

RINGS OF COOPERATIONS FOR HERMITIAN K-THEORY OVER FINITE FIELDS

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ABSTRACT. We compute the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(kq \otimes kq)$ for the very effective Hermitian K-theory over all finite fields \mathbb{F}_q where $\text{char}(\mathbb{F}_q) \neq 2$. To do this, we use the motivic Adams spectral sequence and show that all differentials are determined by the integral motivic cohomology of \mathbb{F}_q . As an application, we compute the E_1 -page of the kq -resolution.

CONTENTS

1. Introduction	1
2. Background	3
3. Trivial Bockstein action	9
4. Nontrivial Bockstein action	19
5. Application to the kq -resolution	27
References	29

1. INTRODUCTION

The stable motivic homotopy groups of spheres are among the most important invariants in stable motivic homotopy theory. For F a field, the bigraded ring $\pi_{**}^F \mathbb{S}$ has many connections with both topological and arithmetic information regarding F . For example, it is a theorem of Morel [Mor04] that there is an isomorphism

$$\pi_{-n,-n}^F \mathbb{S} \cong K_n^{\text{MW}}(F),$$

where $K_i^{\text{MW}}(F)$ is the n^{th} Milnor-Witt K-theory of F , taken to be zero for $i < 0$. As another example, take F to be algebraically closed, and let $e = \text{char}(F)$ for F of positive characteristic and $e = 1$ for F of characteristic 0. Then there is an isomorphism after inverting the characteristic [Lev14; WØ17]

$$(\pi_{n,0}^F \mathbb{S})[e^{-1}] \cong (\pi_n \mathbb{S})[e^{-1}],$$

where the right hand side denotes the classical stable homotopy groups of spheres.

An extremely useful tool for computing stable motivic homotopy groups is the motivic Adams spectral sequence. For any motivic ring spectrum E , there is an E -based motivic Adams spectral sequence computing $\pi_{**}^E \mathbb{S}$ up to some completion. Perhaps the most well studied is the $H\mathbb{F}_p$ -based motivic Adams spectral sequence, where $H\mathbb{F}_p$ is the motivic ring spectrum representing motivic cohomology with mod- p coefficients. We will refer to this spectral sequence simply as the $\mathbf{mASS}_p^F(\mathbb{S})$, which has signature

$$E_2 = \text{Ext}_{\mathcal{A}_p^\vee}^{s,f,w}(\mathbb{M}_p^F, \mathbb{M}_p^F) \implies \pi_{s,w}^F \mathbb{S}_p.$$

This spectral sequence has been well studied over algebraically closed fields [DI10; HKO11a], the real numbers [BI22], and, following the computation of the dual Steenrod algebra \mathcal{A}_p^\vee in positive characteristic [HKØ17], over finite fields [WØ17].

Another perspective which one can take to gain insight into $\pi_{**}^F \mathbb{S}$ is by organizing the elements of $\pi_{**}^F \mathbb{S}$ into periodic families. This is the heart of the chromatic approach to motivic homotopy theory. Instead of using the $\mathbf{mASS}_p^F(\mathbb{S})$ to compute the motivic stable stems one stem degree at a time, one can use different E-based motivic Adams spectral sequences to localize attention to an infinite periodic family in $\pi_{**}^F \mathbb{S}$. One downside to this philosophy is that often these spectral sequences are less computable at the onset, and some amount of genuine work must be put into computing the E_1 -page. These spectral sequences take the form

$$E_1 = \pi_{s+f,w}^F(E \otimes \bar{E}^{\otimes f}) \implies \pi_{s,w}^F \mathbb{S},$$

where \bar{E} is the cofiber of the unit map $\mathbb{S} \rightarrow E$. It is often easier to compute $\pi_{**}^F(E \otimes E)$, known as the ring of cooperations, and then bootstrap up to the E_1 -page.

While this is common practice in classical stable homotopy theory (see [Mah81; Gon98; Bea+20]), in motivic homotopy theory this idea is rather new. For instance, the study of v_1 -periodicity via these techniques is in its nascent stage (although there is work on the subject via the slice spectral sequence due to [BIK24; KQ24]). Two motivic spectra which are good candidates for accessing v_1 -periodicity, in that they contain some power of v_1 in their homotopy, are the very effective Hermitian K-theory spectrum kq [ARØ20] and the truncated motivic Brown–Peterson spectrum $BPGL\langle 1 \rangle$ [HK01]. In [CQ21], Culver–Quigley study the kq -based Adams spectral sequence, known as the kq -resolution, over $F = \mathbb{C}$ at the prime 2. As an application, they determine the v_1 -periodic elements in $\pi_{**}^{\mathbb{C}} \mathbb{S}$. In [Mor25b], we extend the study of the kq -resolution to the $F = \mathbb{R}$ by computing the ring of cooperations $\pi_{**}^{\mathbb{R}}(kq \otimes kq)$ up to v_1 -torsion. In [Li+25], we are geared towards the $BPGL\langle 1 \rangle$ -based motivic Adams spectral sequence. We compute the ring of cooperations and produce spectrum-level splittings over the fields $F = \mathbb{C}, \mathbb{R}, \mathbb{F}_q$ and at all primes p where $\text{char}(\mathbb{F}_q) \neq p$.

1.1. Main Results. The goal of this paper is to extend the study of the kq -resolution to base fields of positive characteristic. Let \mathbb{F}_q be a finite field where $\text{char}(\mathbb{F}_q) \neq 2$. Our first result is a computation of the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(kq \otimes kq)$ up to v_1 -torsion.

Theorem A (Theorem 3.10, Theorem 4.11). *The $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ has signature*

$$E_2 = \bigoplus_{k \geq 0} \Sigma^{4k, 2k} \text{Ext}_{\mathcal{A}(1)^{\vee}}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}, B_0^{\mathbb{F}_q}(k)) \implies \pi_{s,w}^{\mathbb{F}_q}(kq \otimes kq),$$

where $B_0^{\mathbb{F}_q}(k)$ denotes the k^{th} integral motivic Brown–Gitler comodule. We describe the E_{∞} -page, modulo v_1 -torsion, as a module over $\pi_{**}^{\mathbb{F}_q}(kq)$.

Contrary to the classical, \mathbb{C} -motivic, and the \mathbb{R} -motivic analogues of this question, this spectral sequence does not collapse on the E_2 -page. Since $B_0^{\mathbb{F}_q}(0) \cong \mathbb{M}_2^{\mathbb{F}_q}$, our description of the E_2 -page identifies the $k = 0$ -summand as the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$, which also does not collapse at the E_2 -page. Curiously, we make the following observation.

Theorem B (Theorem 3.16, Theorem 4.13). *The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ are determined by the $\text{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(\mathbb{M}_2^{\mathbb{F}_q}, \mathbb{M}_2^{\mathbb{F}_q})$ -module structure of the E_2 -page and the differentials of the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$.*

The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$ are themselves lifted from the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$ along the natural quotient map $kq \rightarrow H\mathbb{Z}$. The above theorem implies that relative to the algebra $\text{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(\mathbb{M}_2^{\mathbb{F}_q}, \mathbb{M}_2^{\mathbb{F}_q})$, the only interesting part of the ring of cooperations for Hermitian K-theory is determined by the integral motivic cohomology of \mathbb{F}_q . We are also able to use this observation to compute $\pi_{**}^{\mathbb{F}_q}(ksp)$ [Lemma 3.7, Lemma 4.7].

As an application, we describe the n -line of the E_1 -page of the kq -resolution.

Theorem C (Proposition 5.3, Theorem 5.4). *The $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes \overline{kq}^{\otimes n})$ has signature*

$$E_2 = \bigoplus_{K \in \mathcal{K}_n} \Sigma^{4|K|, 2|K|} \text{Ext}_{\mathcal{A}(1)^{\vee}}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}, B_0^{\mathbb{F}_q}(K)) \implies \pi_{s,f}^{\mathbb{F}_q}(kq \otimes \overline{kq}^{\otimes n}),$$

where $\mathcal{K}_n = \{K = (k_1, \dots, k_n) : k_j \geq 1 \text{ for all } j\}$, $|K| = \sum_{j=1}^n k_j$, and $B_0^{\mathbb{F}_q}(K) = \bigotimes_{j=1}^n B_0^{\mathbb{F}_q}(k_j)$. For $n = 0$, the differentials are given by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$. For $n > 0$, the differentials are determined by the underlying module structure over the 0-line.

We view the kq -resolution as the key tool to understanding v_1 -periodicity in $\pi_{**}^{\mathbb{F}_q}(\mathbb{S})$. In conjunction with the \mathbb{R} -motivic ring of cooperations computed in [Mor25b], we will use [Theorem C](#) in forthcoming work where we analyze the kq -resolution over \mathbb{R} and \mathbb{F}_q [Mor25a].

1.2. Organization. In [Section 2](#), we introduce relevant background. We recall the stable motivic homotopy category $\mathrm{SH}(\mathbb{F}_q)$, the motivic Adams spectral sequence, the dual Steenrod algebra and Brown–Gitler comodules, and the algebraic Atiyah–Hirzebruch spectral sequence. In [Section 3](#), we compute the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ in the case where $q \equiv 1(4)$. In [Section 4](#), we compute the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ in the case where $q \equiv 3(4)$. Many of the arguments in [Section 3](#) carry over directly to [Section 4](#), but the particular details of the algebra involved are quite different. As such, we separate these cases into two sections. In [Section 5](#), we apply our findings to deduce results about the kq -resolution.

1.3. Notation and Conventions.

- We let \mathbb{F}_q denote the field with q elements where $\mathrm{char}(\mathbb{F}_q) \neq 2$.
- We work in the stable, presentably symmetric monoidal ∞ -category $\mathrm{SH}(\mathbb{F}_q)$. We let $\mathbb{S} := \Sigma_{\mathbb{P}^1}^\infty \mathrm{Spec}(\mathbb{F}_q)_+$ denote the monoidal unit and use $\otimes := \otimes_{\mathbb{S}}$ for the monoidal product.
- We let $H_{**}(-)$ to denote mod-2 motivic homology, and let $\mathbb{M}_2^{\mathbb{F}_q}$ denote the mod-2 motivic homology of a point.
- For B any Hopf algebra over $\mathbb{M}_2^{\mathbb{F}_q}$ and M any B -comodule, we use the common abbreviation

$$\mathrm{Ext}_B^{s,f,w}(M) := \mathrm{Ext}_B^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}, M)$$

to denote the cohomology of B with coefficients in M . Note that this Ext is taken in the category $\mathrm{Comod}(B)$. Typically, B will be a Hopf algebra related to the dual Steenrod algebra \mathcal{A}^\vee .

- Our grading convention for Ext is (s, f, w) , where s is the stem (or total) degree, f is the Adams filtration, and w is the motivic weight. This implies that Adams differentials take the form

$$d_r : E_r^{s,f,w} \rightarrow E_r^{s-1,f+r,w}.$$

We also refer to the the coweight, which is $cw = s - w$. Note that Adams differentials decrease coweight by 1. We display all charts in (s, f) grading with weight suppressed.

- Throughout, all spectra are implicitly 2-complete.

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2. BACKGROUND

Throughout, let \mathbb{F}_q be a finite field with $\mathrm{char}(\mathbb{F}_q) \neq 2$. In this section, we review motivic cohomology and Hermitian K-theory, the dual Steenrod algebra, and Brown–Gitler comodules in the context of $\mathrm{SH}(\mathbb{F}_q)$. Then, we describe the HF_2 and kq -based motivic Adams spectral sequences and outline our plan to computing the ring of cooperations.

2.1. Motivic cohomology and Hermitian K-theory. Spitzweck defined the *integral motivic cohomology spectrum* $H\mathbb{Z} \in \text{SH}(\mathbb{Z})$ representing motivic cohomology with integral coefficients [Spi18]. The unique map $f : \mathbb{Z} \rightarrow \mathbb{F}_q$ induces a pullback $f^* : \text{SH}(\mathbb{F}_q) \rightarrow \text{SH}(\mathbb{Z})$; we abusively denote $H\mathbb{Z} := f^*(H\mathbb{Z}) \in \text{SH}(\mathbb{F}_q)$ the integral motivic cohomology spectrum over \mathbb{F}_q . This represents motivic cohomology in the sense that if $X \in \text{Sm}_{\mathbb{F}_q}$, then

$$[X, \Sigma^{s,w} H\mathbb{Z}] = H^{s,w}(X; \mathbb{Z}),$$

where we let X also denote the motivic suspension spectrum of X . Of particular interest to us is when $X = \text{Spec}(\mathbb{F}_q)$. As we will be computing with the 2-primary motivic Adams spectral sequence, we are only concerned with the 2-completion $H^{p,q}(\mathbb{F}_q; \mathbb{Z})_2^\wedge$. These groups were calculated by Soulé [Sou79]:

$$(2.1) \quad H^{-s,-w}(\mathbb{F}_q; \mathbb{Z})_2^\wedge = \begin{cases} \mathbb{Z}_2 & s = w = 0 \\ \mathbb{Z}/(q^{-w} - 1)_2 & s = -1, w \leq 1 \\ 0 & \text{else.} \end{cases}$$

Notice that $H^{-s,-w}(\mathbb{F}_q; \mathbb{Z}) = \pi_{s,w}^{\mathbb{F}_q}(H\mathbb{Z})$, so this also calculates the homotopy groups of the integral motivic cohomology spectrum.

