Combinatorics Homework 6

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(1) Let (Ω, p) be a finite probability space, and let $X : \Omega \mapsto \mathbb{R}$ be a random variable.

(a) If X is integer-valued, show that

$$\mathbb{E}[X] = \sum_{k \in \mathbb{Z}} k \; p(X = k)$$

Proof. By definiton of linear expectation:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} p(X = \omega) X(\omega)$$

Since $X(\Omega)$ is a set of integer values. Let Ω be the probability space that has the same cardinality as $\mathbb{Z}$ and let X map be a bijection between Ω and $\mathbb{Z}$. In this case the following is true:

$$\sum_{\omega \in \Omega} p(X = \omega) X(\omega) = \sum_{k \in \mathbb{Z}} k \; p(X = k)$$

Since every $X(X = \omega)$ can be mapped to some $k \in \mathbb{Z}$ the above euqlity holds. $\square$

(b) The variance of X is by definition the number $\mathbb{V}[X] = E[(X - \mathbb{E}[X])^2]$. Show that

$$\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Proof.

$$\begin{aligned}
\mathbb{V}[X] &= E[(X - \mathbb{E}[X])^2] \\
\mathbb{V}[X] &= \sum_{x \in X} (x - \mathbb{E}[X])^2 \; p(X = x) \\
\mathbb{V}[X] &= \sum_{x \in X} (x^2 - 2\mathbb{E}[X]x + (\mathbb{E}[X])^2) p(X = x) \\
\mathbb{V}[X] &= \sum_{x \in X} x^2 p(X = x) - 2\mathbb{E}[X] \sum_{x \in X} x p(X = x) + (\mathbb{E}[X])^2 \sum_{x \in X} p(X = x) \\
\mathbb{V}[X] &= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \\
\mathbb{V}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{aligned}$$

$\square$

(c) Show that, for any $k \geq 0$,

$$p(|X - \mathbb{E}[X]| \geq k) \leq \frac{\mathbb{V}[X]}{k^2}$$

Since $g(x) = x^2$ is a monotonic function for $x >= 0$ and $|X - \mathbb{E}[X]|$ satisfies the condition, we have the following:

$$p(|X - \mathbb{E}[X]| \geq k) = p(|X - \mathbb{E}[X]|^2 \geq k^2)$$

$$p(|X - \mathbb{E}[X]|^2 \geq k^2) \leq \frac{\mathbb{E}[|X - \mathbb{E}[X]|^2]}{k^2}$$

$$p(|X - \mathbb{E}[X]| \geq k) \leq \frac{\mathbb{V}[X]}{k^2}$$