Combinatorics: Homework 2

Huize Shi - A92122910

Friday, January 26, 2018

1. *Proof.* Assume towards contradiction that $\log 23$ is a rational number. This implies that $\log_2 3 = \frac{a}{b}$ where $a, b \in \mathbb{Z}$.

$$\begin{aligned} \log_2 3 &= \frac{a}{b} \\ 2^{\frac{a}{b}} &= 3 \\ (2^{\frac{a}{b}})^b &= 3^b \\ 2^a &= 3^b \end{aligned}$$

This is clearly a contradiction because 2 is not a factor of $3^b,\ b \in \mathbb{Z}$, but 2 is a factor of $2^a,\ a \in \mathbb{Z}$. ☐

2. By Legendre's Theorem $\nu_p(n!) = \sum_{k=1} \left\lfloor \frac{n}{p^k} \right\rfloor$, which gives the exact exponent of p in the prime factorization of n!

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ \nu_p\left(\binom{n}{k}\right) &= \nu_p\left(\frac{n!}{k!(n-k)!}\right) \\ &= \nu_p(n!) - (\nu_p(k!) + \nu_p((n-k)!)) \\ &= \sum_{i=1} \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{i=1} \left\lfloor \frac{k}{p^i} \right\rfloor - \sum_{i=1} \left\lfloor \frac{(n-k)}{p^i} \right\rfloor \\ &= \sum_{i=1} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{(n-k)}{p^i} \right\rfloor \right) \end{aligned}$$

The formula for the multiplicity of p in the binomial coefficient $\binom{n}{k}$ is as follows:

$$\sum_{i=1} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{(n-k)}{p^i} \right\rfloor \right)$$

3.

(a) *Proof.* Given $r = \frac{a}{b},\ r! = 0,\ b > 0$, r is a rational number in normal form such that $\frac{a}{b}$ is a unique, irreducible representation of r. if r is not in normal form, normalization of r is trivial by eleminating shared factors between a and b. a and b are coprimes because any shared factor would have been elimenated during normalization of r.
Let $l,\ i,\ j \in \mathbb{Z},\ l >= 1,\ p \in \mathbb{P}$:

$$\begin{aligned} r = \frac{a}{b} &= \frac{\pm \prod_{i=1} p_i^{k_i}}{\prod_{j=1} p_j^{e_j}} \\ &= p_l^{k_l} \frac{\pm \prod_{i=1, i \neq l} p_i^{k_i}}{\prod_{j=1} p_j^{e_j}} \end{aligned}$$

Since this is of the form $p^k\frac{a}{b}$, and a,b are coprimes, p^k is taken out of the prime factors of a, therefore it is not in the prime factor of a or b which is coprime to a, this proves that any nonzero rational number r can be written uniquely in the form $p^k\frac{a}{b}$, where k is an integer, a,b, are coprime integers coprime to p, and b is positive. □

(b)

i. *Proof.* $x = 0 \Rightarrow |x|_p = 0$ is true by definition.
Assume $|x|_p = 0$, show that $x = 0$:

$$\begin{aligned} x = 0 &\Rightarrow |x|_p = 0 \\ x \neq 0 &\Rightarrow |x|_p = p^{-k} \neq 0 \\ |x|_p = 0 &\Rightarrow x = 0 \end{aligned}$$

Hence shown $x = 0 \Leftrightarrow |x|_p = 0$ □

ii. *Proof.* let $x = p^k\frac{a}{b}$, $y = p^l\frac{m}{n}$. Wants to show $|x|_p|y|_p = |xy|_p$

$$xy = p^k p^l \frac{am}{bn}$$

Since p^k is coprime with a and b, p^l is coprime with m and n. This means p is not a factor of m, n, a, b. This means $p^k q^l$ is coprime with am and bn.

$$\begin{aligned} |xy|_p &= (p^k q^l)^{-1} = p^{-k} q^{-l} \\ |x|_p|y|_p &= p^{-k} p^{-l} \\ |xy|_p &= |x|_p|y|_p \end{aligned}$$

If either x, y is 0, then both sides are 0, the case is trivially proved. For all other cases the above processed proved $|xy|_p = |x|_p|y|_p$. □

iii. *Proof.* If either x, y = 0: $y = 0 \Rightarrow |x|_p = |x|_p$, $x = 0 \Rightarrow |y|_p = |y|_p$. This case is trivially proved. Let $x = p^k\frac{a}{b}$, $y = p^l\frac{m}{n}$, assume $k \geq l \Rightarrow p^{-l} = max\{p^{-k}, p^{-l}\}$:

$$\begin{aligned} |x+y|_p &= \left|p^k\frac{a}{b} + p^l\frac{m}{n}\right|_p \\ &= \left|\frac{p^k an + p^l mb}{nb}\right|_p \\ &= \left|p^l\frac{p^{k-l}an + mb}{nb}\right|_p \\ &= \left|p^l\right|_p \cdot \left|\frac{p^{k-l}an + mb}{nb}\right|_p \\ &\leq p^{-l} \cdot 1 = max\{p^{-k}, p^{-l}\} \end{aligned}$$

Hence shown $|x+y|_p \leq max\{p^{-k}, p^{-l}\}$ □

(c) *Proof.* In part (b), $|x+y|_p \leq max\{p^{-k}, p^{-l}\}$ is shown to be true. $max\{p^{-k}, p^{-l}\} \leq p^{-k} + p^{-l}$ is trivially true since neither p^{-k} or p^{-l} are negative. Hence the following ineuqality is true:

$$|x+y|_p \leq max\{p^{-k}, p^{-l}\} \leq p^{-k} + p^{-l}$$

Ergo $|x+y|_p \leq p^{-k} + p^{-l}$ is true. □

(d) *Proof.* From part 3.b.iii, we know the following holds:
Let $x = p^k\frac{a}{b}$, $y = p^l\frac{m}{n}$,

$$|x+y|_p = \left|p^l\right|_p \cdot \left|\frac{p^{k-l}an + mb}{nb}\right|_p$$

Assume $|x|_p \neq |x|_p$, then $k > l0 \Rightarrow p^{-l} = max\{p^{-k}, p^{-l}\}$. This means that $\left|\frac{p^{k-l}an+mb}{nb}\right|_p$ is 1 because p is coprime to an, mb and nb. This means that $|x+y|_p = p^{-l} \cdot 1 = max\{p^{-k}, p^{-l}\}$.
Assume $|x|_p = |x|_p$, $\left|\frac{p^{k-l}an+mb}{nb}\right|_p \leq 1$ because $p^{k-l} = p^0 = 1$. This term becomes $\left|\frac{an+mb}{nb}\right|_p$ which could contain p^x in the numerator. The p-adic of this term could therefore be less than 1. This means that $p^{-l} = max\{p^{-k}, p^{-l}\}$ no longer holds when $|x|_p = |x|_p$.
It is therefore shown that $|x+y|_p = max|x|_p, |x|_p$ whenever $|x|_p \neq |y|_p$ $\square$