095946- ADVANCED ALGORITHMS AND PARALLEL PROGRAMMING

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a.a. 2021-2022

Order Statistics

- Randomized divide and conquer
- Analysis of expected time
- Worst-case linear-time order statistics
- Analysis

- Material adapted from Erik D. Demaine and Charles E. Leiserson slides

Order Statistics

Input: A set of n (distinct) numbers and an integer i, with $1 \leq i \leq n$

Output: The element with *rank* i $x \in A$ that is larger than exactly i-1 other elements of A

- $i = 1$: *minimum*;
- $i = n$: *maximum*;
- $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: *lower or upper median*.

Naive algorithm: Sort and index ith element.

Worst-case running time $= \Theta(n \lg n) + \Theta(1)$
$= \Theta(n \lg n)$,

using merge sort or heapsort (*not* quicksort).

How many comparison are needed to determine the minimum (or maximum) of a set of *n* elements?

```
Minimum (A)
   min ← A[1]
   for i ← 2 to length[A] do
     if min > A[i] then
        min ← A[i]
   return min
```

An upper bound of *n*-1 comparison can be obtained

A dual algorithm for the maximum exist with the same complexity

Lower bound is still n-1 comparisons

Observing that every element excet the winner must lose at least one comparison, we conclude that n-1 comparisons are necessary to dermine the minimum

The Algorithm is optimal w.r.t. the number of comparisons performed

Simultaneous minimum and maximum

How many comparisons are necessary to determine both minimum and maximum

```
MinMax (A)
  if length[A] odd then
    max ← min ← A[1], i ← 2
  else
    if A[1] < A[2] then
      min ← A[1], max ← A[2], i ← 3
    else
      min ← A[2], max ← A[1], i ← 3
```

```
while i ≤ length[A] do
    if A[i] < A[i+1] then
     if min > A[i] then min ← A[i]
     if max < A[i+1] then max ← A[i+1]
    else
     if min > A[i+1] then min ← A[i+1]
     if max < A[i] then max ← A[i]
    i ← i+2
return min,max
```

$3(n-1)/2 = 3\lfloor n/2 \rfloor$ comparisons if n is odd

$1+3(n-2)/2 = 3(n/2)-2$ comparisons if n even

In general we need less than $3\lfloor n/2 \rfloor$ comparisons

Selection in linear time

Select the ith smallest of n elements (the element with *rank* i).

Two versions:

- **Randomized divide-and-conquer algorithm (linear in average)**
- **Deterministic version (derandomization – linear in the worst case)**

Randomized divide-and-conquer algorithm

RAND-SELECT(A, p, q, i) ⊳ i**th smallest of** $A[p \,..\, q]$

if $p = q$ **then return** $A[p]$

$r \leftarrow$ **RAND-PARTITION**(A, p, q)

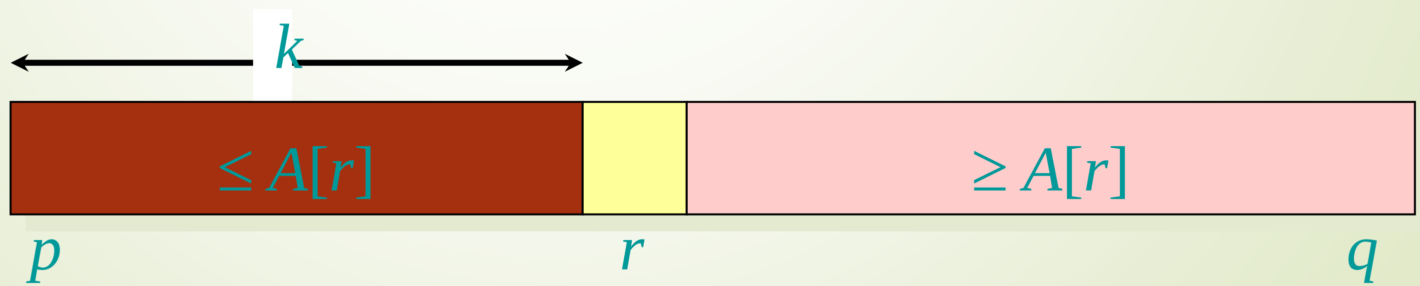
$k \leftarrow r - p + 1$ ⊳ $k = \mathrm{rank}(A[r])$

if $i = k$ **then return** $A[r]$

if $i < k$

then return RAND-SELECT$(A, p, r - 1, i)$

else return RAND-SELECT$(A, r + 1, q, i - k)$



Example

Select the $i = 7$th smallest:

