Optimal Choice

Economics 100A
Winter 2021

1 Overview

1.1 Prolegomenon to Any Future Optimization

Optimization is what you will be doing in this class and in most economics classes. The idea is simple: how do we maximize some good subject to some constraint? This is an idea that permeates all economics courses, since we are obsessed with efficiency, so now is the time to get used to optimization.

How will we be discussing optimization in this specific class, though? So far, we have set up how the consumer can rank bundles (utility function) and how the consumer is limited in what they can afford (budget constraints). We will put these two together to maximize our utility subject to some constraint. Remember, in each section I have been outlining the different components of optimization.

1. **Objective Function**: What are we maximizing/minimizing?
2. **Constraint**: What do we constrain our objective function to?
3. **Choice variables**: What variables do we choose to achieve maximization/minimization?
4. **Exogenous variables**: What variables affect the situation but are taken as given?

By now you can probably guess what our objective function and constraint are. But think a bit about what we want to maximize/minimize right now, and think about what information we are being given. Below, I will outline the specifics for our optimization.

1. **Objective Function**: We maximize consumer utility
2. **Constraint**: We constrain utility to some budget constraint.
3. **Choice variables**: We maximize utility via consuming x and y, so we maximize the quantity of each good.
4. **Exogenous variables**: We take prices and income as given.

Let's write the problem as follows:

$$max u(x, y)$$

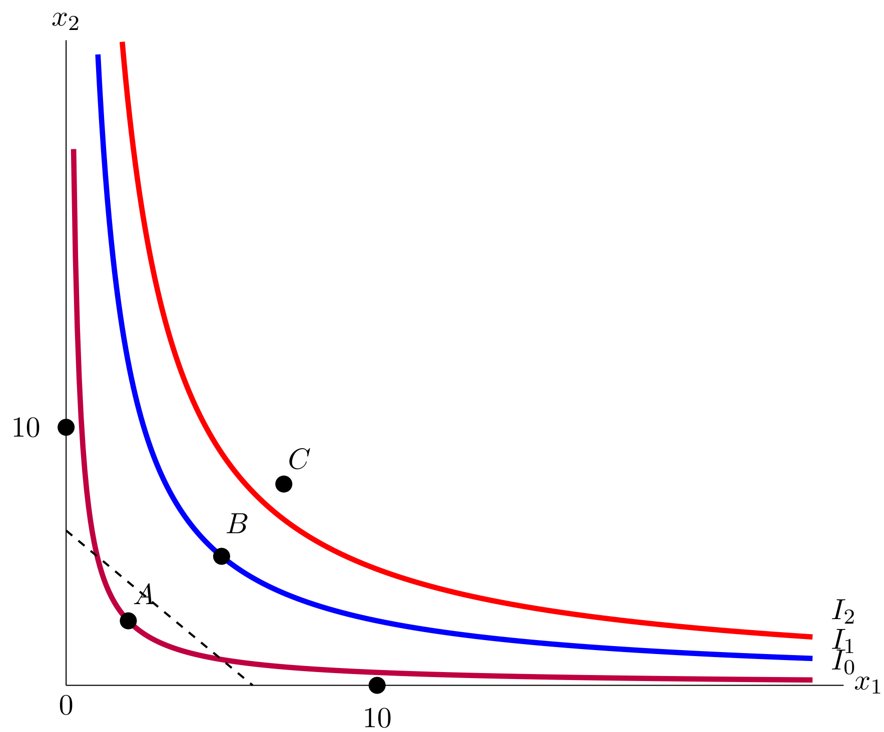
$$\text{s.t.} p_x x + p_y y = I$$

But what does this mean in words? We are trying to find the perfect indifference curve to optimize our utility. Since indifference curves exist everywhere, we want 1) the one which gives the consumer the most utility and 2) one that is feasible. If you had an unlimited budget, this problem would be completely meaningless. However, realistic budgets and prices make this a lot of fun.

In sum, we want to find the optimal bundle x^*, y^* which is feasible (affordable) and gives the consumer the most utility. If there is an indifference curve associated with a high utility, we assume that if the consumer does not take it, it must not be feasible.

2 Optimal Bundles

2.1 Graphical Solutions



The example above shows us why the blue indifference curve I_1 represents the optimal choice for a consumer. We have here that bundle B allows us to reach the highest feasible indifference curve. C is too expensive despite giving us higher utility. A on the other hand is cheaper but provides much less utility.

In summary, we can see that the optimal bundle occurs at a point of tangecy between the budget line and the indifference curve. This is where the slope of the budget constraint $\frac{p_x}{p_y}$ equals the slope of the indifference curve $\frac{MU_x}{MU_y}$. Why does this make sense? Remember that the MRS tells us how much y the consumer is willing to give up for one additional good of x. So, an MRS of 4 means that a consumer is willing to give up 4 units of y for one unit of x. But the slope of the budget constraint is just the opportunity cost of consuming one more of good x. Therefore, if $\frac{p_x}{p_y} < \frac{MU_x}{MU_y}$, the consumer could give up three units of y, get one more x and be better off than before. Since they only lose 3 units of y, they are better off than before, because they could have lost 4 units and been indifferent.