There is also a *mod-2 motivic cohomology spectrum* $H\mathbb{F}_2$. By [Voe03], we have that

$$H^{-s,-w}(\mathbb{F}_q; \mathbb{Z}/2) = \pi_{s,w}^{\mathbb{F}_q}(H\mathbb{F}_2) := M_2^{\mathbb{F}_q} = (K_*^M(\mathbb{F}_q)/2)[\tau],$$

where $K_*^M(\mathbb{F}_q)$ denotes the *Milnor K-theory* of \mathbb{F}_q , and where $|\tau| = (0, -1)$ and $|K_n^M(\mathbb{F}_q)| = (-n, -n)$. Recall that Milnor K-theory $K_*^M(\mathbb{F}_q)$ is defined as the graded tensor algebra over \mathbb{Z} generated by the symbols $[a]$ in degree one for $a \in \mathbb{F}_q^\times$, subject to the relations $[a] \otimes [1-a] = 0$ and $[a] + [b] = [ab]$. It is a routine exercise to show that $K_n^M(\mathbb{F}_q)$ vanishes in degrees $n \geq 2$. In particular, since every element of \mathbb{F}_q^\times is either a square or a non square, this implies that

$$(K_0^M(\mathbb{F}_q)/2) = (K_1^M(\mathbb{F}_q)/2) = \mathbb{Z}/2.$$

To differentiate between the cases of $q \equiv 1(4)$ and $q \equiv 3(4)$, we let u be any generator of $K_1^M(\mathbb{F}_q)$ for $q \equiv 1(4)$, and let $\rho = [-1]$ be a generator of $K_1^M(\mathbb{F}_q)$ for $q \equiv 3(4)$. Notice that if $q \equiv 1(4)$, then -1 has a square root in \mathbb{F}_q , hence $[-1] = 0 \in K_1^M(\mathbb{F}_q)$, so that u is not represented by -1 . Then we have identifications:

$$M_2^{\mathbb{F}_q} = \begin{cases} \mathbb{F}_2[u, \tau]/(u^2) & q \equiv 1(4) \\ \mathbb{F}_2[\rho, \tau]/(\rho^2) & q \equiv 3(4), \end{cases}$$

where $|\tau| = (0, -1)$ and $|u| = |\rho| = (-1, -1)$.

There is a *Hermitian K-theory* spectrum KQ [Hor05; CHN25] which is the main character of our work. The homotopy groups $\pi_{*,*}^{\mathbb{F}_q}(KQ)$ intertwine the classical stable homotopy of the orthogonal, unitary, and symplectic K-groups of \mathbb{F}_q . We will only be interested in the orthogonal K-groups $KO_*(\mathbb{F}_q) := \pi_*(BO(\mathbb{F}_q)^+)$ and the symplectic K-groups $KSp_*(\mathbb{F}_q) := \pi_*(BSp(\mathbb{F}_q)^+)$; for more details, see [Hor05]. These are realized by Hermitian K-theory in the following way:

$$\pi_{s,w}^{\mathbb{F}_q}(KQ) = \begin{cases} KO_{s-2w}(\mathbb{F}_q) & w \equiv 0(4); \\ KSp_{s-2w}(\mathbb{F}_q) & w \equiv 2(4). \end{cases}$$

The Hermitian K-theory spectrum is an \mathbb{E}_∞ motivic ring spectrum [CHN25]. We let kq denote the very effective cover of KQ [ARØ20]. The ring structure on KQ lifts to kq , and the unit map $\mathbb{S} \rightarrow kq$ induces Morel's isomorphism [RSØ19]

$$\pi_{-n,-n}^{\mathbb{F}_q}(\mathbb{S}) \cong K_n^{\text{MW}}(\mathbb{F}_q).$$

In particular, $\pi_{0,0}^{\mathbb{F}_q}(kq) \cong \text{GW}(\mathbb{F}_q)$, the Grothendieck–Witt ring. We will not need all of the details of the very effective slice filtration (see [Ban25, Section 1] for a wonderful recollection). For the purposes of our work, one should treat the very effective cover as an analogue of the connective cover. Indeed, for $s \geq 0$ there is an isomorphism $\pi_{s,w}^{\mathbb{F}_q}(KQ) \cong \pi_{s,w}^{\mathbb{F}_q}(kq)$. However, we will see that the homotopy

groups of kq do not vanish for $s < 0$ (an interpretation of these homotopy groups in negative stem degrees is given in Remark 3.6 and Remark 4.5).

There are cofiber sequences of spectra:

$$\Sigma^{1,1}kq \xrightarrow{\eta} kq \rightarrow kgl,$$

$$\Sigma^{2,1}kgl \xrightarrow{\beta} kgl \rightarrow H\mathbb{Z},$$

where kgl denotes the effective algebraic K-theory spectrum, η is the motivic Hopf map, and β is the Bott periodicity class [ARØ20]. Combining the last morphism of each cofiber sequence gives a map

$$(2.2) \quad kq \rightarrow H\mathbb{Z}$$

which will feature heavily in our arguments.

Related to Hermitian K-theory is the *very effective symplectic K-theory* spectrum ksp , which is defined as the very effective cover of $\Sigma^{4,2}KQ$. The homotopy groups of ksp are a shifted version of the homotopy groups of kq , which we investigate in more detail in later sections. We note that there is a cofiber sequence

$$(2.3) \quad \Sigma^{4,2}kq \rightarrow ksp \rightarrow H\mathbb{Z}$$

coming from the very effective slice filtration on $\Sigma^{4,2}KQ$. For more details on ksp and the very effective slice tower for KQ , see [Mor25b, Section 3].

The Hermitian K-theory of finite fields was calculated by Friedlander [Fri76, Theorem 1.7]. We display the relevant results in Table 1.

n modulo 8	$\text{KO}_n(\mathbb{F}_q)$	$\text{KSp}_n(\mathbb{F}_q)$
0	$\mathbb{Z}/2$	0
1	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
2	$\mathbb{Z}/2$	0
3	$\mathbb{Z}/(q^{(n+1)/2} - 1)$	$\mathbb{Z}/(q^{(n+1)/2} - 1)$
4	0	$\mathbb{Z}/2$
5	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
6	0	$\mathbb{Z}/2$
7	$\mathbb{Z}/(q^{(n+1)/2} - 1)$	$\mathbb{Z}/(q^{(n+1)/2} - 1)$

TABLE 1. Friedlander's calculation of $\text{KO}_n(\mathbb{F}_q)$ and $\text{KSp}_n(\mathbb{F}_q)$, where $\text{char}(\mathbb{F}_q) \neq 2$ and $n > 0$.

2.2. The dual Steenrod algebra and Brown–Gitler comodules. The *dual Steenrod algebra* $\mathcal{A}^\vee := \pi_{**}^{\mathbb{F}_q}(H\mathbb{F}_2 \otimes H\mathbb{F}_2)$ was calculated in positive characteristic by Hoyois–Kelley–Østvær [HKØ17, Proposition 4.3]:

$$\mathcal{A}^\vee = M_2^{\mathbb{F}_q}[\bar{\tau}_0, \bar{\tau}_1, \dots, \bar{\xi}_1, \bar{\xi}_2, \dots]/T,$$

where $|\bar{\tau}_i| = (2^{i+1} - 1, 2^i - 1)$ and $|\bar{\xi}_i| = (2^{i+1} - 2, 2^i - 1)$. The ideal T of relations is dependent on the base field \mathbb{F}_q . In the cases we are concerned with, we have that

$$T = \begin{cases} (\bar{\tau}_i^2 = \tau \bar{\xi}_{i+1}) & q \equiv 1(4) \\ (\bar{\tau}_i^2 = \tau \bar{\xi}_{i+1} + \rho \bar{\tau}_{i+1} + \rho \bar{\tau}_0 \bar{\xi}_{i+1}) & q \equiv 3(4). \end{cases}$$

Note that the pair $(M_2^{\mathbb{F}_q}, \mathcal{A}^\vee)$ is a Hopf algebroid rather than a Hopf algebra [Rav86, Appendix A].

For $n \geq 0$, let $\mathcal{A}(n)^\vee$ be the quotient algebra

$$\mathcal{A}(n)^\vee \cong \mathcal{A}^\vee / (\bar{\xi}_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_n^2, \bar{\xi}_{n+1}, \bar{\xi}_{n+2}, \dots, \bar{\tau}_{n+1}, \bar{\tau}_{n+2}, \dots).$$

There is a sub-Hopf algebra of the motivic Steenrod algebra

$$\mathcal{A}(n) = \langle \text{Sq}^1, \text{Sq}^2, \dots, \text{Sq}^{2^n} \rangle;$$

let $(\mathcal{A} // \mathcal{A}(n))^\vee$ be the subalgebra of \mathcal{A}^\vee given by

$$(\mathcal{A} // \mathcal{A}(n))^\vee = \mathbb{M}_2^{\mathbb{F}_q}[\bar{\xi}_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \dots, \bar{\tau}_{n+1}, \dots] / (\bar{\tau}_i^2 = \rho \bar{\tau}_{i+1} + \rho \bar{\tau}_0 \bar{\xi}_{i+1} + \tau \bar{\xi}_{i+1}).$$

These subalgebras naturally arise via motivic homology: one can show that there are \mathcal{A}^\vee -comodule algebra isomorphisms

$$H_{**}(H\mathbb{Z}) \cong (\mathcal{A} // \mathcal{A}(0))^\vee, \quad H_{**}(kq) \cong (\mathcal{A} // \mathcal{A}(1))^\vee$$

by using the long exact sequence in homology associated to the cofiber sequences [ARØ20]

$$H\mathbb{Z} \xrightarrow{\times 2} H\mathbb{Z} \rightarrow HF_2, \quad \Sigma^{1,1} kq \xrightarrow{\eta} kq \rightarrow kg.$$

We next introduce motivic Brown–Gitler comodules. We define the *Mahowald weight filtration* on \mathcal{A}^\vee by setting

$$\text{wt}(\bar{\xi}_i) = \text{wt}(\bar{\tau}_i) = 2^i, \quad \text{wt}(\tau) = \text{wt}(\rho) = \text{wt}(u) = 0$$

and letting $\text{wt}(xy) = \text{wt}(x) + \text{wt}(y)$. This naturally extends to the subalgebras $(\mathcal{A} // \mathcal{A}(n))^\vee$. The *motivic Brown–Gitler comodule* $B_n^{\mathbb{F}_q}(k)$ is the $\mathcal{A}(n)^\vee$ -comodule

$$B_n^{\mathbb{F}_q}(k) = \langle x \in (\mathcal{A} // \mathcal{A}(n))^\vee : \text{wt}(x) \leq 2^{n+1}k \rangle.$$

A key property of the Brown–Gitler comodules that we will use is the following.

Theorem 2.4. [CQ21, Theorem 3.20] *There is an isomorphism of $\mathcal{A}(1)^\vee$ -comodules*

$$(\mathcal{A} // \mathcal{A}(1))^\vee \cong \bigoplus_{k \geq 0} \Sigma^{4k, 2k} B_0^{\mathbb{F}_q}(k).$$

The 0th Brown–Gitler comodule is given by $B_0^{\mathbb{F}_q}(0) \cong \mathbb{M}_2^{\mathbb{F}_q}$, and the first Brown–Gitler comodule is given by $B_0^{\mathbb{F}_q}(1) \cong \mathbb{M}_2^{\mathbb{F}_q}\{1, \bar{\xi}_1, \bar{\tau}_1\}$. It was shown in [CQ21, Example 3.12] that there is a spectrum $H\mathbb{Z}_1^{\mathbb{F}_q}$ such that

$$H_{**}(H\mathbb{Z}_1^{\mathbb{F}_q}) \cong B_0^{\mathbb{F}_q}(1)$$

as \mathcal{A}^\vee -comodules. Additionally, there is an equivalence of spectra up to 2-completion [Mor25b, Proposition 3.7]:

$$(2.5) \quad ksp \simeq kq \otimes H\mathbb{Z}_1^{\mathbb{F}_q}.$$

In general, the question of finding a motivic spectrum realizing a Brown–Gitler comodule is not simple to answer; see [Mor25b, Remark 2.17].

There are short exact sequences of $\mathcal{A}(1)^\vee$ -comodules relating motivic Brown–Gitler comodules. We refer the reader to [CQ21] for a proof.

Proposition 2.6. [CQ21, Lemma 3.21] *There are short exact sequences of $\mathcal{A}(1)^\vee$ -comodules:*

$$\begin{aligned} 0 \rightarrow \Sigma^{4k, 2k} B_0^{\mathbb{F}_q}(k) &\rightarrow B_0^{\mathbb{F}_q}(2k) \rightarrow B_1^{\mathbb{F}_q}(k-1) \otimes (\mathcal{A}(1) // \mathcal{A}(0))^\vee \rightarrow 0, \\ 0 \rightarrow \Sigma^{4k, 2k} B_0^{\mathbb{F}_q}(k) \otimes B_0^{\mathbb{F}_q}(1) &\rightarrow B_0^{\mathbb{F}_q}(2k+1) \rightarrow B_1^{\mathbb{F}_q}(k-1) \otimes (\mathcal{A}(1) // \mathcal{A}(0))^\vee \rightarrow 0. \end{aligned}$$

Remark 2.7. In [Li+25], we define Brown–Gitler comodules using the Mahowald weight filtration on the subalgebras $(\mathcal{A} // \mathcal{E}(n))^\vee$. This leads to a different family of submodules, except in the case where $n = 0$.