6	10	13	5	8	3	2	11

pivot

$i = 7$

Partition:

2	5	3	6	8	13	10	11

$k = 4$

Select the $7 - 4 = 3$rd smallest recursively.

Intuition for analysis

(All our analyses today assume that all elements are distinct.)

Lucky:

$$T(n) = T(9n/10) + \Theta(n)$$
$$= \Theta(n)$$

$n^{\log_{10/9} 1} = n^0 = 1$

CASE 3

Unlucky:

$$T(n) = T(n - 1) + \Theta(n)$$
$$= \Theta(n^2)$$

arithmetic series

Worse than sorting!

Analysis of expected time

The analysis follows that of randomized quicksort, but it's a little different.

Let $T(n)$ = the random variable for the running time of RAND-SELECT on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the *indicator random variable*

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

Analysis (continued)

To obtain an upper bound, assume that the ith element always falls in the larger side of the partition:

$$T(n) = \begin{cases} T(\max\{0, n-1\}) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(\max\{1, n-2\}) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad \vdots & \\ T(\max\{n-1, 0\}) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k (T(\max\{k, n-k-1\}) + \Theta(n)) \quad .$$

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k(T(\max\{k, n-k-1\}) + \Theta(n))\right]$$

Take expectations of both sides.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k(T(\max\{k, n-k-1\}) + \Theta(n))\right]$$

$$= \sum_{k=0}^{n-1} E[X_k(T(\max\{k, n-k-1\}) + \Theta(n))]$$

Linearity of expectation.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k(T(\max\{k, n-k-1\}) + \Theta(n))\right]$$

$$= \sum_{k=0}^{n-1} E[X_k(T(\max\{k, n-k-1\}) + \Theta(n))]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)]$$

Independence of X_k from other random choices.

Calculating expectation

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E\big[X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\big] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(\max\{k, n-k-1\})] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)
\end{aligned}
$$

Linearity of expectation; $E[X_k] = 1/n$.

Calculating expectation

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k(T(\max\{k, n-k-1\}) + \Theta(n))\right] \\
&= \sum_{k=0}^{n-1} E[X_k(T(\max\{k, n-k-1\}) + \Theta(n))] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(\max\{k, n-k-1\})] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n) \\
&\le \frac{2}{n}\sum_{k=\lfloor n/2\rfloor}^{n-1} E[T(k)] + \Theta(n)
\end{aligned}
$$

Upper terms appear twice.

Hairy recurrence

(But not quite as hairy as the quicksort one.)

$$E[T(n)] = \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

Prove: $E[T(n)] \le cn$ for constant $c > 0$.

- The constant c can be chosen large enough so that $E[T(n)] \le cn$ for the base cases.

Use fact: $$\sum_{k=\lfloor n/2 \rfloor}^{n-1} k \le \frac{3}{8}n^2$$ (exercise).

Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

Substitute inductive hypothesis.

Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\leq \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

Use fact.

Substitution method

$$
\begin{aligned}
E[T(n)] &\leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
&\leq \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n) \\
&= cn - \left(\frac{cn}{4} - \Theta(n)\right)
\end{aligned}
$$

Express as *desired – residual*.

Substitution method

$$
\begin{aligned}
E[T(n)] &\leq \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
&\leq \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n) \\
&= cn - \left(\frac{cn}{4} - \Theta(n)\right) \\
&\leq cn
\end{aligned}
$$

,

if c is chosen large enough so that $cn/4$ dominates the $\Theta(n)$.

Summary of randomized order-statistic selection

- Works fast: linear expected time.
- Excellent algorithm in practice.
- But, the worst case is *very* bad: $\Theta(n^2)$.