2.2 Lagrange and Substitution Methods

Lagrange To solve for the optimal bundle using constrained optimization, we can set up a Lagrangian:

$$\mathcal{L} = u(x, y) + \lambda(I - p_x x - p_y y)$$

For those who may need a calculus review, this equation allows us to maximize the values of x and y subject to a constraint. In this case, we are maximizing our utility from goods x and y and constraining ourselves to a budget. Solving for the optimal bundle first requires us to take three first order conditions.

In case you wanted to see something cool: let's look at the algebraic solution for finding the optimal bundle using the tangency condition and the Lagrange. Remember, we write out the Lagrange by

$$\frac{d_{\mathcal{L}}}{d_x} = \frac{\delta u(x, y)}{\delta x} - p_x \lambda = 0$$

$$\frac{d_{\mathcal{L}}}{d_y} = \frac{\delta u(x, y)}{\delta y} - p_y \lambda = 0$$

We can rearrange this and divide the first by the second:

$$\frac{d_{\mathcal{L}}}{d_x} = \frac{\delta u(x, y)}{\delta x} = p_x \lambda$$

$$\frac{d_{\mathcal{L}}}{d_y} = \frac{\delta u(x, y)}{\delta y} = p_y \lambda$$

$$\frac{\delta u(x, y)/\delta x}{\delta u(x, y)/\delta y} = \frac{p_x}{p_y}$$

And this is just the tangency condition! Let's rewrite this in terms of marginal utilities.

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

From here, we typically solve for the equation in terms of x or y, then we plug that into the budget constraint before solving for x or y.

3 Cobb-Douglas Utility Functions

3.1 General Notes and Results

1. Utility representation: $u(x, y) = x^{\alpha} y^{\beta}$
2. Indifference curves: standard convex curves that never touch the axes
3. Preferences: They prefer mixing goods and having a bit of both.

A Cobb-Douglas utility function typically takes the form

$$u(x, y) = x^{\alpha} y^{\beta}$$

Where x denotes good one and y denotes good two. It can also be written out as

$$u(x, y) = x^{\alpha} y^{\beta}$$

depending on the circumstances (e.g., if $\alpha + \beta > 0$, there is an interesting economic interpretation). I will go over both how to solve for the optimal bundle using a Lagrangian and the substitution method. Most students prefer the substitution method anyway, so it's important to be comfortable with it for exams. The Lagrangian is more useful for harder optimization problems, though, so if you intend on going further into the mathematical side of economics, you should try the Lagrange.

$$\begin{aligned}
\mathcal{L} &= x^{\alpha} y^{\beta} + \lambda(I - p_x - p_y) \\
\frac{d_{\mathcal{L}}}{d_x} &= \alpha x^{(\alpha-1)} y^{(\beta)} - p_x \lambda = 0 \\
\frac{d_{\mathcal{L}}}{d_y} &= (\beta) x^{(\alpha)} y^{(\beta-1)} - p_y \lambda = 0 \\
\frac{d_{\mathcal{L}}}{d_\lambda} &= I - p_x - p_y = 0
\end{aligned}$$

Note 1. We take the first order conditions of each unknown in order to maximize each variable. In your case, you will typically be given the income and prices, meaning you must solve for three unknowns: x, y, and λ.

Once we have our first order conditions, I like to solve for X and Y in terms of *lambda*. We do this by swinging our negative lambda over to the other side of the equation and simplifying until we can equate the two equations.

$$\begin{aligned}
\alpha x^{(\alpha-1)} y^{(\beta)} &= p_x \lambda \\
\beta x^{(\alpha)} y^{(\beta-1)} &= p_y \lambda
\end{aligned}$$

From here, we can simplify x in terms of y or the other way around. Do whatever is easiest for you. In any case, we have to divide by the prices of each to get the following:

$$\frac{\beta x^{(\alpha)} y^{(\beta-1)}}{p_y} = \frac{\alpha x^{(\alpha-1)} y^{(\beta)}}{p_x}$$

Once we have our equality, we can just solve for x or y. In this case, I will solve for y, but I encourage you to solve for X to get better mathematical practice.

$$y = \frac{\beta}{\alpha} \frac{(p_x)x}{p_y}$$

Once we have y, we can find our tangent point on the budget constraint by substituting $\frac{\beta}{\alpha} \frac{(p_x)x}{p_y}$ for y. We would get something like this:

$$I = p_y(\frac{\beta}{\alpha} \frac{(p_x)x}{p_y}) + p_x(x)$$

This thing looks disgusting, I know, but we can take it one step at a time. If you are like me and hate algebra, just slowly simplify until we can write this out as a nice relationship. Luckily for us, the p_y can cancel in the denominator, making things a bit easier.

$$I = \frac{\beta}{\alpha} p_x(x) + p_x(x)$$

Factor out prices and good x.

$$I = (1 + \frac{\beta}{\alpha}) p_x x$$

$$I = (\frac{\alpha + \beta}{\alpha}) p_x x$$

$$I = \frac{p_x x}{\alpha}$$

$$x = \frac{\alpha}{\alpha + \beta} \frac{I}{p_x}$$

We see now that the consumer's consumption of good x relies on three things: income, exponents (α and β), and the price of good x. Notice, then, that a change in the price of good y will not change the consumer's consumption of good x. It only changes the consumption of good y.