An interesting feature in the mod-2 motivic cohomology of finite fields is that, while $\mathbb{M}_2^{\mathbb{F}_q}$ is consistent across all fields of characteristic different from 2, the *Bockstein homomorphism*, which is the connecting homomorphism associated to the coefficient sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0,$$

acts differently on $H^{s,w}(\mathbb{F}_q; \mathbb{Z}/2)$ depending on whether $q \equiv 1(4)$ or $q \equiv 3(4)$. Notice that this Bockstein is represented by Sq^1 in the motivic Steenrod algebra [Voe03]. We can phrase the action of the Bockstein in terms of the action of the motivic Steenrod algebra on $\mathbb{M}_2^{\mathbb{F}_q}$.

Proposition 2.8. [WØ17] *The action of Sq^1 on $\mathbb{M}_2^{\mathbb{F}_q}$ is given by $Sq^1(\tau) = \rho$. In particular, the action is trivial if and only if $q \equiv 1(4)$.*

This implies that $\mathbb{M}_2^{\mathbb{F}_q}$ is central in \mathcal{A}^\vee when $q \equiv 1(4)$, meaning that \mathcal{A}^\vee is a Hopf algebra over $\mathbb{M}_2^{\mathbb{F}_q}$. We will often refer to the case of $q \equiv 1(4)$ as the *trivial Bockstein action* and the case of $q \equiv 3(4)$ as the *nontrivial Bockstein action*. This subtle difference makes a noticeable impact on computation and is the reason we separate our work into distinct sections.

2.3. Motivic Adams spectral sequence. Let $E \in SH(\mathbb{F}_q)$ be a motivic ring spectrum. There is a canonical Adams tower associated to the unit map $\mathbb{S} \rightarrow E$ taking the form

$$\begin{array}{ccccccc} \mathbb{S} & \longleftarrow & \Sigma^{-1,0}\bar{E} & \longleftarrow & \Sigma^{-2,0}\bar{E} \otimes \bar{E} & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ E & & \Sigma^{-1,0}\bar{E} \otimes E & & \Sigma^{-2,0}\bar{E} \otimes \bar{E} \otimes E & & \end{array}$$

where \bar{E} is the cofiber of the unit map $\mathbb{S} \rightarrow E$. For any motivic spectrum X , we can apply the functor $\pi_{**}^{\mathbb{F}_q}(X \otimes -)$. This yields the *E-based motivic Adams spectral sequence for X*. By construction, the E_1 -page of this spectral sequence has signature

$$E_1 = \pi_{s+f,w}^{\mathbb{F}_q}(E \otimes \bar{E}^{\otimes f} \otimes X) \implies \pi_{s,f}^{\mathbb{F}_q}(X_E), \quad d_r : E_r^{s,f,w} \rightarrow E_r^{s-1,f+r,w}$$

where X_E denotes the E -nilpotent completion of X [Bou79]. Convergence is in general not immediate, but we will only be interested in particular cases that are well-understood.

For $E = kq$, the resulting motivic Adams spectral sequence is called the *kq-resolution*. For a general X , this takes the form

$$E_1 = \pi_{s+f,w}^{\mathbb{F}_q}(kq \otimes \bar{kq}^{\otimes f} \otimes X) \implies \pi_{s,f}^{\mathbb{F}_q}(X).$$

Convergence of the kq-resolution was proven for $X = \mathbb{S}$ by Culver–Quigley in [CQ21, Theorem 2.1] over any base field of characteristic different than 2. They use the \mathbb{C} -motivic kq-resolution for the sphere to compute the v_1 -periodic stable stems. We computed the E_1 -page of the kq-resolution in \mathbb{R} -motivic homotopy theory in [Mor25b]. The primary aim of this paper is to compute the E_1 -page of the kq-resolution in \mathbb{F}_q -motivic homotopy theory.

For $E = H\mathbb{F}_2$, the resulting motivic Adams spectral sequence will be referred to as the $\mathbf{mASS}^{\mathbb{F}_q}(X)$. In this case, since $\mathcal{A}^\vee = \pi_{**}^{\mathbb{F}_q}(H\mathbb{F}_2 \otimes H\mathbb{F}_2)$ is flat as a module over $\mathbb{M}_2^{\mathbb{F}_q}$, we are able to pass directly to the E_2 -page. The $\mathbf{mASS}^{\mathbb{F}_q}(X)$ takes the form

$$E_2 = \text{Ext}_{\mathcal{A}^\vee}^{s,f,w}(H_{**}(X)) \implies \pi_{s,f}^{\mathbb{F}_q}(X).$$

Convergence for this motivic Adams spectral sequence was first studied by [HKO11a; DI10]. We will study the ring of cooperations for kq by computing the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$.

Remark 2.9. Another way to extract v_1 -periodicity is by studying the $\text{BPGL}\langle 1 \rangle$ -motivic Adams spectral sequence, where $\text{BPGL}\langle 1 \rangle$ is the first truncated motivic Brown–Peterson spectrum [HK01]. The study of this spectral sequence was initiated in [Li+25], where we computed the ring of cooperations at all primes and over the fields \mathbb{C}, \mathbb{R} , and \mathbb{F}_q . The truncated motivic Brown–Peterson cooperations actually offer a spectrum level decomposition, which is absent in the case of kq (see Remark 3.14). However, we expect that at the prime 2, v_1 -periodicity will be more easily accessible by means of the kq-resolution. It will be interesting to draw comparisons between these two parallel approaches.

2.4. The inductive process to calculate the ring of cooperations. In this section, we describe an inductive process to compute the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$. This involves a delicate analysis of many algebraic Atiyah–Hirzebruch spectral sequences. We refer the reader to [Mor25b, Section 4] for more details.

Observe that the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ has signature

$$E_2^{s,f,w} = \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(H_{**}(kq \otimes kq)) \implies \pi_{s,w}^{\mathbb{F}_q}(kq \otimes kq).$$

The Künneth spectral sequence

$$E_2 = \mathrm{Tor}^{\mathbb{M}_2^{\mathbb{F}_q}}(H_{**}(kq), H_{**}(kq)) \implies H_{**}(kq \otimes kq)$$

collapses on the E_2 -page [Mor25b, Proposition 2.12], giving an isomorphism

$$H_{**}(kq \otimes kq) \cong H_{**}(kq) \otimes H_{**}(kq).$$

Combining this with the change of rings isomorphism [Rav86, Appendix A] coming from the isomorphism $H_{**}(kq) \cong (\mathcal{A} // \mathcal{A}(1))^\vee$ and the Brown–Gitler decomposition of [Theorem 2.4](#), we obtain the following description of the E_2 -page.

Theorem 2.10. [Mor25b, Theorem 4.1] *The E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ is given by*

$$E_2^{s,f,w} = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\Sigma^{4k, 2k} B_0^{\mathbb{F}_q}(k)).$$

This decomposition, combined with the short exact sequences of [Proposition 2.6](#), allows us to compute the E_2 -page inductively. For $k = 0$, we have that $B_0^{\mathbb{F}_q}(0) = \mathbb{M}_2^{\mathbb{F}_q}$, and so the relevant summand of the E_2 -page is $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$. We will review these Ext groups in the coming sections.

For $k = 1$, we have that $B_0^{\mathbb{F}_q}(1) \cong \mathbb{M}_2^{\mathbb{F}_q}\{1, \bar{\xi}_1, \bar{\tau}_1\}$. The relevant summand of the E_2 -page may be calculated using an *algebraic Atiyah–Hirzebruch spectral sequence*, denoted $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$. Filtering $B_0^{\mathbb{F}_q}(1)$ by topological degree and letting $F_i B_0^{\mathbb{F}_q}(1)$ denote the subspace spanned by generators of degree $\leq i$ gives a finite filtration of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_0 B_0^{\mathbb{F}_q}(1) & \longrightarrow & F_1 B_0^{\mathbb{F}_q}(1) & \longrightarrow & F_2 B_0^{\mathbb{F}_q}(1) \longrightarrow F_3 B_0^{\mathbb{F}_q}(1) = B_0^F(1) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{M}_2^{\mathbb{F}_q}\{[1]\} & & 0 & & \mathbb{M}_2^{\mathbb{F}_q}\{[\bar{\xi}_1]\} \\ & & & & & & \mathbb{M}_2^{\mathbb{F}_q}\{[\bar{\tau}_1]\} \end{array}$$

The $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$ comes from applying the functor $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(-)$ to this filtration. This spectral sequence has signature

$$E_1^{s,f,w,a} = \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)),$$

where a denotes the Atiyah–Hirzebruch filtration degree. We let $\alpha[i]$ denote the copy of the class $\alpha \in \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ in Atiyah–Hirzebruch filtration i . The differentials in the $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$ are of the form

$$d_r : E_r^{s,f,w,a} \rightarrow E_r^{s-1,f+1,w,a-r}.$$

Thus the d_r differential lowers Atiyah–Hirzebruch filtration by r . In particular, the structure of $B_0^{\mathbb{F}_q}(1)$ ensures that each differential only takes nonzero value on one Atiyah–Hirzebruch filtration piece. By inspection of the length of the cellular filtration on $B_0^{\mathbb{F}_q}(1)$, we have the following.

Proposition 2.11. *In the $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$, we have that $E_4 = E_\infty$.*

The differentials in the $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$ are determined by the cobar complex; see [Mor25b, Section 4.2]. We recall the behavior of the d_1 and d_2 -differentials.

Proposition 2.12. [Mor25b, Proposition 4.4, Proposition 4.5] In the $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$, the d_1 -differential is determined by

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and the d_2 -differential is determined by

$$d_2(\alpha[2]) = h_1\alpha[0].$$

Since the differentials in the $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$ are linear over $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$, these formulae completely determine the d_1 and d_2 -differentials. By inspection, the only possible nonzero d_3 differential is between Atiyah–Hirzebruch filtrations 3 and 0, and will be given by the Massey product

$$d_3(\alpha[3]) = \langle \alpha, h_0, h_1 \rangle[0].$$

We will see that the nontriviality of this differential depends on whether or not $q \equiv 1(4)$ or $q \equiv 3(4)$.

With the computation of $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(1))$, we may proceed to calculate the rest of the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ by using the short exact sequences of Proposition 2.6. Since we are primarily interested in using the kq -resolution to determine the v_1 -periodic elements in $\pi_{**}^{\mathbb{F}_q}\mathbb{S}$, we will compute the E_2 -page modulo v_1 -torsion, which makes the presentation of our computations significantly more concise.

Following these calculations, we will be prepared to compute the ring of cooperations. First, using Equation (2.1), we will compute the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{HZ})$. Using the quotient map $\mathrm{kq} \rightarrow \mathrm{HZ}$, we will lift the differentials to the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, where we are aided by Friedlander’s calculations of the Hermitian K-theory of finite fields. Then, using the kq -module structure on $\mathrm{kq} \otimes \mathrm{kq}$, we will be able to determine the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$.

Remark 2.13. It is worthwhile to note that, unlike in the classical, \mathbb{C} -motivic, and \mathbb{R} -motivic cases, the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{HZ})$ has room for potential differentials. In fact, by the isomorphism

$$\pi_{s,w}^{\mathbb{F}_q}(\mathrm{HZ}) \cong H^{-s,-w}(\mathbb{F}_q; \mathbb{Z})$$

and the content of Equation (2.1), we will see that there must be infinitely many differentials. Lifting along the quotient map $\mathrm{kq} \rightarrow \mathrm{HZ}$ shows that there are infinitely many differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, and these also persist to the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$.

3. TRIVIAL BOCKSTEIN ACTION

In this section, we compute the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ in the case where there is a trivial Bockstein action on \mathbb{F}_q , that is, when $q \equiv 1(4)$. We first describe the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{HZ})$, then show that the differentials lift to the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, completely determining the behavior of the spectral sequence. After calculating the structure of the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ as a module over the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, we are able to lift differentials again, determining the spectral sequence and the ring of cooperations.

3.1. Integral homology. We begin by computing the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{HZ})$. This spectral sequence has signature

$$E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(H_{**}(\mathrm{HZ})) \implies \pi_{s,w}^{\mathbb{F}_q}(\mathrm{HZ}).$$

Recall that there is an isomorphism of \mathcal{A}^\vee -comodules $H_{**}(\mathrm{HZ}) \cong (\mathcal{A} // \mathcal{A}(0))^\vee$. This allows us to rewrite the E_2 -page of using a change of rings isomorphism as

$$\mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}((\mathcal{A} // \mathcal{A}(0))^\vee) \cong \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}).$$

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.2]

$$\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}) \cong \mathbb{F}_2[u, \tau, h_0]/(u^2),$$

where $|h_0| = (1, 0, 0)$. We depict this E_2 -page in Figure 1. Note that all charts are depicted in (s, f) -grading, and that motivic weight is suppressed.

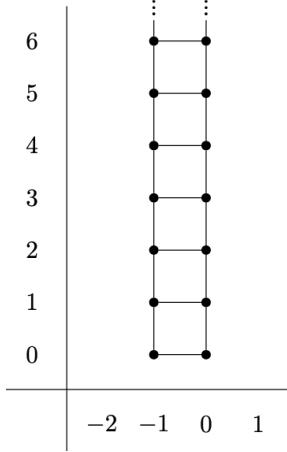


FIGURE 1. $\text{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 1 (4)$. A \bullet denotes $\mathbb{F}_2[\tau]$. A vertical line represents multiplication by h_0 , and a horizontal line represents multiplication by u .

We now determine all of the differentials in the spectral sequence.

