Generalised Pareto Distribution with Predictive Processes – MCMC updates

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Algorithm details

(Filled in below):

Algorithm 1: Gaussian Process Generalised Pareto Distribution Markov chain Monte Carlo

Data: y_{ij}, the declustered threshold excesses at locations $i = 1 \dots n$, with $j = 1 \dots n_i$ excesses at location i;

X_ϕ, X_ξ, $n \times (p+1)$ and $n \times (q+1)$ matrices of p and q covariates at locations $i = 1 \dots n$.

Result: Samples from the posterior distributions of $\phi = \log(\sigma)$ and ξ (the unknown parameters of interest), which can then be used to calculate return level estimates

Initialisation;

Random starting values of $\tilde{\phi}$ and $\tilde{\xi}$;

Projection of ϕ and ξ from $\tilde{\phi}$ and $\tilde{\xi}$ respectively;

Hyper-parameter values of α_ϕ, β_ϕ, ς^2_ϕ, τ^2_ϕ, ν_ϕ, α_ξ, β_ξ, ς^2_ξ, τ^2_ξ, and ν_ξ;

Number of iterations N;

for *iterations k from 1 to N* **do**

    Generate $u \sim U(0,1)$;

    for *locations i from 1 to n* **do**

        Simulate $\tilde{\phi}_{new,i}$;

        Project ϕ_{new} from $\tilde{\phi}_{new}$;

        Set l_{new} = log full conditional of new vector $\tilde{\phi}_{new}$;

        Set l_{old} = log full conditional of old vector $\tilde{\phi}$;

        Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (1);

        if $a > u$ **then**

            Set $\tilde{\phi} = \tilde{\phi}_{new}$;

        end

        Simulate $\tilde{\xi}_{new,i}$;

        Project ξ_{new} from $\tilde{\xi}_{new}$;

        Set l_{new} = log full conditional of new vector $\tilde{\xi}_{new}$;

        Set l_{old} = log full conditional of old vector $\tilde{\xi}$;

        Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (2);

        if $a > u$ **then**

            Set $\tilde{\xi} = \tilde{\xi}_{new}$;

        end

    end

end

for *iterations k from 1 to N* **do**

  for *covariates c from 1 to* $p+1$ **do**

    Simulate $\alpha_{\phi_{new,c}}$;

    Set l_{new} = log full conditional of new vector $\alpha_{\phi_{new}}$;

    Set l_{old} = log full conditional of old vector α_{ϕ};

    Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (3);

    if $a > u$ **then**

      Set $\alpha_{\phi} = \alpha_{\phi_{new}}$;

    end

  end

  for *covariates d from 1 to* $q+1$ **do**

    Simulate $\alpha_{\xi_{new,d}}$;

    Set l_{new} = log full conditional of new vector $\alpha_{\xi_{new}}$;

    Set l_{old} = log full conditional of old vector α_{ξ};

    Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (4);

    if $a > u$ **then**

      Set $\alpha_{\xi} = \alpha_{\xi_{new}}$;

    end

  end

end

for *iterations k from 1 to N* **do**

  for *elements c in the lower-triangle of the matrix β_ϕ* **do**

    Simulate $\beta_{\phi_{new,c}}$;

    Set l_{new} = log full conditional of new matrix $\beta_{\phi_{new}}$;

    Set l_{old} = log full conditional of old matrix β_ϕ;

    Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (5);

    if $a > u$ **then**

      Set $\beta_\phi = \beta_{\phi_{new}}$;

    end

  end

  for *elements d in the lower-triangle of the matrix β_ξ* **do**

    Simulate $\beta_{\xi_{new,d}}$;

    Set l_{new} = log full conditional of new matrix $\beta_{\xi_{new}}$;

    Set l_{old} = log full conditional of old matrix β_ξ;

    Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (6);

    if $a > u$ **then**

      Set $\beta_\xi = \beta_{\xi_{new}}$;

    end

  end

end

for *iterations k from 1 to N* **do**

Simulate $\nu_{\phi_{new}}$;

Set l_{new} = log full conditional of new value $\nu_{\phi_{new}}$;

Set l_{old} = log full conditional of old value ν_{ϕ};

Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (7);

if $a > u$ **then**

Set $\nu_{\phi} = \nu_{\phi_{new}}$;

end

Simulate $\nu_{\xi_{new}}$;

Set l_{new} = log full conditional of new vector $\nu_{\xi_{new}}$;

Set l_{old} = log full conditional of old vector ν_{ξ};

Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (8);

if $a > u$ **then**

Set $\nu_{\xi} = \nu_{\xi_{new}}$;

end

end

for *iterations k from 1 to N* **do**

Simulate $\varsigma^2_{\phi\,new}$;

Set l_{new} = log full conditional of new value $\varsigma^2_{\phi\,new}$;

Set l_{old} = log full conditional of old value ς^2_{ϕ};

Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (9);

if $a > u$ **then**

Set $\varsigma^2_{\phi} = \varsigma^2_{\phi\,new}$;

end

Simulate $\varsigma^2_{\xi\,new}$;

Set l_{new} = log full conditional of new vector $\varsigma^2_{\xi\,new}$;

Set l_{old} = log full conditional of old vector ς^2_{ξ};

Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (10);

if $a > u$ **then**

Set $\varsigma^2_{\xi} = \varsigma^2_{\xi\,new}$;

end

end

for *iterations k from 1 to N* **do**

Simulate ${\tau^2_\phi}_{new}$;

Set l_{new} = log full conditional of new value ${\tau^2_\phi}_{new}$;

Set l_{old} = log full conditional of old value τ^2_ϕ;

Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (11);

if $a > u$ **then**

Set $\tau^2_\phi = {\tau^2_\phi}_{new}$;

end

Simulate ${\tau^2_\xi}_{new}$;

Set l_{new} = log full conditional of new vector ${\tau^2_\xi}_{new}$;

Set l_{old} = log full conditional of old vector τ^2_ξ;

Set $a = \exp(l_{new} - l_{old})$, that is, evaluate equation (12);

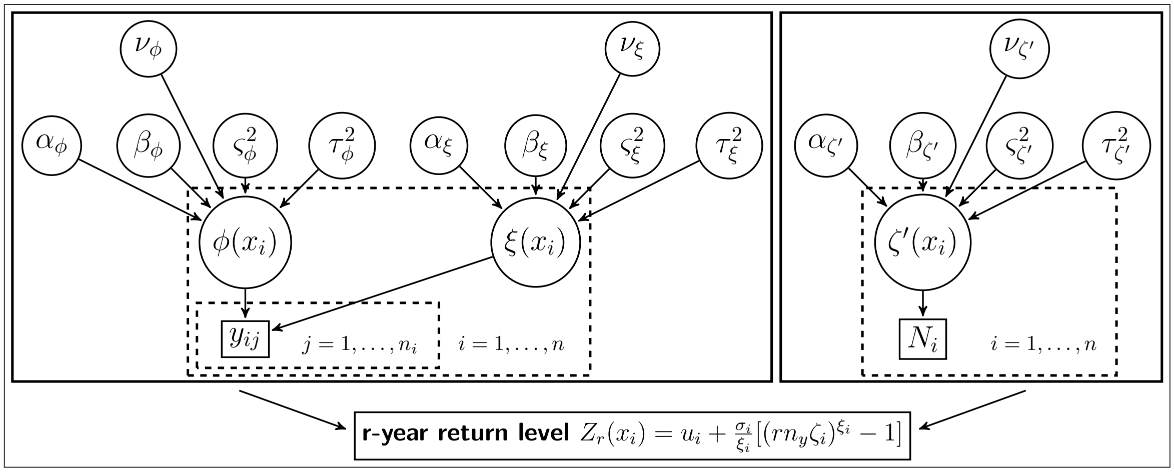
if $a > u$ **then**

Set $\tau^2_\xi = {\tau^2_\xi}_{new}$;

end

end

1 DAG



2 Notation

- x_i Multivariate (bivariate or tri-variate) location values for location i, $i = 1, \ldots, n$. Write the matrix of all locations as just x
- y_{ij} Excess j for observation i, $j = 1, \ldots, n_i$ where n_i is the number of excesses at location i
- $\sigma(x_i)$ Scale parameter for location x_i
- $\phi(x_i) = \log(\sigma(x_i))$ The re-parameterised scale parameter
- $\xi(x_i)$ Shape parameter for location x_i
- z_k Sub-grid locations $k = 1, \ldots, m$. Together written as z
- A, B Projection matrices of dimension $n \times m$
- $\tilde{\phi}, \tilde{\xi}$ Gaussian processes of ϕ and ξ defined on sub-grid z
- $\mu_\phi(z), \mu_\xi(z)$ Means for the Gaussian processes
- Σ, Ψ Auto-covariance matrices for the Gaussian processes
- τ_ϕ^2, τ_ξ^2 Nugget standard deviation parameters
- α_ϕ, α_ξ vectors of coefficients including intercept and slope parameters for Gaussian process means

- $X_\phi, X_\xi, Z_\sigma, Z_\xi$ Matrices of covariates (including column for intercept term) on the x grid and the z sub-grid
- ν_ϕ, ν_ξ Matern smoothness parameters
- β_ϕ, β_ξ Matern length scale matrices
- $\varsigma^2_\phi, \varsigma^2_\xi$ Variance parameters for Gaussian process

