Combinatorics Homework 7

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(1) For each permutation π of $1, \cdots, N$, let $LIS(\pi)$ denote the maximal length of an increasing subsequence in π, and let $LDS(\pi)$ denote the maximal length of a decreasing subsequence in π.

(a) Show that $LIS(\pi)LDS(\pi) \geq N$ for all π.
Computer Erdos-Szekeres number $ES(m, n) = N = (m-1)(n-1) + 1$.

Case 1. LIS *or* $LDS = N$. The condition is trivially satisfied.

Case 2. LIS *or* $LDS \neq N$. The lower bound would be the case when either $LIS(\pi)$ or $LDS(\pi) = 2$ because there must exist a pair of numbers that are out of order otherwise it would be the first case. The other number (the one that's not 2) is as small as possible without violating Erdos-Szekeres theorem. This condiion is reached when $m = n$. Any imbalance would cause either LIS or LDS to be bigger.

$$N = (n-1)(n-1) + 1$$
$$N - 1 = (n-1)^2$$
$$n \geq \frac{N}{2}$$

Since the smallest possible product of $LIS(\pi)LDS(\pi)$ is $\frac{N}{2} * 2$. Hence $LIS(\pi)LDS(\pi) >= N$

(b) Let π_N be a *random* permutation of $1, \cdots, N$. Show that the expected value of $LIS(\pi_N)$ is at least $\sqrt{N}$.

(2) Show that, in any finite gathering of people, there are at least two people who know the same number of people at the gathering (assume that knowing is a mutual relationship).

Proof. First represent the people as vertices V, and "mutually knowing" as edges, E, connecting pairs of people. The group of people can be therefore represented as a graph $G = (V,\ E)$ Wants to show there exists $v_i,\ v_j \in V \mid deg(v_i) = deg(v_j)$

Case 1: G is a connected finite simple graph Suppose $|V| = n$, the vertex set contains vertices $\{v_1,\ v_2, \ldots v_n\}$, the set of degrees for each vertex contains $\{1,\ 2, \ldots n-1\}$. 0 is not in the possible degree set since the graph is connected.
By Pigeonhole Principle, there exist a fiber of the function $f : V \mapsto deg(V)$ that is of size at least 2. This means that there exist at least two people with the same number of "known people."

Case 2: G is not a connected graph If G is a empty graph, then everyone has 0 "known people" which satisifies the condition. If there exist edges in the graph, isolate each connected graph and based on **Case 1**, there exist two people with the same number of "known people." □

(3) Show that, given any five points in a unit square, two of these points are separated by a distance of at most $\frac{1}{\sqrt{2}}$.

Proof. Given a unit square, split the unit square into 4 identicle $\frac{1}{2} \times \frac{1}{2}$ squares by connecting the midpoint of each opposite sides. Given an assignment function f that maps each point to a small square. By pigeonhole principle there exist a fiber of f such that the size of the fiber is at least 2. This means that at least two points must be in the same small square. The maximum distance between this two points would be

$$\begin{aligned}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} &= \sqrt{\frac{1}{4} + \frac{1}{4}} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

□