Lemma 3.1. [Kyl15, Lemma 4.2.1] *Let $q \equiv 1 (4)$. The differentials in the $\text{mASS}^{\mathbb{F}_q}(\text{H}\mathbb{Z})$ are determined by*

$$d_{\nu_2(q-1)+\nu_2(i)}(\tau^i) = u\tau^{i-1}h_0^{\nu_2(q-1)+\nu_2(i)}.$$

Proof. Since $\pi_{0,0}^{\mathbb{F}_q}(\text{H}\mathbb{Z}) \cong \mathbb{Z}_2$, any element in $E_2^{0,f,0}$ must be a permanent cycle for all $f \geq 0$. This implies that 1 and all powers of h_0 are permanent cycles. Thus, if a class x supports a differential, then x must be divisible by τ . Suppose that y is not divisible by τ and that $x = \tau^k y$ for some $k \geq 1$. Then by the Liebniz rule, for any $r \geq 0$ we have

$$d_r(x) = d_r(y)\tau^k + yd_r(\tau^k) = yd_r(\tau^k).$$

Thus all differentials are uniquely determined by their value on τ . Moreover, $\pi_{0,-n}^{\mathbb{F}_q}(\text{H}\mathbb{Z}) = 0$ for any $n \geq 1$, so there can no τ -divisibility in stem 0 on the E_∞ -page. The particular values for the differentials are then a simple consequence of Equation (2.1) combined with the fact that Adams differentials preserve motivic weight. \square

To illustrate the behavior of the differentials in the $\text{mASS}^{\mathbb{F}_q}(\text{H}\mathbb{Z})$, we give an example.

Example 3.2. For example, take $q = 5$. We have that $\pi_{-1,-1}^{\mathbb{F}_5}(\text{H}\mathbb{Z}) \cong \mathbb{Z}/4$, which implies that the class uh_0^2 is the class in stem -1 and weight -1 of lowest Adams filtration which must be the target of a differential. Since τ must die, this forces the differential $d_2(\tau) = uh_0^2$. The Liebniz rule then determines all d_2 -differentials on odd powers of τ . Similarly, we have $\pi_{-1,-2}^{\mathbb{F}_5}(\text{H}\mathbb{Z}) \cong \mathbb{Z}/8$, so we must have that $d_3(\tau^2) = u\tau h_0^3$, and the Liebniz rule determines all d_3 -differentials on τ^{2n} for n odd.

We depict the E_∞ -page in the case of $q = 5$ in Figure 2. An empty \circ denotes \mathbb{F}_2 . A class labeled with τ^n indicates the first nonzero power of τ which exists in that particular bidegree (s, f) . An empty class with k circles around it indicates τ^{2^k} -periodicity. For example, in bidegree $(-1, 2)$ we have $\mathbb{F}_2\{\rho\tau h_0^2\}[\tau^2]$, and in bidegree $(-1, 3)$ we have $\mathbb{F}_2\{\rho\tau^2 h_0^3\}[\tau^4]$.

3.2. Hermitian K-theory. We now compute the $\text{mASS}^{\mathbb{F}_q}(\text{kq})$. This spectral sequence has signature

$$E_2 = \text{Ext}_{\mathcal{A}^\vee}^{s,f,w}(\text{H}_{**}(\text{kq})) \implies \pi_{s,w}^{\mathbb{F}_q}(\text{kq}).$$

Recall that there is an isomorphism of \mathcal{A}^\vee -comodules $\text{H}_{**}(\text{kq}) \cong (\mathcal{A} // \mathcal{A}(1))^\vee$. This allows us to rewrite the E_2 -page using a change of rings isomorphism as

$$\text{Ext}_{\mathcal{A}^\vee}^{s,f,w}((\mathcal{A} // \mathcal{A}(1))^\vee) \cong \text{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}).$$

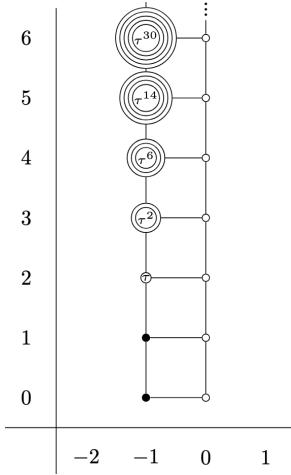


FIGURE 2. The \$E_\infty\$-page of the \$\mathbf{mASS}^{\mathbb{F}_q}(HZ)\$ for \$q=5\$. A \$\bullet\$ denotes \$\mathbb{F}_2[\tau]\$. An empty \$\circ\$ denotes \$\mathbb{F}_2\$. An empty class with \$k\$ circles around it indicates \$\tau^{2^k}\$-periodicity. A vertical line represents multiplication by \$h_0\$, and a horizontal line represents multiplication by \$u\$. See Example 3.2 for more details.

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.2]:

$$(3.3) \quad \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}) = \frac{\mathbb{F}_2[u, \tau, h_0, h_1, a, b]}{(u^2, h_0h_1, \tau h_1^3, h_1a, a^2 = h_0^2b)},$$

where \$|h_0| = (1, 0, 0)\$, \$|h_1| = (1, 1, 1)\$, \$|a| = (4, 3, 2)\$, and \$|b| = (8, 4, 4)\$. We note that this bears similarity to the \$\mathbb{C}\$-motivic analogue [Hil11]:

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{C}}) = \frac{\mathbb{F}_2[\tau, h_0, h_1, a, b]}{(h_0h_1, \tau h_1^3, h_1a, a^2 = h_0^2b)},$$

with generators in the same tridegree as in Equation (3.3). Indeed, there is an abstract isomorphism

$$(3.4) \quad \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}) \cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{C}}) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

We depict this \$E_2\$-page in Figure 3. Be aware that the notation is slightly different than that of Figure 1 in that we suppress \$u\$-multiplication. In particular, the pair of \$h_0\$-towers in Figure 1 appears as a single \$h_0\$-tower in Figure 3.

We will now show that we can lift the differentials from the \$\mathbf{mASS}^{\mathbb{F}_q}(HZ)\$ to the \$\mathbf{mASS}^{\mathbb{F}_q}(kq)\$; see also [Kyl15, Section 4.2].

Proposition 3.5. *Let \$q \equiv 1(4)\$. The differentials in the \$\mathbf{mASS}^{\mathbb{F}_q}(kq)\$ are determined by the differentials in the \$\mathbf{mASS}^{\mathbb{F}_q}(HZ)\$.*

Proof. We first determine the possible Adams differentials on the generators of the \$E_2\$-page. We see that \$u\$ cannot support a differential, as Adams differentials decrease stem degree by 1 and there are no classes in stem -2. Further, Adams differentials preserve weight. Since there are no classes in weight 0 in stem -1, as they are all divisible by \$u\$, we cannot have any differentials on 1 or any power of \$h_0\$. If \$h_1\$ were to support a differential, then degree considerations force it to be of the form \$d_r(h_1) = \tau h_0^{r+1}\$. However, this would imply that \$d_r(h_1h_0) = d_r(h_1)h_0 + h_1d_r(h_0) = \tau h_0^{r+2}\$, which is impossible since \$h_1h_0 = 0\$. Thus \$h_1\$ does not support a differential. Finally, the only chance for \$b\$ to support a differential is \$d_3(b) = h_1^7\$, but since the weight of \$b\$ is 4 and the weight of \$h_1^7\$ is 7, this cannot happen. So, we see that if a class \$x\$ supports a differential, it must be \$\tau\$-divisible.

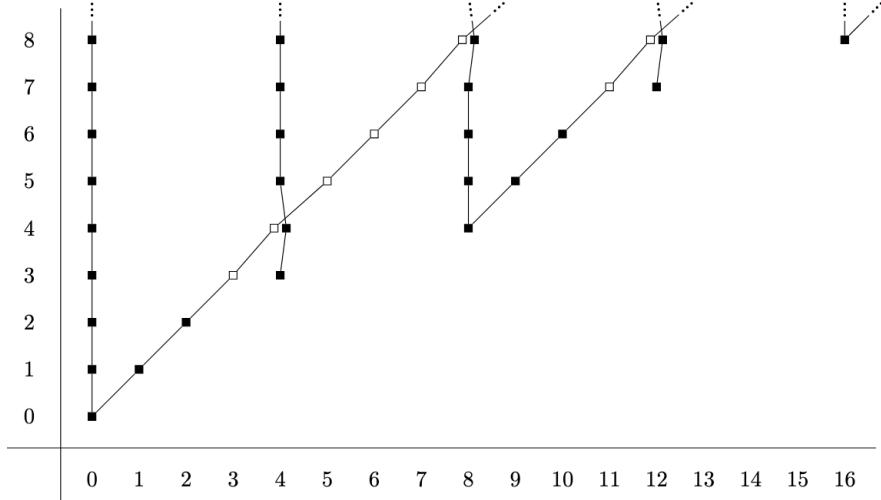


FIGURE 3. $\text{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 1 (4)$. A ■ denotes $\mathbb{F}_2[u, \tau]/(u^2)$, and a □ denotes $\mathbb{F}_2[u]/(u^2)$. A vertical line represents h_0 -multiplication, and a diagonal line represents h_1 -multiplication.

Recall the map $kq \rightarrow H\mathbb{Z}$ of Equation (2.2). This induces a map of spectral sequences $\mathbf{mASS}^{\mathbb{F}_q}(kq) \rightarrow \mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$, which is realized on E_2 -pages as the evident quotient map

$$\text{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}) = \frac{\mathbb{F}_2[u, \tau, h_0, h_1, a, b]}{(u^2, h_0h_1, \tau h_1^3, h_1a, a^2 = h_0^2b)} \rightarrow \mathbb{F}_2[u, \tau, h_0]/(u^2) = \text{Ext}_{\mathcal{A}(0)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}),$$

where $h_1, a, b \mapsto 0$. In particular, this implies that the differentials on powers of τ determined by Lemma 3.1 lift to the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$. Since u, h_0, h_1, a , and b are permanent cycles, the Liebniz rule determines all differentials in the spectral sequence, finishing the proof. \square

Running the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$, we see that

$$\pi_{s,w}^{\mathbb{F}_q}(kq) \cong \begin{cases} \text{KO}_{s-2w}(\mathbb{F}_q) & w \equiv 0 (4) \\ \text{KSp}_{s-2w}(\mathbb{F}_q) & w \equiv 2 (4) \end{cases}$$

for $s \geq 0$, agreeing with Friedlander's calculations given in Table 1.

Remark 3.6. By lifting differentials from the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$, we are able to more clearly identify the values of $\pi_{s,w}^{\mathbb{F}_q}(kq)$ for $s \leq 0$. The quotient map $kq \rightarrow H\mathbb{Z}$ induces an isomorphism of motivic Adams spectral sequence E_2 -pages in stems $s \leq 0$, hence the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$ are isomorphic to the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$ in this range. This gives an isomorphism $\pi_{s,w}^{\mathbb{F}_q}(kq) \cong \pi_{s,w}^{\mathbb{F}_q}(H\mathbb{Z})$ for $s \leq 0$. More generally, there we can express the homotopy of kq in terms of $H\mathbb{Z}$:

$$\pi_{**}^F(kq) \cong (\pi_{**}^F(H\mathbb{Z}))[\eta, \tau^n\eta, \tau^n\eta^2, \alpha, \beta : n \geq 0]/(2\eta, \eta\alpha, \tau^n\eta\alpha, \tau^n\eta^2\alpha, \alpha^2 - 4\beta),$$

where $|\eta| = (1, 1)$, $|\tau^n\eta| = (1, 1-n)$, $|\alpha| = (4, 2)$, and $|\beta| = (8, 4)$. Note that the additional generators $\tau^n\eta, \tau^n\eta^2$ stem from the fact that τ dies in the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$ while $\tau\eta$ survives in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$.

3.3. Brown–Gitler comodules. We begin the inductive process of computing the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$. To start, we will compute $\text{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1))$ by the $\mathbf{AHSS}^{\mathbb{F}_q}(B_0^{\mathbb{F}_q}(1))$. This spectral sequence takes the form

$$E_1^{s,f,w,a} = \text{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \text{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)).$$

We have discussed the differentials in this spectral sequence in [Section 2.4](#). In particular, for $\alpha \in \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$, we have that

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and

$$d_2(\alpha[2]) = h_1\alpha[0].$$

There is a potential third differential between Atiyah-Hirzebruch filtrations 3 and 0, but we see that for degree reasons this is not possible. Since the differentials are module maps over $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$, this determines the spectral sequence. Hidden extensions lift in the same way as in the \mathbb{C} -motivic case; see [\[CQ21, Section 3.4\]](#). We depict the E_∞ -page in [Figure 4](#).

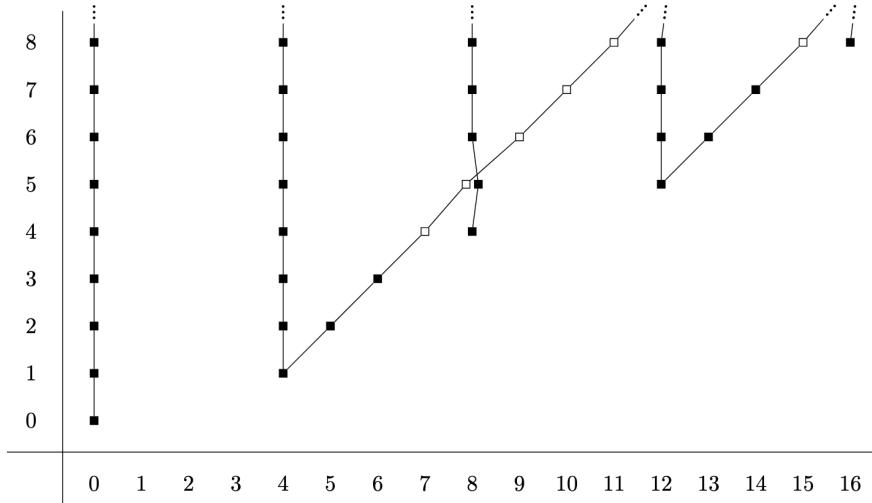


FIGURE 4. The E_∞ -page of the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ for $q \equiv 1 (4)$.