3 Model outline

In hierarchical notation:

$$
\begin{aligned}
y_{ij} &\sim GPD(\sigma(x_i), \xi(x_i)) \\
\log(\sigma(x)) = \phi(x) &= A(x,z)\Sigma^{-1}(z,z)\tilde{\phi}(z) \\
\xi(x) &= B(x,z)\Psi^{-1}(z,z)\tilde{\xi}(z) \\
A(x_i, z_k) &= \varsigma^2_\phi \frac{2^{1-\nu_\phi}}{\Gamma(\nu_\phi)} \left(\sqrt{2\nu_\phi}\frac{\|x_i - z_k\|}{\beta_\phi}\right)^{\nu_\phi} K_{\nu_\phi}\left(\sqrt{2\nu_\phi}\frac{\|x_i - z_k\|}{\beta_\phi}\right) \\
B(x_i, z_k) &= \varsigma^2_\xi \frac{2^{1-\nu_\xi}}{\Gamma(\nu_\xi)} \left(\sqrt{2\nu_\xi}\frac{\|x_i - z_k\|}{\beta_\xi}\right)^{\nu_\xi} K_{\nu_\xi}\left(\sqrt{2\nu_\xi}\frac{\|x_i - z_k\|}{\beta_\xi}\right) \\
\tilde{\phi} &\sim MVN_m(\mu_\phi(z), \tau^2_\phi I_m + \Sigma(z,z)) \\
\tilde{\xi} &\sim MVN_m(\mu_\xi(z), \tau^2_\xi I_m + \Psi(z,z)) \\
\mu_\phi(z) &= Z_\phi \alpha_\phi \\
\mu_\xi(z) &= Z_\xi \alpha_\xi \\
\Sigma(z_k, z_l) &= \varsigma^2_\phi \frac{2^{1-\nu_\phi}}{\Gamma(\nu_\phi)} \left(\sqrt{2\nu_\phi}\frac{\|z_k - z_l\|}{\beta_\phi}\right)^{\nu_\phi} K_{\nu_\phi}\left(\sqrt{2\nu_\phi}\frac{\|z_k - z_l\|}{\beta_\phi}\right) \\
\Psi(z_k, z_l) &= \varsigma^2_\xi \frac{2^{1-\nu_\xi}}{\Gamma(\nu_\xi)} \left(\sqrt{2\nu_\xi}\frac{\|z_k - z_l\|}{\beta_\xi}\right)^{\nu_\xi} K_{\nu_\xi}\left(\sqrt{2\nu_\xi}\frac{\|z_k - z_l\|}{\beta_\xi}\right)
\end{aligned}
$$

Hyper-parameter prior distributions (subject to change):

$$\log(\varsigma_\phi^2), \log(\varsigma_\xi^2), \log(\tau_\phi^2), \log(\tau_\xi^2) \sim N(0, 10)$$
$$\beta_\phi, \beta_\xi, \nu_\phi, \nu_\xi \sim DU(0.001, 0.01, 0.1, 1, 10, 100, 1000)$$
$$\alpha_\phi, \alpha_\xi \sim N(0, 100)$$

4 Posterior distribution

The full posterior distribution is:

$$\begin{aligned}
p(\alpha_\phi, \alpha_\xi, \beta_\phi, \beta_\xi, \varsigma_\phi^2, \varsigma_\xi^2, \tau_\phi^2, \tau_\xi^2, \nu_\phi, \nu_\xi, \tilde{\phi}, \tilde{\xi} | y, x, z, X_\phi, X_\xi, Z_\phi, Z_\xi) \propto & \left[\prod_{i=1}^{n} \prod_{j=1}^{n_i} p(y_{ij} | \sigma(x_i) = \exp(\phi(x_i)), \xi(x_i)) \right] \times \\
& p(\tilde{\phi}(x) | \mu_\phi, \tau_\phi^2, \Sigma) p(\tilde{\xi}(x) | \mu_\xi, \tau_\xi^2, \Psi) \times \\
& p(\alpha_\phi) p(\alpha_\xi) p(\beta_\phi) p(\beta_\xi) p(\nu_\phi) p(\nu_\xi) \times \\
& p(\tau_\phi^2) p(\tau_\xi^2) p(\varsigma_\phi^2) p(\varsigma_\xi^2)
\end{aligned}$$

5 Conditional posterior distributions: Layer 1

Updating the first layer of the DAG - that is, parameters $\tilde{\phi} = \log(\tilde{\sigma})$ and $\tilde{\xi}$. The conditional posterior distribution of $\tilde{\phi}$ is given by:

$$\begin{aligned}
\pi(\tilde{\phi} | y, x, z, X_\phi, X_\xi, Z_\phi, Z_\xi, &\alpha_\phi, \alpha_\xi, \beta_\phi, \beta_\xi, \varsigma_\phi^2, \varsigma_\xi^2, \tau_\phi^2, \tau_\xi^2, \nu_\phi, \nu_\xi, \tilde{\xi}) \propto \\
& p(y | \tilde{\phi}, x, z, X_\phi, X_\xi, Z_\phi, Z_\xi, \alpha_\phi, \alpha_\xi, \beta_\phi, \beta_\xi, \varsigma_\phi^2, \varsigma_\xi^2, \tau_\phi^2, \tau_\xi^2, \nu_\phi, \nu_\xi, \tilde{\xi}) \times \\
& p(\tilde{\phi} | x, z, X_\phi, X_\xi, Z_\phi, Z_\xi, \alpha_\phi, \alpha_\xi, \beta_\phi, \beta_\xi, \varsigma_\phi^2, \varsigma_\xi^2, \tau_\phi^2, \tau_\xi^2, \nu_\phi, \nu_\xi, \tilde{\xi}) \\
\propto & p(y | \sigma, \xi) p(\tilde{\phi} | \mu_\phi, \tau_\phi^2, \Sigma)
\end{aligned}$$

