Mie Theory Sphere in Medium.

Consider Maxwell's equations in a homogeneous non-absorbing medium:

$$(1) \quad \nabla \times \vec{B} = \frac{n^2 \varepsilon_m}{c^2} \frac{\partial \vec{E}}{\partial t} \quad \& \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2)$$

We also have:

$$\nabla \cdot \vec{E} = \rho = 0 \quad \& \quad \nabla \cdot \vec{B} = 0$$

i.e the divergence in electric field vanishes in a homogeneous medium
Apply curl ($\nabla \times$) operator to each of eq^{ns} (1) & (2):

$$\nabla \times (\nabla \times \vec{B}) = \frac{n^2 \varepsilon_m}{c^2} \frac{\partial (\nabla \times \vec{E})}{\partial t} = -\frac{n^2 \varepsilon_m}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial (\nabla \times \vec{B})}{\partial t} = -\frac{n^2 \varepsilon_m}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Also consider

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla \cdot \nabla \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$\therefore$

$$\nabla \times (\nabla \times \vec{E}) = \nabla \underbrace{(\nabla \cdot \vec{E})}_{= 0} - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla \underbrace{(\nabla \cdot \vec{B})}_{= 0} - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$$

So we have

$$\nabla^2 \vec{B} = \frac{n^2 \varepsilon_m}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} \; ; \quad \nabla^2 \vec{E} = \frac{n^2 \varepsilon_m}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

The electromagnetic wave vector equation

$$\nabla^2 \vec{A} = \frac{n^2 \varepsilon_m}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \quad , \quad \underline{\underline{\vec{A} = \vec{E} \text{ or } \vec{B}}}$$

each component of $\vec{A}$ must satisfy the scalar wave equation...

$$\nabla^2 \varphi = \frac{n^2 \epsilon_m}{c^2} \frac{\partial^2 \varphi}{\partial t^2}$$

assume φ to be separable:

$$\varphi(x, t) = X(x) T(t)$$

$$\nabla^2 X(x) T(t) = \frac{n^2 \epsilon_m}{c^2} \cdot \frac{\partial^2 X(x) T(t)}{\partial t^2}$$

$$\frac{\nabla^2 X(x)}{X(x)} = \frac{n^2 \epsilon_m}{c^2} \frac{1}{T(t)} \frac{\partial^2 T(t)}{\partial t^2}$$

Let $\beta = \frac{2\omega c}{n} = k$ $\qquad \frac{\partial^2 T(t)}{\partial t^2} + \omega^2 T(t) = 0$

$$T \sim \begin{Bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{Bmatrix}$$

and the spatial component

$$\nabla^2 X + k^2 \epsilon_m X = 0 \qquad (\text{Helmholtz Eq}^{n})$$

write in spherical coordinates:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}\right] X + k^2 \epsilon_m X = 0 \quad (3)$$

~~(3) × r² sin θ~~ assume X is separable ie $X(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$(4) \quad \frac{1}{r^2 R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{r^2 \Theta \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \frac{1}{r^2 \Phi \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2} + k^2 \epsilon_m = 0$$

$(4) \times r^2 \sin^2\theta$

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} + k^2\varepsilon_m r^2\sin^2\theta = 0$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = \text{constant} = -m^2$$

$$\frac{\partial^2\Phi}{\partial\phi^2} + m^2\Phi = 0$$

$$\Phi \sim \begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix}$$

$\Phi(0) = \Phi(2\pi) \therefore m \in \mathbb{Z}$

so eq^n (4) becomes

(5) $$\frac{1}{r^2R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{r^2\sin^2\theta} + k^2\varepsilon_m = 0$$

(5) $\times r^2$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} + k^2\varepsilon_m r^2 = 0$$

(5a) $$\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} = \text{constant} = \ell$$

so eq^n (5) becomes

(6) $$\frac{1}{r^2R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\ell}{r^2} + k^2\varepsilon_m = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \left[\varepsilon_m k^2 + \frac{l}{r^2}\right]R = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \left[\varepsilon_m k^2 r^2 + l\right]R = 0$$

$$r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} + \left[\varepsilon_m k^2 r^2 + l\right]R = 0$$

if we let $l = -l(l+1)$

$$r^2\frac{\partial^2 R}{\partial r^2} + 2r\frac{\partial R}{\partial r} + \left[\varepsilon_m k^2 r^2 - l(l+1)\right]R = 0$$