In [\[Mor25b, Theorem 3.7\]](#), we show that there is an equivalence of spectra

$$\mathrm{ksp} \simeq \mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q} \otimes \mathrm{kq},$$

where ksp is the very effective symplectic K-theory spectrum and $\mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q}$ is the spectrum realizing $B_0^{\mathbb{F}_q}(1)$ described in [Section 2.2](#). The Künneth spectral sequence computing $\mathrm{H}_{**}(\mathrm{ksp})$ collapses, and so the change of rings isomorphism implies that the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$ is isomorphic to $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(1))$. However, there are differentials in this spectral sequence.

Lemma 3.7. *Let $q \equiv 1 (4)$. In the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$, the differentials are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ and the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$.*

Proof. Recall the cofiber sequence of [Equation \(2.3\)](#)

$$\Sigma^{4,2}\mathrm{kq} \rightarrow \mathrm{ksp} \rightarrow \mathrm{H}\mathbb{Z}.$$

By applying $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathrm{H}_{**}(-))$ to this cofiber sequence, we get a long exact sequence in Ext groups. Notice that these Ext groups are the E_2 -pages for the respective motivic Adams spectral sequences. For degree reasons, the connecting homomorphism

$$\delta : \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(\mathrm{H}_{**}(\mathrm{H}\mathbb{Z})) \rightarrow \mathrm{Ext}_{\mathcal{A}^\vee}^{s-1,f+1,w}(\mathrm{H}_{**}(\Sigma^{4,2}\mathrm{kq}))$$

is trivial. Thus, after a few change of rings isomorphisms, we have a decomposition of the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$:

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)) \cong \Sigma^{4,2}\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q})\langle 1 \rangle \oplus \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}),$$

where $\langle - \rangle$ denotes a shift in the Adams filtration degree. The projection $ksp \rightarrow H\mathbb{Z}$ induces evident quotient map on E_2 -pages which lift the differentials from [Lemma 3.1](#). The inclusion $\Sigma^{4,2}kq \rightarrow ksp$ induces the evident inclusion on E_2 -page, lifting the differentials from [Proposition 3.5](#) to the $\mathbf{mASS}^{\mathbb{F}_q}(ksp)$ to this summand. For degree reasons there are no other differentials possible, finishing the proof. \square

By running the $\mathbf{mASS}^{\mathbb{F}_q}(ksp)$ and comparing with [Table 1](#), we recover the expected isomorphism for $s \geq 4$

$$\pi_{s,w}^{\mathbb{F}_q}(ksp) \cong \begin{cases} KSp_{s-2w}(\mathbb{F}_q) & w \equiv 0 \pmod{4}; \\ KO_{s-2w}(\mathbb{F}_q) & w \equiv 2 \pmod{4}. \end{cases}$$

We also obtain an isomorphism $\pi_{s,w}^{\mathbb{F}_q}(ksp) \cong \pi_{s,w}^{\mathbb{F}_q}(kq)$ for $s \leq 0$.

The following is immediate from our description.

Lemma 3.8. *Let $q \equiv 1 \pmod{4}$. There is an isomorphism*

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)) \cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{C}}(1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Proposition 3.9. *Let $q \equiv 1 \pmod{4}$. For all $i \geq 0$, there is an isomorphism*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)^{\otimes i})}{v_1\text{-torsion}} \cong \frac{\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{C}}(1)^{\otimes i})}{v_1\text{-torsion}} \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Proof. The case $i = 0$ was shown in [Equation \(3.4\)](#), and the case $i = 1$ was shown in [Lemma 3.8](#). Assume the result holds for all $n \leq i$. We may calculate the Ext group in question by another algebraic Atiyah–Hirzebruch spectral sequence. For the rest of this proof, we implicitly compute modulo v_1 -torsion. By applying the functor $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(1)^{\otimes i} \otimes -)$ to the cellular filtration of $B_0^{\mathbb{F}_q}(1)$, we obtain a spectral sequence with signature

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)^{\otimes i}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)^{\otimes i+1}).$$

By induction, we can rewrite the E_1 -page as

$$\left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{C}}(1)^{\otimes i}) \otimes \mathbb{M}_2^{\mathbb{C}}\{1, [\bar{\xi}_1], [\bar{\tau}_1]\} \right) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

As our spectral sequence is induced by the same filtration as the $\mathbf{AHSS}(B_0^{\mathbb{F}_q}(1))$, the differentials are determined by the same formulae. For degree reasons, we see that there are no d_3 -differentials. Hidden extensions follow as in the classical case. Since the differentials are the same as in the \mathbb{C} -motivic case by [\[CQ21, Lemma 3.36\]](#), we see that the abutment is isomorphic to

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{C}}(1)^{\otimes i+1}) \otimes \mathbb{M}_2^{\mathbb{F}_q},$$

which is what we wanted to show. \square

Theorem 3.10. *Let $q \equiv 1 \pmod{4}$. For all $k \geq 0$, there is an isomorphism*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(k))}{v_1\text{-torsion}} \cong \frac{\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{C}}(k))}{v_1\text{-torsion}} \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Proof. We use the short exact sequences of $\mathcal{A}(1)^\vee$ -comodules described in [Proposition 2.6](#). For convenience, all Ext groups are implicitly computed modulo v_1 -torsion in this proof. The case of $k = 1$ was shown in [Lemma 4.2](#). Now, suppose that the theorem is true for all $i < k$.

Suppose that k is even. There is a short exact sequence of Brown–Gitler comodules

$$0 \rightarrow \Sigma^{2k,k} B_0^{\mathbb{F}_q}(\frac{k}{2}) \rightarrow B_0^{\mathbb{F}_q}(k) \rightarrow B_1^{\mathbb{F}_q}(\frac{k}{2}-1) \otimes (\mathcal{A}(1) // \mathcal{A}(0))^\vee \rightarrow 0.$$

Applying the functor $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(-)$ gives a long exact sequence of $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ -modules. We can use a change of rings isomorphism to rewrite the cokernel in Ext:

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_1^{\mathbb{F}_q}(\frac{k}{2}-1) \otimes (\mathcal{A}(1) // \mathcal{A}(0))^\vee) \cong \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(B_1^{\mathbb{F}_q}(\frac{k}{2}-1)).$$

As we are working mod v_1 -torsion, the connecting homomorphism must be trivial. Thus, the Ext group in question decomposes into the Ext group of the kernel and the Ext group of the cokernel. The Ext group of the kernel, $\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\Sigma^{2k,k} B_0^{\mathbb{F}_q}(\frac{k}{2}))$, is handled by the induction hypothesis. A simple computation shows that the Ext group of the cokernel consists solely of a direct sum of $\mathbb{F}_2[u, \tau, h_0]/(u^2)$ indexed along the generators of $B_1^{\mathbb{F}_q}(\frac{k}{2} - 1)$ as an $\mathcal{A}(0)^\vee$ -comodule. In other words, since these summands are in the same stem as their \mathbb{C} -motivic counterpart [CQ21, Theorem 3.38], we have an isomorphism

$$\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(B_1^{\mathbb{F}_q}(\frac{k}{2} - 1)) \cong \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(B_1^{\mathbb{C}}(\frac{k}{2} - 1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Altogether, we have

$$\begin{aligned} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(k)) &\cong \left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\Sigma^{2k,k} B_0^{\mathbb{C}}(\frac{k}{2})) \oplus \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(B_1^{\mathbb{C}}(\frac{k}{2} - 1)) \right) \otimes \mathbb{M}_2^{\mathbb{F}_q} \\ &\cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{C}}(k)) \otimes \mathbb{M}_2^{\mathbb{F}_q} \end{aligned}$$

where the last equality is due to [Mor25b, Theorem 5.6].

Suppose that k is odd. There is a short exact sequence of Brown–Gitler comodules

$$0 \rightarrow \Sigma^{2(k-1), k-1} B_0^{\mathbb{F}_q}(\frac{k-1}{2}) \otimes B_0^{\mathbb{F}_q}(1) \rightarrow B_0^{\mathbb{F}_q}(k) \rightarrow B_1^{\mathbb{F}_q}(\frac{k-1}{2} - 1) \otimes (\mathcal{A}(1) // \mathcal{A}(0))^\vee \rightarrow 0.$$

After applying $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(-)$ and using a change of rings isomorphism on the cokernel, the same argument as given above implies that the connecting homomorphism in the long exact sequence of Ext groups is trivial modulo v_1 -torsion. Thus, we only need to understand the Ext group of the kernel and the Ext group of the cokernel. The Ext group of the cokernel follows from the same style of computation as in the previous case, giving an isomorphism

$$\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(B_1^{\mathbb{F}_q}(\frac{k-1}{2} - 1)) \cong \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(B_1^{\mathbb{C}}(\frac{k-1}{2} - 1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

The Ext group of the kernel may be calculated by another algebraic Atiyah–Hirzebruch spectral sequence. By applying the functor $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(\frac{k-1}{2}) \otimes -)$ to the cellular filtration on $B_0^{\mathbb{F}_q}(1)$, we obtain a spectral sequence with signature

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(\frac{k-1}{2})) \otimes \mathbb{M}_2^{\mathbb{F}_q}[[1], [\bar{\xi}_1], [\bar{\tau}_1]] \implies \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(\frac{k-1}{2}) \otimes B_0^{\mathbb{F}_q}(1)).$$

By the induction hypothesis, we have that

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(\frac{k-1}{2})) \cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{C}}(\frac{k-1}{2})) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

The differentials are induced by the cellular filtration on $B_0^{\mathbb{F}_q}(1)$, so by the same argument as given in Proposition 3.9, the differentials are the same as the \mathbb{C} -motivic case determined in [CQ21, Theorem 3.38] and there is no room for a d_3 -differential. Thus the abutment is isomorphic to

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{C}}(\frac{k-1}{2}) \otimes B_0^{\mathbb{C}}(1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Therefore, we have

$$\begin{aligned} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(k)) &\cong \left(\mathrm{Ext}_{\mathcal{A}^\vee}^{***}(\Sigma^{2(k-1), (k-1)} B_0^{\mathbb{C}}(\frac{k}{2} - 1) \otimes B_0^{\mathbb{C}}(1)) \oplus \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(B_1^{\mathbb{C}}(\frac{k-1}{2} - 1)) \right) \otimes \mathbb{M}_2^{\mathbb{F}_q} \\ &\cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{C}}(k)) \otimes \mathbb{M}_2^{\mathbb{F}_q} \end{aligned}$$

where the last equivalence is again due to [Mor25b, Theorem 5.6]. \square

We can be more descriptive, but we must introduce some notation. For M an (s, f, w) -trigraded module (such as an Ext group over the dual Steenrod algebra), let $\tau_{\geq n} M$ denote the truncation in the s -degree, so that $\tau_{\geq n} M = M$ for $s \geq n$ and $\tau_{\geq n} M = 0$ for $s < n$. Let $E \in \mathrm{SH}(\mathbb{F}_q)$ be any motivic spectrum. Whenever E has a factor of $(\mathcal{A} // \mathcal{A}(1))^\vee$ in its mod-2 homology, we will use the notation $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(E)$ for the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(E)$. Let $E^{\langle m \rangle}$ denote the m^{th} Adams cover of E , which is

the m^{th} term in a minimal HF_2 -Adams resolution for E . Recall that a minimal Adams resolution is a diagram of the form [Rav86, Definition 2.1.3]

$$\begin{array}{ccccccc} E^{\langle 0 \rangle} := E & \longleftarrow & E^{\langle 1 \rangle} & \longleftarrow & E^{\langle 2 \rangle} & \longleftarrow & \cdots \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

and applying homotopy groups to this diagram gives the $\text{mASS}^{\mathbb{F}_q}(E)$. Here $K_n := \bigoplus_B \text{HF}_2$ for a basis B of the homology group $H_{**}(E^{\langle n \rangle})$, and $E^{\langle n \rangle}$ is the fiber of the map $E^{\langle n-1 \rangle} \rightarrow K_{n-1}$, defined inductively by taking $E^{\langle 0 \rangle} := E$. The group $\text{Ext}_{\mathcal{A}^\vee}^{***}(H_{**}(E^{\langle n \rangle}))$ can be obtained by described as

$$\text{Ext}_{\mathcal{A}^\vee}^{s,f,w}(H_{**}(E^{\langle n \rangle})) \cong \begin{cases} \text{Ext}_{\mathcal{A}^\vee}^{s,f+n,w}(H_{**}(E)) & f \geq 0 \\ 0 & f < 0. \end{cases}$$

One may interpret this Ext group in charts by taking the charts for $\text{Ext}_{\mathcal{A}^\vee}^{***}(H_{**}(E))$, relabeling the filtration $f = n$ -line as $f = 0$, and then omitting anything below this now filtration 0 line. In other words, one slides the s -axis up to filtration $f = n$ and truncates all information in lower filtration.