That is, the conditional posterior of $\tilde{\phi}$ is proportional to the product of the likelihood of the data y given $\tilde{\phi}$ and all other parameters, and the prior probability density of $\tilde{\phi}$ given all of the other parameters. In the final line, most parameters have dropped out from the right-hand side, as the densities are independent of these, given the remaining terms (see the DAG for this). Of the remaining terms σ, ξ, μ_ϕ and Σ are all deterministic given the other parameters. These will have been calculated using the formulae in section 3.

In briefer notation (to be used from now on):

$$
\begin{aligned}
\pi(\tilde{\phi}|\dots) \propto & p(y|\phi,\xi)p(\tilde{\phi}|\mu_\phi,\tau_\phi^2,\Sigma) \\
= & \left[\prod_{i=1}^{n}\prod_{j=1}^{n_i} p(y_{ij}|\sigma(x_i),\xi(x_i))\right] p(\tilde{\phi}|\mu_\phi,\tau_\phi^2,\Sigma) \\
= & \left[\prod_{i=1}^{n}\prod_{j=1}^{n_i} \frac{1}{\sigma(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)^{-(1/\xi(x_i)+1)}\right] \times \\
& \frac{1}{\sqrt{\det(2\pi(\Sigma+\tau_\phi^2 I_m))}} e^{-\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)}
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\pi(\tilde{\xi}|\dots) \propto & p(y|\phi,\xi)p(\tilde{\xi}|\mu_\xi,\tau_\xi^2,\Psi) \\
= & \left[\prod_{i=1}^{n}\prod_{j=1}^{n_i} p(y_{ij}|\sigma(x_i),\xi(x_i))\right] p(\tilde{\xi}|\mu_\xi,\tau_\xi^2,\Psi) \\
= & \left[\prod_{i=1}^{n}\prod_{j=1}^{n_i} \frac{1}{\sigma(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)^{-(1/\xi(x_i)+1)}\right] \times \\
& \frac{1}{\sqrt{\det(2\pi(\Psi+\tau_\xi^2 I_m))}} e^{-\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)}
\end{aligned}
$$

5.1 Metropolis-Hastings Markov chain Monte Carlo sampling

In order to sample from these conditional posterior distributions, we use the Metropolis-Hastings Markov chain Monte Carlo (MCMC) algorithm. New samples are accepted or rejected at random according to the algorithm outlined below.

A new value of $\tilde{\phi}_i$ is suggested: $\tilde{\phi}_i{}'$. The new vector ϕ' is then calculated using the projection formula.

Though $\tilde{\phi}'$ differs from $\tilde{\phi}$ in just one location i, ϕ and ϕ' can be different from each other in many locations

i due to this projection.

Suggested updates are drawn from a Normal distribution centred on the old value and with a variance of a manually set tuning parameter used to control the size of the proposed steps.

We calculate:

$$\rho(\tilde{\phi}_i, \tilde{\phi}_i') = \min\left(1, \frac{\pi(\tilde{\phi}'|\dots)q_t(\tilde{\phi}_i' \to \tilde{\phi}_i)}{\pi(\tilde{\phi}|\dots)q_t(\tilde{\phi}_i \to \tilde{\phi}_i')}\right)$$

where $\pi(\tilde{\phi}|\dots)$ is as defined above, and $q_t(a \to b)$ is the transition probability of proposing value b given value a. Since updates are proposed using a Normal distribution, these transition probabilites above and below the line will always cancel, so the above simplifies to:

$$\rho(\tilde{\phi}_i, \tilde{\phi}_i') = \min\left(1, \frac{\pi(\tilde{\phi}'|\dots)}{\pi(\tilde{\phi}|\dots)}\right)$$

Following this calculation, we always accept proposed value $\tilde{\phi}_i'$ when $\rho(\tilde{\phi}_i, \tilde{\phi}_i')$ is bigger than 1 and we reject accordingly when the ratio is smaller than 1 by simulating a random variable $u \sim U[0,1]$ and accepting proposed value $\tilde{\phi}_i'$ when $u \leq \rho(\tilde{\phi}_i, \tilde{\phi}_i')$.

Evaluating $\rho(\tilde{\phi}_i, \tilde{\phi}_i')$ typically involves products and quotients of many terms which may be close to 0. In order to work with something far more stable, we use the property that $x = \exp(\log(x))$.

