# KODAIRA DIMENSION OF MODULI OF SPECIAL $K3^{[2]}$-FOURFOLDS OF DEGREE 2

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ABSTRACT. We study the Noether-Lefschetz locus of the moduli space $\mathcal{M}$ of $K3^{[2]}$-fourfolds with a polarization of degree 2. Following Hassett's work on cubic fourfolds, Debarre, Iliev, and Manivel have shown that the Noether-Lefschetz locus in $\mathcal{M}$ is a countable union of special divisors $\mathcal{M}_d$, where the discriminant $d$ is a positive integer congruent to 0, 2, or 4 modulo 8. We compute the Kodaira dimensions of these special divisors for all but finitely many discriminants; in particular, we show that for $d > 224$ and for many other small values of $d$, the space $\mathcal{M}_d$ is a variety of general type.

## 1. INTRODUCTION

The aim of this paper is to study the internal geometry of some moduli spaces of hyperkähler fourfolds. Let $\mathcal{M}$ denote the moduli space of complex four-dimensional polarized hyperkähler (HK) manifolds of $K3^{[2]}$ type with polarization of degree 2. It is an irreducible quasi-projective variety of dimension 20. Recall that for any HK manifold $X$, the Picard group Pic X injects (via the exponential exact sequence) into the singular cohomology group $H^2(X, \mathbf{Z})$. The *Beauville-Bogomolov form* $q_X \colon H^2(X, \mathbf{Z}) \to \mathbf{Z}$ equips $H^2(X, \mathbf{Z})$ with the structure of an even integral lattice. A point $p \in \mathcal{M}$ is represented by a pair $(X, H)$ where $X$ is an HK fourfold of deformation type $K3^{[2]}$ and $H \in \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbf{Z})$ is a primitive, ample divisor with $q_X(H) = H^2 = 2$.

1.1. **Statement of main theorem.** A very general $X \in \mathcal{M}$ has the property that $X$ has Picard rank 1. The locus where this property fails is the *Noether-Lefschetz locus* $NL(\mathcal{M})$ of $\mathcal{M}$:
$$NL(\mathcal{M}) = \{(X, H) \in \mathcal{M}(\mathbf{C}) : \ \mathrm{rk} \, \mathrm{Pic} \, X \geq 2\}.$$
A polarized HK fourfold $(X, H)$ is said to be *special* if $(X, H) \in NL(\mathcal{M})$. The fourfold $X$ together a polarization $H$ and a sublattice $K \subseteq \mathrm{Pic} \, X$ of rank 2 containing $H$ form the data of a *special labelling of discriminant $d$* for $X$ (or more precisely, for $(X, H)$), where $d = |D(K^{\perp}_{H^2(X, \mathbf{Z})})|$ (cf. [DM, §4]).

For each $d$, there is a moduli space $\mathcal{M}_d \subset \mathcal{M}$ of polarized special $K3^{[2]}$-fourfolds of discriminant $d$. The nonempty $\mathcal{M}_d$ are hypersurfaces in $\mathcal{M}$, first studied by Debarre, Iliev, and Manivel in [DIM15] as the locus of Hodge structures possessing a special labelling in the period domain for prime Fano fourfolds of index 10 and degree 2 with special labelling of discriminant $d$ (such Fano fourfolds are known as *Gushel-Mukai fourfolds*). They prove that the moduli space $\mathcal{M}_d$ is nonempty if and only if $d \notin \{2, 8\}$ and $d \equiv 0, 2, 4 \bmod 8$.

Furthermore, the divisor $\mathcal{M}_d$ is irreducible if $d \equiv 0, 4 \bmod 8$ or $d = 10$; otherwise, when $d \equiv 2 \bmod 8$, the hypersurface $\mathcal{M}_d$ of special fourfolds of discriminant $d$ is the union of two irreducible divisors, denoted $\mathcal{M}'_d$ and $\mathcal{M}''_d$, which are birationally isomorphic (see [DM, Theorem 6.1]).

In this paper, we determine the Kodaira dimension of $\mathcal{M}_d$ for nearly every value of $d$. We show $\mathcal{M}_d$ is of general type for almost all $d$:

$$d > 224 \implies \kappa(\mathcal{M}_d) = 19.$$

Moreover, we push our methods to determine the Kodaira dimension for many other small values of $d$. Our results, together with the additional inputs to be discussed in §1.2, determine information about the birational type of $\mathcal{M}_d$ for all but 34 discriminants.[1]

Our goal is to prove the following theorem:

**Theorem 1.1.** *Let $\mathcal{M}$ denote the moduli space of hyperkähler fourfolds of degree 2 of $K3^{[2]}$-type, and let $\mathcal{M}_d \subset \mathcal{M}$ denote the moduli space of special $K3^{[2]}$-fourfolds with a special labelling of discriminant $d$.*

(1) *Suppose that $d = 8m$ with $m \geq 11$. Then $\mathcal{M}_d$ is of general type for $m \notin \{11, 12, 13, 14, 16, 17, 22, 25, 28\}$. Furthermore, for $m \notin \{14, 16, 22\}$, the variety $\mathcal{M}_d$ has nonnegative Kodaira dimension.*

(2) *Suppose that $d = 8m + 2$ with $m \geq 12$. Then $\mathcal{M}_d$ has two birationally isomorphic irreducible components, $\mathcal{M}'_d$ and $\mathcal{M}''_d$, both of which are of general type when $m \notin \{12, 13, 14, 15, 16, 17, 21, 23\}$. Furthermore, for $m \notin \{14, 16\}$, the varieties $\mathcal{M}'_d$ and $\mathcal{M}''_d$ have nonnegative Kodaira dimension.*

(3) *Suppose that $d = 8m + 4$ with $m \geq 14$. Then $\mathcal{M}_d$ is of general type if $m \notin \{15, 17, 21, 25, 27\}$. Furthermore, for $m \neq 15$, the variety $\mathcal{M}_d$ has nonnegative Kodaira dimension.*

The idea of the proof is to work with the global period domain $\mathcal{D}_d$, an irreducible quasi-projective variety. The Torelli theorem for $\mathcal{M}$ shows that $\mathcal{M}_d$ is a Zariski open subset of $\mathcal{D}_d$. Then we use automorphic techniques developed by Gritsenko-Hulek-Sankaran in [GHS07] and [GHS13] to study the Kodaira dimension of $\mathcal{D}_d$. This requires the construction of special odd weight modular forms on certain quotients of type IV Hermitian symmetric domains of the form $\widetilde{O}^+(L) \backslash \Omega_L^+$ (see 2 for the relevant definitions).

We note that by a result of Ma, there are only finitely many even integral lattices $L$ of signature $(2, n)$ such that $\widetilde{O}^+(L) \backslash \Omega_L^+$ is of general type ([Ma18, Theorem 1.3]). Ma's result implies that each nonempty $\mathcal{D}_d$ is of general type for $d \geq D_0$, where $D_0$ is some constant $D_0 \geq 5.5 \cdot 10^{16}$. In this note, we find a smaller upper bound, $d_0 = 224$, such that each nonempty $\mathcal{D}_d$ is of general type for $d > d_0$.

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[1]The 34 discriminants for which we have no information on the Kodaira dimension of $\mathcal{M}_d$ at the present time are: $12, 16, 18, 24, 28, 32, 36, 40, 42, 48, 50, 52, 56, 58, 60, 64, 66, 68, 72, 74, 76, 80, 82, 84, 90, 92, 100, 108, 112, 114, 124, 128, 130, 176$.

1.2. **Relationship to $\mathcal{K}_d$ and $\mathcal{C}_d$.** There are 40 values of $d$ for which the techniques used to prove Theorem 1.1 do not yield any information about $\mathcal{M}_d$. However, it is possible to use results on the Kodaira dimension of the moduli space of degree $d$ polarized $K3$ surfaces $\mathcal{K}_d$ to conclude something about $\mathcal{M}_d$ for some of these discriminants. For $d = 2k$ with $1 \leq k \leq 13$ or $k \in \{15, 16, 17, 19\}$, it is known that $\mathcal{K}_d$ has negative Kodaira dimension, and in fact $\mathcal{K}_d$ is unirational ([GHS13, Theorem 4.1] and [Nue16]). Since $\mathcal{K}_d$ dominates $\mathcal{M}_d$ whenever $d$ is not divisible by a prime 3 mod 4 and $\mathcal{M}_d \neq \emptyset$ ([DIM15, Proposition 6.5]), we conclude that $\mathcal{M}_d$ has negative Kodaira dimension and is in fact unirational when $d \in \{4, 10, 20, 26, 34\}$.

Similarly, the moduli space $\mathcal{C}_d$ of special cubic fourfolds of discriminant $d$ dominates $\mathcal{M}_d$ whenever $d \equiv 2$ or 20 mod 24 and the only odd primes dividing $d$ are congruent to $\pm 1$ mod 12 ([DIM15, Proposition 6.5]). The only new information this yields about the Kodaira dimension of $\mathcal{M}_d$ is that $\mathcal{M}_{44}$ has negative Kodaira dimension, since $\mathcal{C}_{44}$ is uniruled by work of Nuer (see [Nue16]).

1.3. **EPW double sextics and $\mathcal{M}_d$.** O'Grady has shown that a general $(X, H) \in \mathcal{M}$ is a smooth EPW double sextic (see [O'G06]). Precisely, there is a Zariski open subset $\mathcal{U}$ of $\mathcal{M}$ parametrizing pairs $(X, H)$ with ample and base-point free $H$ such that $|H|: X \to \mathbf{P}^5$ realizes $X$ as a ramified double cover of an EPW sextic in $\mathbf{P}^5$. We can consider the subvariety $\mathcal{U}_d = \mathcal{M}_d \cap \mathcal{U} \subset \mathcal{M}$ in $\mathcal{U}$ parametrizing EPW double sextics which have a special labelling of discriminant $d$. It is possible that $\mathcal{U}_d = \emptyset$ but $\mathcal{M}_d \neq \emptyset$; for example, there is no special EPW sextic of discriminant 4, so $\mathcal{U}_4 = \emptyset$ (see [Deb18, Example 6.3]). Still, for $d$ sufficiently large, it holds that $\mathcal{U}_d$ is birational to $\mathcal{M}_d$. Thus, we may conclude that $\mathcal{U}_d$ is of general type for such $d$. It would follow from a conjecture of O'Grady [DM, Example 6.3] that $\mathcal{U}_d$ is birational to $\mathcal{M}_d$ for all $d \neq 4$ (where $d$ is such that $\mathcal{M}_d$ is nonempty).

**Corollary 1.2.** *Let $\mathcal{U}_d$ denote the moduli space of smooth EPW double sextics that possess a special labelling of discriminant $d$. Then for all sufficiently large $d$ the following conclusions hold:*

- *If $d \equiv 0, 4$ mod 8, the space $\mathcal{U}_d$ is of general type.*
- *If $d \equiv 2$ mod 8, both irreducible components of $\mathcal{U}_d$ are of general type.*

If O'Grady's conjecture is true, then one can take $d > 224$ in Corollary 1.2, but at present the result remains ineffective.