Since we already have a ratio of x to y, we can just solve for y by plugging in the variables. Remember, we said that:

$$y = \frac{\beta}{\alpha + \beta} \frac{(p_x)x}{p_y}$$

So we can find y once we have x, which is $\frac{I\alpha}{p_x}$. I will leave finding y as an exercise for you.

Substitution The tangency condition tells us that

$$MRS = MRT$$
$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

As we saw from the graph above, the tangency requires the slopes to be equal. In this case, we set the slope of the indifference curve equal to the slope of the budget constraint. Let's do a quick example for a simple utility function subject to a general budget constraint:

$$u(x, y) = xy$$

$$I = p_x x + p_y y$$

Hopefully by now you are comfortable with finding the MRS. We write the MRS as the ratio of marginal utilities:

$$MRS = \frac{MU_x}{MU_y} = \frac{y}{x}$$

After we have the MRS, we equate it to the slope of the budget constraint:

$$\frac{y}{x} = \frac{p_x}{p_y}$$

Now what we want to do is solve a system of two equations with two unknowns, x and y (assume that prices are given to us and therefore we cannot solve them). We must now solve for this equation in terms of either x or y.

$$y = \frac{p_x x}{p_y}$$

After this, we have to plug this into the constraint to maximize the value of either x or y. Let's do that now:

$$I = p_y(\frac{p_x x}{p_y}) + p_x x$$

See that we can cancel p_y out and factor like terms:

$$I = p_x x + p_x x$$
$$I = x(p_x + p_x)$$
$$I = x(2p_x)$$

And now we divide to solve for x.

$$x = \frac{I}{2p_x}$$

And now since income and prices are exogenous, we just plug those in to solve for the optimal quantity. But what about y? We know from previous algebra that

$$y = \frac{p_x x}{p_y}$$

so we can plug in $x = \frac{I}{2p_x}$ so solve for y. I will leave this as an exercise for you. You should get that

$$y = \frac{I}{2p_x}$$

3.2 Example

Suppose we have

1. $u(x, y) = x^3 y^2$
2. $I = 80, p_x = 2, p_y = 4$

You can do a Lagrangian but it would take a long time. Instead, set up MRS = MRT. Let's start by

$$MRS = \frac{3}{2}\frac{y}{x}$$

$$MRT = \frac{2}{4} = \frac{1}{2}$$

$$\frac{3}{2}\frac{y}{x} = \frac{1}{2}$$

$$\frac{2x}{3y} = 2$$

$$x = 3y$$

Plug this into the budget constraint:

$$p_x x + p_y y = I$$

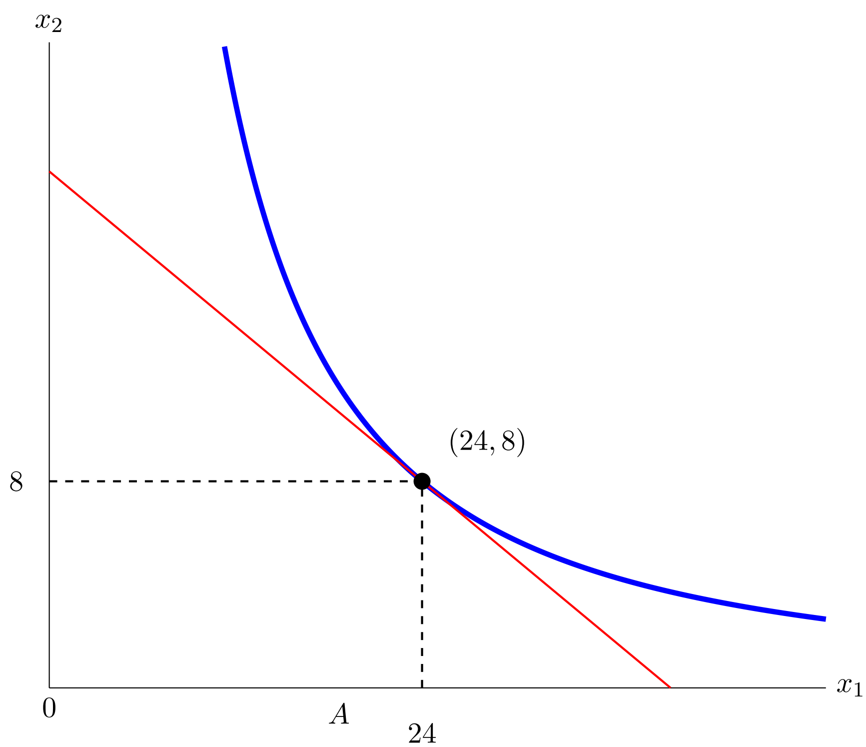
$$2(3y) + 4y = 80$$

$$2(3y) + 4y = 80$$

$$10y = 80$$

$$y = 8$$

C



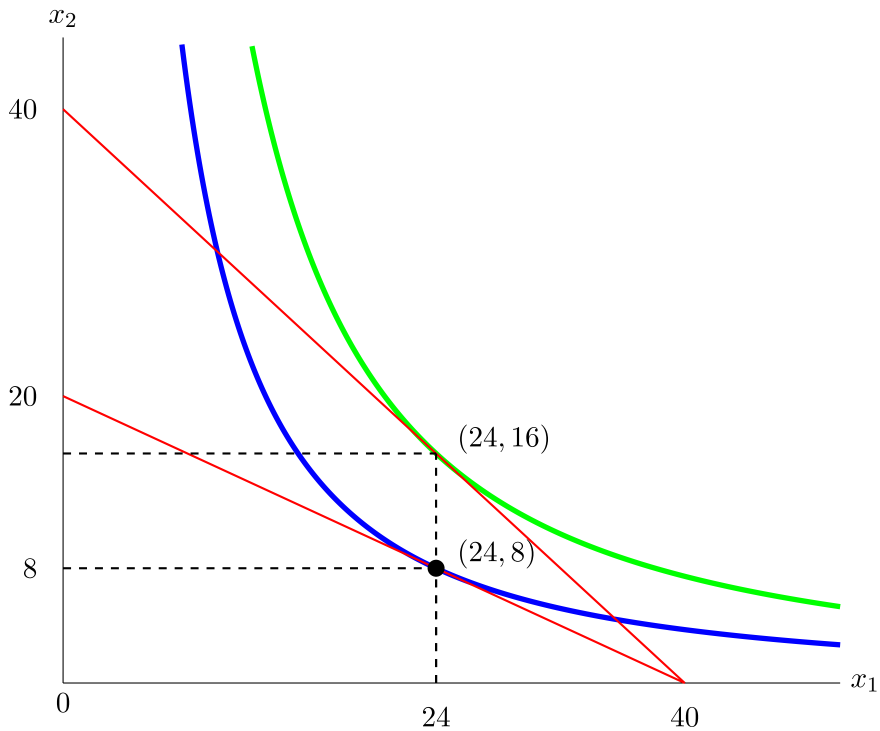
And through the $x = 3y$ relationship, we know that $x = 3(8) = 24$.

But let's take this a step further: what if p_y fell to 2? How would this change the optimal bundle? We know from above that $y = \frac{\beta I}{\alpha + \beta p_y}$, so I will skip the optimization steps for now and write the results:

$$y = \frac{\beta}{\alpha + \beta} \frac{I}{p_y} = \frac{2}{5} \frac{80}{2}$$

$$y = 16$$

And there we have it! When the price of y halves, the quantity demanded doubles. And since neither I nor p_x changed, x is still 24. A graph of this change would look something like this:



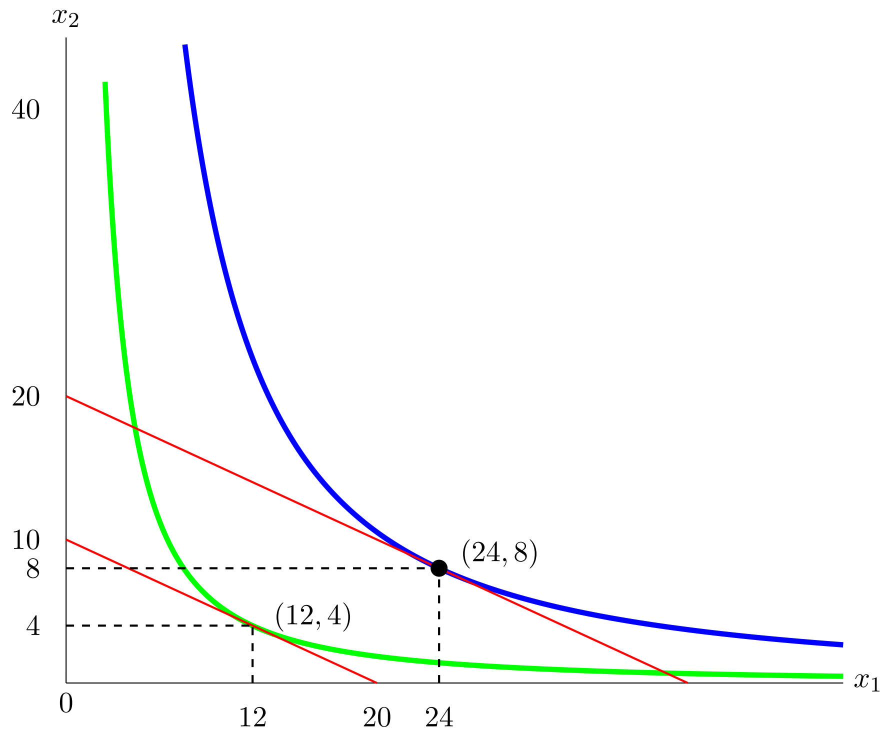
So this just shows that our consumption of good y changes whereas our consumption of good x remains constant. But what if instead of a change in prices, we faced a change in income? Let's say that income decreases from 80 to 40. What happens to x^* and y^*?

Using the shortcut for both, we see that

$$x^* = \frac{\alpha}{\alpha + \beta}\frac{I}{p_x} = \frac{3}{5}x\frac{40}{2} = 12$$

$$y^* = \frac{\beta}{\alpha + \beta}\frac{I}{p_y} = \frac{2}{5}x\frac{40}{4} = 4$$

When our income halves, our consumption of each good also halves. Let's look at this graphically:



This graph shows the contraction of income and the subsequent decrease in consumption of both goods. Consumption of x and y changed proportionally with income.

4 Perfect Substitutes

4.1 General Notes and Results

1. General form: $u(x, y) = \frac{1}{\alpha}x + \frac{1}{\beta}y or u(x, y) = \alpha x + \beta y$
2. Indifference Curves: Linear indifference curves
3. Preferences: A consumer is as happy with α units of good x as they are with β units of y.

