mathsf{copyShape}}$, given by Figure 4.6. The implementation recursively follows the structure of the type, replicating the constructors, reaching the crucial case where a polymorphically type $z : \alpha$ is mapped to $((), z)$ in the third equation.

Granule implements both these derived combinators in a similar way to *push*/*pull* providing `copyShape` and `drop` which can be derived for a type `T` via type application, e.g. `drop @T : T → ()` if it can be derived. Otherwise, the type checker produces an error, explaining why `drop` is not derivable at type `T`.

4.5 RELATED WORK

In this section we consider some of the wider related work as they relate to the ideas presented in this chapter.

4.5.1 *Generic Programming Methodology*

The deriving mechanism for Granule is based on the methodology of generic functional programming [Hinze, 2000], where functions may be defined generically for all possible data types in the language; generic functions are defined inductively on the structure of the types. This technique has notably been used before in Haskell, where there has been a strong interest in deriving type class instances

$$
\begin{aligned}
[\![C^w]\!]^{\Sigma}_{\mathsf{copyShape}} z &= ((),\ z)\\
[\![1]\!]^{\Sigma}_{\mathsf{copyShape}} z &= \textbf{case}\ z\ \textbf{of}\ () \to ((),\ ())\\
[\![\alpha]\!]^{\Sigma}_{\mathsf{copyShape}} z &= ((), z)\\
[\![X]\!]^{\Sigma}_{\mathsf{copyShape}} z &= \Sigma(X) z\\
[\![A \oplus B]\!]^{\Sigma}_{\mathsf{copyShape}} z &= \textbf{case}\ z\ \textbf{of}\\
&\qquad\qquad \mathsf{inl}\ x \to \textbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{copyShape}}(x)\ \textbf{of}\ (s,\ x') \to\\
&\qquad\qquad\qquad\qquad (\mathsf{inl}\ s,\ \mathsf{inr}\ x')\\
&\qquad\qquad \mathsf{inr}\ y \to \textbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{copyShape}}(y)\ \textbf{of}\ (s,\ y') \to\\
&\qquad\qquad\qquad\qquad (\mathsf{inr}\ s,\ \mathsf{inr}\ y')\\
[\![A \otimes B]\!]^{\Sigma}_{\mathsf{copyShape}} z &= \textbf{case}\ z\ \textbf{of}\ (x, y) \to\\
&\qquad\qquad \textbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{copyShape}}(x)\ \textbf{of}\ (s,\ x') \to\\
&\qquad\qquad\qquad \textbf{case}\ [\![B]\!]^{\Sigma}_{\mathsf{copyShape}}(y)\ \textbf{of}\\
&\qquad\qquad\qquad\qquad (s',\ y') \to ((s,\ s'),\ (x',\ y'))\\
[\![\mu X.A]\!]^{\Sigma}_{\mathsf{copyShape}} z &= \textbf{letrec}\ f = [\![A]\!]^{\Sigma, X \mapsto f : A \multimap 1 \otimes A}_{\mathsf{copyShape}}\ \textbf{in}\ f\ z
\end{aligned}
$$

Figure 4.6: Interpretation rules for $[\![A]\!]_{\mathsf{copyShape}}$

automatically. Particularly relevant to this work is the work on generic deriving [Magalhães et al., 2010], which allows Haskell programmers to automatically derive arbitrary class instances using standard datatype-generic programming techniques as described above.

4.5.2 *Non-graded Distributive Laws*

Distributive laws are standard components in abstract mathematics. Distributive laws between categorical structures used for modelling modalities (like monads and comonads) are well explored. For example, Brookes and Stone [1993] defined a categorical semantics using monads combined with comonads via a distributive law capturing both intensional and effectful aspects of a program. Power and Watanabe [2002] study in detail different ways of combining comonads and monads via distributive laws. Such distributive laws have been applied in the programming languages literature, e.g., for modelling streams of partial elements [Uustalu and Vene, November 2006].

4.5.3 *Graded Distributive Laws*

Gaboardi et al. define families of graded distributive laws for graded monads and comonads [Gaboardi et al., 2016b]. They include the ability to interact the grades, e.g., with operations such as $\Box_{\iota(r,f)}\Diamond_f A \to \Diamond_{\kappa(r,f)}\Box_r A$ between a graded comonad $\Box_r$ and graded monad $\Diamond_f$ where ι and κ capture information about the distributive law in the grades. In comparison, our distributive laws here are more prosaic since they involve only a graded comonad (semiring graded necessity) distributed over a functor and vice versa. That said, the scheme of Gaboardi et al. suggests that there might be interesting graded distributive laws between $\Box_r$ and the indexed types, for example, $\Box_r(\mathsf{Vec}\, n\, A) \to \mathsf{Vec}\, (r * n)\, (\Box_1 A)$ which internally replicates a vector. However, it is less clear how useful such combinators would be in general or how systematic their construction would be. In contrast, the distributive laws explained here appear frequently and have a straightforward uniform calculation.

We noted in Section 4.2 that neither of our distributive laws can be derived over graded modalities themselves, i.e., we cannot derive $push : \Box_r\Box_s A \to \Box_s\Box_r A$. Such an operation would itself be a distributive law between two graded modalities, which may have further semantic and analysis consequences beyond the normal derivations here for regular types. Exploring this is future work, for which the previous work on graded distributive laws can provide a useful scheme for considering the possibilities here. Furthermore, Granule has both graded comonads and graded monads so there is scope for exploring possible graded distributive laws between these in the future following Gaboardi et al. [Gaboardi et al., 2016b].

4.6 CONCLUSION

The work described here addresses the practical aspects of applying these techniques in real-world programming. Our hope is that this aids the development of the next generation of programming languages with rich type systems for high-assurance programming.

The calculus of this chapter serves somewhat as a bridge between the systems in Chapters 3 and 5. In the next chapter, we continue on from Chapter 3, presenting a synthesis calculus for a fully graded typing calculus. This system, however, is significantly more expressive than the calculus of Section 3.1, incorporating ADTs, pattern matching, recursion, and polymorphism.

5

AN EXTENDED SYNTHESIS CALCULUS

So far, we have considered a language with a several basic types, but which falls short of the full expressive power of Granule. In this chapter, we consider a target language which is significantly more expressive than what we have seen thus far, constituting a fully-fledged functional programming language.

The above features could all have been added to the calculi from Chapter 3, however, we take the inclusion of these new language features as an opportunity to also explore synthesis in a fully graded type system. As mentioned in 2, the fully-graded linear-lambda calculus is one of the two dominant flavours of quantitative type system. This approach is common amongst implementations of quantitative type systems, such as Idris 2, the Linear Types language extension to GHC, and the *GradedBase* language extension of Granule.

We begin by extending the calculus of Section 2.3.2, adding the synthesis of recursion, polymorphic user-defined algebraic data types, as well as the synthesis of recursive function definitions from polymorphic type schemes. Section 5.1 provides a full formal description of our target language.

We then provide a program synthesis calculus with an additive resource management scheme for this language in Section 5.2, describing the rules in turn, introducing some additional features such as the ability to specify input-output examples for a desired synthesis problem (Section 5.3), and additional post-synthesis refactoring procedures(Section 5.4). We then apply focusing to this calculus in Section 5.5, as we did in Chapter 3, using this focused form as the basis of the implementation of our synthesis tool.

The synthesis tool is evaluated We evaluate our implementation on a set of 46 benchmarks (Section 5.6), including several non-trivial programs which use algebraic data types and recursion.

In Section 5.7, to demonstrate the practicality and versatility of our approach, we apply our algorithm to synthesising programs in Haskell from type signatures that use GHC's *linear types* extension.

5.1 A FULLY GRADED TARGET LANGUAGE

We now formally define our target language, extending the fully graded λ-calculus of Chapter 2. Our language comprises the λ-calculus extended with grades and a graded necessity modality, arbitrary user-defined recursive algebraic data types (ADTs), as well as rank-1 polymorphism. The syntax of types is given by:

$$A, B ::= A^r \to B \mid K \mid A\, B \mid \Box_r A \mid \mu X.A \mid X \mid \alpha \qquad (\textit{types})$$

$$K ::= 1 \mid \otimes \mid \oplus \qquad (\textit{type constructors})$$

$$\tau ::= \forall \overline{\alpha : \kappa}.A \qquad (\textit{type schemes})$$

Recursive types $\mu X.A$ are equi-recursive (although we also provide explicit typing rules) with type recursion variables X. Data constructors and other top-level definitions are typed by type schemes τ (rank-1 polymorphic types), which bind a set of kind-annotated universally quantified type variables $\overline{\alpha : \kappa}$ à la ML [Milner, 1978]. Thus, types may contain type variables α. Kinds κ are standard, given by Figure 5.1.

The syntax of terms is given by:

$$t ::= x \mid \lambda x.t \mid t_1\, t_2 \mid [t] \mid C\, t_1 \ldots t_n \mid \textbf{case}\ t\ \textbf{of}\ p_1 \mapsto t_1; \ldots; p_n \mapsto t_n \qquad (\textit{terms})$$

$$p ::= x \mid \_ \mid [p] \mid C\, p_1 \ldots p_n \qquad (\textit{patterns})$$

Terms consist of a graded λ-calculus, a *promotion* construct $[t]$ which introduces a graded modality explicitly, as well as data constructor introduction ($C\, t_1 \ldots t_n$) and elimination via **case** expressions with patterns, which are defined via the syntax of patterns p.

Typing judgements have the form $\Sigma; \Gamma \vdash t : A$ assigning a type A to a term t under type variables Σ and variable context Γ, given by:

$$\Delta, \Gamma ::= \emptyset \mid \Gamma, x :_r A \qquad (\textit{contexts})$$

That is, a context may be empty $\emptyset$ or extended with a *graded* assumption $x :_r A$. Graded assumptions must be used in a way which adheres to the constraints of the grade r. Structural exchange is permitted, allowing a context to be arbitrarily reordered. A global context D parametrises the system, containing top-level definitions and data constructors annotated with type schemes. A context of kind annotated type variables Σ is used for kinding and when instantiating a type scheme from D.

Given a typing judgment $\Sigma; \Gamma \vdash t : A$ we say that t is both *well typed* and *well resourced* to highlight the role of grading in accounting for resource use via the semiring information. Another judgment types top-level terms (definitions) with polymorphic type schemes:

$$\frac{\overline{\alpha : \kappa}; \emptyset \vdash t : A}{\emptyset; \emptyset \vdash t : \forall \overline{\alpha : \kappa}.A} \ \textsc{TopLevel}$$

This rule takes the type scheme and adds its universally quantified type variables to Σ, where they can be used subsequently in the typing rules. The rule's premise then types the body at A, using the typing rules for terms of Figure 5.2, whose rules help explain the meaning of the syntax with reference to their static semantics.

$$\frac{}{\Sigma, \alpha : \kappa \vdash \alpha : \kappa} \ \kappa_{\textsc{Var}} \qquad \frac{\Sigma \vdash A : \mathsf{Type}}{\Sigma \vdash \Box_r A : \mathsf{Type}} \ \kappa_{\Box}$$

$$\frac{\Sigma \vdash A : \mathsf{Type} \qquad \Sigma \vdash B : \mathsf{Type}}{\Sigma \vdash A^r \to B : \mathsf{Type}} \ \kappa_{\to}$$

$$\frac{\Sigma \vdash A : \kappa_1 \to \kappa_2 \qquad \Sigma \vdash B : \kappa_1}{\Sigma \vdash A\,B : \kappa_2} \ \kappa_{\textsc{App}}$$

$$\frac{}{\Sigma \vdash 1 : \mathsf{Type}} \ \kappa_{\textsc{Unit}} \qquad \frac{\Sigma \vdash A : \kappa \qquad \Sigma \vdash B : \kappa}{\Sigma \vdash A \otimes B : \kappa} \ \kappa_{\otimes}$$

$$\frac{\Sigma \vdash A : \kappa \qquad \Sigma \vdash B : \kappa}{\Sigma \vdash A \oplus B : \kappa} \ \kappa_{\oplus}$$

$$\frac{\Sigma \vdash A : \mathsf{Type}}{\Sigma \vdash \mu X.A : \mathsf{Type}} \ \kappa_{\mu 1} \qquad \frac{\Sigma \vdash A : \mathsf{Type}}{\Sigma \vdash A[\mu X.A/X] : \mathsf{Type}} \ \kappa_{\mu 2}$$

Figure 5.1: Kinding rules for the fully graded typing calculus

Top-level definitions are typed by the Def rule. The definition x must be present in the global definition context D, with the type scheme $\forall \overline{\alpha : \kappa}.A'$. The type A results from instantiating all of the universal variables to types via the judgment $\Sigma \vdash A = \mathsf{inst}(\forall \overline{\alpha : \kappa}.A')$ in a standard way as in Algorithm W [Milner, 1978].

The graded λ-calculus fragment of our language remains mostly the same as in Section 2.3.2. However, the Var rule also checks the kind of the assumption x's type in the premise.

Recursion is typed via the μ_1 and μ_2 rules, in a standard way.

Introduction and elimination of data constructors is given by the Con and Case rules respectively, with Case also handling graded modality elimination via pattern matching. For Con, we may type

$$\frac{\Sigma \vdash A : \mathsf{Type}}{\Sigma; 0 \cdot \Gamma, x :_1 A \vdash x : A}\ \textsc{Var}$$

$$\frac{(x : \forall \overline{\alpha : \kappa}.A') \in D \qquad \Sigma \vdash A = \mathsf{inst}(\forall \overline{\alpha : \kappa}.A')}{\Sigma; 0 \cdot \Gamma \vdash x : A}\ \textsc{Def}$$

$$\frac{\Sigma; \Gamma, x :_r A \vdash t : B}{\Sigma; \Gamma \vdash \lambda x.t : A^r \to B}\ \textsc{Abs} \qquad \frac{\Sigma; \Gamma_1 \vdash t_1 : A^r \to B \qquad \Gamma_2 \vdash t_2 : A}{\Sigma; \Gamma_1 + r \cdot \Gamma_2 \vdash t_1\, t_2 : B}\ \textsc{App}$$

$$\frac{\Sigma; \Gamma \vdash t : A}{\Sigma; r \cdot \Gamma \vdash [t] : \Box_r A}\ \textsc{Pr} \qquad \frac{\Sigma; \Gamma, x :_r A, \Gamma' \vdash t : B \qquad r \sqsubseteq s}{\Sigma; \Gamma, x :_s A, \Gamma' \vdash t : B}\ \textsc{Approx}$$

$$\frac{\begin{array}{l}(C : \forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A}') \in D \\ \Sigma \vdash B_1{}^{q_1} \to \ldots \to B_n{}^{q_n} \to K\vec{A} = \mathsf{inst}(\forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A}')\end{array}}{\Sigma; 0 \cdot \Gamma \vdash C : B_1{}^{q_1} \to \ldots \to B_n{}^{q_n} \to K\vec{A}}\ \textsc{Con}$$

$$\frac{\Sigma; \Gamma \vdash t : A \qquad \Sigma; r \vdash p_i : A \rhd \Delta_i \qquad \Sigma; \Gamma', \Delta_i \vdash t_i : B}{\Sigma; r \cdot \Gamma + \Gamma' \vdash \textbf{case}\ t\ \textbf{of}\ p_1 \mapsto t_1; \ldots; p_n \mapsto t_n : B}\ \textsc{Case}$$

$$\frac{\Sigma; \Gamma \vdash t : A[\mu X.A/X]}{\Sigma; \Gamma \vdash t : \mu X.A}\ \mu_1 \qquad \frac{\Sigma; \Gamma \vdash t : \mu X.A}{\Sigma; \Gamma \vdash t : A[\mu X.A/X]}\ \mu_2$$

Figure 5.2: Typing rules for the fully graded polymorphic calculus

$$\frac{0 \sqsubseteq r \qquad \Sigma \vdash A : \mathsf{Type}}{\Sigma; r \vdash \_ : A \rhd \emptyset}\ \textsc{PWild}$$

$$\frac{\Sigma \vdash A : \mathsf{Type}}{\Sigma; r \vdash x : A \rhd x :_r A}\ \textsc{PVar} \qquad \frac{\Sigma; r \cdot s \vdash p : A \rhd \Gamma}{\Sigma; r \vdash [p] : \Box_s A \rhd \Gamma}\ \textsc{PBox}$$

$$\frac{\begin{array}{l}(C : \forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A}') \in D \\ \Sigma \vdash B_1{}^{q_1} \to \ldots \to B_n{}^{q_n} \to K\vec{A} = \mathsf{inst}(\forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A}') \\ \Sigma; q_i \cdot r \vdash p_i : B_i \rhd \Gamma_i \qquad |K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq r\end{array}}{\Sigma; r \vdash C\, p_1 \ldots p_n : K\vec{A} \rhd \vec{\Gamma_i}}\ \textsc{PCon}$$

Figure 5.3: Pattern typing rules of for the fully graded typing calculus

a data constructor C of some data type $K\vec{A}$ (with zero or more type parameters represented by $\vec{A}$) if it is present in the global context of data constructors D. Data constructors are closed requiring our context Γ to have zero-use grades, thus we scale Γ by 0. Elimination of data constructors take place via pattern matching over a constructor. Patterns p are typed by the judgement $r \vdash p : A \rhd \Delta$ which states that a pattern p has type A and produces a context of typed binders Δ. The grade r to the left of the turnstile represents the grade information arising from usage in the context generated by this pattern match. The pattern typing rules are given by Figure 5.3.

Variable patterns are typed by PVar, which simply produces a singleton context containing an assumption $x :_r A$ from the variable pattern with any grade r. A wildcard pattern _, typed by the PWild rule, is only permissible with grades that allow for weakening, i.e., where $0 \sqsubseteq r$. Pattern matching over data constructors is handled by the PCON rule. A data constructor may have up to zero or more sub-patterns ($p_1 ... p_n$), each of which is typed under the grade $q_i \cdot r$ (where q_i is the grade of corresponding argument type for the constructor, as defined in D). Additionally, we have the constraint $|K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq r$ which witnesses the fact that if there is more than one data constructor for the data type (written $|K\vec{A}| > 1$), then r must approximate 1 because pattern matching on a data constructor incurs some usage since it reveals information about that constructor.[1] By contrast, pattern matching on a type with only one constructor cannot convey any information by itself and so no usage requirement is imposed. Finally, elimination of a graded modality (often called *unboxing*) takes place via the PBOX rule, with syntax $[p]$. Like PCON, this rule propagates the grade information of the box pattern's type s to the enclosed sub-pattern p, yielding a context with the grades $r \cdot s$. One may observe that PBOX (and by extension PR) could be considered as special cases of PCON (and CON respectively), if we were to treat our promotion construct as a data constructor with the type $A^r \multimap \Box_r A$. We find it helpful to keep explicit modality introduction and elimination distinct from constructors, however, particularly with regard to synthesis.

5.1.1 *Metatheory*

Lastly we note that the fully graded system also enjoys admissibility of substitution [Abel and Bernardy, 2020] which is critical in type preservation proofs, and is needed in our proof of soundness for synthesis:

Lemma 5.1.1 (Admissibility of substitution). *Let* $\Delta \vdash t' : A$, *then: If* $\Gamma, x :_r A, \Gamma' \vdash t : B$ *then* $\Gamma + (r \cdot \Delta) + \Gamma' \vdash [t'/x]t : B$

5.2 A FULLY GRADED SYNTHESIS CALCULUS

Having defined the target language, we define our synthesis calculus, which uses the *additive* approach to resource management (see Section 3.3.2), with judgements:

$$\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta$$

That is, given an input context Γ, for goal type A we can synthesise the term t with the output context Δ describing how variables were used in t. As with the typing rules, top-level definitions and data constructors in scope are contained in a set D, which parametrises the system. Σ is a context of kind-annotated type variables, which we elide

1 A discussion of this additional constraint on grades for case expressions is given by Hughes et al. [2020] comparing how this manifests in various approaches.

in rules where it is passed inductively to the premise(s). The graded context Δ need not use all the variables in Γ, nor with exactly the same grades. Instead, the relationship between synthesis and typing is given by the central soundness result, which we state up-front:

Theorem 5.2.1 (Soundness of synthesis)**.** *Given a particular pre-ordered semiring $\mathcal{R}$ parametrising the calculi, then:*

1. *For all contexts Γ and Δ, types A, terms t:*

$$\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta \quad \implies \quad \Sigma; \Delta \vdash t : A$$

i.e. t has type A under context Δ whose grades capture variable use in t.

2. *At the top-level, for all type schemes $\forall \overline{\alpha : \kappa}.A$ and terms t then:*

$$\emptyset; \emptyset \vdash \forall \overline{\alpha : \kappa}.A \Rightarrow t \mid \emptyset \quad \implies \quad \emptyset; \emptyset \vdash t : \forall \overline{\alpha : \kappa}.A$$

The first part of soundness on its own does not guarantee that a synthesised program t is *well resourced*, i.e., the grades in Δ may not be approximated by the grades in Γ. For example, a valid judgement (whose more general rule is seen shortly) under semiring $\mathbb{N}_{\equiv}$ is:

$$x :_2 A \vdash A \Rightarrow^+ x; x :_1 A$$

i.e., for goal A, if x has type A in the context then we synthesis x as the result program, regardless of the grades. A synthesis judgement such as this may be part of a larger derivation in which the grades eventually match, i.e., this judgement forms part of a larger derivation which has a further sub-derivation in which x is used again and thus the total usage for x is eventually 2 as prescribed by the input context. However, at the level of an individual judgement we do not guarantee that the synthesised term is well-resourced. A reasonable *pruning condition* that could be used to assess whether any synthesis judgement is *potentially* well-resourced is $\exists \Delta'.(\Delta + \Delta') \sqsubseteq \Gamma$, i.e., there is some additional usage Δ' (that might come from further on in the synthesis process) that 'fills the gap' in resource use to produce $\Delta + \Delta'$ which is overapproximated by Γ. In this example, $\Delta' = x :_1 A$ would satisfy this constraint, explaining that there is some further possible single usage which will satisfy the incoming grade. However, Chapter 3 showed that excessive pruning at every step becomes too costly in a general setting. Instead, we apply such pruning more judiciously, only requiring that variable use is well-resourced at the point of synthesising binders. Therefore synthesised closed terms are always well-resourced (second part of the soundness theorem).

Section B.3.1 of Appendix B provides the soundness proof, which in part resembles a translation from sequent calculus to natural deduction, but also with the management of grades between synthesis and type checking.

$$\frac{(x : \forall \overline{\alpha : \kappa}.A') \in D \qquad \Sigma \vdash A = \mathrm{inst}(\forall \overline{\alpha : \kappa}.A')}{\Sigma; \Gamma \vdash A \Rightarrow x \mid 0 \cdot \Gamma}\ \textsc{Def}$$

$$\frac{\overline{\alpha : \kappa}; \emptyset \vdash A \Rightarrow t \mid \emptyset}{\emptyset; \emptyset \vdash \forall \overline{\alpha : \kappa}.A \Rightarrow t \mid \emptyset}\ \textsc{TopLevel} \qquad \frac{\Sigma \vdash A : \mathrm{Type}}{\Sigma; \Gamma, x :_r A \vdash A \Rightarrow x \mid 0 \cdot \Gamma, x :_1 A}\ \textsc{Var}$$

$$\frac{\Sigma; \Gamma, x :_q A \vdash B \Rightarrow t \mid \Delta, x :_r A \qquad r \sqsubseteq q}{\Sigma; \Gamma \vdash A^q \to B \Rightarrow \lambda x.t \mid \Delta}\ \to_{\mathrm{R}}$$

$$\frac{\begin{array}{c}\Sigma; \Gamma, x_1 :_{r_1} A^q \to B, x_2 :_{r_1} B \vdash C \Rightarrow t_1 \mid \Delta_1, x_1 :_{s_1} A^q \to B, x_2 :_{s_2} B \\ \Sigma; \Gamma, x_1 :_{r_1} A^q \to B \vdash A \Rightarrow t_2 \mid \Delta_2, x_1 :_{s_3} A^q \to B \qquad \Sigma \vdash A^q \to B : \mathrm{Type}\end{array}}{\Sigma; \Gamma, x_1 :_{r_1} A^q \to B \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + s_2 \cdot q \cdot \Delta_2), x_1 :_{s_2 + s_1 + (s_2 \cdot q \cdot s_3)} A^q \to B}\ \to_{\mathrm{L}}$$

$$\frac{\begin{array}{l}(C : \forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A}') \in D \\ \Sigma \vdash B_1{}^{q_1} \to \ldots \to B_n{}^{q_n} \to K\vec{A} = \mathrm{inst}(\forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A}') \\ \Sigma; \Gamma \vdash B_i \Rightarrow t_i \mid \Delta_i\end{array}}{\Sigma; \Gamma \vdash K\vec{A} \Rightarrow C\, t_1 \ldots t_n \mid 0 \cdot \Gamma + (q_1 \cdot \Delta_1) + \ldots + (q_n \cdot \Delta_n)}\ \mathrm{C_R}$$

$$\frac{\begin{array}{l}(C_i : \forall \overline{\alpha : \kappa}.B_1'^{\,q_1^i} \to \ldots \to B_n'^{\,q_n^i} \to K\vec{A}') \in D \qquad \Sigma \vdash K\vec{A} : \mathrm{Type} \\ \Sigma \vdash B_1{}^{q_1} \to \ldots \to B_n{}^{q_n} \to K\vec{A} = \mathrm{inst}(\forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A}') \\ \Sigma; \Gamma, x :_r K\vec{A}, y_1^i :_{r \cdot q_1^i} B_1, \ldots, y_n^i :_{r \cdot q_1^i} B_n \vdash B \Rightarrow t_i \mid \Delta_i, x :_{r_i} K\vec{A}, y_1^i :_{s_1^i} B_1, \ldots, y_n^i :_{s_n^i} B_n \\ \exists s'^i_j.\, s_j^i \sqsubseteq s'^i_j \cdot q_j^i \sqsubseteq r \cdot q_j^i \qquad s_i = s'^i_1 \sqcup \ldots \sqcup s'^i_n \qquad |K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq s_1 \sqcup \ldots \sqcup s_m\end{array}}{\Sigma; \Gamma, x :_r K\vec{A} \vdash B \Rightarrow \textbf{case}\ x\ \textbf{of}\ \overline{C_i\, y_1^i \ldots y_n^i \mapsto t_i} \mid (\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1 \sqcup \ldots \sqcup r_m) + (s_1 \sqcup \ldots \sqcup s_m)} K\vec{A}}\ \mathrm{C_L}$$

$$\frac{\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta}{\Sigma; \Gamma \vdash \Box_r A \Rightarrow [t] \mid r \cdot \Delta}\ \Box_{\mathrm{R}}$$

$$\frac{\begin{array}{c}\Sigma; \Gamma, y :_{r \cdot q} A, x :_r \Box_q A \vdash B \Rightarrow t \mid \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \\ \exists s_3.\, s_1 \sqsubseteq s_3 \cdot q \sqsubseteq r \cdot q \qquad \Sigma \vdash \Box_q A : \mathrm{Type}\end{array}}{\Sigma; \Gamma, x :_r \Box_q A \vdash B \Rightarrow \textbf{case}\ x\ \textbf{of}\ [y] \to t \mid \Delta, x :_{s_3 + s_2} \Box_q A}\ \Box_{\mathrm{L}}$$

$$\frac{D; \Sigma; \Gamma \vdash A[\mu X.A/X] \Rightarrow t \mid \Delta}{D; \Sigma; \Gamma \vdash \mu X.A \Rightarrow t \mid \Delta}\ \mu_{\mathrm{R}} \qquad \frac{D; \Sigma; \Gamma, x :_r A[\mu X.A/X] \vdash B \Rightarrow t \mid \Delta}{D; \Sigma; \Gamma, x :_r \mu X.A \vdash B \Rightarrow t \mid \Delta}\ \mu_{\mathrm{L}}$$

Figure 5.4: Collected rules of the fully graded synthesis calculus

For open terms, the implementation checks that from a user-given top-level goal A for which $\Gamma \vdash A \Rightarrow t \mid \Delta$ is derivable then t is only provided as a valid (well-typed and well-resourced) result if $\Delta \sqsubseteq \Gamma$.

We next present the synthesis calculus in stages. Each type former of the core calculus (with the exception of type variables) has two corresponding synthesis rules: a right rule for introduction (labelled R) and a left rule for elimination (labelled L). We frequently apply the algorithmic reading of the judgements, where meta-level terms to the left of $\Rightarrow$ are inputs (i.e., context Γ and goal type A) and terms to the right of $\Rightarrow$ are outputs (i.e., the synthesised term t and the usage context Δ). The synthesis calculus is non-deterministic, i.e., for any Γ and A there may be many possible t and Δ such that $\Gamma \vdash A \Rightarrow^{+} t; \Delta$.

5.2.1 *Core Synthesis Rules*

5.2.1.1 *Top-level*

We begin with the TopLevel rule, which is for a judgment form with a type scheme goal instead of just a type, providing the entry-point to synthesis:

$$\frac{\overline{\alpha : \kappa}; \emptyset \vdash A \Rightarrow t \mid \emptyset}{\emptyset; \emptyset \vdash \forall \overline{\alpha : \kappa}.A \Rightarrow t \mid \emptyset}\ \textsc{TopLevel}$$

This rule takes the universally quantified type variables $\overline{\alpha : \kappa}$ from the type scheme and adds them to the type variable context Σ; type variables are only equal to themselves. This rule corresponds to the generalisation step of typing polymorphic definitions [Milner, 1978].

5.2.1.2 *Variables*

For any goal type A, if there is a variable in the context matching this type then it can be synthesised for the goal, given by the terminal rule:

$$\frac{\Sigma \vdash A : \text{Type}}{\Sigma; \Gamma, x :_r A \vdash A \Rightarrow x \mid 0 \cdot \Gamma, x :_1 A}\ \textsc{Var}$$

Said another way, to synthesise the use of a variable x, we require that x be present in the input context Γ. The output context here then explains that only variable x is used: it consists of the entirety of the input context Γ scaled by grade 0 (using Definition 2.3.2), extended with $x :_1 A$, i.e. a single usage of x as denoted by the 1 element of the semiring. Maintaining this zeroed Γ in the output context simplifies subsequent rules by avoiding excessive context membership checks.

The Var rule permits the synthesis of terms which may not be well-resourced, e.g., if $r = 0$, the rule still synthesises a use of x. This is locally ill-resourced, but is acceptable at the global level as we check that an assumption has been used correctly in the rule where the assumption is bound. This does leads us to consider some branches of synthesis that are guaranteed to fail: at the point of synthesising a usage of a variable in the additive scheme, isolated from information about how else the variable is used, there is no way of knowing if such a usage will be permissible in the final synthesised program. However, it also reduces the amount of intermediate theorems that need solving, which can significantly effect performance as shown in Chapter 3, especially since the variable rule is applied very frequently.

5.2.1.3 *Functions*

Synthesis of programs from function types is handled by the $\rightarrow_{\text{R}}$ and $\rightarrow_{\text{L}}$ rules, which synthesise abstraction and application terms, respectively. An abstraction is synthesised like so:

$$\frac{\Gamma, x :_q A \vdash B \Rightarrow t \mid \Delta, x :_r A \qquad r \sqsubseteq q}{\Gamma \vdash A^q \rightarrow B \Rightarrow \lambda x.t \mid \Delta} \rightarrow_{\text{R}}$$

Reading bottom up, to synthesise a term of type $A^q \rightarrow B$ in context Γ we first extend the context with a fresh variable assumption $x :_q A$ and synthesise a term of type B that will ultimately become the body of the function. The type $A^q \rightarrow B$ conveys that A must be used according to q in our term for B. The fresh variable x is passed to the premise of the rule using the grade of the binder: q. The x must then be used to synthesise a term t with q usage. In the premise, after synthesising t we obtain an output context $\Delta, x :_r A$. As mentioned, the Var rule ensures that x is present in this context, even if it was not used in the synthesis of t (e.g., $r = 0$). The rule ensures the usage of bound term (r) in t does not violate the input grade q via the requirement that $r \sqsubseteq q$ i.e. that r *approximates* q. If met, Δ becomes the output context of the rule's conclusion.

The counterpart to abstraction synthesises an application from the occurrence of a function in the context (a left rule):

$$\frac{\begin{array}{c}\Gamma, x_1 :_{r_1} A^q \rightarrow B, x_2 :_{r_1} B \vdash C \Rightarrow t_1 \mid \Delta_1, x_1 :_{s_1} A^q \rightarrow B, x_2 :_{s_2} B \\ \Gamma, x_1 :_{r_1} A^q \rightarrow B \vdash A \Rightarrow t_2 \mid \Delta_2, x_1 :_{s_3} A^q \rightarrow B\end{array}}{\Gamma, x_1 :_{r_1} A^q \rightarrow B \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + s_2 \cdot q \cdot \Delta_2), x_1 :_{s_2 + s_1 + (s_2 \cdot q \cdot s_3)} A^q \rightarrow B} \rightarrow_{\text{L}}$$

Reading bottom up again, the input context contains an assumption with a function type $x_1 :_{r_1} A^q \rightarrow B$. We may attempt to use this assumption in the synthesis of a term with the goal type C, by applying some argument to it. We do this by synthesising the argument from the input type of the function A, and then binding the result of this application as an assumption of type B in the synthesis of C. This is decomposed into two steps corresponding to the two premises (though in the implementation the first premise is considered first):

1. The first premise synthesises a term t_1 from the goal type C under the assumption that the function x_1 has been applied and its result is bound to x_2. This placeholder assumption is bound with the same grade as x_1.

2. The second premise synthesises an argument t_2 of type A for the function x_1. In the implementation, this synthesis step occurs only after a term t_1 is found for the goal C as a heuristic to avoid possibly unnecessary work if no term can be synthesised for C.

In the conclusion of the rule, a term is synthesised which substitutes in t_1 the result placeholder variable x_2 for the application $x_1\,t_2$.

The first premise yields an output context $\Delta_1, x_1 :_{s_1} A^q \rightarrow B, x_2 :_{s_2} B$. The output context of the conclusion is obtained by taking the context addition of Δ_1 and $s_2 \cdot q \cdot \Delta_2$. The output context Δ_2 is first scaled by q since t_2 is used according to q when applied to x_1 (as per the type of x_1). We then scale this again by s_2 which represents the usage of the entire application $x_1\,t_2$ inside t_1.

The output grade of x_1 follows a similar pattern since this rule permits the re-use of x_1 inside both premises of the application (which differs from our treatment of synthesis in a linear setting). As x_1's input grade r_1 may permit multiple uses both inside the synthesis of the application argument t_2 and in t_1 itself, the total usage of t_1 across both premises must be calculated. In the first premise x_1 is used according to s_1, and in the second according to s_3. As with Δ_2, we take the semiring multiplication of s_3 and q and then multiply this by s_2 to yield the final usage of x_1 in t_2. We then add this to $s_2 + s_1$ to yield the total usage of x_1 in t_1.

5.2.1.4 *Using polymorphic definitions*

Programs can be synthesised from a polymorphic type scheme (the previously shown TopLevel rule), treating universally-quantified type variables at the top-level of our goal type as logical atoms which cannot be unified with and are only equal to themselves. The Def rule synthesises a *use* of a top-level polymorphic function via instantiation:

$$\frac{(x : \forall \overline{\alpha : \kappa}.A') \in D \qquad \Sigma \vdash A = \mathrm{inst}(\forall \overline{\alpha : \kappa}.A')}{\Sigma; \Gamma \vdash A \Rightarrow x \mid 0 \cdot \Gamma}\ \textsc{Def}$$

For example, in the following we have a polymorphic function `flip` that we want to use to synthesise a monomorphic function:

```
flip : ∀ c d  . (c, d) %1 → (d, c)
flip (x, y) = (y, x)
f : (Int, Int) %1 → (Int, Int)
f x = ? -- synthesis to flip x trivially
```

To synthesise the term `flip x`, the type scheme of `flip` is instantiated via DEF with $\emptyset \vdash (\mathit{Int} \otimes \mathit{Int})^1 \rightarrow (\mathit{Int} \otimes \mathit{Int}) = \text{inst}(\forall c : \text{Type}, d : \text{Type}.(c \otimes d)^1 \rightarrow (d \otimes c))$.

5.2.1.5 *Graded modalities*

Graded modalities are introduced and eliminated explicitly through the $\Box_\text{R}$ and $\Box_\text{L}$ rules, respectively. In the $\Box_\text{R}$ rule, we synthesise a promotion $[t]$ for some graded modal goal type $\Box_r A$:

$$\frac{\Gamma \vdash A \Rightarrow t \mid \Delta}{\Gamma \vdash \Box_r A \Rightarrow [t] \mid r \cdot \Delta} \; \Box_\text{R}$$

In the premise, we synthesise from A, yielding the sub-term t and an output context Δ. In the conclusion, Δ is scaled by the grade of the goal type r: as $[t]$ must use t as r requires.

Grade elimination (*unboxing*) takes place via pattern matching in a **case** statement:

$$\frac{\begin{array}{l}\Gamma, y :_{r \cdot q} A, x :_r \Box_q A \vdash B \Rightarrow t \mid \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \\ \exists s_3 .\, s_1 \sqsubseteq s_3 \cdot q \sqsubseteq r \cdot q\end{array}}{\Gamma, x :_r \Box_q A \vdash B \Rightarrow \textbf{case}\ x\ \textbf{of}\ [y] \rightarrow t \mid \Delta, x :_{s_3 + s_2} \Box_q A} \; \Box_\text{L}$$

To eliminate the assumption x of graded modal type $\Box_q A$, we bind a fresh assumption in the synthesis of the premise: $y :_{r \cdot q} A$. This assumption is graded with $r \cdot q$: the grade from the assumption's type multiplied by the grade of the assumption itself. As with previous elimination rules, x is rebound in the rule's premise. A term t is then synthesised resulting in the output context $\Delta, y :_{s_1} A, x :_{s_2} \Box_q A$, where s_1 and s_2 describe how y and x were used in t. The second premise ensures that the usage of y is well-resourced. The grade s_3 represents how much the usage of y inside t contributes to the overall usage of x. The constraint $s_1 \sqsubseteq s_3 \cdot q$ conveys the fact that q uses of y constitutes a single use of x, with the constraint $s_3 \cdot q \sqsubseteq r \cdot q$ ensuring that the overall usage does not exceed the binding grade. For the output context of the conclusion, we simply remove the bound y from Δ and add x, with the grade $s_2 + s_3$: representing the total usage of x in t.

5.2.1.6 *Data types*

The synthesis of introduction forms for data types is by the CONrule:

$$\frac{\begin{array}{l}(C : \forall \overline{\alpha : \kappa}. {B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K \vec{A'}) \in D \\ \Sigma \vdash {B_1}^{q_1} \rightarrow \ldots \rightarrow {B_n}^{q_n} \rightarrow K \vec{A} = \text{inst}(\forall \overline{\alpha : \kappa}. {B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K \vec{A'}) \\ \Sigma; \Gamma \vdash B_i \Rightarrow t_i \mid \Delta_i\end{array}}{\Sigma; \Gamma \vdash K \vec{A} \Rightarrow C\, t_1 \ldots t_n \mid 0 \cdot \Gamma + (q_1 \cdot \Delta_1) + \ldots + (q_n \cdot \Delta_n)} \; \text{C}_\text{R}$$

where D is the set of data constructors in global scope, e.g., coming from ADT definitions, including here products, unit, and coproducts with $(,) : A^1 \to B^1 \to A \otimes B$, $Unit : 1$, $\mathsf{inl} : A^1 \to A \oplus B$, and $\mathsf{inr} : B^1 \to A \oplus B$.