**Remark 1.3.** There is an remarkable geometric association, first appearing in [IM11], between Gushel-Mukai fourfolds and EPW double sextics. Hence there is a morphism from the 24-dimensional moduli stack of GM fourfolds to the 20-dimensional moduli stack of EPW double sextics; in particular, the image of a special Gushel-Mukai fourfolds of discriminant $d$ is a special EPW double sextic of discriminant $d$ (cf. [DIM15], [DK18]), and hence the image of the locus of special Gushel-Mukai fourfolds lies in $\mathcal{U}_d$.

1.4. **Overview and contributions.** In §2 we review some basic notions of lattice theory and the definitions of the moduli spaces $\mathcal{M}$ and $\mathcal{M}_d$. Let $L$ be an integral lattice of signature

$(2, n)$. In §2.1 we will define the complex analytic period space $\Omega_L^+$ and the groups

$$\widetilde{O}^+(L) \subseteq O^+(L)$$

which are finite index subgroups of $O(L)$ acting properly discontinuously on $\Omega_L^+$. These objects will appear many times in subsequent sections. We will focus especially on the $K3^{[2]}$-*lattice*

$$M := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus (-2). \tag{1.1}$$

The period spaces $\Omega_L^+$ that we study will come from certain sublattices $L \subset M$.

In §2.2, we discuss the moduli and periods of HK fourfolds of $K3^{[2]}$-type with degree 2 polarization. For a $K3^{[2]}$-fourfold $X \in \mathcal{M}$ with polarization $H$ and $q_{BB}(H) = 2$, there is a lattice isomorphism $H^2(X, \mathbf{Z}) \cong M$ with $H \mapsto h = u + v \in U \subset M$ where $\{u, v\}$ is an isotropic basis for $U$. Next, we consider $\Lambda := (h)_M^\perp$, a sublattice of $M$ of signature $(2, 20)$, and use this new lattice to define the (global) period domain $\mathcal{D}$ for degree 2 $K3^{[2]}$-fourfolds:

$$\mathcal{D} := \widetilde{O}^+(\Lambda) \backslash \Omega_\Lambda^+.$$

The space $\mathcal{D}$ is an irreducible algebraic variety of dimension 20 parametrizing weight 2 Hodge structures of $K3$-type on $\Lambda$ and has a natural morphism $\tau \colon \mathcal{M} \to \mathcal{D}$. By the Torelli theorem of Verbitsky and Markman, the morphism $\tau$ is an open immersion. This lets us realize $\mathcal{M}$ as a Zariski open subvariety of $\mathcal{D}$.

Next, in §2.3 we begin to study Noether-Lefschetz locus

$$NL(\mathcal{M}) = \bigcup_d \mathcal{M}_d,$$

already introduced in §1.1 as the locus in $\mathcal{M}$ of points where $\mathrm{rk}(\mathrm{Pic}\, X) > 1$. Applying the aforementioned Torelli theorem, we may study $\mathcal{M}_d$ by working in the period space. In [DIM15, Section 5], Debarre, Iliev, and Manivel classify the rank 2 lattices $K \subset M$ which contain $h$ (their results are written in the language of Gushel-Mukai varieties, cf. [DK18]). In particular, they show that the integer $d = |D(K^\perp)|$ must be congruent to 0 mod 4 or 2 mod 8; if $d$ satisfies these congruence conditions, then either $d \equiv 0$ mod 4 and there is a unique $\widetilde{O}^+(\Lambda)$-orbit of special rank 2 lattices $K$ containing $h$, or $d \equiv 2$ mod 8 and there are two such orbits, exchanged by an involution. Letting $[K_d]$ denote the $\widetilde{O}^+(\Lambda)$ orbit of such a special lattice $K_d$ of discriminant $d = |D(K_d^\perp)|$, we will consider hypersurfaces $\mathcal{D}_{[K_d]} \subset \mathcal{D}$ defined by the image of the locus

$$\Omega_{[K_d]}^+ = \{\omega \in \Omega_L^+ \ : \ \omega^\perp \supseteq K_d\}$$

under the quotient map $\Omega_L^+ \to \mathcal{D}$. If $d \equiv 0$ mod 4 then we will write $\mathcal{D}_{[K_d]} = \mathcal{D}_d$, while if $d \equiv 2$ mod 8 then we have two irreducible hypersurfaces exchanged by an involution, denoted $\mathcal{D}_d'$ and $\mathcal{D}_d''$, and we let $\mathcal{D}_d = \mathcal{D}_d' \cup \mathcal{D}_d''$. Crucially for us, the varieties $\mathcal{M}_d$ and $\mathcal{D}_d$ are birational (with $d > 10$ or $d = 4$; if $d = 10$ then $\mathcal{M}_{10}$ maps to one component of $\mathcal{D}_{10}$ (Theorem 2.3)). We thus set out to prove our theorem by studying the Kodaira dimension of $\mathcal{D}_d$.

In §2.4, we show that each of the irreducible components of the varieties $\mathcal{D}_d$ is birational to a certain *orthogonal modular variety* $\mathcal{F}_d = \Gamma_d \backslash \Omega_{K_d^\perp}^+$, where $\Gamma_d$ is a certain subgroup $O^+(K_d^\perp)$

of finite index, the *stable orthogonal group* (see Proposition 2.6). This discussion is followed by a review of the "low-weight cusp form trick" (Theorem 2.7) for determining the Kodaira dimension of $\mathcal{F}_d$ due to Gritsenko, Hulek and Sankaran, a technique originating in [GHS07], and upon which much of our proof is based. Briefly, if we can find a nonzero cusp form on $\Omega^{+,\bullet}_{K_d^\perp}$ of weight $a$ for $12 < a < 19$ which is modular with respect to $\Gamma_d$, then Theorem 2.7 guarantees that the variety $\mathcal{F}_d$ is of general type.

In order to construct such a nonzero cusp form of prescribed weight, we use the quasi-pullback method (described in [GHS13, Section 8]) to pullback the Borcherds form $\Phi_{12}$ along the inclusion $\Omega^{+,\bullet}_{K_d^\perp} \to \Omega^{+,\bullet}_{L_{2,26}}$ induced by a lattice embedding $K_d^\perp \hookrightarrow L_{2,26}$. Here, the lattice $L_{2,26}$ denotes the unique (up to isometry) even unimodular lattice of signature $(2,26)$. The lattice embedding determines the number $N(K_d^\perp)$ of pairs of $(-2)$-roots in $(K_d^\perp)^\perp_{L_{2,26}}$, which in turn determines the weight $a = 12 + N$. Thus, the strategy of our proof is to establish the existence of lattice embeddings $K_d^\perp \hookrightarrow L_{2,26}$ containing at least one and no more than six pairs of $(-2)$-roots.

The systematic study of these lattice embeddings is taken up in §3 where we recall a classification of the types of $(-2)$-roots from [TVA19, Section 4] and find that we need to impose additional conditions on the discriminant to ensure this classification holds. We also use a slightly modified version of the "lattice engineering" trick from [TVA19, Section 4] which gives great control on the number $N(K_d^\perp)$ of pairs of $(-2)$-roots orthogonal to $K_d^\perp$. But to use this trick, one has to impose inequalities bounding $d$ from below. The main challenge here is to compute some $d_0$ such that we can show $\mathcal{M}_d$ is of general type for $d \geq d_0$, and then to analyze all $d < d_0$ with the aid of a computer.

Finally, in §4, we take up the cases $d = 8m$, $8m + 2$, and $8m + 4$ in three separate but similar analyses (see §4.1, §4.2, §4.3). We first compute $d_0$ such that $\mathcal{M}_d$ is of general type if $d \geq d_0$ by writing down a systematic way to embed $K_d^\perp \hookrightarrow L_{2,26}$ for each such $d$. We then reduce the problem of constructing embeddings with $0 < N(K_d^\perp) < 7$ to a number-theoretic problem about the integer valued points on a diagonal quadric subject to certain conditions. To guarantee the existence of such points for sufficiently large $d$, we invoke a classical result of Halter-Koch on the sums of three squares. Our computations allow us to compute $d_0$ effectively in each of the three cases. We end in §4.4 with a discussion of the low discriminant analysis by computer. A list of embeddings for these low discriminant cases is available in the arXiv version of this article and is also available on the author's webpage. We also provide code, written in the `Magma` language [BCP97], to verify that these embeddings have the desired properties.

## 2. BASIC NOTIONS AND DEFINITIONS

In this section we define the main objects of the paper, starting with a review of lattice theory in §2.1 and the moduli and periods of our hyperkähler fourfolds in §2.2. The special divisors $\mathcal{M}_d$ and $\mathcal{D}_d$ are discussed in §2.3, and the orthogonal modular varieties $\mathcal{F}_d$ are discussed in §2.4.

2.1. **Lattices.** (References:[CS99], [Ser73].) Let $R$ be a ring. An $R$-*lattice* is a free $R$-module $L$ of finite rank together with a nondegenerate symmetric $R$-bilinear form

$$(\cdot, \cdot) \colon L \times L \to R.$$

In the present work, the relevant cases are $R = \mathbf{Z}$ and $R = \mathbf{Q}$. When $R$ is a subring of $\mathbf{R}$, we may consider the signature $(r, s)$ of $L$, which is the signature of the Gram matrix for a basis of $L$. If $L$ is an integral lattice (i.e., a $\mathbf{Z}$-lattice) with $(x, x) := x^2 \in 2\mathbf{Z}$ for all $x \in L$, we say that $L$ is an *even* integral lattice. An element $x \in L$ is *primitive* if it is not an integral multiple of any other vector in $L$. We say that $r \in L$ is an $(n)$-root if $r$ is a primitive vector of square-length $r^2 = n$.

An embedding $L \hookrightarrow M$ of integral lattices is said to be *primitive* if the quotient group $M/L$ is torsion-free. The orthogonal complement of $L$ in $M$ is denoted $L_M^\perp$, or simply $L^\perp$ with the ambient lattice understood from context. To every even integral lattice $L$, there is the associated dual lattice $L^\vee = \operatorname{Hom}(L, \mathbf{Z})$ with an embedding $L \hookrightarrow L^\vee$ given by $x \mapsto (x, \cdot)$. The group $D(L) := L^\vee/L$ is a finite abelian group, called the *discriminant group*. The natural extension of $(\cdot, \cdot)$ to $L^\vee$ endows $L^\vee$ with the structure of a $\mathbf{Q}$-lattice. This in turn gives rise to a $\mathbf{Q}/2\mathbf{Z}$-valued bilinear form $b_L$ on $L^\vee/L$, called the *discriminant form*. An integral lattice is said to be *unimodular* if it has trivial discriminant group. Let $O(L)$ denote the group of automorphisms of $L$ preserving $(\cdot, \cdot)$, and let $\widetilde{O}(L)$ denote the subgroup of automorphisms which preserve the discriminant form; that is,

$$\widetilde{O}(L) := \ker(O(L) \to O(L^\vee/L)).$$

The group $\widetilde{O}(L)$ is a finite index subgroup of $O(L)$ and is known as the *stable orthogonal group*. In this work, the notation $(n)$ for a nonzero integer $n$ will denote a rank 1 integral lattice with a generator $x$ of length $n$. Following standard practice, the lattice $A_1$ denotes the lattice $(2)$. If $L$ is a lattice, then $L(n)$ denotes the lattice with the same underlying abelian group as $L$ with pairing given by

$$(x, y)_{L(n)} = n \cdot (x, y)_L.$$

Often, we will write down a lattice by writing down a Gram matrix for a basis of the lattice. The lattices $U$ and $E_8$ denote, respectively, the hyperbolic plane given by the Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the unique unimodular positive-definite even lattice of rank 8. Later, when perform explicit computation involving $E_8$, we make use of the Gram matrix for $E_8$ ([CS99,

§8]):

$$E_8 \cong \begin{pmatrix} 2 & 0 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 \\ -2 & 0 & 4 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We also need the "checkerboard" lattice $D_6$ ([CS99, §7]): let $e_1, \ldots, e_6$ denote the standard basis of $\mathbf{Z}^6 \subset \mathbf{R}^6$ with the usual dot product. Then we define an even integral lattice $D_6$ by $D_6 = \{\sum c_i e_i \in \mathbf{Z}^6 : \sum c_i \equiv 0 \bmod 2\} \subset \mathbf{Z}^6$. The 2-roots of $D_6$ (i.e. the square-length 2 vectors) are given by $S \cup -S$, where $S = \{e_i \pm e_j : i \neq j\}$. The dual lattice $D_6^\vee$ is the $\mathbf{Z}$-span of $\mathbf{Z}^6$ and the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

**Remark 2.1.** If $A_1^{\oplus 2} \hookrightarrow E_8$ is primitive embedding of lattices, then $(A_1^{\oplus 2})^\perp \cong D_6$. This can be verified by direct computation, first on a single embedding, and then by using that embeddings $A_1^{\oplus 2} \hookrightarrow E_8$ are unique up to isometry (see [Nik79, Theorem 1.14.4]).