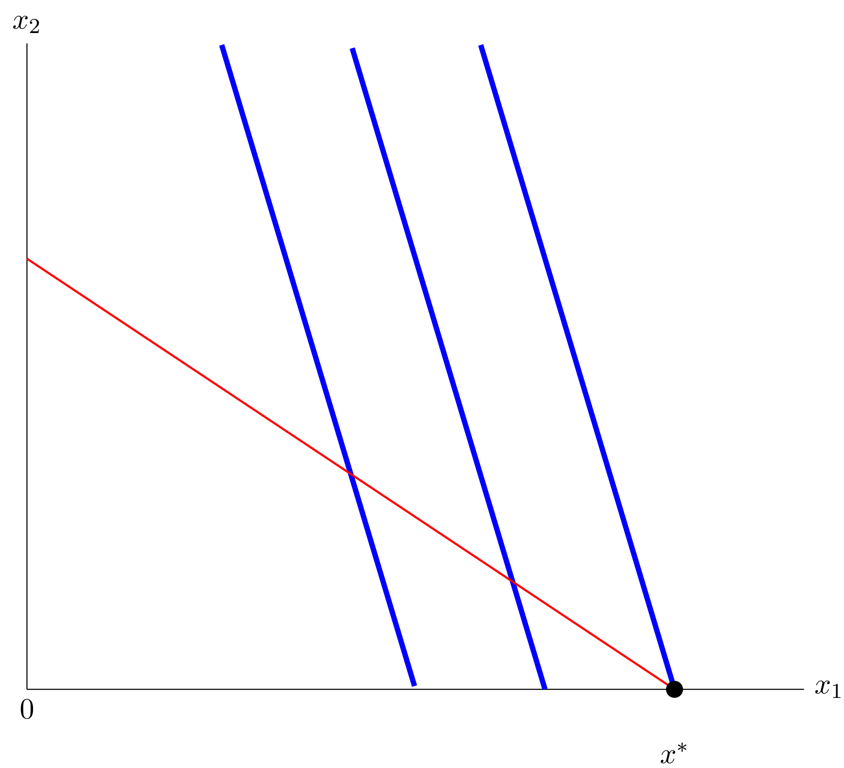
Perfect substitutes are utility functions whose MRS are some constant. Typically, you can identify a perfect substitutes function with ease; however, there are some exceptions. (Try taking the MRS of $\sqrt{2x - y}$ for example.) Using constrained optimization for perfect substitutes is a complete waste of time, so don't even bother. Remember, since the MRS is constant, we are never going to find a point of tangency. Instead, we are going to find a corner solution.

Since both marginal utilities are positive and constant, and prices are constant, we should just find which marginal utility constantly yields higher utiliy per dollar spent. In short, we compare

$$\frac{MU_x}{p_x} vs \frac{MU_y}{p_y}$$

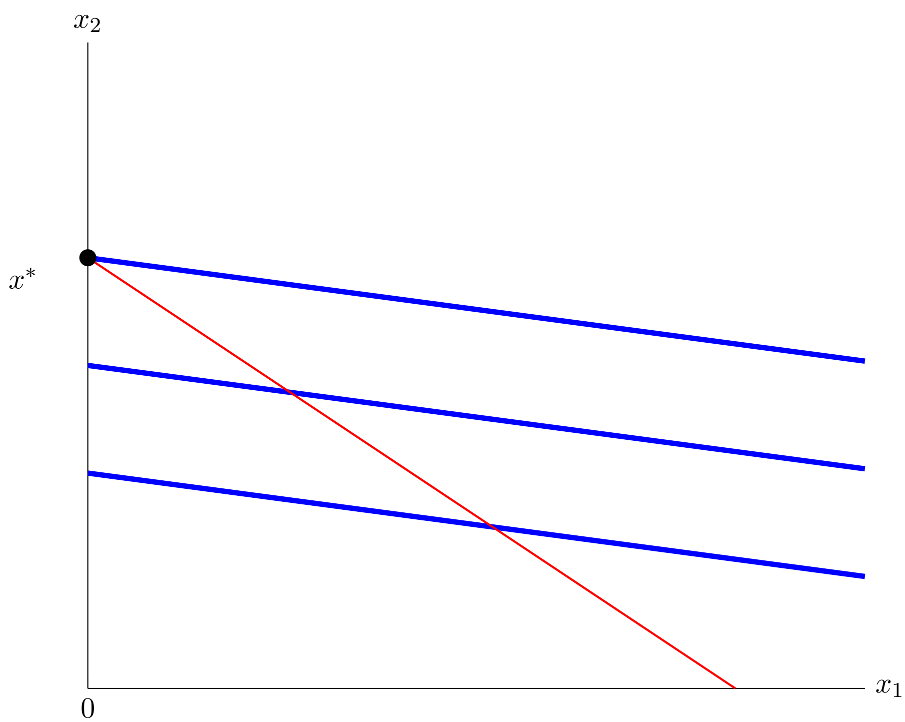
We must compare the marginal utility per dollar in order to find the optimal bundle. The best bundle will be the one that provides higher utility per dollar.