Notice that $\tau_{\geq n}(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{kq}^{\langle m \rangle}))$ and $\tau_{\geq n}(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{ksp}^{\langle m \rangle}))$ have isolated non-nilpotent h_1 -towers which are not connected to an h_0 -tower. In these cases, we will use the modified truncation $\tau_{\geq n}^{h_1}(-)$ to indicate taking the truncation and then omitting out these isolated h_1 -towers. As an example, we depict $\tau_{\geq 4}(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{kq}^{\langle 2 \rangle}))$ in Figure 5 and $\tau_{\geq 4}^{h_1}(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{kq}^{\langle 2 \rangle}))$ in Figure 6.

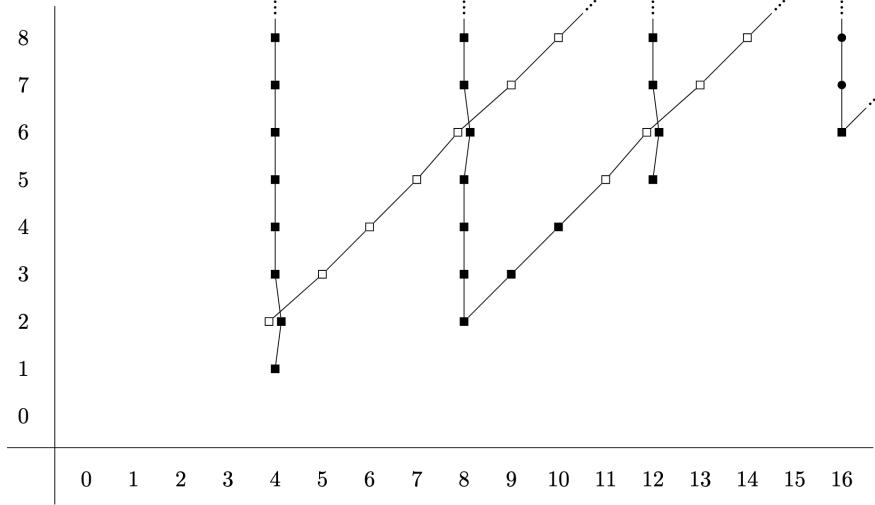


FIGURE 5. $\tau_{\geq 4}(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{kq}^{\langle 2 \rangle}))$, with an isolated h_1 -tower.

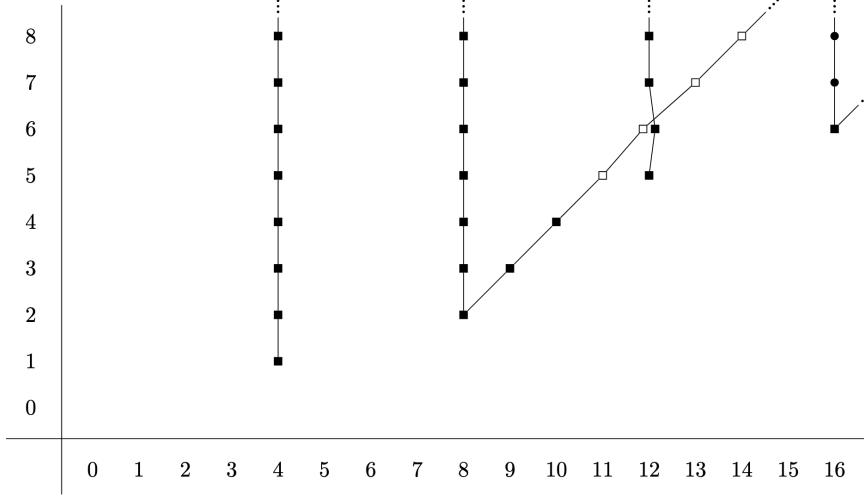
Definition 3.11. For $i \geq 0$, let $Z_i^{\mathbb{C}}$ be the $\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{C}})$ -module defined as follows.

For $i \equiv 0 \pmod{4}$, let

$$Z_i^{\mathbb{C}} = \tau_{\geq 2i}^{h_1} \left(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{kq}^{\langle i \rangle}) \right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j, 2j} \text{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{C}}).$$

For $i \equiv 1 \pmod{4}$, let

$$Z_i^{\mathbb{C}} = \tau_{\geq 2i-2}^{h_1} \left(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{ksp}^{\langle i-1 \rangle}) \right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j, 2j} \text{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{C}}).$$

FIGURE 6. $\tau_{\geq 4}^{h_1}(\mathrm{Ext}_{\mathcal{A}(1)\vee}^{***}(\mathrm{kq}^{\langle 2 \rangle}))$, with no isolated h_1 -towers.

For $i \equiv 2 \pmod{4}$, let

$$Z_i^{\mathbb{C}} = \tau_{\geq 2i}^{h_1} \left(\mathrm{Ext}_{\mathcal{A}(1)\vee}^{***}(\mathrm{kq}^{\langle i \rangle}) \right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j, 2j} \mathrm{Ext}_{\mathcal{A}(0)\vee}^{***}(\mathbb{M}_2^{\mathbb{C}}) \oplus \Sigma^{2i-2, i} \mathbb{F}_2[\tau].$$

For $i \equiv 3 \pmod{4}$, let

$$Z_i^{\mathbb{C}} = \tau_{\geq 2i-2}^{h_1} \left(\mathrm{Ext}_{\mathcal{A}(1)\vee}^{***}(\mathrm{ksp}^{\langle i-1 \rangle}) \right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j, 2j} \mathrm{Ext}_{\mathcal{A}(0)\vee}^{***}(\mathbb{M}_2^{\mathbb{C}}) \oplus \Sigma^{2i-3, i-2} \mathbb{F}_2[\tau, h_1]/(h_1^2).$$

Corollary 3.12. *Let $q \equiv 1 \pmod{4}$. For all $k \geq 0$, there is an isomorphism:*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)\vee}^{***}(B_0^{\mathbb{F}_q}(k))}{v_1\text{-torsion}} \cong \left(\Sigma^{4k-4, 2k-2} Z_{\alpha(k)}^{\mathbb{C}} \oplus \bigoplus_{j=0}^{k-2} \mathbb{M}_2^{\mathbb{C}}[h_0] \right) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathbb{M}_2^{\mathbb{F}_q},$$

where $\alpha(k)$ is the number of 1's in the dyadic expansion of k , and the right hand summand is taken to be empty in the case of $k=1$.

Proof. By [Mor25b, Theorem 5.6], there is an isomorphism

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)\vee}^{***}(B_0^{\mathbb{C}}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4, 2k-2} Z_{\alpha(k)}^{\mathbb{C}} \oplus \bigoplus_{j=0}^{k-2} \mathbb{M}_2^{\mathbb{C}}[h_0].$$

The result then follows by Theorem 3.10 □

Remark 3.13. The notation in Definition 3.11, although equivalent, differs from the notation in [Mor25b, Section 4]. We hope that this presentation in terms of Adams covers is more well suited for generalization.

Remark 3.14. As is true over \mathbb{C} and \mathbb{R} , it is in this computation that we deviate from the classical story. Mahowald is able to express the groups $\mathrm{Ext}_{\mathcal{A}(1)\vee}^{**}(B_0(1)^{\otimes i})$ solely in terms of Adams covers of bo and bsp [LM87]. The analogous statement in motivic homotopy is not true. If one were to express $\mathrm{Ext}_{\mathcal{A}(1)\vee}^{***}(B_0^{\mathbb{F}_q}(1)^{\otimes i})$ in terms of Adams covers of kq or ksp , then one would see summands of non-nilpotent h_1 -towers appearing in the formulae for $Z_i^{\mathbb{C}}$, reflecting that η is non-nilpotent in $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq}^{\langle n \rangle})$ and $\pi_{**}^{\mathbb{F}_q}(\mathrm{ksp}^{\langle n \rangle})$. However, these formulas do not allow for this, and indeed one can see through

the **aAHSS** that such h_1 -towers cannot be obtained. This is the reason behind the sophisticated truncation functors we introduced in the formulae given for $Z_i^{\mathbb{C}}$.

3.4. The ring of cooperations. We now assemble our findings to compute $\pi_{**}^{\mathbb{F}_q}(kq \otimes kq)$. First, we describe the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$.

Corollary 3.15. *Let $q \equiv 1(4)$. The E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$, modulo v_1 -torsion, is given by*

$$E_2 = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(\Sigma^{4k, 2k} B_0^{\mathbb{F}_q}(k)) \cong \bigoplus_{k \geq 0} \Sigma^{4k, 2k} \left(\Sigma^{4k-4, 2k-2} Z_{\alpha(k)}^{\mathbb{C}} \oplus \bigoplus_{j=0}^{4k-8} \Sigma^{4j, 2j} \mathbb{M}_2^{\mathbb{C}}[h_0] \right) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Proof. The first equivalence follows from [Theorem 2.10](#), and the second follows from [Corollary 3.12](#). \square

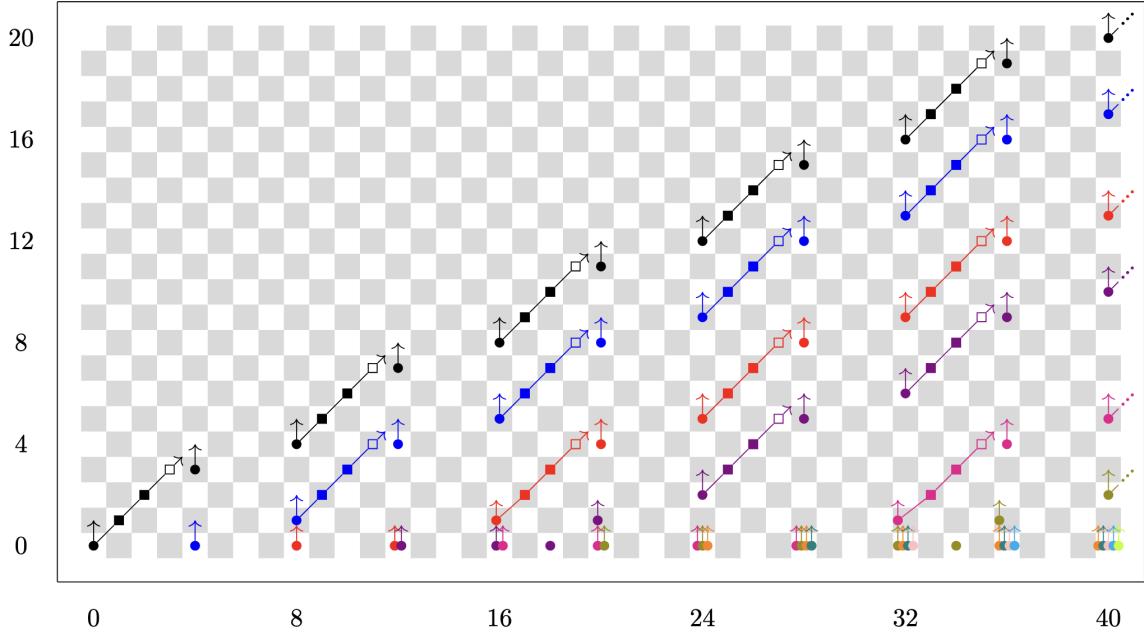
Notice that this description of the E_2 -page is as a module over the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$, that is, as a module over the algebra $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$. We leverage this to determine the differentials in our spectral sequence.

Theorem 3.16. *The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$.*

Proof. The inclusion map $kq \rightarrow kq \otimes kq$ induces a map of E_2 -pages which pushes the differentials of [Proposition 3.5](#) to the $k = 0$ summand of $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(kq \otimes kq)$. The $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(kq)$ -module structure of $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(kq \otimes kq)$ and degree considerations lift all of the other differentials. To be precise, each summand of the E_2 -page contains a submodule which is isomorphic to either $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(kq)$ or $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(ksp)$. Let x be a generator for this submodule. Then $d_r(x) = 0$ for degree reasons for all r as all classes in the prior stem have motivic weight at least 1 lower than x , as they are classes of the form $u\tau^n h_0^m$ for some $m, n \geq 0$, and Adams differentials preserve weight. Letting 1 denote the generator for $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(kq)$ of $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(ksp)$, this is just the same as why $d_r(1) = 0$ for all r in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$ or the $\mathbf{mASS}^{\mathbb{F}_q}(ksp)$. The rest of the differentials on this submodule are now determined by the Liebniz rule and the underlying differentials in the $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(kq)$, following the proof of [Proposition 3.5](#).

If y is a generator for any submodule of the form $\mathbb{F}_2[u, \tau, h_0]/(u^2)$, then we must have that $d_r(y) = 0$ for all r as all classes in the prior stem have weight at least 1 lower than y . Thus, the Liebniz rule implies that all the differentials will be given by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$. Finally there are no differentials on any ssummand of the form $\mathbb{F}_2[\tau]$ or $\mathbb{F}_2[\tau, h_1]/(h_1)^2$ for the same reason there is no differential on h_1 in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$, following the proof in [Proposition 3.5](#). \square

[Figure 7](#) depicts a chart for the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ in the style of [[Beh+19](#), Figure 3.2]. Different colors denote different summands of the E_2 -page. For example, the $k = 0$ summand is given by $\mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(kq)$. This is depicted in black, and starts in bidegree $(0, 0)$. The $k = 1$ summand is given by $\Sigma^{4, 2} \mathrm{Ext}_{\mathcal{A}(1)^{\vee}}^{***}(ksp)$. This is depicted in blue, and starts in bidegree $(4, 0)$. Since no classes which are divisible by h_1 support a differential, one can interpret these charts as either the E_2 -page or the E_{∞} -page. For the E_2 -page, let a \bullet of any color with a vertical arrow denote $\mathbb{F}_2[\tau, u, h_0]/(u^2)$. For the E_{∞} -page, let a \bullet of any color with a vertical arrow denote the E_{∞} -page of the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$ shifted to that tridegree. A \blacksquare denotes $\mathbb{F}_2[\tau, u]/(u^2)$, and a \square denotes $\mathbb{F}_2[u]/(u^2)$.