When $L$ has signature $(2, m)$, we also define the subgroup $O^+(L)$ of automorphisms which preserve the orientation on the positive-definite part of $L$. Note that $O^+(L)$ is a finite index subgroup of $O(L)$ and that $O^+(L)$ acts on the period space for $L$:

$$\Omega_L^+ := \{x \in \mathbf{P}(L \otimes \mathbf{C}) : (x, x) = 0, (x, \overline{x}) > 0\}^+$$

where the $+$ notation indicates that we are taking one component of the two-component set $\{x \in \mathbf{P}(L \otimes \mathbf{C}) : (x, x) = 0, (x, \overline{x}) > 0\}$ (the two components are exchanged by complex conjugation). For any primitive vector $r \in L$ of square length $r^2 < 0$, there is a *rational quadratic divisor* in $\Omega_L^+$ defined by

$$\Omega_L^+(r) := \{Z \in \Omega_L^+ : (Z, r) = 0\}.$$

We will also need the group

$$\widetilde{O}^+(L) := O^+(L) \cap \widetilde{O}(L)$$

which is a finite index subgroup of the groups $O(L), O^+(L)$, and $\widetilde{O}(L)$, and acts properly and discontinuously on $\Omega_L^+$ (as does any finite index subgroup $\Gamma \subseteq O^+(L)$). For a sublattice $K \subset L$, define

$$O(L, (K)) = \{g \in O(L) : g(K) = K\}$$

and define

$$O(L, K) = \{g \in O(L, (K)) : g|_K = \mathrm{id}_K\}.$$

We will write $O(L, v) := O(L, \mathbf{Z}v)$ for $v \in L$. One can also define $O^+(L, (K)), \widetilde{O}^+(L, K)$, and so on.

## 2.2. Moduli and periods of hyperkähler fourfolds of $K3^{[2]}$-type. (Reference: [Deb18]).

Let $X$ be a complex algebraic variety which is deformation equivalent to the Hilbert scheme

$S^{[2]}$ of length-two zero-dimensional subschemes of a $K3$ surface $S$ (or the Douady space, for $S$ a non-algebraic $K3$). Then $X$ is a four-dimensional hyperkähler (HK) manifold — meaning $X$ is a simply connected with a nowhere degenerate 2-form $\omega$ such that $H^0(X, \Omega^2 X) = \mathbf{C}\omega$. We say that such HK manifolds are *of $K3^{[2]}$-type*. One can show that any HK manifold has $H^r(X, \mathcal{O}_X) = 0$ for any $r$ odd, so the exponential exact sequence shows that $\operatorname{Pic} X$ injects into $H^2(X, \mathbf{Z})$. The second integral singular cohomology also underlies a Hodge structure of weight 2 of $K3$-type. The gives another realization of the Picard group as $\operatorname{Pic} X = H^{1,1}(X) \cap H^2(X, \mathbf{Z})$.

The group $H^2(X, \mathbf{Z})$ and (its subgroup $\operatorname{Pic} X$) inherits the structure of a quadratic space from the *Beauville-Bogomolov-Fujiki (BBF) form* $q_X$, a certain canonically defined nondegenerate integral quadratic form of signature $(3, b_2(X) - 3)$. For more on $q_X$ we refer the reader to [Bea83]. For $S$ a $K3$ surface, the second cohomology with the BBF form $(H^2(S^{[2]}, \mathbf{Z}), q_S)$ is isomorphic to $H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\delta$ with $\delta^2 = -2$. The summand $H^2(S, \mathbf{Z})$ is the K3 lattice and carries an intersection form given by the cup product, with $s \cdot s = q(s)$. The class $2\delta$ is corresponds to the divisor in $S^{[2]}$ parametrizing nonreduced subschemes of $S$ of length two. Since $q(H^2(S^{[2]}, \mathbf{Z})) = 2\mathbf{Z}$, the cohomology group $H^2(S^{[2]}, \mathbf{Z})$ has the structure of an even, integral lattice.

The second integral cohomology with the BBF form is deformation invariant. As $H^2(S, \mathbf{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ for any $K3$ surface $S$, it follows for $X$ a fourfold of $K3^{[2]}$-type that $H^2(X, \mathbf{Z})$ is isomorphic to the lattice

$$M = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus (-2).$$

Let $u, v$ denote a null basis for the first copy of $U$ in the decomposition of $M$:

$$u^2 = v^2 = 0, (u, v) = 1.$$

Let $u', v'$ denote a null-basis for the second copy of $U$, and let $w$ denote the $(-2)$ factor in the decomposition above.

A *polarized* HK fourfold is a pair $(X, H)$ where $H \in \operatorname{Pic} X$ is a primitive, ample divisor with $q(H) = e > 0$. The integer $e$ is called the *degree* of the polarized fourfold. In this work we consider the lowest possible polarization degree $K3^{[2]}$-type fourfolds, those with degree $e = 2$. There is a coarse quasi-projective moduli space $\mathcal{M}$, which is irreducible and has dimension 20, parametrizing polarized $K3^{[2]}$-type fourfolds of degree 2 up to isomorphism; O'Grady showed that this moduli space is unirational (see [O'G06, Theorem 1.1]). A *marking* of an HK fourfold of $K3^{[2]}$-type is an isomorphism

$$\varphi \colon H^2(X, \mathbf{Z}) \cong M.$$

Every $(X, H)$ with marking is isomorphic to $(X, H)$ with a marking $\varphi \colon H^2(X, \mathbf{Z}) \to M$ sending $H \mapsto h := u + v$. One computes that

$$h^\perp = \Lambda = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus (-2)^{\oplus 2}.$$

We briefly recall some relevant Hodge theory for our degree 2 $K3^{[2]}$-fourfolds. The *period* of a point $(X, H) \in \mathcal{M}$ together with marking $\varphi$ is the line

$$\varphi_{\mathbf{C}}(H^{2,0}(X)) \in \Lambda \otimes \mathbf{C}.$$

A period determines, via the Hodge-Riemann relations, a weight 2 Hodge structure on $\Lambda$ of $K3$-type. The global and local period domains for $\Lambda$ are spaces that parametrize these Hodge structures. There exists a map to the *local period domain* $\Omega_{\Lambda}^{+}$,

$$\{(X, H, \varphi) \,:\, (X, H) \in \mathcal{M}, \ \varphi \colon H^2(X, \mathbf{Z}) \to M, \ \varphi(H) = h\} \longrightarrow \Omega_{\Lambda}^{+},$$

which sends a triple $(X, H, \varphi)$ to its period; after quotienting out by isomorphism of these triples, one gets a map into the *global period domain*

$$\tau \colon \mathcal{M} \to \mathcal{D} := \widetilde{O}^{+}(\Lambda)\backslash\Omega_{\Lambda}^{+}.$$

Applying well-known results of Baily-Borel [BB66], the arithmetic quotient $\mathcal{D}$ is a quasi-projective, irreducible, normal variety. By the global Torelli theorem for polarized HK fourfolds, due to Verbitsky and Markman (see [Mar11, Theorem 8.4]), the morphism $\tau$ is algebraic and is an open immersion. We note for later use that

$$\widetilde{O}^{+}(\Lambda) = \{\gamma \in O^{+}(\Lambda) \,:\, \gamma \in O(M, h)|_{\Lambda}\},$$

an equality which follows from [Nik79, Corollary 1.5.2].

2.3. **Noether-Lefschetz locus.** We say that $X$ *possesses a special labelling of discriminant* $d$ if there exists a lattice $K \subset \operatorname{Pic} X$ of rank 2 with $H \in K$ such that $|D(K^{\perp})| = d$. A very general fourfold $X$ in $\mathcal{M}$ has $\operatorname{Pic} X = 1$ and thus does not possess any special labelling (see [Zar90, Section 5.1] for a standard argument for this fact). The following result of Debarre, Iliev, and Manivel classifies all possible special labellings.

**Theorem 2.2.** [DIM15, Proposition 6.2] *A special sublattice $K$, i.e. a rank 2 sublattice $K \subset M$ with $u + v \in K$ of signature $(1,1)$, must have discriminant $d \equiv 0, 2, 4 \bmod 8$. Furthermore, the orbits of $O^{+}(\Lambda)$ acting on the set of special rank 2 sublattices are as follows:*

(1) *If $d = 8m$, there is just one orbit for each $m > 0$, represented by $K_d$ with $K_d \cong$ $\begin{pmatrix} 2 & 0 \\ 0 & -2m \end{pmatrix}$ and $K_d \cap \Lambda = \mathbf{Z}(u' - mv')$.*

(2) *If $d = 8m + 2$, there are two orbits for each $m > 0$, exchanged by an automorphism of $\Lambda$ switching $w$ and $u - v$. Both of these orbits have $K_d \cong \begin{pmatrix} 2 & 0 \\ 0 & -2 - 8m \end{pmatrix}$. One of these orbits has representative $K_d$ with $K_d \cap \Lambda = \mathbf{Z}(u - v + 2u' - 2mv')$. The other has representative $K_d$ such that $K_d \cap \Lambda = \mathbf{Z}(w + 2u' - 2mv')$.*

(3) *If $d = 8m + 4$, there is just one orbit for each $m > 0$. This orbit has a representative $K_d$ with $K_d \cong \begin{pmatrix} 2 & 0 \\ 0 & -4 - 8m \end{pmatrix}$, and $K_d \cap \Lambda = \mathbf{Z}(u - v + w + 2u' - 2mv')$.*

Using [Nik79, Corollary 1.5.2] once again, we observe that
$$\widetilde{O}^+(\Lambda, K_d \cap \Lambda)|_{K_d^\perp} = O^+(M, K_d)|_{K_d^\perp} = \widetilde{O}^+(K_d^\perp)$$
and
$$\Gamma'_d := \widetilde{O}^+(\Lambda, (K_d \cap \Lambda))|_{K_d^\perp} = (O^+(M, h) \cap O^+(M, (K_d)))|_{K_d^\perp} = \langle \widetilde{O}^+(K_d^\perp), -\mathrm{id}_{K_d^\perp} \rangle. \quad (2.1)$$

In particular, the group $\widetilde{O}^+(K_d^\perp)$ is an index 2 subgroup of $\Gamma'_d$.

