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Discussion: A04
Homework: 9

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1. (a) Prove that $x^3 - x + 1$ is irreducible in $\mathbb{Z}_3[x]$.
   By lemma, since $x^3 - x + 1$ is of degree 3, having no zero in $\mathbb{Z}_3$, which is a field, means irreducible in $\mathbb{Z}_4$:

x	$x^3 - x + 1$
0	1
1	1
2	$7 \overset{3}{\equiv} 1$

   (b) Prove that ${\mathbb{Z}_3[x]}/{\langle x^3 - x + 1 \rangle}$ is a field.
   By part a we know that $x^3 - x + 1$ is irreducible, since $\mathbb{Z}_3$ is a field $\mathbb{Z}_3[x]$ is a PID, $\langle x^3 - x + 1 \rangle$ is therefore maximal. This means that ${\mathbb{Z}_3[x]}/{\langle x^3 - x + 1 \rangle}$ is a field.

   (c) Prove that there is a field F that has 27 elements, and it has a zero α of $x^3 - x + 1$.
   By part b, we know that ${\mathbb{Z}_3[x]}/{\langle x^3 - x + 1 \rangle}$ is a field. By lemma, it is of size $3^3 = 27$. $0 + \langle x^3 - x + 1 \rangle$ is a zero of $x^3 - x + 1$.

2. Suppose $f(x) = x^5 - 6x^4 + 30x + 12$.

   (a) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

$$\begin{aligned} p &= 3 \\ 3 &\nmid 1 \\ 3 &\mid -6,\ 30,\ 12 \\ 3^2 = 9 &\nmid 12 \end{aligned}$$

   By Eisenstein's irreducibility criterion, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

   (b) Suppose $\alpha \in \mathbb{C}$ is a zero of f. Prove that $\{a_0 + a_1\alpha + \cdots + a_4\alpha^4 \mid a_0, \ldots, a_4 \in \mathbb{Q}\}$ is a field.
   Since $f(x)$ is irreducible in $\mathbb{Q}[x]$, let $i : \mathbb{Q} \hookrightarrow E$, $E \subseteq \mathbb{C}$ by $i(a) = a$, i is an injective ring homomorphism. By theorem, $\exists \alpha,\ i(f)(\alpha) = 0,\ E = \{a_0, a_1\alpha, \ldots, a_4\alpha^4 \mid a_0, \ldots, a_4 \in \mathbb{Q}\}$.

   (c) Prove that $1, \alpha, \ldots, \alpha^4$ are linearly independent over $\mathbb{Q}$; That means: if $a_0 + a_1\alpha + \cdots + a_4\alpha^4 = 0$ for some $a_i \in \mathbb{Q}$, then $a_0 = a_1 = \cdots = a_4 = 0$.

3. Suppose p is an odd prime. Prove that $x^{p-1} - x^{p-2} + \cdots + x^2 - x + 1$ is irreducible in $\mathbb{Q}[x]$. (Consider $f(-x)$)

4. Let $\alpha = \sqrt{1 + \sqrt{3}}$

(a) Prove that $x^4 - 2x^2 - 2$ is a minimal polynomial of α over $\mathbb{Q}$.

(b) $\{a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{Q}\}$ is a field.

5. Show that there is a finite field of order 25.