For a goal type $K\vec{A}$ where K is a data type with zero or more type arguments (denoted by the vector $\vec{A}$), then a constructor term $C\, t_1 \mathinner{..} t_n$ for $K\vec{A}$ is synthesised. The type scheme of the constructor in D is first instantiated (similar to DEF rule), yielding a type ${B_1}^{q_1} \to \ldots \to {B_n}^{q_n} \to K\vec{A}$. A sub-term is then synthesised for each of the constructor's arguments t_i in the third premise (which is repeated for each instantiated argument type B_i), yielding output contexts Δ_i. The output context for the rule's conclusion is obtained by performing a context addition across all the output contexts generated from the premises, where each context Δ_i is scaled by the corresponding grade q_i from the data constructor in D capturing the fact that each argument t_i is used according to q_i.

Dual to the above, constructor elimination synthesises **case** statements with branches pattern matching on each data constructor of the target data type $K\vec{A}$, with various associated constraints on grades which require some explanation:

$$\dfrac{\begin{array}{l}(C_i : \forall \overline{\alpha : \kappa}.{B'_1}^{q^i_1} \to \ldots \to {B'_n}^{q^i_n} \to K\vec{A}') \in D \qquad \Sigma \vdash K\vec{A} : \mathrm{Type} \\ \Sigma \vdash {B_1}^{q_1} \to \ldots \to {B_n}^{q_n} \to K\vec{A} = \mathrm{inst}(\forall \overline{\alpha : \kappa}.{B'_1}^{q_1} \to \ldots \to {B'_n}^{q_n} \to K\vec{A}') \\ \Sigma; \Gamma, x :_r K\vec{A}, y^i_1 :_{r \cdot q^i_1} B_1, \ldots, y^i_n :_{r \cdot q^i_1} B_n \vdash B \Rightarrow t_i \mid \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, \ldots, y^i_n :_{s^i_n} B_n \\ \exists {s'}^i_j . s^i_j \sqsubseteq {s'}^i_j \cdot q^i_j \sqsubseteq r \cdot q^i_j \qquad s_i = {s'}^i_1 \sqcup \ldots \sqcup {s'}^i_n \qquad |K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq s_1 \sqcup \ldots \sqcup s_m\end{array}}{\Sigma; \Gamma, x :_r K\vec{A} \vdash B \Rightarrow \textbf{case}\ x\ \textbf{of}\ \overline{C_i\, y^i_1 \ldots y^i_n} \mapsto t_i \mid (\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1 \sqcup \ldots \sqcup r_m) + (s_1 \sqcup \ldots \sqcup s_m)} K\vec{A}}\ \mathrm{C_L}$$

where $1 \leq i \leq m$ is used to index the data constructors of which there are m (i.e., $m = |K\vec{A}|$) and $1 \leq j \leq n$ is used to index the arguments of the i^{th} data constructor. For brevity, the rule focuses n-ary data constructors where $n > 0$.

As with constructor introduction, the relevant data constructors are retrieved from the global scope D in the first premise. A data constructor type is a function type from the constructor's arguments $B_1 \ldots B_n$ to a type constructor applied to zero or more type parameters $K\vec{A}$. However, in the case of nullary data constructors (e.g., for the unit type), the data constructor type is simply the type constructor's type with no arguments. For each data constructor C_i, we synthesise a term t_i from the result type of the data constructor's type in D, binding the data constructor's argument types as fresh assumptions to be used in the synthesis of t_i.

To synthesise the body for each branch i, the arguments of the data constructor are bound to fresh variables in the premise, with the grades from their respective argument types in D multiplied by the r. This follows the pattern typing rule for constructors; a pattern match under some grade r must bind assumptions that have the capability to be used according to r.

The assumption being eliminated $x :_r K\vec{A}$ is also included in the premise's context (as in $\rightarrow_{\mathrm{L}}$) as we may perform additional eliminations on the current assumption subsequently if the grade r allows us. If successful, this will yield both a term t_i and an output context for the pattern match branch. The output context can be broken down into three parts:

1. Δ_i contains any assumptions from Γ were used to construct t_i
2. $x :_{r_i} K\,A$ describes how the assumption x was used
3. $y^i_1 :_{s^i_1} B_1, ..., y^i_n :_{s^i_n} B_n$ describes how each assumption y^i_j bound in the pattern match was used in t_i.

This leaves the question of how we calculate the final grade to attribute to x in the output context of the rule's conclusion. For each bound assumption, we generate a fresh grade variable s'^i_j which represents how that variable was used in t_i after factoring out the multiplication by q^i_j. This is done via the constraint in the third premise that $\exists s'^i_j. s^i_j \sqsubseteq s'^i_j \cdot q^i_j \sqsubseteq r \cdot q^i_j$. The join of each s'^i_j (for each assumption) is then taken to form a grade variable s_i which represents the total usage of x for this branch that arises from the use of assumptions which were bound via the pattern match (i.e. not usage that arises from reusing x explicitly inside t_i). For the output context of the conclusion, we then take the join of output context from the constructors used. This is extended with the original x assumption with the output grade consisting of the join of each r_i (the usages of x directly in each branch) plus the join of each s_i (the usages of the assumptions that were bound from matching on a constructor of x).

Example 5.2.1 (Example of **case** synthesis)**.** Consider two possible synthesis results:

$$x :_r 1 \oplus A, y :_s A, z :_{r \cdot q_1} A \vdash A \Rightarrow^+ z;\ x :_0 1 \oplus A, y :_0 A, z :_1 A \tag{5.1}$$

$$x :_r 1 \oplus A, y :_s A \vdash A \Rightarrow^+ y;\ x :_0 1 \oplus A, y :_1 A \tag{5.2}$$

We will plug these into the rule for generating case expressions as follows where in the following instead of using the above concrete grades we have used the abstract form of the rule (the two will be linked by equations after):

$$\dfrac{\begin{array}{c}\mathsf{Just} : \forall \overline{\alpha : \kappa}.A'^1 \rightarrow A' \oplus 1 \in D \qquad\qquad \mathsf{Nothing} : \forall \overline{\alpha : \kappa}.1^1 \rightarrow A' \oplus 1 \in D \\ \Sigma \vdash A^1 \rightarrow A \oplus 1 = \mathsf{inst}(\forall \overline{\alpha : \kappa}.A'^1 \rightarrow A' \oplus 1) \\ \Sigma \vdash 1^1 \rightarrow A \oplus 1 = \mathsf{inst}(\forall \overline{\alpha : \kappa}.1^1 \rightarrow A' \oplus 1) \\ (5.1) \qquad \Sigma; x :_r 1 \oplus A, y :_s A, z :_{r \cdot q_1} A \vdash A \Rightarrow z \mid x :_0 1 \oplus A, y :_0 A, z :_{s_1} A \\ (5.2) \qquad \Sigma; x :_r 1 \oplus A, y :_s A \vdash A \Rightarrow y \mid x :_0 1 \oplus A, y :_1 A \\ \exists s'_1. s_1 \sqsubseteq s'_1 \cdot q_1 \sqsubseteq r \cdot q_1 \qquad\qquad s' = s'_1\end{array}}{\Sigma; x :_r 1 \oplus A, y :_s A \vdash A \Rightarrow (\mathbf{case}\ x\ \mathbf{of}\ \mathsf{Just}\, z \rightarrow z; \mathsf{Nothing} \rightarrow y) \mid x :_{(0 \sqcup 0)+s'} 1 \oplus A, y :_{0 \sqcup 1} A}\ \textsc{Case}$$

Thus, to unify (5.1) and (5.2) with the rule format we have that $s_1 = 1$ and $q_1 = 1$. Applying these two equalities as rewrites to the remaining constraint, we have:

$$\exists s'_1. 1 \sqsubseteq s'_1 \cdot 1 \sqsubseteq r \cdot 1 \qquad \Longrightarrow \qquad \exists s'_1. 1 \sqsubseteq s'_1 \sqsubseteq r$$

These constraints can be satisfied with the natural-number intervals semiring where y has grade 0..1 and x has grade 1..1.

Deep pattern matching, over nested data constructors, is handled via inductively applying the Case rule but with a post-synthesis refactoring procedure substituting the pattern match of the inner case statement into the outer pattern match. For example, nested matching on pairs becomes a single **case** with nested pattern matching, simplifying the program:

$$\begin{aligned} & \mathbf{case}\ x\ \mathbf{of}\ (y_1, y_2) \rightarrow \mathbf{case}\ y_1\ \mathbf{of}\ (z_1, z_2) \rightarrow z_2 \\ \textit{(rewritten to)} \rightsquigarrow\ & \mathbf{case}\ x\ \mathbf{of}\ ((z_1, z_2), y_2) \rightarrow z_2 \end{aligned}$$

5.2.1.7 *Recursion*

Synthesis permits recursive definitions, as well as programs which may make use of calls to functions from a user-supplied context of function definitions in scope (see Section 5.3). Synthesis of non-recursive function applications may take place arbitrarily, however, synthesising a recursive function definition application requires more care. To ensure that a synthesised programs terminates, we only permit synthesis of terms which are *structurally recursive*, i.e., those which apply the recursive definition to a sub-term of the function's inputs [Osera, 2015].

Synthesis rules for recursive types (μ-types) are straightforward:[2]

$$\frac{\Gamma \vdash A[\mu X.A/X] \Rightarrow t \mid \Delta}{\Gamma \vdash \mu X.A \Rightarrow t \mid \Delta}\ \mu_{\mathrm{R}} \qquad \frac{\Gamma, x :_r A[\mu X.A/X] \vdash B \Rightarrow t \mid \Delta}{\Gamma, x :_r \mu X.A \vdash B \Rightarrow t \mid \Delta}\ \mu_{\mathrm{L}}$$

This μ_{R} rule states that to synthesise a recursive data structure of type $\mu X.A$, we must be able to synthesise A with $\mu X.A$ substituted for the recursion variables X in A. For example, if we wish to synthesise a list data type `List a` with constructors `Nil` and `Cons a (List a)`, then when choosing the `Cons` constructor in the μ_{R} rule, the type of this constructor requires us to re-apply the μ_{R} rule, to synthesise the recursive part of `Cons`. Elimination of a recursive data structure may be synthesised using the μ_{L} rule. In this rule, we have some recursive data type $\mu X.A$ in our context which we may wish to pattern match

2 Though μ types are equi-recursive, we make explicit the synthesis rules here which maps more closely to the implementation where iterative deepening information needs to be tracked at the points of using μ_{L} and μ_{R}.

on via the C_L rule. To do this, the assumption is bound in the premise with the type A, substituting $\mu X.A$ for the recursion variables X in A.

Recursive data structures present a challenge in the implementation. For our list data type, how do we prevent our synthesis tool from simply applying the μ_L rule, followed by the C_L rule on the `Cons` constructor ad infinitum? We resolve this issue using an *iterative deepening* approach to synthesis similar to the approach used by MYTH [Osera, 2015]. Programs are synthesised with elimination (and introduction) forms of constructors restricted up to a given depth. If no program is synthesised within these bounds, then the depth limits are incremented. Combined with focusing (see Section 5.5), this provides the basis for an efficient implementation of the above rules.

5.3 INPUT-OUTPUT EXAMPLES

When specifying the synthesis context of top-level definitions, the user may also supply a series of input-output examples showcasing desired behaviour. Our approach to examples is deliberately naïve; we evaluate a fully synthesised candidate program against the inputs and check that the results match the corresponding outputs. Unlike many sophisticated example-driven synthesis tools, the examples here do not themselves influence the search procedure, and are used solely to allow the user to clarify their intent. This lets us consider the effectiveness of basing the search primarily around the use of grade information. An approach to synthesis of resourceful programs with examples closely integrated into the search as well is further work.

We augmented the Granule language with first-class syntax for specifying input-output examples, both as a feature for aiding synthesis but also for aiding documentation that is type checked (and therefore more likely to stay consistent with a code base as it evolves). Synthesis specifications are written in Granule directly above a program hole (written using `?`) using the `spec` keyword. The input-output examples are then listed per-line.

```
tail : ∀ { a : Type } . List a %0..1 → List a
spec
  tail (Cons 1 Nil) = Nil;
  tail (Cons 1 (Cons 2 Nil)) = Cons 2 Nil;
tail = ?
```

Any synthesised term must then behave according to the supplied examples. This `spec` structure can also be used to describe additional synthesis components that the user wishes the tool to make use of. These components comprise a list of in-scope definitions separated by commas. The user can choose to annotate each component with a grade, describing the required usage in the synthesised term. This defaults to a 1 grade if not specified. For example, the specification for a function which returns the length of a list might be:

```
length : ∀ { a : Type } . List a %0..∞. → N
spec
    length Nil = Z;
    length (Cons 1 Nil) = S Z;
    length (Cons 1 (Cons 1 Nil)) = S (S Z);
    length %0..∞.
length = ?
```

with the following resulting program produced by our synthesis algorithm (on average in about 400ms on a standard laptop, see Section 5.6 where this is one of the benchmarks for evaluation):

```
length Nil = Z;
length (Cons y z) = S (length z)
```

5.4 POST-SYNTHESIS REFACTORING

In Section 3.4, we considered how our synthesised terms could be refactored into a more idiomatic programming style. Those same refactoring transformations also apply to our calculus here, along with the additional treatment relating to case statements.

Recall that the C_L binds a data constructor's patterns as a series of variables. Synthesising a pattern match over a nested data structure therefore yields a term such as:

```
case x of
  C₁ y →
    case y of
      D₁ z → ...
      D₂ z → ...
  C₂ y →
    case y of
      D₁ z → ...
      D₂ z → ...
```

which would be rather unnatural for a programmer to write. Nested case statements are therefore folded together to yield a single case statement which pattern matches over all combination of patterns from each statement. The above cases are then transformed into the much more compact and readable single case:

```
case x of
  C₁ (D₁ z) → ...
  C₁ (D₂ z) → ...
  C₂ (D₁ z) → ...
  C₂ (D₂ z) → ...
```

Furthermore, pattern matches over a function's arguments in the form of case statements are refactored such that a new function equation is created for each unique combination of pattern match. In this way, a refactored program should only contain case statements that arise from pattern matching over the result of an application.

```
neg : Bool %1 → Bool %1
neg x = case x of
          True → False;
          False → True
```

is refactored into:

```
neg : Bool %1 → Bool %1
neg True = False;
neg False = True
```

The exception to this is where the scrutinee of a case statement is re-used inside one of the case branches, in which case refactoring would cause us to throw away the binding of the scrutinee's name and so it cannot be folded into the head pattern match, for example:

```
last : ∀ { a : Type } . (List a) %0..∞ → Maybe a
spec
    last Nil = Nothing;
    last (Cons 1 Nil) = Just 1;
    last (Cons 1 (Cons 2 Nil)) = Just 2;
    last %0..∞
last Nil = Nothing;
last (Cons y z) =
    (case z of
      Nil → Just y;
      Cons u v → last z)
```

A final minor refactoring procedure is to refactor a variable pattern into a wildcard pattern, in the case that the bound variable is not used inside the body of the case branch:

```
throwAway : ∀ { a : Type } . a %0..∞ → ()
throwAway x = ()
```

is refactored into:

```
throwAway : ∀ { a : Type } . a %0..∞ → ()
throwAway _ = ()
```

For such a scenario to occur a pattern must typed with a grade that is approximatable by 0.

5.5 FOCUSING

As in Chapter 3, the synthesis rules as we have presented them thus far are far too non-deterministic to form the basis of an implementation through direct translation into code. Again, we apply the technique of focusing [Andreoli, 1992] to our calculus, yielding a *focused* synthesis calculus which imposes an ordering on when rules may be applied at each stage of synthesis.

We have already outlined the principles behind focusing in Section 3.5, and thus opt not to repeat ourselves here. Instead, we merely present the focused form of the fully graded synthesis calculus in Figures 5.5, 5.6, 5.7, 5.8, and 5.9, and state that focusing is sound (Lemma 5.5.1). The proof of soundness of focusing can be found in Section B.3.2 of Appendix B.

Lemma 5.5.1 (Soundness of focusing for graded-base synthesis)**.** *For all contexts* Γ, Ω *and types* A*:*

$$\begin{array}{lllll}
1.\ \textit{Right Async}: & D;\Sigma;\Gamma;\Omega \vdash A \Uparrow \Rightarrow t \mid \Delta & \Longrightarrow & D;\Sigma;\Gamma,\Omega \vdash A \Rightarrow t \mid \Delta \\
2.\ \textit{Left Async}: & D;\Sigma;\Gamma;\Omega \Uparrow \vdash B \Rightarrow t \mid \Delta & \Longrightarrow & D;\Sigma;\Gamma,\Omega \vdash B \Rightarrow t \mid \Delta \\
3.\ \textit{Right Sync}: & D;\Sigma;\Gamma;\emptyset \vdash A \Downarrow \ \Rightarrow t \mid \ \Delta & \Longrightarrow & D;\Sigma;\Gamma \vdash A \Rightarrow t \mid \Delta \\
4.\ \textit{Left Sync}: & D;\Sigma;\Gamma;x :_r A \Downarrow \vdash B \Rightarrow t \mid \Delta & \Longrightarrow & D;\Sigma;\Gamma,x :_r A \vdash B \Rightarrow t \mid \Delta \\
5.\ \textit{Focus Right}: & D;\Sigma;\Gamma;\emptyset \Uparrow \vdash B \Rightarrow t \mid \ \Delta & \Longrightarrow & D;\Sigma;\Gamma \vdash B \Rightarrow t \mid \Delta \\
6.\ \textit{Focus Left}: & D;\Sigma;\Gamma,x :_r A;\emptyset \Uparrow \vdash B \Rightarrow t \mid \ \Delta & \Longrightarrow & D;\Sigma;\Gamma,x :_r A \vdash B \Rightarrow t \mid \Delta
\end{array}$$

i.e. t *has type* A *under context* Δ*, which contains variables with grades reflecting their use in* t*.*

$$\frac{D;\Sigma;\Gamma;\Omega,x :_q A \vdash B \Uparrow \Rightarrow t \mid \Delta,x :_r A \qquad r \sqsubseteq q}{D;\Sigma;\Gamma;\Omega \vdash A^q \to B \Uparrow \Rightarrow \lambda x.t \mid \Delta}\ \to_{\text{R}}$$

$$\frac{D;\overline{\alpha:\kappa};\emptyset;\emptyset \vdash A \Uparrow \Rightarrow t \mid \emptyset}{D;\emptyset;\emptyset;\emptyset \vdash \forall\overline{\alpha:\kappa}.A \Uparrow \Rightarrow t \mid \emptyset}\ \textsc{TopLevel}$$

$$\frac{D;\Sigma;\Gamma;\Omega \Uparrow \vdash B \Rightarrow t \mid \Delta \qquad B \text{ not right async}}{D;\Sigma;\Gamma;\Omega \vdash B \Uparrow \Rightarrow t \mid \Delta}\ \Uparrow_{\text{R}}$$

Figure 5.5: Right Async rules of the focused fully graded synthesis calculus

$$\dfrac{\begin{array}{c}(C_i : \forall \overline{\alpha : \kappa}.{B'_1}^{q^i_1} \to \ldots \to {B'_n}^{q^i_n} \to K\vec{A'}) \in D \qquad \Sigma \vdash K\vec{A} : \mathsf{Type} \\ \Sigma \vdash {B_1}^{q_1} \to \ldots \to {B_n}^{q_n} \to K\vec{A} = \mathsf{inst}(\forall \overline{\alpha : \kappa}.{B'_1}^{q_1} \to \ldots \to {B'_n}^{q_n} \to K\vec{A'}) \\ D;\Sigma;\Gamma;\Omega, x :_r K\vec{A}, y^i_1 :_{r \cdot q^i_1} B_1, \ldots, y^i_n :_{r \cdot q^i_1} B_n \Uparrow \vdash B \Rightarrow t_i \mid \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, \ldots, y^i_n :_{s^i_n} B_n \\ \exists {s'}^i_j . s^i_j \sqsubseteq {s'}^i_j \cdot q^i_j \sqsubseteq r \cdot q^i_j \qquad s_i = {s'}^i_1 \sqcup \ldots \sqcup {s'}^i_n \qquad |K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq s_1 \sqcup \ldots \sqcup s_m\end{array}}{D;\Sigma;\Gamma;\Omega, x :_r K\vec{A} \Uparrow \vdash B \Rightarrow \mathbf{case}\ x\ \mathbf{of}\ \overline{C_i\ y^i_1 \ldots y^i_n \mapsto t_i} \mid (\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1 \sqcup \ldots \sqcup r_m) + (s_1 \sqcup \ldots \sqcup s_m)} K\vec{A}}\ \mathrm{C_L}$$

$$\dfrac{D;\Sigma;\Gamma;\Omega, x :_r A[\mu X.A/X] \Uparrow \vdash B \Rightarrow t \mid \Delta}{D;\Sigma;\Gamma;\Omega, x :_r \mu X.A \Uparrow \vdash B \Rightarrow t \mid \Delta}\ \mu_\mathrm{L}$$

$$\dfrac{\begin{array}{c}D;\Sigma;\Gamma;\Omega, y :_{r \cdot q} A, x :_r \Box_q A \Uparrow \vdash B \Rightarrow t \mid \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \\ \exists s_3 . s_1 \sqsubseteq s_3 \cdot q \sqsubseteq r \cdot q \qquad \Sigma \vdash \Box_q A : \mathsf{Type}\end{array}}{D;\Sigma;\Gamma;\Omega, x :_r \Box_q A \Uparrow \vdash B \Rightarrow \mathbf{case}\ x\ \mathbf{of}\ [y] \to t \mid \Delta, x :_{s_3 + s_2} \Box_q A}\ \Box_\mathrm{L}$$

$$\dfrac{D;\Sigma;\Gamma, x :_r A;\Omega \Uparrow \vdash B \Rightarrow t \mid \Delta \qquad \text{A not left async}}{D;\Sigma;\Gamma;\Omega, x :_r A \Uparrow \vdash B \Rightarrow t \mid \Delta}\ \Uparrow_\mathrm{L}$$

Figure 5.6: Left Async rules of the focused fully graded synthesis calculus

$$\dfrac{D;\Sigma\Gamma;\emptyset \vdash B \Downarrow \Rightarrow t \mid \Delta \qquad \text{B not atomic}}{D;\Sigma;\Gamma;\emptyset \Uparrow \vdash B \Rightarrow t \mid \Delta}\ \mathrm{Foc_R}$$

$$\dfrac{D;\Sigma;\Gamma; x :_r A \Downarrow \vdash B \Rightarrow t \mid \Delta}{D;\Sigma;\Gamma, x :_r A;\emptyset \Uparrow \vdash B \Rightarrow t \mid \Delta}\ \mathrm{Foc_L}$$

Figure 5.7: Focus rules of the focused fully graded synthesis calculus

$$\dfrac{\begin{array}{c}(C : \forall \overline{\alpha : \kappa}.{B'_1}^{q_1} \to \ldots \to {B'_n}^{q_n} \to K\vec{A'}) \in D \\ \Sigma \vdash {B_1}^{q_1} \to \ldots \to {B_n}^{q_n} \to K\vec{A} = \mathsf{inst}(\forall \overline{\alpha : \kappa}.{B'_1}^{q_1} \to \ldots \to {B'_n}^{q_n} \to K\vec{A'}) \\ D;\Sigma;\Gamma;\emptyset \vdash B_i \Downarrow \Rightarrow t_i \mid \Delta_i\end{array}}{D;\Sigma;\Gamma;\emptyset \vdash K\vec{A} \Downarrow \Rightarrow C\, t_1 \ldots t_n \mid \Delta_1 + \ldots + \Delta_n}\ \mathrm{Con_R}$$

$$\dfrac{D;\Sigma;\Gamma;\emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta}{D;\Sigma;\Gamma;\emptyset \vdash \Box_r A \Downarrow \Rightarrow [t] \mid r \cdot \Delta}\ \Box_\mathrm{R}$$

$$\dfrac{D;\Sigma;\Gamma;\emptyset \vdash A[\mu X.A/X] \Downarrow \Rightarrow t \mid \Delta}{D;\Sigma;\Gamma;\emptyset \vdash \mu X.A \Downarrow \Rightarrow t \mid \Delta}\ \mu_\mathrm{R}$$

$$\dfrac{D;\Sigma;\Gamma;\emptyset \vdash A \Uparrow \Rightarrow t \mid \Delta}{D;\Sigma;\Gamma;\emptyset \vdash A \Downarrow t \mid \Delta}\ \Downarrow_\mathrm{R}$$

Figure 5.8: Right Sync rules of the focused fully graded synthesis calculus

$$\frac{\begin{array}{c} D;\Sigma;\Gamma;x_1 :_{r_1} A^q \rightarrow B, x_2 :_{r_1} B \Downarrow \vdash C \Rightarrow t_1 \mid \Delta_1, x_1 :_{s_1} A^q \rightarrow B, x_2 :_{s_2} B \\ D;\Sigma;\Gamma;x_1 :_{r_1} A^q \rightarrow B \Downarrow \vdash A \Rightarrow t_2 \mid \Delta_2, x_1 :_{s_3} A^q \rightarrow B \qquad \Sigma \vdash A^q \rightarrow B : \text{Type} \end{array}}{D;\Sigma;\Gamma;x_1 :_{r_1} A^q \rightarrow B \Downarrow \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + s_2 \cdot q \cdot \Delta_2), x_1 :_{s_2+s_1+(s_2 \cdot q \cdot s_3)} A^q \multimap B} \rightarrow_{\mathrm{L}}$$

$$\frac{\Sigma \vdash A : \text{Type}}{D;\Sigma;\Gamma;x :_r A \Downarrow \vdash A \Rightarrow x \mid 0 \cdot \Gamma, x :_1 A} \textsc{Var}$$

$$\frac{\Sigma \vdash A = \text{inst}(\forall \overline{\alpha : \kappa}.A')}{D, x : \forall \overline{\alpha : \kappa}.A';\Sigma;\Gamma;\emptyset \Downarrow \vdash A \Rightarrow x \mid 0 \cdot \Gamma} \textsc{Def}$$

$$\frac{D;\Sigma;\Gamma;x :_r A \Uparrow \vdash B \Rightarrow t \mid \Delta \qquad \text{A not atomic and not left sync}}{D;\Sigma;\Gamma;x :_r A \Downarrow \vdash B \Rightarrow t \mid \Delta} \Downarrow_{\mathrm{L}}$$

Figure 5.9: Left Sync, Var, and Def rules of the focused fully graded synthesis calculus

5.6 EVALUATING THE SYNTHESIS CALCULUS

In evaluating our fully graded synthesis approach and tool, we made the following hypotheses:

H1. (**Expressivity; less consultation**) The use of grades in synthesis results in a synthesised program that is more likely to have the behaviour desired by the user; the user needs to request fewer alternate synthesised results (*retries*) and thus is consulted less in order to arrive at the desired program.

H2. (**Expressivity; fewer examples**) Grade-and-type directed synthesis requires fewer input-output examples to arrive at the desired program compare with a purely type-driven approach.

H3. (**Performance; more pruning**) The ability to prune resource-violating candidate programs from the search tree leads to a synthesised program being found more quickly when synthesised from a graded type compared with the same type but without grades (purely type-driven approach).

5.6.1 *Methodology*

To evaluate our approach, we collected a suite of benchmarks comprising graded type signatures for common transformations on data structures such as lists, streams, booleans, option ('maybe') types, unary natural numbers, and binary trees. We draw many of these from the benchmark suite of the MYTH synthesis tool [Osera and Zdancewic, 2015]. Benchmarks are divided into classes based on the main data type, with an additional category of miscellaneous programs. The type schemes for the full suite of benchmarks can be found in Table 5.2

while 5.1 lists the data types used by the benchmarking problems. The complete synthesised programs, including examples used and synthesis contexts can be found in Section A.2 of Appendix A.

To compare, in various ways, our grade-and-type-directed synthesis to traditional type-directed synthesis, each benchmark signature is also "de-graded" by replacing all grades in the goal with `Any` which is the only element of the singleton `Cartesian` semiring in Granule. When synthesising in this semiring, we can forgo discharging grade constraints in the SMT solver entirely. Thus, synthesis for Cartesian grades degenerates to typed-directed synthesis following our rules.

To assess hypothesis 1 (grade-and-type directed leads to less consultation / more likely to synthesise the intended program) we perform grade-and-type directed synthesis on each benchmark problem and type-directed synthesis on the corresponding de-graded version. For the de-graded versions, we record the number of retries N needed to arrive at a well-resourced answer by type checking the output programs against the original graded type signature, retrying if the program is not well-typed (essentially, not well-resourced). This provides a means to check whether a program may be as intended without requiring user input. In each case we also compared whether the resulting programs from synthesis via graded-and-type directed vs. type-directed with retries (on non-well-resourced outputs) were equivalent.

To assess hypothesis 2 (graded-and-type directed requires fewer examples than type-directed), we run the de-graded (Cartesian) synthesis with the smallest set of examples which leads to the model program being synthesised (without any retries). To compare across approaches to the state-of-the-art type-directed approach, we also run a separate set of experiments comparing the minimal number of examples required to synthesise in Granule (with grades) vs. MYTH.

To assess hypothesis 3 (grade-and-type-directed faster than type-directed) we compare performance in the graded setting to the non-graded Cartesian setting. Comparing our tool for speed against another type-directed (but not graded-directed) synthesis tool is likely to be largely uninformative due to differences in implementation approach obscuring any meaningful comparison. Thus, we instead compare timings for the graded approach and de-graded approach within Granule. We also record the number of search paths taken (over all retries) to assess the level of pruning in graded vs non-graded.

We ran our synthesis tool on each benchmark for both the graded type and the de-graded Cartesian case, computing the mean after 10 trials for timing data. Benchmarking was carried out using version 4.12.1 of Z3 on an M1 MacBook Air with 16 GB of RAM. A timeout limit of 10 seconds was set for synthesis.

5.6.2 *Results and Analysis*

Table 5.3 records the results comparing grade-and-type synthesis vs. the Cartesian (de-graded) type-directed synthesis. The left column gives the benchmark name, number of top-level definitions in scope that can be used as components (size of the synthesis context) labelled Ctxt, and the minimum number of examples needed (#/Exs) to synthesise the Graded and Cartesian programs. In the Cartesian setting, where grade information is not available, if we forgo type checking a candidate program against the original graded type then additional input-output examples are required to provide a strong enough specification such that the correct program is synthesised (see H3). The number of additional examples is given in parentheses for those benchmarks which required these additional examples to synthesise a program in the Cartesian setting.

Each subsequent results column records: whether a program was synthesised successfully ✓ or not × (due to timeout or no solution found), the mean synthesis time (μT) or if timeout occurred, and the number branching paths (Paths) explored in the search space.

The first results column (Graded) contains the results for graded synthesis. The second results column (Cartesian + Graded type-check) contains the results for synthesising a program in the Cartesian (degraded) setting, using the same examples set as the Graded column, and recording the number of retries (consultations of the type-checker) N needed to reach a well-resourced program. In all cases, this resulting program in the Cartesian column was equivalent to that generated by the graded synthesis, none of which needed any retries (i.e., implicitly $N = 0$ for graded synthesis). H1 is confirmed by the fact that N is greater than 0 in 29 out of 46 benchmarks (60%), i.e., the Cartesian case does not synthesis the correct program first time and needs multiple retries to reach a well-resource program, with a mean of 19.60 retries and a median of 4 retries.

For each row, we highlight the column which synthesised a result the fastest in yellow. The results show that in 17 of the 46 benchmarks (37%) the graded approach out-performed non-graded synthesis. This contradicts hypothesis 3 somewhat: whilst type-directed synthesis often requires multiple retries (versus no retries) it still outperforms the graded equivalent. This appears to be due to the cost of our SMT solving procedure which must first compile a first-order theorem on grades into the SMT-lib file format, start up Z3, and then run the solver. Considerable amounts of system overhead are incurred in this procedure. A more efficient implementation calling Z3 directly (e.g., via a dynamic library call) may give more favourable results here. However, H3 is still somewhat supported: the cases in which the graded does outperform the Cartesian are those which involve considerable complexity in their use of grades, such as `stutter`, `inc`, and `bind` for lists, as well as `sum` for both lists and trees. In each of these cases, the Cartesian column is significantly slower, even timing out for `stutter`; this shows the power of the graded approach. Furthermore, we highlight the column with the smallest number of synthesis paths explored in blue, observing that the number of paths in the graded case is always the same or less than that those in the Cartesian+graded type check case (apart from Tree stutter).

Interestingly the paths explored are sometimes the same because we use backtracking search in the Cartesian+Graded type check case where, if an output program fails to type check against the graded type signature, the search backtracks rather than starting again.

Confirming H2, we find that for the non-graded setting without graded type checking, further examples are required to synthesise the same program as the graded in 20 out of 46 (43%) cases. In these cases, an average of 1.25 additional examples was required.

Data Type		Type Scheme
List	Cons	$\forall a.a^1 \rightarrow \text{List } a^1 \rightarrow \text{List } a$
	Nil	$\forall a.\text{List } a$
Stream	Next	$\forall a.a^1 \rightarrow \text{Stream } a^1 \rightarrow \text{Stream } a$
Bool	True	Bool
	False	Bool
Maybe	Just	$\forall a.a^1 \rightarrow \text{Maybe } a$
	Nothing	$\forall a.\text{Maybe } a$
N	S	$\text{N}^1 \rightarrow \text{N}$
	Z	N
Tree	Node	$\forall a.\text{Tree } a^1 \rightarrow a^1 \rightarrow \text{Tree } a^1 \rightarrow \text{Tree } a$
	Leaf	$\forall a.\text{Tree } a$

Table 5.1: Data types used in synthesis benchmarking problems

	Problem	Type Scheme
List	append	$\forall a.\text{List } a^{1} \rightarrow a^{1} \rightarrow \text{List } a$
	concat	$\forall a.\text{List } a^{1..\infty} \rightarrow \text{List } a^{1..\infty} \rightarrow \text{List } a$
	empty	$\forall a.\text{Unit}^{1} \rightarrow \text{List } a$
	snoc	$\forall a.\text{List}^{1..\infty} \rightarrow a^{1..\infty} \rightarrow \text{List } a$
	drop	$\forall a.\text{N}^{0..\infty} \rightarrow \text{List}^{0..\infty} \rightarrow \text{List } a$
	flatten	$\forall a.\text{List } (\text{List } a)^{0..\infty} \rightarrow \text{List } a$
	bind	$\forall a\ b.\text{List } a^{1..\infty} \rightarrow (a^{1..\infty} \rightarrow \text{List } b)^{0..\infty} \rightarrow \text{List } b$
	return	$\forall a.a^{1} \rightarrow \text{List } a$
	inc	$\text{List } \text{N}^{1..\infty} \rightarrow \text{List N}$
	head	$\forall a.\text{List } a^{0..1} \rightarrow a^{0..1} \rightarrow at$
	tail	$\forall a.\text{List } a^{0..1} \rightarrow \text{List } a$
	last	$\forall a.\text{List } a^{0..\infty} \rightarrow \text{Maybe } a$
	length	$\forall a.\text{List } a \rightarrow \text{N}$
	map	$\forall a\ b.(a^{1..\infty} \rightarrow b)^{0..\infty} \rightarrow \text{List } a^{1..\infty} \rightarrow \text{List } b$
	replicate5	$\forall a.a^{5} \rightarrow \text{List } a$
	replicate10	$\forall a.a^{10} \rightarrow \text{List } a$
	replicateN	$\forall a.\text{N}^{0..\infty} \rightarrow a^{0..\infty} \rightarrow \text{List}a$
	stutter	$\forall a.\text{List } (a\ [2])^{1..\infty} \rightarrow \text{List } a$
	sum	$\text{List } \text{N}^{0..\infty} \rightarrow \text{N}$
Stream	build	$\forall a.a^{1..1} \rightarrow \text{Stream } a^{1..1} \rightarrow \text{Stream } a$
	map	$\forall a\ b.\text{Stream } a^{1..\infty} \rightarrow (a^{1..\infty} \rightarrow b)^{1..\infty} \rightarrow \text{Stream } b$
	take1	$\forall a.\text{Stream } a^{0..1} \rightarrow a$
	take2	$\forall a.\text{Stream } a^{0..1} \rightarrow (a,\ a)$
	take3	$\forall a.\text{Stream } a^{0..1} \rightarrow (a,\ (a,\ a))$
Bool	neg	$\text{Bool}^{1} \rightarrow \text{Bool}$
	and	$\text{Bool}^{1} \rightarrow \text{Bool}^{1} \rightarrow \text{Bool}$
	impl	$\text{Bool}^{1} \rightarrow \text{Bool}^{1} \rightarrow \text{Bool}$
	or	$\text{Bool}^{1} \rightarrow \text{Bool}^{1} \rightarrow \text{Bool}$
	xor	$\text{Bool}^{1} \rightarrow \text{Bool}^{1} \rightarrow \text{Bool}$
Maybe	bind	$\forall a\ b.\text{Maybe } a^{1..1} \rightarrow (a^{1..1} \rightarrow \text{Maybe } b)^{0..1} \rightarrow \text{Maybe } b$
	fromMaybe	$\forall a.\text{Maybe } a^{1..1} \rightarrow a^{0..1} \rightarrow a$
	return	$\forall a.a^{1} \rightarrow \text{Maybe } a$
	isJust	$\forall a.\text{Maybe } a^{1} \rightarrow \text{Bool}$
	isNothing	$\forall a.\text{Maybe } a^{1} \rightarrow \text{Bool}$
	map	$\forall a\ b.(a^{1..1} \rightarrow b)^{0..1} \rightarrow \text{Maybe } a^{1..1} \rightarrow \text{Maybe } b$
	mplus	$\forall a\ b.\text{Maybe } a^{1} \rightarrow \text{Maybe } b^{1} \rightarrow \text{Maybe } (a,\ b)$
Nat	isEven	$\text{N}^{1..\infty} \rightarrow \text{Bool}$
	pred	$\text{N}^{1} \rightarrow \text{N}$
	succ	$\text{N}^{1} \rightarrow \text{N}$
	sum	$\text{N}^{1..\infty} \rightarrow \text{N}^{1..\infty} \rightarrow \text{N}$
Tree	map	$\forall a\ b.(a^{1..\infty} \rightarrow b)^{0..\infty} \rightarrow \text{Tree } a^{1..\infty} \rightarrow \text{Tree } b$
	stutter	$\forall a.\text{Tree } (a\ [2])^{1..\infty} \rightarrow \text{Tree } (a,\ a)$
	sum	$\text{Tree } \text{N}^{0..\infty} \rightarrow \text{N}$
Misc	compose	$\forall k : \text{Coeffect}, n\ m : k, a\ b\ c : \text{Type}.(a^{m} \rightarrow b)^{n} \rightarrow (b^{n} \rightarrow c)^{1:k} \rightarrow a^{n \cdot m} \rightarrow c$
	copy	$\forall a.a^{2} \rightarrow (a,\ a)$
	push	$\forall k : \text{Coeffect}, c : k, a\ b : \text{Type}.(a^{1} \rightarrow b)^{c} \rightarrow a^{c} \rightarrow b\ [c]$