We define the divisor $\mathcal{D}_d \subset \mathcal{D}$ for each $d \equiv 0, 2, 4$ as in Theorem 2.2 as follows: For $d \equiv 0, 4 \bmod 8$, define
$$\Omega_d^+ := \{\omega \in \Omega_\Lambda^+ : \omega^\perp \supseteq K_d \cap \Lambda\};$$
Then $\mathcal{D}_d$ is the image of $\Omega_d^+$ under the projection map $\Omega_\Lambda^+ \to \mathcal{D}_\Lambda$, and is an irreducible divisor. The divisors $\mathcal{M}_d$ are now defined to be $\mathcal{M}_d := \tau^{-1}(\mathcal{D}_d)$. Note that $\mathcal{M}_d$ parameterizes the $(X, H) \in \mathcal{M}$ that possess a special labelling of discriminant $d$. For $d \equiv 2 \bmod 8$, the irreducible divisors $\mathcal{D}'_d, \mathcal{D}''_d \subseteq \mathcal{D}$ and $\mathcal{M}_d \subseteq \mathcal{M}$ are similarly defined.

The following theorem of Debarre and Macrì, a consequence of [DM, Proposition 4.1 and Theorem 6.1], gives the image of $\tau$:

**Theorem 2.3** (Debarre-Macrì). *The image of the Torelli map $\tau \colon \mathcal{M} \to \mathcal{D}$ meets exactly the following divisors ($d > 0$):*

(1) *If $d \equiv 0, 4 \bmod 8$, the image meets $\mathcal{D}_d$ except for $d = 4$ and $d = 8$.*

(2) *If $d \equiv 2 \bmod 8$, the image meets $\mathcal{D}'_d$ and $\mathcal{D}''_d$, except for: $d = 2$, and one of $\mathcal{D}'_d, \mathcal{D}''_d$ for $d = 10$.*

To prove Theorem 1.1, it suffices to compute the Kodaira dimension for $\mathcal{D}_d$, since $\mathcal{M}_d$ and $\mathcal{D}_d$ are birational.

**Notational Convention 2.4.** For $d \equiv 2 \bmod 8$, we will set $\mathcal{D}_d = \mathcal{D}'_d$, as we only care about Kodaira dimension, and $\mathcal{D}'_d$ is isomorphic to $\mathcal{D}''_d$. We will also set $K_d = K'_d$.

2.4. **Orthogonal modular varieties.** Let us now relate $\mathcal{D}_d$ via a birational map to an *orthogonal modular variety*, a quotient of the form $\Gamma \backslash \Omega_L^+$ for any $\Gamma \subseteq O^+(L)$ of finite index. Our approach to finding an appropriate orthogonal modular variety $\mathcal{F}_d$ birational to $\mathcal{D}_d$ is inspired by Hassett's work ([Has96], [Has00]) on the analogous problem for special cubic fourfolds, which is lucidly explained in [Huy18] and in [Bra18]. Then we discuss how to apply the low-weight cusp form trick.

Recall that $K_d^\perp$ denotes the orthogonal complement (in $M$) of the representative $K_d$ given in Theorem 1.1. We defined (2.1) a group $\Gamma_d \subset O^+(K_d^\perp)$ which contains $\widetilde{O}^+(K_d^\perp)$ as an index 2 subgroup. We have natural morphisms of algebraic varieties:
$$\mathcal{G}_d := \widetilde{O}^+(K_d^\perp) \backslash \Omega_{K_d^\perp}^+ \to \mathcal{F}_d := \Gamma_d \backslash \Omega_{K_d^\perp}^+ \to \widetilde{O}^+(\Lambda) \backslash \Omega_\Lambda^+ = \mathcal{D} \quad (2.2)$$

By definition, the image of the second morphism in 2.2 is $\mathcal{D}_d$, so we may rewrite these morphisms as
$$\mathcal{G}_d \xrightarrow{\phi} \mathcal{F}_d \xrightarrow{\psi} \mathcal{D}_d. \quad (2.3)$$

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The variety $\mathcal{G}_d$ parametrizes *marked* special weight 2 Hodge structures of $K3$ type on $K_d^\perp$ (a Hodge structure on $K_d^\perp$ together with the data of a lattice embedding $K_d \hookrightarrow M$) , while $\mathcal{F}_d$ parametrizes *labelled* weight 2 Hodge structures of $K3$ type on $K_d^\perp$ (Hodge structures on $M$ together with the data of the *image* of a lattice embedding $K_d \hookrightarrow M$).

**Remark 2.5.** We note that since $-\,\mathrm{id}$ acts as the identity on $\Omega^+_{K_d^\perp}$, we have that $\mathcal{F}_d = \mathcal{G}_d$. We choose to work with $\mathcal{F}_d$ because the property that $-\,\mathrm{id} \in \Gamma_d$ will be useful in §3.2.

The next proposition, whose proof we mirror on similar arguments appearing in [Huy18, Corollary 2.5] and [Bra18], gives us some key properties of the morphism $\psi$ appearing in 2.3:

**Proposition 2.6.** *The morphism $\psi$ is the normalization of $\mathcal{D}_d$.*

*Proof.* We show $\psi$ is finite of degree 1. We begin by showing the properness of $\psi$: start with observation that the morphisms (in the complex analytic category) $\Omega^+_\Lambda \to \mathcal{D}_\Lambda$, $\Omega^+_{K_d^\perp} \to \Omega^+_\Lambda$, and $\Omega^+_{K_d^\perp} \to \mathcal{F}_d$ are *closed*, and that the composition $\Omega^+_{K_d^\perp} \to \Omega^+_\Lambda \to \mathcal{D}_d$ is closed as well. Since we can further factor this closed morphism into the composition of of two other morphisms with the first being closed,

$$\Omega^+_{K_d^\perp} \to \mathcal{F}_d \to \mathcal{D}_d,$$

it follows that $\mathcal{F}_d \to \mathcal{D}_d$ is closed. Since each fiber is a compact set - indeed a finite set- this is a proper morphism. Furthermore, as $\psi$ is quasi-finite and proper, it follows that $\psi$ is finite.

Let $n$ denote the degree of $\psi$, i.e. there is an open set $U \subseteq \mathcal{F}_d$ such that, for any $x \in U$, the fiber $\psi^{-1}(x)$ has cardinality $n$. Since a very general $(X, H) \in \mathcal{M}_d$ has $\mathrm{rk}(\mathrm{Pic}\, X) = 2$ (again by the reasoning in [Zar90, Section 5.1]), a very general fiber must consist of a single point. Therefore, we have $n = 1$ and so $\psi$ is a birational morphism. By [BB66], the variety $\mathcal{F}_d$ is normal, so $\mathcal{F}_d$ must be the normalization of $\mathcal{D}_d$. $\qquad\square$

Since $\psi$ is a birational map, we may conclude

$$\kappa(\mathcal{F}_d) = \kappa(\mathcal{D}_d) = \kappa(\mathcal{M}_d).$$

To use the low-weight cusp-form trick to compute $\kappa(\mathcal{F}_d) = \kappa(\mathcal{M}_d)$, we review a little theory of modular forms on orthogonal groups. Let $L$ be a signature $(2, n)$ lattice, let $\Gamma \subseteq O^+(L)$ be a finite index subgroup, let $\chi \colon \Gamma \to \mathbf{C}^\times$ be a character, and let $\Omega^{+\bullet}_L$ denote the affine cone over $\Omega^+_L$. A *modular form of weight $k$ with character $\chi$ for the group $\Gamma$* is a holomorphic function $F \colon \Omega^{+\bullet}_L \to \mathbf{C}$ satisfying the following properties for all $z \in \Omega^{+\bullet}_L$:

(1) For every $\gamma \in \Gamma$, we have $F(\gamma z) = \chi(\gamma)F(z)$

(2) For every $t \in \mathbf{C}^\times$, we have $F(tz) = t^{-k}F(z)$.

Let us denote by $M_k(\Gamma, \chi)$ the collection of all such modular forms of fixed weight $k$ with character $\chi$ for the group $\Gamma$. A *cusp form* is a modular form $F \in M_k(\Gamma, \chi)$ vanishing at the cusps of the Baily-Borel compactification of the variety $\Gamma \backslash \mathcal{D}_L$, and all such forms form a vector space denoted $S_k(\Gamma, \chi)$. The low-weight cusp form trick is summarized in the following theorem of Gritsenko, Hulek, and Sankaran:

**Theorem 2.7.** [GHS07, Theorem 1.1] *Let $L$ be a lattice of signature $(2, n)$ with $n \geq 9$ and $\Gamma \subseteq O^+(L)$ a subgroup of finite index. The variety $\Gamma \backslash \Omega_L^+$ is of general type if there exists a cusp form $F$ for the group $\Gamma$ with weight $a < n$ and character $\chi$ such that $F$ vanishes along the divisor of ramification of the projection map $\Omega_L^+ \to \Gamma \backslash \Omega_L^+$. If there is a nonzero cusp form of weight $n$ for $\Gamma$ with character $\det$, then $\kappa(\Gamma \backslash \Omega_L^+) \geq 0$.*

To apply Theorem 2.7 to compute the Kodaira dimension of $\widetilde{O}^+(K_d^\perp) \backslash \Omega_{K_d^\perp}^+$, one needs a supply of modular forms which are modular with respect to $\widetilde{O}^+(K_d^\perp)$. For us, these are provided by *quasi-pullbacks* of modular forms with respect to some higher rank orthogonal group, which we now describe. Let $L_{2,26}$ denote the unique even unimodular lattice of signature $(2, 26)$:

$$L_{2,26} = U^{\oplus 2} \oplus E_8(-1)^{\oplus 3}$$

It is known ([Bor95]) that $M_{12}(O^+(L_{2,26}), \det)$ is a one-dimensional complex vector space spanned by a modular form $\Phi_{12}$, called the *Borcherds form*. The divisor of zeros of $\Phi_{12}$ is the union

$$\text{div}(\Phi_{12}) = \bigcup_{r \in L_{2,26}, r^2 = -2} \Omega_{L_{2,26}}^+(r), \tag{2.4}$$

where $\Omega_{L_{2,26}}(r)$ denotes a rational quadratic divisor as in 2.1 Given a primitive embedding of lattices $\iota \colon L \hookrightarrow L_{2,26}$, with $L$ of signature $(2, n)$, let

$$R_{-2}(\iota) := \{r \in L_{2,26} \colon r^2 = -2, (r, K_d^\perp) = 0\}.$$

When the embedding is clear from context, we may sometimes write $R_{-2}(L)$. To construct a modular form for some subgroup of $O^+(L)$, one might try to pullback $\Phi_{12}$ along the naturally induced closed immersion $\Omega_L^{+\bullet} \to \Omega_{L_{2,26}}^{+\bullet}$. But for any $r \in R_{-2}(L)$, one has $\Omega_{K_d^\perp}^{+\bullet} \subset \Omega_r^\bullet$, and hence $\Phi_{12}$ vanishes identically on $\Omega_{K_d^\perp}^{+\bullet}$. The method of the quasi-pullback, due to Gritsenko, Hulek, and Sankaran, deals with this issue by dividing out by appropriate linear factors:

**Theorem 2.8.** [GHS13, Theorem 8.2] *Let $L$ be a lattice of signature $(2, n)$, with $3 \leq n \leq 26$. Given primitive embedding of lattices $\iota \colon L \hookrightarrow L_{2,26}$ and the naturally induced embedding $\Omega_L^{+\bullet} \to \Omega_{L_{2,26}}^{+\bullet}$, the set $R_{-2}(L)$ of $(-2)$-vectors of $L_{2,26}$ orthogonal to $L$ is a finite set. The quasi-pullback of $\Phi_{12}$ with respect to this embedding*

$$\Phi|_{\iota(L)} := \frac{\Phi_{12}(Z)}{\prod_{r \in R_{-2}(L)/\pm 1}(Z, r)} \Big|_{\Omega_L^{+\bullet}}$$

*is a nonzero modular form in $M_{N(\iota(L))+12}(\widetilde{O}^+(L), \det)$ where $N(\iota(L)) := \#R_{-2}(\iota)/2$. If $N(\iota(L)) > 0$, then $\Phi|_{\iota(L)}$ is a cusp form.*

*Proof.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Throughout this paper, when an underlying embedding $\iota \colon K_d^\perp$ is clear from context, we will adopt the notations $\Phi|_{K_d^\perp} = \Phi|_{\iota_d}$ and $N(K_d^\perp) = N(\iota_d)$.