FIGURE 7. The $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ for $q \equiv 1 \pmod{4}$.

4. NONTRIVIAL BOCKSTEIN ACTION

In this section, we compute the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(kq \otimes kq)$ in the case where there is a nontrivial Bockstein action, that is, for $q \equiv 3 \pmod{4}$. We follow the same process as was done in the previous section, and often refer to the previous section when we encounter identical proofs, but the Bockstein action makes the particular computations in Ext more complicated in this case. In particular, when computing over $\mathcal{A}(1)^\vee$, we display our data using multiple charts organized by coweight.

4.1. Integral homology. We begin by computing the $\mathbf{mASS}^{\mathbb{F}_q}(\mathbb{H}\mathbb{Z})$. This spectral sequence has signature

$$E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(\mathrm{H}_{**}(\mathbb{H}\mathbb{Z})) \implies \pi_{s,w}^{\mathbb{F}_q}(\mathbb{H}\mathbb{Z}).$$

Recall that there is an isomorphism of \mathcal{A}^\vee -comodules $\mathrm{H}_{**}(\mathbb{H}\mathbb{Z}) \cong (\mathcal{A} // \mathcal{A}(0))^\vee$. This allows us to rewrite the E_2 -page using a change of rings isomorphism as

$$\mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}((\mathcal{A} // \mathcal{A}(0))^\vee) \cong \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}).$$

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.3]

$$(4.1) \quad \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}) \cong \mathbb{F}_2[\rho, \tau^2, h_0, \rho\tau]/(\rho^2, \rho h_0, \rho(\rho\tau), (\rho\tau)^2),$$

where $|h_0| = (1, 0, 0)$ and $\rho\tau$ is indecomposable. We depict this E_2 -page in Figure 8. Note that, although at first glance this chart looks nearly identical to Figure 8, it is much sparser in the motivic weight degree. For example, in stem 0 there are no classes of odd motivic weight, and in stem -1 the only classes of odd motivic weight are those of the form $\rho\tau^{2n}$. In particular, neither of the h_0 towers contain classes of odd motivic weight. To highlight the difference between these two sets of charts, we do not depict multiplication by $\rho\tau$.

We now determine all of the differentials in the spectral sequence.

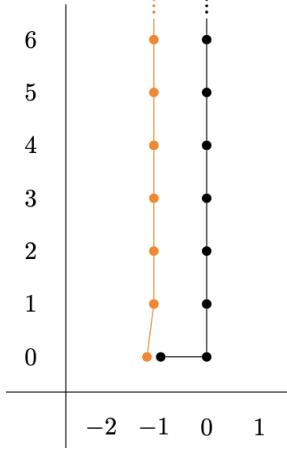


FIGURE 8. $\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 3 (4)$. A \bullet denotes $\mathbb{F}_2[\tau^2]$, and the color orange denotes $\rho\tau$ -divisibility. A vertical line represents multiplication by h_0 , and a horizontal line represents multiplication by ρ .

Lemma 4.2. [Kyl15, Lemma 4.2.2] Let $q \equiv 3 (4)$. The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathbb{H}\mathbb{Z})$ are determined by

$$d_{\nu_2(q^2-1)+i}(\tau^{2^i}) = \rho\tau^{2^i-1}h_0^{\nu_2(q^2-1)+i}.$$

Proof. The same proof as was given in Lemma 3.1 works by comparing with Equation (2.1), noting that since $\tau = 0$ on the E_2 -page, the first class in stem 0 which must support a differential is τ^2 . \square

We depict the E_∞ -page in the case of $q = 3$ in Figure 9, which can be deduced using the same techniques as in Example 3.2. An empty \circ denotes \mathbb{F}_2 . A class labeled with τ^n indicates the first nonzero power of τ which exists in that particular bidegree. An empty class with k circles around it indicates τ^{2^k} -periodicity.

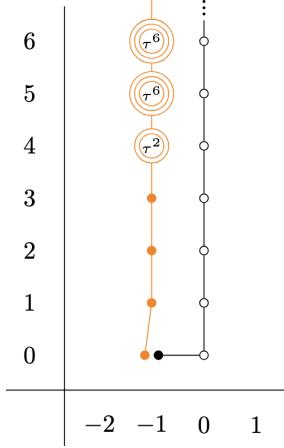


FIGURE 9. The E_∞ -page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathbb{H}\mathbb{Z})$ for $q \equiv 3 (4)$. A \bullet denotes $\mathbb{F}_2[\tau^2]$. An empty \circ denotes \mathbb{F}_2 . A class labeled with τ^n indicates the first nonzero power of τ which exists in that particular bidegree. An empty class with k circles around it indicates τ^{2^k} -periodicity. A vertical line represents multiplication by h_0 , and a horizontal line represents multiplication by ρ .

4.2. Hermitian K-theory. We now compute the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$. This spectral sequence has signature

$$E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(\mathrm{H}_{**}(\mathrm{kq})) \implies \pi_{s,w}^{\mathbb{F}_q}(\mathrm{kq}).$$

Recall that there is an isomorphism of \mathcal{A}^\vee -comodules $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} // \mathcal{A}(1))^\vee$. This allows us to rewrite the E_2 -page using a change of rings isomorphism as

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}((\mathcal{A} // \mathcal{A}(1))^\vee) \cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}).$$

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.5]:

$$(4.3) \quad \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_3}) \cong \mathbb{F}_2[\rho, \tau^2, h_0, h_1, a, b, \tau h_1, \rho\tau]/I.$$

The degrees and coweights of the generators are listed in [Table 2](#), where coweight refers to the difference between stem and weight. The ideal of relations I is described in [Table 3](#).

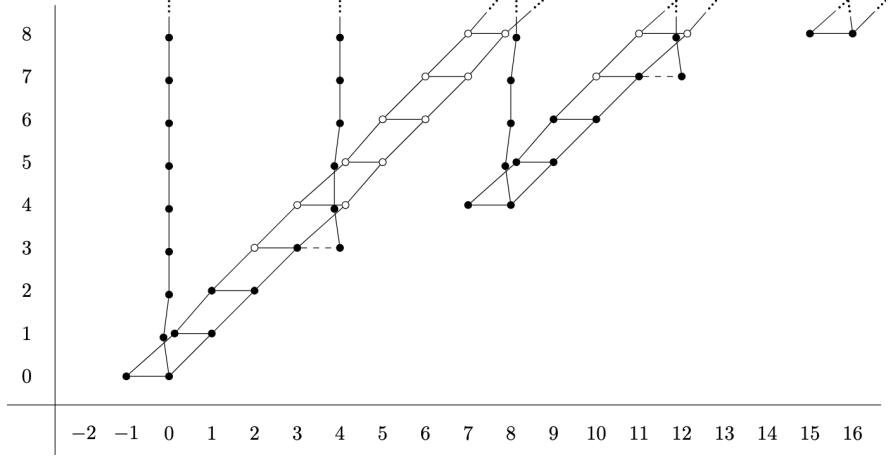
Generator	(s, f, w)	Coweight
ρ	$(-1, 0, -1)$	0
h_0	$(0, 1, 0)$	0
h_1	$(1, 1, 1)$	0
b	$(8, 4, 4)$	4
$\rho\tau$	$(-1, 0, -2)$	1
τh_1	$(1, 1, 0)$	1
τ^2	$(0, 0, -2)$	2
a	$(4, 3, 2)$	2

TABLE 2. Generators for $\mathrm{Ext}_{\mathcal{A}(1)_{\mathbb{F}_3}^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 3(4)$.

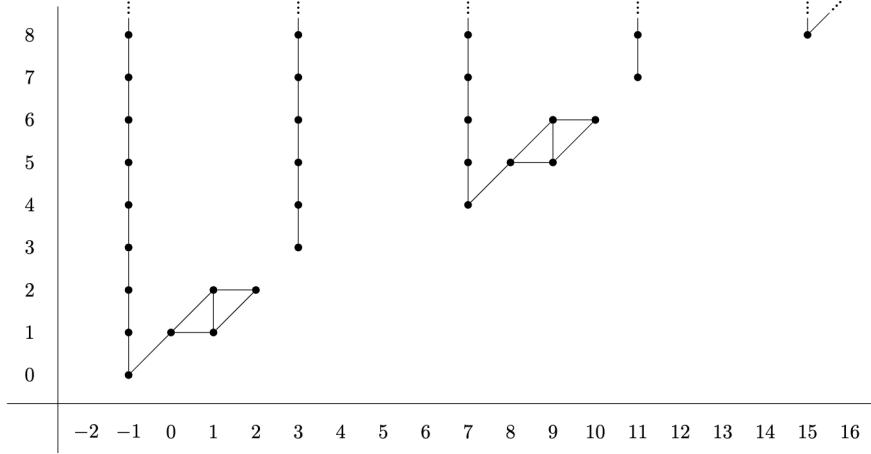
Relation	(s, f, w)	Coweight
ρh_0	$(-1, 1, -1)$	0
$h_0 h_1$	$(1, 2, 1)$	0
ρ^2	$(-2, 0, -2)$	0
$a^2 + h_0^2 b$	$(8, 6, 4)$	4
$h_0 \cdot \tau h_1 + \rho h_1 \cdot \tau h_1$	$(1, 2, 0)$	1
$h_1^2 \cdot \tau h_1$	$(3, 3, 2)$	1
$\rho\tau \cdot h_1^3$	$(2, 3, 1)$	1
$\rho \cdot \tau h_1 + h_1 \cdot \rho\tau$	$(0, 1, -)$	1
$\rho \cdot \rho\tau$	$(-2, 0, -3)$	1
$\tau^2 h_1^3 + \rho a$	$(3, 3, 1)$	2
$h_1 a$	$(5, 4, 3)$	2
$(\tau h_1)^2 = \tau^2 h_1^2$	$(2, 2, 0)$	2
$(\rho\tau)^2$	$(-2, 0, -4)$	2
$\tau h_1 \cdot \rho\tau + \rho\tau^2 h_1$	$(0, 1, -2)$	2
$\tau h_1 \cdot a$	$(5, 4, 2)$	3

TABLE 3. Relations for $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_3})$ for $q \equiv 3(4)$.

To clearly present the data of [Equation \(4.3\)](#), we give individual charts for each coweight modulo 2. Note that the element τ^2 gives a coweight periodicity operator of degree 2. [Figure 10](#) depicts the coweight $cw \equiv 0(2)$ piece. [Figure 11](#) depicts the coweight $cw \equiv 1(2)$ piece. The notation agrees with the notation used earlier in this section.



[FIGURE 10.](#) $\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ in coweight $cw \equiv 0(2)$ for $q \equiv 3(4)$. A \bullet denotes $\mathbb{F}_2[\tau^2]$. An empty \circ denotes \mathbb{F}_2 . A vertical line represents multiplication by h_0 , a horizontal line represents multiplication by ρ , and a diagonal line represents multiplication by h_1 . A dashed line indicates that multiplication by ρ hits a τ^2 -multiple of the indicated generator. For example, we have $\rho a = \tau^2 h_1^3$, which is indicated by the dashed line from $(4, 3)$ to $(3, 3)$.



[FIGURE 11.](#) $\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ in coweights $cw \equiv 1(2)$ for $q \equiv 3(4)$. An empty \circ denotes \mathbb{F}_2 . A vertical line represents multiplication by h_0 , a horizontal line represents multiplication by ρ , and a diagonal line represents multiplication by h_1 .

As in the case of a trivial Bockstein action, we can lift the differentials from the $\text{mASS}^{\mathbb{F}_q}(\mathbb{H}\mathbb{Z})$ to completely determine the behavior of this spectral sequence. The proof follows from the same argument given in [Proposition 3.5](#).

Proposition 4.4. *Let $q \equiv 3(4)$. The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$ are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$.*

As in the previous section, by running the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$ we see that

$$\pi_{s,w}^{\mathbb{F}_q}(kq) \cong \begin{cases} KO_{s-2w}(\mathbb{F}_q) & w \equiv 0(4) \\ KSp_{s-2w}(\mathbb{F}_q) & w \equiv 2(4) \end{cases}$$

for $s \geq 0$, agreeing with Friedlander's calculations given in [Table 1](#).

Remark 4.5. The same argument given in [Remark 3.6](#) shows that there is an isomorphism $\pi_{s,w}^{\mathbb{F}_q}(kq) \cong \pi_{s,w}^{\mathbb{F}_q}(H\mathbb{Z})$ for $s \leq 0$. More generally, we can express the homotopy of kq in terms of $H\mathbb{Z}$:

$$\pi_{**}^{\mathbb{F}_q}(kq) \cong (\pi_{**}^{\mathbb{F}_q}(H\mathbb{Z}))[\eta, \tau^n \eta, \tau^n \eta^2, \alpha, \beta : n \geq 0] / (2\eta, \eta\alpha, \tau^n \eta\alpha, \tau^n \eta^2 \alpha, \alpha^2 - 4\beta).$$

Notice that since τh_1 survives the spectral sequence, the homotopy of kq will have classes of the form $\tau^n \eta$ for all $n \geq 0$, even though $\tau = 0$ on the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$.