Following this observation we need to evaluate:

$$
\begin{aligned}
&\exp\left(\log\left(\frac{\pi(\tilde{\phi}'|\dots)}{\pi(\tilde{\phi}|\dots)}\right)\right)\\
&= \exp\left[\log(\pi(\tilde{\phi}'|\dots)) - \log(\pi(\tilde{\phi}|\dots))\right]\\
&= \exp\Bigg[\log\Bigg(\Big[\prod_{i=1}^{n}\prod_{j=1}^{n_i} p(y_{ij}|\sigma'(x_i),\xi(x_i))\Big] p(\tilde{\phi}'|\mu_\phi,\tau_\phi^2,\Sigma)\Bigg) -\\
&\qquad \log\Bigg(\Big[\prod_{i=1}^{n}\prod_{j=1}^{n_i} p(y_{ij}|\sigma(x_i),\xi(x_i))\Big] p(\tilde{\phi}|\mu_\phi,\tau_\phi^2,\Sigma)\Bigg)\Bigg]
\end{aligned}
$$

Filling in the specific distribution from above, this becomes:

$$
\begin{aligned}
&\exp\Bigg[\log\Bigg(\Big[\prod_{i=1}^{n}\prod_{j=1}^{n_i}\frac{1}{\sigma'(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma'(x_i)}\right)^{-(1/\xi(x_i)+1)}\Big]\times\\
&\qquad \frac{1}{\sqrt{\det(2\pi(\Sigma+\tau_\phi^2 I_m))}} e^{-\frac{1}{2}(\tilde{\phi}'-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}'-\mu_\phi)}\Bigg)\\
&\quad -\log\Bigg(\Big[\prod_{i=1}^{n}\prod_{j=1}^{n_i}\frac{1}{\sigma(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)^{-(1/\xi(x_i)+1)}\Big]\times\\
&\qquad \frac{1}{\sqrt{\det(2\pi(\Sigma+\tau_\phi^2 I_m))}} e^{-\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)}\Bigg)\Bigg]\\
&= \exp\Bigg[\sum_{i=1}^{n}\sum_{j=1}^{n_i}\log\Bigg(\Big[\frac{1}{\sigma'(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma'(x_i)}\right)^{-(1/\xi(x_i)+1)}\Big]\Bigg)+\\
&\qquad \log\left(\frac{1}{\sqrt{\det(2\pi(\Sigma+\tau_\phi^2 I_m))}}\right) - \frac{1}{2}(\tilde{\phi}'-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}'-\mu_\phi)\\
&\quad -\sum_{i=1}^{n}\sum_{j=1}^{n_i}\log\Bigg(\Big[\frac{1}{\sigma(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)^{-(1/\xi(x_i)+1)}\Big]\Bigg)-\\
&\qquad \log\left(\frac{1}{\sqrt{\det(2\pi(\Sigma+\tau_\phi^2 I_m))}}\right) + \frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)\Bigg]\\
&= \exp\Bigg[\sum_{i=1}^{n}\sum_{j=1}^{n_i}\log\Bigg(\Big[\frac{1}{\sigma'(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma'(x_i)}\right)^{-(1/\xi(x_i)+1)}\Big]\Bigg) - \frac{1}{2}(\tilde{\phi}'-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}'-\mu_\phi)\\
&\quad -\sum_{i=1}^{n}\sum_{j=1}^{n_i}\log\Bigg(\Big[\frac{1}{\sigma(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)^{-(1/\xi(x_i)+1)}\Big]\Bigg) + \frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)\Bigg]
\end{aligned}
$$

The common term in both numerator and denominator above dropped out as it will be equal in both (it has no dependence on $\tilde{\phi}$).

The GPD can clearly be simplified further using the properties of logs. Just taking one of the functions on its own for clarity:

$$\sum_{i=1}^{n}\sum_{j=1}^{n_i}\log\left(\left[\frac{1}{\sigma'(x_i)}\left(1+\xi(x_i)\frac{y_{ij}}{\sigma'(x_i)}\right)^{-(1/\xi(x_i)+1)}\right]\right)=$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left[\log\Big(\frac{1}{\sigma'(x_i)}\Big)+\log\left(1+\xi(x_i)\frac{y_{ij}}{\sigma'(x_i)}\right)^{-(1/\xi(x_i)+1)}\right]=$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left[-\log(\sigma'(x_i))-\left(\frac{1}{\xi(x_i)}+1\right)\log\left(1+\xi(x_i)\frac{y_{ij}}{\sigma'(x_i)}\right)\right]$$