Table 5.2: Type schemes for synthesis benchmarking results

				Graded			Cartesian + Graded type-check			
	Problem	Ctxt	#/Exs.		μT (ms)	Paths		μT (ms)	N	Paths
List	append	0	0 (+1)	✓	115.35 (5.13)	130	✓	105.24 (0.36)	8	130
	concat	1	0 (+3)	✓	1104.76 (1.60)	1354	✓	615.29 (1.43)	12	1354
	empty	0	0	✓	5.31 (0.02)	17	✓	1.20 (0.01)	0	17
	snoc	1	1	✓	2137.28 (2.14)	2204	✓	1094.03 (4.75)	8	2278
	drop	1	1	✓	1185.03 (2.53)	1634	✓	445.95 (1.71)	8	1907
	flatten	2	1	✓	1369.90 (2.60)	482	✓	527.64 (1.04)	8	482
	bind	2	0 (+2)	✓	62.20 (0.21)	129	✓	622.84 (0.95)	18	427
	return	0	0 (+1)	✓	19.71 (0.18)	49	✓	22.00 (0.08)	4	49
	inc	1	1	✓	708.23 (0.69)	879	✓	2835.53 (7.69)	24	1664
	head	0	1	✓	68.23 (0.53)	34	✓	20.78 (0.10)	4	34
	tail	0	1	✓	84.23 (0.20)	33	✓	38.59 (0.06)	8	33
	last	1	1 (+1)	✓	1298.52 (1.17)	593	✓	410.60 (6.25)	4	684
	length	1	1	✓	464.12 (0.90)	251	✓	127.91 (0.58)	4	251
	map	1	0 (+1)	✓	550.10 (0.61)	3075	✓	249.42 (0.73)	4	3075
	replicate5	0	0 (+1)	✓	372.23 (0.70)	1295	✓	435.78 (1.06)	4	1295
	replicate10	0	0 (+1)	✓	2241.87 (4.74)	10773	✓	2898.93 (1.47)	4	10773
	replicateN	1	1	✓	593.86 (1.68)	772	✓	108.98 (0.65)	4	772
	stutter	1	0	✓	1325.36 (1.77)	1792	×	Timeout	-	-
	sum	2	1 (+1)	✓	84.09 (0.25)	208	✓	3236.74 (0.87)	192	3623
Stream	build	0	0 (+1)	✓	61.27 (0.45)	75	✓	84.44 (0.49)	4	75
	map	1	0 (+1)	✓	351.93 (0.91)	1363	✓	153.01 (0.37)	0	1363
	take1	0	0 (+1)	✓	34.02 (0.23)	22	✓	19.32 (0.05)	0	22
	take2	0	0 (+1)	✓	110.18 (0.31)	204	✓	89.10 (0.18)	0	208
	take3	0	0 (+1)	✓	915.39 (1.42)	1139	✓	631.47 (1.14)	0	1172
Bool	neg	0	2	✓	209.09 (0.31)	42	✓	168.37 (0.56)	0	42
	and	0	4	✓	3129.30 (2.82)	786	✓	7069.14 (15.91)	0	2153
	impl	0	4	✓	1735.09 (4.31)	484	✓	3000.48 (4.65)	0	1214
	or	0	4	✓	1213.86 (1.02)	374	✓	2867.74 (3.52)	0	1203
	xor	0	4	✓	2865.79 (4.33)	736	✓	7251.38 (32.06)	0	2229
Maybe	bind	0	0 (+1)	✓	159.87 (0.52)	237	✓	55.33 (0.33)	0	237
	fromMaybe	0	0 (+2)	✓	54.27 (0.35)	18	✓	11.58 (0.10)	0	18
	return	0	0	✓	9.89 (0.02)	17	✓	11.49 (0.04)	4	17
	isJust	0	2	✓	69.33 (0.17)	48	✓	22.07 (0.09)	0	48
	isNothing	0	2	✓	102.42 (0.32)	49	✓	31.89 (0.22)	0	49
	map	0	0 (+1)	✓	54.90 (0.22)	120	✓	22.01 (0.10)	0	120
	mplus	0	1	✓	319.64 (0.47)	318	✓	70.98 (0.05)	0	318
Nat	isEven	1	2	✓	1027.79 (1.28)	466	✓	313.77 (0.92)	8	468
	pred	0	1	✓	46.20 (0.18)	33	✓	48.04 (0.13)	8	33
	succ	0	1	✓	115.16 (0.91)	76	✓	156.02 (0.50)	8	76
	sum	1	1 (+2)	✓	1582.23 (3.60)	751	✓	734.38 (1.41)	12	751
Tree	map	1	0 (+1)	✓	1168.60 (1.21)	4259	✓	525.47 (1.31)	4	4259
	stutter	1	0 (+1)	✓	693.44 (1.21)	832	✓	219.91 (1.02)	4	674
	sum	2	3	✓	1477.83 (1.28)	3230	✓	3532.24 (7.19)	192	3623
Misc	compose	0	0	✓	40.27 (0.08)	38	✓	14.53 (0.09)	2	38
	copy	0	0	✓	5.24 (0.04)	21	✓	6.16 (0.10)	2	21
	push	0	0	✓	26.66 (0.18)	45	✓	14.23 (0.13)	2	45

Table 5.3: Results. μT in *ms* to 2 d.p. with standard sample error in brackets

We briefly examine some of the more complex benchmarks which make use of almost all of our synthesis rules in one program. The `stutter` case from the List class of benchmarks is specified as:

```
stutter : ∀ a . List (a [2]) %1..∞ → List a
spec
    stutter % 0..∞
stutter = ?
```

This is a function which takes a list of values of type `a`, where each element in the list is explicitly graded by `2`, indicating that each element must be used twice. The return type of `stutter` is a list of type `a`. The argument list itself must be used at least once with potential usage extending up to infinity, suggesting that some recursion will be necessary in the program. This is further emphasised by the `spec`, which states that we can use the definition of `stutter` inside the function body in an unrestricted way. From this, we synthesise:

```
stutter Nil = Nil;
stutter (Cons [u] z) = (Cons u) ((Cons u) (stutter z))
```

in 1325ms (∼1.3 seconds). We also have a `stutter` case in the Tree class of benchmarks, which instead performs the above transformation over a binary tree data type, which yields the following program in 693ms (∼0.7 seconds):

```
stutter : ∀ a b . Tree (a [2]) % 1..∞ → Tree (a, a)
spec
    stutter %0..∞
stutter Leaf = Leaf;
stutter (Node y [v] u) = ((Node (stutter y)) (v, v)) (stutter
     u)
```

Lastly, we compare between the number of examples required by Granule (using grades) and the Myth program synthesis tool (based on pruning by examples). We take the cases from our benchmark set which have an equivalent in the Myth benchmark suite [Osera and Zdancewic, 2015]. Table 5.4 shows the number of examples used in Granule, and the number required for the equivalent Myth case. In both Granule and Myth this number represents the minimal number of examples required to synthesise the correct program.

For most of the problems (15 out of 20), Granule required fewer examples to identify the desired program in synthesis. The disparity in the number of examples required is quite significant in some cases: with 13 examples required by Myth to synthesise the *concat* problem but only 1 example for Granule. This shows the pruning power of graded information in synthesis, confirming H2.

5.7 SYNTHESIS OF LINEAR HASKELL PROGRAMS

As part of a growing trend of resourceful types being added to more mainstream languages, Haskell has introduced support for linear types as of GHC 9, using an underlying graded type system which can be enabled as a language extension of GHC's existing type system [Bernardy et al., 2018] (the `LinearType` extension). This system is closely related to Granule, but limited only to one semiring for its grades. This however presents an exciting opportunity: can we leverage our tool to synthesise (linear) Haskell programs?

Like Granule, grades in Linear Haskell can be expressed as "multiplicities" on function types: `a %r -> b`. The multiplicity r can be either 1 or ω (or polymorphic), with 1 denoting linear usage (also written as `'One`) and ω for (`'Many`) unrestricted usage. Likewise, in Granule, we can model linear types using the 0-1-ω semiring [Hughes et al., 2021].

Synthesising Linear Haskell programs then simply becomes a task of parsing a Haskell type into a Granule equivalent, synthesising a term from this, and compiling the synthesised term back into Haskell. The close syntactic correspondence between Granule and Haskell makes this translation straightforward.

Our implementation includes a prototype synthesis tool using this approach. A synthesis problem takes the form of a Linear Haskell program with a hole, e.g.

```
{-# LANGUAGE LinearTypes #-}
swap :: (a, b) %One -> (b, a)
swap = _
```

We invoke the tool with `gr --linear-haskell swap.hs` producing:

```
swap (z, y) = (y, z)
```

Haskell tuples are converted into a Granule data type, generated based on the tuple's arity. This data type is given unique name based on this size, e.g. `,4` for a tuple of arity 4. Using the special character `,` in the constructor name prevents name clashes with other in-scope data constructors, as this character would not be permitted in a typical user-defined constructor.

Users may make use of lists, tuples, `Maybe` and `Either` data types from Haskell's prelude, as well as user-defined ADTs. Further integration of the tool, as well as support for additional Haskell features such as GADTs is left as future work.

This tool can be especially useful as a programming aid when writing linear versions of Haskell libraries. To conclude this section, we showcase some of the programs synthesised via our Linear Haskell tool with the goal of writing a Linear Haskell library for Haskell's `Maybe` data type.

```
{-# LANGUAGE LinearTypes #-}
```

```
maybe :: b %Many -> (a %One -> b) %Many -> (Maybe a) %One -> b
maybe x y Nothing = x
maybe x y (Just u) = y u

bind :: Maybe a %One -> (a %One -> Maybe b) %Many -> Maybe b
bind Nothing y = Nothing
bind (Just z) y = y z

map :: (a %One -> b) %Many -> Maybe a %One -> Maybe b
map x Nothing = Nothing
map x (Just z) = Just (x z)

maybeToList :: Maybe a %One -> [a]
maybeToList Nothing = []
maybeToList (Just y) = [y]

fromMaybe :: Maybe a %One -> a %Many -> a
fromMaybe Nothing y = y
fromMaybe (Just z) y = z
```

5.8 conclusion

In this chapter we presented a synthesis calculus for a feature-rich type system based on the fully graded λ-calculus of section 2.3.2, which complements our graded linear system of Chapter 3 to allow us to synthesise programs for the major approaches to graded type systems. We found through our experimental evaluation that synthesising programs in this calculus requires the exploration of fewer synthesis paths when using the information provided by grades to prune the search space of candidate programs. In terms of pure speed, un-graded type-directed program synthesis outperformed our implementation, largely due to the overhead of discharging constraints to an SMT solver.

This discrepancy in speed necessitates that our synthesis tool seek a more efficient means of interfacing with the solver. However, the theoretical advantage of grade-based pruning remains clear: the search space was the same or reduced in all but one of our benchmarking examples in Table 5.3. In the next chapter, we conclude our synthesis journey with some future directions that this work might take, including how to improve interaction with the solver.

		Granule	Myth
Problem		#/Exs	#/Exs
List	append	0	6
	concat	1	6
	snoc	1	8
	drop	1	13
	inc	1	4
	head	1	3
	tail	1	3
	last	1	6
	length	1	3
	map	0	8
	stutter	0	3
	sum	1	3
Bool	neg	2	2
	and	4	4
	impl	4	4
	or	4	4
	xor	4	4
Nat	isEven	2	4
	pred	1	3
Tree	map	0	7

Table 5.4: Number of examples needed for synthesis, comparing Granule vs. Myth

6 CONCLUSION

In this dissertation, we have provided a framework for designing program synthesis tool based on linear and graded type systems, overcoming the challenges imposed by treating program values as resources, and leveraging the type systems' properties to build two efficient synthesis tools, targeting two common flavours of graded type system. In summary, in our work we have succeeded in:

- Building several core calculi for a two varieties of quantitative type systems, encompassing the major approaches in the literature, with both styles differing in expressivity and imposing unique requirements.
- Implementing these synthesis calculi as tools for Granule, as fully integrated components of the Granule toolchain.
- Evaluating each of these systems in a variety of criteria, our main result being that synthesis with graded types is an effective way of reducing the program synthesis search space.
- Showing that programs require fewer examples than in a purely example-driven synthesis setting, such as MYTH.
- Providing an alternative approach to generating graded combinators, based on a generic programming methodology.
- Adapting our synthesis technique to Linear Haskell, showing the viability of our framework as a foundation for building synthesis tools for other quantitative type systems.
- Proved the soundness of each of these systems, reasoning about the behavioural properties of our synthesis calculi.

Essentially, we have shown that program synthesis from linear and graded types is a feasible and effective approach to reducing the search space of a program synthesis task. The expressivity of linear and graded types in describing the static semantics of a program is highly valued by many programmers, so harnessing this expressivity in synthesis is a something that we find will be useful to programs in this context, allowing the synthesis of programs often without the need for additional specification such as examples.

There remain many avenues for future exploration, which intend to pursue, and hope that others will also find use in this approach when designing synthesis tools for their own type systems.

6.1 FUTURE DIRECTIONS

Our goal was to create a program synthesis tool for Granule which assists the programmer in writing programs with resource-sensitive types. We intend to pursue further improvements to our tool which serve this end, including reducing the overhead of SMT solving, integrating examples into the search algorithm itself in the style of MYTH [Osera and Zdancewic, 2015] and Leon [Kneuss et al., 2013], as well as considering possible semiring-dependent optimisations that may be applicable.

6.1.1 *Subtractive Resource Management*

Ultimately, we opted to focus on the additive resource management scheme for Chapter 5 due to concerns regarding the efficiency of its implementation for a fully graded typing calculus.

In the subtractive scheme, the VAR rule generates a constraint which determines if the use of a variable is permissible based on the rest of the partially synthesised program:

$$\frac{\exists s.\ r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash A \Rightarrow x \mid \Gamma, x :_s A}\ \textsc{Var}$$

i.e., r must overapproximate one use of x plus the future use of x given by the existential s.

Our comparative evaluation of the additive and subtractive schemes in Section 3.6 showed that these, and other associated constraints from the subtractive approach, are larger, typically more complex, and are discharged more frequently than their counterparts in the additive system (every time a variable usage is being considered in the above case). Their evaluation concluded that the only situation where subtractive decisively outperformed additive was on purely linear programs. This, coupled with the fact that the subtractive approach has limitations in the presence of polymorphic grades, influenced our decision to adopt the additive scheme, especially as we considered much more complex programs than in the calculi of Chapter 3, e.g., targeting recursion.

6.1.2 *SMT Solving*

Related to the above, there is scope for improving the interaction between the synthesis tool and the SMT solver, to make synthesis more efficient. Both evaluations showed that the bulk of synthesis time is spent solving constraints in the SMT solver. Making this interaction more streamlined is therefore particularly appealing. One approach would be to run an "online" SMT solving procedure. This would reduce the overhead of discharging constraints to the solver by eliminating the need to re-solve already solved constraints.

Alternatively, one could imagine implementing custom solvers for the more commonly used semirings. From our lists of benchmarking examples in Tables 3.1 and 5.2, we can see the natural number, and intervals over the natural numbers semirings appear frequently, more than any other semiring. An SMT solver which focuses purely on solving in these semirings, sacrificing the generality provided by solvers such as Z3 de Moura and Bjørner [2008], may be a worthwhile avenue of exploration.

6.1.3 *Generalised Algebraic Data Types*

A logical next step is to incorporate GADTs (Generalised ADTs), i.e., indexed types, into the synthesis algorithm. Granule provides support for user-defined GADTs, and the interaction between grades and type indices is a key contributor to Granule's expressive power [Orchard et al., 2019]. Consider our list type benchmarks for example. In most cases, when we want to synthesise a recursive function definition which takes a list as input, we have to give the list a 0..∞ interval grade to account for potentially unlimited usage. Take for example the `map` benchmark:

```
map : ∀ a b . (a % 1..∞ → b) % 0..∞ → List a % 1..∞ →
    List b
spec
    map % 0..∞
map = ?
```

With indexed types, we can be much more precise. Consider a size-indexed vector type, defined as:

```
data Vec (n : Nat) t where
  Nil  : Vec 0 t;
  Cons : t → Vec n t → Vec (n+1) t
```

The corresponding `map` function can be given a much tighter specification, connecting the usage of the input function to the length of the vector, from which we could synthesise the program:

```
vmap : ∀ {a b : Type, n : Nat} . (a → b) %n → Vec n a →
    Vec n b
```

```
spec
    vmap % n
vmap f Nil = Nil;
vmap f (Cons x xs) = Cons (f x) (vmap f xs)
```

The latter type not only provides us with a greater opportunity to prune grade-violating programs, its type is also much more descriptive of the user's intent. Adapting our approach to GADTs is future work, and mostly consists of extending the typing for our synthesis rule for **case** statements to handle GADT specialisation.

GADTs are sometimes referred to as *lightweight* dependent types. Dependent type systems allow arbitrary program properties to be expressed and verified during type checking, with types being indexed by arbitrary program terms. Several works have endeavoured to integrate fully-dependent types into a quantitative setting [Choudhury et al., 2021, Abel et al., 2023, McBride, 2016, Krishnaswami et al., 2015]. One such examples, Moon et al. [2021]'s GERTY, is based on Granule and has a prototype implementation, making it a particularly appealing target for experimenting with the synthesis in a fully-dependent graded setting, using the foundations laid by the work in this thesis.

6.1.4 *Large Language Models*

With the rise in LLMs (Large Language Models) showing their power at program synthesis tasks [Austin et al., 2021, Jain et al., 2021], the deductive approach still has something to contribute: it provides correct-by-construction synthesis based on specifications, rather than predicted programs which may violate more fine-grained type constraints (e.g., as provided by grades). Future approaches may combine both LLM approaches with deductive approaches, where the logical engine of the deductive approach can guide prediction. Exploring this is further work and a general opportunity and challenge for the deductive synthesis community.

6.2 FINAL REMARKS

Type-directed program synthesis has been long been an attractive field in computer science partially due to the potential it offers: the ability to write programs that are correct by construction, with significant help from the computer.

We feel that type-directed program synthesis complements linear and graded types very well: as richer types further constrain the number of possible inhabitants, theoretically there is less work to be done by the computer to identify the program which behaves according to the user's intent.

Programming with linear and graded types can require significantly more forethought than standard functional programming, and programs which seem correct at first glance to a user might actually be resource-violating. Our hope is that the tools developed in this work may be useful in reducing this cognitive overhead, and that the ideas we have developed may be useful to those in both the program synthesis, and the quantitative types communities.

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Part II

APPENDIX

A BENCHMARKING SUITES

A.1 SYNTHESISED PROGRAMS FOR THE GRADED LINEAR SYNTHESIS CALCULI

This section includes the synthesised programs fro the benchmarking problems in Section 3.6.

A.1.1 *Hilbert*

1. ⊗-Intro:

```
and : ∀ a b . a → b → (a, b)
and x y = (x, y)
```

2. ⊗-Elim:

```
and1 : ∀ a b .  (a, b [0]) → a
and1 (y, [u]) = y

and2 : ∀ a b . (a [0], b) → b
and2 ([u], z) = z
```

3. ⊕-Intro:

```
data Either a b = Left a | Right b

or1 : ∀ a b . a → Either a b
or1 x = Left x

or2 : ∀ a b . b → Either a b
or2 x = Right x
```

4. ⊕-Elim:

```
data Either a b = Left a | Right b

or3 : ∀ a b c
    . (a → c) [0..1]
    → (b → c) [0..1]
    → (Either a b)
    → c
or3 [u] [v] (Right x6) = v x6;
or3 [u] [v] (Left x5) = u x5
```

5. SKI:

```
s : ∀ a b c
  . (a → (b → c))
  → (a → b)
  → a [2]
  → c
s x y [u] = (x u) (y u)

k : ∀ a b . a → b [0..1] → a
k x [z] = x

i : ∀ a . a → a
i x = x
```

A.1.2 *Comp*

1. 0/1:

```
comp-01 : ∀ a b c
        . (a [] → b []) []
        → (b [] → c []) []
        → (a [] → c) []
comp-01 [z] [u] =
    [λv →
        let [w] = v in
        let [q] = z [w] in
        let [x9] = u [q] in
        let [x11] = z [w] in x9]
```

2. CBN:

```
comp-cbn : ∀ a b c
         . (a [] → b) []
         → (b [] → c) [] → a [] → c
comp-cbn [u] [w] [v] = w [u [v]]
```

3. CBV:

```
comp-cbv : ∀ a b c  .
         (a [] → b []) []
         → (b [] → c []) []
         → (a [] → c []) []
comp-cbv [z] [u] =
    [λv →
        (λ [w] →
            [(λ [q] → q) (u [(λ [x9] → x9) (z [w])])]
    ) v]
```

4. coK-$\mathcal{R}$:

```
compGen : ∀ {k : Coeffect, n m : k, a b c : Type}
        . (a [m] → b) [n]
        → (b [n] → c)
        → a [n * m]
        → c
compGen [u] y [v] = y [u [v]]
```

5. coK-$\mathbb{N}$:

```
compNat : ∀ {n m : Nat, a b c : Type}
     . (a [m] → b) [n]
     → (b [n] → c)
     → a [n * m]
     → c
compNat [u] y [v] = y [u [v]]
```

6. mult:

```
mult : ∀ a b c
     . (a → b)
     → (b → c)
     → a
     → c
mult x y z = y (x z)
```

A.1.3 *Dist*

1. $\otimes$-!:

```
pull : ∀ a b . (a [], b []) → (a, b) []
pull ([u], [v]) = [(u, v)]
```

2. $\otimes$-$\mathbb{N}$:

```
pull : ∀ {a b : Type, n : Nat}
     . (a [n], b [n])
     → (a, b) [n]
pull ([u], [v]) = [(u, v)]
```

3. $\otimes$-$\mathcal{R}$:

```
pull : ∀ {a b : Type, k : Coeffect, c : k}
     . (a [c], b [c])
     → (a, b) [c]
pull ([u], [v]) = [(u, v)]
```

4. $\oplus$-!:

```
data Either a b = Left a | Right b

pull : ∀ a b
```

```
     . Either (a []) (b [])
     → (Either a b) []
pull (Right [v]) = [Right v];
pull (Left [z]) = [Left z]
```

5. ⊕-ℕ:

```
data Either a b = Left a | Right b

pull : ∀ {a b : Type, n : Nat}
     . Either (a [n]) (b [n])
     → (Either a b) [n]
pull (Right [v]) = [Right v];
pull (Left [z]) = [Left z]
```

6. ⊕-$\mathcal{R}$:

```
data Either a b = Left a | Right b

pull : ∀ {a b : Type, k : Coeffect, c : k}
     . Either (a [c]) (b [c])
     → (Either a b) [c]
pull (Right [v]) = [Right v];
pull (Left [z]) = [Left z]
```

7. ⊸-!:

```
push : ∀ a b
     . (a → b) []
     → a []
     → b []
push [z] [u] = [z u]
```

8. ⊸-ℕ:

```
push : ∀ {a b : Type, c : Nat}
     . (a → b) [c]
     → a [c]
     → b [c]
push [z] [u] = [z u]
```

9. ⊸-$\mathcal{R}$:

```
push : ∀ {a b : Type, k : Coeffect, c : k}
     . (a → b) [c]
     → a [c]
     → b [c]
push [z] [u] = [z u]
```

A.1.4 *Vec*

1. vec5:

```
vec5 : ∀ a b
     . (a → b) [5]
     → ((((a, a), a), a), a)
     → ((((b, b), b), b), b)
vec5 [z] ((((x9, x10), x8), x6), x4) =
    ((((z x9, z x8), z x6), z x4), z x10)
```

2. vec10:

```
vec10 : ∀ a b
      . (a → b) [10]
      → (((((((((a, a), a), a), a), a), a), a), a), a)
      → (((((((((b, b), b), b), b), b), b), b), b), b)
vec10 [z] (((((((((x19, x20), x18), x16), x14), x12),
    x10), x8), p), v) =
    (((((((((z x20, z v), z p), z x8), z x10), z x12), z
     x14), z x16), z x18), z x19)
```

3. vec15:

```
vec15 : ∀ a b
      . (a → b) [15]
      → ((((((((((((((a, a), a), a), a), a), a), a), a)
    , a), a), a), a), a), a)
      → ((((((((((((((b, b), b), b), b), b), b), b), b)
    , b), b), b), b), b), b)
vec15 [z] ((((((((((((((x29, x30), x28), x26), x24), x22
    ), x20), x18), x16), x14), x12), x10), x8), p), v) =
    ((((((((((((((z x30, z v), z p), z x8), z x10), z
    x12), z x14), z x16), z x18), z x20), z x22), z x24)
    , z x26), z x28), z x29)
```

4. vec20:

```
vec20 : ∀ a b
      . (a → b) [20]
      → (((((((((((((((((((a, a), a), a), a), a), a), a
    ), a), a), a), a), a), a), a), a), a), a), a), a)
      → (((((((((((((((((((b, b), b), b), b), b), b), b
    ), b), b), b), b), b), b), b), b), b), b), b), b)
vec20 [z] (((((((((((((((((((x39, x40), x38), x36), x34)
    , x32), x30), x28), x26), x24), x22), x20), x18),
    x16), x14), x12), x10), x8), p), v) =
    (((((((((((((((((((z x40, z v), z p), z x8), z x10),
     z x12), z x14), z x16), z x18), z x20), z x22), z
    x24), z x26), z x28), z x30), z x32), z x34), z x36)
    , z x38), z x39)
```

A.1.5 *Misc*

1. split⊕:

```
data Either a b where Left a | Right b

splitPlus : ∀ a b c
          . b [2..3]
          → Either a c
          → Either (a, b [2..2]) (c, b [3..3])
splitPlus [z] (Right x4) = Right (x4, [z]);
splitPlus [z] (Left x3) = Left (x3, [z])
```

2. split⊗:

```
splitPair : ∀ a
          . (a → a → a) [0..2]
          → a [10..10]
          → (a [2..2], a [6..6])
splitPair [z] [u] = ([(z u) u], [u])
```

3. share:

```
share : ∀ a
      . (a → a → a) [0..2]
      → a [10..10]
      → (a [2..2], a [6..6])
share [z] [u] = ([(z u) u], [u])
```

4. Exm. 3.1.2:

```
dontLeak : ∀ a b
         . (a [Public], a [Private])
         → ((a, ()) [Public] → b)
         → b
dontLeak ([w], [v]) y = y [(w, ())]
```

A.2 SYNTHESISED PROGRAMS FOR THE FULLY GRADED SYNTHESIS CALCULUS

This section includes the synthesised programs, their synthesis contexts, and examples used for the benchmarking problems in Section 5.6. The context of each synthesised program is listed by the program's `spec`. If the program has no `spec`, then synthesis is occurring in an empty context.

For each program we list the examples required to synthesise the correct result in the Graded case, and the additional examples required to synthesise the same program without grades (i.e. in the Cartesian setting). See Section 5.6 for further details.

A.2.1 *List*

1. append:

```
language GradedBase

data List a = Nil | Cons a (List a)

append : ∀ a
       . (List a) %1
       → a %1
       → List a
append x y = (Cons y) x
```

with no Graded examples and Cartesian example(s):

```
append (Cons 1 Nil) 2 = (Cons 2 (Cons 1 Nil));
```

2. concat:

```
language GradedBase

data List a = Cons a (List a) | Nil

concat : ∀ a
       . (List a) %1..∞
       → (List a) %1..∞
       →  List a
spec
    concat %0..∞
concat Nil y = y;
concat (Cons z u) y = (Cons z) ((concat u) y)
```

with no Graded examples and Cartesian examples(s):

```
concat (Cons 0 Nil) Nil = (Cons 0 Nil);
concat (Cons 0 Nil) (Cons 1 Nil) = (Cons 0 (Cons 1 Nil))
    ;
concat (Cons 0 (Cons 1 (Cons 2 Nil))) (Cons 3 (Cons 4
    Nil)) = Cons 0 (Cons 1 (Cons 2 (Cons 3 (Cons 4 Nil))
    ));
```

3. empty:

```
language GradedBase

data List a = Nil | Cons a (List a)

empty : ∀ a b . () → List a
empty () = Nil
```

4. snoc:

```
language GradedBase

data List a = Nil | Cons a (List a)

data N = S N | Z
```

```
snoc : ∀ a
     . (List a) %1..∞
     → a %1..∞
     → List a
spec
    snoc %0..∞
snoc x y =
    (case x of
      Nil → (Cons y) x;
      Cons z u → (Cons z) ((snoc u) y))
```

with Graded example(s):

```
snoc (Cons Z Nil) (S Z) = (Cons Z (Cons (S Z) Nil));
```

5. drop:

```
language GradedBase

data List a = Cons a (List a) | Nil

data N = S N | Z

drop : ∀ a . N %0..∞ → List (a) %0..∞ → List a
spec
    drop % 0..∞
drop x y =
    (case y of
        Nil →
    (case x of
        Z → y;
        S z → (drop z) y);
        Cons p q →
    (case x of
        Z → q;
        S x8 → q))
```

with Graded example(s):

```
drop (S Z) (Cons (S Z) (Cons Z Nil)) = Cons Z Nil;
```

6. flatten:

```
language GradedBase

data List a = Cons a (List a) | Nil

append : ∀ {a : Type}
       . (List a) %1..∞
       → (List a) %1..∞
       →  List a
append Nil y = y;
append (Cons z u) y = (Cons z) ((append u) y)

flatten : ∀ a . (List (List a)) %0..∞ → (List a)
spec
    flatten % 0..∞ , append %0..∞
flatten Nil = (append Nil) Nil;
flatten (Cons u v) = (append u) (flatten v)
```

with Graded example(s):

```
flatten (Cons (Cons 1 Nil) (Cons (Cons 1 Nil) Nil)) =
    Cons 1 (Cons 1 Nil);
```

7. bind:

```
language GradedBase

data List a = Nil | Cons a (List a)

data N = S N | Z

data Bool = True | False;

append : ∀ {a : Type}
       . (List a) %1..∞
       → (List a) %1..∞
       →  List a
append Nil y = y;
append (Cons z u) y = (Cons z) ((append u) y)

concat : ∀ a . (List (List a)) %1..∞ → (List a)
concat Nil = Nil;
concat (Cons u v) = (append u) (concat v)

map : ∀ a b
    . (a %1..∞ → b) %0..∞
    → List a %1..∞
    → List b
map x Nil = Nil;
map x (Cons z u) = (Cons (x z)) ((map x) u)

isEven : N %1..∞ → List N
isEven Z = Nil;
isEven (S Z) = Cons (S Z) Nil;
isEven (S (S z)) = concat (Cons (isEven z) Nil)

bind : ∀ a b
     . List a %1..∞
     → (a %1..∞ → List b) %0..∞
     → List b
spec
    map %1..∞, concat %1..∞
bind x y = concat ((map (λw → w)) x)
```

with no Graded examples and Cartesian example(s):

```
bind (Cons Z Nil) isEven = Nil;
bind (Cons (S Z) Nil) isEven = (Cons (S Z) Nil);
```

8. return:

```
language GradedBase

data List a =  Cons a (List a) | Nil

return : ∀ { a b : Type} . a → List a
return x = (Cons x) Nil
```

with no Graded examples and Cartesian example(s):

```
return 1 = Cons 1 Nil;
```

9. inc:

```
language GradedBase

data List a = Cons a (List a) | Nil

data N = S N | Z

map : (List N) %1..∞
    → (N %1..∞ → N) %0..∞
    → (List N)
map Nil f = Nil;
map (Cons x xs) f = (Cons (f x) (map xs f))

inc : ∀ a . (List N) %1..∞ → (List N)
spec
    map %1..∞
inc x = (map x) (λu → S u)
```

with Graded example(s):

```
inc (Cons (S Z) Nil) = (Cons (S (S Z)) Nil);
```

10. head:

```
language GradedBase

data List a = Cons a (List a) | Nil

head : ∀ a . (List a) %0..1 → a %0..1 → a
head Nil y = y;
head (Cons z u) y = z
```

with Graded example(s):

```
head (Cons 1 (Cons 2 Nil)) 2 = 1;
```

11. tail:

```
language GradedBase

data List a = Cons a (List a) | Nil

tail : ∀ a . List a %0..1 → List a
tail Nil = Nil;
tail (Cons y z) = z
```

with Graded example(s):

```
tail (Cons 1 (Cons 2 Nil)) = Cons 2 Nil;
```

12. last:

```
language GradedBase

data List a = Cons a (List a) | Nil

data Maybe a = Just a | Nothing

last : ∀ a . (List a) %0..∞ → Maybe a
spec
    last %0..∞
last Nil = Nothing;
last (Cons y z) =
    (case z of
        Nil → last z;
        Cons v w → Just v)
```

with Graded example(s):

```
last (Cons 1 (Cons 2 Nil)) = Just 2;
```

and Cartesian example(s):

```
last (Cons 1 Nil) = Just 1;
```

13. length:

```
language GradedBase

data List a = Cons a (List a) | Nil

data N = S N | Z

length : ∀ a . List a %0..∞ → N
spec
    length %0..∞
length Nil = Z;
length (Cons y z) = S (length z)
```

with Graded example(s):

```
length (Cons 1 (Cons 1 Nil)) = S (S Z);
```

14. map:

```
language GradedBase

data List a = Nil | Cons a (List a)

data Bool = True | False;

neg : Bool %1..∞ → Bool
neg True = False;
neg False = True

map : ∀ a b
    . (a % 1..∞ → b) %0..∞
    → List a %1..∞
    → List b
spec
    map % 0..∞
map x Nil = Nil;
map x (Cons z u) = (Cons (x z)) ((map x) u)
```

with no Graded examples and Cartesian example(s):

```
map neg (Cons True Nil) = Cons False Nil;
```

15. replicate5:

```
language GradedBase

data List a =  Cons a (List a) | Nil

replicate5 : ∀ { a : Type} . a %5 → List a
replicate5 x = (Cons x) ((Cons x) ((Cons x) ((Cons x) ((
    Cons x) Nil))))
```

with no Graded examples and Cartesian example(s):

```
replicate5 1 = Cons 1 (Cons 1 (Cons 1 (Cons 1 (Cons 1
    Nil))));
```

16. replicate10:

```
language GradedBase

data List a =  Cons a (List a) | Nil

replicate10 : ∀ { a : Type} . a %10 → List a
replicate10 x = (Cons x) ((Cons x) ((Cons x) ((Cons x)
    ((Cons x) ((Cons x) ((Cons x) ((Cons x) ((Cons x) ((
    Cons x) Nil)))))))))
```

with no Graded examples and Cartesian example(s);

```
replicate10 1 = (Cons 1 (Cons 1 (Cons 1 (Cons 1 (Cons 1
    (Cons 1 (Cons 1 (Cons 1 (Cons 1 (Cons 1 Nil))))))))))
    );
```

17. replicateN:

```
language GradedBase

data List a =  Cons a (List a) | Nil

data N = S N | Z

replicateN : ∀ a
           . N %0..∞
           → a %0..∞
           → List a
spec
    replicateN % 0..∞
replicateN Z y = Nil;
replicateN (S z) y = (Cons y) ((replicateN z) y)
```

with Graded example(s):

```
replicateN (S (S Z)) 2 = (Cons 2 (Cons 2 Nil));
```

18. stutter:

```
language GradedBase

data List a = Cons a (List a) | Nil

stutter : ∀ a . List (a [2]) %1..∞ → List a
spec
    stutter % 0..∞
stutter Nil = Nil;
stutter (Cons [u] z) = (Cons u) ((Cons u) (stutter z))
```

19. sum:

```
language GradedBase

data N = S N | Z

data List a = Cons a (List a) | Nil

fold : List N %0..∞
     → (N % 1..∞
     → N % 1..∞ → N) %0..∞
     → N %0..∞ → N
fold Nil f acc = acc;
fold (Cons x xs) f acc =  fold xs f (f acc x)

add : N %1..∞ → N %1..∞ → N
add Z n2 = n2;
add (S n1) n2 = S (add n1 n2)

sum : List N %0..∞ → N
spec
    fold %0..∞, add %0..∞
sum x = ((fold x) (λp → λq → (add p) q)) ((add Z) Z)
```

with Graded example(s):

```
sum (Cons (S Z) (Cons (S Z) Nil)) = (S (S Z));
```

and Cartesian example(s):

```
sum (Cons (S Z) (Cons (S Z) (Cons (S Z) Nil))) = (S (S (
    S Z)));
```

A.2.2 *Stream*

1. build:

```
language CBN
language GradedBase

data Stream a = Next a (Stream a)

ones : () %1..1 → Stream Int
ones () = Next 1 (ones ())

head : ∀ a . Stream a %0..1 → a
head (Next x xs) = x

build : ∀ a . a %1..1 → (Stream a) %1..1 → Stream a
build x y = (Next x) y
```

with no Graded examples and Cartesian example(s):

```
head (build 2 (ones ())) = 2;
```

2. map:

```
language CBN
language GradedBase

data Stream a = Next a (Stream a)

data Bool = True | False

trues : () %1..∞ → Stream Bool
trues () = Next True (trues ())

neg : Bool %1..∞ → Bool
neg True = False;
neg False = True

head : ∀ a . Stream a %0..∞ → a
head (Next x xs) = x

map : ∀ a b
    . Stream a %1..∞
    → (a %1..∞ → b) %1..∞
    → Stream b
spec
    map % 1..∞
map (Next z u) y = (Next (y z)) ((map_stream u) y)
```

with no Graded examples and Cartesian example(s):

```
head (map_stream (trues ()) neg) = False;
```

3. take1:

```
language CBN
language GradedBase

data Stream a = Next a (Stream a)

data Bool = True | False

ones : () %1..1 → Stream Int
ones () = Next 1 (ones ())

head : ∀ a . (Stream a) %0..1 → a
head (Next x _) = x

take1 : ∀ a . Stream a %0..1 → a
take1 (Next y z) = y
```

with no Graded examples and Cartesian example(s):

```
take1 (Next 2 (ones ())) = 2;
```

4. take2:

```
language CBN
language GradedBase

data Stream a = Next a (Stream a)
data Bool = True | False

ones : () %1..1 → Stream Int
ones () = Next 1 (ones ())

head : ∀ a . (Stream a) %0..1 → a
head (Next x _) = x
```

```
take2 : ∀ a . Stream a %0..1 → (a, a)
take2 (Next y (Next u v)) = (u, y)
```

with no Graded examples and Cartesian example(s):

```
take2 (Next 2 (ones ())) = (2, 1);
```

5. take3:

```
language CBN
language GradedBase

data Stream a = Next a (Stream a)

data Bool = True | False

ones : () %1..1 → Stream Int
ones () = Next 1 (ones ())

head : ∀ a . (Stream a) %0..1 → a
head (Next x _) = x

take3 : ∀ a . Stream a %0..1 → (a, (a, a))
take3 (Next y (Next u (Next w p))) = (y, (w, u))
```

with no Graded examples and Cartesian example(s):

```
take3 (Next 3 (Next 2 (ones ()))) = (3, (2, 1));
```

A.2.3 *Bool*

1. neg:

```
language GradedBase

data Bool = True | False

neg : Bool %1 → Bool
neg True = False;
neg False = True
```

with Graded example(s):

```
neg True = False;
neg False = True;
```

2. and:

```
language GradedBase

data Bool = True | False

and : Bool %1 → Bool %1 → Bool
and True True = True;
and False True = False;
and True False = False;
and False False = False
```

with Graded example(s):

```
and True True = True;
and False True = False;
and True False = False;
and False False = False;
```