Q. Is there an algorithm that runs in linear time in the worst case?

A. Yes, due to Blum, Floyd, Pratt, Rivest, and Tarjan [1973].

IDEA: Generate a good pivot recursively.

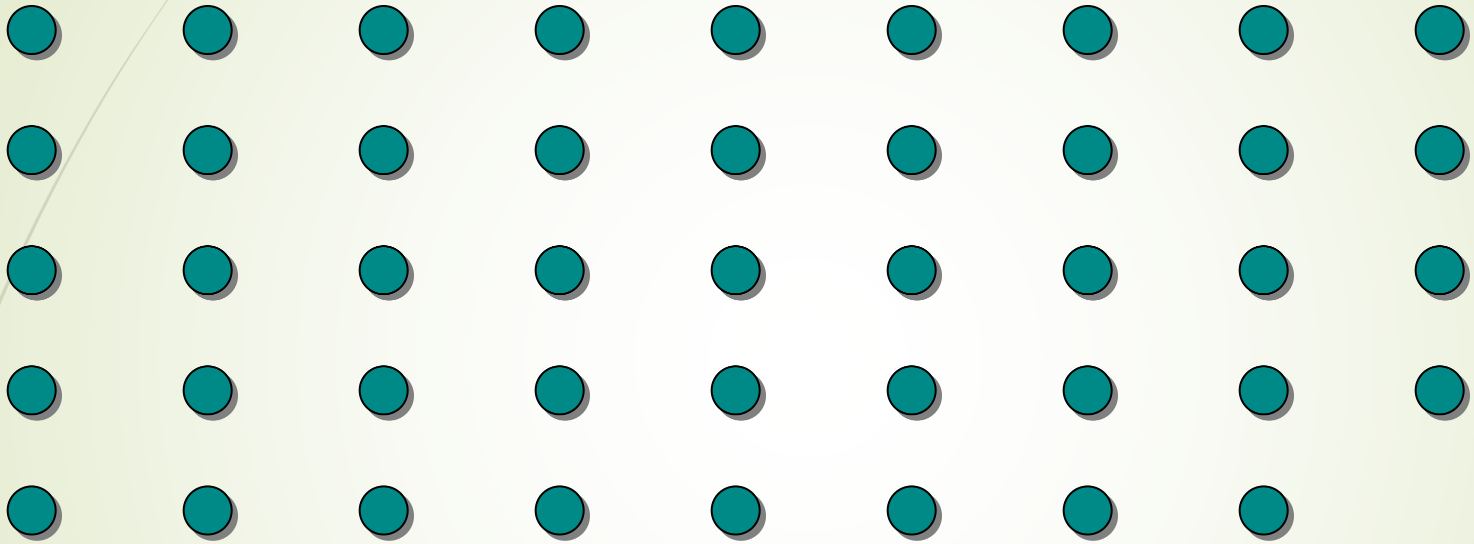
Worst-case linear-time order statistics

SELECT(i, n)