Let's go through some scenarios.If $\frac{MU_x}{p_x} > \frac{MUy}{p_y}$, we will have a graph that looks something like this:



This means that we have a solution on the x-axis, where $x^* = \frac{I}{p_x}$ and $y^* = 0$. Look how additionally the indifference curves are steeper than the budget constraint. This should give you another deeper intuition behind why we have a corner solution: we can reach a higher indifference curve by consuming only x.

If $\frac{MU_x}{p_x} < \frac{MUy}{p_y}$, we will have a graph that looks something like this:



This means that we have a solution on the y-axis, where $y^* = \frac{I}{p_y}$ and $x^* = 0$. Look how additionally the indifference curves are less steep than the budget constraint. We can reach a higher indifference curve by consuming only y.

The only possibility is that $\frac{MU_x}{p_x} = \frac{MUy}{p_y}$, which would mean that the slopes of the budget constraint and the indifference curves are the same. Additionally, the marginal utility per dollar spent is the same for both goods. Any point on the indifference curve is optimal.

4.2 Examples

Say we have $u(x, y) = 10x + 5y$ and $p_x = 5, p_y = 2$. Additionally, $I = 20$. Comparing the marginal utility per dollar is simple at this point, so let's just do it:

$$\frac{10}{5} < \frac{5}{2}$$

So the consumer consumes $y = \frac{20}{2} = 10$ units of y and zero x.

If the price of y increases from 2 to 4, though, we have a different story. Comparing marginal utility per dollar:

$$\frac{10}{5} > \frac{5}{4}$$

so the consumer buys $\frac{20}{5} = 4$ and 0 units of y.

5 Perfect Compliments

5.1 General Notes and Results

1. General form: $u(x, y) = min\frac{1}{\alpha}a\frac{1}{\beta}y$
2. Indifference Curves: L-shaped indifference curves
3. Preferences: A consumer who must consume bundles in specific proportions. Specifically, they consume α units of x with β units of y.

Now, we do not use the tangency condition with perfect complements. Think about it: what is the slope (MRS) of the indifference curve? The vertical line has a slope of $-\infty$, whereas the flat line has a slope of 0. And what is the slope of the kink? It's very strange to think about. This function is not differentiable, so we have to approach it with a different strategy.

This is why we have to think about optimal consumption. Think about this kink in terms of *optimal* bundles. If we consume at each corner, we are not wasting any goods, so to speak. If we deviate from the kink, we have an MRS of either ∞ or 0. We are essentially buying goods that give us no marginal utility. Now, let's think about where the kinks are. They occur when we equate the two inner components:

$$\frac{1}{\alpha}x = \frac{1}{\beta}y$$

$$\beta x = \alpha y$$

$$y = \frac{\beta}{\alpha}x$$

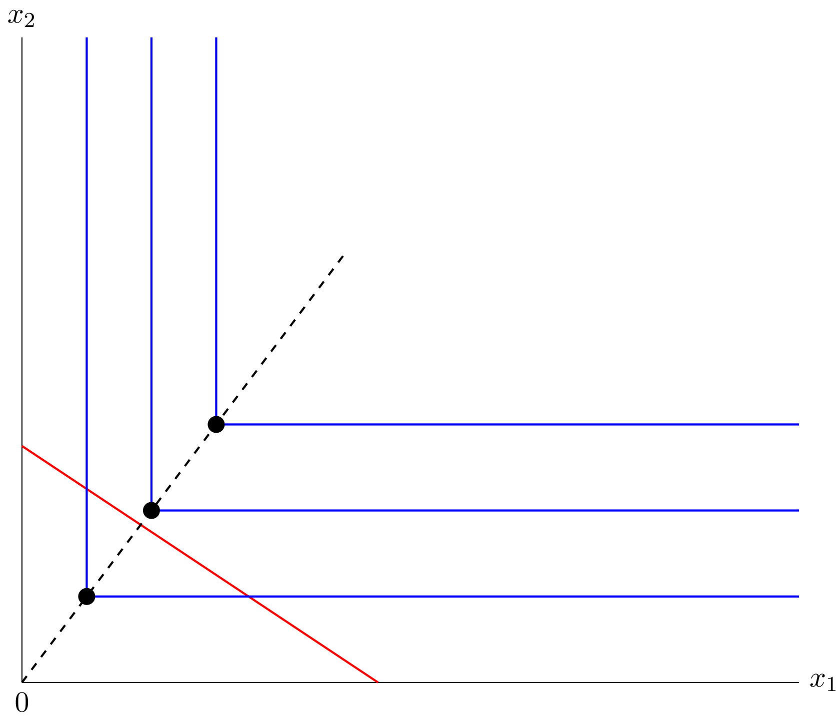
How wonderful is it that we can plot this line? On this line, all the optimal kink points will occur. Okay, so let's just solve for the optimal bundles by plugging that value of y into the budget constraint.

