Combinatorics Homework 7

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(1) Give an example showing that the Friendship Theorem does not hold for infinite graphs.
Construct the following graph:

(i) Start with a regular pantagon $G_0 = G_5$.
(ii) Add a vertex that is adjacent (connected) to each pair of vertices in G_i to form G_{i+1}.
(iii) Since the graph can be infinite, if this process were to be repeated for infinite steps, each pair of vertex shares a common neighbor because of step (ii).

However, there does not exist a hub vertex since the original pantagon does not get connected to a single point. Hence shown Friendship Theorem does not hold for infinite graphs.

(2) Let G be the graph with vertex set $V = \{1,\ 2\dots\ n\}$ and edge set $\{\{1,\ j\} : 2 \leq j \leq n\}$. Calculate the number of r-step walks on G which:

(a) Begin and end at 1

$$r - step - walk(r, n) = \begin{cases} 0, & \text{if } (r) \text{ is odd} \\ (n-1)^{\frac{r}{2}}, & \text{otherwise} \end{cases}$$

(b) Begin and end at 2

$$r - step - walk(r, n) = \begin{cases} 0, & \text{if } (r) \text{ is odd} \\ (n-1)^{\frac{r-2}{2}}, & \text{otherwise} \end{cases}$$

(c) Begin at 1 and end at 2

$$r - step - walk(r, n) = \begin{cases} 0, & \text{if } (r) \text{ is even} \\ (n-1)^{\frac{r-1}{2}}, & \text{otherwise} \end{cases}$$

(d) Begin at 2 and end at 3

$$r - step - walk(r, n) = \begin{cases} 0, & \text{if } (r) \text{ is odd} \\ (n-1)^{\frac{r-2}{2}}, & \text{otherwise} \end{cases}$$

(3) Prove that any group of six people contains either three mutual friends, or three mutual strangers. Does the same statement hold for groups of five? Explain.

Compute $R(3,3)$: Given a complete graph $G = (V, E)$ on 6 vertices. Let 3 edges be blue and 3 be red. Given $v \in V$, since G is complete, there are 5 edges with v: $(v, v_1),\ (v, v_2),\ (v, v_3),\ (v, v_4),\ (v, v_5)$. Since half of the edges are colored differently from the other half, at least 3 edges in the aforementioned set are of same color. Lets pick $(v, v_1),\ (v, v_2),\ (v, v_3)$ are blue (if not possible, just switch color). This means if $(v_1, v_2),\ (v_2, v_3),\ (v_1, v_3)$ are blue, there exist a blue triangle, hence we have three mutual firends. If none of the $(v_1, v_2),\ (v_2, v_3),\ (v_1, v_3)$ edges are blue, we have an all red triangle and no blue triangle which means we have a group of three strangers.

Group of 5: No it does not. With group of five, draw a complete graph G_5 and color all outside edges red and inside edges blue. In this case we do not have any triangles of same color, therefore we can not state that there exist either a group of 3 strangers or a group of 3 friends.