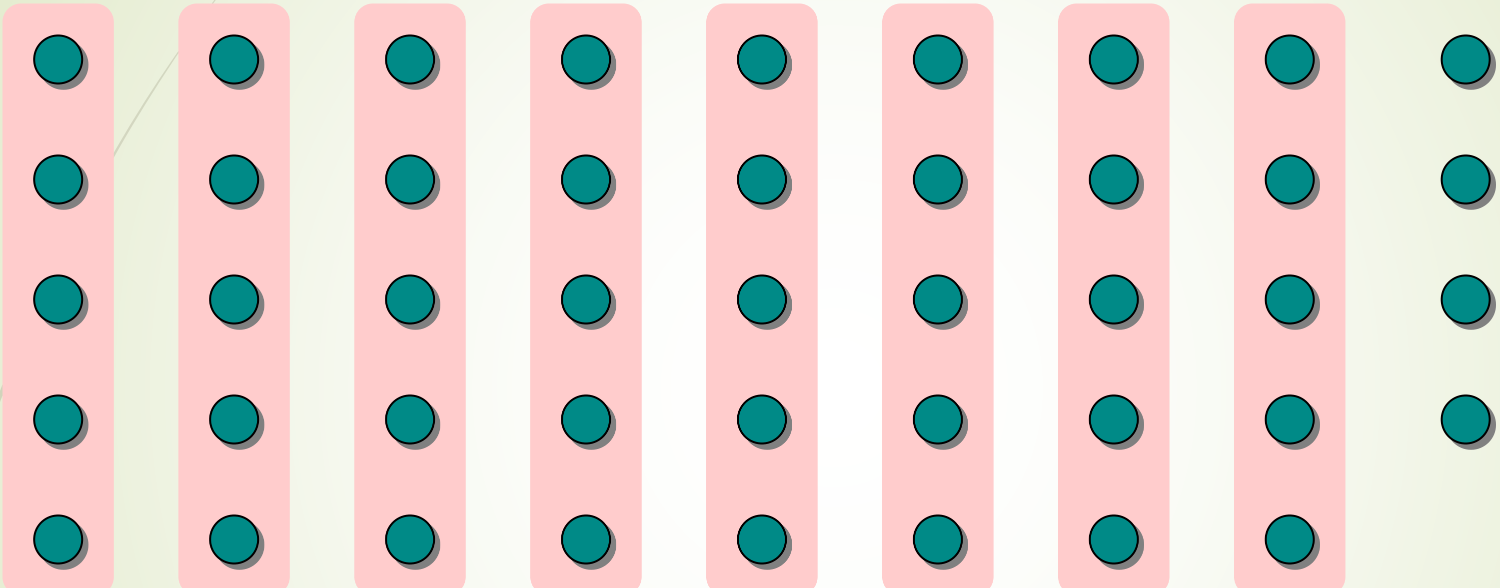
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
3. Partition around the pivot x. Let $k = \text{rank}(x)$.
4. if $i = k$ then return x
   elseif $i < k$
   then recursively SELECT the ith smallest element in the lower part
   else recursively SELECT the $(i–k)$th smallest element in the upper part