Following this, the full term we need to evaluate in order to update $\tilde{\phi}$ is:

$$\begin{aligned}=\exp\Bigg[&\sum_{i=1}^{n}\sum_{j=1}^{n_i}\Bigg[-\log(\sigma'(x_i))-\left(\frac{1}{\xi(x_i)}+1\right)\log\left(1+\xi(x_i)\frac{y_{ij}}{\sigma'(x_i)}\right)\Bigg]-\frac{1}{2}(\tilde{\phi}'-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}'-\mu_\phi)\\&-\sum_{i=1}^{n}\sum_{j=1}^{n_i}\Bigg[-\log(\sigma(x_i))-\left(\frac{1}{\xi(x_i)}+1\right)\log\left(1+\xi(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)\Bigg]+\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)\Bigg]\end{aligned}\tag{1}$$

Similar reasoning leads to the update for $\tilde{\xi}$. We need to evaluate:

$$\rho(\tilde{\xi}_i,\tilde{\xi}_i')=\min\left(1,\frac{\pi(\tilde{\xi}'|\ldots)q_t(\tilde{\xi}_i'\to\tilde{\xi}_i)}{\pi(\tilde{\xi}|\ldots)q_t(\tilde{\xi}_i\to\tilde{\xi}_i')}\right)$$

The full term we need to evaluate is:

$$\begin{aligned}=\exp\Bigg[&\sum_{i=1}^{n}\sum_{j=1}^{n_i}\Bigg[-\log(\sigma(x_i))-\left(\frac{1}{\xi'(x_i)}+1\right)\log\left(1+\xi'(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)\Bigg]-\frac{1}{2}(\tilde{\xi}'-\mu_\xi)'(\Psi+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}'-\mu_\xi)\\&-\sum_{i=1}^{n}\sum_{j=1}^{n_i}\Bigg[-\log(\sigma(x_i))-\left(\frac{1}{\xi(x_i)}+1\right)\log\left(1+\xi(x_i)\frac{y_{ij}}{\sigma(x_i)}\right)\Bigg]+\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)\Bigg]\end{aligned}\tag{2}$$

6 Conditional posterior distributions: Layer 2

6.1 α

Updating the second layer of the DAG - that is, all hyper-parameters - starting with α_ϕ:

$$\begin{aligned}
\pi(\alpha_\phi|y, x, z, X_\phi, X_\xi, Z_\phi, Z_\xi, &\alpha_\xi, \beta_\phi, \beta_\xi, \varsigma_\phi^2, \varsigma_\xi^2, \tau_\phi^2, \tau_\xi^2, \nu_\phi, \nu_\xi, \tilde{\phi}, \tilde{\xi}) \propto \\
&\quad p(y|\alpha_\phi, x, z, X_\phi, X_\xi, Z_\phi, Z_\xi, \alpha_\phi, \alpha_\xi, \beta_\phi, \beta_\xi, \varsigma_\phi^2, \varsigma_\xi^2, \tau_\phi^2, \tau_\xi^2, \nu_\phi, \nu_\xi, \tilde{\phi}, \tilde{\xi}) \times \\
&\quad p(\alpha_\phi|x, z, X_\phi, X_\xi, Z_\phi, Z_\xi, \alpha_\phi, \alpha_\xi, \beta_\phi, \beta_\xi, \varsigma_\phi^2, \varsigma_\xi^2, \tau_\phi^2, \tau_\xi^2, \nu_\phi, \nu_\xi, \tilde{\phi}, \tilde{\xi}) \\
&\propto p(y|\sigma, \xi) p(\tilde{\phi}|\mu_\phi, \tau_\phi^2, \Sigma) p(\alpha_\phi) \\
&\propto p(\tilde{\phi}|\mu_\phi, \tau_\phi^2, \Sigma) p(\alpha_\phi)
\end{aligned}$$

(I'm not sure my steps are correct there, notation-wise...) but essentially the GPD component is independent of α_ϕ once the other parameters are known, and so can be absorbed into the constant of proportionality. I'm not sure of the correct step to bring in the $p(\tilde{\phi}|\dots)$ piece. What we're left with is the MVN piece (since α_ϕ features in the calculation of μ_ϕ) and the prior on α_ϕ.

Then we have:

$$\begin{aligned}
\pi(\alpha_\phi|\dots) &\propto p(\tilde{\phi}|\mu_\phi, \tau_\phi^2, \Sigma) p(\alpha_\phi) \\
&\propto \frac{1}{\sqrt{\det(2\pi(\Sigma + \tau_\phi^2 I_m))}} e^{-\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)} \times \\
&\quad \frac{1}{\sqrt{\det(2\pi\Lambda_\phi)}} e^{-\frac{1}{2}(\alpha_\phi-\lambda_\alpha)'\Lambda_\phi^{-1}(\alpha_\phi-\lambda_\alpha)}
\end{aligned}$$

where Λ_ϕ is the covariance matrix for the prior distribution of α_ϕ and λ_ϕ is the prior mean.