3. impl:

```
language GradedBase

data Bool = True | False

impl : Bool %1 → Bool %1 → Bool
impl True True = True;
impl True False = False;
impl False True = True;
impl False False = True
```

with Graded example(s):

```
impl True True = True;
impl True False = False;
impl False True = True;
impl False False = True;
```

4. or:

```
language GradedBase

data Bool = True | False

or : Bool %1 → Bool %1 → Bool
or True True = True;
or False True = True;
or True False = True;
or False False = False
```

with Graded example(s):

```
or True True = True;
or False True = True;
or True False = True;
or False False = False
```

5. xor:

```
language GradedBase
```

```
data Bool = True | False

xor : Bool %1 → Bool %1 → Bool
xor True True = False;
xor False True = True;
xor True False = True;
xor False False = False
```

with Graded example(s):

```
xor True True = False;
xor False True = True;
xor True False = True;
xor False False = False
```

A.2.4 *Maybe*

1. bind:

```
language GradedBase

data Maybe a = Just a | Nothing

data N = S N | Z

data Bool = True | False;

isEven : N %1..∞ → Maybe N
isEven Z = Nothing;
isEven (S Z) = Just (S Z);
isEven (S (S z)) =
    case isEven z of
        Nothing → Nothing;
        Just (S Z) → Just (S Z)

bind : ∀ a b
     . Maybe a %1..1
     → (a %1..1 → Maybe b) %0..1
     → Maybe b
bind Nothing y = Nothing;
bind (Just z) y = y z
```

with no Graded example(s) and Cartesian example(s):

```
bind (Just (S Z)) isEven = Just (S Z);
```

2. fromMaybe:

```
language GradedBase

data Maybe a = Nothing | Just a

```

```
fromMaybe : ∀ a . Maybe a %(1..1) → a %(0..1) → a
fromMaybe Nothing y = y;
fromMaybe (Just z) y = z
```

with no Graded example(s) and Cartesian example(s):

```
fromMaybe  Nothing 1 = 1;
fromMaybe (Just 1) 2 = 1;
```

3. return:

```
language GradedBase

data Maybe a = Nothing | Just a

return : ∀ a . a → Maybe a
return x = Just x
```

4. isJust:

```
language GradedBase

data Maybe a = Just a | Nothing

data Bool = True | False

isJust : ∀ a . (Maybe a) %0..1 → Bool
isJust Nothing = False;
isJust (Just y) = True
```

with Graded example(s):

```
isJust (Just 1) = True;
isJust Nothing = False;
```

5. isNothing:

```
language GradedBase

data Maybe a = Just a | Nothing

data Bool = True | False

isNothing : ∀ a . (Maybe a) %0..1 → Bool
isNothing Nothing = True;
isNothing (Just y) = False
```

with Graded example(s):

```
isNothing (Just 1) = False;
isNothing Nothing = True;
```

6. map:

```
language GradedBase

data Maybe a = Nothing | Just a

data N = S N | Z
data Bool = True | False

isOne : N %0..1 → Bool
isOne (S Z) = True;
isOne _ = False

map : ∀ a b
    . (a → b) %(0..1)
    → (Maybe a) %(1..1)
    → Maybe b
map x Nothing = Nothing;
map x (Just z) = Just (x z)
```

with no Graded examples and Cartesian example(s):

```
map isOne (Just (S Z)) = Just True;
```

7. mplus:

```
language GradedBase

data Maybe a = Nothing | Just a


mplus : ∀ a b
      . Maybe a %(0..1)
      → Maybe b %(0..1)
      → Maybe (a, b)
mplus Nothing Nothing = Nothing;
mplus (Just z) Nothing = Nothing;
mplus Nothing (Just v) = Nothing;
mplus (Just w) (Just v) = Just (w, v)
```

with Graded example(s):

```
mplus (Just 1) (Just 2) = Just (1, 2);
```

A.2.5 *Nat*

1. isEven:

```
language GradedBase

data N = S N | Z

data Bool = True | False

isEven : N %1..∞ → Bool
```

```
spec
    isEven % 0..∞
isEven Z = True;
isEven (S Z) = False;
isEven (S (S z)) = isEven z
```

with Graded example(s):

```
isEven (S (S Z)) = True;
isEven (S (S (S Z))) = False;
```

2. pred:

```
language GradedBase

data N = S N | Z

pred : N %1 → N
pred Z = Z;
pred (S y) = y
```

with Graded example(s):

```
pred (S (S Z)) = (S Z);
```

3. succ:

```
language GradedBase

data N = S N | Z

succ : N %1 → N
succ x = S x
```

with Graded example(s):

```
succ Z = (S Z);
```

4. sum:

```
language GradedBase

data N = S N | Z

sum : N %1..∞ → N %1..∞ → N
spec
    sum % 0..∞
sum Z y = y;
sum (S z) y = S ((sum z) y)
```

with Graded example(s):

```
sum (S Z) (S Z) = (S (S Z));
```

and Cartesian example(s):

```
sum (S Z) Z = (S Z);
sum (S (S (S (S Z)))) (S (S Z)) = (S (S (S (S (S (S Z)))
    )));
```

A.2.6 *Tree*

1. map:

```
language GradedBase

data Tree a = Leaf | Node (Tree a) a (Tree a)

data Bool = True | False

neg : Bool %1..∞ → Bool
neg True = False;
neg False = True

map : ∀ a b
    . Tree a %1..∞
    → (a %1..∞ → b) %0..∞
    → Tree b
spec
    map %0..∞
map Leaf y = Leaf;
map (Node z u v) y = ((Node ((map z) y)) (y u)) ((map v)
     y)
```

with no Graded examples and Cartesian example(s):

```
map (Node Leaf True Leaf) neg = (Node Leaf False Leaf);
```

2. sutter:

```
language GradedBase

data Tree a = Leaf | Node (Tree a) a (Tree a)

stutter : ∀ a b . Tree (a [2]) %1..∞ → Tree (a, a)
spec
    stutter %0..∞
stutter Leaf = Leaf;
stutter (Node y [v] u) = ((Node (stutter y)) (v, v)) (
    stutter u)
```

with no Graded examples and Cartesian example(s):

```
stutter (Node Leaf [1] Leaf) = (Node Leaf (1, 1) Leaf);
```

3. sum:

```
language GradedBase
```

```
data N = S N | Z
:W
data Tree a = Leaf | Node (Tree a) a (Tree a)

add : N %1..∞ → N %1..∞ → N
add Z y = y;
add (S z) y = S ((add z) y)

fold : Tree N %0..∞
     → (N %1..∞ → N %1..∞ → N) %0..∞
     → N %0..∞
     → N
fold Leaf f acc = acc;
fold (Node l x r) f acc =
    fold l f (f x (fold r f acc))

sum : ∀ a . Tree N %0..∞ → N
spec
    fold %1..∞, add %1..∞
sum x = (add Z) (((fold x) (λp → λq → (add p) q)) Z)
```

with Graded example(s):

```
sum Leaf = Z;
sum (Node Leaf (S (S Z)) Leaf) = (S (S Z));
sum (Node (Node Leaf (S Z) (Node Leaf (S Z) Leaf)) (S (S
     Z)) Leaf) = (S (S (S (S Z))));
```

A.2.7 *Misc*

1. compose:

```
language GradedBase

comp : ∀ {k : Coeffect, n m : k, a b c : Type}
     . (a %m → b) %n
     → (b %n → c) %(1 : k)
     → a %(n * m)
     → c
comp x y z = y (x z)
```

2. copy:

```
language GradedBase

copy : ∀ a . a %2 → (a, a)
copy x = (x, x)
```

3. push:

```
language GradedBase
```

```
push : ∀ {a b : Type, k : Coeffect, c : k}
     . (a → b) %c
     → a %c
     → b [c]
push x y = [x y]
```

B
PROOFS

B.1 PROOFS FOR THE LINEAR-BASE CALCULI

This section gives the proofs of Lemma 3.3.1 and Lemma 3.3.2, along with soundness results for the additive pruning variant.

We first state and prove some intermediate results about context manipulations which are needed for the main lemmas.

Definition B.1.1 (Context approximation)**.** For contexts Γ_1, Γ_2 then:

$$\frac{}{\emptyset \sqsubseteq \emptyset} \qquad \frac{\Gamma_1 \sqsubseteq \Gamma_2}{\Gamma_1, x : A \sqsubseteq \Gamma_2, x : A}$$

$$\frac{\Gamma_1 \sqsubseteq \Gamma_2 \qquad r \sqsubseteq s}{\Gamma_1, x :_r A \sqsubseteq \Gamma_2, x :_s A} \qquad \frac{\Gamma_1 \sqsubseteq \Gamma_2 \qquad 0 \sqsubseteq s}{\Gamma_1 \sqsubseteq \Gamma_2, x :_s A}$$

This is actioned in type checking by iterative application of Approx.

Lemma B.1.1 $(\Gamma + (\Gamma' - \Gamma'') \sqsubseteq (\Gamma + \Gamma') - \Gamma'')$**.**

Proof. Induction over the structure of both Γ' and Γ''. The possible forms of Γ' and Γ'' are considered in turn:

1. $\Gamma' = \emptyset$ and $\Gamma'' = \emptyset$

   We have:

   $$(\Gamma + \emptyset) - \emptyset = \Gamma + (\emptyset - \emptyset)$$

   From definitions 2.3.1 and 3.3.1, we know that on the left hand side:

   $$\begin{aligned}(\Gamma + \emptyset) - \emptyset &= \Gamma + \emptyset \\ &= \Gamma\end{aligned}$$

   and on the right-hand side:

   $$\begin{aligned}\Gamma + (\emptyset - \emptyset) &= \Gamma + \emptyset \\ &= \Gamma\end{aligned}$$

   making both the left and right hand sides equivalent:

   $$\Gamma = \Gamma$$

2. $\Gamma' = \Gamma', x : A$ and $\Gamma'' = \emptyset$
   We have

   $$(\Gamma + \Gamma', x : A) - \emptyset = \Gamma + (\Gamma, x : A - \emptyset)$$

   From definitions 2.3.1 and 3.3.1, we know that on the left hand side we have:

   $$\begin{aligned}(\Gamma + \Gamma', x : A) - \emptyset &= (\Gamma, \Gamma'), x : A - \emptyset \\ &= (\Gamma, \Gamma'), x : A\end{aligned}$$

   and on the right hand side:

   $$\begin{aligned}\Gamma + (\Gamma, x : A - \emptyset) &= \Gamma + \Gamma', x : A \\ &= (\Gamma, \Gamma', x : A)\end{aligned}$$

   making both the left and right hand sides equal:

   $$(\Gamma, \Gamma'), x : A = (\Gamma, \Gamma'), x : A$$

3. $\Gamma' = \Gamma', x : A$ and $\Gamma'' = \Gamma'', x : A$
   We have

   $$(\Gamma + \Gamma', x : A) - \Gamma'', x : A = \Gamma + (\Gamma', x : A - \Gamma'', x : A)$$

   From definitions 2.3.1 and 3.3.1, we know that on the left hand side we have:

   $$\begin{aligned}(\Gamma + \Gamma', x : A) - \Gamma'', x : A &= (\Gamma, \Gamma'), x : A - \Gamma'', x : A \\ &= \Gamma, \Gamma' - \Gamma''\end{aligned}$$

   and on the right hand side:

   $$\begin{aligned}\Gamma + (\Gamma', x : A - \Gamma'', x : A) &= \Gamma + (\Gamma' - \Gamma'') \\ &= \Gamma, \Gamma' - \Gamma''\end{aligned}$$

   making both the left and right hand sides equivalent:

   $$\Gamma, \Gamma' - \Gamma'' = \Gamma, \Gamma' - \Gamma''$$

4. $\Gamma' = \Gamma', x :_r A$ and $\Gamma'' = \emptyset$
   We have

   $$(\Gamma + \Gamma', x :_r A) - \emptyset = \Gamma + (x :_r A - \emptyset)$$

   From definitions 2.3.1 and 3.3.1, we know that on the left hand side we have:

   $$\begin{aligned}(\Gamma + \Gamma', x :_r A) - \emptyset &= (\Gamma + \Gamma', x :_r A) \\ &= (\Gamma, \Gamma'), x :_r A\end{aligned}$$

and on the right hand side:

$$\Gamma + (\Gamma', x :_r A - \emptyset) = \Gamma + (\Gamma', x :_r A) \qquad = (\Gamma, \Gamma'), x :_r A$$

making both the left and right hand sides equivalent:

$$(\Gamma, \Gamma'), x :_r A = (\Gamma, \Gamma'), x :_r A$$

5. $\Gamma' = \Gamma', x :_r A$ and $\Gamma'' = \Gamma'', x :_s A$

   Thus we have (for the LHS of the inequality term):

   $$\Gamma + (\Gamma', x :_r A - \Gamma'', x :_s A)$$

   which by context subtraction yields:

   $$\Gamma + (\Gamma', x :_r A - \Gamma'', x :_s A) = \Gamma + (\Gamma' - \Gamma''), x :_{q'} A$$

   where:

   $$\exists q'.r \sqsupseteq q' + s \quad \forall \hat{q}'.r \sqsupseteq \hat{q}' + s \implies q' \sqsupseteq \hat{q}' \qquad (2)$$

   And for the LHS of the inequality, from definitions 2.3.1 and 3.3.1 we have:

   $$\begin{aligned}(\Gamma + \Gamma', x :_r A) - \Gamma'', x :_s A &= (\Gamma + \Gamma'), x :_r A - \Gamma'', x :_s A \\ &= ((\Gamma + \Gamma') - \Gamma''), x :_r A - x :_s A \\ &= ((\Gamma + \Gamma') - \Gamma''), x :_q A\end{aligned}$$

   where:

   $$\exists q.r \sqsupseteq q + s \quad \forall \hat{q}.r \sqsupseteq \hat{q} + s \implies q \sqsupseteq \hat{q} \qquad (1)$$

   Applying $\exists q.r \sqsupseteq q + s$ to maximality (2) (at $\hat{q}' = q$) then yields that $q \sqsubseteq q'$.

   Therefore, applying induction, we derive:

   $$\frac{(\Gamma + (\Gamma' - \Gamma'')) \sqsubseteq ((\Gamma + \Gamma') - \Gamma'') \qquad q \sqsubseteq q'}{(\Gamma + (\Gamma' - \Gamma'')), x :_q A \sqsubseteq ((\Gamma + \Gamma') - \Gamma''), x :_{q'} A}$$

   satisfying the lemma statement.

□

Lemma B.1.2 $((\Gamma - \Gamma') + \Gamma' \sqsubseteq \Gamma)$**.**

Proof. The proof follows by induction over the structure of Γ'. The possible forms of Γ' are considered in turn:

1. $\Gamma' = \emptyset$
   We have:
   $$(\Gamma - \emptyset) + \emptyset = \Gamma$$
   From definition 3.3.1, we know that:
   $$\Gamma - \emptyset = \Gamma$$
   and from definition 2.3.1, we know:
   $$\Gamma + \emptyset = \Gamma$$
   giving us:
   $$\Gamma = \Gamma$$

2. $\Gamma' = \Gamma'', x : A$
   and let $\Gamma = \Gamma', x : A$.
   $$(\Gamma', x : A - \Gamma'', x : A) + \Gamma'', x : A = \Gamma$$
   From definition 2.3.1, we know that:
   $$\begin{aligned}(\Gamma', x : A - \Gamma'', x : A) + \Gamma'', x : A &= ((\Gamma' - \Gamma'') + \Gamma''), x : A \\ \mathit{induction} &= \Gamma', x : A \\ &= \Gamma\end{aligned}$$
   thus satisfying the lemma statement by equality.

3. $\Gamma' = \Gamma'', x :_r A$
   and let $\Gamma = \Gamma', x :_s A$.
   We have:
   $$(\Gamma', x :_s A - \Gamma'', x :_r A) + \Gamma'', x :_r A$$
   From definition 3.3.1, we know that:
   $$\begin{aligned}&(\Gamma', x :_s A - \Gamma'', x :_r A) + \Gamma'', x :_r A \\ = &\ (\Gamma' - \Gamma''), x :_q A + \Gamma'', x :_r A \\ = &\ ((\Gamma' - \Gamma'') + \Gamma''), x :_{q+r} A\end{aligned}$$
   where $s \sqsupseteq q + r$ and $\forall q'.s \sqsupseteq q' + r \implies q \sqsupseteq q'$.
   Then by induction we derive the ordering:
   $$\frac{((\Gamma' - \Gamma'') + \Gamma'') \sqsubseteq \Gamma' \qquad q + r \sqsubseteq s}{((\Gamma' - \Gamma'') + \Gamma''), x :_{q+r} A \sqsubseteq \Gamma', x :_s A}$$
   which satifies the lemma statement.

□

Lemma B.1.3 (Context negation). *For all contexts* Γ*:*

$$\emptyset \sqsubseteq \Gamma - \Gamma$$

Proof. By induction on the structure of Γ:

- $\Gamma = \emptyset$ Trivial.
- $\Gamma = \Gamma', x : A$ then $(\Gamma', x : A) - (\Gamma', x : A) = \Gamma' - \Gamma'$ so proceed by induction.
- $\Gamma = \Gamma', x :_r A$ then $\exists q.\ (\Gamma', x :_r A) - (\Gamma', x :_r A) = (\Gamma - \Gamma'), x :_q A$ such that $r \sqsupseteq q + r$ and $\forall q'.r \sqsupseteq q' + r \implies q \sqsupseteq q'$.

  Instantiating maximality with $q' = 0$ and reflexivity then we have $0 \sqsubseteq q$. From this, and the inductive hypothesis, we can construct:

$$\frac{\emptyset \sqsubseteq (\Gamma - \Gamma') \quad 0 \sqsubseteq q}{\emptyset \sqsubseteq (\Gamma - \Gamma'), x :_q A}$$

□

Lemma B.1.4. *For all contexts* Γ_1*,* Γ_2*, where* $[\Gamma_2]$ *(i.e.,* Γ_2 *is all graded) then:*

$$\Gamma_2 \sqsubseteq \Gamma_1 - (\Gamma_1 - \Gamma_2)$$

Proof. By induction on the structure of Γ_2.

- $\Gamma_2 = \sqsubseteq$

  Then $\Gamma_1 - (\Gamma_1 - \emptyset) = \Gamma_1 - \Gamma_1$.

  By Lemma B.1.3, then $\emptyset \sqsubseteq (\Gamma_1 - \Gamma_1)$ satisfying this case.
- $\Gamma_2 = \Gamma_2', x :_s A$

  By the premises $\Gamma_1 \sqsubseteq \Gamma_2$ then we can assume $x \in \Gamma_1$ and thus (by context rearrangement) $\Gamma_1', x :_r A$.

  Thus we consider $(\Gamma_1', x :_r A) - ((\Gamma_1', x :_r A) - (\Gamma_2', x :_s A))$.

$$\begin{aligned} &(\Gamma_1', x :_r A) - ((\Gamma_1', x :_r A) - (\Gamma_2', x :_s A)) \\ =\ &(\Gamma_1', x :_r A) - ((\Gamma_1' - \Gamma_2'), x :_q A) \\ =\ &(\Gamma_1' - (\Gamma_1' - \Gamma_2')), x :_{q'} A \end{aligned}$$

  where (1) $\exists q. r \sqsupseteq q + s$ with (2) $(\forall \hat{q}.r \sqsupseteq \hat{q} + s \implies q \sqsupseteq \hat{q})$ and (3) $\exists q'. r \sqsupseteq q' + q$ with (4) $(\forall \hat{q'}.r \sqsupseteq \hat{q'} + s \implies q' \sqsupseteq \hat{q'})$.

  Apply (1) to (4) by letting $\hat{q'} = s$ and by commutativity of $+$ then we get that $q' \sqsupseteq s$.

By induction we have that

$$\Gamma_1' \sqsubseteq \Gamma_1' - (\Gamma_1' - \Gamma_2') \tag{ih}$$

Thus we get that:

$$\frac{s \sqsubseteq q' \quad \Gamma_1' \sqsubseteq \Gamma_1' - (\Gamma_1' - \Gamma_2')}{\Gamma_1', x :_s A \sqsubseteq (\Gamma_1' - (\Gamma_1' - \Gamma_2')), x :_{q'} A}$$

- $\Gamma_2 = \Gamma_2', x : A$ Trivial as it violates the grading condition of the premise.

□

B.1.1 *Soundness of the Subtractive Graded Linear Typing Calculus*

Lemma 3.3.1 (Subtractive synthesis soundness). *For all Γ and A then:*

$$\Gamma \vdash A \Rightarrow^{-} t \mid \Delta \quad \implies \quad \Gamma - \Delta \vdash t : A$$

i.e. t has type A under context $\Gamma - \Delta$, that contains just those linear and graded variables with grades reflecting their use in t.

Proof. Structural induction over the synthesis rules. Each of the possible synthesis rules are considered in turn.

1. Case LinVar^{-}
   In the case of linear variable synthesis, we have the derivation:

   $$\frac{}{\Gamma, x : A \vdash A \Rightarrow^{-} x \mid \Gamma}\ \textsc{LinVar}^{-}$$

   By the definition of context subtraction, $(\Gamma, x : A) - \Gamma = x : A$, thus we can construct the following typing derivation, matching the conclusion:

   $$\frac{}{x : A \vdash x : A}\ \textsc{Var}$$

2. Case GrVar^{-}
   Matching the form of the lemma, we have the derivation:

   $$\frac{\exists s. r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash A \Rightarrow^{-} x \mid \Gamma, x :_s A}\ \textsc{GrVar}^{-}$$

   By the definition of context subtraction, $(\Gamma, x :_r A) - (\Gamma, x :_s A) = x :_q A$ where (1) $\exists q. r \sqsupseteq q + s$ and $\forall q'.r \sqsupseteq q' + s \implies q \sqsupseteq q'$.

   Applying maximality (1) with $q = 1$ then we have that $1 \sqsubseteq q$ (*)

Thus, from this we can construct the typing derivation, matching the conclusion:

$$\dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}\,\textsc{Var}}{x :_1 A \vdash x : A} \,\textsc{Der} \qquad 1 \sqsubseteq q \; (*)}{x :_q A \vdash x : A}\,\textsc{Approx}$$

3. Case $\multimap^-_R$
   We thus have the derivation:

$$\dfrac{\Gamma, x : A \vdash B \Rightarrow^- t \mid \Delta \qquad x \notin |\Delta|}{\Gamma \vdash A \multimap B \Rightarrow^- \lambda x.t \mid \Delta}\,\multimap^-_R$$

   By induction we then have that:

$$(\Gamma, x : A) - \Delta \vdash t : B$$

   Since $x \notin |\Delta|$ then by the definition of context subtraction we have that $(\Gamma, x : A) - \Delta = (\Gamma - \Delta), x : A$. From this, we can construct the following derivation, matching the conclusion:

$$\dfrac{(\Gamma - \Delta), x : A \vdash t : B}{\Gamma - \Delta \vdash \lambda x.t : A \multimap B}\,\textsc{Abs}$$

4. Case $\multimap^-_L$
   Matching the form of the lemma, the application derivation is:

$$\dfrac{\Gamma, x_2 : B \vdash C \Rightarrow^- t_1 \mid \Delta_1 \qquad x_2 \notin |\Delta_1| \qquad \Delta_1 \vdash A \Rightarrow^- t_2 \mid \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^- [(x_1\, t_2)/x_2]t_1 \mid \Delta_2}\,\multimap^-_L$$

   By induction, we have that:

$$(\Gamma, x_2 : B) - \Delta_1 \vdash t_1 : C \qquad \text{(ih1)}$$
$$\Delta_1 - \Delta_2 \vdash t_2 : A \qquad \text{(ih2)}$$

   By the definition of context subtraction and since $x_2 \notin |\Delta_1|$ then (ih1) is equal to:

$$(\Gamma - \Delta_1), x_2 : B \vdash t_1 : C \qquad \text{(ih1}'\text{)}$$

   We can thus construct the following typing derivation, making use of of the admissibility of linear substitution (Lemma 5.1.1):

$$\dfrac{\dfrac{(\Gamma - \Delta_1), x_2 : B \multimap C \vdash t_1 : C}{\Gamma - \Delta_1 \vdash \lambda x_2.t_1 : B \multimap C}\,\textbf{ABS} \qquad \dfrac{\dfrac{}{x_1 : A \multimap B \vdash x_1 : A \multimap B}\,\textbf{VAR} \qquad \Delta_1 - \Delta_2 \vdash t_2 : A}{(\Delta_1 - \Delta_2), x_1 : A \multimap B \vdash x_1\, t_2 : B}\,\textbf{APP}}{(\Gamma - \Delta_1) + (\Delta_1 - \Delta_2), x_1 : A \multimap B \vdash [(x_1\, t_2)/x_2]t_1 : C}\,\textbf{APP}$$

From Lemma B.1.1, we have that

$$((\Gamma - \Delta_1) + (\Delta_1 - \Delta_2)), x_1 : A \multimap B \sqsubseteq (((\Gamma - \Delta_1) + \Delta_1) - \Delta_2), x_1 : A \multimap B$$

and from Lemma B.1.2, that:

$$(((\Gamma - \Delta_1) + \Delta_1) - \Delta_2), x_1 : A \multimap B \sqsubseteq (\Gamma - \Delta_2), x_1 : A \multimap B$$

which, since x_1 is not in Δ_2 (as x_1 is not in Γ) $(\Gamma - \Delta_2), x_1 : A \multimap B = (\Gamma, x_1 : A \multimap B) - \Delta_2$. Applying these inequalities with Approx then yields the lemma's conclusion $(\Gamma, x_1 : A \multimap B) - \Delta_2 \vdash [(x_1\, t_2)/x_2]t_1 : C$.

5. Case $\Box^-_R$
The synthesis rule for boxing can be constructed as:

$$\frac{\Gamma \vdash A \Rightarrow^- t \mid \Delta}{\Gamma \vdash \Box_r A \Rightarrow^- [t] \mid \Gamma - r \cdot (\Gamma - \Delta)}\ \Box^-_R$$

By induction on the premise we get:

$$\Gamma - \Delta \vdash t : A$$

Since we apply scalar multipication ih the conclusion of the rule to $\Gamma - \Delta$ then we know that all of $\Gamma - \Delta$ must be graded assumptions.

From this, we can construct the typing derivation:

$$\frac{[\Gamma - \Delta] \vdash t : A}{r \cdot [\Gamma - \Delta] \vdash [t] : \Box_r A}\ \textsc{Pr}$$

Via Lemma B.1.4, we then have that $(r \cdot \Gamma - \Delta) \sqsubseteq (\Gamma - (\Gamma - (r \cdot (\Gamma - \Delta))))$ thus, we can derived:

$$\frac{\dfrac{[\Gamma - \Delta] \vdash t : A}{r \cdot [\Gamma - \Delta] \vdash [t] : \Box_r A}\ \textsc{Pr} \quad \text{Lem. B.1.4}}{\Gamma - (\Gamma - (r \cdot (\Gamma - \Delta))) \vdash [t] : \Box_r A}\ \textsc{Approx}$$

Satisfying the goal of the lemma.

6. Case $\Box^-_L$
The synthesis rule for unboxing has the form:

$$\frac{\Gamma, x_2 :_r A \vdash B \Rightarrow^- t \mid \Delta, x_2 :_s A \qquad 0 \sqsubseteq s}{\Gamma, x_1 : \Box_r A \vdash B \Rightarrow^- \textbf{let}\, [x_2] = x_1\, \textbf{in}\, t \mid \Delta}\ \Box^-_L$$

By induction on the premise we have that:

$$(\Gamma, x_2 :_r A) - (\Delta, x_2 :_s A) \vdash t : B$$

By the definition of context subtraction we get that $\exists q$ and:

$$(\Gamma, x_2 :_r A) - (\Delta, x_2 :_s A) = (\Gamma - \Delta), x_2 :_q A$$

such that $r = q + s$

We also have that $0 \sqsubseteq s$.

By monotonicity with $q \sqsubseteq q$ (reflexivity) and $0 \sqsubseteq s$ then $q \sqsubseteq q + s$.

By context subtraction we have $r = q + s$ therefore $q \sqsubseteq r$ (*).

From this, we can construct the typing derivation:

$$\dfrac{\dfrac{\dfrac{}{x_1 : \Box_r A \vdash x_1 : \Box_r A}\textsc{Var} \qquad \dfrac{(\Gamma - \Delta), x_2 :_q A \vdash t : B \quad (*)}{(\Gamma - \Delta), x_2 :_r A \vdash t : B}\textsc{approx}}{}}{(\Gamma - \Delta), x_1 : \Box_r A \vdash \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t : B}\textsc{Let}$$

Which matches the goal.

7. Case $\otimes_R^-$

The synthesis rule for pair introduction has the form:

$$\dfrac{\Gamma \vdash A \Rightarrow^- t_1 \mid \Delta_1 \qquad \Delta_1 \vdash B \Rightarrow^- t_2 \mid \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^- (t_1, t_2) \mid \Delta_2}\otimes_R^-$$

By induction we get:

$$\Gamma - \Delta_1 \vdash t_1 : A \qquad \text{(ih1)}$$
$$\Delta_1 - \Delta_2 \vdash t_2 : B \qquad \text{(ih2)}$$

From this, we can construct the typing derivation:

$$\dfrac{\Gamma - \Delta_1 \vdash t_1 : A \qquad \Delta_1 - \Delta_2 \vdash t_2 : B}{(\Gamma - \Delta_1) + (\Delta_1 - \Delta_2) \vdash (t_1, t_2) : A \otimes B}\textsc{Pair}$$

From Lemma B.1.1, we have that:

$$(\Gamma - \Delta_1) + (\Delta_1 - \Delta_2) \sqsubseteq ((\Gamma - \Delta_1) + \Delta_1) - \Delta_2$$

and from Lemma B.1.2, that:

$$((\Gamma - \Delta_1) + \Delta_1) - \Delta_2 \sqsubseteq \Gamma - \Delta_2$$

From which we then apply Approx to the above derivation, yielding the goal $\Gamma - \Delta_2 \vdash (t_1, t_2) : A \otimes B$.

8. Case $\otimes_L^-$
The synthesis rule for pair elimination has the form:

$$\frac{\Gamma, x_1 : A, x_2 : B \vdash C \Rightarrow^- t_2 \mid \Delta \qquad x_1 \notin |\Delta| \qquad x_2 \notin |\Delta|}{\Gamma, x_3 : A \otimes B \vdash C \Rightarrow^- \textbf{let}\,(x_1, x_2) = x_3\,\textbf{in}\,t_2 \mid \Delta}\ \otimes_L^-$$

By induction we get:

$$(\Gamma, x_1 : A, x_2 : B) - \Delta \vdash t_2 : C$$

since $x_1 \notin |\Delta| \wedge x_2 \notin |\Delta|$ then $(\Gamma, x_1 : A, x_2 : B) - \Delta = (\Gamma - \Delta), x_1 : A, x_2 : B$.

From this, we can construct the following typing derivation, matching the conclusion:

$$\frac{\dfrac{}{x_3 : A \otimes B \vdash x_3 : A \otimes B}\ \textsc{Var} \qquad (\Gamma - \Delta), x_1 : A, x_2 : B \vdash t_2 : C}{(\Gamma - \Delta), x_3 : A \otimes B \vdash \textbf{let}\,(x_1, x_2) = x_3\,\textbf{in}\,t_2 : C}\ \textsc{Case}$$

which matches the conclusion since $(\Gamma - \Delta), x_3 : A \otimes B = (\Gamma, x_3 : A \otimes B) - \Delta$ since $x_3 \notin |\Delta|$ by its disjointness from Γ.

9. Case $\oplus1_R^-$and $\oplus2_R^-$
The synthesis rules for sum introduction are straightforward. For $\oplus1_R^-$we have the rule:

$$\frac{\Gamma \vdash A \Rightarrow^- t \mid \Delta}{\Gamma \vdash A \oplus B \Rightarrow^- \textbf{inl}\,t \mid \Delta}\ \oplus1_R^-$$

By induction we have:

$$\Gamma - \Delta \vdash t : A \qquad \text{(ih1)}$$

from which we can construct the typing derivation, matching the conclusion:

$$\frac{\Gamma - \Delta \vdash t : A}{\Gamma - \Delta \vdash \textbf{inl}\,t : A \oplus B}\ \oplus1_R^-$$

Matching the goal. And likewise for $\oplus2_R^-$.

10. Case $\oplus_L^-$The synthesis rule for sum elimination has the form:

$$\frac{\Gamma, x_2 : A \vdash C \Rightarrow^- t_1 \mid \Delta_1 \quad \Gamma, x_3 : B \vdash C \Rightarrow^- t_2 \mid \Delta_2 \quad x_2 \notin |\Delta_1| \quad x_3 \notin |\Delta_2|}{\Gamma, x_1 : A \oplus B \vdash C \Rightarrow^- \textbf{case}\ x_1\ \textbf{of inl}\ x_2 \rightarrow t_1;\ \textbf{inr}\ x_3 \rightarrow t_2 \mid \Delta_1 \sqcap \Delta_2}\ \oplus_L^-$$

By induction:

$$(\Gamma, x_2 : A) - \Delta_1 \vdash t_1 : C \qquad \text{(ih)}$$
$$(\Gamma, x_3 : B) - \Delta_2 \vdash t_2 : C \qquad \text{(ih)}$$

From this we can construct the typing derivation, matching the conclusion:

$$\dfrac{\dfrac{}{x_1 : A \oplus B \vdash t_1 : A \oplus B}\textsc{Var} \qquad (\Gamma - \Delta_1), x_2 : A \vdash t_2 : C \qquad (\Gamma - \Delta_2), x_3 : B \vdash t_3 : C}{(\Gamma, x_1 : A \oplus B) - (\Delta_1 \sqcap \Delta_2) \vdash \textbf{case}\ x_1\ \textbf{of inl}\ x_2 \to t_1;\ \textbf{inr}\ x_3 \to t_2 : C}\textsc{Case}$$

11. Case 1_R^-

$$\dfrac{}{\Gamma \vdash 1 \Rightarrow^- () \mid \Gamma}1_R^-$$

By Lemma B.1.3 we have that $\emptyset \sqsubseteq \Gamma - \Gamma$ then we have:

$$\dfrac{\dfrac{}{\emptyset \vdash () : 1}1}{\Gamma - \Gamma \vdash () : 1}\textsc{Approx}$$

Matching the goal

12. Case 1_L^-

$$\dfrac{\Gamma \vdash C \Rightarrow^- t \mid \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^- \textbf{let}\ () = x\ \textbf{in}\ t \mid \Delta}1_L^-$$

By induction we have:

$$\Gamma - \Delta \vdash t : C \qquad \text{(ih)}$$

Then we make the derivation:

$$\dfrac{\dfrac{}{x : 1 \vdash x : 1}\textsc{Var} \qquad \Gamma - \Delta \vdash t : C}{(\Gamma - \Delta), x : 1 \vdash \textbf{let}\ () = x\ \textbf{in}\ t : C}\textsc{Let1}$$

where the context is equal to $(\Gamma, x : 1) - \Delta$.

13. Case $\textsc{der}^-$

$$\dfrac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^- t \mid \Delta, x :_{s'} A \qquad y \notin |\Delta| \qquad \exists s.\, r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash B \Rightarrow^- [x/y]t \mid \Delta, x :_{s'} A}\textsc{der}^-$$

By induction:

$$(\Gamma, x :_s A, y : A) - (\Delta, x :_{s'} A) \vdash t : B \qquad \text{(ih)}$$

By the definition of context subtraction we have (since also $y \notin |\Delta|$)

$$\begin{aligned}&(\Gamma, x :_s A, y : A) - (\Delta, x :_{s'} A)\\ &= (\Gamma - \Delta), x :_q A, y : A\end{aligned}$$

where $\exists q. s \sqsupseteq q + s'$ (1) and $\forall \hat{q}. s \sqsupseteq \hat{q} + s' \implies q \sqsupseteq \hat{q}$ (2)

The goal context is computed by:

$$\begin{aligned}&(\Gamma, x :_r A) - (\Delta, x :_{s'} A)\\ &= (\Gamma - \Delta), x :_{q'} A\end{aligned}$$

where $r \sqsupseteq q' + s'$ (3) and $\forall \hat{q'}. r \sqsupseteq \hat{q'} + s' \implies q' \sqsupseteq \hat{q'}$ (4)

From the premise of DER$^-$we have $r \sqsupseteq (s + 1)$.

$$\begin{aligned}
\text{congruence of + and (1)} &\implies s + 1 \sqsupseteq q + s' + 1 && (5)\\
\text{transitivity with DER}^-\text{premise and (5)} &\implies r \sqsupseteq q + s' + 1 && (6)\\
\text{+ assoc./comm. on (6)} &\implies r \sqsupseteq q + 1 + s' && (7)\\
\text{apply (8) to (4) with } \hat{q'} = q + 1 &\implies q' \sqsupseteq q + 1 && (8)
\end{aligned}$$

Using this last result we derive:

$$\dfrac{\dfrac{\dfrac{(\Gamma - \Delta), x :_q A, y : A \vdash t : B}{(\Gamma - \Delta), x :_q A, y :_1 A \vdash t : B}\,\text{Der}}{(\Gamma - \Delta), x :_{q+1} A \vdash [x/y]t : B}\,\text{CONTRACTION} \quad (8)}{(\Gamma - \Delta), x :_{q'} A \vdash [x/y]t : B}\,\text{APPROX}$$

Which matches the goal.