Same as RAND-SELECT

Choosing the pivot

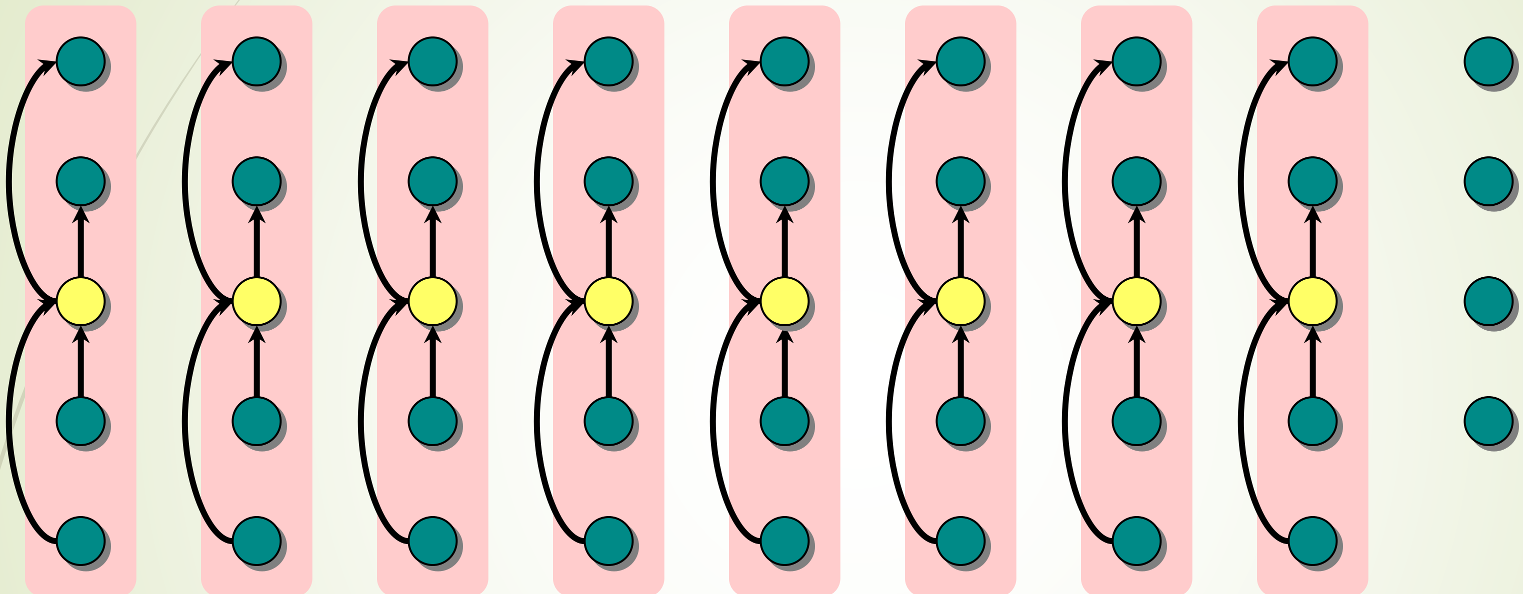


Choosing the pivot

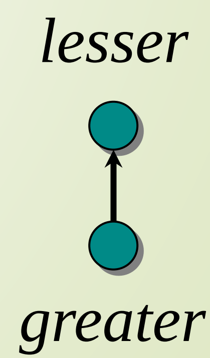


1. Divide the n elements into groups of 5.

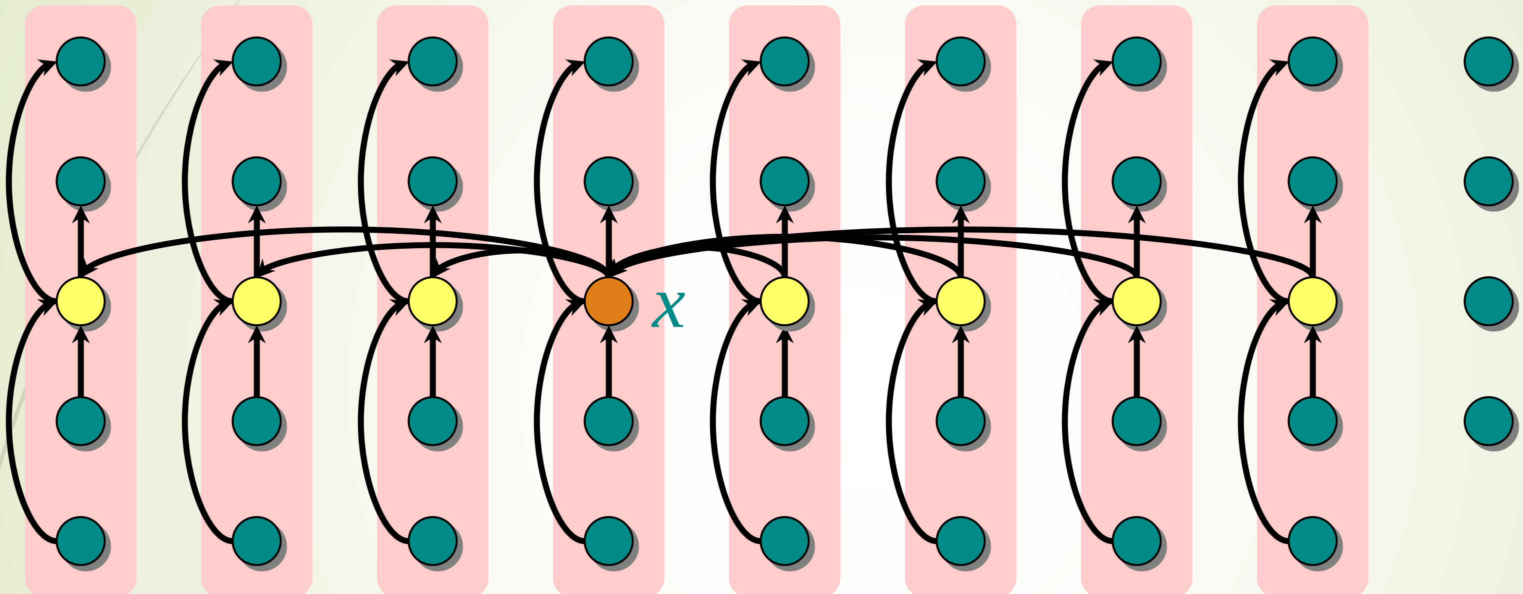
Choosing the pivot



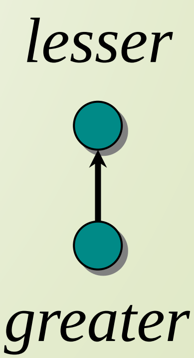
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.