$$p_1 x + p_y y = I$$

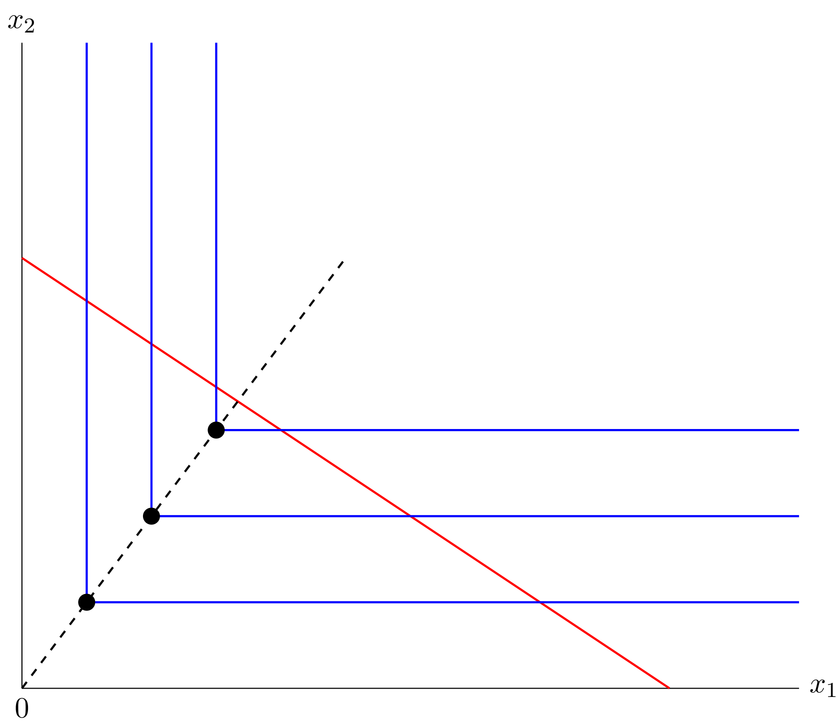
$$p_1 x + p_y(\frac{\beta}{\alpha}x) = I$$

$$x(\frac{\alpha p_1 + \beta p_2}{\alpha}) = I$$

$$x = (\frac{\alpha I}{\alpha p_1 + \beta p_2})$$



Notice in the above case, the budget line is not tangent with any of the kink points. Additionally, the slope of the dashed line is just $y = \frac{\beta}{\alpha}x$. Let's look at a case where the line is tangent.



I should note that technically there is not a "tangency" in the strict sense. The slope is undefined at the kink point. However, it is the optimal point.

5.2 Example

Let's assume that we have $u = min(x, 5y)$ and $p_1 = 3, p_2 = 1, I = 15$. How do we find the optimal bundle? Let's first equate the inner terms:

$$x = 5y$$
$$y = \frac{1}{5}x$$

Plugging x into the budget constraint:

$$x = 5y$$
$$p_x x + p_y y = I$$
$$p_x(5y) + p_y y = I$$
$$y(5p_x + p_y) = I$$
$$y = \frac{I}{(5p_x + p_y)}$$

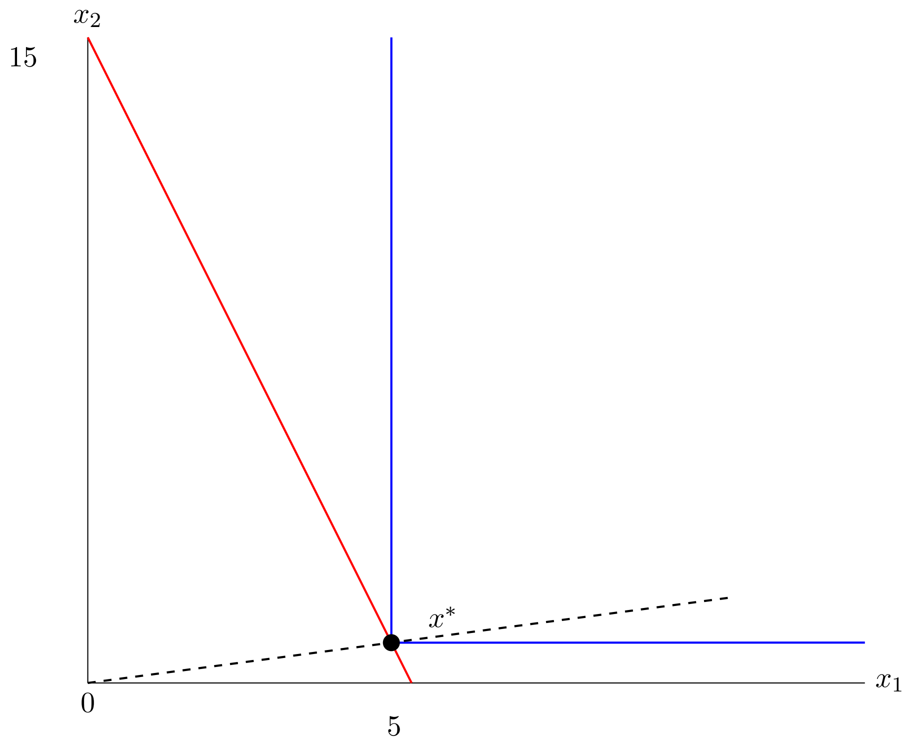
And from the previous $x = 5y$, we know that

$$x = \frac{5I}{(5p_x + p_y)}$$

Now all we have to do is plug in the parameters p_x, p_y, and I.

$$x = \frac{5I}{(5p_x + p_y)} = \frac{5(15)}{5(3) + 1} = \frac{75}{16}$$
$$y = \frac{I}{(5p_x + p_y)} = \frac{15}{5(3) + 1} = \frac{15}{16}$$

