Question 1 (10pts). Let $X_1 \sim \text{Gamma}(r_1, \lambda)$ and $X_2 \sim \text{Gamma}(r_2, \lambda)$ be independent random variables, and let $Y = X_1 + X_2$ and let $Z = X_1/(X_1 + X_2)$.

Find the joint density of Y and Z, and find the marginal densities of Y and Z. Identify the distributions of Y and Z. Note that $\text{Gamma}(r, \lambda)$ distribution has density function

$$f(x) = \frac{\lambda^r}{\Gamma(r)} e^{-\lambda x} x^{r-1}, \quad x > 0$$

Answer:

Since $X_1 \perp\!\!\!\perp X_2$, their joint density is the product of their marginal densities.

$$f_{X_1,X_2}(x_1, x_2) = \frac{\lambda^{r_1+r_2}}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot e^{-\lambda(x_1+x_2)} \cdot x_1^{r_1-1} \cdot x_2^{r_2-1} \tag{1.1}$$

Let $X = [X_1, X_2]^\intercal$. Define

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = H(X) = \begin{bmatrix} X_1 + X_2 \\ \frac{X_1}{X_1+X_2} \end{bmatrix} \tag{1.2}$$

The transformation $H(X)$ has Jacobian matrix

$$J(H) = \begin{bmatrix} 1 & 1 \\ \frac{X_2}{(X_1+X_2)^2} & \frac{-X_1}{(X_1+X_2)^2} \end{bmatrix} \tag{1.3}$$

the determinant of which is $-1/(X_1 + X_2)$. So

$$F_{Y,Z}(y, z) = \sum_{X:H(X)=[y,z]} \frac{\lambda^{r_1+r_2}}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot e^{-\lambda(x_1+x_2)} \cdot x_1^{r_1-1} \cdot x_2^{r_2-1} \cdot \underbrace{(x_1 + x_2)}_{=1/|\det(J(H))|} \tag{1.4}$$

$$= \frac{\lambda^{r_1+r_2}}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot e^{-\lambda(x_1+x_2)} \cdot x_1^{r_1} \cdot x_2^{r_2} \cdot \frac{x_1 + x_2}{x_1x_2} \tag{1.5}$$

$$= \frac{\lambda^{r_1+r_2}}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot e^{-\lambda y} \cdot (yz)^{r_1} \cdot (y(1-z))^{r_2} \cdot \frac{1}{yz(1-z)} \tag{1.6}$$

$$= \underbrace{\frac{\lambda^{r_1+r_2}}{\Gamma(r_1+r_2)} \cdot e^{-\lambda y} \cdot y^{r_1+r_2-1}}_{\text{Gamma}(r_1+r_2,\lambda)} \cdot \underbrace{\frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot z^{r_1-1}(1-z)^{r_2-1}}_{\text{Beta}(r_1,r_2)} \tag{1.7}$$

So 1.7 is the joint probability density of Y and Z. Then, we integrate out y and z to obtain their marginal densities. First, note that y has support in $[0, \infty)$, whereas z has support in $(0, 1)$. Next, as the underbraced portions of 1.7 note, the joint probability density of Y and Z is the product of two expressions, one in y, the other in z, each of which is a known probability density. So

$$f_Y(y) = \frac{\lambda^{r_1+r_2}}{\Gamma(r_1+r_2)} \cdot e^{-\lambda y} \cdot y^{r_1+r_2-1} \cdot \int_0^1 \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot z^{r_1-1}(1-z)^{r_2-1} dz \tag{1.8}$$

$$= \frac{\lambda^{r_1+r_2}}{\Gamma(r_1+r_2)} \cdot e^{-\lambda y} \cdot y^{r_1+r_2-1} \tag{1.9}$$

and

$$f_Z(z) = \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot z^{r_1-1}(1-z)^{r_2-1} \cdot \int_0^\infty \frac{\lambda^{r_1+r_2}}{\Gamma(r_1+r_2)} \cdot e^{-\lambda y} \cdot y^{r_1+r_2-1} dy \tag{1.10}$$

$$= \frac{\Gamma(r_1+r_2)}{\Gamma(r_1)\cdot\Gamma(r_2)} \cdot z^{r_1-1}(1-z)^{r_2-1} \tag{1.11}$$

So $Y \sim \text{Gamma}(r_1 + r_2, \lambda)$ and $Z \sim \text{Beta}(r_1, r_2)$. Furthermore, since $f_{Y,Z}(y, z) = f_Y(y) \cdot f_Z(z)$, $Y \perp\!\!\!\perp Z$.

```python
def egyptian_multiplication(a, n):
    """
    Returns the product `a * n`.
    Assume n is a nonnegative integer
    """

    def is_odd(n):
        """
        Returns True if n is odd.
        """
        return n & 0x1 == 1

    if n == 1:
        return a
    if n == 0:
        return 0

    if is_odd(n):
        return egyptian_multiplication(a + a, n // 2) + a
    else:
        return egyptian_multiplication(a + a, n // 2)
```