Homework 1

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Problem 1:

Proove $\sqrt{p}$ is irrational for $p \in \mathcal{P}$, where $\mathcal{P}$ is the set containing all prime numbers.

Proof. Assume $\sqrt{p}$ is rational. This implies that $\exists\ a,\ b \in \mathbb{Z},\ \sqrt{p} = \frac{a}{b}$. Define $\Omega(x)$ as the number of prime factors for $x,\ x \in \mathbb{Z}$.

$$\begin{aligned}
\sqrt{p} &= \frac{a}{b} \\
b\sqrt{p} &= a \\
b^2 p &= a^2 \\
\Omega(b^2 p) &\stackrel{?}{=} \Omega(a^2) \\
(\Omega(b) * \Omega(b)) + \Omega(p) &\stackrel{?}{=} \Omega(a) * \Omega(a) \\
(\Omega(b) * \Omega(b)) + 1 &\neq (\Omega(a) * \Omega(a))
\end{aligned}$$

This statement is false because $(\Omega(x) \in \mathbb{Z}) \wedge (\Omega(p) = 1) \wedge (\Omega(x) * \Omega(x))$ is even. The number of prime factors for the left hand side is odd, and the number of prime factor for the right hand is even. This is a contradiction. The assumption is false, $\sqrt{p}$ is irrational. $\square$

Problem 2:

a. Let $n \in \mathbb{Z}$, and $n = \prod_{i=1}^{k} p_i^{e_i}$, its decomposition into primes. Find a formula for the number of divisors of n.

Let s be a factor of n, $n = st,\ t \in \mathbb{Z}$:

$$n = \prod_{i=1}^{k} p_i^{e_i}$$
$$st = \prod_{i=1}^{k} p_i^{e_i}$$
$$Let\ s = \prod_{i=1}^{k} p_i^{f_i}$$
$$t = \prod_{i=1}^{k} p_i^{e_i - f_i}$$

From the above expansion, it is aparent that for all prime factors p_i of n, the factor s can choose f_i such that $e_i - f_i \geq 0$ (because otherwise n is no longer an integer). This means f_i can be $\{0, 1 \dots e_i\}$, which has $e_i + 1$ elements. The number of divisors for n is defined as follows:

$$\tau(n) = \prod_{i=1}^{k} (e_i + 1)$$

b. Find number/numbers with the highest number of divisors in [1, 33].

$$\begin{array}{cccccc}
\tau(1)=1 & \tau(2)=2 & \tau(3)=2 & \tau(4)=3 & \tau(5)=2 & \tau(6)=4 \\
\tau(7)=2 & \tau(8)=4 & \tau(9)=3 & \tau(10)=4 & \tau(11)=2 & \tau(12)=6 \\
\tau(13)=2 & \tau(14)=4 & \tau(15)=4 & \tau(16)=5 & \tau(17)=2 & \tau(18)=6 \\
\tau(19)=2 & \tau(20)=6 & \tau(21)=4 & \tau(22)=4 & \tau(23)=2 & \tau(24)=8 \\
\tau(25)=3 & \tau(26)=4 & \tau(27)=4 & \tau(28)=6 & \tau(29)=2 & \tau(30)=8 \\
\tau(31)=2 & \tau(32)=6 & \tau(33)=4 & & &
\end{array}$$

The integers 24 and 30 have the highest number of divisors (8) in [1, 33].

Problem 3:

Fix a real number $x \geq 1$, and let $\mathbb{N}_x$, denote the set of positive integers with no prime factor exceeding x. Prove the inequality:

$$\sum_{m \in N,\ m \leq x} \frac{1}{m} \leq \sum_{m \in \mathbb{N}_x} \frac{1}{m}$$

Proof. Wants to show $(m \in N,\ m \leq x) \subseteq (m \in \mathbb{N}_x)$ because $\frac{1}{m} > 0$, therefore the sumation of $\frac{1}{m}$ is greater if one $\sum \frac{1}{m}$ has a greater range than another.

Set cardinality: $\{m \mid m \in N,\ m \leq x\}$ contains all integers greater than 0, less than or equals to x. This implies that the set contains all integers greater than 0 less than or equals to x *that is composed of prime factors less than x* because. By being less than or equals to x, there cannot be any factors that is greater than x. It is easy to see that $(m \in \mathbb{N}_x)$ is not constrained to be less than or equals to x. This implies that $(m \in N,\ m \leq x) \subseteq (m \in \mathbb{N}_x)$. For reasons stated above, this implies the following inequality holds:

$$\sum_{m \in N,\ m \leq x} \frac{1}{m} \leq \sum_{m \in \mathbb{N}_x} \frac{1}{m}$$

□

Problem 4:

a. Show that the following holds:

$$\binom{2n}{0} < \binom{2n}{1} < \cdots < \binom{2n}{n} > \cdots \binom{2n}{2n-1} > \binom{2n}{2n}$$

Proof. Let $x \in (\mathbb{Z}^+ \cup \{0\})$, proove $\binom{2n}{n}$ is the largest item in the inequality above:

$$\begin{aligned}\binom{2n}{x} < \binom{2n}{x+1} &\Leftrightarrow \frac{(2n)!}{x!(2n-x)!} < \frac{(2n)!}{(x+1)!(2n-x-1)!} \\ &\Leftrightarrow \frac{1}{2n-x} < \frac{1}{x+1} \\ &\Leftrightarrow x+1 < 2n-x \\ &\Leftrightarrow x < n - \frac{1}{2}\end{aligned}$$

The largest integer value for x for the statement to hold is n-1. The largest item according to the inequality is $\binom{2n}{x+1}$, which is $\binom{2n}{n}$ given x = n-1. Hence prooved the inequality holds. □

b. Deduce that $\binom{2n}{n} \leq \frac{4^n}{2n}$.

$$\begin{aligned}(1+1)^{2n} &= 4^n \\ \sum_{k=0}^{2n} \binom{2n}{k} 1^{n-k} 1^k &= 4^n \\ \frac{\sum_{k=0}^{2n} \binom{2n}{k}}{2n} &= \frac{4^n}{2n}\end{aligned}$$

Since $\frac{\sum_{k=0}^{2n}\binom{2n}{k}}{2n}$ is the average value of the sum of $\binom{2n}{k}$ with $k = 0 \ldots 2n$, from **part a**, we proved that $\binom{2n}{n}$ is the greatest element in the series. It is the case that $\binom{2n}{n}$ equals to the mean of the series if n=0. Hence the following is shown:

$$\binom{2n}{n} \leq \frac{\sum_{k=0}^{2n}\binom{2n}{k}}{2n} \Leftrightarrow \binom{2n}{n} \leq \frac{4^n}{2n}$$