Homework 5

Due: 11:59pm, Thursday, March 31

Instruction: Please scan or typeset your solutions and upload them as a single pdf file to Canvas. Do not just take a picture of your solutions.

0. Readings: Notes 6, Section 6.3, Section 9.1, Section 9.2. You are encouraged to read Chapters 6, 7 and 8.

1. Let

$$Y \sim N(\mu, \sigma^2)$$

write down its pdf.

2. Let $Y_1, Y_2, \ldots, Y_n \sim_{iid} N(\mu, \sigma^2)$, then write down the joint distribution $f(y_1, \ldots, y_n)$ when $Y_1 = y_1, \ldots, Y_n = y_n$ are observed.

3. This joint distribution is a function of μ and σ, when $Y_1 = y_1, \ldots, Y_n = y_n$ are observed. It is called the Likelihood function. Write down this Likelihood function $L(\mu, \sigma)$

4. Write down the log likelihood function, $l(\mu, \sigma) = \log L(\mu, \sigma)$, and negative log likelihood function $-l(\mu, \sigma)$.

5. The maximum likelihood estimator of μ and σ is

$$(\hat{\mu}, \hat{\sigma}) = \operatorname{argmax} L(\mu, \sigma)$$

explain that it is equivalent to the followings

$$\begin{aligned}(\hat{\mu}, \hat{\sigma}) &= \operatorname{argmax} l(\mu, \sigma) \\ &= \operatorname{argmin} -l(\mu, \sigma)\end{aligned}$$

6. Explain that the maximum likelihood estimator and least squared estimator of μ are the same.

7. Consider the carprice example, and let Y be the price (in hundreds). Assume $Y \sim N(\mu, \sigma^2)$. (`carprice.xlsx` or `carprice.csv` are on Canvas)

(a) Read the data in `R`. (Check `Notes7` for data input)

(b) You may use the following optimization procedure to obtain the $\hat{\mu}$ and $\hat{\sigma}$, show your results.

```
fn.logL <- function(par,y){
  n <- length(y)
  Ln.l <- 0
  mu.l <- par[1]
  sigma.l <- par[2]
  for(i in 1:n){
    Ln.l <- Ln.l + log(dnorm(y[i],mean=mu.l,sd=sigma.l))
  }
  return(-Ln.l)
}

opt.out <- optim(c(100,30),fn.logL,y=y)

mu.hat <- opt.out$par[1]
sigma.hat <- opt.out$par[2]
```

(Remark. Here we obtain the estimates through the maximizing the log likelihood. You may alter the code by maximizing the likelihood directly or by minimizing the squared loss. We use the initial values of 100 and 30 in `optim()` procedure, you may alter that as well. Another thing is you may try to use `nlminb()`)

8. (Optional) Use calculus to show that the MLE

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$