Abstract Algebra: Homework 3

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Section 5:

11.

Let H be a $n \times n$ matrix with determinant -1. This is not a subgroup of G because H is not closed. Let $h_1, h_2 \in H$:

$$det(h_1 \cdot h_2) = det(h_1) \cdot det(h_2) = -1 \cdot -1 = 1$$

Since matrix with determinant 1 is not in H, H is not a subgroup of G.

39.

a. True. Associativity is a part of the definition of Group.

b. False. Every element in a group must have a inverse. This means that the cancellation law must hold.

c. True. Every group is a subset of itself which is a group.

d. False. Only the group itself is the improper subgroup.

e. False. Only the element(s) whoes absolute value is the smallest are the generator(s).

f. False. $< 2 >=< -2 >$

g. False. Set of negative numbers is closed under addition but not closed under multiplicaiton.

h. False. Subgroup has additional constraints such as closed under opteration.

i. True. It's generated by 1 and 3

j. False. Subset of a group may not contain identity.

40.

Given $G = \langle \mathbb{Z}, \cdot \rangle$

$$x^2 = e$$

$$x^2 = 1$$

$$x_1 = 1,\ x_2 = -1$$

41.

Let $g_1,\ g_2 \in \langle G, * \rangle$, $h_1,\ h_2 \in \langle H, * \rangle$, $H \leq G \Rightarrow h \in G$.

$$h \in G \Rightarrow \phi(h) \in G'$$

Since H is a group, $\phi(H)$ is a group. Since $\phi(h) \in H \wedge \phi(h) \in G$, $\phi(H) \subseteq \phi(G)$

43.

Proof. Show that $\{hk \mid h \in H,\ k \in K\}$ is closed under *, contain identity element and inverses for all elements: Let $A = \{hk \mid h \in H,\ k \in K\}$

Closure: Let $a, b \in A \Rightarrow a = hk,\ b = h'k' \mid h,\ h' \in H \wedge k,\ k' \in K$. Wants to show $a * b \in A$

$$\begin{aligned} a * b &\in A \\ hkh'k' &\in A \end{aligned}$$

Since A is Abelian, the following is true:

$$\begin{aligned} hh'kk' &\in A \\ hh' \in H \wedge kk' &\in K \\ a * b &\in A \end{aligned}$$

Left identity: Show that $\exists\ e_A \in A \mid e_A * a = a,\ \forall a \in A$
Let $e_A = e_G \mid e_G * g = g,\ \forall g \in G$. Wants to show $e_A * a = a$, since $H, K \subseteq G \Rightarrow h, k \in G$ the following holds:

$$\begin{aligned} (e_G)(hk) &= (e_G h)k \\ &= hk \end{aligned}$$

Ergo $\exists e_A \in A \mid e_A * a = a, \forall a \in A$

Left inverse: Show that $\exists\ a^{-1} \in A \mid a^{-1} * a = e_A = e_G,\ \forall a \in A$
Since $H,\ K \leq G$, $h^{-1}h = k^{-1}k = e_G = e_A$, let $a^{-1} = h^{-1}k^{-1}$:

$$\begin{aligned} a^{-1}a &= (h^{-1}k^{-1})(hk) \\ &= (h^{-1}h)(k^{-1}k) \\ &= e_A e_A = e_A \end{aligned}$$

Ergo the inverse $a^{-1}\ \exists \forall a \in A, a^{-1}a = e_A$

Since we have now shown that the set $A = \{hk \mid h \in H,\ k \in K\}$ is closed under *, associative, has left identity and has left inverse, it can be concluded that the set $A = \{hk \mid h \in H,\ k \in K\}$ is a subgroup of G. ☐

51.

Proof. $H_a = \{x \in G \mid xa = ax\}$

Closure: Let $x, y \in H_a \mid ax = xa,\ ay = ya$

$$\begin{aligned} a(xy) &= (ax)y \\ &= (xa)y \\ &= x(ay) \\ &= x(ya) \\ &= (xy)a \end{aligned}$$

Since $a(xy) = (xy)a$ holds, $xy \in H_a$, H_a is closed.

Left identity: Let e be the identity in G. Since $a \in G$, The following holds $e * a = a$.

Left inverse: Let $x \in H_a \mid ax = xa$

$$\begin{aligned} ax &= xa \\ a(xx^{-1}) &= xax^{-1} \\ a &= xax^{-1} \\ x^{-1}a &= (x^{-1}x)ax^{-1} \\ x^{-1}a &= ax^{-1} \end{aligned}$$

Ergo, the inverse of a is also in H_a Hence shown that $H_a = \{x \in G \mid xa = ax\}$ is a subgroup of G. ☐

Section 6:

18.

$$\begin{aligned} gcd(30,42) &= 6 \\ \frac{42}{6} &= 7 \end{aligned}$$

There are 7 elements in $\mathbb{Z}_{42}$ manufactured by 30.

27.

$$\begin{aligned} gcd(0,12) &= 0 \\ gcd(1,12) &= 1 \\ gcd(2,12) &= 2 \\ gcd(3,12) &= 3 \\ gcd(4,12) &= 4 \\ gcd(5,12) &= 1 \\ gcd(6,12) &= 6 \\ gcd(7,12) &= 1 \\ gcd(8,12) &= 4 \\ gcd(9,12) &= 3 \\ gcd(10,12) &= 2 \\ gcd(11,12) &= 1 \end{aligned}$$

Subgroups for $\mathbb{Z}_{12}$ are $< 0 >$, $< 1 >$, $< 2 >$, $< 3 >$, $< 4 >$, $< 6 >$

32.

a. True. Cyclic group can be expressed as ¡a¿. Since a^n is a repeatedly operated on itself, cyclic group is trivially Abelian.

b. False. Klein 4-Group is Abelian but not cyclic.

c. False. There is always a number whose absolute distance is closer to 0 than the generator.

d. False The group $2\mathbb{Z}$ is only generated by $< 2 >$ and $< -2 >$, 4 is generated by 2 but it is not a generator of $2\mathbb{Z}$.

e. True. Diagnal symmetric tabel exists for every dimension.

f. False. Klein 4-Group is not cyclic and it is of order 4.

g. False. 9 is coprime to 20, therefore 9 generates $\mathbb{Z}_{20}$, but 9 is not a prime.

h. False. $G \cap G$ does not necessarily contain the identity element.

i. True.

j. True. There are at least two number (1 and n-1) are coprime to n when n ¿ 2.

33.

The Klein 4-Group is Abelian but not cyclic.

44.

Given a subgroup H, of cyclic group G, H is trivially cyclical if $H = \{e\}$. Otherwise let a be a generator of the cyclic group H. Let n be the smallest element $\in \mathbb{Z}^+$ such that $a^n \in H$. For any other member, $a^m \in H$, division algorithm for n divided by m states that $n = qm \mid q \in \mathbb{Z}$. Hence shown $a^m = (a^n)^q$.

46.

Let $a, b \in G$, show