Homework 8

Due: 11:59pm, Thursday, April 21

Instruction: Please scan or typeset your solutions and upload them as a single pdf file to Canvas. Do not just take a picture of your solutions.

0. Readings: Sections 9.3, 9.4, 10.1 and Notes 9 and 10.

Let $Y_1, Y_2, \ldots, Y_n \sim_{iid} N(\theta, \sigma^2)$ where σ^2 is known. Consider

$$H_0 : \theta \in \Omega_0 \quad \text{vs} \quad H_1 : \theta \in \Omega_1$$

where

$$\Omega_0 = (-\infty, \theta_0], \quad \Omega_1 = (\theta_0, \infty), \quad \Omega = (-\infty, \infty) = \Omega_0 \cup \Omega_1.$$

1. Write down the joint distribution $f(y_1, \ldots, y_n)$ and the likelihood function $L(\theta)$.

2. Given that the MLE is

$$\hat{\theta} = \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i,$$

then what is the

$$\max_{\theta \in \Omega} L(\theta)?$$

3. Under $H_0 : \theta \leq \theta_0$, the restricted MLE is

$$\hat{\theta}_R = \begin{cases} \bar{y}, & \bar{y} \leq \theta_0 \\ \theta_0, & \bar{y} > \theta_0 \end{cases}$$

then what is

$$\max_{\theta \in \Omega_0} L(\theta)?$$

4. The likelihood ratio test statistic is

$$\lambda(y_1, \ldots, y_n) = \frac{\max_{\theta \in \Omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$

Obtain this likelihood ratio test statistic.

5. The likelihood ratio test procedure is that we will reject H_0 if

$$\lambda(y_1, \ldots, y_n) \leq k,$$

where $k < 1$. Explain that this is equivalent to

$$z \geq \sqrt{-2 \log k}, \quad \text{where} \quad z = \frac{\bar{y} - \theta_0}{\sigma/\sqrt{n}}.$$

6. Why $k < 1$?

7. (Optional) Derive both MLE $\hat{\theta}$ and restricted MLE $\hat{\theta}_R$.