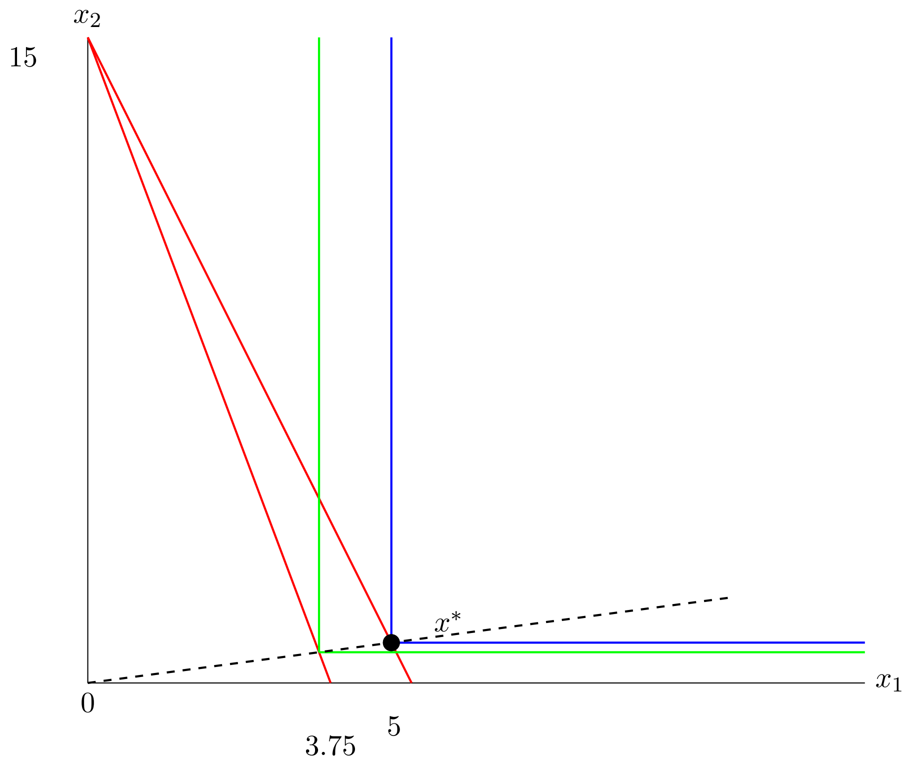
Sometimes your numbers won't be clean, but that's okay! In any event, we can visual these results:



Quickly, let's assume that p_1 increases to 4. Let's solve for the optimal bundles.

$$x = \frac{5I}{(5p_x + p_y)} = \frac{5(15)}{5(4)+1} = \frac{75}{21}$$

$$y = \frac{I}{(5p_x + p_y)} = \frac{15}{5(4)+1} = \frac{15}{21}$$



The above graph is not the best, I recognize, but the intuition is just to see how changes in the budget constraint changes the optimal bundle.

5.3 Quasi-Linear

We write quasi-linear utility functions in the following form:

$$u = \phi(x) + y$$

These are called quasi-linear because of the linear term for good y. We will explore the components of this function in due time. However, let's first return to the fun that is constrained optimization.

$$\mathcal{L} = \phi(x) + y + \lambda(I - p_x x - p_y y)$$

First order conditions:

$$\mathcal{L} = \phi(x) + y + \lambda(I - p_x x - p_y y)$$
$$\frac{d_{\mathcal{L}}}{d_x} = \phi'(x) - \lambda p_x = 0$$
$$\frac{d_{\mathcal{L}}}{d_y} = 1 - p_y \lambda = 0$$

$$\frac{d_{\mathcal{L}}}{d_{\lambda}} = I - p_x - p_y = 0$$

Anyone else see just how beautiful this result is? What is the difference between the first order conditions here versus the Cobb-Douglas function? Here, $\lambda = 1/p_y$. Why is this remarkable? What does λ normally represent? It represents the marginal effect of income on utility. In this case, any change in utility is constant. This means that, as income increases, the agent simply purchases more of good y.

From here, we need to solve for x. That's typically a pain to do in the abstract, general case. Plus, it may help to go through an actual example. So let's say that $u = x^{\frac{1}{2}} + y$. How would we solve this? I will skip a few steps, but I will provide key steps.

$$\lambda = \frac{\frac{1}{2}x^{\frac{-1}{2}}}{p_x}$$

$$\lambda = \frac{1}{p_y}$$

$$\frac{1}{p_y} = \frac{\frac{1}{2}x^{\frac{-1}{2}}}{p_x}$$

$$\frac{p_x}{p_y} = \frac{1}{2}x^{\frac{-1}{2}}$$

$$x = (\frac{2p_x}{p_y})^{-2}$$

Wow! x is only reliant on the prices! This means that the consumer buys a "fixed" number of x (based on the price ratio) and spends the rest of their income on good Y. Any increase in income goes directly to good y. However, a change in prices will lead to an increase in both goods. This is a really important point, so hammer it home.

I suggest solving for y yourself for practice. However, you should get the result:

$$y = \frac{I}{p_y} - \frac{1p_y}{4p_x}$$

A rise in p_y is associated with less y. Is this surprising? Let's solve for I. We can do some clever algebra and get the following:

$$I = \frac{1p_y^2}{4p_x}$$

If income is greater than $\frac{1p_y^2}{4p_x}$, the consumer will buy both goods. However, if income is less than that, the consumer will only purchase good one and we will be left with a corner solution.