□

B.1.2 *Soundness of the Additive Graded Linear Typing Calculus*

Lemma 3.3.2 (Additive synthesis soundness). *For all* Γ *and* A*:*

$$\Gamma \vdash A \Rightarrow^+ t; \Delta \quad \implies \quad \Delta \vdash t : A$$

Proof. 1. Case LinVar$^+$

In the case of linear variable synthesis, we have the derivation:

$$\dfrac{}{\Gamma, x : A \vdash A \Rightarrow^+ x; x : A}\,\text{LinVar}^+$$

Therefore we can construct the following typing derivation, matching the conclusion:

$$\dfrac{}{x : A \vdash x : A}\,\text{VAR}$$

2. Case $\textsc{GrVar}^+$
   Matching the form of the lemma, we have the derivation:

$$\dfrac{}{\Gamma, x :_r A \vdash A \Rightarrow^+ x;\ x :_1 A}\ \textsc{GrVar}^+$$

   From this we can construct the typing derivation, matching the conclusion:

$$\dfrac{\dfrac{}{x : A \vdash x : A}\ \textsc{Var}}{x :_1 A \vdash x : A}\ \textsc{Der}$$

3. Case $\multimap_R^+$
   We thus have the derivation:

$$\dfrac{\Gamma, x : A \vdash B \Rightarrow^+ t;\ \Delta, x : A}{\Gamma \vdash A \multimap B \Rightarrow^+ \lambda x.t;\ \Delta}\ \multimap_R^+$$

   By induction on the premise we then have:

$$\Delta, x : A \vdash t : B$$

   From this, we can construct the typing derivation, matching the conclusion:

$$\dfrac{\Delta, x : A \vdash t : B}{\Delta \vdash \lambda x.t : A \multimap B}\ \textsc{abs}$$

4. Case $\multimap_L^+$
   Matching the form of the lemma, the application derivation can be constructed as:

$$\dfrac{\Gamma, x_2 : B \vdash C \Rightarrow^+ t_1;\ \Delta_1, x_2 : B \qquad \Gamma \vdash A \Rightarrow^+ t_2;\ \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^+ [(x_1\, t_2)/x_2]t_1;\ (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \multimap_L^+$$

   By induction on the premises we then have the following typing judgments:

$$\begin{aligned} \Delta_1, x_2 : B &\vdash t_1 : C \\ \Delta_2 &\vdash t_2 : A \end{aligned}$$

   We can thus construct the following typing derivation, making use of the admissibility of linear substitution (Lemma 5.1.1):

$$\dfrac{\dfrac{\dfrac{}{x_1 : A \multimap B \vdash x_1 : A \multimap B}\ \textsc{var} \qquad \Delta_2 \vdash t_2 : A}{\Delta_2, x_1 : A \multimap B \vdash x_1\, t_2 : B}\ \textsc{app} \qquad \Delta_1, x_2 : B \vdash t_1 : C}{(\Delta_1 + \Delta_2), x_1 : A \multimap B \vdash [(x_1\, t_2)/x_2]t_1 : C}\ \text{(L. 5.1.1)}$$

5. Case $\Box_R^+$
The synthesis rule for boxing can be constructed as:

$$\frac{\Gamma \vdash A \Rightarrow^+ t;\, \Delta}{\Gamma \vdash \Box_r A \Rightarrow^+ [t];\, r \cdot \Delta}\ \Box_R^+$$

By induction we then have:

$$\Delta \vdash t : A$$

In the conclusion of the above derivation we know that $r \cdot \Delta$ is defined, therefore it must be that all of Δ are graded assumptions, i.e., we have that $[\Delta]$ holds. We can thus construct the following typing derivation, matching the conclusion:

$$\frac{[\Delta] \vdash t : A}{r \cdot [\Delta] \vdash [t] : \Box_r A}\ \textsc{Pr}$$

6. Case $\textsc{der}^+$
From the dereliction rule we have:

$$\frac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^+ t;\, \Delta, y : A}{\Gamma, x :_s A \vdash B \Rightarrow^+ [x/y]t;\, \Delta + x :_1 A}\ \textsc{der}^+$$

By induction we get:

$$\Delta, y : A \vdash t : B \qquad \text{(ih)}$$

Case on $x \in \Delta$

- $x \in \Delta$, i.e., $\Delta = \Delta', x :_{s'} A$.

  Then by admissibility of contraction we can derive:

$$\frac{\dfrac{\Delta', x :_{s'} A, y : A \vdash t : B}{\Delta', x :_{s'} A, y :_1 A \vdash t : B}\ \textsc{Der}}{(\Delta', x :_{s'} A) + x :_1 A \vdash [x/y]t : B}$$

  Satisfying the lemma statment.

- $x \notin \Delta$. Then again from the admissiblity of contraction, we derive the typing:

$$\frac{\dfrac{\Delta, y : A \vdash t : B}{\Delta, y :_1 A \vdash t : B}\ \textsc{Der}}{\Delta + x :_1 A \vdash [x/y]t : B}$$

  which is well defined as $x \notin \Delta$ and gives the lemma conclusion.

7. Case $\Box_L^+$
The synthesis rule for unboxing has the form:

$$\frac{\Gamma, x_2 :_r A \vdash B \Rightarrow^+ t;\ \Delta \qquad \textit{if } x_2 :_s A \in \Delta \textit{ then } s \sqsubseteq r \textit{ else } 0 \sqsubseteq r}{\Gamma, x_1 : \Box_r A \vdash B \Rightarrow^+ \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t;\ (\Delta \backslash x_2), x_1 : \Box_r A}\ \Box_L^+$$

By induction we have that:

$$\Delta \vdash t : B \tag{ih}$$

Case on $x_2 :_s A \in \Delta$

- $x_2 :_s A \in \Delta$, i.e., $s \sqsubseteq r$.
From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\dfrac{}{x_1 : \Box_r A \vdash x_1 : \Box_r A}\ \textsc{var} \qquad \Delta, x_2 :_r A \vdash t : B}{\Delta, x_1 : \Box_r A \vdash \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t : B}\ \textsc{let}\Box$$

- $x_2 :_s A \notin \Delta$, i.e., $0 \sqsubseteq r$.
From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\begin{array}{c}\dfrac{}{x_1 : \Box_r A \vdash x_1 : \Box_r A}\ \textsc{var} \\ \dfrac{\dfrac{\Delta \vdash t : B}{\Delta, x_2 :_0 A \vdash t : B}\ \textsc{Weak} \qquad 0 \sqsubseteq r}{\Delta, x_2 :_r A \vdash t : B}\ \textsc{Approx}\end{array}}{\Delta, x_1 : \Box_r A \vdash \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t : B}\ \textsc{let}\Box$$

8. Case $\otimes_R^+$

The synthesis rule for pair introduction has the form:

$$\frac{\Gamma \vdash A \Rightarrow^+ t_1;\ \Delta_1 \qquad \Gamma \vdash B \Rightarrow^+ t_2;\ \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^+ (t_1, t_2);\ \Delta_1 + \Delta_2}\ \otimes_R^+$$

By induction on the premises we have that:

$$\Delta_1 \vdash t_1 : A \tag{ih1}$$
$$\Delta_2 \vdash t_2 : B \tag{ih2}$$

From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\Delta_1 \vdash t_1 : A \qquad \Delta_2 \vdash t_2 : B}{\Delta_1 + \Delta_2 \vdash (t_1, t_2) : A \otimes B}\ \textsc{pair}$$

9. Case $\otimes_L^+$
The synthesis rule for pair elimination has the form:

$$\frac{\Gamma, x_1 : A, x_2 : B \vdash C \Rightarrow^+ t_2;\ \Delta, x_1 : A, x_2 : B}{\Gamma, x_3 : A \otimes B \vdash C \Rightarrow^+ \mathbf{let}\,(x_1, x_2) = x_3\,\mathbf{in}\,t_2;\ \Delta, x_3 : A \otimes B}\ \otimes_L^+$$

By induction on the premises we have that:

$$\Delta_1 \vdash t_1 : A \tag{ih1}$$
$$\Delta_2 \vdash t_2 : B \tag{ih2}$$

From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\dfrac{}{x_3 : A \otimes B \vdash x_3 : A \otimes B}\ \textsc{Var} \qquad \Delta, x_1 : A, x_2 : B \vdash t_2 : C}{\Delta, x_3 : A \otimes B \vdash \mathbf{let}\,(x_1, x_2) = x_3\,\mathbf{in}\,t_2 : C}\ \textsc{LetPair}$$

10. Case $\oplus 1_R^+$and $\oplus 2_R^+$
The synthesis rules for sum introduction are straightforward. For $\oplus 1_R^+$we have the rule:

$$\frac{\Gamma \vdash A \Rightarrow^+ t;\ \Delta}{\Gamma \vdash A \oplus B \Rightarrow^+ \mathbf{inl}\,t;\ \Delta}\ \oplus 1_R^+$$

By induction on the premises we have that:

$$\Delta \vdash t : A \tag{ih}$$

From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\Delta \vdash t : A}{\Delta \vdash \mathbf{inl}\,t : A \oplus B}\ \textsc{Inl}$$

Likewise, for the $\oplus 2_R^+$we have the synthesis rule:

$$\frac{\Gamma \vdash B \Rightarrow^+ t;\ \Delta}{\Gamma \vdash A \oplus B \Rightarrow^+ \mathbf{inr}\,t;\ \Delta}\ \oplus 2_R^+$$

By induction on the premises we have that:

$$\Delta \vdash t : B \tag{ih}$$

From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\Delta \vdash t : B}{\Delta \vdash \mathbf{inl}\,t : A \oplus B}\ \textsc{Inr}$$

11. Case $\oplus_L^+$
The synthesis rule for sum elimination has the form:

$$\frac{\Gamma, x_2 : A \vdash C \Rightarrow^+ t_1;\, \Delta_1, x_2 : A \qquad \Gamma, x_3 : B \vdash C \Rightarrow^+ t_2;\, \Delta_2, x_3 : B}{\Gamma, x_1 : A \oplus B \vdash C \Rightarrow^- \textbf{case}\ x_1\ \textbf{of inl}\ x_2 \to t_1;\ \textbf{inr}\ x_3 \to t_2 \mid \Delta_1 \sqcup \Delta_2, x_1 : A \oplus B}\ \oplus_L^+$$

By induction on the premises we have that:

$$\Delta_1, x_2 : A \vdash t_1 : C \qquad \text{(ih1)}$$
$$\Delta_2, x_3 : B \vdash t_2 : C \qquad \text{(ih2)}$$

From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\dfrac{}{x_1 : A \oplus B \vdash x_1 : A \oplus B}\ \textsc{Var} \qquad \Delta_1, x_2 : A \vdash t_1 : C \qquad \Delta_2, x_3 : B \vdash t_2 : C}{(\Delta_1 \sqcup \Delta_2), x_1 : A \oplus B \vdash \textbf{case}\ x_1\ \textbf{of inl}\ x_2 \to t_1;\ \textbf{inr}\ x_3 \to t_2 : C}\ \textsc{Case}$$

12. Case 1_R^+
The synthesis rule for unit introduction has the form:

$$\frac{}{\Gamma \vdash 1 \Rightarrow^+ ();\, \emptyset}\ 1_R^+$$

From this, we can construct the typing derivation, matching the conclusion:

$$\frac{}{\emptyset \vdash () : 1}\ 1$$

13. Case 1_L^+
The synthesis rule for unit elimination has the form:

$$\frac{\Gamma \vdash C \Rightarrow^+ t;\, \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^+ \textbf{let}\,() = x\,\textbf{in}\,t;\, \Delta, x : 1}\ 1_L^+$$

By induction on the premises we have that:

$$\Delta \vdash t : C \qquad \text{(ih)}$$

From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\dfrac{}{x : 1 \vdash x : 1}\ \textsc{Var} \qquad \Delta \vdash t : C}{\Delta, x : 1 \vdash \textbf{let}\,() = x\,\textbf{in}\,t : C}\ \textsc{Let1}$$

□

B.1.3 *Soundness of the Additive Pruning Graded Linear Typing Calculus*

Lemma 3.3.3 (Additive pruning synthesis soundness)**.** *For all* Γ *and* A*:*

$$\Gamma \vdash A \Rightarrow^{+} t; \Delta \quad\Longrightarrow\quad \Delta \vdash t : A$$

Proof. The cases for the rules in the additive pruning synthesis calculus are equivalent to lemma (3.3.2), except for the cases of the $\multimap_L'^{+}$ and $\otimes_R'^{+}$ rules which we consider here:

1. Case $\multimap_L'^{+}$
   Matching the form of the lemma, the application derivation can be constructed as:

$$\dfrac{\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1; \Delta_1, x_2 : B \qquad \Gamma - \Delta_1 \vdash A \Rightarrow^{+} t_2; \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^{+} [(x_1\, t_2)/x_2]t_1; (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \multimap_L'^{+}$$

   By induction on the premises we then have the following typing judgments:

$$\begin{aligned} \Delta_1, x_2 : B &\vdash t_1 : C \\ \Delta_2 &\vdash t_2 : A \end{aligned}$$

   We can thus construct the following typing derivation, making use of the admissibility of linear substitution (Lemma 5.1.1):

$$\dfrac{\dfrac{\dfrac{}{x_1 : A \multimap B \vdash x_1 : A \multimap B}\ \textsc{var} \qquad \Delta_2 \vdash t_2 : A}{\Delta_2, x_1 : A \multimap B \vdash x_1\, t_2 : B}\ \textsc{app} \qquad \Delta_1, x_2 : B \vdash t_1 : C}{(\Delta_1 + \Delta_2), x_1 : A \multimap B \vdash [(x_1\, t_2)/x_2]t_1 : C}\ (\text{L. } 5.1.1)$$

2. Case $\otimes_R'^{+}$

   The synthesis rule for the pruning alternative for pair introduction has the form:

$$\dfrac{\Gamma \vdash A \Rightarrow^{+} t_1; \Delta_1 \qquad \Gamma - \Delta_1 \vdash B \Rightarrow^{+} t_2; \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^{+} (t_1, t_2); \Delta_1 + \Delta_2}\ \otimes_R'^{+}$$

   By induction on the premises we have that:

$$\Delta_1 \vdash t_1 : A \qquad \text{(ih1)}$$
$$\Delta_2 \vdash t_2 : B \qquad \text{(ih2)}$$

   From this, we can construct the typing derivation, matching the conclusion:

$$\frac{\Delta_1 \vdash t_1 : A \qquad \Delta_2 \vdash t_2 : B}{\Delta_1 + \Delta_2 \vdash (t_1, t_2) : A \otimes B}\ \textsc{pair}$$

□

B.1.4 *Soundness of Focusing for the Subtractive Linear Graded Synthesis Calculus*

Lemma 3.5.1 (Soundness of focusing for subtractive synthesis). *For all contexts* Γ, Ω *and types* A *then:*

1. *Right Async* :	$\Gamma;\Omega \vdash A \Uparrow \Rightarrow^- t \mid \Delta$	$\Longrightarrow$	$\Gamma,\Omega \vdash A \Rightarrow^- t \mid \Delta$
2. *Left Async* :	$\Gamma;\Omega \Uparrow \vdash C \Rightarrow^- t \mid \Delta$	$\Longrightarrow$	$\Gamma,\Omega \vdash C \Rightarrow^- t \mid \Delta$
3. *Right Sync* :	$\Gamma;\emptyset \vdash A \Downarrow \Rightarrow^- t \mid \Delta$	$\Longrightarrow$	$\Gamma \vdash A \Rightarrow^- t \mid \Delta$
4. *Left Sync* :	$\Gamma; x : A \Downarrow \vdash C \Rightarrow^- t \mid \Delta$	$\Longrightarrow$	$\Gamma, x : A \vdash C \Rightarrow^- t \mid \Delta$
5. *Focus Right* :	$\Gamma;\Omega \Uparrow \vdash C \Rightarrow^- t \mid \Delta$	$\Longrightarrow$	$\Gamma \vdash C \Rightarrow^- t \mid \Delta$
6. *Focus Left* :	$\Gamma, x : A;\Omega \Uparrow \vdash C \Rightarrow^- t \mid \Delta$	$\Longrightarrow$	$\Gamma \vdash C \Rightarrow^- t \mid \Delta$

i.e. t has type A under context Δ*, which contains assumptions with grades reflecting their use in t.*

Proof. 1. Case 1. Right Async:

a) Case $\multimap_R^-$
In the case of the right asynchronous rule for abstraction introduction, the synthesis rule has the form:

$$\frac{\Gamma;\Omega, x : A \vdash B \Uparrow \Rightarrow^- t \mid \Delta \quad x \notin |\Delta|}{\Gamma;\Omega \vdash A \multimap B \Uparrow \Rightarrow^- \lambda x.t \mid \Delta}\ \multimap_R^-$$

By induction on the first premise, we have that:

$$(\Gamma,\Omega), x : A \vdash A \Rightarrow^- t \mid \Delta \qquad \text{(ih)}$$

from case 1 of the lemma. From which, we can construct the following instantiation of the $\multimap_R^-$synthesis rule in the non-focusing calculus:

$$\frac{(\Gamma,\Omega), x : A \vdash B \Rightarrow^- t \mid \Delta \quad x \notin |\Delta|}{\Gamma,\Omega \vdash A \multimap B \Rightarrow^- \lambda x.t \mid \Delta}\ \multimap_R^-$$

b) Case $\Uparrow_R^-$
In the case of the right asynchronous rule for transition to a left asynchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma;\Omega \Uparrow \vdash C \Rightarrow^- t \mid \Delta \qquad C \text{ not right async}}{\Gamma;\Omega \vdash C \Uparrow \Rightarrow^- t \mid \Delta}\ \Uparrow_R^-$$

By induction on the first premise, we have that:

$$\Gamma, \Omega \vdash C \Rightarrow^{-} t \mid \Delta$$

from case 2 of the lemma.

2. Case 2. Left Async:

   a) Case $\otimes_L^-$
   In the case of the left asynchronous rule for pair elimination, the synthesis rule has the form:

$$\frac{\begin{array}{c}\Gamma; \Omega, x_1 : A, x_2 : B \Uparrow \vdash C \Rightarrow^{-} t_2 \mid \Delta \\ x_1 \notin |\Delta| \qquad x_2 \notin |\Delta|\end{array}}{\Gamma; \Omega, x_3 : A \otimes B \Uparrow \vdash C \Rightarrow^{-} \mathbf{let}\,(x_1, x_2) = x_3\,\mathbf{in}\,t_2 \mid \Delta} \otimes_L^-$$

   By induction on the first premise, we have that:

$$(\Gamma, \Omega), x_1 : A, x_2 : B \vdash C \Rightarrow^{-} t \mid \Delta \qquad \text{(ih)}$$

   from From which, we can construct the following instantiation of the $\otimes_R^-$ synthesis rule in the non-focusing calculus:

$$\frac{\begin{array}{c}(\Gamma, \Omega), x_1 : A, x_2 : B \vdash C \Rightarrow^{-} t \mid \Delta \\ x_1 \notin |\Delta| \qquad x_2 \notin |\Delta|\end{array}}{\Gamma, (\Omega, x_3 : A \otimes B) \vdash C \Rightarrow^{-} \mathbf{let}\,(x_1, x_2) = x_3\,\mathbf{in}\,t \mid \Delta_2} \otimes_L^-$$

   b) Case $\oplus_L^-$
   In the case of the left asynchronous rule for sum elimination, the synthesis rule has the form:

$$\frac{\Gamma; \Omega, x_2 : A \Uparrow \vdash C \Rightarrow^{-} t_1 \mid \Delta_1 \quad \Gamma; \Omega, x_3 : B \Uparrow \vdash C \Rightarrow^{-} t_2 \mid \Delta_2 \quad x_2 \notin |\Delta_1| \quad x_3 \notin |\Delta_2|}{\Gamma; \Omega, x_1 : A \oplus B \Uparrow \vdash C \Rightarrow^{-} \mathbf{case}\ x_1\ \mathbf{of}\ \mathbf{inl}\ x_2 \to t_1;\ \mathbf{inr}\ x_3 \to t_2 \mid \Delta_1 \sqcap \Delta_2} \oplus_L^-$$

   By induction on the first and second premises, we have that:

$$(\Gamma, \Omega), x_2 : A \vdash C \Rightarrow^{-} t_1 \mid \Delta_1 \qquad \text{(ih1)}$$

$$(\Gamma, \Omega), x_3 : B \vdash C \Rightarrow^{-} t_2 \mid \Delta_2 \qquad \text{(ih2)}$$

   from case 2 of the lemma. From which, we can construct the following instantiation of the $\oplus_L^-$ synthesis rule in the non-focusing calculus:

$$\frac{(\Gamma, \Omega), x_2 : A \vdash C \Rightarrow^{-} t_1 \mid \Delta_1 \quad (\Gamma, \Omega), x_3 : B \vdash C \Rightarrow^{-} t_2 \mid \Delta_2 \quad x_2 \notin |\Delta_1| \quad x_3 \notin |\Delta_2|}{\Gamma, (\Omega, x_1 : A \oplus B) \vdash C \Rightarrow^{-} \mathbf{case}\ x_1\ \mathbf{of}\ \mathbf{inl}\ x_2 \to t_1;\ \mathbf{inr}\ x_3 \to t_2 \Delta_1 \sqcap \Delta_2} \oplus_L^-$$

c) Case 1_L^-
In the case of the left asynchronous rule for unit elimination, the synthesis rule has the form:

$$\frac{\Gamma; \emptyset \vdash C \Rightarrow^- t \mid \Delta}{\Gamma; x : 1 \vdash C \Rightarrow^- \textbf{let}\,() = x\,\textbf{in}\,t \mid \Delta}\ 1_L^-$$

By induction on the premise, we have that:

$$\Gamma \vdash C \Rightarrow^- t \mid \Delta \qquad \text{(ih)}$$

from case 2 of the lemma. From which, we can construct the following instantiation of the 1_L^- synthesis rule in the non-focusing calculus matching the conclusion:

$$\frac{\Gamma \vdash C \Rightarrow^- t \mid \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^- \textbf{let}\,() = x\,\textbf{in}\,t \mid \Delta}\ 1_L^-$$

d) Case $\Box_L^-$
In the case of the left asynchronous rule for graded modality elimination, the synthesis rule has the form:

$$\frac{\Gamma; \Omega, x_2 :_r A \Uparrow \vdash B \Rightarrow^- t \mid \Delta, x_2 :_s A \qquad 0 \sqsubseteq s}{\Gamma; \Omega, x_1 : \Box_r A \Uparrow \vdash B \Rightarrow^- \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t \mid \Delta}\ \Box_L^-$$

By induction on the first premise, we have that:

$$(\Gamma, \Omega), x_2 :_r A \vdash B \Rightarrow^- t \mid \Delta, x_2 :_s A \qquad \text{(ih)}$$

from case 2 of the lemma. From which, we can construct the following instatiation of the $\Box_L^-$synthesis rule in the non-focusing calculus:

$$\frac{(\Gamma, \Omega), x_2 :_r A \vdash B \Rightarrow^- t \mid \Delta, x_2 :_s A \qquad 0 \sqsubseteq s}{\Gamma, (\Omega, x_1 : \Box_r A) \vdash B \Rightarrow^- \textbf{let}\,[x_2] = x_1\,\textbf{in}\,t \mid \Delta}\ \Box_L^-$$

e) Case DER$^-$
In the case of the left asynchronous rule for dereliction, the synthesis rule has the form:

$$\frac{\begin{array}{c}\Gamma; x :_s A, y : A \Uparrow \vdash B \Rightarrow^- t \mid \Delta, x :_{s'} A \\ y \notin |\Delta| \qquad \exists s.\, r \sqsupseteq s + 1\end{array}}{\Gamma; x :_r A \Uparrow \vdash B \Rightarrow^- [x/y]t \mid \Delta, x :_{s'} A}\ \text{DER}^-$$

By induction on the first premise, we have that:

$$\Gamma, x :_s A, y : A \vdash B \Rightarrow^- t \mid \Delta, x :_{s'} A \qquad \text{(ih)}$$

from case 2 of the lemma. From which, we can construct the following instatiation of the DER$^-$synthesis rule in the non-focusing calculus:

$$\frac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^- t \mid \Delta, x :_{s'} A \qquad y \notin |\Delta| \qquad \exists s.\, r \sqsupseteq s + 1}{\Gamma, x :_r A \vdash B \Rightarrow^- [x/y]t \mid \Delta, x :_{s'} A}\ \text{DER}^-$$

f) Case $\Uparrow_L^-$
In the case of the left asynchronous rule for transitioning an assumption from the focusing context Ω to the non-focusing context Γ, the synthesis rule has the form:

$$\frac{\Gamma, x : A; \Omega \Uparrow \vdash C \Rightarrow^- t \mid \Delta \qquad \text{A not left async}}{\Gamma; \Omega, x : A \Uparrow \vdash C \Rightarrow^- t \mid \Delta}\ \Uparrow_L^-$$

By induction on the first premise, we have that:

$$\Gamma, x : A, \Omega \vdash C \Rightarrow^- t \mid \Delta \tag{ih}$$

from case 2 of the lemma.

3. Case 3. Right Sync:

a) Case $\otimes_R^-$
In the case of the right synchronous rule for pair introduction, the synthesis rule has the form:

$$\frac{\Gamma; \varnothing \vdash A \Downarrow \Rightarrow^- t_1 \mid \Delta_1 \qquad \Delta_1; \varnothing \vdash B \Downarrow \Rightarrow^- t_2 \mid \Delta_2}{\Gamma; \varnothing \vdash A \otimes B \Downarrow \Rightarrow^- (t_1, t_2) \mid \Delta_2}\ \otimes_R^-$$

By induction on the first and second premises, we have that:

$$\Gamma \vdash A \Rightarrow^- t_1 \mid \Delta_1 \tag{ih1}$$

$$\Delta_1 \vdash B \Rightarrow^- t_2 \mid \Delta_2 \tag{ih2}$$

from case 3 of the lemma. From which, we can construct the following instatiation of the $\otimes_R^-$ synthesis rule in the non-focusing calculus:

$$\frac{\Gamma \vdash A \Rightarrow^- t_1 \mid \Delta_1 \qquad \Delta_1 \vdash B \Rightarrow^- t_2 \mid \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^- (t_1, t_2) \mid \Delta_2}\ \otimes_R^-$$

b) Case $\oplus 1_R^-$ and $\oplus 2_R^-$
In the case of the right synchronous rules for sum introduction, the synthesis rules has the form:

$$\frac{\Gamma; \varnothing \vdash A \Downarrow \Rightarrow^- t \mid \Delta}{\Gamma; \varnothing \vdash A \oplus B \Downarrow \Rightarrow^- \mathbf{inl}\, t \mid \Delta}\ \oplus 1_L^+$$

$$\frac{\Gamma;\emptyset \vdash B \Downarrow \Rightarrow^{-} t \mid \Delta}{\Gamma;\emptyset \vdash A \oplus B \Downarrow \Rightarrow^{-} \mathbf{inr}\, t \mid \Delta}\ \oplus 2_L^{+}$$

By induction on the premises of these rules, we have that:

$$\Gamma \vdash A \Rightarrow^{-} t \mid \Delta \tag{ih1}$$

$$\Gamma \vdash B \Rightarrow^{-} t \mid \Delta \tag{ih2}$$

from case 3 of the lemma. From which, we can construct the following instatiations of the $\oplus 1_R^{-}$ and $\oplus 2_R^{-}$ rule in the non-focusing calculus, respectively:

$$\frac{\Gamma \vdash A \Rightarrow^{-} t \mid \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{-} \mathbf{inl}\, t \mid \Delta}\ \oplus 1_R^{-}$$

$$\frac{\Gamma \vdash B \Rightarrow^{-} t \mid \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{-} \mathbf{inr}\, t \mid \Delta}\ \oplus 2_R^{-}$$

c) Case 1_R^{-}
In the case of the right synchronous rule for unit introduction, the synthesis rule has the form:

$$\frac{}{\Gamma \vdash 1 \Rightarrow^{-} () \mid \Gamma}\ 1_R^{-}$$

From which, we can construct the following instatiation of the 1_R^{-} synthesis rule in the non-focusing calculus:

$$\frac{}{\Gamma, \Omega \vdash 1 \Rightarrow^{-} () \mid \Gamma}\ 1_R^{-}$$

d) Case $\Box_R^{-}$
In the case of the right synchronous rule for graded modality introduction, the synthesis rule has the form:

$$\frac{\Gamma;\emptyset \vdash A \Uparrow \Rightarrow^{-} t \mid \Delta}{\Gamma;\emptyset \vdash \Box_r A \Downarrow \Rightarrow^{-} t \mid \Gamma - r \cdot (\Gamma - \Delta)}\ \Box_R^{-}$$

By induction on the premise, we have that:

$$\Gamma \vdash A \Rightarrow^{-} t \mid \Delta \tag{ih}$$

from case 1 of the lemma. From which, we can construct the following instatiation of the $\Box_R^{-}$synthesis rule in the non-focusing calculus:

$$\frac{\Gamma \vdash A \Rightarrow^{-} t \mid \Delta}{\Gamma \vdash \Box_r A \Rightarrow^{-} [t] \mid \Gamma - r \cdot (\Gamma - \Delta)}\ \Box_R^{-}$$

e) Case $\Downarrow_R^-$
In the case of the right synchronous rule for transitioning back to an asynchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma; \varnothing \vdash A \Uparrow \Rightarrow^- t \mid \Delta}{\Gamma; \varnothing \vdash A \Downarrow \Rightarrow^- t \mid \Delta} \Downarrow_R^-$$

By induction on the premise, we have that:

$$\Gamma \vdash A \Rightarrow^- t \mid \Delta \tag{ih}$$

from case 1 of the lemma.

4. Case 4. Left Sync

a) Case $\multimap_L^-$
In the case of the left synchronous rule for application, the synthesis rule has the form:

$$\frac{\Gamma; x_2 : B \Downarrow \vdash C \Rightarrow^- t_1 \mid \Delta_1 \qquad x_2 \notin |\Delta_1| \qquad \Delta_1; \varnothing \vdash A \Downarrow \Rightarrow^- t_2 \mid \Delta_2}{\Gamma; x_1 : A \multimap B \Downarrow \vdash C \Rightarrow^- [(x_1\, t_2)/x_2]t_1 \mid \Delta_2} \multimap_L^-$$

By induction on the first premise, we have that:

$$\Gamma, x_2 : B \vdash C \Rightarrow^- t_1 \mid \Delta_1 \tag{ih1}$$

from case 4 of the lemma. By induction on the third premise, we have that:

$$\Delta_1 \vdash A \Rightarrow^- t_2 \mid \Delta_2 \tag{ih2}$$

from case 3 of the lemma. From which, we can construct the following instatiation of the $\multimap_L^-$ synthesis rule in the non-focusing calculus:

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^- t_1 \mid \Delta_1 \qquad x_2 \notin |\Delta_1| \qquad \Delta_1 \vdash A \Rightarrow^- t_2 \mid \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^- [(x_1\, t_2)/x_2]t_1 \mid \Delta_2} \multimap_L^-$$

b) Case LinVar$^-$
In the case of the left synchronous rule for linear variable synthesis, the synthesis rule has the form:

$$\frac{}{\Gamma; x : A \Downarrow \vdash A \Rightarrow^- x \mid \Gamma} \textsc{LinVar}^-$$

From which, we can construct the following instatiation of the LinVar$^-$ synthesis rule in the non-focusing calculus:

$$\frac{}{\Gamma, x : A \vdash A \Rightarrow^- x \mid \Gamma} \textsc{LinVar}^-$$

c) Case GrVar$^-$
In the case of the left synchronous rule for graded variable synthesis, the synthesis rule has the form:

$$\frac{\exists s.\, r \sqsubseteq s + 1}{\Gamma; x :_r A \Downarrow \vdash A \Rightarrow^- x \mid \Gamma, x :_s A}\ \textsc{GrVar}^-$$

From which, we can construct the following instatiation of the GrVar$^-$ synthesis rule in the non-focusing calculus:

$$\frac{\exists s.\, r \sqsubseteq s + 1}{\Gamma, x :_r A \vdash A \Rightarrow^- x \mid \Gamma, x :_s A}\ \textsc{GrVar}^-$$

d) Case $\Downarrow_L^-$
In the case of the left synchronous rule for transitioning back to an asynchronous judgement, the synthesis rule has the form:

$$\frac{\begin{array}{c}\Gamma; x : A \Uparrow \vdash C \Rightarrow^- t \mid \Delta \\ \text{A not atomic and not left sync}\end{array}}{\Gamma; x : A \Downarrow \vdash C \Rightarrow^- t \mid \Delta}\ \Downarrow_L^-$$

By induction on the premise, we have that:

$$\Gamma, x : A \vdash C \Rightarrow^- t \mid \Delta \qquad \text{(ih)}$$

from case 2 of the lemma.

5. Case 5. Focus Right: focus$_R^-$
In the case of the focusing rule for transitioning from a left asynchronous judgement to a right synchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma; \emptyset \vdash C \Downarrow \Rightarrow^- t \mid \Delta \qquad \text{C not atomic}}{\Gamma; \emptyset \Uparrow \vdash C \Rightarrow^- t \mid \Delta}\ \textsc{focus}_R^-$$

By induction on the first premise, we have that:

$$\Gamma \vdash C \Rightarrow^- t \mid \Delta \qquad \text{(ih)}$$

from case 2 of the lemma.

6. Case 6. Focus Left focus$_L^-$
In the case of the focusing rule for transitioning from a left asynchronous judgement to a left synchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma; x : A \Downarrow \vdash C \Rightarrow^- t \mid \Delta}{\Gamma, x : A; \emptyset \Uparrow \vdash C \Rightarrow^- t \mid \Delta}\ \textsc{focus}_L^-$$

By induction on the first premise, we have that:

$$\Gamma, x : A \vdash C \Rightarrow^- t \mid \Delta \qquad \text{(ih)}$$

from case 2 of the lemma.

□

B.1.5 *Soundness of Focusing for the Additive Linear Graded Synthesis Calculus*

Lemma 3.5.2 (Soundness of focusing for additive synthesis). *For all contexts* Γ, Ω *and types* A *then:*

$$\begin{array}{llcl}
1.\ \textit{Right Async}: & \Gamma;\Omega \vdash A \Uparrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash A \Rightarrow^{+} t;\ \Delta \\
2.\ \textit{Left Async}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash C \Rightarrow^{+} t;\ \Delta \\
3.\ \textit{Right Sync}: & \Gamma;\emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash A \Rightarrow^{+} t;\ \Delta \\
4.\ \textit{Left Sync}: & \Gamma;x : A \Downarrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,x : A \vdash C \Rightarrow^{+} t;\ \Delta \\
5.\ \textit{Focus Right}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t;\ \Delta \\
6.\ \textit{Focus Left}: & \Gamma,x : A;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t;\ \Delta
\end{array}$$

i.e. t has type A under context Δ*, which contains assumptions with grades reflecting their use in t.*

Proof. 1. Case 1. Right Async:

a) Case $\multimap^{+}_{R}$

In the case of the right asynchronous rule for abstraction introduction, the synthesis rule has the form:

$$\frac{\Gamma;\Omega,x : A \vdash B \Uparrow \Rightarrow t \mid \Delta,x : A}{\Gamma;\Omega \vdash A \multimap B \Uparrow \Rightarrow \lambda x.t \mid \Delta}\ \multimap^{+}_{R}$$

By induction on the premise, we have that:

$$(\Gamma,\Omega),x : A \vdash B \Rightarrow^{+} t;\ \Delta,x : A \tag{ih}$$

from case 1 of the lemma. From which, we can construct the following instatiation of the $\multimap^{+}_{R}$ synthesis rule in the non-focusing calculus:

$$\frac{(\Gamma,\Omega),x : A \vdash B \Rightarrow^{+} t;\ \Delta,x : A}{\Gamma,\Omega \vdash A \multimap B \Rightarrow^{+} \lambda x.t;\ \Delta}\ \text{R}{\multimap^{+}}$$

b) Case $\Uparrow^{+}_{R}$In the case of the right asynchronous rule for transition to a left asynchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta \qquad C \text{ not right async}}{\Gamma;\Omega \vdash C \Uparrow \Rightarrow t \mid \Delta}\ \Uparrow^{+}_{R}$$

By induction on the first premise, we have that:

$$\Gamma,\Omega \vdash C \Rightarrow^{+} t;\ \Delta$$

from case 2 of the lemma.

2. Case 2. Left Async:

   a) Case $\otimes_L^+$
   In the case of the left asynchronous rule for pair elimination, the synthesis rule has the form:

$$\dfrac{\Gamma;\Omega,x_1:A,x_2:B\vdash C\Rightarrow t_2\mid\Delta,x_1:A,x_2:B}{\Gamma;\Omega,x_3:A\otimes B\vdash C\Rightarrow\textbf{let}\,(x_1,x_2)=x_3\,\textbf{in}\,t_2\mid\Delta,x_3:A\otimes B}\ \otimes_L^+$$

   By induction on the premise, we have that:

$$(\Gamma,\Omega),x_1:A,x_2:B\vdash C\Rightarrow^+ t_2;\ \Delta,x_1:A,x_2:B \qquad \text{(ih)}$$

   from case 2 of the lemma. From which, we can construct the following instatiation of the $\otimes_L^+$ synthesis rule in the non-focusing calculus:

$$\dfrac{(\Gamma,\Omega),x_1:A,x_2:B\vdash C\Rightarrow^+ t_2;\ \Delta,x_1:A,x_2:B}{\Gamma,(\Omega,x_3:A\otimes B)\vdash C\Rightarrow^+\textbf{let}\,(x_1,x_2)=x_3\,\textbf{in}\,t_2;\ \Delta,x_3:A\otimes B}\ \text{L}\otimes^+$$

   b) Case $\oplus_L^+$
   In the case of the left asynchronous rule for sum elimination, the synthesis rule has the form:

$$\dfrac{\begin{array}{c}\Gamma;\Omega,x_2:A\Uparrow\vdash C\Rightarrow t_1\mid\Delta_1,x_2:A\\ \Gamma;\Omega,x_3:B\Uparrow\vdash C\Rightarrow t_2\mid\Delta_2,x_3:B\end{array}}{\Gamma;\Omega,x_1:A\oplus B\Uparrow\vdash C\Rightarrow^-\textbf{case}\ x_1\ \textbf{of}\ \textbf{inl}\ x_2\to t_1;\ \textbf{inr}\ x_3\to t_2\mid\Delta_1\sqcup\Delta_2,x_1:A\oplus B}\ \oplus_L^+$$

   By induction on the premises, we have that:

$$(\Gamma,\Omega),x_2:A\vdash C\Rightarrow^+ t_1;\ \Delta_1,x_2:A \qquad \text{(ih1)}$$

$$(\Gamma,\Omega),x_3:B\vdash C\Rightarrow^+ t_2;\ \Delta_2,x_3:B \qquad \text{(ih2)}$$

   from case 2 of the lemma. From which, we can construct the following instatiation of the $\oplus_L^+$ synthesis rule in the non-focusing calculus:

$$\dfrac{\begin{array}{c}(\Gamma,\Omega),x_2:A\vdash C\Rightarrow^+ t_1;\ \Delta_1,x_2:A\\ (\Gamma,\Omega),x_3:B\vdash C\Rightarrow^+ t_2;\ \Delta_2,x_3:B\end{array}}{\Gamma,(\Omega,x_1:A\oplus B)\vdash C\Rightarrow^+\textbf{case}\ x_1\ \textbf{of}\ \textbf{inl}\ x_2\to t_1;\ \textbf{inr}\ x_3\to t_2\mid(\Delta_1\sqcup\Delta_2),x_1:A\oplus B}\ \text{L}\oplus^+$$

   c) Case 1_L^+
   In the case of the left asynchronous rule for unit elimination, the synthesis rule has the form:

$$\dfrac{\Gamma;\varnothing\vdash C\Rightarrow t\mid\Delta}{\Gamma;x:1\vdash C\Rightarrow\textbf{let}\,()=x\,\textbf{in}\,t\mid\Delta,x:1}\ 1_L^+$$

By induction on the premise, we have that:

$$\Gamma \vdash C \Rightarrow^{+} t;\, \Delta \tag{ih}$$

from case 2 of the lemma. From which, we can construct the following instatiation of the 1^{+}_{L} synthesis rule in the non-focusing calculus:

$$\dfrac{\Gamma \vdash C \Rightarrow^{+} t;\, \Delta}{\Gamma, x : 1 \vdash C \Rightarrow^{+} \mathbf{let}\,() = x\,\mathbf{in}\,t;\, \Delta, x : 1}\;\text{L1}^{+}$$

d) Case $\Box^{+}_{L}$
In the case of the left asynchronous rule for graded modality elimination, the synthesis rule has the form:

$$\dfrac{\begin{array}{c}\Gamma; \Omega, x_2 :_r A \Uparrow \vdash B \Rightarrow t \mid \Delta \\ \textit{if } x_2 :_s A \in \Delta \textit{ then } s \sqsubseteq r \textit{ else } 0 \sqsubseteq r\end{array}}{\Gamma; \Omega, x_1 : \Box_r A \vdash B \Rightarrow \mathbf{let}\,[x_2] = x_1\,\mathbf{in}\,t \mid (\Delta \backslash x_2), x_1 : \Box_r A}\;\Box^{+}_{L}$$

By induction on the first premise, we have that:

$$(\Gamma, \Omega), x_2 :_r A \vdash B \Rightarrow^{+} t;\, \Delta \tag{ih}$$

from case 2 of the lemma. From which, we can construct the following instatiation of the $\Box^{+}_{L}$ synthesis rule in the non-focusing calculus:

$$\dfrac{\begin{array}{c}(\Gamma, \Omega), x_2 :_r A \vdash B \Rightarrow^{+} t;\, \Delta \\ \textit{if } x_2 :_s A \in \Delta \textit{ then } s \sqsubseteq r \textit{ else } 0 \sqsubseteq r\end{array}}{\Gamma, (\Omega, x_1 : \Box_r A) \vdash B \Rightarrow^{+} \mathbf{let}\,[x_2] = x_1\,\mathbf{in}\,t;\, (\Delta \backslash x_2), x_1 : \Box_r A}\;\text{L}\Box^{+}$$

e) Case DER^{+}
In the case of the left asynchronous rule for dereliction, the synthesis rule has the form:

$$\dfrac{\Gamma; x :_s A, y : A \Uparrow \vdash B \Rightarrow t \mid \Delta, y : A}{\Gamma; x :_s A \Uparrow \vdash B \Rightarrow [x/y]t \mid \Delta + x :_1 A}\;\text{DER}^{+}$$

By induction on the premise, we have that:

$$\Gamma, x :_s A, y : A \vdash B \Rightarrow^{+} t;\, \Delta, y : A \tag{ih}$$

from case 2 of the lemma. From which, we can construct the following instantiation of the DER^{+} synthesis rule in the non-focusing calculus:

$$\dfrac{\Gamma, x :_s A, y : A \vdash B \Rightarrow^{+} t;\, \Delta, y : A}{\Gamma, x :_s A \vdash B \Rightarrow^{+} [x/y]t;\, \Delta + x :_1 A}\;\text{DER}^{+}$$

f) Case $\Uparrow_L^+$
In the case of the left asynchronous rule for transitioning an assumption from the focusing context Ω to the non-focusing context Γ, the synthesis rule has the form:

$$\frac{\Gamma, x : A; \Omega \Uparrow \vdash C \Rightarrow t \mid \Delta \qquad \text{A not left async}}{\Gamma; \Omega, x : A \Uparrow \vdash C \Rightarrow t \mid \Delta} \Uparrow_L^+$$

By induction on the first premise, we have that:

$$\Gamma, x : A, \Omega \vdash C \Rightarrow^+ t; \Delta \tag{ih}$$

from case 2 of the lemma.