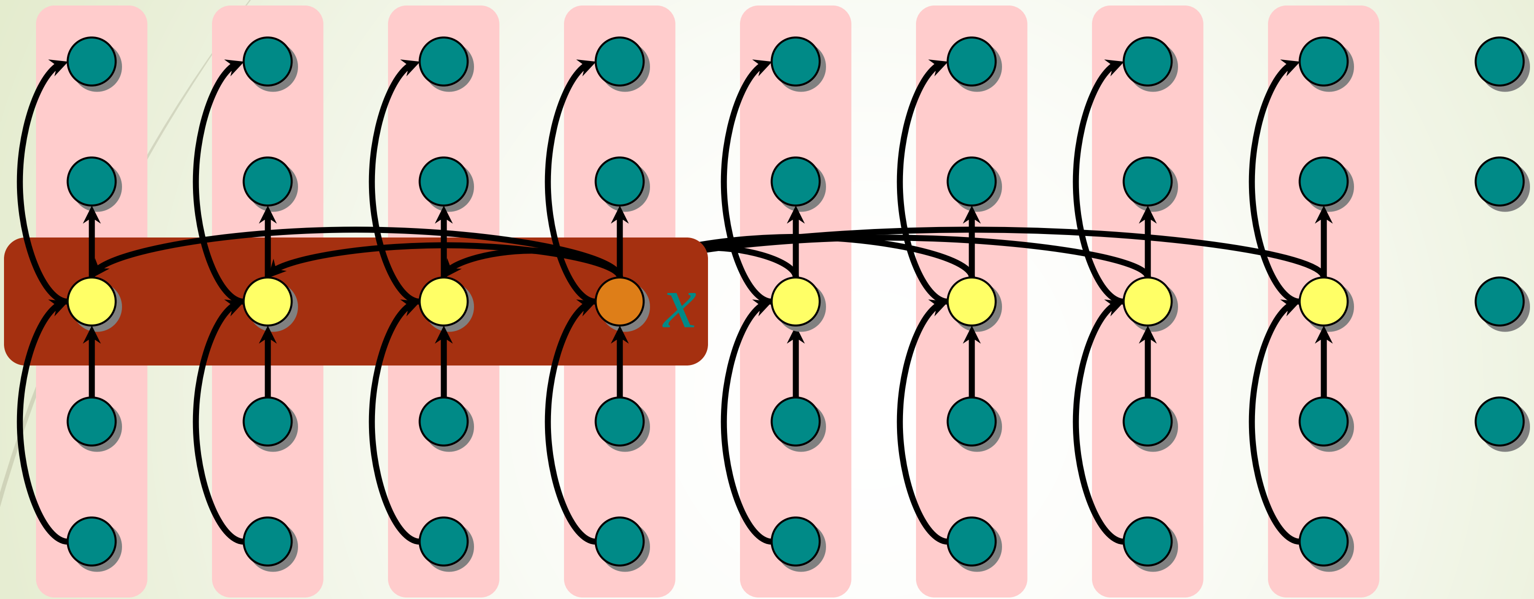
Choosing the pivot



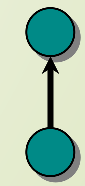
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.



