Abstract Algebra Homework 8

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Section 10

34.

A proper subgroup can conly be of cardinality p, q and 1. Since p and q are primes, groups of prime order is syclic and groups of order 1 contains only the identity which is cyclic. This means every proper subgroup of a group of order pq must be cyclic.

35.

Define a bijection $\phi(aH) = Ha^{-1}$:

Well defined: Show that $aH = bH \Rightarrow Ha^{-1} = Hb^{-1}$
Let $\phi(aH) = a^{-1}$, $\phi(bH) = b^{-1}$, $aH = bH$. Wants to show $Ha^{-1} = Hb^{-1}$.
Let $a \in aH$. Since $aH = bH$, $a \in bH$, which implies $\exists h \in H \mid a = bh$. If we take the inverse of both sides, we have $a^{-1} = h^{-1}b^{-1}$ Since H is a subgroup $h^{-1} \in H$. This implies $a^{-1} \in Hb^{-1}$. Since $a = ah \mid h = e \in H$, $a^{-1} = h^{-1}a^{-1}$, $a^{-1} \in Ha^{-1}$, this means that $Ha^{-1} = Hb^{-1}$.

Bijective: $\phi^{-1}(Ha^{-1}) = aH$

39.

The left cosets partition G into two cells: $aH, eH = H$. Since cells form a parition of G, $aH = G \setminus H$. The right cosets parition G also into two cells $Ha, eH = H$. These cells also form a partiton of G which means $Ha = G \setminus H$. This means $aH = Ha$.

41.

Identity: identity of $(R,\ +)$ is 0. The left coset of $(\mathbb{Z},\ +)$ containing identity holds this property because it is simply $\mathbb{Z}$ and only $0 \in \mathbb{Z} \mid 0 \leq 0 < 1$.
Consider $r \in \mathbb{R}$, $z_i \in \mathbb{Z} = \{\cdots z_{-1}, z_0, z_1 \cdots\}$ in increasing order, let $r + z_i$ satisfy the condition $0 \leq r + z_i < 1$. Consider $r + z_{i\pm1}$, since $z_{i\pm1} - z_1 = \pm1$. This means $r + z_{i\pm1}$ will add or subtract 1 from $r + z_i$ which position them out of the interval. Since $\mathbb{Z}$ is ordered as shown above and it monotolically increase, this means only z_i is in the interval.

43.

a.

Reflexive:

$$a = a$$
$$eae = eae$$
$$a \sim a$$

Symmetric:

$$\begin{aligned} a &\sim b \\ a &= hbk \\ hbk &= b \\ h^{-1}hbkk^{-1} &= h^{-1}bk^{-1} \\ b &= h^{-1}bk^{-1} \end{aligned}$$

Since $h^{-1} \in H \wedge k^{-1} \in K$, $b \sim a$.

Transitive:

$$\begin{aligned} &a \sim b \wedge b \sim c \\ &a = hbkwedgeb = h'ck' \\ &a = hh'ck'k \end{aligned}$$

Since H and K are sub groups, $hh' \in H \wedge kk' \in K$ therefore $a \sim c$.

b. The equivalence class contains HaK. This set contains all right coset of H on a, and left coset of K on a.

Section 11

1.

element: order

$$\begin{aligned} (0,0) &: \ 1 \\ (1,0) &: \ 2 \\ (0,1) &: \ 4 \\ (1,1) &: \ 6 \\ (0,2) &: \ 2 \\ (1,2) &: \ 4 \\ (0,3) &: \ 4 \\ (1,3) &: \ 6 \end{aligned}$$

The group is cyclic because the highest order is 6.

2.

element: order

$$
\begin{aligned}
(0,0) &: \ 1 \\
(0,1) &: \ 4 \\
(0,2) &: \ 2 \\
(0,3) &: \ 4 \\
(1,0) &: \ 3 \\
(1,1) &: \ 12 \\
(1,2) &: \ 6 \\
(1,3) &: \ 12 \\
(2,0) &: \ 3 \\
(2,1) &: \ 12 \\
(2,2) &: \ 6 \\
(2,3) &: \ 12
\end{aligned}
$$

3.

order of $(2,6)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ is $lcm(2,2) = 2$.

4.

order of $(2,3)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{15}$ is $lcm(3,5) = 15$.

7.

order of $(3,6,12,16)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$ is $lcm(4,2,5,3) = 60$.

46.

Proof. Given abelian groups $G_1, G_2, \cdots, G_n$. Let their cartesian product $G_1 \times G_2 \times \cdots \times G_n = (g_1, g_2, \cdots, g_n)$. Consider the direct product $(g_1, g_2, \cdots, g_n)(g_1', g_2', \cdots, g_n') = (g_1 g_1', g_2 g_2', \cdots, g_n g_n')$. Since each group $G_i \mid i \in \mathbb{Z}, 1 \leq i \leq n$ is Abelian, the following is true:

$$
\begin{aligned}
(g_1, g_2, \cdots, g_n)(g_1', g_2', \cdots, g_n') &= (g_1 g_1', g_2 g_2', \cdots, g_n g_n') \\
&= (g_1' g_1, g_2' g_2, \cdots, g_n' g_n) \\
&= (g_1', g_2', \cdots, g_n')(g_1, g_2, \cdots, g_n)
\end{aligned}
$$

Hence shown a direct product of abelian group is abelian. □