hw8-report

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Let $Y_1, Y_2, \ldots, Y_n \sim_{iid} N(\theta, \sigma^2)$ where σ^2 is known. Consider $H_0 : \theta \in \Omega_0$ vs $H_1 : \theta \in \Omega_1$ where

$$\Omega_0 = (-\infty, \theta_0], \quad \Omega_1 = (\theta_0, \infty), \quad \Omega = (-\infty, \infty) = \Omega_0 \cup \Omega_1$$

1. Write down the joint distribution $f(y_1, \ldots, y_n)$ and likelihood function $L(\theta)$.

$$\boxed{= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\theta)^2}}$$

The two equations have the same form, but the unknowns are not the same. For the joint distribution, the y values have not yet been observed and are unknown. On the other hand, the joint distribution *becomes* the likelihood function once these y values have been observed and the value of θ is the primary unknown of the equation.

2. Given that the MLE is

$$\hat{\theta} = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i,$$

then what is the $max_{\theta\in\Omega} L(\theta)$?

$$maxL(\theta) = L(\bar{y}) = \boxed{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i-\bar{y})^2}}$$

3. Under $H_0 : \theta \le \theta_0$, the restricted MLE is

$$\hat{\theta_R} = \begin{cases} \bar{y}, & \bar{y} \le \theta_0 \\ \theta_0, & \bar{y} > \theta_0 \end{cases}$$

then what is $max_{\theta \in \Omega_0} L(\theta)$?

$$max_{H_0} L(\theta) = L(\theta_0) = \begin{cases} \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \bar{y})^2}, & \bar{y} \leq \theta_0 \\ \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \theta_0)^2}, & \bar{y} > \theta_0 \end{cases}$$
$$= \boxed{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \theta_0)^2}}$$

We can dismiss the case where $\bar{y}$ is less than θ_0 because the case invalidates our likelihood comparison in the first place. If ever $\bar{y}$ is less than θ_0, your ratio evaluates to one, a value you will never reject. Hence, the only hypotheses worth evaluating with our comparison are those that underestimate $\bar{y}$.

4. The likelihood ratio test statistic is

$$\lambda(y_1, \ldots, y_n) = \frac{max_{\theta \in \Omega_0} L(\theta)}{max_{\theta \in \Omega} L(\theta)}$$

Obtain this likelihood ratio test statistic.

$$\lambda(y_i, \ldots, y_n) = \frac{max_{\theta \in \Omega_0} L(\theta)}{max_{\theta \in \Omega} L(\theta)}$$
$$= \frac{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \theta_0)^2}}{\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \bar{y})^2}}$$
$$= e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \theta_0)^2 + \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \bar{y})^2}$$
$$= e^{-\frac{1}{2\sigma^2}[\sum_{i=1}^n (y_i - \theta_0)^2 - \sum_{i=1}^n (y_i - \bar{y})^2]}$$
$$\boxed{= e^{-\frac{1}{2\sigma^2} n(\bar{y} - \theta_0)^2}}$$

Note:

$$\sum_{i=1}^n (y_i - \theta_0)^2 - \sum_{i=1}^n (y_i - \bar{y})^2$$
$$= \sum_{i=1}^n (y_i - \bar{y} + \bar{y} - \theta_0)^2 - \sum_{i=1}^n (y_i - \bar{y})^2$$
$$= \sum_{i=1}^n (y_i - \bar{y})^2 + 2(y_i - \bar{y})(\bar{y} - \theta_0)^2 + (\bar{y} - \theta_0)^2 - \sum_{i=1}^n (y_i - \bar{y})^2$$
$$= n(\bar{y} - \theta_0)^2$$

5. The likelihood ratio test procedure is that we will reject H_0 if $\lambda(y_1, \ldots, y_n) \leq k$, where $k < 1$. Explain that this is equivalent to $z \geq \sqrt{-2logk}$, where

$$z = \frac{\bar{y} - \theta_0}{\sigma/\sqrt{n}}$$

We must start from our likelihood ratio metric (i.e. $e^{-\frac{1}{2\sigma^2}n(\bar{y}-\theta_0)^2} \leq k$) and from there, we can derive our value for z.

$$
\begin{aligned}
& e^{-\frac{1}{2\sigma^2}n(\bar{y}-\theta_0)^2} \leq k \\
\Rightarrow \quad & -\frac{1}{2\sigma^2}n(\bar{y}-\theta_0)^2 \leq log(k) \\
\Rightarrow \quad & \frac{n(\bar{y}-\theta_0)^2}{\sigma^2} \geq -2log(k) \\
\Rightarrow \quad & \frac{|\bar{y}-\theta_0|}{\sigma/\sqrt{n}} \geq \sqrt{-2log(k)}
\end{aligned}
$$

At this point, if we assign z to the value shown in the equation, we can see that comparing our likelihood ratio to k is equivalent to comparing our z metric to $\sqrt{-2log(k)}$.

6. Why must k < 1?

One can tell that the value for comparison, k, must be less than 1 simply by understanding the terms in the ratio it is compared with. The denominator of $\lambda(y_1, \ldots, y_n)$ is $max_{\theta\in\Omega}L(\theta)$. This term represents the global maximum of the likelihood with no restriction on the values that θ can take. The numerator of $\lambda(y_1, \ldots, y_n)$ is the maximum of the same likelihood function, but with a restriction on the values that θ can take. It therefore follows that the value numerator will be less than that of the denominator, leading to a value wherein k < 1.