the sol^{n} to this is defined as a spherical Bessel function

$$R \sim \begin{Bmatrix} j_l(\hat{n}_m k r) \\ y_l(\hat{n}_m k r) \end{Bmatrix} \quad \text{where } \hat{n}_m = \sqrt{\varepsilon_m} \leftarrow \text{refractive index of medium.}$$

and
$$j_l = \sqrt{\frac{\pi}{2\hat{n}_m k r}}\, J_{(l+\frac{1}{2})}(\hat{n}_m k r)$$

$$y_l = \sqrt{\frac{\pi}{2\hat{n}_m k r}}\, Y_{(l+\frac{1}{2})}(\hat{n}_m k r)$$

with $J_{(l+\frac{1}{2})}(x)$ / $Y_{(l+\frac{1}{2})}(x)$ are the ordinary Bessel functions.

consider (5b)
$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) - m^2/\sin^2\theta = -l(l+1)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + l(l+1) - m^2/\sin^2\theta = 0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \Theta\left[l(l+1) - \frac{m^2}{\sin^2\theta}\right] = 0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \Theta\left[l(l+1) - \frac{m^2}{1-\cos^2\theta}\right] = 0$$

$$\frac{1}{\sin\theta}\left\{\sin^2\theta\frac{\partial^2\Theta}{\partial\theta^2} - 2\cos\theta\frac{\partial\Theta}{\partial\theta}\right\} + \Theta\left[l(l+1) - \frac{m^2}{1-\cos^2\theta}\right] = 0$$

$$(1-\cos^2\theta)\frac{\partial^2\Theta}{\partial(\cos\theta)^2} - 2\cos\theta\frac{\partial\Theta}{\partial(\cos\theta)} + \Theta\left[l(l+1) - \frac{m^2}{1-\cos^2\theta}\right] = 0$$

these eq's satisfy the Legendre Polynomials

$$\Theta \sim \begin{Bmatrix} P_{l,m}(\cos\theta) \\ Q_{l,m}(\cos\theta) \end{Bmatrix}$$

so we now have φ

$$\varphi \sim \begin{Bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{Bmatrix}\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}\begin{Bmatrix} j_l(\hat{n}_m kr)\\ y_l(\hat{n}_m kr)\end{Bmatrix}\begin{Bmatrix}P_{l,m}(\cos\theta)\\ Q_{l,m}(\cos\theta)\end{Bmatrix}$$

$Q_{l,m}(\cos\theta)$ has singularities in $\theta \in [0, 2\pi]$ $\quad y_l(\hat{n}_m kr) \to \infty$ as $r \to 0$
$\Rightarrow$ both dropped
$h_l^{(2)}(\hat{n}_m kr) \sim \frac{i^{l-1}}{\hat{n}_m kr} e^{-i\hat{n}_m kr}$ at $\infty \Rightarrow$ ~~ellipticalle~~ spherical wave
✓✓ Huygen's principle.

so $\quad \varphi \sim e^{\mp i\omega t}\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}\begin{Bmatrix}j_l(\hat{n}_m kr)\\ y_l(\hat{n}_m kr)\end{Bmatrix}P_{l,m}(\cos\theta)$

so we now have sol^{n} to

$\nabla^2 X + k^2\varepsilon_m X = 0$ but need sol^{n} to vector WE so...

$$X \sim \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \begin{Bmatrix} j_l(\hat{n}_m kr) \\ y_l(\hat{n}_m kr) \end{Bmatrix} P_{l,m}(\cos\theta)$$

Theorem: If X satisfies the scalar wave equation then the vectors $\vec{M}_x$ & $\vec{N}_x$ must satisfy the vectorial Helmholtz equation where $\vec{M}_x$ & $\vec{N}_x$ are defined by...

$$\vec{M}_x = \nabla \times (\vec{r} X)$$
$$\hat{n}_m k \vec{N}_x = \nabla \times \vec{M}_x$$

and related by...

$$\hat{n}_m k \vec{M}_x = \nabla \times \vec{N}_x$$

Aside 1: Check this...

$$\nabla \times (\vec{r} \nabla^2 X) + \varepsilon_m k^2 \vec{M}_x = 0$$

using

$$\nabla [\nabla \cdot (\nabla X \otimes \vec{r})] = \vec{r}(\nabla \otimes \nabla X) - \nabla X(\nabla \otimes \vec{r}) = 0$$

we get...