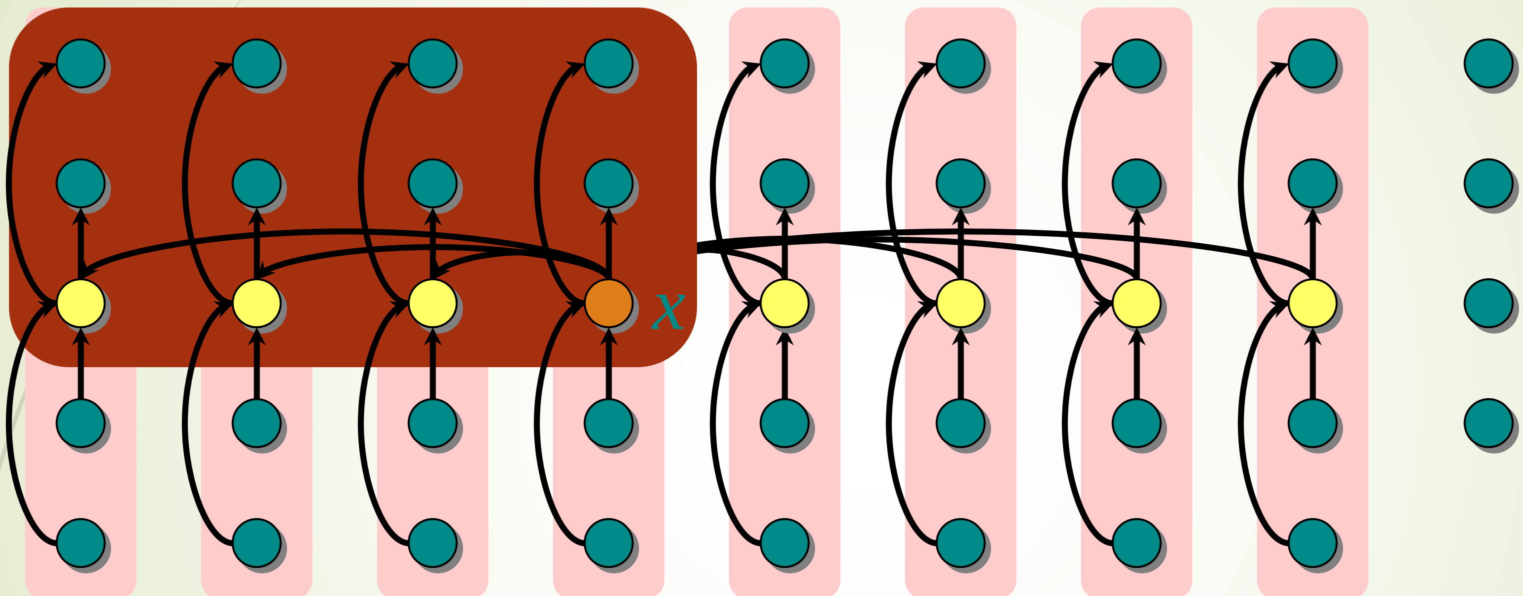
Analysis



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

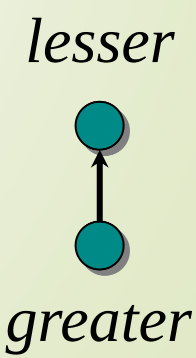


Analysis (Assume all elements are distinct.)