Similarly:

$$
\begin{aligned}
\pi(\alpha_\xi|\dots) \propto & p(\tilde{\xi}|\mu_\xi, \tau_\xi^2, \Psi)p(\alpha_\xi) \\
& \propto \frac{1}{\sqrt{\det(2\pi(\Psi + \tau_\xi^2 I_m))}} e^{-\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)} \times \\
& \frac{1}{\sqrt{\det(2\pi\Lambda_\xi)}} e^{-\frac{1}{2}(\alpha_\xi-\omega_\alpha)'\Lambda_\xi^{-1}(\alpha_\xi-\omega_\alpha)}
\end{aligned}
$$

where Λ_ξ is the covariance matrix for the prior distribution of α_ξ and ω_α is the prior mean.

As before, updates are suggested for alpha element-wise: $\alpha_{\phi,k} \to \alpha'_{\phi,k}$. The ratio $\rho(\alpha_{\phi,k}, \alpha'_{\phi,k})$ is then calculated. Similar manipulations to those used in the previous section lead to the following calculation needed to update α_ϕ:

$$
\begin{aligned}
\exp\Big[& -\frac{1}{2}\log(\det(2\pi(\Sigma + \tau_\phi^2 I_m))) - \frac{1}{2}(\tilde{\phi} - \mu_\phi)'(\Sigma + \tau_\phi^2 I_m)^{-1}(\tilde{\phi} - \mu_\phi) \\
& - \frac{1}{2}\log(\det(2\pi\Lambda_\phi)) - \frac{1}{2}(\alpha'_\phi - \lambda_\alpha)'\Lambda_\phi^{-1}(\alpha'_\phi - \lambda_\alpha)) \\
& + \frac{1}{2}\log(\det(2\pi(\Sigma + \tau_\phi^2 I_m))) + \frac{1}{2}(\tilde{\phi} - \mu_\phi)'(\Sigma + \tau_\phi^2 I_m)^{-1}(\tilde{\phi} - \mu_\phi) \\
& + \frac{1}{2}\log(\det(2\pi\Lambda_\phi)) + \frac{1}{2}(\alpha_\phi - \lambda_\alpha)'\Lambda_\phi^{-1}(\alpha_\phi - \lambda_\alpha))\Big] = \\
\exp\Big[& -\frac{1}{2}(\tilde{\phi} - \mu_\phi)'(\Sigma + \tau_\phi^2 I_m)^{-1}(\tilde{\phi} - \mu_\phi) - \frac{1}{2}(\alpha'_\phi - \lambda_\alpha)'\Lambda_\xi^{-1}(\alpha'_\phi - \lambda_\alpha)) \\
& + \frac{1}{2}(\tilde{\phi} - \mu_\phi)'(\Sigma + \tau_\phi^2 I_m)^{-1}(\tilde{\phi} - \mu_\phi) + \frac{1}{2}(\alpha_\phi - \lambda_\alpha)'\Lambda_\phi^{-1}(\alpha_\phi - \lambda_\alpha))\Big] \qquad (3)
\end{aligned}
$$

And for α_ξ:

$$
\begin{aligned}
\exp\Big[& -\frac{1}{2}(\tilde{\xi} - \mu_\xi)'(\Psi + \tau_\xi^2 I_m)^{-1}(\tilde{\xi} - \mu_\xi) - \frac{1}{2}(\alpha'_\xi - \omega_\alpha)'\Lambda_\xi^{-1}(\alpha'_\xi - \omega_\alpha)) \\
& + \frac{1}{2}(\tilde{\xi} - \mu_\xi)'(\Psi + \tau_\xi^2 I_m)^{-1}(\tilde{\xi} - \mu_\xi) + \frac{1}{2}(\alpha_\xi - \omega_\alpha)'\Lambda_\xi^{-1}(\alpha_\xi - \omega_\alpha))\Big] \qquad (4)
\end{aligned}
$$

Updates for the other hyper-parameters are very similar - though a little bit shorter as none feature the MVN prior that α_ϕ and α_ξ do. The remaining hyper-parameters either have a discrete update (in which case the prior probability will be $^1/_{\#discrete.values}$ and will cancel above and below the line, or have a univariate Normal prior on them (or their log).

6.2 β

For β_ϕ we have:

$$\pi(\beta_\phi | \dots) \propto p(\tilde{\phi} | \mu_\phi, \tau_\phi^2, \Sigma) p(\beta_\phi)$$

Since β_ϕ has a discrete prior, to update β_ϕ we simply need to evaluate:

$$\exp\Big[-\frac{1}{2}(\tilde{\phi} - \mu_\phi)'(\Sigma' + \tau_\phi^2 I_m)^{-1}(\tilde{\phi} - \mu_\phi) + \frac{1}{2}(\tilde{\phi} - \mu_\phi)'(\Sigma + \tau_\phi^2 I_m)^{-1}(\tilde{\phi} - \mu_\phi)\Big] \tag{5}$$

where Σ' has been formed using β'_ϕ, the new proposal value.

