Homework 1

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Section 0

16.

a. $\mathcal{P}(\emptyset) = \Big\{\emptyset\Big\} \qquad |\mathcal{P}(\emptyset)| = 1$

b. $\mathcal{P}(\{a\}) = \Big\{\emptyset,\ a\Big\} \qquad |\mathcal{P}(\{a\})| = 2$

c. $\mathcal{P}(\{a,\ b\}) = \Big\{\emptyset,\ \{a\},\ \{b\},\ \{a,b\}\Big\} \qquad |\mathcal{P}(\{a\})| = 4$

d. $\mathcal{P}(\{a,\ b,\ c\}) = \Big\{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,\ b\},\ \{b,\ c\},\ \{a,\ c\},\ \{a,\ b,\ c\}\Big\}$
$|\mathcal{P}(\{a\})| = 8$

18.

$\forall$ set A and $B = \{1,\ 0\}$, $B^A = \{f \mid f : A \mapsto B\}$, find bijection T such that $T : B^A \mapsto \mathcal{P}(A)$

Let T be defined as $T(f) = \{a \in A \mid f(a) = 1\}$

Proof of bijectivity:

Proof of surjectivity: Proove that $\forall X \in \mathcal{P}(a)$, $\exists\, f$ such that $T(f) = X$.
Since $X \in \mathcal{P}(A) \Rightarrow X \subseteq A$, define f such that $f(a) = \begin{cases} 1 & a \in X \\ 0 & a \notin X \end{cases}$
This implies that $T(f) = X$. T is therefore surjective.

Proof of injectivity: Proove that $\forall\ f_1, f_2 \in B^A,\ T(f_1) = T(f_2) = X \Rightarrow f_1(a) = f_2(a),\ a \in A$
Case 1: $a \in X \Rightarrow f_1(a) = 1 \wedge f_2(a) = 1$
Case 2: $a \notin X \Rightarrow f_1(a) = 0 \wedge f_2(a) = 0$
Since $f_1(a) = f_2(a),\ \forall a \in A,\ f_1 = f_2$, T is injective.

Since T is both surjective and injective, T is also bijective. Since there exists bijection T, such that $T : B^A \mapsto \mathcal{P}(A)$, $|B^A| = |\mathcal{P}(A)|$

30.

$x\mathcal{R}y$ in $\mathbb{R}$ if $x \leq y$ is not a equivalence relation because it is not symmetric. For example $(0 \leq 1) \wedge (1 \nleq 0)$.

31.

Given $x\mathcal{R}y$ in $\mathbb{R}$ if $|x| = |y|$

Proove R is an equivalence relation:

Reflexive: $|x| = |x|$ because
$$\begin{cases} x = x & x > 0 \\ -x = -x & x < 0 \end{cases}$$

Symmetric: $(|x| = |y|) \Rightarrow (|y| = |x|)$ because
$$\begin{cases} x = x & (x > 0) \wedge (y > 0) \\ -x = y & (x < 0) \wedge (y > 0) \\ x = -y & (x > 0) \wedge (y < 0) \\ -x = -y & (x < 0) \wedge (y < 0) \end{cases}$$

Transitive: $(|x| = |y| \wedge |y| = |z|) \Rightarrow (|x| = |z|)$ because
$$\begin{cases} x = y = z & (x > 0) \wedge (y > 0) \wedge (z > 0) \\ -x = y = z & (x < 0) \wedge (y > 0) \wedge (z > 0) \\ x = -y = z & (x > 0) \wedge (y < 0) \wedge (z > 0) \\ x = y = -z & (x > 0) \wedge (y > 0) \wedge (z < 0) \\ -x = -y = z & (x < 0) \wedge (y < 0) \wedge (z > 0) \\ x = -y = -z & (x > 0) \wedge (y < 0) \wedge (z < 0) \\ -x = y = -z & (x < 0) \wedge (y > 0) \wedge (z < 0) \\ -x = -y = -z & (x < 0) \wedge (y < 0) \wedge (z < 0) \end{cases}$$

Since $\mathcal{R}$ is reflexive, symmetric, transitive, $\mathcal{R}$ is a equivalence relation.