3. Case 3. Right Sync:

a) Case $\otimes_R^+$
In the case of the right synchronous rule for pair introduction, the synthesis rule has the form:

$$\frac{\Gamma; \varnothing \vdash A \Downarrow \Rightarrow t_1 \mid \Delta_1 \qquad \Gamma; \varnothing \vdash B \Downarrow \Rightarrow t_2 \mid \Delta_2}{\Gamma; \varnothing \vdash A \otimes B \Downarrow \Rightarrow (t_1, t_2) \mid \Delta_1 + \Delta_2} \otimes_R^+$$

By induction on the premises, we have that:

$$\Gamma \vdash A \Rightarrow^+ t_1; \Delta_1 \tag{ih1}$$

$$\Gamma \vdash B \Rightarrow^+ t_2; \Delta_2 \tag{ih2}$$

from case 3 of the lemma. From which, we can construct the following instantiation of the $\otimes_R^+$ synthesis rule in the non-focusing calculus:

$$\frac{\Gamma \vdash A \Rightarrow^+ t_1; \Delta_1 \qquad \Gamma \vdash B \Rightarrow^+ t_2; \Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^+ (t_1, t_2); \Delta_1 + \Delta_2} \mathrm{R}\otimes^+$$

b) Case $\oplus 1_R^+$ and $\oplus 2_R^+$
In the case of the right synchronous rules for sum introduction, the synthesis rules have the form:

$$\frac{\Gamma; \varnothing \vdash A \Downarrow \Rightarrow t \mid \Delta}{\Gamma; \varnothing \vdash A \oplus B \Downarrow \Rightarrow \textbf{inl}\, t \mid \Delta} \oplus 1_L^+$$

$$\frac{\Gamma; \varnothing \vdash B \Downarrow \Rightarrow t \mid \Delta}{\Gamma; \varnothing \vdash A \oplus B \Downarrow \Rightarrow \textbf{inr}\, t \mid \Delta} \oplus 2_L^+$$

By induction on the premises of the rules, we have that:

$$\Gamma \vdash A \Rightarrow^+ t; \Delta \tag{ih1}$$

$$\Gamma \vdash B \Rightarrow^{+} t;\, \Delta \qquad \text{(ih2)}$$

from case 3 of the lemma. From which, we can construct the following instantiations of the $\oplus 1^{+}_{R}$ and $\oplus 2^{+}_{R}$ synthesis rules in the non-focusing calculus, respectively:

$$\frac{\Gamma \vdash A \Rightarrow^{+} t;\, \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{+} \mathbf{inl}\, t;\, \Delta}\ \mathrm{R}\oplus^{+}_{1}$$

$$\frac{\Gamma \vdash B \Rightarrow^{+} t;\, \Delta}{\Gamma \vdash A \oplus B \Rightarrow^{+} \mathbf{inr}\, t;\, \Delta}\ \mathrm{R}\oplus^{+}_{2}$$

c) Case 1^{+}_{R}
In the case of the right synchronous rule for unit introduction, the synthesis rule has the form:

$$\frac{}{\Gamma; \emptyset \vdash 1 \Rightarrow () \mid \emptyset}\ 1^{+}_{R}$$

From which, we can construct the following instantiation of the 1^{+}_{R} synthesis rule in the non-focusing calculus:

$$\frac{}{\Gamma \vdash 1 \Rightarrow^{+} ();\, \emptyset}\ \mathrm{R}1^{+}$$

d) Case $\Box^{+}_{R}$
In the case of the right synchronous rule for graded modality introduction, the synthesis rule has the form:

$$\frac{\Gamma; \emptyset \vdash A \Uparrow \Rightarrow t \mid \Delta}{\Gamma; \emptyset \vdash \Box_{r} A \Downarrow \Rightarrow [t] \mid r \cdot \Delta}\ \Box^{+}_{R}$$

By induction on the premise, we have that:

$$\Gamma \vdash A \Rightarrow^{+} t;\, \Delta \qquad \text{(ih)}$$

from case 1 of the lemma. From which, we can construct the following instantiation of the $\Box^{+}_{R}$ synthesis rule in the non-focusing calculus:

$$\frac{\Gamma \vdash A \Rightarrow^{+} t;\, \Delta}{\Gamma \vdash \Box_{r} A \Rightarrow^{+} [t];\, r \cdot \Delta}\ \mathrm{R}\Box^{+}$$

e) Case $\Downarrow^{+}_{R}$
In the case of the right synchronous rule for transitioning back to an asynchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma; \emptyset \vdash A \Uparrow \Rightarrow t \mid \Delta}{\Gamma; \emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta}\ \Downarrow^{+}_{R}$$

By induction on the premise, we have that:

$$\Gamma \vdash A \Rightarrow^{+} t;\ \Delta \tag{ih}$$

from case 1 of the lemma.

4. Case 4. Left Sync

   a) Case $\multimap^{+}_{L}$
   In the case of the left synchronous rule for application, the synthesis rule has the form:

$$\frac{\Gamma; x_2 : B \Downarrow \vdash C \Rightarrow t_1 \mid \Delta_1, x_2 : B \qquad \Gamma; \emptyset \vdash A \Downarrow \Rightarrow t_2 \mid \Delta_2}{\Gamma; x_1 : A \multimap B \Downarrow \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \multimap^{+}_{L}$$

   By induction on the first premise, we have that:

$$\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1;\ \Delta_1, x_2 : B \tag{ih1}$$

   from case 4 of the lemma. By induction on the second premise, we have that:

$$\Gamma \vdash A \Rightarrow^{+} t_2;\ \Delta_2 \tag{ih2}$$

   from case 3 of the lemma. From which, we can construct the following instantiation of the $\multimap^{+}_{L}$ synthesis rule in the non-focusing calculus:

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1;\ \Delta_1, x_2 : B \qquad \Gamma \vdash A \Rightarrow^{+} t_2;\ \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^{+} [(x_1\, t_2)/x_2]t_1;\ (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \text{L}\multimap^{+}$$

   b) Case LinVar^{+}
   In the case of the left synchronous rule for linear variable synthesis, the synthesis rule has the form:

$$\frac{}{\Gamma; x : A \vdash A \Rightarrow x \mid x : A}\ \textsc{LinVar}^{+}$$

   From which, we can construct the following instantiation of the LinVar^{+} in the non-focusing calculus:

$$\frac{}{\Gamma, x : A \vdash A \Rightarrow^{+} x;\ x : A}\ \textsc{LinVar}^{+}$$

   c) Case GrVar^{+}
   In the case of the left synchronous rule for graded variable synthesis, the synthesis rule has the form:

$$\frac{}{\Gamma; x :_r A \vdash A \Rightarrow x \mid x :_1 A}\ \textsc{GrVar}^{+}$$

From which, we can construct the following instantiation of the GrVar$^+$ synthesis rule in the non-focusing calculus:

$$\frac{}{\Gamma, x :_r A \vdash A \Rightarrow^+ x;\, x :_1 A}\ \textsc{GrVar}^+$$

d) Case $\Downarrow_L^+$
In the case of the left synchronous rule for transitioning back to an asynchronous judgement, the synthesis rule has the form:

$$\frac{\begin{array}{c}\Gamma; x : A \Uparrow \vdash C \Rightarrow t \mid \Delta \\ \text{A not atomic and not left sync}\end{array}}{\Gamma; x : A \Downarrow \vdash C \Rightarrow t \mid \Delta}\ \Downarrow_L^+$$

By induction on the premise, we have that:

$$\Gamma, x : A \vdash C \Rightarrow^+ t;\, \Delta \qquad \text{(ih)}$$

from case 2 of the lemma.

5. Case 5. Focus Right: focus$_R^+$
In the case of the focusing rule for transitioning from a left asynchronous judgement to a right synchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma; \varnothing \vdash C \Downarrow \Rightarrow t \mid \Delta \qquad \text{C not atomic}}{\Gamma; \varnothing \Uparrow \vdash C \Rightarrow t \mid \Delta}\ \textsc{focus}_R^+$$

By induction on the first premise, we have that:

$$\Gamma \vdash C \Rightarrow^+ t;\, \Delta \qquad \text{(ih)}$$

from case 2 of the lemma.

6. Case 6. Focus Left: focus$_L^+$
In the case of the focusing rule for transitioning from a left asynchronous judgement to a left synchronous judgement, the synthesis rule has the form:

$$\frac{\Gamma; x : A \Downarrow \vdash C \Rightarrow t \mid \Delta}{\Gamma, x : A; \varnothing \Uparrow \vdash C \Rightarrow t \mid \Delta}\ \textsc{focus}_L^+$$

By induction on the first premise, we have that:

$$\Gamma, x : A \vdash C \Rightarrow^+ t;\, \Delta \qquad \text{(ih)}$$

from case 2 of the lemma.

□

B.1.6 *Soundness of Focusing for the Additive Pruning Linear Graded Synthesis Calculus*

Lemma 3.5.3 (Soundness of focusing for additive pruning synthesis)**.**
For all contexts Γ, Ω *and types A then:*

$$\begin{array}{llll}
1.\ \textit{Right Async}: & \Gamma;\Omega \vdash A \Uparrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash A \Rightarrow^{+} t;\Delta \\
2.\ \textit{Left Async}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,\Omega \vdash C \Rightarrow^{+} t;\Delta \\
3.\ \textit{Right Sync}: & \Gamma;\emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash A \Rightarrow^{+} t;\Delta \\
4.\ \textit{Left Sync}: & \Gamma;x:A \Downarrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma,x:A \vdash C \Rightarrow^{+} t;\Delta \\
5.\ \textit{Focus Right}: & \Gamma;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t;\Delta \\
6.\ \textit{Focus Left}: & \Gamma,x:A;\Omega \Uparrow \vdash C \Rightarrow t \mid \Delta & \Longrightarrow & \Gamma \vdash C \Rightarrow^{+} t;\Delta
\end{array}$$

i.e. t has type A under context Δ*, which contains assumptions with grades reflecting their use in t.*

Proof. 1. Case: 1. Right Async: The proofs for right asynchronous rules are equivalent to those of lemma (3.5.2)

2. Case 2. Left Async: The proofs for left asynchronous rules are equivalent to those of lemma (3.5.2)

3. Case 3. Right Sync: The proofs for right synchronous rules are equivalent to those of lemma (3.5.2), except for the case of the $\otimes_R'^{+}$rule:

   a) Case $\otimes_R'^{+}$
   In the case of the right synchronous rule for pair introduction, the synthesis rule has the form:

$$\dfrac{\Gamma;\emptyset \vdash A \Rightarrow t_1 \mid \Delta_1 \qquad \Gamma - \Delta_1;\emptyset \vdash B \Rightarrow t_2 \mid \Delta_2}{\Gamma;\emptyset \vdash A \otimes B \Rightarrow (t_1,t_2) \mid \Delta_1 + \Delta_2}\ \otimes_R'^{+}$$

   By induction on the premises, we have that:

$$\Gamma \vdash A \Rightarrow^{+} t_1;\Delta_1 \qquad \text{(ih1)}$$

$$\Gamma - \Delta_1 \vdash B \Rightarrow^{+} t_2;\Delta_2 \qquad \text{(ih2)}$$

   from case 3 of the lemma. From which, we can construct the following instantiation of the $\otimes_R'^{+}$ synthesis rule in the non-focusing calculus:

$$\dfrac{\Gamma \vdash A \Rightarrow^{+} t_1;\Delta_1 \qquad \Gamma - \Delta_1 \vdash B \Rightarrow^{+} t_2;\Delta_2}{\Gamma \vdash A \otimes B \Rightarrow^{+} (t_1,t_2);\Delta_1 + \Delta_2}\ \mathrm{R}'\otimes^{+}$$

4. Case 4. Left Sync: The proofs for left synchronous rules are equivalent to those of lemma (3.5.2), except for the case of the $\multimap_L'^{+}$ rule:

   a) Case $\multimap_L'^{+}$
   In the case of the left synchronous rule for application, the synthesis rule has the form:

$$\frac{\Gamma; x_2 : B \vdash C \Rightarrow t_1 \mid \Delta_1, x_2 : B \qquad \Gamma - \Delta_1; \emptyset \vdash A \Rightarrow t_2 \mid \Delta_2}{\Gamma; x_1 : A \multimap B \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \multimap_L'^{+}$$

   By induction on the first premise, we have that:

$$\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1;\ \Delta_1, x_2 : B \qquad \text{(ih1)}$$

   from case 4 of the lemma. By induction on the second premise, we have that:

$$\Gamma \vdash A \Rightarrow^{+} t_2;\ \Delta_2 \qquad \text{(ih2)}$$

   from case 3 of the lemma. From which, we can construct the following instantiation of the $\multimap_L'^{+}$ synthesis rule in the non-focusing calculus:

$$\frac{\Gamma, x_2 : B \vdash C \Rightarrow^{+} t_1;\ \Delta_1, x_2 : B \qquad \Gamma - \Delta_1 \vdash A \Rightarrow^{+} t_2;\ \Delta_2}{\Gamma, x_1 : A \multimap B \vdash C \Rightarrow^{+} [(x_1\, t_2)/x_2]t_1;\ (\Delta_1 + \Delta_2), x_1 : A \multimap B}\ \mathrm{L}' \multimap^{+}$$

5. Case 5. Right Focus: focus$_R^{+}$ - The proof for right focusing rule is equivalent to that of lemma (3.5.2)

6. Case 6. Left Focus: focus$_L^{+}$ - The proof for left focusing rule is equivalent to that of lemma (3.5.2)

□

B.2 PROOFS FOR THE DERIVING MECHANISM

This section contains the proofs relating to Chapter 4. We begin by presenting a typed equational theory for use in the proofs that *push* and *pull* are the inverse of each other (Section B.2.1). The type soundness proofs of *push* and *pull* then follow in Section B.2.3, followed by the aforementioned inverse proofs in Section B.2.4.

B.2.1 *Typed Equational Theory*

Figure B.1 defines an equational theory for GRMINI^P. The equational theory is typed, but for brevity we elide the typing for most cases since it follows exactly from the structure of the terms. The fully typed rules are provided in the appendix **?**. For those rules which are type restricted, we include the full typed-equality judgment here. The typability of these equations relies on previous work on the Granule language which proves that pattern matching and substitution are well typed Orchard et al. [2019].

$$\begin{aligned} (\lambda x.t_2)\,t_1 &\equiv t_2[t_1/x] && \beta \\ \lambda x.t\,x &\equiv t && (x\#t)\ \eta \end{aligned}$$

$$\begin{aligned} \textbf{letrec}\ x = t_1\ \textbf{in}\ t_2 &\equiv t_2[\textbf{letrec}\ x = t_1\ \textbf{in}\ t_1/x] && \beta_{letrec} \\ f\,(\textbf{letrec}\ x = t_1\ \textbf{in}\ t_2) &\equiv \textbf{letrec}\ x = t_1\ \textbf{in}\ (f\,t_2) && \textsc{LetRecDsitrib} \end{aligned}$$

$$\begin{aligned} \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i} &\equiv (t \rhd p_j)t_j && (\mathsf{minimal}(j))\ \beta_{case} \\ \textbf{case}\ t_1\ \textbf{of}\ \overline{p_i \to [p_i/z]t_2} &\equiv [t_1/z]t_2 && \eta_{case} \\ \textbf{case}\ (\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i})\ \textbf{of}\ \overline{p'_i \to t'_i} &\equiv \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to (\textbf{case}\ t_i\ \textbf{of}\ \overline{p'_i \to t'_i})} && \textsc{Equiv\_caseAssoc} \\ f\,(\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i}) &\equiv \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to (f\,t_i)} && \textsc{CaseDistrib} \\ \textbf{case}\ [\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i}]\ \textbf{of}\ \overline{[p'_i] \to t'_i} &\equiv \textbf{case}\ [t]\ \textbf{of}\ \overline{[p_i] \to \textbf{case}\ [t_i]\ \textbf{of}\ \overline{[p'_i] \to t'_i}} && (\mathsf{lin}(p_i))\ \textsc{Equiv\_caseBoxAssoc} \end{aligned}$$

$$\frac{\Gamma \vdash t : \Box_r A \qquad r \vdash p_i : A \rhd \Delta_i \qquad \Delta_i \vdash p_i : A \qquad 1 \sqsubseteq r}{\Gamma \vdash \textbf{case}\ t\ \textbf{of}\ \overline{[p_i] \to p_i} \equiv \textbf{case}\ t\ \textbf{of}\ [x] \to x : A}\ \textsc{Equiv\_caseGen}$$

Figure B.1: Equational theory for GRMINI^P

The β and η rules follow the standard rules from the λ-calculus, where # is a *freshness* predicate, denoting that variable x does not appear inside term t.

For recursive **letrec** bindings, the β_{letrec} rule substitutes any occurrence of the bound variable x in t_2 with **letrec** $x = t_1$ **in** t_1, ensuring that recursive uses of x inside t_1 can be substituted with t_1 through subsequent β_{letrec} reduction. The LETRECDISTRIB rule allows distributivity of functions over **letrec** expressions, stating that if a function f can be applied to the entire **letrec** expression, then this is equivalent to applying f to just the body term t_2.

Term elimination is via **case**, requiring rules for both β- and η-equality on case expressions, as well as rules for associativity and distributivity. In β_{case}, a term t is matched against a pattern p_j in the context of the term t_j through the use of the partial function

$(t \rhd p_j)t_j = t'$ which may substitute terms bound in p_j into t_j to yield t' if the match is successful. This partial function is defined inductively:

$$\frac{}{(t \rhd \_)t' = t'}\ \rhd_{\_} \qquad \frac{}{(t \rhd x)t' = [t/x]t'}\ \rhd_{var}$$

$$\frac{(t \rhd p)t' = t''}{([t] \rhd [p])t' = t''}\ \rhd_{\square} \quad \frac{(t_i \rhd p_i)t'_i = t'_{i+1}}{(C\ t_1 ... t_n \rhd C\, p_1 ... p_n)t'_1 = t'_{n+1}}\ \rhd_C$$

As a predicate to the β_{case} rule, we require that j be minimal, i.e. the first pattern p_j in $p_1 ... p_n$ for which $(t \rhd p_j)t_j = t'$ is defined. Rule η_{case} states that if all branches of the case expression share a common term t_2 which differs between branches only in the occurrences of terms that match the pattern used, then we can substitute t_1 for the pattern inside t_2.

Associativity of case expressions is provided by the CaseAssoc rule. This rule allows us to restructure nested case expressions such that the output terms t_i of the inner case may be matched against the patterns of the outer case, to achieve the same resulting output terms t'_i. The [CaseAssoc] rule provides a graded alternative to the CaseAssoc rule, where the nested case expression is graded, provided that the patterns p'_i of the outer case expression are also graded. Notably, this rule only holds when the patterns of the inner case expression are linear (i.e., variable or constant) so that there are no nested box patterns, represented via the $\mathsf{lin}(p_i)$ predicate. As with **letrec**, distributivity of functions over a case expression is given by CaseDistrib.

Lastly generalisation of an arbitrary boxed pattern to a variable is permitted through the CaseGen rule. Here, a boxed pattern $[p_i]$ and the output term of the case may be converted to a variable if the output term is equivalent to the pattern inside the box. The term t being matched against must therefore have a grade approximatable by 1, as witnessed by the predicate $1 \sqsubseteq r$ in the typing derivation.

$$\frac{\Gamma_1, x : A \vdash t_2 : B \qquad \Gamma_2 \vdash t_1 : A}{\Gamma_1 + \Gamma_2 \vdash (\lambda x.t_2)\, t_1 \equiv [t_1/x]t_2 : B}\ \textsc{Equiv\_beta}$$

$$\frac{\Gamma \vdash t : A \multimap B \qquad [x\#t]}{\Gamma \vdash \lambda x.t\, x \equiv t : A \multimap B}\ \textsc{Equiv\_eta}$$

$$\frac{\Gamma_1, x : A \vdash t_1 : A \qquad \Gamma_2, x : A \vdash t_2 : B}{\Gamma_1 + \Gamma_2 \vdash \textbf{letrec}\ x\ =\ t_1\ \textbf{in}\ t_2 \equiv [\textbf{letrec}\ x\ =\ t_1\ \textbf{in}\ t_1/x]t_2 : B}\ \textsc{Equiv\_letrecBeta}$$

$$\frac{\Gamma_1, x : A \vdash t_1 : A \qquad \Gamma_2 \vdash t_2 : B \qquad \Gamma_3 \vdash f : B \multimap W}{\Gamma_1 + \Gamma_2 + \Gamma_3 \vdash f\,(\textbf{letrec}\ x\ =\ t_1\ \textbf{in}\ t_2) \equiv \textbf{letrec}\ x\ =\ t_1\ \textbf{in}\ (f\, t_2) : W}\ \textsc{Equiv\_letrecDist}$$

$$\frac{\begin{array}{c}\Gamma_1 \vdash t : A \qquad - \vdash p_i : A \rhd \Delta_i \\ \Gamma_2, \Delta_i \vdash t_i : B\end{array}}{\Gamma_1 + \Gamma_2 \vdash \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i} \equiv (t \rhd p_j)t_j : B}\ \textsc{Equiv\_caseBeta}$$

$$\frac{\begin{array}{c}\Gamma_1 \vdash t_1 : A \qquad - \vdash p_i : A \rhd \Delta_i \\ \Gamma_2, z : A \vdash t_2 : B\end{array}}{\Gamma_1 + \Gamma_2 \vdash \textbf{case}\ t_1\ \textbf{of}\ \overline{p_i \to [p_i/z]t_2} \equiv [t_1/z]t_2 : B}\ \textsc{Equiv\_caseEta}$$

$$\frac{\Gamma \vdash t : \Box_r A \qquad r \vdash p_i : A \rhd \Delta_i \qquad \Delta_i \vdash p_i : A \qquad 1 \sqsubseteq r}{\Gamma \vdash \textbf{case}\ t\ \textbf{of}\ \overline{[p_i] \to p_i} \equiv \textbf{case}\ t\ \textbf{of}\ [x] \to x : A}\ \textsc{Equiv\_caseGen}$$

$$\frac{\Gamma \vdash t : A \qquad - \vdash p_i : A \rhd \Delta_i \qquad \Gamma', \Delta_i' \vdash t_i : B \qquad - \vdash p_i' : B \rhd \Delta_i' \qquad \Gamma'', \Delta_i' \vdash t_i' : W}{\Gamma + \Gamma' + \Gamma'' \vdash \textbf{case}\ (\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i})\ \textbf{of}\ \overline{p_i' \to t_i'} \equiv \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to (\textbf{case}\ t_i\ \textbf{of}\ \overline{p_i' \to t_i'})} : W}\ \textsc{Equiv\_caseAssoc}$$

$$\frac{\begin{array}{c}\Gamma \vdash t : A \qquad - \vdash p_i : A \rhd \Delta_i \qquad \Gamma', \Delta_i' \vdash t_i : B \\ r \vdash p_i' : B \rhd \Delta_i' \qquad \Gamma'', \Delta_i' \vdash t_i' : W \qquad \mathsf{lin}(p)\end{array}}{r \cdot (\Gamma + \Gamma') + \Gamma'' \vdash \textbf{case}\ [\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i}]\ \textbf{of}\ \overline{[p_i'] \to t_i'} \equiv \textbf{case}\ [t]\ \textbf{of}\ \overline{[p_i] \to \textbf{case}\ [t_i]\ \textbf{of}\ \overline{[p_i'] \to t_i'}} : W}\ \textsc{Equiv\_caseBoxAssoc}$$

$$\frac{\Gamma_1 \vdash t : A \qquad - \vdash p_i : A \rhd \Delta_i \qquad \Gamma_2, \Delta_i \vdash t_i : B \qquad \Gamma_3 \vdash f : B \multimap W}{\Gamma_1 + \Gamma_2 + \Gamma_3 \vdash f\,(\textbf{case}\ t\ \textbf{of}\ \overline{p_i \to t_i}) \equiv \textbf{case}\ t\ \textbf{of}\ \overline{p_i \to (f\, t_i)} : W}\ \textsc{Equiv\_caseDist}$$

In Equiv_caseAssoc the predicate $\mathsf{lin}(p)$ classifies those patterns which are *linear*, which are those which are variables or constructor patterns only.

B.2.1.1 *Derived Rules*

Proposition B.2.1 ('Case push' property).

$$\frac{\Gamma \vdash t : A \qquad r \vdash p_i : A \rhd \Delta_i \qquad \Gamma', \Delta_i \vdash t_i : B}{s \cdot r \cdot \Gamma + s \cdot \Gamma' \vdash [\textbf{\textit{case}}\ t\ \textbf{\textit{of}}\ \overline{[p_i] \to t_i}] \equiv \textbf{\textit{case}}\ [t]\ \textbf{\textit{of}}\ \overline{[p_i] \to [t_i]} : \Box_s B}\ \textsc{Equiv\_casePushDerived}$$

Proof. Applying β_{case} and congruence over promotion, to the left-hand side of the case push equation yields:

$$[\textbf{case}\ t\ \textbf{of}\ [p_i] \to t_i] = [(t \rhd p_j)t_j]$$

for the smallest j. Applying β_{case} to the right-hand side of the case push equation yields:

$$\textbf{case}\ [t]\ \textbf{of}\ [p_i] \to [t_i] = ([t] \rhd [p_i])[t_j]$$

for the same smallest j (since the patterns pi are the same).

By PATSEMUNBOX, then we have the derivation of pattern matching:

$$\frac{(t \rhd p_i)[t_j] = t''}{([t] \rhd [p_i])[t_j] = t''}\ \textsc{patSemunbox}$$

therefore **case** $[t]$ **of** $[p_i] \to [t_i] = ([t] \rhd [p_i])[t_j] = (t \rhd p_i)[t_j]$.

Then by Proposition B.2.2 (below), $(t \rhd p_i)[t_j] = [(t \rhd p_j)t_j]$, yielding case push. □

Proposition B.2.2 (Pattern matching distributes with promotion)**.** *For all t, p, t' then:*

$$(t \rhd p)[t'] = [(t \rhd p)t']$$

Proof. By induction on syntactic pattern matching:

- (wild) $(t \rhd \_)[t'] = [t']$ and $[(t \rhd \_)t'] = [t']$.
- (var) $(t \rhd x)[t'] = [t/x][t'] = [[t/x]t']$ and $[(t \rhd x)t'] = [[t/x]t']$
- (unbox)

  $$\frac{(t \rhd p)t' = t''}{([t] \rhd [p])t' = t''}\ \textsc{patSemunbox}$$

  By induction then $(t \rhd p)[t'] = [(t \rhd p)t']$ therefore $([t] \rhd [p])[t'] = [([t] \rhd [p])t']$ since this rule preserves its result in the conclusion.
- (constr)

  $$\frac{(t_i \rhd p_i)t_i = t_{i+1}}{(C\, t_0 \mathinner{..} t_n \rhd C\, p_0 \mathinner{..} p_n)t_0 = t_{n+1}}\ \textsc{patSemconstr}$$

  By induction, similarly to the above case, but across multiple terms.

□

B.2.2 *Functor Derivation*

Definition B.2.1 (Deriving functor). Given a function $f : \alpha \multimap \beta$ then there is a function $\llbracket \mathsf{F}\overline{\alpha} \rrbracket_{\mathsf{fmap}}(f) : \mathsf{F}\ \alpha \multimap \mathsf{F}\ \beta$ derived from the type $\mathsf{F}\overline{\alpha}$ as follows:

$$
\begin{array}{lcl}
\llbracket 1 \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ () \rightarrow () \\
\llbracket \alpha \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & f\ z \\
\llbracket X \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & (\Sigma(X)\ f)\ z \\
\llbracket \Box_r A \rrbracket_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ [y] \rightarrow [\llbracket A \rrbracket_{\mathsf{fmap}}(f)\ y] \\
\llbracket A \oplus B \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ \ \mathsf{inl}\ x \rightarrow \mathsf{inl}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ x; \\
 & & \qquad\qquad\quad \mathsf{inr}\ y \rightarrow \mathsf{inr}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ y \\
\llbracket A \otimes B \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{case}\ z\ \textbf{of}\ (x, y) \rightarrow (\llbracket A \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ x, \llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ y) \\
\llbracket A \multimap B \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \lambda x. \llbracket B \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ (z\ x) \\
\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{fmap}}(f)\ z & = & \textbf{letrec}\ g = \llbracket A \rrbracket^{\Sigma, X \mapsto g : (\alpha \multimap \beta) \multimap \mu X.A \multimap (\mu X.A)\overrightarrow{[\alpha/\beta]}}_{\mathsf{fmap}}(f)\ \textbf{in}\ g\ z
\end{array}
$$

B.2.3 *Type Soundness of* push *and* pull

The following shows that the calculation of *push* and *pull* distributive laws is well-typed.

Proposition 1. *Type soundness of* $\llbracket \mathsf{F}\ \overline{\alpha_i} \rrbracket_{\mathsf{pull}} : \mathsf{F}\ (\overline{\Box_{r_i}\alpha_i}) \rightarrow \Box_{\bigwedge_{i=1}^{n} r_i}(\mathsf{F}\ \overline{\alpha_i})$.

Proof.

- $\llbracket 1 \rrbracket^{\Sigma}_{\mathsf{pull}} : 1 \rightarrow \Box_{(\bigwedge_{i=1}^{n} r_i)} 1$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = 1$).

$$
\dfrac{
\dfrac{\dfrac{}{\varnothing \vdash () : 1}\ \textsc{con}}{\varnothing \vdash [()] : \Box_{\bigwedge_{i=1}^{n} r_i} 1}\ \textsc{pr}
\qquad
\dfrac{|1 = 1|}{- \vdash () : 1 \rhd \varnothing}\ \textsc{Pcon}
}{
z : 1 \vdash \textbf{case}\ z\ \textbf{of}\ () \rightarrow [()] : \Box_{\bigwedge_{i=1}^{n} r_i} 1
}\ \textsc{case}
$$

- $\llbracket X \rrbracket^{\Sigma}_{\mathsf{pull}} : X \rightarrow \Box_{(\bigwedge_{i=1}^{n} r_i)} X$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = X$).

$$
\dfrac{
\dfrac{X : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A) \in \Sigma}{\Sigma \vdash \Sigma(X)\ : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}\ \textsc{lookup}
\qquad
\dfrac{}{z : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash z : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\ \textsc{var}
}{
\Sigma, z : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \Sigma(X)\ z : \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)
}\ \textsc{app}
$$

- $\llbracket \alpha_j \rrbracket_{\mathsf{pull}} : \Box_{r_j}\alpha_j \rightarrow \Box_{(\bigwedge_{i=1}^{n} r_i)}\alpha_j$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \alpha$).

$$
\dfrac{
\dfrac{}{\Sigma, z : \Box_{r_j}\alpha_j \vdash z : \Box_{r_j}\alpha_j}\ \textsc{var}
}{
\Sigma, z : \Box_{r_j}\alpha_j \vdash z : \Box_{(\bigwedge_{i=1}^{n} r_i)}\alpha_j
}\ \textsc{approx}
$$

- $\llbracket (A \oplus B)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rrbracket^{\Sigma}_{\mathsf{pull}} : A \oplus B \to \Box_{(\bigwedge_{i=1}^{n} r_i)} A \oplus B$ (i.e. $\mathsf{F}\,\overline{\alpha_i} = A \oplus B$).

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x' : A \vdash x' : A}\,\textsc{var}}{x' :_1 A \vdash x' : A}\,\textsc{der}}{x' :_1 A \vdash \mathbf{inl}\,(x') : A \oplus B}\,\textsc{Con}}{x' :_{\bigwedge_{i=1}^{n} r_i} A \vdash [\mathbf{inl}\,(x')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\textsc{pr} \tag{B.1}$$

$$\cfrac{\cfrac{\cfrac{}{\emptyset \vdash \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}} : A \multimap \Box_{\bigwedge_{i=1}^{n} r_i} A}\,\textsc{pull} \qquad \cfrac{}{x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\textsc{var}}{x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}(x) : \Box_{\bigwedge_{i=1}^{n} r_i} A}\,\textsc{app} \qquad (B.1) \qquad \cfrac{\cfrac{}{\bigwedge_{i=1}^{n} r_i \vdash x' : A \rhd x' :_{\bigwedge_{i=1}^{n} r_i} A}\,[\textsc{pvar}]}{- \vdash [x'] : \Box_{\bigwedge_{i=1}^{n} r_i} A \rhd x' :_{\bigwedge_{i=1}^{n} r_i} A}\,[\textsc{pbox}]}{x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}(x)\ \mathbf{of}\ [x'] \to [\mathbf{inl}\,(x')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\textsc{case} \tag{B.2}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{y' : B \vdash y' : B}\,\textsc{var}}{y' :_1 B \vdash y' : B}\,\textsc{der}}{y' :_1 B \vdash \mathbf{inr}\,(y') : A \oplus B}\,\textsc{Con}}{y' :_{\bigwedge_{i=1}^{n} r_i} B \vdash [\mathbf{inr}\,(y')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\textsc{pr} \tag{B.3}$$

$$\cfrac{\cfrac{\cfrac{}{\emptyset \vdash \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}} : B \multimap \Box_{\bigwedge_{i=1}^{n} r_i} B}\,\textsc{pull} \qquad \cfrac{}{y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\textsc{var}}{y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \llbracket B \rrbracket_{\mathsf{pull}}(y) : \Box_{\bigwedge_{i=1}^{n} r_i} B}\,\textsc{app} \qquad (B.3) \qquad \cfrac{\cfrac{}{\bigwedge_{i=1}^{n} r_i \vdash y' : B \rhd y' :_{\bigwedge_{i=1}^{n} r_i} B}\,[\textsc{pvar}]}{- \vdash [y'] : \Box_{\bigwedge_{i=1}^{n} r_i} B \rhd y' :_{\bigwedge_{i=1}^{n} r_i} B}\,[\textsc{pbox}]}{y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}(y)\ \mathbf{of}\ [y'] \to [\mathbf{inr}\,(x_2')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\textsc{case} \tag{B.4}$$

$$\cfrac{\cfrac{}{- \vdash x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rhd x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\textsc{pvar}}{- \vdash \mathbf{inl}\,(x) : (A \oplus B)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rhd x : A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\textsc{Pcon} \tag{B.5}$$

$$\cfrac{\cfrac{}{- \vdash y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rhd y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\textsc{pvar}}{- \vdash \mathbf{inr}\,(x) : (A \oplus B)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rhd y : B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\,\textsc{Pcon} \tag{B.6}$$

$$\cfrac{(B.2) \qquad (B.4) \qquad\qquad (B.5) \qquad (B.6)}{z : A \oplus B\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash \mathbf{case}\ z\ \mathbf{of}\ \mathbf{inl}\,(x) \to (\mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}(x)\ \mathbf{of}\ [x'] \to [\mathbf{inl}\,(x')]); \mathbf{inr}\,(y) \to (\mathbf{case}\ \llbracket B \rrbracket^{\Sigma}_{\mathsf{pull}}(y)\ \mathbf{of}\ [y'] \to [\mathbf{inr}\,(y')]) : \Box_{\bigwedge_{i=1}^{n} r_i} A \oplus B}\,\textsc{case}$$

- $[\![A \otimes B]\!]^{\Sigma}_{\mathsf{pull}} : (A \otimes B)[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \to \Box_{(\bigwedge_{i=1}^{n} r_i)}(A \otimes B)$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = A \otimes B$).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x' : A \vdash x' : A}\textsc{var}}{x' :_1 A \vdash x' : A}\textsc{der} \qquad \dfrac{\dfrac{}{y' : B \vdash y' : B}\textsc{var}}{y' :_1 B \vdash y' : B}\textsc{der}}{x' :_1 A, y' :_1 B \vdash (x', y') : A \otimes B}\textsc{Con}}{x' :_{\bigwedge_{i=1}^{n} r_i} A, y' :_{\bigwedge_{i=1}^{n} r_i} B \vdash [(x', y')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \otimes B}\textsc{pr} \tag{B.7}$$

$$\dfrac{\dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash x' : A \rhd x' :_{\bigwedge_{i=1}^{n} r_i} A}[\textsc{pvar}]}{- \vdash [x'] : \Box_{\bigwedge_{i=1}^{n} r_i} A \rhd x' :_{\bigwedge_{i=1}^{n} r_i} A}[\textsc{pbox}] \qquad \dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash y' : B \rhd y' :_{\bigwedge_{i=1}^{n} r_i} B}[\textsc{pvar}] \quad |A \otimes B| = 1}{- \vdash [y'] : \Box_{\bigwedge_{i=1}^{n} r_i} B \rhd y' :_{\bigwedge_{i=1}^{n} r_i} B}[\textsc{pbox}]}{- \vdash ([x'], [y']) : (\Box_{\bigwedge_{i=1}^{n} r_i} A) \otimes (\Box_{\bigwedge_{i=1}^{n} r_i} B) \rhd x' :_{\bigwedge_{i=1}^{n} r_i} A, y' :_{\bigwedge_{i=1}^{n} r_i} B}[\textsc{Pcon}] \tag{B.8}$$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}} : A \multimap \Box_{\bigwedge_{i=1}^{n} r_i} A}\textsc{pull} \qquad \dfrac{}{x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\textsc{var}}{x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}}(x) : \Box_{\bigwedge_{i=1}^{n} r_i} A}\textsc{app} \tag{B.9}$$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{pull}} : B \multimap \Box_{\bigwedge_{i=1}^{n} r_i} B}\textsc{pull} \qquad \dfrac{}{y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\textsc{var}}{y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash [\![B]\!]^{\Sigma}_{\mathsf{pull}}(y) : \Box_{\bigwedge_{i=1}^{n} r_i} B}\textsc{app} \tag{B.10}$$

$$\dfrac{\dfrac{\text{(B.9)} \qquad \text{(B.10)}}{x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}], y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash ([\![A]\!]^{\Sigma}_{\mathsf{pull}}(x), [\![B]\!]^{\Sigma}_{\mathsf{pull}}(y)) : (\Box_{\bigwedge_{i=1}^{n} r_i} A) \otimes (\Box_{\bigwedge_{i=1}^{n} r_i} B)}\textsc{pair} \qquad \text{(B.7)} \qquad \text{(B.8)}}{x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}], y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash \mathbf{case}\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}} x, [\![B]\!]^{\Sigma}_{\mathsf{pull}} y)\ \mathbf{of}\ ([x'], [y']) \to [(x', y')] : \Box_{\bigwedge_{i=1}^{n} r_i} A \otimes B}\textsc{case} \tag{B.11}$$

$$\dfrac{\dfrac{\dfrac{}{- \vdash x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \rhd x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\textsc{pvar} \qquad \dfrac{}{- \vdash y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \rhd y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\textsc{pvar}}{- \vdash (x, y) : (A \otimes B)[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \rhd x : A[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}], y : B[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}]}\textsc{pcon} \qquad \text{(B.11)}}{z : (A \otimes B)[\overrightarrow{\Box_{r_i}\alpha_i/\alpha_i}] \vdash \mathbf{case}\ z\ \mathbf{of}\ (x, y) \to (\mathbf{case}\ ([\![A]\!]^{\Sigma}_{\mathsf{pull}} x, [\![B]\!]^{\Sigma}_{\mathsf{pull}} y)\ \mathbf{of}\ ([x'], [y']) \to [(x', y')]) : \Box_{\bigwedge_{i=1}^{n} r_i} A \otimes B}\textsc{case}$$

- $[\![\mu X.A]\!]^{\Sigma}_{\mathsf{pull}} : (\mu X.A)\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \rightarrow \Box_{(\bigwedge_{i=1}^{n} r_i)}(\mu X.A)$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \mu X.A$).

$$\dfrac{}{\Sigma, f : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A) \vdash f : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}\ \textsc{var}$$

$$\dfrac{\dfrac{}{\Sigma, z : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash z : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}}\ \textsc{var}}{\Sigma, f : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A), z : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \vdash f\, z : \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}\ \textsc{app} \tag{B.12}$$

$$\dfrac{\dfrac{}{\Sigma \vdash [\![A]\!]^{\Sigma, X \mapsto f \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}} : \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}\ \textsc{pull} \qquad (\text{B.25})}{\Sigma, z : (\mu X.A)\overrightarrow{[r_i\,\alpha_i/\alpha_i]} \vdash \textbf{letrec}\, f \ = \ [\![A]\!]^{\Sigma, X \mapsto f \mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}} \ \textbf{in}\, f\, z : \Box_{\bigwedge_{i=1}^{n} r_i}\mu X.A}\ \textsc{letrec}$$

□

Proposition 2. *Type soundness of* $[\![\mathsf{F}\ \overline{\alpha_i}]\!]^{\Sigma}_{\mathsf{push}}$: $[\![\mathsf{F}\overline{\alpha_i}]\!]^{\Sigma}_{\mathsf{push}} : \Box_r \mathsf{F}\overline{\alpha_i} \rightarrow \mathsf{F}(\overline{\Box_r\alpha_i})$.