Thus, to show that $\kappa(\mathcal{M}_d) = 19$, we must first construct embeddings $\iota_d \colon K_d^\perp \to L_{2,26}$ such that $0 < N(\iota_d) < 7$, producing the quasi-pulledback form $\Phi|_{\iota_d(K_d^\perp)}$ of weight $12 +$

$N(K_d^\perp)$ (If an embedding of $K_d^\perp$ satisfies $N(K_d^\perp) = 7$, we may still use this embedding in a proof that $\kappa(\mathcal{M}_d) \geq 0$). These embeddings will automatically be modular with respect to $\widetilde{O}^+(K_d^\perp)$. However, there is nothing in Theorem 2.8 to guarantee that $\Phi|_{K_d^\perp}$ vanish along the ramification divisor. To deal with this, we $\Phi|_{K_d^\perp}$ is modular with respect to $\Gamma_d$. We deal with this issue by constructing our embeddings to be modular with respect to the additional isometry $-\operatorname{id}$ from the start.

## 3. Constructing embeddings: generalities

In this section, we begin constructing embeddings $K_d^\perp \hookrightarrow L_{2,26}$ such that $N(K_d^\perp) < 7$. Let us first write down the lattices $K_d^\perp$ we are studying. Using the representatives from Theorem 2.2, we compute the lattices $K_d^\perp$. The results of this straightforward computation are summarized in the following proposition. We introduce for ease of notation lattices $M_d$ for each $m \in \mathbf{Z}$ defined by their Gram matrices:

$$d = 8m, \ M_d := \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2m \end{pmatrix}$$

$$d = 8m + 2, \ M_d := \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 2m \end{pmatrix}$$

$$d = 8m + 4, \ M_d := \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 2m \end{pmatrix}$$

**Proposition 3.1.** *Let $K_d$ be the representative rank 2 lattice from Theorem 2.2. Then*

$$K_d^\perp \cong M_d \oplus U \oplus E_8^{\oplus 2}(-1).$$

Note that in every $M_d$, there is a primitively embedded copy of the lattice $A_1(-1)^{\oplus 2}$ corresponding to the upper-left $2 \times 2$ block in the Gram matrix of $M_d$, so from here on we will refer to a sublattice $A_1(-1)^{\oplus 2} \cong A \subset M_d$.

We want to consider as many embeddings as possible. We will label the factors in our decomposition of $L_{2,26}$ as follows:

$$L_{2,26} = U_1 \oplus U_2 \oplus E_8(-1)^{(1)} \oplus E_8(-1)^{(2)} \oplus E_8(-1)^{(3)}.$$

By Nikulin's analog of Witt's theorem (see [Nik79, Theorem 1.14.4]), a primitive embedding $U \oplus E_8(-1)^{\oplus 2} \hookrightarrow L_{2,26}$ is unique up to isometry of $L_{2,26}$, and the same is true for any primitive embedding $A_1(-1)^{\oplus 2} \hookrightarrow U \oplus E_8(-1)$. Thus, without loss of generality, we will from now on assume that all of our embeddings:

(1) identify the factor $U \oplus E_8^{\oplus 2}(-1)$ appearing in our decomposition of $K_d^\perp$ in Proposition 3.1 with $U_1 \oplus E_8^{(1)}(-1) \oplus E_8(-1)^{(2)} \subset L_{2,26}$ ; and

(2) Isometrically embed $A_1(-1)^{\oplus 2} \subset M_d$ into $E_8^{(3)}$. Let $a_1, a_2$ denote the images of generators of the two $A_1(-1)$ summands.

So the problem of writing down embeddings to prove Theorem 1.1 is reduced to choosing $\ell \in U_3 \oplus E_8(-1)^{(3)}$ such that $\ell^2 = 2m$ and

$$
\begin{cases}
(\ell, a_1) = (\ell, a_2) = 0 & \text{if } d = 8m, \\
(\ell, a_1) = 1, (\ell, a_2) = 0 & \text{if } d = 8m + 2 \\
(\ell, a_1) = (\ell, a_2) = 1 & \text{if } d = 8m + 4.
\end{cases}
\tag{3.1}
$$

We will say that a vector $\ell = \alpha e + \beta f + v$ of length $2m$ is *admissible for $d$* if one of the three equations in (3.1) holds. Note that if a vector $\ell$ is admissible, there is a unique associated discriminant $d \in \{8m, 8m + 2, 8m + 4\}$ such that (3.1) is true. For admissible $\ell$ and its associated discriminant $d$, we introduce the following notations:

- $\iota_\ell \colon K_d^\perp \hookrightarrow L_{2,26}$ is the embedding associated to $\ell$
- $R_\ell$ is the set $R_{-2}(\iota_\ell(K_d^\perp))$
- $N_\ell = \#R_\ell / 2$.
- $\Phi_\ell$ is the modular form $\Phi|_{\iota_\ell(K_d^\perp)}$.

**Remark 3.2.** Every primitive embedding $K_d^\perp \hookrightarrow L_{2,26}$ is isometric to $\iota_\ell$ for some admissible $\ell$ (the converse is false). Note that $\iota_\ell$ is primitive whenever $\alpha$ and $\beta$ are coprime.

For each $d$, we wish to find admissible $\ell$ such that the following hold:

(1) $\iota_\ell$ is primitive and $0 < N_\ell < 7$ (or $0 < N_\ell \leq 7$ if attempting to prove $\kappa(\mathcal{M}_d) \geq 0$).

(2) $\Phi_\ell$ is modular with respect to $\Gamma_d$.

(3) $\Phi_\ell$ vanishes along the ramification locus of the projection $\Omega_L^+ \to \Gamma \backslash \Omega_L^+$.

We take up each of these issues in the next three subsections.

3.1. **Controlling the size of $R_\ell$.** Since $D_6(-1) = \langle a_1, a_2 \rangle_{E_8(-1)}^\perp$, it folows that

$$
R_\ell = \{r \in U \oplus D_6(-1) : r^2 = -2, (r, \ell) = 0\}.
$$

The next two lemmas from [TVA19, Section 4], which we state in a slightly more general form, will help us count the number of roots $R_\ell$.

**Lemma 3.3.** *Let $L = U \oplus E_8(-1)$ where $U = \langle e, f \rangle$ with $e^2 = f^2 = 0$ and $(e, f) = 1$, and let $L_0$ be a primitive rank 2 sublattice of $E_8(-1)$. Let $\ell \in L$ have length $\ell^2 = 2m$, for some $m > 0$ a positive integer, such that $\ell = \alpha e + \beta f + v$ with $\alpha, \beta \in \mathbf{Z}$ and $v \in E_8(-1)$, and suppose further that $\alpha \neq \beta$ and $n < \alpha\beta < 2n$. Let $R_\ell$ denote the finite set*

$$
\{r \in U \oplus (L_0)_{E_8(-1)}^\perp : r^2 = -2, (r, \ell) = 0\}.
$$

*Let $r = \alpha' e + \beta' f + v' \in R_\ell$. Then $\alpha'\beta' = 0$ and there are three types of vectors $r \in R_\ell$:*

(1) *Type I vectors $r = v'$. In this case $\alpha' = \beta' = 0$ and $r \in (L_0)_{E_8(-1)}^\perp$.*

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(2) *Type II vectors $r = \alpha'e + v'$, $\alpha' \neq 0$. In this case, $(v, v') \equiv 0 \bmod \beta$.*

(3) *Type III vectors $r = \beta'f + v'$, $\beta \neq 0$. In this case, $(v, v') \equiv 0 \bmod \alpha$.*

*Proof.* See [TVA19, Lemma 4.1] and [TVA19, Remark 4.2]. The proof there works for this slightly more general statement, as it only relies on the Cauchy-Schwarz inequality and on the negative definiteness of $L_0$. $\square$

Applying Lemma 3.3 to $L_0 = \langle a_1, a_2 \rangle$, we get a tidy classification for the vectors in $R_\ell$, provided that $\alpha \neq \beta$ and $m < \alpha\beta < 2m$. Imposing slightly stronger inequalities, we get an even stronger statement:

**Lemma 3.4.** [TVA19, Lemma 4.3] *Suppose we are in the situation of Lemma 3.3, and suppose furthermore that the following three inequalities hold:*

$$\alpha > \sqrt{n}, \ \beta > \sqrt{n}, \ \alpha\beta < \frac{5n}{4}.$$

*Then every $r \in R_\ell$ is a vector of Type I, i.e. $r \in (L_0)^{\perp}_{E_8(-1)}$.*

*Proof.* Let $r = \alpha'e + \beta'f + v' \in R_\ell$. Since $\alpha'\beta' = 0$ by Lemma 3.3, it follows that $(v')^2 = -2$. Then by Cauchy-Schwarz,

$$(v, v') \leq \sqrt{2}|v^2| = \sqrt{4(\alpha\beta - n)} < \sqrt{4\left(\frac{5n}{4} - n\right)} = \sqrt{n}.$$

But then $(v, v')$ is not divisible by $\alpha$, nor by $\beta$, by the first two inequalities in the hypotheses above. So $r$ is of Type I. $\square$

**Remark 3.5.** In fact, for our embeddings, we will want to impose a stronger condition for $\alpha$ and $\beta$, for some $\epsilon > 0$ to be determined later:

$$\sqrt{(1 + \epsilon)m} < \alpha < \sqrt{\frac{5m}{4}}, \ \sqrt{(1 + \epsilon)m} < \beta < \sqrt{\frac{5m}{4}} \tag{3.2}$$

The upshot of Lemma 3.4 is that, for any admissible $\ell = \alpha e + \beta f + v$ such that $\alpha$ and $\beta$ satisfy the inequalities (3.2), the set $R_\ell$ is contained entirely in $D_6(-1)$:

$$R_\ell = \{r \in D_6(-1) \ : \ r^2 = -2, (r, \ell) = 0\}.$$

Our strategy is to determine a lower bound on the discriminants $d$ such that there exists admissible $\ell$ associated with $d$ with $0 < N_\ell < 7$. In order to do this, we stipulate that $\alpha$ and $\beta$ satisfy inequalities (3.2), and apply Lemma 3.4 to construct $\ell$ with $N_\ell$ bounded as desired when $d$ is sufficiently large. For discriminants below this bound, we use a computer to aid in writing down embeddings. In §4, these lower bounds are calculated, and we also discuss the details of the computer-assisted search for low discriminant lattice embeddings.

