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**Discussion:** A04
**Homework:** 7

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1. Find all the primes $p$ such that $x+2$ is a factor of $f(x) = x^6 + x^4 + x^3 - x + 1 \in \mathbb{Z}_p[x]$. Suppose $f(x) = (x+2)g(x)$, since $\mathbb{Z}_p$ is a field by theorem, $-2$ is a zero of $f(x)$.

$$\begin{aligned} f(-2) &\overset{p}{\equiv} (-2)^6 + (-2)^4 + (-2)^3 + 2 + 1 \\ &\overset{p}{\equiv} 75 \end{aligned}$$

   Since $75 = 5 \cdot 5 \cdot 3$, $p \in \{5, 3\}$.

2. Find a zero of $x^3 - 2x + 1$ in $\mathbb{Z}_5$ and express it as a product of a degree 1 and a degree 2 polynomial.

| $x$ | $x^3 - 2x + 1$ |
|---|---|
| 0 | 1 |
| 1 | 0 |
| 2 | 5 |
| 3 | 22 |
| 4 | 57 |

   $x^3 - 2x + 1 = (x-1)(x^2 + x - 1)$

3. How many degree 2 and degree 3 polynomials with no zeros in $\mathbb{Z}_2[x]$ are there?

   Since the leading coefficient has to be 1 in order for polynomial to have the right degrees, and the constant has to be 1 since otherwise 0 would be a zero of the polynomial, there is only two options for degree 2 polynomial and 4 options for degree three polynomials. One top of that, $x = 0$ is not a zero for any of these functions since all have constant of 1. In case of 1, it is not zero for any polynomial with odd terms since it would result in odd number which is 1 in $\mathbb{Z}_p$. We have $x^2+x+1$, $x^3+x^2+1$, $x^3+x+1$.

   There are three polynomials with no zeros in $\mathbb{Z}_2[x]$.

4. We are told that $x^4 - 2x^2 - 2$ is irreducible in $\mathbb{Q}[x]$.

   (a) Prove that $\mathbb{Q}[x]/\langle x^4 - 2x^2 - 2\rangle \simeq \{c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \mid c_i \in \mathbb{Q}\}$ where $\alpha = \sqrt{1+\sqrt{3}} \in \mathbb{R}$.

   Let $\phi : \mathbb{Q}[x] \mapsto \{c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \mid c_i \in \mathbb{Q}\}$ by $\phi(f) = \phi(\alpha)$. We know evaluation map is a homomorphism.

$ker\ \phi = \langle x^4 - 2x^2 - 2\rangle$ To show this, first show that $\langle x^4 - 2x^2 - 2\rangle \subseteq ker\ \phi$

$$\begin{aligned}
\phi(x^4 - 2x^2 - 2) &= \left(\sqrt{1+\sqrt{3}}\right)^4 - 2\left(\sqrt{1+\sqrt{3}}\right)^2 - 2 \\
&= (1+\sqrt{3})^2 - 2(1+\sqrt{3}) - 2 \\
&= 1 + 2\sqrt{3} + 3 - 2 - 2\sqrt{3} - 2 \\
&= 0
\end{aligned}$$

Since we know $x^4 - 2x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ and $\mathbb{Q}[x]$ is a PID since $\mathbb{Q}$ is a field, $\langle x^4 - 2x^2 - 2\rangle$ is maximal in $\mathbb{Q}[x]$. Since $\langle x^4 - 2x^2 - 2\rangle \subseteq ker\ \phi$, and $\langle x^4 - 2x^2 - 2\rangle$ is maximal, we know $ker\ \phi = \langle x^4 - 2x^2 - 2\rangle$.
Since $\mathbb{Q}$ is a field, $\mathbb{Q}[x]$ is a euclidean domain, hence $f(x) = (x^4 - 2x^2 - 2)q(x) + r(x),\ \forall f \in \mathbb{Q}[x],\ r(x) = c_0 + c_1x + c_2x^2 + c_3x^3 \mid c_i \in \mathbb{Q}$

$$\begin{aligned}
\phi(f) &= (\alpha^4 - 2\alpha^2 - 2)q(\alpha) + r(\alpha) \\
&= 0 \cdot q(\alpha) + r(\alpha) \\
&= c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3
\end{aligned}$$

Hence shown $Im\ \phi(f) = \{c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \mid c_i \in \mathbb{Q}\}$.
By $1^{st}$ isomorphism theorem,

$$\mathbb{Q}[x] \big/ \langle x^4 - 2x^2 - 2\rangle \simeq \{c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \mid c_i \in \mathbb{Q}\}$$

(b) Write $\alpha^{-1}$ in the form $c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \mid c_i \in \mathbb{Q}$.
Assume $\alpha^{-1} = c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 \mid c_i \in \mathbb{Q}$

$$\begin{aligned}
\alpha(c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3) &= 1 \\
c_0\alpha + c_1\alpha^2 + c_2\alpha^3 + c_3\alpha^4 &= 1 \\
c_0\sqrt{1+\sqrt{3}} + c_1\left(\sqrt{1+\sqrt{3}}\right)^2 + \left(c_2\sqrt{1+\sqrt{3}}\right)^3 + c_3\left(\sqrt{1+\sqrt{3}}\right)^4 &= 1 \\
c_0\sqrt{1+\sqrt{3}} + c_1(1+\sqrt{3}) + \left(c_2\sqrt{1+\sqrt{3}}\right)^3 + c_3(1+2\sqrt{3}+3) &= 1 \\
c_0\sqrt{1+\sqrt{3}} + c_1 + c_1\sqrt{3} + \left(c_2\sqrt{1+\sqrt{3}}\right)^3 + c_3 + 2c_3\sqrt{3} + 3c_3) &= 1 \\
c_0\sqrt{1+\sqrt{3}} + \left(c_2\sqrt{1+\sqrt{3}}\right)^3 + c_1 + \sqrt{3}(2c_3 + c_1) + 4c_3) &= 1
\end{aligned}$$

$$\begin{aligned}
c_0 = 0, c_2 &= 0 \\
c_1 + 4c_3 &= 1 \\
2c_3 + c_1 &= 0 \\
2c_3 = 1 \Rightarrow c_3 &= \frac{1}{2} \\
c_1 &= -1
\end{aligned}$$

$a^{-1} = -\alpha + \frac{1}{2}\alpha^3$