Proof.

- $[\![1]\!]^{\Sigma}_{\mathsf{push}} : \Box_r 1 \rightarrow 1$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = 1$).

$$\dfrac{\dfrac{}{\varnothing \vdash () : 1}\ \textsc{con} \qquad \dfrac{\dfrac{|1| = 1}{r \vdash\ () : 1 \rhd \varnothing}\ [\textsc{pcon}]}{- \vdash\ [()] : 1 \rhd \varnothing}\ [\textsc{pbox}]}{z : \Box_r 1 \vdash \textbf{case}\ z\ \textbf{of}\ [()] \rightarrow () : 1}\ \textsc{case}$$

- $[\![X]\!]^{\Sigma}_{\mathsf{push}} : \Box_r X \rightarrow X\overrightarrow{[\Box_r\alpha_i/\alpha_i]}$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = X$).

$$\dfrac{\dfrac{X : \Box_r(\mu X.A) \multimap (\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]} \in \Sigma}{\Sigma \vdash \Sigma(X)\ : \Box_r(\mu X.A) \multimap (\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}\ \textsc{lookup} \qquad \dfrac{}{z : \Box_r(\mu X.A) \vdash z : \Box_r(\mu X.A)}\ \textsc{var}}{\Sigma, z : \Box_r(\mu X.A) \vdash \Sigma(X)\ \ z : (\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}\ \textsc{app}$$

- $[\![\alpha_j]\!]^{\Sigma}_{\mathsf{push}} : \Box_{r_j}\alpha_j \rightarrow \Box_{r_j}\alpha_j$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \alpha$).

$$\dfrac{}{z : \Box_{r_j}\alpha_j \vdash z : \Box_{r_j}\alpha_j}\ \textsc{var}$$

- $[\![A \oplus B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r(A \oplus B) \rightarrow (A\overrightarrow{[\Box_r\alpha_i/\alpha_i]} \oplus B\overrightarrow{[\Box_r\alpha_i/\alpha_i]})$

$$\frac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}} : \Box_r A \multimap A[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{push}$$

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}\,\textsc{var}}{x :_1 A \vdash x : A}\,\textsc{der}}{x :_r A \vdash [x] : \Box_r A}\,\textsc{Pr}}{x :_r A \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]) : A[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{app}}{x :_r A \vdash \mathbf{inr}\, [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]) : A[\overrightarrow{\Box_r\alpha_i/\alpha_i}] \oplus B[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{Con} \tag{B.13}$$

$$\frac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r B \multimap B[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{push}$$

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{y : B \vdash y : B}\,\textsc{var}}{y :_1 B \vdash y : B}\,\textsc{der}}{y :_r B \vdash [y] : \Box_r B}\,\textsc{Pr}}{y :_r B \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : B[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{app}}{y :_r B \vdash \mathbf{inr}\, [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : A[\overrightarrow{\Box_r\alpha_i/\alpha_i}] \oplus B[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{Con} \tag{B.14}$$

$$\frac{\dfrac{\dfrac{}{r \vdash\ x : A \rhd x :_r A}\,[\textsc{Pvar}] \qquad |A \oplus B| > 1 \Rightarrow 1 \sqsubseteq r}{r \vdash\ \mathbf{inl}\,(x) : A \oplus B \rhd x :_r A}\,[\textsc{Pcon}]}{- \vdash\ [\mathbf{inl}\,(x)] : \Box_r A \oplus B \rhd x :_r A}\,[\textsc{Pbox}] \tag{B.15}$$

$$\frac{\dfrac{\dfrac{}{r \vdash\ y : B \rhd y :_r B}\,[\textsc{Pvar}] \qquad |A \oplus B| > 1 \Rightarrow 1 \sqsubseteq r}{r \vdash\ \mathbf{inr}\,(y) : A \oplus B \rhd y :_r B}\,[\textsc{Pcon}]}{- \vdash\ [\mathbf{inr}\,(y)] : \Box_r A \oplus B \rhd y :_r B}\,[\textsc{Pbox}] \tag{B.16}$$

$$\frac{(\text{B.13}) \qquad (\text{B.14}) \qquad (\text{B.15}) \qquad (\text{B.16})}{z : \Box_r(A \oplus B) \vdash \mathbf{case}\ z\ \mathbf{of}\ [\mathbf{inl}\,(x)] \to \mathbf{inl}\, [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]); [\mathbf{inr}\,(y)] \to \mathbf{inr}\, [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : A[\overrightarrow{\Box_r\alpha_i/\alpha_i}] \oplus B[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{Case}$$

- $[\![A \otimes B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r(A \otimes B) \to (A[\overrightarrow{\Box_r\alpha_i/\alpha_i}] \otimes B[\overrightarrow{\Box_r\alpha_i/\alpha_i}])$

$$\frac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}} : \Box_r A \multimap A[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{push} \qquad \dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}\,\textsc{var}}{x :_1 A \vdash x : A}\,\textsc{der}}{x :_r A \vdash [x] : \Box_r A}\,\textsc{pr}}{x :_r A \vdash [\![A]\!]^{\Sigma}_{\mathsf{push}}([x]) : A[\overrightarrow{\Box_r\alpha_i/\alpha_i}]}\,\textsc{app} \tag{B.17}$$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r B \multimap B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{push} \qquad \dfrac{\dfrac{\dfrac{}{y : B \vdash y : B}\;\textsc{var}}{y :_1 B \vdash y : B}\;\textsc{der}}{y :_r B \vdash [y] : \Box_r B}\;\textsc{pr}}{y :_r B \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}}([y]) : B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{app} \tag{B.18}$$

$$\dfrac{(\text{B.17}) \qquad (\text{B.18})}{x :_r A, y :_r B \vdash ([\![A]\!]^{\Sigma}_{\mathsf{push}}([x]), [\![B]\!]^{\Sigma}_{\mathsf{push}}([y])) : A[\overrightarrow{\Box_r \alpha_i/\alpha_i}] \otimes B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{Con} \tag{B.19}$$

$$\dfrac{\dfrac{\dfrac{}{r \vdash\ x : A \rhd x :_r A}\;[\textsc{PVar}] \qquad \dfrac{}{r \vdash\ y : B \rhd y :_r B}\;[\textsc{PVar}] \qquad |A \otimes B| = 1}{r \vdash\ (x, y) : A \otimes B \rhd x :_r A, y :_r B}\;[\textsc{PCon}]}{- \vdash\ [(x, y)] : \Box_r A \otimes B \rhd x :_r A, y :_r B}\;[\textsc{PBox}] \tag{B.20}$$

$$\dfrac{(\text{B.19}) \qquad (\text{B.20})}{z : \Box_r(A \otimes B) \vdash \textbf{case}\ z\ \textbf{of}\ [(x, y)] \to ([\![A]\!]^{\Sigma}_{\mathsf{push}}([x]), [\![B]\!]^{\Sigma}_{\mathsf{push}}([y])) : A[\overrightarrow{\Box_{r_i} \alpha_i/\alpha_i}] \otimes B[\overrightarrow{\Box_{r_i} \alpha_i/\alpha_i}]}\;\textsc{Case}$$

- $[\![A \multimap B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r(A \multimap B) \to (A[\overrightarrow{\Box_r \alpha_i/\alpha_i}] \multimap B[\overrightarrow{\Box_{r_i} \alpha_i/\alpha_i}])$

$$\dfrac{\dfrac{}{\emptyset \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}} : \Box_r B \multimap B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{push} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{f : A \multimap B \vdash f : A \multimap B}\;\textsc{var}}{f :_1 A \multimap B \vdash f : A \multimap B}\;\textsc{der} \qquad \dfrac{\dfrac{}{x : A \vdash x : A}\;\textsc{var}}{x :_1 A \vdash x : A}\;\textsc{der}}{f :_1 A \multimap B, x :_1 A \vdash f\,x : B}\;\textsc{app}}{f :_r A \multimap B, x :_{\bigwedge_{i=1}^{n} r_i} A \vdash [f\,x] : \Box_r B}\;\textsc{Pr}}{f :_r A \multimap B, x :_{\bigwedge_{i=1}^{n} r_i} A \vdash [\![B]\!]^{\Sigma}_{\mathsf{push}}[f\,x] : B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{app} \tag{B.21}$$

$$\dfrac{\dfrac{}{\bigwedge_{i=1}^{n} r_i \vdash\ x : A \rhd x :_{\bigwedge_{i=1}^{n} r_i} A}\;[\textsc{PVar}]}{- \vdash\ [x] : \Box_r A \rhd x :_{\bigwedge_{i=1}^{n} r_i} A}\;[\textsc{PBox}] \tag{B.22}$$

$$\dfrac{\dfrac{\dfrac{}{\emptyset \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}} : A \multimap \Box_{\bigwedge_{i=1}^{n} r_i} A}\;\textsc{pull} \qquad \dfrac{}{y : A \vdash y : A}\;\textsc{var}}{y : A \vdash [\![A]\!]^{\Sigma}_{\mathsf{pull}}(y) : \Box_{\bigwedge_{i=1}^{n} r_i} A}\;\textsc{app} \qquad (\text{B.21}) \qquad (\text{B.22})}{y : A, f :_r A \multimap B \vdash \textbf{case}\ [\![A]\!]^{\Sigma}_{\mathsf{pull}}(y)\ \textbf{of}\ [x] \to [\![B]\!]^{\Sigma}_{\mathsf{push}}[f\,x] : B[\overrightarrow{\Box_r \alpha_i/\alpha_i}]}\;\textsc{Case}$$

(B.23)

$$\dfrac{\dfrac{}{r \vdash f : (A \multimap B) \rhd f :_r A \multimap B}\,[\textsc{PVar}]}{- \vdash [f] : \Box_r(A \multimap B) \rhd f :_r A \multimap B}\,[\textsc{PBox}] \tag{B.24}$$

$$\dfrac{\dfrac{(\text{B.23}) \qquad (\text{B.24})}{z : \Box_r(A \multimap B), y : A \vdash \mathbf{case}\ z\ \mathbf{of}\ [f] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}(y)\ \mathbf{of}\ [x] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[f\,x] : B\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}\,\textsc{Case}}{z : \Box_r(A \multimap B) \vdash \lambda y.\mathbf{case}\ z\ \mathbf{of}\ [f] \to \mathbf{case}\ \llbracket A \rrbracket^{\Sigma}_{\mathsf{pull}}(y)\ \mathbf{of}\ [x] \to \llbracket B \rrbracket^{\Sigma}_{\mathsf{push}}[f\,x] : A\overrightarrow{[\Box_r\alpha_i/\alpha_i]} \multimap B\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}\,\text{ABS}$$

- $\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}} : (\mu X.\Box_r A) \to (\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]})$ (i.e. $\mathsf{F}\ \overline{\alpha_i} = \mu X.A$).

$$\dfrac{\dfrac{}{\Sigma, f : \mu X.\Box_r A \multimap (\mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]}) \vdash f : \mu X.\Box_r A \multimap (\mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]})}\,\text{VAR} \qquad \dfrac{}{\Sigma, z : \mu X.\Box_r A \vdash z : \mu X.\Box_r A}\,\text{VAR}}{\Sigma, f : \mu X.\Box_r A \multimap (\mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]}), z : \mu X.\Box_r A \vdash f\,z : (\mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]})}\,\text{APP} \tag{B.25}$$

$$\dfrac{\dfrac{}{\Sigma \vdash \llbracket A \rrbracket^{\Sigma, X \mapsto f\,\mu X.\Box_r A \multimap (\mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]})}_{\mathsf{push}} : \mu X.\Box_r A \multimap (\mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]})}\,\text{PUSH} \qquad (\text{B.25})}{\Sigma, z : (\mu X.\Box_r A) \vdash \mathbf{letrec}\, f \ = \ \llbracket A \rrbracket^{\Sigma, X \mapsto f\,\mu X.\Box_r A \multimap (\mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]})}_{\mathsf{push}}\ \mathbf{in}\ f\,z : \mu X.A\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}\,\text{LETREC}$$

□

B.2.4 *Inverse Property of the Distributive Laws*

Proposition 4.2.1 (Pull is right inverse to push)**.** *For all n-arity types* F *which do not contain function types, then for type variables $(\alpha_i)_{i \in [1..n]}$ and for all grades $r \in \mathcal{R}$ where $1 \sqsubseteq r$ if $|\mathsf{F}\overline{\alpha_i}| > 1$, then:*

$$\llbracket \mathsf{F}\ \overline{\alpha_i} \rrbracket_{\mathsf{pull}}(\llbracket \mathsf{F}\ \overline{\alpha_i} \rrbracket_{\mathsf{push}}) = id \ : \Box_r \mathsf{F}\overline{\alpha_i} \multimap \Box_r \mathsf{F}\overline{\alpha_i}$$

Proof. By induction on the syntax of the type $\mathsf{F}\overline{\alpha_i}$ which we denote by T in the following. We first prove a subresult that for $\llbracket T \rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket T \rrbracket^{\Sigma}_{\mathsf{push}}\ z) \equiv z$, which by function extensionality then gives us $\llbracket T \rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket T \rrbracket^{\Sigma}_{\mathsf{push}}) \equiv id$, under the assumption that for all X, every $f \in \Sigma(X)$ and $g \in \Sigma'(X)$ then $g \circ f = id$, in order to apply the recursive argument.

- $T = 1$

$$\begin{array}{lll}
 & \llbracket 1 \rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket 1 \rrbracket^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & \llbracket 1 \rrbracket^{\Sigma'}_{\mathsf{pull}}(\mathbf{case}\ z\ \mathbf{of}\ [()] \to ()) & \{\mathit{defn.}\ \llbracket 1 \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ [()] \to ())\ \mathbf{of}\ () \to [()] & \{\mathit{defn.}\ \llbracket 1 \rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [()] \to \mathbf{case}\ ()\ \mathbf{of}\ () \to [()] & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ [()] \to [()] & \{\beta_{\mathit{case}}\} \\
\equiv & z & \{\eta_{\mathit{case}}\}
\end{array}$$

- $T = \alpha$

$$\begin{aligned}
& \llbracket\alpha\rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket\alpha\rrbracket^{\Sigma}_{\mathsf{push}}\ z) \\
\equiv\ & \llbracket\alpha\rrbracket^{\Sigma'}_{\mathsf{pull}}(z) && \{\textit{defn. } \llbracket\alpha\rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv\ & z && \{\textit{defn. } \llbracket\alpha\rrbracket^{\Sigma'}_{\mathsf{pull}}\}
\end{aligned}$$

- $T = X$

$$\begin{aligned}
& \llbracket X\rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket X\rrbracket^{\Sigma}_{\mathsf{push}}\ z) \\
\equiv\ & \llbracket X\rrbracket^{\Sigma'}_{\mathsf{pull}}(\Sigma(X)z) && \{\textit{defn. } \llbracket X\rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv\ & \Sigma'(X)(\Sigma(X)z) && \{\textit{defn. } \llbracket X\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv\ & z && \{\textit{recursion assumption}\}
\end{aligned}$$

- $T = A \oplus B$:

$$\begin{aligned}
& \llbracket A\oplus B\rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket A\oplus B\rrbracket^{\Sigma}_{\mathsf{push}}\ z) \\
\equiv\ & \llbracket A\oplus B\rrbracket^{\Sigma'}_{\mathsf{pull}}(\textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x];\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]\ ) && \{\textit{defn. } \llbracket A\oplus B\rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv\ & \ \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x];\ \ [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]\ )\ \textbf{of} && \{\textit{defn. } \llbracket A\oplus B\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
& \quad \mathsf{inl}\ x \to \textbf{case}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; \\
& \quad \mathsf{inr}\ y \to \textbf{case}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \textbf{case}\ \mathsf{inl}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x]\ \textbf{of}\ \ \mathsf{inl}\ x \to \textbf{case}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x\ \textbf{of}\ [u] \to [\mathsf{inl}\ u];\ ; && \{\textit{case assoc.}\} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \mathsf{inr}\ y \to \textbf{case}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] \\
& \qquad\qquad\quad [\mathsf{inr}\ y] \to \textbf{case}\ \mathsf{inr}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]\ \textbf{of}\ \ \mathsf{inl}\ x \to \textbf{case}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \mathsf{inr}\ y \to \textbf{case}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \textbf{case}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x]\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; && \{\beta_{case}\} \\
& \qquad\qquad\quad [\mathsf{inr}\ y] \to \textbf{case}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to \textbf{case}\ [x]\ \textbf{of}\ [u] \to [\mathsf{inl}\ u]; && \{\textit{induction}\} \\
& \qquad\qquad\quad [\mathsf{inr}\ y] \to \textbf{case}\ [y]\ \textbf{of}\ [v] \to [\mathsf{inr}\ v] \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ \ [\mathsf{inl}\ x] \to [\mathsf{inl}\ x];\ \ [\mathsf{inr}\ y] \to [\mathsf{inr}\ y] && \{\beta_{case}\} \\
\equiv\ & z && \{\eta_{case}\}
\end{aligned}$$

$T = A \otimes B$:

$$\begin{aligned}
& \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{push}}\ z) \\
\equiv\ & \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}(\textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y])) && \{\textit{defn. } \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv\ & \textbf{case}\ (\textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]))\ \textbf{of}\ (x,y) \to \\
& \qquad \textbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y)\ \textbf{of}\ ([u],[v]) \to [(u,v)] && \{\textit{defn. } \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to \textbf{case}\ (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y])\ \textbf{of}\ (x,y) \to \\
& \qquad\qquad\qquad\qquad \textbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y)\ \textbf{of}\ ([u],[v]) \to [(u,v)] && \{\textit{case assoc.}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\textbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}\ [x], \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}\ [y])\ \textbf{of}\ ([u],[v]) \to [(u,v)]) && \{\beta_{case}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to (\textbf{case}\ ([x],[y])\ \textbf{of}\ ([u],[v]) \to [(u,v)]) && \{\textit{induction}\} \\
\equiv\ & \textbf{case}\ z\ \textbf{of}\ [(x,y)] \to [(x,y)] && \{\beta_{case}\} \\
\equiv\ & z && \{\eta_{case}\}
\end{aligned}$$

$T = \mu X.A$:

$$\begin{array}{lll}
& \llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{pull}}(\llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}\ z) & \\
\equiv & \llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{pull}}(\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in}\ f\ z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f' : \mu X.A\overrightarrow{[\Box_{r_i} \alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in} & \\
& \quad f'(\mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in}\ f\ z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f' : \mu X.A\overrightarrow{[\Box_{r_i} \alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in} & \\
& \quad \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in}\ f'(f\ z) & \{\mathit{let\ dist.}\ \} \\
\equiv & \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f' : \mu X.A\overrightarrow{[\Box_{r_i} \alpha_i/\alpha_i]} \multimap \Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in} & \\
& \quad \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in}\ f'(\llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ z) & \{\beta_{letrec}\} \\
\equiv & \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ \mathbf{in} & \\
& \quad \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma, X \mapsto f : \mu X.\Box_r A \multimap (\mu X.A)\overrightarrow{[\Box_r \alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in}\ \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}(\llbracket A \rrbracket^{\Sigma, X \mapsto f}_{\mathsf{push}}\ z) & \{\beta_{letrec}\} \\
\equiv & \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ \mathbf{in} & \\
& \quad \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}(\llbracket A \rrbracket^{\Sigma, X \mapsto \mathbf{letrec}\ f\ =\ \llbracket A \rrbracket^{\Sigma, X \mapsto f}_{\mathsf{push}}\ \mathbf{in}\ f}_{\mathsf{push}}\ z) & \{\beta_{letrec}\} \\
\equiv & \llbracket A \rrbracket^{\Sigma', X \mapsto \mathbf{letrec}\ f'\ =\ \llbracket A \rrbracket^{\Sigma', X \mapsto f'}_{\mathsf{pull}}\ \mathbf{in}\ f'}_{\mathsf{pull}}(\llbracket A \rrbracket^{\Sigma, X \mapsto \mathbf{letrec}\ f\ =\ \llbracket A \rrbracket^{\Sigma, X \mapsto f}_{\mathsf{push}}\ \mathbf{in}\ f}_{\mathsf{push}}\ z) & \{\beta_{letrec}\} \\
\equiv & z & \{\mathit{induction}\}
\end{array}$$

$\Box$

Proposition 4.2.2 (Pull is left inverse to push). *For all n-arity types* F *which do not contain function types, then for type variables* $(\alpha_i)_{i \in [1..n]}$ *and for all grades* $r \in \mathcal{R}$ *where* $1 \sqsubseteq r$ *if* $|\mathsf{F}\overline{\alpha_i}| > 1$, *then:*

$$\llbracket \mathsf{F}\ \overline{\alpha_i} \rrbracket_{\mathsf{push}}(\llbracket \mathsf{F}\ \overline{\alpha_i} \rrbracket_{\mathsf{pull}}) = id\ : \mathsf{F}(\Box_r \overline{\alpha_i}) \multimap \mathsf{F}(\Box_r \overline{\alpha_i})$$

Proof. By induction on the syntax of the type $\mathsf{F}\overline{\alpha_i}$ which we denote by T in the following. The following proof is for $\llbracket T \rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket T \rrbracket^{\Sigma}_{\mathsf{pull}}\ z) \equiv z$, which by function extensionality then gives us $\llbracket T \rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket T \rrbracket^{\Sigma}_{\mathsf{pull}}) \equiv id$, under the assumption that for all X, every $f \in \Sigma(X)$ and $g \in \Sigma'(X)$ then $g \circ f = id$, in order to apply the recursive argument.

- $T = 1$

$$\begin{array}{lll}
& \llbracket 1 \rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket 1 \rrbracket^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & \llbracket 1 \rrbracket^{\Sigma}_{\mathsf{push}}(\mathbf{case}\ z\ \mathbf{of}\ () \to [()]) & \{\mathit{defn.}\ \llbracket 1 \rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ () \to [()])\ \mathbf{of}\ [()] \to () & \{\mathit{defn.}\ \llbracket 1 \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ () \to \mathbf{case}\ [()]\ \mathbf{of}\ [()] \to () & \{\mathit{case\ assoc.}\} \\
\equiv & \mathbf{case}\ z\ \mathbf{of}\ () \to () & \{\beta_{case}\} \\
\equiv & z & \{\eta_{case}\}
\end{array}$$

- $T = \alpha$

$$\begin{array}{lll} & \llbracket\alpha\rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket\alpha\rrbracket^{\Sigma'}_{\mathsf{pull}}\ z) & \\ \equiv & \llbracket\alpha\rrbracket^{\Sigma}_{\mathsf{push}}(z) & \{\mathit{defn.}\ \llbracket\alpha\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\ \equiv & z & \{\mathit{defn.}\ \llbracket\alpha\rrbracket^{\Sigma}_{\mathsf{push}}\} \end{array}$$

- $T = X$

$$\begin{array}{lll} & \llbracket X\rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket X\rrbracket^{\Sigma'}_{\mathsf{pull}}\ z) & \\ \equiv & \llbracket X\rrbracket^{\Sigma}_{\mathsf{push}}(\Sigma'(X)z) & \{\mathit{defn.}\ \llbracket X\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\ \equiv & \Sigma(X)(\Sigma'(X)z) & \{\mathit{defn.}\ \llbracket X\rrbracket^{\Sigma}_{\mathsf{push}}\} \\ \equiv & z & \{\mathit{recursion\ assumption}\} \end{array}$$

- $T = A \oplus B$

$$\begin{array}{llll} & \llbracket A\oplus B\rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket A\oplus B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ z) & & \\ \equiv & \llbracket A\oplus B\rrbracket^{\Sigma}_{\mathsf{push}}\mathbf{case}\ z\ \mathbf{of} & \mathsf{inl}\ x \to \mathbf{case}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]; & \{\mathit{defn.}\ \llbracket A\oplus B\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\ & & \mathsf{inr}\ y \to \mathbf{case}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] & \\ \equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of} & \mathsf{inl}\ x \to \mathbf{case}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u];\ )\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x]; & \{\mathit{defn.}\ \llbracket A\oplus B\rrbracket^{\Sigma}_{\mathsf{push}}\} \\ & & \mathsf{inr}\ y \to \mathbf{case}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v] \qquad\quad [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y] & \\ \equiv & \mathbf{case}\ z\ \mathbf{of} & \mathsf{inl}\ x \to \mathbf{case}\ \mathbf{case}\ \llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x\ \mathbf{of}\ [u] \to [\mathsf{inl}\ u]\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x];\ ; & \{\mathit{case\ assoc.}\} \\ & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y] & \\ & & \mathsf{inr}\ y \to \mathbf{case}\ \mathbf{case}\ \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y\ \mathbf{of}\ [v] \to [\mathsf{inr}\ v]\ \mathbf{of}\ \ [\mathsf{inl}\ x] \to \mathsf{inl}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x]; & \\ & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad [\mathsf{inr}\ y] \to \mathsf{inr}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y] & \\ \equiv & \mathbf{case}\ z\ \mathbf{of} & \mathsf{inl}\ x \to \mathsf{inl}\ \llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x; & \{\beta_{\mathit{case}}\} \\ & & \mathsf{inr}\ y \to \mathsf{inr}\ \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}\llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y & \\ \equiv & \mathbf{case}\ z\ \mathbf{of} & \mathsf{inl}\ x \to \mathsf{inl}\ x; & \{\mathit{induction}\} \\ & & \mathsf{inr}\ y \to \mathsf{inr}\ y & \\ \equiv & z & & \{\eta_{\mathit{case}}\} \end{array}$$

- $T = A \otimes B$

$$\begin{array}{lll} & \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ z) & \\ \equiv & \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{push}}(\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)]) & \{\mathit{defn.}\ \llbracket A\otimes B\rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\ \equiv & \mathbf{case}\ (\mathbf{case}\ z\ \mathbf{of}\ (x,y) \to & \\ & \quad \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to [(u,v)])\ \mathbf{of}\ [(x,y)] \to (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\mathit{defn.}\ \llbracket A\otimes B\rrbracket^{\Sigma}_{\mathsf{push}}\} \\ \equiv & \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to & \\ & \quad \mathbf{case}\ (\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y)\ \mathbf{of}\ ([u],[v]) \to \mathbf{case}\ [(u,v)]\ \mathbf{of}\ [(x,y)] \to (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}[x], \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}[y]) & \{\mathit{case\ assoc.}\} \\ \equiv & \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to (\llbracket A\rrbracket^{\Sigma}_{\mathsf{push}}\llbracket A\rrbracket^{\Sigma'}_{\mathsf{pull}}\ x, \llbracket B\rrbracket^{\Sigma}_{\mathsf{push}}\llbracket B\rrbracket^{\Sigma'}_{\mathsf{pull}}\ y) & \{\beta_{\mathit{case}}\} \\ \equiv & \mathbf{case}\ z\ \mathbf{of}\ (x,y) \to (x,y) & \{\mathit{induction}\} \\ \equiv & z & \{\eta_{\mathit{case}}\} \end{array}$$

- $T = \mu X.A$

$$
\begin{array}{lll}
& \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}(\llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{pull}}\ z) & \\
\equiv & \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}(\mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma',X\mapsto f':\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}\multimap\Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f'\ z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma'}_{\mathsf{pull}}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma,X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \qquad f\ (\mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma',X\mapsto f':\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}\multimap\Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f'\ z) & \{\mathit{defn.}\ \llbracket \mu X.A \rrbracket^{\Sigma}_{\mathsf{push}}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma,X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \qquad \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma',X\mapsto f':\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}\multimap\Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ (f'\ z) & \{\mathit{let\ dist.}\ \} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma,X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \qquad \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma',X\mapsto f':\mu X.A\overrightarrow{[\Box_{r_i}\alpha_i/\alpha_i]}\multimap\Box_{\bigwedge_{i=1}^{n} r_i}(\mu X.A)}_{\mathsf{pull}}\ \mathbf{in}\ f\ (\llbracket A \rrbracket^{\Sigma',X\mapsto f'}_{\mathsf{pull}}\ z) & \{\beta_{\mathit{letrec}}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma,X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \qquad \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma',X\mapsto f'}_{\mathsf{pull}}\ \mathbf{in}\ \llbracket A \rrbracket^{\Sigma,X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ (\llbracket A \rrbracket^{\Sigma',X\mapsto f'}_{\mathsf{pull}}\ z) & \{\beta_{\mathit{letrec}}\} \\
\equiv & \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma,X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ \mathbf{in} & \\
& \qquad \llbracket A \rrbracket^{\Sigma,X\mapsto f:\mu X.\Box_r A\multimap(\mu X.A)\overrightarrow{[\Box_r\alpha_i/\alpha_i]}}_{\mathsf{push}}\ (\llbracket A \rrbracket^{\Sigma',X\mapsto \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma',X\mapsto f'}_{\mathsf{pull}}\ \mathbf{in}\ f'}_{\mathsf{pull}}\ z) & \{\beta_{\mathit{letrec}}\} \\
\equiv & \llbracket A \rrbracket^{\Sigma,X\mapsto \mathbf{letrec}\ f = \llbracket A \rrbracket^{\Sigma,X\mapsto f}_{\mathsf{push}}\ \mathbf{in}\ f}_{\mathsf{push}}\ (\llbracket A \rrbracket^{\Sigma',X\mapsto \mathbf{letrec}\ f' = \llbracket A \rrbracket^{\Sigma',X\mapsto f'}_{\mathsf{pull}}\ \mathbf{in}\ f'}_{\mathsf{pull}}\ z) & \{\beta_{\mathit{letrec}}\} \\
\equiv & z & \{\mathit{induction}\}
\end{array}
$$

□

B.3 PROOFS FOR THE FULLY GRADED SYNTHESIS CALCULUS

This section contains the soundness proof for the fully graded synthesis calculus presented in Chapter 5, as well as the proof of soundness for focusing this calculus.

B.3.1 *Soundness of the Fully Graded Synthesis Calculus*

Theorem 5.2.1 (Soundness of synthesis). *Given a particular pre-ordered semiring $\mathcal{R}$ parametrising the calculi, then:*

1. *For all contexts Γ and Δ, types A, terms t:*

$$\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta \quad \Longrightarrow \quad \Sigma; \Delta \vdash t : A$$

   i.e. t has type A under context Δ whose grades capture variable use in t.

2. *At the top-level, for all type schemes $\forall \overline{\alpha : \kappa}.A$ and terms t then:*

$$\emptyset; \emptyset \vdash \forall \overline{\alpha : \kappa}.A \Rightarrow t \mid \emptyset \quad \Longrightarrow \quad \emptyset; \emptyset \vdash t : \forall \overline{\alpha : \kappa}.A$$

Proof. Induction on the synthesis rules. We consider the cases of the lemma in order, first proving soundness for synthesis of open terms from types, followed by soundness of synthesis for closed term from type schemes.

