# Combinatorics: Homework 4

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February 9, 2018

**1**. Let $\mathcal{P}$ be a point configuration in $\mathbb{R}^3$. Let $\mathcal{L}$ denote the corresponding line configuration. Show that either $\mathcal{L}$ consists of a single line, or it contains an ordinary line (Hint: reduce to the two-dimensional case).

Sylvester-Gallai Theorem:

*Proof.* Consider $\mathcal{S} = \{(L, P) \mid L \in \mathcal{L},\ P \in \mathcal{P},\ P \notin L$. This contains all pairs of line and point such that the point P is not on the line L.

$\mathcal{S} = \emptyset$**:** There does not exist a pair of point and line such that the point is not on the line. This means that all the points in $\mathcal{P}$ are colinear. $\mathcal{L} = \{L\}$, such that L contains all points in $\mathcal{P}$.

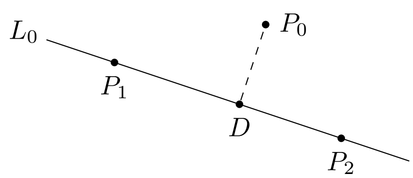
$\mathcal{S}$ **is non-empty:** Construct the following function:

$$d : f \mapsto R_{>0}$$
$$d(L, P) = distance(P, L)$$

Let $(L_0, P_0) = \underset{(L,P)}{argmin}\ distance(P, L)$:

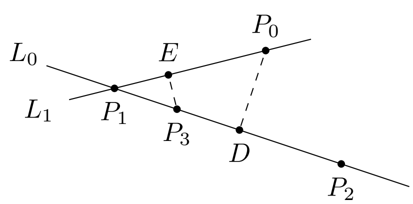
$$d(L_0, P_0) = min\{d(L, P) \mid (L, P) \in \mathcal{S}\}$$

**Claim:** $L_0$ is an ordinary line. Since $L_0 \in \mathcal{L}$ it passes through two points $P_1, P_2 \in \mathcal{P}$. Since we can define a plane with any three points, let the following diagram be a plane in $\mathbb{R}^3$ that contains $P_0$, $P_1$, $P_3$:

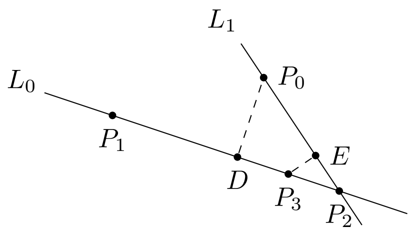



Let $\overline{P_0D}$ is the shortest distance between $P_0$ and $D$. Assume towards a contrary that $L_0$ contains a third point $P_3$. Since this is now compressed to a plane, the line $L_1$ and point $P_3$ can be draw as follows:

**Case 1:** $P_3$ is between D and $P_1$



**Case 2:** $P_3$ is between D and $P_2$

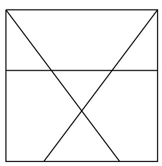



Hence shown $\mathcal{S}$ a non-empty, set. Any minimizer $(L_0, P_0) = \underset{(L,P)}{argmin\ distance}(P, L)$ result in a line $L_0$ that pass through exactly two points within $\mathcal{P}$

$\square$

**2.** A singel line drawn in the plane divides the plane into two regions. Two lines divide the pane into four regions, provided they are not parallel.

**a.** Determine the maximum number of regions into which three lines can divide the plane.



3 lines can divide the plane into a maximum of 7 regions.

**b.** Find a formula for the maximum number of regions into which n lines can divide the plane. Proove the correctness of the formula.

$$R(3) = 7$$
$$R(n) = R(n-1) + n \qquad \forall n > 3$$

This is because to maximize the number of regions generated by a new line, the new line should intersect with all old lines and also form a region by including the new line itself. This means the new line should add $(n-1)+1$ regions on top of previous ones. The closed form of this formula is as follows:

$$R(n) = 1 + \frac{n(n+1)}{2}$$

*Proof.* Show that the algorithm $R(n) = 1 + \frac{n(n+1)}{2}$ is correct by induction:

**Base Case:**

$$\begin{aligned} R(3) &= 1 + \frac{3(3+1)}{2} \\ &= 1 + 6 \\ &= 7 \end{aligned}$$

**Induction Hypothesis:** $R(n) = R(n-1) + n = 1 + \frac{n(n+1)}{2}$ yields the maximum numbers of regions that a plane can be divided into by n lines.

**Induction Step:** Wants to show $R(n+1)$ holds:

$$\begin{aligned} R(n+1) &= 1 + \frac{(n+1)(n+2)}{2} \\ &= 1 + \frac{n^2+3n+2}{2} \\ &= 1 + \frac{n^2+n+2n+2}{2} \\ &= \left(1 + \frac{n^2+n}{2}\right) + \left(\frac{2n}{2} + \frac{2}{2}\right) \\ &= \left(1 + \frac{n(n+1)}{2}\right) + (n+1) \\ &= R(n) + (n+1) \end{aligned}$$

By the recursive definition $R(n) = R(n-1) + n$, $R$ holds for $n+1$, therefore $R(n)$ yields the maximum number of regions that a plane can be divided into by n lines. $\square$