Describe patition araising from $\mathcal{R}$: Since $\overline{a}$ and $\overline{-a}$, $a \in \mathbb{R}$, is consisted of $\{a,\ -a\}$, the set $\mathbb{R}$ is therefore partitioned into $\{x, y\},\ x, y \in \mathbb{R} \wedge x = -y$

32.

$x\mathcal{R}y$ in $\mathbb{R}$ if $|x - y| \leq 3$ is not a equivalence relation because it fails transitivity. $0\mathcal{R}1 \wedge 1\mathcal{R}4 \nRightarrow 0\mathcal{R}4$ since $|0 - 1| \leq 3$ and $|1 - 4| \leq 3$, however, $|0 - 4| \nleq 3$.

Section 2

1.

$b * d = e$,
$c * c = b$,

$$
\begin{aligned}
[(a * c) * e] * a &= [c * e] * a \\
&= a * a \\
&= a
\end{aligned}
$$

2.

$$
\begin{aligned}
(a * b) * c &\stackrel{?}{=} a * (b * c) \\
b * c &\stackrel{?}{=} a * a \\
a &= a
\end{aligned}
$$

It is possible that operation * is associative. To assert that * is associative, it must be prooven that such operation holds true for all other triples. Therefore this computation is not sufficient.

3.

$$
\begin{aligned}
(b * d) * c &\stackrel{?}{=} b * (d * c) \\
e * c &\stackrel{?}{=} b * b \\
a &\neq c
\end{aligned}
$$

Operation * is not associative.

4.

$$
\begin{aligned}
b * e &\stackrel{?}{=} e * b \\
c &\neq b
\end{aligned}
$$

Operation * is not commutative.

18.

Operation * on $\mathbb{Z}^+$ as $a * b = a^b$ is a binary operation since $a^b \in \mathbb{Z}^+$ is exactly one element in $\mathbb{Z}^+$.

24.

a. False **b.** True **c.** False **d.**False **e.** False
f. True **g.** True **h.** True **i.** True **j.** False

26.

The following holds if * is an associative and commutative binary operation on a set S:

$$\begin{aligned}(a * b) * (c * d) &\stackrel{?}{=} [(d * c) * a] * b \\ &\stackrel{?}{=} (d * c) * (a * b) \\ &\stackrel{?}{=} (a * b) * (d * c) \\ &= (a * b) * (c * d)\end{aligned}$$

35.

Statement is false.
Let * be +, *' be ·, and S be ℤ. If a, b, c = 1, 2, 3 respctively,

$$\begin{aligned}1 + (2 \cdot 3) &\stackrel{?}{=} (1 + 2) * (1 + 3) \\ 1 + 6 &\stackrel{?}{=} 3 * 4 \\ 7 &\neq 12\end{aligned}$$

36.

Given $a, b \in H$, wants to show $(a, b) \in H$.
By definition of H, $(a * x = x * a) \wedge (b * x = x * b),\ \forall x \in S$.
Since * is associative, the following holds:

$$\begin{aligned}(a * b) * x &= a * (b * x) \\ &= a * (x * b) \\ &= (a * x) * b \\ &= (x * a) * b \\ &= x * (a * b)\end{aligned}$$

By definition of H, $(a * b) \in H.$ Hence H is closed under *.

37.

Given $a, b \in H$, wants to show $(a, b) \in H$.
By definition of H, $(a * a = a) \land (b * b = b)$.
Since * is associative and commutative, the following holds:

$$\begin{aligned}(a * b) * (a * b) &= a * (b * a) * b \\ &= a * (a * b) * b \\ &= (a * a) * (b * b) \\ &= (a * b)\end{aligned}$$

By definition of H, $(a * b) \in H$. Hence H is closed under *.