1. a) Case VAR
   For synthesis of a variable term, we have the derivation:

$$\frac{\Sigma \vdash A : \text{Type}}{\Sigma; \Gamma, x :_r A \vdash A \Rightarrow x \mid 0 \cdot \Gamma, x :_1 A}\ \textsc{Var}$$

From the premise, we have that:

$$\Sigma \vdash A : \text{Type}$$

from which we can construct the following typing derivation, matching the above conclusion:

$$\frac{\Sigma \vdash A : \text{Type}}{\Sigma; 0 \cdot \Gamma, x :_1 A \vdash x : A}\ \textsc{Var}$$

   b) Case DEF
   For synthesis of a top-level definition usage, we have the derivation:

$$\frac{(x : \forall \overline{\alpha : \kappa}.A') \in D \qquad \Sigma \vdash A = \text{inst}(\forall \overline{\alpha : \kappa}.A')}{\Sigma; \Gamma \vdash A \Rightarrow x \mid 0 \cdot \Gamma}\ \textsc{Def}$$

From the premise, we have that:

$$\Sigma \vdash A = \text{inst}(\forall \overline{\alpha : \kappa}.A')$$

from which we can construct the following typing derivation, matching the above conclusion:

$$\frac{(x : \forall \overline{\alpha : \kappa}.A') \in D \qquad \Sigma \vdash A = \text{inst}(\forall \overline{\alpha : \kappa}.A')}{\Sigma; 0 \cdot \Gamma \vdash x : A}\ \textsc{Def}$$

   c) Case $\rightarrow_{\text{R}}$
   For synthesis of an abstraction term, we have the derivation:

$$\frac{\Sigma; \Gamma, x :_q A \vdash B \Rightarrow t \mid \Delta, x :_r A \qquad r \sqsubseteq q}{\Sigma; \Gamma \vdash A^q \rightarrow B \Rightarrow \lambda x.t \mid \Delta}\ \rightarrow_{\text{R}}$$

By induction on the premise, we have:

$$\Sigma; \Delta, x :_r A \vdash t : B \qquad \text{(ih)}$$

and that:

$$r \sqsubseteq q$$

from which we can construct the following typing derivation, matching the conclusion:

$$\dfrac{\dfrac{\Sigma; \Delta, x :_r A \vdash t : B \qquad r \sqsubseteq q}{\Sigma; \Delta, x :_q A \vdash t : B}\textsc{Approx}}{\Sigma; \Delta \vdash \lambda x.t : A^q \to B}\textsc{Abs}$$

d) Case $\to_L$

For synthesising an application, we have the derivation:

$$\dfrac{\begin{array}{c}\Sigma; \Gamma, x_1 :_{r_1} A^q \to B, x_2 :_{r_1} B \vdash C \Rightarrow t_1 \mid \Delta_1, x_1 :_{s_1} A^q \to B, x_2 :_{s_2} B \\ \Sigma; \Gamma, x_1 :_{r_1} A^q \to B \vdash A \Rightarrow t_2 \mid \Delta_2, x_1 :_{s_3} A^q \to B \qquad \Sigma \vdash A^q \to B : \text{Type}\end{array}}{\Sigma; \Gamma, x_1 :_{r_1} A^q \to B \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + s_2 \cdot q \cdot \Delta_2), x_1 :_{s_2 + s_1 + (s_2 \cdot q \cdot s_3)} A^q \to B}\to_L$$

By induction on the premises, we obtain the following typing judgements:

$$\Sigma; \Delta_1, x_1 :_{s_1} A^q \to B, x_2 :_{s_2} B \vdash t_1 : C \qquad \text{(ih)}$$
$$\Sigma; \Delta_2, x_1 :_{s_3} A^q \to B \vdash t_2 : A \qquad \text{(ih)}$$

and from the premises, we have that:

$$\Sigma \vdash A^q \to B : \text{Type}$$

from which we can construct the following derivation, making use of the admissibility of substitution:

$$\dfrac{\dfrac{\Sigma \vdash A^q \to B : \text{Type}}{\Sigma; x_1 :_1 A^q \to B \vdash x_1 : A^q \to B}\textsc{Var} \qquad \Sigma; \Delta_2, x_1 :_{s_3} A^q \to B \vdash t_2 : A}{\Sigma; q \cdot \Delta_2, x_1 :_{1 + (q \cdot s_3)} A^q \to B \vdash x_1\, t_2 : B}\textsc{App} \tag{B.26}$$

$$\dfrac{\text{(B.26)} \qquad \Sigma; \Delta_1, x_1 :_{s_1} A^q \to B, x_2 :_{s_2} B \vdash t_1 : C}{\Sigma; (\Delta_1 + s_2 \cdot q \cdot \Delta_2), x_1 :_{s_2 + s_1 + (s_2 \cdot q \cdot s_3)} A^q \to B \vdash [(x_1\, t_2)/x_2]t_1 : C}\textsc{Subst}$$

making use of the distributivity property of semirings, along with unitality of 1 and commutativity of $+$, such that $s_1 + s_2 \cdot (1 + (q \cdot s_3)) = s_1 + (s_2 \cdot 1) + (s_2 \cdot q \cdot s_3) = s_2 + s_1 + (s_2 \cdot q \cdot s_3)$.

e) Case $\textsc{Con}_R$

For synthesising a constructor introduction, we have the derivation:

$$\dfrac{\begin{array}{l}(C : \forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A'}) \in D \\ \Sigma \vdash B_1{}^{q_1} \to \ldots \to B_n{}^{q_n} \to K\vec{A} = \text{inst}(\forall \overline{\alpha : \kappa}.B_1'^{\,q_1} \to \ldots \to B_n'^{\,q_n} \to K\vec{A'}) \\ \Sigma; \Gamma \vdash B_i \Rightarrow t_i \mid \Delta_i\end{array}}{\Sigma; \Gamma \vdash K\vec{A} \Rightarrow C\, t_1 \ldots t_n \mid 0 \cdot \Gamma + (q_1 \cdot \Delta_1) + \ldots + (q_n \cdot \Delta_n)}\mathrm{C_R}$$

By induction on the premises, we obtain the following typing judgements:

$$\Sigma;\Delta_1 \vdash t_1 : B_1 \quad ,\ldots, \quad \Sigma;\Delta_n \vdash t_n : B_n \tag{ih}$$

from which we can construct the following derivation, matching the above conclusion:

$$\dfrac{\begin{array}{c}(C : \forall\overline{\alpha : \kappa}.{B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K\vec{A}') \in D \\ \Sigma \vdash {B_1}^{q_1} \rightarrow \ldots \rightarrow {B_n}^{q_n} \rightarrow K\vec{A} = \mathrm{inst}(\forall\overline{\alpha : \kappa}.{B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K\vec{A}')\end{array}}{\Sigma;0\cdot\Gamma \vdash C : {B_1}^{q_1} \rightarrow \ldots \rightarrow {B_n}^{q_n} \rightarrow K\vec{A}}\ \textsc{Con} \tag{B.27}$$

$$\dfrac{\dfrac{(\text{B.27}) \qquad \Sigma;\Delta_1 \vdash t_1 : B_1}{\Sigma;0\cdot\Gamma + q_1\cdot\Delta_1 \vdash C\,t_1 : {B_2}^{q_1} \rightarrow \ldots \rightarrow {B_n}^{q_n} \rightarrow K\vec{A}}\ \textsc{App} \\ \vdots}{\Sigma;0\cdot\Gamma + q_1\cdot\Delta_1 + \ldots + q_{n-1}\cdot\Delta_{n-1} \vdash C\,t_1\ldots t_{n-1} : {B_n}^{q_n} \rightarrow K\vec{A}}\ \textsc{App} \tag{B.28}$$

$$\dfrac{(\text{B.28}) \qquad \Sigma;\Delta_n \vdash t_n : B_n}{\Sigma;0\cdot\Gamma + q_1\cdot\Delta_1 + \ldots + q_n\cdot\Delta_n \vdash C\,t_1\ldots t_n : K\vec{A}}\ \textsc{App}$$

f) Case $\textsc{Con}_\mathrm{L}$

For synthesising a case statement, we have the derivation:

$$\dfrac{\begin{array}{l}(C_i : \forall\overline{\alpha : \kappa}.{B'_1}^{q^i_1} \rightarrow \ldots \rightarrow {B'_n}^{q^i_n} \rightarrow K\vec{A}') \in D \qquad \Sigma \vdash K\vec{A} : \mathrm{Type} \\ \Sigma \vdash {B_1}^{q_1} \rightarrow \ldots \rightarrow {B_n}^{q_n} \rightarrow K\vec{A} = \mathrm{inst}(\forall\overline{\alpha : \kappa}.{B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K\vec{A}') \\ \Sigma;\Gamma, x :_r K\vec{A}, y^i_1 :_{r\cdot q^i_1} B_1, \ldots, y^i_n :_{r\cdot q^i_1} B_n \vdash B \Rightarrow t_i \mid \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, \ldots, y^i_n :_{s^i_n} B_n \\ \exists {s'}^i_j . s^i_j \sqsubseteq {s'}^i_j \cdot q^i_j \sqsubseteq r\cdot q^i_j \qquad s_i = {s'}^i_1 \sqcup \ldots \sqcup {s'}^i_n \qquad |K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq s_1 \sqcup \ldots \sqcup s_m\end{array}}{\Sigma;\Gamma, x :_r K\vec{A} \vdash B \Rightarrow \textbf{case}\ x\ \textbf{of}\ \overline{C_i\ y^i_1\ldots y^i_n \mapsto t_i} \mid (\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1\sqcup\ldots\sqcup r_m)+(s_1\sqcup\ldots\sqcup s_m)} K\vec{A}}\ \mathrm{C_L}$$

By induction on the premises we obtain the following typing judgements:

$$\Sigma;\Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, \ldots, y^i_n :_{s^i_n} B_n \vdash t_i : B \tag{ih}$$

We have by the definition of $\sqcup$:

i. $\Delta_i \sqsubseteq (\Delta_1 \sqcup \ldots \sqcup \Delta_m)$

ii. $r_i \sqsubseteq r_1 \sqcup \ldots \sqcup r_m$

and from the premises of the synthesis rule:

iii. ${s'}^i_j \sqsubseteq s_1 \sqcup \ldots \sqcup s_m$

iv. $s^i_j \sqsubseteq s'^i_j \cdot q^i_j$

v. $|KA| > 1 \Rightarrow 1 \sqsubseteq s_1 \sqcup \ldots \sqcup s_m$

vi. $\Sigma \vdash K\vec{A} : \mathsf{Type}$

and from rule $\kappa_{\rightarrow}$ we have that

$$\Sigma \vdash K\vec{A} : \mathsf{Type} \Rightarrow {B_1}^{q^i_1} \rightarrow \ldots \rightarrow {B_n}^{q^i_n} \rightarrow K\vec{A} : \mathsf{Type}$$

thus we have that

vii. $\Sigma \vdash B_j : \mathsf{Type}$

We then construct the following three derivations towards the goal:

$$\dfrac{(g)}{\Sigma; q^i_j \cdot s_1 \sqcup \ldots \sqcup s_m \vdash y^i_j : B_j \rhd y^i_j :_{q^i_j \cdot s_1 \sqcup \ldots \sqcup s_m} B_j}\ \textsc{PVar} \qquad \text{(B.29)}$$

$$\dfrac{\begin{array}{c}(C : \forall \overline{\alpha : \kappa}.{B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K\vec{A'}) \in D \\ \Sigma \vdash {B_1}^{q_1} \rightarrow \ldots \rightarrow {B_n}^{q_n} \rightarrow K\vec{A} = \mathsf{inst}(\forall \overline{\alpha : \kappa}.{B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K\vec{A'}) \\ (\text{B.29}) \qquad (e)\end{array}}{\Sigma; s_1 \sqcup \ldots \sqcup s_m \vdash C_i\ y^i_1 \ldots y^i_n : K\vec{A} \rhd y^i_j :_{q^i_j \cdot s_1 \sqcup \ldots \sqcup s_m} B_j, \ldots, y^i_n :_{q^i_n \cdot s_1 \sqcup \ldots \sqcup s_m} B_n}\ \textsc{PCon}$$

(B.30)

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Sigma; \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, \ldots, y^i_n :_{s^i_n} B_n \vdash t_i : B}\ \text{INDUCTION} \quad (d)}{\Sigma; \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{q^i_1 \cdot s'^i_j} B_1, \ldots, y^i_n :_{q^i_n \cdot s'^i_j} B_n \vdash t_i : B \quad (c)}\ \textsc{Approx}}{\Sigma; \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{q^i_1 \cdot s_1 \sqcup \ldots \sqcup s_m} B_1, \ldots, y^i_n :_{q^i_n \cdot s_1 \sqcup \ldots \sqcup s_m} B_n \vdash t_i : B \quad (b)}\ \textsc{Approx}}{\Sigma; \Delta_i, x :_{(r_1 \sqcup \ldots \sqcup r_m)} K\vec{A}, y^i_1 :_{q^i_1 \cdot s_1 \sqcup \ldots \sqcup s_m} B_1, \ldots, y^i_n :_{q^i_n \cdot s_1 \sqcup \ldots \sqcup s_m} B_n \vdash t_i : B \quad (a)}\ \textsc{Approx}}{\Sigma; (\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1 \sqcup \ldots \sqcup r_m)} K\vec{A}, y^i_1 :_{q^i_1 \cdot s_1 \sqcup \ldots \sqcup s_m} B_1, \ldots, y^i_n :_{q^i_n \cdot s_1 \sqcup \ldots \sqcup s_m} B_n \vdash t_i : B}\ \textsc{Approx}$$

(B.31)

$$\dfrac{\dfrac{\dfrac{(f)}{\Sigma; x :_1 K\vec{A} \vdash x : KA}\ \textsc{Var} \qquad (\text{B.30}) \qquad (\text{B.31})}{\Sigma; ((\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1 \sqcup \ldots \sqcup r_m)} K\vec{A}) + x :_{(s_1 \sqcup \ldots \sqcup s_m \cdot 1)} K\vec{A} \vdash \mathbf{case}\ x\ \mathbf{of}\ \overline{C_i\ y^i_1 \ldots y^i_n \mapsto t_i} : B}\ \textsc{Case}}{\Sigma; (\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1 \sqcup \ldots \sqcup r_m) + (s_1 \sqcup \ldots \sqcup s_m)} K\vec{A} \vdash \mathbf{case}\ x\ \mathbf{of}\ \overline{C_i\ y^i_1 \ldots y^i_n \mapsto t_i} : B}\ \equiv$$

g) Case $\Box_{\mathrm{R}}$

For synthesising a promotion, we have the derivation:

$$\dfrac{\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta}{\Sigma; \Gamma \vdash \Box_r A \Rightarrow [t] \mid r \cdot \Delta}\ \Box_{\mathrm{R}}$$

By induction on the premise we have:

$$\Sigma ; \Delta \vdash t : A$$

From which we can construct the following derivation, matching the above conclusion:

$$\dfrac{\Sigma ; \Delta \vdash t : A}{r \cdot \Delta \vdash [t] : \Box_r A}\;\textsc{Pr}$$

h) Case $\Box_{\mathrm{L}}$
For synthesising an unboxing, we have the derivation:

$$\dfrac{\begin{array}{c}\Sigma ; \Gamma, y :_{r \cdot q} A, x :_r \Box_q A \vdash B \Rightarrow t \mid \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \\ \exists s_3 . s_1 \sqsubseteq s_3 \cdot q \sqsubseteq r \cdot q \qquad \Sigma \vdash \Box_q A : \mathsf{Type}\end{array}}{\Sigma ; \Gamma, x :_r \Box_q A \vdash B \Rightarrow \textbf{case}\ x\ \textbf{of}\ [y] \rightarrow t \mid \Delta, x :_{s_3 + s_2} \Box_q A}\;\Box_{\mathrm{L}}$$

By induction on the premise we have:

$$\Sigma ; \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \vdash t : B \tag{ih}$$

and from the premises we have that:

viii. $s_1 \sqsubseteq s_3 \cdot q$

ix. $\Sigma \vdash \Box_q A : \mathsf{Type}$

and through the $\kappa_\Box$ rule, we have that:

$$\Sigma \vdash \Box_q A : \mathsf{Type} \Rightarrow \Sigma \vdash A : \mathsf{Type}$$

From this we can construct the following derivation, towards the goal:

$$\dfrac{(i)}{\Sigma ; x :_1 \Box_q A \vdash x : \Box_q A}\;\textsc{Var} \tag{B.32}$$

$$\dfrac{(\text{B.32}) \qquad \begin{array}{c}\dfrac{\dfrac{\Sigma \vdash A : \mathsf{Type}}{\Sigma ; s_3 \cdot q \vdash y : A \rhd y :_{s_3 \cdot q} A}\;\textsc{PVar}}{\Sigma ; s_3 \vdash [y] : \Box_q A \rhd y :_{s_3 \cdot q} A}\;\textsc{PBox} \\ \dfrac{\Sigma ; \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \vdash t : B \qquad s_1 \sqsubseteq s_3 \cdot q}{\Sigma ; \Delta, y :_{s_3 \cdot q} A, x :_{s_2} \Box_q A \vdash t : B}\;\textsc{Approx}\end{array}}{\Sigma ; \Delta, x :_{s_3 + s_2} \Box_q A \vdash \textbf{case}\ x\ \textbf{of}\ [y] \rightarrow t : B}\;\textsc{Case}$$

i) Case μR
For synthesising a recursive data type introduction form, we have the derivation:

$$\dfrac{D ; \Sigma ; \Gamma \vdash A[\mu X . A / X] \Rightarrow t \mid \Delta}{D ; \Sigma ; \Gamma \vdash \mu X . A \Rightarrow t \mid \Delta}\;\mu_{\mathrm{R}}$$

By induction on the premise we have:

$$D; \Sigma; \Delta \vdash t : A[\mu X.A/X] \qquad \text{(ih)}$$

from which we can construct the derivation, matching the form of the lemma, leveraging the equirecursivity of our types:

$$\dfrac{D; \Sigma; \Delta \vdash t : A[\mu X.A/X]}{D; \Sigma; \Delta \vdash t : \mu X.A}\ \mu_1$$

j) Case μL
For synthesising a recursive data type elimination form, we have the derivation:

$$\dfrac{D; \Sigma; \Gamma, x :_r A[\mu X.A/X] \vdash B \Rightarrow t \mid \Delta}{D; \Sigma; \Gamma, x :_r \mu X.A \vdash B \Rightarrow t \mid \Delta}\ \mu_{\text{L}}$$

By induction on the premise we have:

$$D; \Sigma; \Delta, x :_r A[\mu X.A/X] \vdash t : B \qquad \text{(ih)}$$

from which we can construct the derivation:

$$\dfrac{\dfrac{\Sigma \vdash \mu X.A : \text{Type}}{D; \Sigma; x :_r \mu X.A \vdash x :_1 \mu X.A}\ \text{Var}}{D; \Sigma; x :_r \mu X.A \vdash x :_1 A[\mu X.A]}\ \mu_2$$

and by using lemma 5.1.1 on :

$$D; \Sigma; x : A[\mu X.A/X] \vdash t : B$$

we obtain the following, matching the above conclusion:

$$D; \Sigma; D, x : \mu X.A \vdash [x/x]t : B = D; \Sigma; D, x : \mu X.A \vdash t : B$$

2. a) Case TopLevel
For the top-level of synthesis, we have the derivation:

$$\dfrac{\overline{\alpha : \kappa}; \emptyset \vdash A \Rightarrow t \mid \emptyset}{\emptyset; \emptyset \vdash \forall \overline{\alpha : \kappa}.A \Rightarrow t \mid \emptyset}\ \text{TopLevel}$$

from induction on the premise, we have that:

$$\overline{\alpha : \kappa}; \emptyset \vdash t : A \qquad \text{(ih)}$$

from which we can construct the following typing derivation, matching the above conclusion:

$$\dfrac{\overline{\alpha : \kappa}; \emptyset \vdash t : A}{\emptyset; \emptyset \vdash t : \forall \overline{\alpha : \kappa}.A}\ \text{TopLevel}$$

□

B.3.2 *Soundness of Focusing for the Fully Graded Synthesis Calculus*

Lemma B.3.1 (Soundness of focusing for graded-base synthesis)**.** *For all contexts* Γ*,* Ω *and types* A *then:*

1. $TopLevel$:	$\emptyset;\emptyset;\emptyset \vdash \forall\overline{\alpha:\kappa}.A \Rightarrow t \mid \emptyset$	$\Longrightarrow$	$\emptyset;\emptyset \vdash \forall\overline{\alpha:\kappa}.A \Rightarrow t \mid \emptyset$
1. $Right\ Async$:	$\Sigma;\Gamma;\Omega \vdash A \Uparrow \Rightarrow t \mid \Delta$	$\Longrightarrow$	$\Sigma;\Gamma,\ ,\Omega \vdash A \Rightarrow t \mid \Delta$
2. $Left\ Async$:	$\Sigma;\Gamma;\Omega \Uparrow \vdash B \Rightarrow t \mid \Delta$	$\Longrightarrow$	$\Sigma;\Gamma,\ ,\Omega \vdash B \Rightarrow t \mid \Delta$
3. $Right\ Sync$:	$\Sigma;\Gamma;\emptyset \vdash A \Downarrow\ \Rightarrow t \mid\ \Delta$	$\Longrightarrow$	$\Sigma;\Gamma \vdash A \Rightarrow t \mid \Delta$
4. $Left\ Sync$:	$\Sigma;\Gamma;x :_r A \Downarrow \vdash B \Rightarrow t \mid \Delta$	$\Longrightarrow$	$\Sigma;\Gamma,x :_r A \vdash B \Rightarrow t \mid \Delta$
5. $Focus\ Right$:	$\Sigma;\Gamma;\emptyset \vdash B \Rightarrow t \mid\ \Delta$	$\Longrightarrow$	$\Sigma;\Gamma \vdash B \Rightarrow t \mid \Delta$
6. $Focus\ Left$:	$\Sigma;\Gamma,x :_r A;\emptyset \vdash B \Rightarrow t \mid\ \Delta$	$\Longrightarrow$	$\Sigma;\Gamma,x :_r A \vdash B \Rightarrow t \mid \Delta$

Proof.

1. Case: 1. TopLevel:

   a) Case TopLevel

      In the case of the right asynchronous rule for abstraction introduction, the synthesis rule has the form:

      $$\frac{D;\overline{\alpha:\kappa};\emptyset;\emptyset \vdash A \Uparrow \Rightarrow t \mid \emptyset}{D;\emptyset;\emptyset;\emptyset \vdash \forall\overline{\alpha:\kappa}.A \Uparrow \Rightarrow t \mid \emptyset}\ \textsc{TopLevel}$$

      By induction on the first premise, we have that:

      $$\overline{\alpha:\kappa};\emptyset \vdash A \Rightarrow t \mid \emptyset \qquad \text{(ih)}$$

      from case 1 of the lemma. From which, we can construct the following instatiation of the TopLevel synthesis rule in the non-focusing calculus:

      $$\frac{\overline{\alpha:\kappa};\emptyset \vdash A \Rightarrow t \mid \emptyset}{\emptyset;\emptyset \vdash \forall\overline{\alpha:\kappa}.A \Rightarrow t \mid \emptyset}\ \textsc{TopLevel}$$

2. Case: 2. Right Async:

   a) Case $\rightarrow_{\mathrm{R}}$

      In the case of the right asynchronous rule for abstraction introduction, the synthesis rule has the form:

      $$\frac{D;\Sigma;\Gamma;\Omega,x :_q A \vdash B \Uparrow \Rightarrow t \mid \Delta,x :_r A \qquad r \sqsubseteq q}{D;\Sigma;\Gamma;\Omega \vdash A^q \rightarrow B \Uparrow \Rightarrow \lambda x.t \mid \Delta}\ \rightarrow_{\mathrm{R}}$$

      By induction on the premise, we have that:

      $$\Sigma;(\Gamma,\Omega),x :_q A \vdash B \Rightarrow t \mid \Delta,x :_r A \qquad \text{(ih)}$$

from case 2 of the lemma. From which, we can construct the following instatiation of the $\rightarrow_{\mathrm{R}}$ synthesis rule in the non-focusing calculus:

$$\dfrac{\Sigma; (\Gamma, \Omega), x :_q A \vdash B \Rightarrow t \mid \Delta, x :_r A \qquad r \sqsubseteq q}{\Sigma; \Gamma, \Omega \vdash A \rightarrow B \Rightarrow \lambda x.t \mid \Delta}\ \rightarrow_{\mathrm{R}}$$

b) Case $\Uparrow_{\mathrm{R}}$
In the case of the right asynchronous rule for transition to a left asynchronous judgement, the synthesis rule has the form:

$$\dfrac{D; \Sigma; \Gamma; \Omega \Uparrow \vdash B \Rightarrow t \mid \Delta \qquad B \text{ not right async}}{D; \Sigma; \Gamma; \Omega \vdash B \Uparrow \Rightarrow t \mid \Delta}\ \Uparrow_{\mathrm{R}}$$

By induction on the first premise, we have that:

$$\Sigma; \Gamma, \Omega \vdash B \Rightarrow t \mid \Delta \qquad \text{(ih)}$$

from case 3 of the lemma.

3. Case 3. Left Async:

a) Case $\mathrm{Con}_{\mathrm{L}}$
In the case of the left asynchronous rule for constructor elimination, the synthesis rule has the form:

$$\dfrac{\begin{array}{l}(C_i : \forall \overline{\alpha : \kappa}.{B'_1}^{q^i_1} \rightarrow \ldots \rightarrow {B'_n}^{q^i_n} \rightarrow K\vec{A'}) \in D \qquad \Sigma \vdash K\vec{A} : \mathrm{Type} \\ \Sigma \vdash {B_1}^{q_1} \rightarrow \ldots \rightarrow {B_n}^{q_n} \rightarrow K\vec{A} = \mathrm{inst}(\forall \overline{\alpha : \kappa}.{B'_1}^{q_1} \rightarrow \ldots \rightarrow {B'_n}^{q_n} \rightarrow K\vec{A'}) \\ D; \Sigma; \Gamma; \Omega, x :_r K\vec{A}, y^i_1 :_{r \cdot q^i_1} B_1, \ldots, y^i_n :_{r \cdot q^i_1} B_n \Uparrow \vdash B \Rightarrow t_i \mid \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, \ldots, y^i_n :_{s^i_n} B_n \\ \exists {s'}^i_j . s^i_j \sqsubseteq {s'}^i_j \cdot q^i_j \sqsubseteq r \cdot q^i_j \qquad s_i = {s'}^i_1 \sqcup \ldots \sqcup {s'}^i_n \qquad |K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq s_1 \sqcup \ldots \sqcup s_m\end{array}}{D; \Sigma; \Gamma; \Omega, x :_r K\vec{A} \Uparrow \vdash B \Rightarrow \textbf{case } x \textbf{ of } \overline{C_i\ y^i_1 \ldots y^i_n \mapsto t_i} \mid (\Delta_1 \sqcup \ldots \sqcup \Delta_m), x :_{(r_1 \sqcup \ldots \sqcup r_m) + (s_1 \sqcup \ldots \sqcup s_m)} K\vec{A}}$$

By induction on the second premise, we have that:

$$\Sigma; (\Gamma, \Omega), x :_r K\vec{A}, y^i_1 :_{r \cdot q^i_1} B_1, \ldots, y^i_n :_{r \cdot q^i_1} B_n \Uparrow \vdash B \Rightarrow t_i \mid \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, \ldots, y^i_n :_{s^i_n} B_n \qquad \text{(ih)}$$

from case 3 of the lemma. From the second premise, we have that:

$$\Sigma \vdash K\vec{A} : \mathrm{Type}$$

From which we can construct the following instantiation of the $\textsc{Con}_{\text{L}}$ rule in the non-focusing calculus:

$$\dfrac{\begin{array}{c}(C_i : \forall \overline{\alpha : \kappa}.B'^{\,q^i_1}_1 \rightarrow ... \rightarrow B'^{\,q^i_n}_n \rightarrow K\vec{A}') \in D \\ \Sigma \vdash K\vec{A} : \text{Type} \\ \Sigma \vdash B_1{}^{q_1} \rightarrow ... \rightarrow B_n{}^{q_n} \rightarrow K\vec{A} = \text{inst}(\forall \overline{\alpha : \kappa}.B'^{\,q_1}_1 \rightarrow ... \rightarrow B'^{\,q_n}_n \rightarrow K\vec{A}') \\ \Sigma; (\Gamma, \Omega), x :_r K\vec{A}, y^i_1 :_{r \cdot q^i_1} B_1, ..., y^i_n :_{r \cdot q^i_1} B_n \vdash B \Rightarrow t_i \mid \Delta_i, x :_{r_i} K\vec{A}, y^i_1 :_{s^i_1} B_1, ..., y^i_n :_{s^i_n} B_n \\ \exists s'^i_j . s^i_j \sqsubseteq s'^i_j \cdot q^i_j \sqsubseteq r \cdot q^i_j \\ s_i = s'^i_1 \sqcup ... \sqcup s'^i_n \qquad |K\vec{A}| > 1 \Rightarrow 1 \sqsubseteq s_1 \sqcup ... \sqcup s_m\end{array}}{\Sigma; (\Gamma, \Omega), x :_r K\vec{A} \vdash B \Rightarrow \textbf{case } x \textbf{ of } \overline{C_i\ y^i_1 ... y^i_n \mapsto t_i} \mid (\Delta_1 \sqcup ... \sqcup \Delta_m), x :_{(r_1 \sqcup ... \sqcup r_m) + (s_1 \sqcup ... \sqcup s_m)} K\vec{A}}\ \textsc{Con}_{\text{L}}$$

b) Case $\Box_{\text{L}}$
In the case of the left asynchronous rule for graded modality elimination, the synthesis rule has the form:

$$\dfrac{\begin{array}{c}D; \Sigma; \Gamma; \Omega, y :_{r \cdot q} A, x :_r \Box_q A \Uparrow \vdash B \Rightarrow t \mid \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \\ \exists s_3 . s_1 \sqsubseteq s_3 \cdot q \sqsubseteq r \cdot q \qquad \Sigma \vdash \Box_q A : \text{Type}\end{array}}{D; \Sigma; \Gamma; \Omega, x :_r \Box_q A \Uparrow \vdash B \Rightarrow \textbf{case } x \textbf{ of } [y] \rightarrow t \mid \Delta, x :_{s_3 + s_2} \Box_q A}\ \Box_{\text{L}}$$

By induction on the first premise, we have that:

$$\Sigma; (\Gamma, \Omega), y :_{r \cdot q} A, x :_r \Box_q A \vdash B \Rightarrow t \mid \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \tag{ih}$$

from case 3 of the lemma. From the third premise, we have that:

$$\Sigma \vdash \Box_q A : \text{Type}$$

From which, we can construct the following instantiation of the $\Box_{\text{L}}$ synthesis rule in the non focusing calculus:

$$\dfrac{\begin{array}{c}\Sigma; (\Gamma, \Omega), y :_{r \cdot q} A, x :_r \Box_q A \vdash B \Rightarrow t \mid \Delta, y :_{s_1} A, x :_{s_2} \Box_q A \\ \exists s_3 . s_1 \sqsubseteq s_3 \cdot q \sqsubseteq r \cdot q \qquad \Sigma \vdash \Box_q A : \text{Type}\end{array}}{\Sigma; (\Gamma, \Omega), x :_r \Box_q A \vdash B \Rightarrow \textbf{case } x \textbf{ of } [y] \rightarrow t \mid \Delta, x :_{s_3 + s_2} \Box_q A}\ \Box_{\text{L}}$$

c) Case μ_{L}
In the case of the left asynchronous rule for recursive data type elimination, the synthesis rule has the form:

$$\dfrac{D; \Sigma; \Gamma; \Omega, x :_r A[\mu X.A/X] \Uparrow \vdash B \Rightarrow t \mid \ \Delta}{D; \Sigma; \Gamma; \Omega, x :_r \mu X.A \Uparrow \vdash B \Rightarrow t \mid \ \Delta}\ \mu_{\text{L}}$$

By induction on the first premise, we have that:

$$\Sigma; (\Gamma, \Omega), x :_r A[\mu X.A/X] \vdash B \Rightarrow t \mid \ \Delta \tag{ih}$$

from case 3 of the lemma. From which, we can construct the following instantiation of the μ_L synthesis rule in the non focusing calculus:

$$\frac{\Sigma; (\Gamma, \Omega), x :_r A[\mu X.A/X] \vdash B \Rightarrow t \mid \Delta}{\Sigma; (\Gamma, \Omega), x :_r \mu X.A \vdash B \Rightarrow t \mid \Delta}\ \mu_L$$

d) Case $\Uparrow_L$
In the case of the left asynchronous rule for transitioning an assumption from the focusing context Ω to the non-focusing context Γ, the synthesis rule has the form:

$$\frac{D; \Sigma; \Gamma, x :_r A; \Omega \Uparrow \vdash B \Rightarrow t \mid \Delta \qquad \text{A not left async}}{D; \Sigma; \Gamma; \Omega, x :_r A \Uparrow \vdash B \Rightarrow t \mid \Delta}\ \Uparrow_L$$

By induction on the first premise, we have that:

$$\Sigma; \Gamma, x : A, \Omega \vdash C \Rightarrow t \mid \Delta \qquad \text{(ih)}$$

from case 3 of the lemma.

4. Case 4. Right Sync:

a) Case $\textsc{Con}_R$
In the case of the right synchronous rule for constructor introduction, the synthesis rule has the form:

$$\frac{\begin{array}{c}(C : \forall \overline{\alpha : \kappa}.B_1'^{q_1} \to \ldots \to B_n'^{q_n} \to K\vec{A'}) \in D \\ \Sigma \vdash B_1^{q_1} \to \ldots \to B_n^{q_n} \to K\vec{A} = \text{inst}(\forall \overline{\alpha : \kappa}.B_1'^{q_1} \to \ldots \to B_n'^{q_n} \to K\vec{A'}) \\ \Sigma; \Gamma; \emptyset \vdash B_i \Downarrow \Rightarrow t_i \mid \Delta_i\end{array}}{\Sigma; \Gamma; \emptyset \vdash K\vec{A} \Downarrow \Rightarrow C\,t_1 \ldots t_n \mid \Delta_1 + \ldots + \Delta_n}\ \textsc{Con}_R$$

By induction on the second premise, we have that:

$$\Sigma; \Gamma \vdash B_i \Rightarrow t_i \mid \Delta_i \qquad \text{(ih)}$$

from case 4 of the lemma. From which, we can construct the following instantiation of the $\textsc{Con}_R$ synthesis rule in the non-focusing calculus:

$$\frac{\begin{array}{c}(C : \forall \overline{\alpha : \kappa}.B_1^{q_1} \to \ldots \to B_n^{q_n} \to K\vec{A}) \in D \\ \Sigma \vdash B_1^{q_1} \to \ldots \to B_n^{q_n} \to K\vec{A} = \text{inst}(\forall \overline{\alpha : \kappa}.B_1'^{q_1} \to \ldots \to B_n'^{q_n} \to K\vec{A'}) \\ \Sigma; \Gamma \vdash B_i \Rightarrow t_i \mid \Delta_i\end{array}}{\Sigma; \Gamma \vdash K\vec{A} \Rightarrow C\,t_1 \ldots t_n \mid 0 \cdot \Gamma + (q_1 \cdot \Delta_1) + \ldots + (q_n \cdot \Delta_n)}\ \textsc{Con}_R$$

b) Case $\Box_R$

In the case of the right synchronous rule for graded modality introduction, the synthesis rule has the form:

$$\frac{\Sigma; \Gamma; \emptyset \vdash A \Downarrow \Rightarrow t \mid \Delta}{\Sigma; \Gamma; \emptyset \vdash \Box_r A \Downarrow \Rightarrow [t] \mid r \cdot \Delta} \Box_R$$

By induction on the premises, we have that:

$$\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta \qquad \text{(ih)}$$

from case 4 of the lemma. From which, we can construct the following instantiation of the μR synthesis rule in the non-focusing calculus:

$$\frac{\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta}{\Sigma; \Gamma \vdash \Box_r A \Rightarrow [t] \mid r \cdot \Delta} \Box_R$$

c) Case μ_R
In the case of the right synchronous rule for recursive data type introduction, the synthesis rule has the form:

$$\frac{D; \Sigma; \Gamma; \emptyset \vdash A[\mu X.A/X] \Downarrow \Rightarrow t \mid \Delta}{D; \Sigma; \Gamma; \emptyset \vdash \mu X.A \Downarrow \Rightarrow t \mid \Delta} \mu_R$$

By induction on the premises, we have that:

$$\Sigma; \Gamma \vdash A[\mu X.A/X] \Rightarrow t \mid \Delta \qquad \text{(ih)}$$

from case 4 of the lemma. From which, we can construct the following instantiation of the μ_R synthesis rule in the non-focusing calculus:

$$\frac{D; \Sigma; \Gamma \vdash A[\mu X.A/X] \Rightarrow t \mid \Delta}{D; \Sigma; \Gamma \vdash \mu X.A \Rightarrow t \mid \Delta} \mu_R$$

d) Case $\Downarrow_R$
In the case of the right synchronous rule for transitioning from the right focusing phase to an asynchronous right phase, the synthesis rule has the form:

$$\frac{D; \Sigma; \Gamma; \emptyset \vdash A \Uparrow \Rightarrow t \mid \Delta}{D; \Sigma; \Gamma; \emptyset \vdash A \Downarrow t \mid \Delta} \Downarrow_R$$

By induction on the first premise, we have that:

$$\Sigma; \Gamma \vdash A \Rightarrow t \mid \Delta \qquad \text{(ih)}$$

from case 4 of the lemma.

5. Case 5. Left Sync:

   a) Case $\multimap_L$
   In the case of the left synchronous rule for application, the synthesis rule has the form:

$$\frac{\begin{array}{c}\Sigma;\Gamma,x_1 :_{r_1} A^q \multimap B; x_2 :_{r_1} B \Downarrow\vdash C \Rightarrow t_1 \mid \Delta_1, x_1 :_{s_1} A^q \multimap B, x_2 :_{s_2} B \\ \Sigma;\Gamma,x_1 :_{r_1} A^q \multimap B;\emptyset \vdash A \Downarrow \Rightarrow t_2 \mid \ \Delta_2, x_1 :_{s_3} A^q \multimap B \\ \Sigma \vdash A^q \multimap B : \text{Type}\end{array}}{\Sigma;\Gamma;x_1 :_{r_1} A^q \multimap B \Downarrow\vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + s_2 \cdot q \cdot \Delta_2), x_1 :_{s_2+s_1+(s_2 \cdot q \cdot s_3)} A^q \multimap B}$$

   By induction on the first premise, we have that:

$$\Sigma;\Gamma,x_1 :_{r_1} A^q \multimap B, x_2 :_{r_1} B \vdash C \Rightarrow t_1 \mid \Delta_1, x_1 :_{s_1} A^q \multimap B, x_2 :_{s_2} B$$

   from case 5 of the lemma. By induction on the second premise, we have that:

$$\Sigma;\Gamma,x_1 :_{r_1} A^q \multimap B \vdash A \Rightarrow t_2 \mid \ \Delta_2, x_1 :_{s_3} A^q \multimap B \qquad \text{(ih)}$$

   from case 4 of the lemma. From the third premise, we have that:

$$\Sigma \vdash A^q \multimap B : \text{Type}$$

   From which, we can construct the following instantiation of the $\multimap_L$ synthesis rule in the non-focusing calculus:

$$\frac{\begin{array}{c}\Sigma;\Gamma,x_1 :_{r_1} A^q \multimap B, x_2 :_{r_1} B \vdash C \Rightarrow t_1 \mid \Delta_1, x_1 :_{s_1} A^q \multimap B, x_2 :_{s_2} B \\ \Sigma;\Gamma,x_1 :_{r_1} A^q \multimap B \vdash A \Rightarrow t_2 \mid \Delta_2, x_1 :_{s_3} A^q \multimap B \\ \Sigma \vdash A^q \multimap B : \text{Type}\end{array}}{\Sigma;\Gamma,x_1 :_{r_1} A^q \multimap B \vdash C \Rightarrow [(x_1\, t_2)/x_2]t_1 \mid (\Delta_1 + s_2 \cdot q \cdot \Delta_2), x_1 :_{s_2+s_1+(s_2 \cdot q \cdot s_3)} A^q \multimap B} \multimap_L$$

   b) Case Var
   In the case of the left synchronous rule for variable synthesis, the synthesis rule has the form:

$$\frac{\Sigma \vdash A : \text{Type}}{D;\Sigma;\Gamma;x :_r A \Downarrow \vdash A \Rightarrow x \mid 0 \cdot \Gamma, x :_1 A} \text{Var}$$

   From the premise, we have that:

$$\Sigma \vdash A : \text{Type}$$

   from which, we can construct the following instantiation of the Var synthesis rule in the non-focusing calculus:

$$\frac{\Sigma \vdash A : \text{Type}}{\Sigma;\Gamma,x :_r A \vdash A \Rightarrow x \mid 0 \cdot \Gamma, x :_1 A} \text{Var}$$

c) Case Def

In the case of the left synchronous rule for synthesis of a top-level definition usage, the synthesis rule has the form:

$$\frac{\Sigma \vdash A = \text{inst}(\forall \overline{\alpha : \kappa}.A')}{D, x : \forall \overline{\alpha : \kappa}.A'; \Sigma; \Gamma; \emptyset \Downarrow \vdash A \Rightarrow x \mid 0 \cdot \Gamma} \textsc{Def}$$

From the premise, we have that:

$$\Sigma \vdash A : \text{Type}$$

from which, we can construct the following instantiation of the Var synthesis rule in the non-focusing calculus:

$$\frac{(x : \forall \overline{\alpha : \kappa}.A') \in D \qquad \Sigma \vdash A = \text{inst}(\forall \overline{\alpha : \kappa}.A')}{\Sigma; \Gamma \vdash A \Rightarrow x \mid 0 \cdot \Gamma} \textsc{Def}$$

d) Case $\Downarrow_L$

In the case of the left synchronous rule for transitioning from the right focusing phase to an asynchronous left phase, the synthesis rule has the form:

$$\frac{\begin{array}{c} D; \Sigma; \Gamma; x :_r A \Uparrow \vdash B \Rightarrow t \mid \Delta \\ \text{A not atomic and not left sync} \end{array}}{D; \Sigma; \Gamma; x :_r A \Downarrow \vdash B \Rightarrow t \mid \Delta} \Downarrow_L$$

By induction on the first premise, we have that:

$$\Sigma; \Gamma, x :_r A \vdash B \Rightarrow t \mid \Delta \tag{ih}$$

from case 5 of the lemma.

6. Case 6. Right Focus:
   In the case of the focusing rule for transitioning from a left asynchronous judgement to a right synchronous judgement, the synthesis rule has the form:

$$\frac{D; \Sigma; \Gamma; \emptyset \vdash B \Downarrow \Rightarrow t \mid \ \Delta \qquad \text{B not atomic}}{D; \Sigma; \Gamma; \emptyset \Uparrow \vdash B \Rightarrow t \mid \Delta} \textsc{Foc}_R$$

   By induction on the first premise, we have that:

$$\Sigma; \Gamma \vdash C \Rightarrow t \mid \Delta \tag{ih}$$

   from case 3 of the lemma.

7. Case 7. Left Focus:
   In the case of the focusing rule for transitioning from a left

asynchronous judgement to a left synchronous judgement, the synthesis rule has the form:

$$\frac{D; \Sigma; \Gamma; x :_r A \Downarrow \vdash B \Rightarrow t \mid \Delta}{D; \Sigma; \Gamma, x :_r A; \emptyset \Uparrow \vdash B \Rightarrow t \mid \Delta}\ \textsc{Foc}_{\mathrm{L}}$$

By induction on the first premise, we have that:

$$\Sigma; \Gamma, x :_r A \vdash C \Rightarrow t \mid \Delta \tag{ih}$$

from case 3 of the lemma.

□

DECLARATION

Put your declaration here.

Canterbury, November 2023

Jack Oliver Hughes

COLOPHON

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Final Version as of November 21, 2023 (classicthesis v4.6).