3.2. **Modularity with respect to $\Gamma_d$.** The quasi-pullback $\Phi_\ell$ along our embeddings is already modular with respect to $\widetilde{O}^+(K_d^\perp)$. Since $\mathcal{F}_d = \mathcal{G}_d$, we could simply work with $\mathcal{F}_d$ and verify that $\Phi_\ell$ vanishes along the ramification divisor along the lines of [GHS13, Proposition

8.13]. We offer an alternative approach to the vanishing along the ramification divisor using modularity with respect to $\Gamma_d$.

**Remark 3.6.** Our results are unchanged whether we consider modularity with respect to $\widetilde{O}^+(K_d^\perp)$ or $\Gamma_d$, since all modular forms for $\widetilde{O}^+(K_d^\perp)$ computed in the small discriminant range in §4 are also modular with respect to $\Gamma_d$.

We would like to choose $\ell$ such that $\Phi|_\ell$ is in addition modular with respect to $-\operatorname{id} \in O(K_d^\perp)$. Then $\Phi|_{K_d^\perp}$ will be modular with respect to $\Gamma_d$ since $-\operatorname{id}$ and $\widetilde{O}(K_d^\perp)^+$ generate $\Gamma_d$. But since $\Phi|_{K_d^\perp}$ is *a priori* $\widetilde{O}^+(K_d^\perp)$-modular, we know that

$$\Phi_\ell(-\operatorname{id} Z) = \Phi_\ell(-Z) = (-1)^{N_\ell}\Phi_\ell(Z).$$

As a consequence, we have shown the following important lemma:

**Lemma 3.7.** *Let $L \hookrightarrow L_{2,26}$ be a primitive embedding of lattices as in Theorem 2.8 . Then $\Phi|_L$ is modular with respect to $-\operatorname{id} \in O^+(L^\perp)$ if and only if $N(L)$ is odd.*

Thus, we want to be certain that each embedding $\iota_\ell$ which we construct has $N(\iota_\ell(K_d^\perp))$ odd.

3.3. **Vanishing along the ramification divisor.** For $r \in L$ such that $r^2 < 0$, we say that $r$ is *reflective* whenever the reflection

$$\sigma_r \colon v \mapsto v - \frac{(v,r)}{(r,r)}r$$

is an isometry of $L$, i.e. $\sigma_r \in O(L)$. A rational quadratic divisor $\Omega_L^+(r)$ is said to be a *reflective divisor* if $r$ is reflective. The following proposition of Gritsenko, Hulek, and Sankaran describes the ramification divisor of the projection $\Omega_L^+ \to \Gamma\backslash\Omega_L^+$ as a union of certain reflective divisors:

**Proposition 3.8.** *(see* [GHS07, Corollary 2.13]*) Let $L$ be a lattice of signature $(2,n)$ and $\Gamma$ be a finite index subgroup of $O^+(L)$. Then the ramification divisor $Bdiv(\pi_\Gamma)$ of the projection $\pi_\Gamma \colon \Omega_L^+ \to \Gamma\backslash\Omega_L^+$ is given as the countable union*

$$Bdiv(\pi_\Gamma) = \bigcup_{\substack{\mathbf{Z}r \subset L, r^2 < 0 \\ \pm\sigma_r \in \Gamma}} \Omega_L^+(r).$$

Let us now apply the above proposition to a modular form $\Phi \in M_k(\Gamma_d, \det)$. We first observe that $-\sigma_r \in \Gamma_d \iff \sigma_r \in \Gamma_d$. Thus, to prove $\Phi$ vanishes along $\mathrm{Bdiv}(\pi_{\Gamma_d})$, it suffices to show that $\Phi$ vanishes on all reflective divisors $\Omega_{K_d^\perp}^+(r)$ with $\sigma_r \in \Gamma_d$. By modularity, we have $\det(\sigma_r)\Phi(Z) = \Phi(\sigma_r Z)$ for all $Z \in \Omega_{K_d^\perp}^{+,\bullet}$. We observe that $\det(\sigma_r) = -1$ and $(\sigma_r)|_{\Omega_{K_d^\perp}^+(r)^\bullet} = \operatorname{id}$. It follows that $\Phi$ vanishes on $\Omega_{K_d^\perp}^+(r)^\bullet$. This yields the following proposition:

**Proposition 3.9.** *Every modular form for $\Gamma_d$ with character $\det$ vanishes along the ramification divisor.*

# 4. CONSTRUCTING EMBEDDINGS: SPECIFICS

In this section, we prove Theorem 1.1.

*Proof of Theorem 1.1.* We break our analysis into the three cases of discriminant congruent to $0, 2$, or $4$ modulo 8 in sections §4.1, §4.2, and §4.3. We first construct primitive embeddings $\iota_\ell$ associated to $\ell = \alpha e + \beta f + v$ such that $0 < N_\ell < 7$ and $N_\ell$ is odd, provided certain conditions on $\alpha, \beta$ are satisfied. These conditions come from (3.2) and Lemma 4.2. We then compute a lower bound on the discriminants for which these conditions can always be met. This leaves us with a finite list of discriminants to analyze. We handle these cases with a computer, giving a summary of this procedure in §4.4.

4.1. **Analysis:** $d = 8m$. For the case $d = 8m$, we are searching for $\alpha, \beta$, and $v$ such that $\ell = \alpha e + \beta f + v$ of length $2m$ is admissible for $d = 8m$. For the admissibility of $\ell$, it is necessary and sufficient that $(\ell, a_1) = (\ell, a_2) = 0$ (by (3.1)), which amounts to requiring $v \in D_6(-1)$. The next lemma gives a way to construct $\ell$ such that the associated embedding has $N_\ell \in \{1, 3\}$:

**Lemma 4.1.** *Let $\ell = \alpha e + \beta f + v \in U \oplus D_6(-1)$. Suppose that $\alpha, \beta$ satisfy the inequalities (3.2), and that $v$ is of the form*

$$v = x_1 e_1 + x_2 e_2 + x_3 e_3 + e_4 + e_5 \tag{4.1}$$

*with $x_i > 0$ for all $i$.*

(1) *Suppose that integers $x_1, x_2, x_3$ are distinct integers all greater than 1. Then $R_\ell = \{\pm(e_4 - e_5)\}$.*

(2) *Suppose that the integers $x_1, x_2, x_3$ are distinct positive integers with $x_j = 1$ for exactly one index. Then $R_\ell = \{\pm(e_4 - e_5), \pm(e_4 - e_j), \pm(e_5 - e_j)\}$.*

*Proof.* By hypothesis, all vectors in $R_\ell$ are of Type I (Lemma 3.3). We shall write $x_4 = x_5 = 1$ and $x_6 = 0$. The roots of $D_6(-1)$ are $\pm e_i \pm e_j$, $1 \le i, j \le 6$, $i \ne j$. We have for all such roots $r \in D(-1)$, $(r, v) = \pm(e_i \pm e_j, v) = \pm(x_i \pm x_j)$. The latter expression is equal to zero if and only if $r \in \{\pm(e_3 - e_4)\}$ in the case of (1), or $r \in \{\pm(e_3 + (-1)^h e_j), \pm(e_4 + (-1)^h e_j)\}$ in the case of (2). $\square$

Thus, to find $v$ as in the lemma, it would suffice to pick $\alpha, \beta$ satisfying 3.2 such that $2(\alpha\beta - m - 1)$ is a sum of three distinct nonvanishing squares: any solution $(x_1, x_2, x_3) \in \mathbf{Z}_{>0}^3$ to

$$x_1^+ x_2^2 + x_3^2 = 2(\alpha\beta - m - 1), \quad x_1 x_2 x_3 \ne 0 \tag{4.2}$$

and with $x_1, x_2, x_3$ distinct yields $v$ satisfying the hypothesis of the lemma.

$$x_1^2 + x_2^2 + x_3^2 = 2(\alpha\beta - m - 1)$$

The next lemma, describing the existence of these solutions, is from [HK82, Section 1, Korollar 1]:

**Lemma 4.2.** *Every integer $\Delta \not\equiv 0, 4, 7 \bmod 8$ with*

$$\Delta \notin \{1, 2, 3, 6, 9, 11, 18, 19, 22, 27, 33, 43, 51, 57, 67, 99, 102, 123, 163, 177, 187, 267, 627\} \cup \{N\}$$

*may be written as the sum of three distinct, coprime, nonvanishing squares. If GRH is true, we may as well take $N = 1$, but if GRH is false, then $N > 5 \cdot 10^{10}$.*

We also have the following lemma to give us more flexibility in our choice of $\alpha$ and $\beta$ beyond $(\alpha, \beta) = 1$

**Lemma 4.3.** *Assume that $\ell = \alpha e + \beta f + v \in U \oplus E_8(-1)$ has square length $\ell^2 = 2m$, with $v$ primitive in $D_6(-1) = \langle a_1, a_2 \rangle^{\perp}_{E_8(-1)}$, and furthermore assume that $2 \nmid (\alpha, \beta)$, i.e. $\alpha$ and $\beta$ are both odd. Then the embedding $\iota_\ell \colon K^{\perp}_{8m}(-1) \hookrightarrow L_{2,26}$ is primitive.*

*Proof.* It is enough to check that $M_d = A_1(-1)^{\oplus 2} \oplus \langle 2m \rangle$ embeds primitively into $U \oplus E_8(-1)$. To show an embedding is primitive, it suffices to show the image of every primitive vector is primitive. Thus, we check that $xu + y\ell$ is primitive in $U \oplus E_8(-1)$ for any relatively prime integers $x$ and $y$ and any primitive vector $u \in \langle a_1, a_2 \rangle$. Suppose that there is a positive integer $n$ dividing $xu + y\ell$ in $U \oplus E_8(-1)$. Then $n | y(\alpha, \beta)$ and $n | xu + yv$ in $E_8(-1)$. As $E_8(-1)/(A_1(-1)^{\oplus 2} \oplus D_6(-1)) \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$, we must have $n|2$. It follows that $n|y$, so $n|x$ as well (as $A_1(-1)^{\oplus 2}$ is primitively embedded in $E_8(-1)$). As $x$ and $y$ are coprime, we must have $n = 1$, so $xu + y\ell$ is indeed primitive under the embedding $\iota_\ell$, and we conclude that $\iota_\ell$ is primitive. $\qquad\square$

To build our desired embeddings, it suffices to pick $\alpha$, $\beta$ so that: (a) Either $2 \nmid (\alpha, \beta)$ or $(\alpha, \beta) = 1$, (b) the inequalities (3.2) hold, and (c) $2(\alpha\beta - m - 1)$ is a sum of three distinct nonzero squares. We remark that (a) ensures primitivity (Lemma 4.3), while (b) and (c) guarantee $N_\ell$ is small and odd (Lemma 4.1). We begin by observing that it is necessary and sufficient for (c) to hold that $\alpha\beta - m - 1$ be odd and avoid some finite set of exceptiional values (see Lemma 4.2); in particular, we will ask that

$$\alpha\beta - m - 1 > 51, \alpha\beta - m - 1 \neq N. \tag{4.3}$$

We further ask that the inequality

$$\sqrt{\frac{5m}{4}} - \sqrt{(1 + \epsilon)m} > 2 \tag{4.4}$$

holds, thereby giving us freedom to pick $\alpha$ to be in any residue class mod 2 If $m \equiv 0 \bmod 2$, we are able to pick $\alpha$ and $\beta = \alpha + 1$ satisfying (3.2), thanks to (4.4). Then $\alpha\beta - m - 1 = \alpha^2 + \alpha - m - 1$ is odd. On the other hand, if $m \equiv 1 \bmod 2$, we again can use (4.4) to pick $\alpha \equiv 1 \bmod 2$ satisfying the inequality for $\alpha$ in 3.2, and set $\alpha = \beta$, guaranteeing in any case that (a), (b), and (c) all hold.