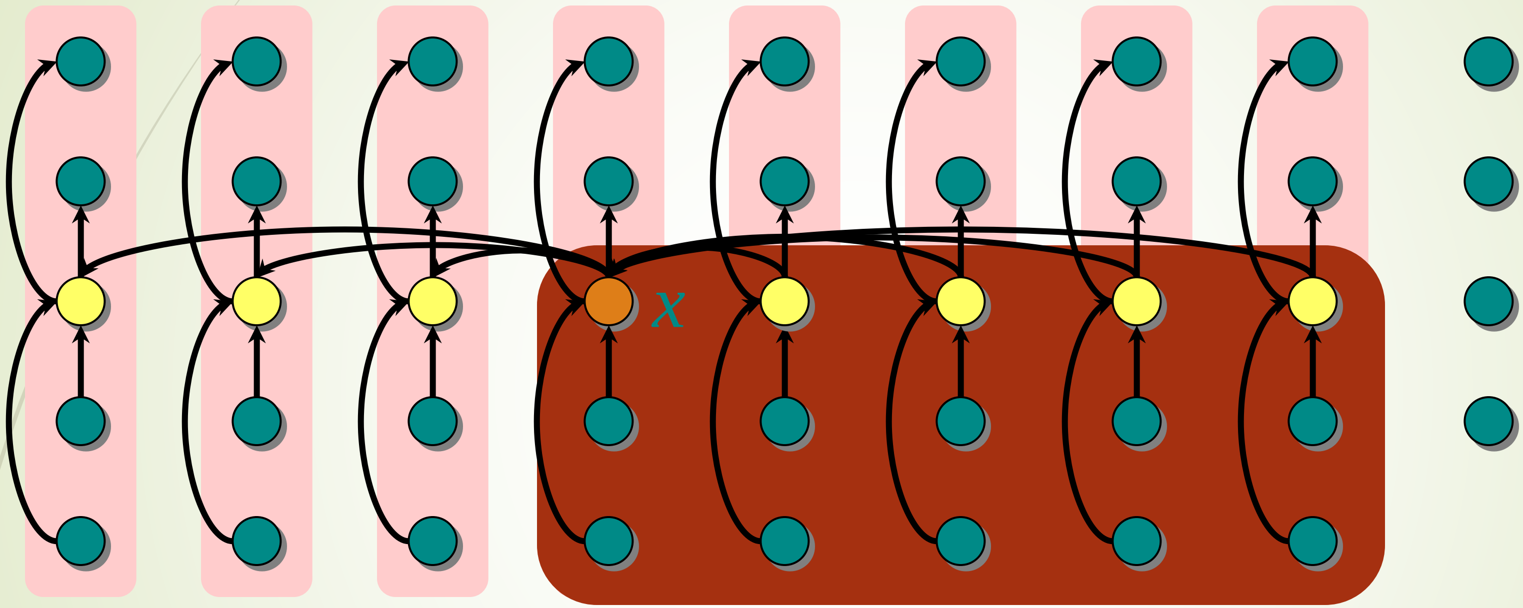


At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\leq x$.

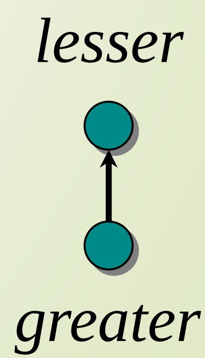


Analysis (Assume all elements are distinct.)



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\leq x$.
- Similarly, at least $3\lfloor n/10 \rfloor$ elements are $\geq x$.



Minor simplification

- For $n \geq 50$, we have $3\lfloor n/10 \rfloor \geq n/4$.
- Therefore, for $n \geq 50$ the recursive call to SELECT in Step 4 is executed recursively on $\leq 3n/4$ elements.
- Thus, the recurrence for running time can assume that Step 4 takes time $T(3n/4)$ in the worst case.
- For $n < 50$, we know that the worst-case time is $T(n) = \Theta(1)$.

Developing the recurrence

$T(n)$	SELECT(i, n)
$\Theta(n)$	1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
$T(n/5)$	2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
$\Theta(n)$	3. Partition around the pivot x. Let $k = \text{rank}(x)$.
$T(3n/4)$	4. if $i = k$ then return x elseif $i < k$ then recursively SELECT the ith smallest element in the lower part else recursively SELECT the $(i–k)$th smallest element in the upper part

Solving the recurrence

$$T(n) = T\left(\frac{1}{5}n\right) + T\left(\frac{3}{4}n\right) + \Theta(n)$$

Substitution:

$T(n) \le cn$

$$\begin{aligned} T(n) &\le \frac{1}{5}cn + \frac{3}{4}cn + \Theta(n) \\ &= \frac{19}{20}cn + \Theta(n) \\ &= cn - \left(\frac{1}{20}cn - \Theta(n)\right) \\ &\le cn \end{aligned}$$

,

if c is chosen large enough to handle both the $\Theta(n)$ and the initial conditions.

Conclusions

- Since the work at each level of recursion is a constant fraction (19/20) smaller, the work per level is a geometric series dominated by the linear work at the root.
- In practice, this algorithm runs slowly, because the constant in front of n is large.
- The randomized algorithm is far more practical.

Exercise: *Why not divide into groups of 3?*