Combinatorics: Homework 3

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1. For each positive integer n, prove that there exists a $2^n \times 2^n$ Hadamard matrix.

Proof. Show $m \times m$ Hadamard matrices exist for all $m = 2^n$.
Let $X = \{x \in \mathbb{N} \mid x \leq n\}$. Since $|X| = n$, $|P(X)| = 2^n$. Let C be subsets of X indexed as $\{C_1,\ C_2,\ \cdots, C_{2^n}\}$.
Let matrix $A = (a_{ij})$ defined as follows:

$$a_{ij} = (-1)^{|C_i \cap C_j|}$$

To claim that A is a Hadamard matrix, $\langle r_i, r_j \rangle = 0$ for $i \neq j$ must be true. From definition of A, the following holds:

$$\langle r_i, r_j \rangle = \sum_k (-1)^{|C_i \cap C_k| + |C_j \cap C_k|}$$

Since $C_i \neq C_j$, there exists an element $x \in X$ such that $x \in C_i \setminus C_j$ or $x \in C_j \setminus C_i$. Let $x \in C_i \setminus C_j$. Since x is a arbituary element in X, half of the subsets contain x, the other half does not. Let C_x be all the subsets that contain x, $C_{\bar{x}}$ be all the subsets that does not contain x.
Given $\langle r_i, r_j \rangle = \sum_k (-1)^{|C_i \cap C_k| + |C_j \cap C_k|}$, for half of the sum $C_k \in C_x$, for the other half $C_k \in C_{\bar{x}}$. These two halves have different parity because x is in half of the C_k of the sum and not in C_k for the other half. Since half of the sum has $(-1)^{odd}$, the other half have $(-1)^{even}$, this result in $(-1)+(1)$ which is 0.

Hence shown for any number n, there is a $m \times m$ Hadamard matrix such that $m = 2^n$ □

2. *Proof.* Wants to show:

$$min\{a_1, a_2, \ldots a_n\} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq max\{a_1, a_2, \ldots a_n\}$$

let a_{min} be the minimum element in series $\{a_1, a_2, \ldots a_n\}$, a_{max} be the maximum element in series $\{a_1, a_2, \ldots a_n\}$.

$$\begin{aligned}
a_{min} \cdot n &\leq a_1 + a_2 + \cdots + a_n \leq a_{max} \cdot n \\
\frac{a_{min} \cdot n}{n} &\leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \frac{a_{max} \cdot n}{n} \\
a_{min} &\leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq a_{max} \\
min\{a_1, a_2, \ldots a_n\} &\leq \frac{a_1 + a_2 + \cdots + a_n}{n} \leq max\{a_1, a_2, \ldots a_n\}
\end{aligned}$$

□

3. For each positive integer n, prove that there exists an $n \times n$ matrix A with ± 1 entries such that $|detA| \geq \sqrt{n!}$

Proof. There are 2^{n^2} matrices with ± 1 entries. Let D_n be the mean square average:

$$D_n = \sqrt{\frac{\sum_A (det\ A)^2}{2^{n^2}}}$$

$$\Rightarrow max_A\ det\ A \geq D_n$$

This is true as we have shown in previous proof that average is in between min and max. By squaring both sides of D and subsituting the definition of determinant, the following is true:

$$D_n = \sqrt{\frac{\sum_A (det\ A)^2}{2^{n^2}}} D_n{}^2 = \frac{1}{2^{n^2}} \sum_A \left(\sum_\pi (sign\ \pi) a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)} \right)^2$$

By expanding the squared summation, splitting π into σ and τ, the following holds:

$$D_n{}^2 = \frac{1}{2^{n^2}} \sum_A \left(\sum_\sigma \sum_\tau (sign\ \sigma)(sign\ \tau) a_{1\sigma(1)} a_{1\tau(1)} \dots a_{n\sigma(n)} a_{n\tau(n)} \right)$$

By applying interchage of summation, we have:

$$D_n{}^2 = \frac{1}{2^{n^2}} \sum_{\sigma,\ \tau} (sign\ \sigma)(sign\ \tau) \left(\sum_A a_{1\sigma(1)} a_{1\tau(1)} \dots a_{n\sigma(n)} a_{n\tau(n)} \right)$$

Since $\sum_A X$ is equivalent as the following:

$$\sum_{a_{11}, a_{12}, \dots a_{nn}} X$$

$$\sum_{a_{11}=\pm 1} \sum_{a_{12}=\pm 1} \cdots \sum_{a_{nn}=\pm 1} X$$

Let $X = a_{1\sigma(1)} a_{1\tau(1)} \dots a_{n\sigma(n)} a_{n\tau(n)}$ we have the following:

$$\sum_{a_{11}=\pm 1} \sum_{a_{12}=\pm 1} \cdots \sum_{a_{nn}=\pm 1} a_{1\sigma(1)} a_{1\tau(1)} \dots a_{n\sigma(n)} a_{n\tau(n)}$$

Suppose $\sigma(i) = k \neq \tau(i)$. Every sum contains a_{ik}. This implies that the entire sum has the factor $\sum_{a_{ik}=\pm 1} a_{ik} = 0$. If $\sigma = \tau$ the sum is no longer 0. The long product in this case converges to 1, $(sign\ \sigma)^2$ also converges to 1. Hence we have the following:

$$\sum_{a_{11}=\pm 1} \cdots \sum_{a_{nn}=\pm 1} 1 = 2^{n^2}$$

$$D_n^2 = \frac{1}{2^{n^2}} \sum_\sigma 2^{n^2} = n! D_n = \sqrt{n!}$$

Hence shown there exists an $n \times n$ matrix A with entries ± 1 such that $|det\ A| \geq \sqrt{n!}$ $\square$