We now use the conditions (4.4) together with (4.3) to determine a lower bound $m_0$ such that $\mathcal{M}_{8m}$ is of general type for $m \geq m_0$. First, note that

$$\alpha^2 + \alpha - m - 1 > \alpha^2 - m - 1 > \epsilon m - 1 \tag{4.5}$$

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holds for all $m, \alpha$ for which 3.2 holds. So given our choices of $\alpha$ and $\beta$ from the previous paragraph, we have the inequality

$$2(\alpha\beta - m - 1) > 2\epsilon m - 2.$$

Now, we impose the additional constraint that

$$\epsilon m > 52 \qquad (4.6)$$

guaranteeing that $2(\alpha\beta - m - 1) > 102$ and thereby avoiding the exceptional values of Lemma 4.2, except perhaps $N$. If $2(\alpha\beta - m - 1) = N$, then the inequalities

$$\beta^2 - \beta - m - 1 < \beta^2 - m - 1 < \beta^2 - m < \frac{m}{4}$$

hold under our continuing assumption of 3.2 , so

$$N < \frac{m}{2}.$$

Therefore, we have $m > 10 \cdot 10^{10}$. If we take $\epsilon$ to be sufficiently small and $m$ is large enough, then

$$\sqrt{\frac{5m}{4}} - \sqrt{(1+\epsilon)m} > 4 \qquad (4.7)$$

so we can adjust $\alpha$ by $\pm 2$ to avoid $N$.

At this point, we have demonstrated that whenever $m$ and $\epsilon$ satisfy the inequalities (4.6) and (4.4), it is possible to pick $\alpha$ and $\beta$ and $v$ to prove $\mathcal{M}_d$ is of general type. A simple optimization for (4.4) and (4.6) yields $m \geq 2055$, $\epsilon = .1533$. If $m > 10 \cdot 10^{10}$, then (4.7) holds, so $\alpha$ may be adjusted to avoid $N$ if necessary. Putting everything together, we have proved the following theorem:

**Theorem 4.4.** *For $m \geq 2055$, the moduli space $\mathcal{M}_{8m}$ is a variety of general type.*

*Proof.* The above discussion shows that for $m \geq 2055$, there exists primitive embedding $\iota_\ell \colon K_{8m} \hookrightarrow L_{2,26}$ such that $N_\ell \in \{1, 3\}$. Using Lemma 3.7 and Proposition 3.9, we see that $\iota_\ell$ satisfies the hypotheses of Theorem 1.1, proving the claim, $\qquad \square$

To study the Kodaira dimension for $m < 2055$ and thereby conclude Theorem 1.1(1), we make use of a computer to find explicit embeddings. See §4.4 for details.

4.2. **Analysis:** $d = 8m + 2$. As in the $d = 8m$ case, we are searching for $\alpha, \beta$, and $v$ such that the square-length $2m$ vector $\ell = \alpha e + \beta f + v$ is admissible for $d = 8m + 2$ and yields a small, odd value for $N_\ell$. For the admissibility of $\ell$, it is necessary and sufficient that the vector $v \in E_8(-1)$ may be written as

$$v = \frac{-a_2}{2} + v' \in (\langle a_1, a_2 \rangle \oplus D_6(-1))^\vee = (\langle a_1, a_2 \rangle^\vee \oplus D_6(-1))^\vee,$$

where $v' \in D_6(-1)^\vee = \langle a_1, a_2 \rangle$.

For each $m$ greater than the lower bound that is to be determined, our argument is written in a way that relies on the choice of $a_1, a_2 \in E_8(-1)$; precisely, for each $m$, we will construct $E_8(-1)$ as specific overlattice of $A_1(-1)^{\oplus 2} \oplus D_6(-1)$, and then consider embeddings

for which $a_1, a_2$ generate image of the summand $A_1(-1)^{\oplus 2}$. The theory of overlattices is explained in [Nik79, Section 1.4], a consequence of which is the following: there are exactly two unimodular negative definite even integral sublattices of rank 8 (necessarily isomorphic to $E_8$) contained in the $\mathbf{Q}$-lattice $(A_1(-1)^{\oplus 2})^\vee \oplus D_6(-1)^\vee$, each of which corresponds to one of the two maximal isotropic subgroups of $D((A_1(-1)^{\oplus 2})^\vee \oplus D_6(-1)^\vee)$. Let us call these two sublattices $L_1$ and $L_2$. To describe them, let $h_1, h_2$ each denote a generator of an orthogonal summand of $A_1(-1)^{\oplus 2}$, and define elements $b_1, b_{2,p}$ in $\langle h_1, h_2 \rangle^\vee \oplus D_6(-1)^\vee$ by

$$b_1 := e_1 + \frac{h_1 + h_2}{2}$$

$$b_{2,p} := \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + \frac{h_p}{2}.$$

Then $L_p$ is generated as an sublattice of $A_1(-1)^{\oplus 2} \vee \oplus D_6(-1)^\vee$ by $b_1, b_{2,p}$, and $\langle h_1, h_2 \rangle \oplus D_6(-1)$.

We now prove two simple lemmas: one will help ensure our eventual choice for $v'$ actually gives an embedding, and the other controls the size of $N_\ell$.

**Lemma 4.5.** *Suppose that $v' \in \mathbf{Q}^6(-1)$ is of the form*

$$v' = \frac{1}{2}(x_1 e_1 + x_2 e_2 + x_3 e_3 + 3e_4 + 3e_5 + 3e_6) \tag{4.8}$$

*with $x_i \in \mathbf{Z}$ all positive odd. Then $v' \in D_6(-1)^\vee$ and there is always a choice of $p \in \{1, 2\}$ such that $v' - \frac{h_2}{2} \in L_p$.*

*Proof.* We have $v' \in D_6(-1)^\vee$ because all the coefficients with respect to the $\{e_1, \dots, e_6\}$ basis are half-integers. For the other statement, we calculate inner products among the vectors $b_1, b_{2,p}$, and $v = v' - \frac{h_2}{2}$ in $\langle h_1, h_2 \rangle^\vee \oplus D_6(-1)^\vee$:

$$(v, b_1) = \frac{-3 + 1}{2} = -1$$

$$(v, b_{2,p}) = -\frac{1}{4}(9 + x_1 + x_2 + x_3) - \frac{1}{4}(h_2, h_p).$$

These inner products are integer-valued if and only if $v \in L_p$. By taking $p = 1$ when $x_1 + x_2 + x_3 \equiv 3 \bmod 4$ or choosing $p = 2$ otherwise, we see there is always $p$ such that $v \in L_p$. $\qquad\square$

**Lemma 4.6.** *Suppose that*

- *$\alpha, \beta$, and $m$ are positive integers satisfying the inequalities (3.2),*
- *$v' = \frac{1}{2}(x_1 e_1 + x_2 e_2 + x_3 e_3 + 3e_4 + 3e_5 + 3e_6) \in D_6(-1)^\vee$, as in Lemma 4.5.*
- *$(v')^2 = (2m - \alpha\beta) + \frac{1}{2}$*
- *the integers $x_1, x_2, x_3$ in $v'$ are distinct integers, none of which are equal to 3.*

*Choose $p \in \{1, 2\}$ so that $v' - \frac{h_2}{2} \in L_p$, and fix an identification of $L_p$ with $E_8(-1)$. Let $\iota_\ell$ be the embedding defined by $a_1 = h_1, a_2 = h_2$, and $\ell = \alpha e + \beta f + v' - \frac{a_2}{2}$. Then $R_\ell = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3)\}$.*

*Proof.* Omitted, as is completely similar to the proof of Lemma 4.1. $\qquad\square$

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Assuming we have chosen $\alpha, \beta$, and $m$ satisfying the inequalities 3.2, we show that it is always possible pick $v' \in D_6(-1)$ satisfying the hypothesis of the lemma. A vector $v'$ as in (4.8) satisfies

$$-(2v')^2 = x_1^3 + x_2^2 + x_3^6 + 27 = -(8(m - \alpha\beta) + 2) = 8(\alpha\beta - m) - 2.$$

So it suffices to find a solution to

$$x_1^2 + x_2^2 + x_3^2 = 8(\alpha\beta - m) - 29 \tag{4.9}$$

subject to certain conditions; precisely, we want *distinct* positive, coprime integer solutions $(x_1, x_2, x_3)$, such that $3 \notin \{x_1, x_2, x_3\}$. Since every square is 0 or 1 mod 4, it follows that any solution satisfying these conditions is a triple of *odd* integers. As $8(\alpha\beta - m) - 29 \equiv 3 \bmod 8$, we can always solve (4.9) away from the finite list of exceptional values, by Lemma 4.2. Suppose that we arrange, by appropriately choosing $\alpha$ and $\beta$, that $3 | 8(\alpha\beta - n) - 29$. Then each of the pairwise coprime integers $x_1, x_2, x_3$ coming from a solution to (4.9) must be distinct from 3, or else we would have $3 | \text{GCD}(x_1, x_2, x_3)$. Therefore, if we impose the additional condition on $\alpha, \beta$, and $m$ that $3 | 8(\alpha\beta - m) - 29$, then there exists a $v'$ satisfying the hypotheses of Lemma 4.6.

To build our embeddings, it suffices to arrange that: (a) $(\alpha, \beta) = 1$ (to guarantee primitivity), (b) the inequalities (3.2) hold, and (c) $8(\alpha\beta - m) - 29$ is a sum of three distinct nonzero squares. We have already seen that (c) holds if

$$8(\alpha\beta - m) - 29 > 627, 8(\alpha\beta - m) - 29 \neq N \tag{4.10}$$

and

$$3 | 8(\alpha\beta - m) - 29. \tag{4.11}$$

If the inequality

$$\sqrt{\frac{5m}{4}} - \sqrt{(1 + \epsilon)m} > 6 \tag{4.12}$$

holds, then there must exist relatively prime $\alpha, \beta$ satisfying (3.2) such that both $\beta = \alpha + g$ for some $g \in \{1, 3\}$ *and* $3 | 8(\alpha\beta - m) - 29$.