$$\cancel{\nabla \otimes \nabla \otimes (\nabla X \otimes \vec{r}) = \nabla \times [(\vec{r} \cdot \nabla)\nabla X - \vec{r} \nabla^2 X}$$

$$\nabla \otimes \nabla \otimes (\nabla X \otimes \vec{r}) = \nabla [\nabla \cdot (\nabla X \otimes \vec{r})] - \nabla^2 (\nabla X \otimes \vec{r}) = -\nabla^2 (\nabla X \otimes$$

also

$$\nabla \otimes \nabla \otimes (\nabla X \otimes \vec{r}) = \nabla \otimes [(\vec{r} \cdot \nabla)\nabla X - \vec{r} \nabla^2 X - (\nabla X \cdot \nabla)\vec{r} + \nabla X(\nabla \cdot \vec{r})]$$

$$= \nabla \otimes [(\vec{r} \cdot \nabla)\nabla X] - \nabla \otimes [\vec{r} \nabla^2 X] - \nabla \otimes [(\nabla X \cdot \nabla)\vec{r}] + \nabla \otimes [\nabla X \otimes (\nabla \cdot \vec{r})]$$

using

$$\vec{r} \cdot \nabla = r\frac{\partial}{\partial r} \Rightarrow \nabla \times [(\vec{r} \cdot \nabla)\nabla X] - \nabla \times \left(r \frac{\partial}{\partial r} \nabla X\right) = 0$$

$$(\nabla X \cdot \nabla)\vec{r} = \nabla X \Rightarrow \nabla \times [(\nabla X \cdot \nabla)\vec{r}] = \nabla \times \nabla X = 0$$

$$\nabla \cdot \vec{r} \Rightarrow \nabla \times [\nabla X (\nabla \vec{r})] = 0$$

giving
$$\nabla \otimes \nabla \otimes (\nabla \chi \otimes \vec{r}) = -\nabla \otimes (\vec{r} \nabla^2 \chi)$$

so we can say...

$$\nabla \times (\vec{r} \nabla^2 \chi) = \nabla^2 (\nabla \chi \times \vec{r})$$

∴

$$\nabla \times (\vec{r} \nabla^2 \chi) = \nabla^2 (\nabla \chi \times \vec{r})$$

merging

$$\nabla^2 \vec{M}_x + \varepsilon_m k^2 \vec{M}_x = 0 \quad \checkmark\checkmark \quad \vec{M}_x \text{ satisfy Helmholtz}$$

$$\vec{E} = M_v + i N_u$$

$$\cancel{B =}$$

$$\vec{B} = \hat{n}_m (-M_u + i N_v)$$

where u, v are solutions to Helmholtz eq$^{\underline{n}}$

Consider a linearly polarised incoming plane wave

$$\vec{E} = \vec{a}_x e^{-ikz + i\omega t}$$

$$\vec{H} = \vec{a}_y e^{-ikz + i\omega t}$$

Outside incident wave $m = 1$

$$\cancel{u = e^{i\omega t} \sum \cos\varphi \sum_{l=1}^{\infty} (-i)^l \frac{2l+1}{l(l+1)} P_l^1(\cos\theta) j_l(\hat{n}_m k r)}$$

$$\cancel{v = e^{i\omega t} \sin\varphi \sum_{l=1}^{\infty} (-i)^l \, 2l+1}$$

$$u = e^{i\omega t} \cos\phi \sum_{l=1}^{\infty} (-i)^l \frac{2l+1}{l(l+1)} P_l(\cos\theta) j_l(\hat{n}_m k r)$$

$$v = e^{i\omega t} \sin\phi \sum_{l=1}^{\infty} (-i)^l \frac{2l+1}{l(l+1)} P_l(\cos\theta) j_l(\hat{n}_m k r)$$

Outside wave scattered ⇒ use Hankel function for Huygen's Principle satisfaction.

$$u = -e^{i\omega t} \cos\phi\, E_m \sum_{l=1}^{\infty} a_n (-i)^l \frac{2l+1}{l(l+1)} P_l(\cos\theta)\, h_l^2(\hat{n}_m k r)$$

$$v = -e^{i\omega t} \sin\phi\, E_m \sum_{l=1}^{\infty} b_n (-i)^l \frac{2l+1}{l(l+1)} P_l(\cos\theta)\, h_l^2(\hat{n}_m k r)$$

Inside Now follow Van de Hulst... pg 122.

un