Homework 6

Due: 11:59pm, Thursday, April 6

Instruction: Please scan or typeset your solutions and upload them as a single pdf file to Canvas. Do not just take a picture of your solutions.

0. Readings: Notes 7 and Chapter 15.

We consider the regression model for the `carprice` data set. (`carprice.xlsx` or `carprice.csv` are on Canvas)

1. Let y_i be the car price (in **hundreds**), x_{i1} be the age, and x_{i2} be the mileage (in **thousands**) of the ith car, consider the following model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i, \quad i = 1, \ldots, n$$

where ϵ_i is the random error component and is assumed to be iid $N(0, \sigma^2)$. Write down the pdf $f(y_i)$.

2. Write down the joint distribution $f(y_1, \ldots, y_n)$.

3. What is the likelihood function $L(\beta_0, \beta_1, \beta_2, \sigma)$.

4. Write down the log likelihood function, $l(\beta_0, \beta_1, \beta_2, \sigma) = \log L(\beta_0, \beta_1, \beta_2, \sigma)$, and negative log likelihood function $-l(\beta_0, \beta_1, \beta_2, \sigma)$.

5. The maximum likelihood estimator of $\beta_0, \beta_1, \beta_2$ and σ is

$$(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\sigma}) = \operatorname{argmax} L(\beta_0, \beta_1, \beta_2, \sigma)$$

explain that it is equivalent to the followings

$$\begin{aligned}(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\sigma}) &= \operatorname{argmax} l(\beta_0, \beta_1, \beta_2, \sigma) \\ &= \operatorname{argmin} \, - l(\beta_0, \beta_1, \beta_2, \sigma)\end{aligned}$$

6. Explain that the maximum likelihood estimator and least squared estimator of $\beta_0, \beta_1, \beta_2$ are the same.

7. In class, we show that $\hat{\beta} = (X^T X)^{-1} X^T y$ (see class notes), carry out this computation in `R`, what is your $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$?

8. How do you interpret $\hat{\beta}_1$ and $\hat{\beta}_2$?

9. Given your estimates, what is your price prediction of a used car with 4 years old, and $50,000$ miles?

10. Use `lm` command, what is your $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$? Are they the same as your answers in Question 7?

11. (Optional) Show that

$$\hat{\beta} = (X^T X)^{-1} X^T y.$$