By considering the conditions (4.12) and (4.10), we can now successfully determine a lower bound $m_0$ such that $\mathcal{M}_{8m+2}$ is of general type for $m \geq m_0$. First, note that for $\alpha$, $\beta = \alpha + g$, and $m$ satisfying (3.2), we have the inequality

$$\alpha\beta - m = \alpha^2 + g\alpha - m > \alpha^2 - m > \epsilon m \tag{4.13}$$

and, as an immediate consequence,

$$8(\alpha\beta - m) - 29 > 8\epsilon m - 29.$$

Thus, taking

$$\epsilon m > 82 \tag{4.14}$$

will ensure that $8(\alpha\beta - m) - 29 > 627$. If $8(\alpha\beta - m) - 29 = N$, where recall that $N$ is as defined in Lemma 4.2, then the inequalities

$$\alpha\beta - m = \beta^2 - g\beta - m < \beta^2 - m < \frac{m}{4}$$

hold under our continuing assumptions on $\alpha, \beta = \alpha + g$, and $m$. Therefore for such $N$ we must have

$$N < 2m - 29.$$

So we would like to ensure that for $m > (N + 29)/2$, the quantity $\epsilon > 0$ is small enough so that the difference

$$\sqrt{\frac{5m}{4}} - \sqrt{(1 + \epsilon)m} \tag{4.15}$$

is large enough to adjust $\alpha, \beta$ by $\pm 3$ in order to avoid $N$.

As before, optimization for (4.12) and (4.14) yields $m \geq 3238$ and $\epsilon = .025328$. In the range $m \geq 3238$ for this $\epsilon$, one checks that

$$\sqrt{\frac{5m}{4}} - \sqrt{(1 + \epsilon)m} > 16000 \tag{4.16}$$

so we are always able to adjust $\alpha$ to avoid $N$ . As in §4.1, we now have proven the following theorem:

**Theorem 4.7.** *For $m \geq 3238$, the moduli space $\mathcal{M}_{8m+2}$ is a variety of general type.*

The remaining cases are for $d = 8m + 2$ and are handled by computer (see §4.4).

4.3. **Analysis:** $d = 8m + 4$. Our argument for $d = 8m + 4$ is nearly identical to the case for $d = 8m + 2$, but we must write it the details in order to compute an explicit lower bound. To precisely state the problem, we wish to show that for all but finitely many positive integers $m$, there are positive integers $\alpha, \beta$, and $v \in U \oplus E_8(-1)$ such that the square-length $2m$ vector $\ell = \alpha e + \beta f + v$ is admissible for $d = 8m + 2$ and yields a small, odd value for $N_\ell$. For the admissibility of $\ell$, it is necessary and sufficient that the vector $v \in E_8(-1)$ may be written as

$$v = \frac{-a_1 - a_2}{2} + v' \in (\langle a_1, a_2 \rangle \oplus D_6(-1))^\vee = (\langle a_1, a_2 \rangle^\vee \oplus D_6(-1))^\vee,$$

where $v' \in D_6(-1)^\vee = \langle a_1, a_2 \rangle$.

The following two lemmas adapt Lemmas 4.5 and 4.6 to the case of $8m + 4$. Recall the vectors $h_1, h_2$ are an orthogonal basis for $A_1(-1)$ and $b_1, b_{2,p}$ for $p \in \{1, 2\}$ are vectors in $(A_1(-1)^{\oplus 2} \oplus D_6(-1))^\vee)$.

**Lemma 4.8.** *Suppose that $v' \in \mathbf{Q}^6(-1)$ is of the form*

$$v' = x_1 e_1 + x_2 e_2 + x_3 e_3 + 3e_4 + 3e_5 + 3e_6 \tag{4.17}$$

*with $x_i \in \mathbf{Z}$ all positive integers such that $\sum x_i \equiv 0 \bmod 2$ . Then $v' \in D_6(-1)^\vee$; furthermore, for any isometrically embedded sublattice $A_1(-1)^{\oplus 2} \oplus D_6(-1) \hookrightarrow E_8(-1)$, the image of $v' - \frac{h_1 + h_2}{2}$ is an element of $E_8(-1)$.*

*Proof.* We have $v' \in D_6(-1)^\vee$ because all the coefficients with respect to the $\{e_1, \ldots, e_6\}$ basis are integers. For the other statement, we recall that for some $p \in \{1, 2\}$, $E_8(-1)$ is formed by the span of the isometric image of $\langle h_1, h_2 \rangle \oplus D_6(-1)$ and $b_1, b_{2p}$. Taking inner products among the vectors $b_1, b_{2,p}$, and $v = v' - \frac{h_1 + h_2}{2}$ in $\langle h_1, h_2 \rangle^\vee \oplus D_6(-1)^\vee$:

$$(v, b_1) = -2x_1 + 1$$

$$(v, b_{2,p}) = -\frac{1}{2}(9 + x_1 + x_2 + a_3) - \frac{1}{2}.$$

By hypothesis, these inner two quantities are integers, and therefore $v \in E_8(-1)$ □

**Lemma 4.9.** *Suppose that*

- *$\alpha, \beta$, and $m$ are positive integers satisfying the inequalities (3.2),*
- *$v' = x_1 e_1 + x_2 e_2 + x_3 e_3 + 3e_4 + 3e_5 + 3e_6 \in D_6(-1)^\vee$, as in Lemma 4.8,*
- *$(v')^2 = (2m - \alpha\beta) + 1$,*
- *the integers $x_1, x_2, x_3$ in $v'$ are distinct integers, none of which are equal to 3.*

*Pick any $a_1, a_2$ orthogonal $(-2)$-roots of $E_8(-1)$, and let $\iota_\ell$ be the embedding defined by $a_1 = h_1, a_2 = h_2$, and $\ell = \alpha e + \beta f + v' - \frac{a_2}{2}$. Then $R_\ell = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3)\}$.*

*Proof.* Omitted, as is completely similar to the proof of Lemma 4.1. □

Assuming we have chosen $\alpha, \beta$, and $m$ satisfying the inequalities 3.2, we show that it is always possible pick $v' \in D_6(-1)$ satisfying the hypothesis of Lemma 4.9. A vector $v'$ as in (4.17) satisfies

$$-(v)^2 = a_2^3 + a_5^2 + a_6^6 + 27 = 2(\alpha\beta - m) - 1$$

So it suffices to find a solution to

$$x_1^2 + x_2^2 + x_3^2 = 2(\alpha\beta - m) - 28 \tag{4.18}$$

subject to certain conditions; precisely, we want *distinct* positive, coprime integer solutions $(x_1, x_2, x_3)$, such that $3 \notin \{x_1, x_2, x_3\}$. Suppose we have arranged that $2(\alpha\beta - n) - 28 \equiv 2 \bmod 4$, or, equivalently, that $\alpha\beta - n$ is odd. Then we can always solve (4.18) (by Lemma 4.2), away from the finite list of exceptional values. Suppose that we have additionally arranged, by appropriately choosing $\alpha$ and $\beta$, that $3 | 3(\alpha\beta - n) - 28$. Then each of the pairwise coprime integers $x_1, x_2, x_3$ coming from a solution to (4.18) must be distinct from 3, or else we would have $3 | \mathrm{GCD}(x_1, x_2, x_3)$. Therefore, if we impose the additional conditions on $\alpha, \beta$, and $m$ that $3 | 3(\alpha\beta - m) - 28$ and that $\alpha\beta - m$ is odd, then there exists a $v'$ satisfying the hypotheses of Lemma 4.9.

To build our embeddings, it suffices to arrange that: (a) $(\alpha, \beta) = 1$ (to guarantee primitivity), (b) the inequalities (3.2) hold, and (c) $2(\alpha\beta - m) - 28$ is a sum of three distinct nonzero squares. We have already seen that (c) holds if

$$2(\alpha\beta - m) - 28 > 2, 8(\alpha\beta - m) - 29 \neq N, \tag{4.19}$$

$$3 | 8(\alpha\beta - m) - 28, \tag{4.20}$$

$$\alpha\beta - m \equiv 1 \bmod 2. \tag{4.21}$$

If we insist that the inequality

$$\sqrt{\frac{5m}{4}} - \sqrt{(1+\epsilon)m} > 12 \tag{4.22}$$

holds, then there must exist relatively prime $\alpha, \beta$ satisfying (3.2) such that (c) holds: the inequality (4.22) lets us pick $\alpha, \beta$ with $\beta = \alpha + g$ for some $g \in \{1, 2, 3, 6\}$ satisfying such that $3|2(\alpha\beta - n) - 2$ and $\alpha\beta - n$ is odd. Specifically, if $n$ is odd, pick appropriate $\alpha$ and $\beta = \alpha + g$ for $g \in \{1, 3\}$, while if $n$ is even pick $\beta = \alpha + g$ with $g \in \{2, 6\}$.

By considering the conditions (4.22) and (4.10), we can now successfully determine a lower bound $m_0$ such that $\mathcal{M}_{8m+2}$ is of general type for $m \geq m_0$. First, note that for $\alpha$, $\beta = \alpha + g$, and $m$ satisfying (3.2), we have the inequality

$$\alpha\beta - m = \alpha^2 + g\alpha - m > \alpha^2 - m > \epsilon m \tag{4.23}$$

and, as an immediate consequence,

$$2(\alpha\beta - m) - 28 > 2\epsilon m - 28.$$

Thus, taking

$$\epsilon m > 15 \tag{4.24}$$

will ensure that $2(\alpha\beta - m) - 28 > 2$. If $2(\alpha\beta - m) - 28 = N$, where recall that $N$ is as defined in Lemma 4.2, then the inequalities

$$\alpha\beta - m = \beta^2 - g\beta - m < \beta^2 - m < \frac{m}{4}$$

hold under our continuing assumptions on $\alpha, \beta = \alpha + g$, and $m$. Therefore for such $N$ we must have

$$N < n/2 - 28.$$

So we would like to ensure that for $m > 4(N + 28)$, the quantity $\epsilon > 0$ is small enough so that the difference

$$\sqrt{\frac{5m}{4}} - \sqrt{(1+\epsilon)m}$$

is large enough to adjust $\alpha, \beta$ by $\pm 6$ in order to avoid $N$.

As before, optimization for (4.22) and (4.24) yields $m \geq 10463$ and $\epsilon = 0.0014337$.

$$\sqrt{\frac{5m}{4}} - \sqrt{(1+\epsilon)m} > 50000 \tag{4.25}$$

so we are always able to adjust $\alpha$ to avoid $N$. As in §4.1, we now have proven the following theorem:

**Theorem 4.10.** *For $m \geq 10463$, the moduli space $\mathcal{M}_{8m+4}$ is a variety of general type.*

The remaining cases are for $d = 8m + 4$ and are handled by computer (see §4.4).

4.4. **Searching for embeddings by computer.** Included in the electronic distribution of this article is a list of embeddings for the values of $m$ less than the lower bounds we calculated above. To find these embeddings, we used a simple transplantation of the algorithm

given in [TVA19, §5]. Our search for these embeddings was exhaustive: we include in our list every $m$ for which there exists an embedding $K_d^{\perp} \to L_{2,26}$ with our desired properties. We include this list along with `Magma` code [BCP97] to certify that the embeddings in our list produce modular forms of the correct weight[2]. To count the size of $R_{-2}$ corresponding for each embedding, we count by their Type from Lemma 3.3 (see Step (iv) of the algorithm in [TVA19, §5]). Our list of explicit embeddings, taken together with the analyses in §§4.1, 4.2, 4.3, prove Theroem 1.1.

$\square$

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