Then for β_ξ we have:

$$\pi(\beta_\xi | \dots) \propto p(\tilde{\xi} | \mu_\xi, \tau_\xi^2, \Psi) p(\beta_\xi)$$

To update β_ξ we need to evaluate:

$$\exp\Big[-\frac{1}{2}(\tilde{\xi} - \mu_\xi)'(\Psi' + \tau_\xi^2 I_m)^{-1}(\tilde{\xi} - \mu_\xi) + \frac{1}{2}(\tilde{\xi} - \mu_\xi)'(\Psi + \tau_\xi^2 I_m)^{-1}(\tilde{\xi} - \mu_\xi)\Big] \tag{6}$$

where Ψ' has been formed using β'_ξ, the new proposal value.

6.3 ν

ν_ϕ also has a discrete update and so will look identical to the update for β_ϕ. It has conditional posterior distribution of:

$$\pi(\nu_\phi|\dots) \propto p(\tilde{\phi}|\mu_\phi, \tau_\phi^2, \Sigma)p(\nu_\phi)$$

We need to evaluate:

$$\exp\Big[-\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma' + \tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi) + \frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma + \tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)\Big] \tag{7}$$

where Σ' has been formed using ν'_ϕ, the new proposal value.

In order to update ν_ξ we have:

$$\pi(\nu_\xi|\dots) \propto p(\tilde{\xi}|\mu_\xi, \tau_\xi^2, \Psi)p(\nu_\xi)$$

We therefore need to evaluate:

$$\exp\Big[-\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi' + \tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi) + \frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi + \tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)\Big] \tag{8}$$

where Ψ' has been formed using ν'_ξ, the new proposal value.

6.4 ς^2

ς_ϕ^2 has a conditional posterior distribution of:

$$\pi(\varsigma_\phi^2|\dots) \propto p(\tilde{\phi}|\mu_\phi, \tau_\phi^2, \Sigma)p(\varsigma_\phi^2)$$

The prior distribution of the log of ς_ϕ^2 is a univariate Normal. So to update ς_ϕ^2 we need to evaluate:

$$\exp\bigg[-\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma'+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)-\frac{(\log(\varsigma_\phi^{2'})-m)^2}{2s^2}$$
$$+\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)+\frac{(\log(\varsigma_\phi^{2})-m)^2}{2s^2}\bigg] \tag{9}$$

where Σ' has been formed using $\varsigma_\phi^{2'}$, the new proposal value, and m and s are the prior mean and standard deviation repectively.

ς_ξ^2 has a conditional posterior distribution of:

$$\pi(\varsigma_\xi^2|\dots) \propto p(\tilde{\xi}|\mu_\xi,\tau_\xi^2,\Psi)p(\varsigma_\xi^2)$$

To update ς_ξ^2 we need to evaluate:

$$\exp\bigg[-\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi'+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)-\frac{(\log(\varsigma_\xi^{2'})-m)^2}{2s^2}$$
$$+\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)+\frac{(\log(\varsigma_\xi^{2})-m)^2}{2s^2}\bigg] \tag{10}$$

where Ψ' has been formed using $\varsigma_\xi^{2'}$, the new proposal value, and m and s are the prior mean and standard deviation repectively.

6.5 τ^2

τ_ϕ^2 has a conditional posterior distribution of:

$$\pi(\tau_\phi^2|\dots) \propto p(\tilde{\phi}|\mu_\phi,\tau_\phi^2,\Sigma)p(\tau_\phi^2)$$

The prior distribution of the log of τ_ϕ^2 is a univariate Normal. So to update τ_ϕ^2 we need to evaluate:

$$\exp\Bigg[-\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma'+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)-\frac{(\log(\tau_\phi^{2'})-m)^2}{2s^2}$$
$$+\frac{1}{2}(\tilde{\phi}-\mu_\phi)'(\Sigma+\tau_\phi^2 I_m)^{-1}(\tilde{\phi}-\mu_\phi)+\frac{(\log(\tau_\phi^{2})-m)^2}{2s^2}\Bigg] \quad (11)$$

where Σ' has been formed using $\tau_\phi^{2'}$, the new proposal value, and m and s are the prior mean and standard deviation repectively.

τ_ξ^2 has a conditional posterior distribution of:

$$\pi(\tau_\xi^2|\dots) \propto p(\tilde{\xi}|\mu_\xi, \tau_\xi^2, \Psi)p(\tau_\xi^2)$$

To update τ_ξ^2 we need to evaluate:

$$\exp\Bigg[-\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi'+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)-\frac{(\log(\tau_\xi^{2'})-m)^2}{2s^2}$$
$$+\frac{1}{2}(\tilde{\xi}-\mu_\xi)'(\Psi+\tau_\xi^2 I_m)^{-1}(\tilde{\xi}-\mu_\xi)+\frac{(\log(\tau_\xi^{2})-m)^2}{2s^2}\Bigg] \quad (12)$$

where Ψ' has been formed using $\tau_\xi^{2'}$, the new proposal value, and m and s are the prior mean and standard deviation repectively.