4.3. Brown–Gitler comodules. We begin the inductive process of computing the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$. To start, we will compute $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1))$ by the $\mathbf{aAHSS}^{\mathbb{F}_q}(B_0^{\mathbb{F}_q}(1))$. This spectral sequence takes the form

$$E_1^{s,f,w,a} = \mathrm{Ext}_{\mathcal{A}(1)}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\bar{\xi}_1], [\bar{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)}^{s,f,w}(B_0^{\mathbb{F}_q}(1)).$$

We have discussed the differentials in this spectral sequence in [Section 2.4](#). In particular, for $\alpha \in \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$, we have that

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and

$$d_2(\alpha[2]) = h_1\alpha[0].$$

There is a potential third differential between Atiyah-Hirzebruch filtrations 3 and 0, which we saw did not exist for degree reasons in the case of a trivial Bockstein action on \mathbb{F}_q . However, this is not the case here.

Lemma 4.6. *Let $q \equiv 3(4)$. There is a d_3 -differential in the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$:*

$$d_3(\rho[3]) = (\tau h_1)[0].$$

Proof. This differential is given by the Massey product $\langle \rho, h_0, h_1 \rangle[0]$. A simple computation in the cobar complex shows that this is equal to τh_1 with no indeterminacy. \square

Since the differentials in the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ are module maps over $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$, this determines the spectral sequence. Hidden extensions lift as in the previous section. Note that the d_1 and d_2 -differentials are given by multiplication with an element of coweight 0, while the d_3 -differential is given by multiplication with an element of coweight 1.

We now provide charts for the E_∞ -page. [Figure 12](#) depicts the coweight $cw \equiv 0(2)$ piece. [Figure 13](#) depicts the coweight $cw \equiv 1(2)$ piece.

The equivalence of motivic spectra $ksp \simeq H\mathbb{Z}_1^{\mathbb{F}_q} \otimes kq$ implies that the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(ksp)$ is given by $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(1))$. Our computations with the $\mathbf{aAHSS}^{\mathbb{F}_q}(B_0^{\mathbb{F}_q}(1))$ show that

$$\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(1)) \cong \Sigma^{4,2} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q})\langle 1 \rangle \oplus \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}).$$

The same argument as in [Lemma 3.7](#) gives the following lemma.

Lemma 4.7. *Let $q \equiv 3(4)$. In the $\mathbf{mASS}^{\mathbb{F}_q}(ksp)$, the differentials are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$ and the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$.*

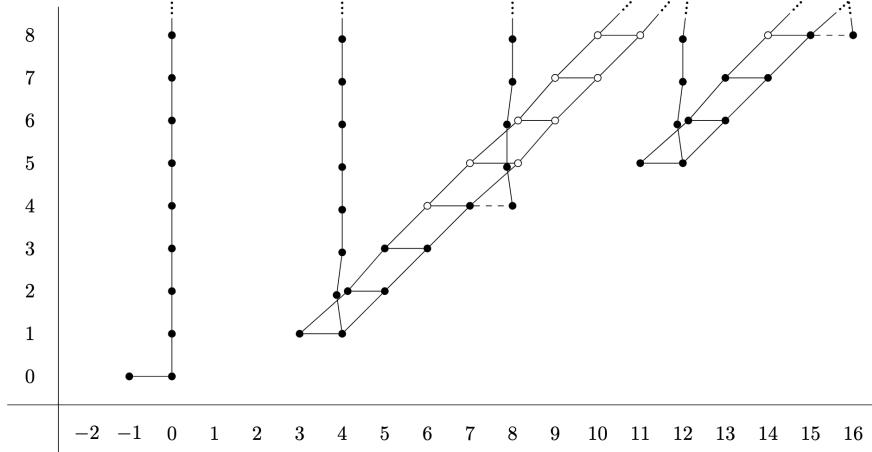


FIGURE 12. The E_∞ -page of the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ in coweight $cw \equiv 0 \pmod{2}$ for $q \equiv 3 \pmod{4}$.

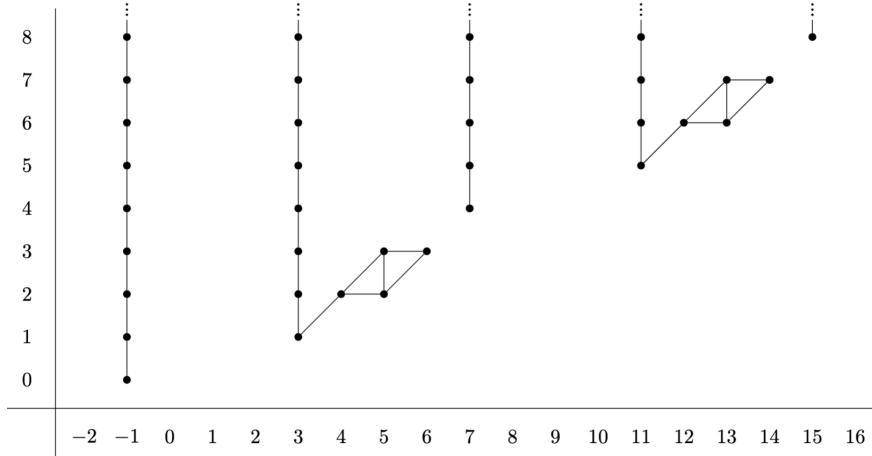


FIGURE 13. The E_∞ -page of the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ in coweight $cw \equiv 1 \pmod{2}$ for $q \equiv 3 \pmod{4}$.

By running the **mASS** $^{\mathbb{F}_q}(\text{ksp})$ and comparing with [Table 1](#), we recover the expected isomorphism for $s \geq 4$

$$\pi_{s,w}^{\mathbb{F}_q}(\text{ksp}) \cong \begin{cases} \text{KSp}_{s-2w}(\mathbb{F}_q) & w \equiv 0 \pmod{4}; \\ \text{KO}_{s-2w}(\mathbb{F}_q) & w \equiv 2 \pmod{4}. \end{cases}$$

We also obtain an isomorphism $\pi_{s,w}^{\mathbb{F}_q}(\text{ksp}) \cong \pi_{s,w}^{\mathbb{F}_q}(\text{kq})$ for $s \leq 0$.

We now compute the group $\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(1)^{\otimes i})$. To do this, we introduce a family of trigraded groups and use the notation described in the paragraphs preceding [Definition 3.11](#).

Definition 4.8. For $i \geq 0$, let $Z_i^{\mathbb{F}_q}$ be the $\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ -module defined as follows.

For $i \equiv 0 \pmod{4}$, let

$$Z_i^{\mathbb{F}_q} = \tau_{\geq 2i}^{h_1} \left(\text{Ext}_{\mathcal{A}(1)^\vee}^{***}(\text{kq}^{\langle i \rangle}) \right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j, 2j} \text{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}).$$

For $i \equiv 1 \pmod{4}$, let

$$Z_i^{\mathbb{F}_q} = \tau_{\geq 2i-2}^{h_1} \left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***} (\mathrm{ksp}^{\langle i-1 \rangle}) \right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j, 2j} \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***} (\mathbb{M}_2^{\mathbb{F}_q}).$$

For $i \equiv 2 \pmod{4}$, let

$$Z_i^{\mathbb{F}_q} = \tau_{\geq 2i}^{h_1} \left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***} (\mathrm{kq}^{\langle i \rangle}) \right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j, 2j} \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***} (\mathbb{M}_2^{\mathbb{F}_q}) \oplus \Sigma^{2i-2, i} \mathbb{F}_2[\tau^2].$$

For $i \equiv 3 \pmod{4}$, let

$$Z_i^{\mathbb{F}_q} = \tau_{\geq 2i-2}^{h_1} \left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***} (\mathrm{ksp}^{\langle i-1 \rangle}) \right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j, 2j} \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***} (\mathbb{M}_2^{\mathbb{F}_q}) \oplus \Sigma^{2i-3, i-2} \mathbb{F}_2[\tau^2, h_1]/(h_1^2).$$

Proposition 4.9. *Let $q \equiv 3 \pmod{4}$. For all $i \geq 0$, there is an isomorphism*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***} (B_0^{\mathbb{F}_q}(1)^{\otimes i})}{v_1\text{-torsion}} \cong Z_i^{\mathbb{F}_q}.$$

Proof. The proof is similar to the proof of [Proposition 3.9](#). The case of $i = 0$ was shown in [Equation \(4.3\)](#), and the case of $i = 1$ was shown in [Lemma 4.7](#). In the general case, consider the algebraic Atiyah-Hirzebruch spectral sequence

$$E_1 = \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s, f, w} (B_0^{\mathbb{F}_q}(1)^{\otimes i}) \otimes \mathbb{M}_2^{\mathbb{F}_q} \{[1], [\bar{\epsilon}_1], [\bar{\tau}_1]\} \implies \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{s, f, w} (B_0^{\mathbb{F}_q}(1)^{\otimes i+1}).$$

The differentials are determined by the cellular filtration on $B_0^{\mathbb{F}_q}(1)$ in the exact same way as before, and hidden extensions follow as in the classical case. The result follows. \square

Before computing the groups $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***} (B_0^{\mathbb{F}_q}(k))$, we make a preliminary calculation.

Lemma 4.10. *Let $q \equiv 3 \pmod{4}$. There is an isomorphism*

$$\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***} (B_1^{\mathbb{F}_q}(k)) \cong \bigoplus_{j=0}^k \Sigma^{4j, 2j} \mathbb{F}_2[\rho, \tau^2, h_0, \rho\tau]/(\rho^2, \rho h_0, \rho(\rho\tau), (\rho\tau)^2).$$

Proof. There is an equivalence of $\mathcal{A}(0)^\vee$ -comodules

$$B_1^{\mathbb{F}_q}(k) \cong \bigoplus_{j=0}^k \Sigma^{4j, 2j} \mathbb{M}_2^{\mathbb{F}_q}.$$

The result now follows from [Equation \(4.1\)](#). Alternatively, one may filter the cobar complex $C_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{F}_q}(k))$ by powers of ρ to yield a ρ -Bockstein spectral sequence computing the Ext group in question. Since $\rho^2 = 0 \in \mathbb{M}_2^{\mathbb{F}_q}$, the only differential is $d_1(\tau) = \rho h_0$, giving the desired structure. \square

Theorem 4.11. *Let $q \equiv 3 \pmod{4}$. There is an isomorphism*

$$\frac{\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***} (B_0^{\mathbb{F}_q}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4, 2k-2} Z_{\alpha(k)}^{\mathbb{F}_q} \oplus \bigoplus_{j=0}^{k-2} \Sigma^{4j, 2j} \mathbb{F}_2[\tau^2, h_0, \rho\tau]/((\rho\tau)^2),$$

where $\alpha(k)$ is the number of 1's in the dyadic expansion of k , and the righthand summand is taken to be empty in the case of $k = 1$.

Proof. The same argument as given in [Theorem 3.10](#) works here as well. To be clear, one uses the short exact sequences of [Proposition 2.6](#) to induce long exact sequences in Ext groups and induction on k , where the case of $k = 1$ was performed in [Lemma 4.7](#). The result follows by noticing that, since we are working modulo v_1 -torsion, the connecting homomorphism is trivial, and the class ρ coming from $\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***} (B_1^{\mathbb{F}_q}(k))$ is v_1 -torsion for degree reasons. \square

4.4. The ring of cooperations. We now assemble our findings to compute $\pi_{**}^{\mathbb{F}_q}(kq \otimes kq)$. First, we describe the E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$.

Corollary 4.12. *Let $q \equiv 3(4)$. The E_2 -page of the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$, modulo v_1 -torsion, is given by*

$$E_2 = \bigoplus_{k \geq 0} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\Sigma^{4k, 2k} B_0^{\mathbb{F}_q}(k)) \cong \bigoplus_{k \geq 0} \Sigma^{4k, 2k} \left(\Sigma^{4k-4, 2k-2} Z_{\alpha(k)}^{\mathbb{F}_q} \oplus \bigoplus_{j=0}^{k-2} \Sigma^{4j, 2j} \mathbb{F}_2[\tau^2, h_0, \rho\tau]/((\rho\tau)^2) \right).$$

Proof. The first equivalence follows from [Theorem 2.10](#), and the second follows from [Theorem 4.11](#). \square

Theorem 4.13. *The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(kq)$.*

Proof. The same proof as was given in [Theorem 3.16](#) works here. On each summand of the E_2 -page, there is a submodule given by an Adams cover of kq or ksp . The differentials on this submodule are determined by the module structure over $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$. On each of the $\mathbb{F}_2[\tau^2, h_0, \rho\tau]/((\rho\tau)^2)$, the differentials are determined by the formulas for the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(H\mathbb{Z})$ in the same way we determined the differentials on the $\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ -summand of the $\mathbf{mASS}^{\mathbb{F}_q}(ksp)$. \square

[Figure 14](#) depicts a chart for the $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$. A \bullet indicates an h_0 -tower. Note that an h_0 -tower supports $\rho\tau$ -multiplication. A \square indicates a class which supports ρ and $\rho\tau$ -multiplication. A filled isosceles triangle indicates a class which supports $\rho, \rho\tau$, and τh_1 -multiplication. Any shaded class is polynomial on τ^2 . As in [Figure 7](#), one may interpret this chart as either the E_2 -page of the E_∞ -page of the spectral sequence since no classes which are divisible by h_1 support a differential.

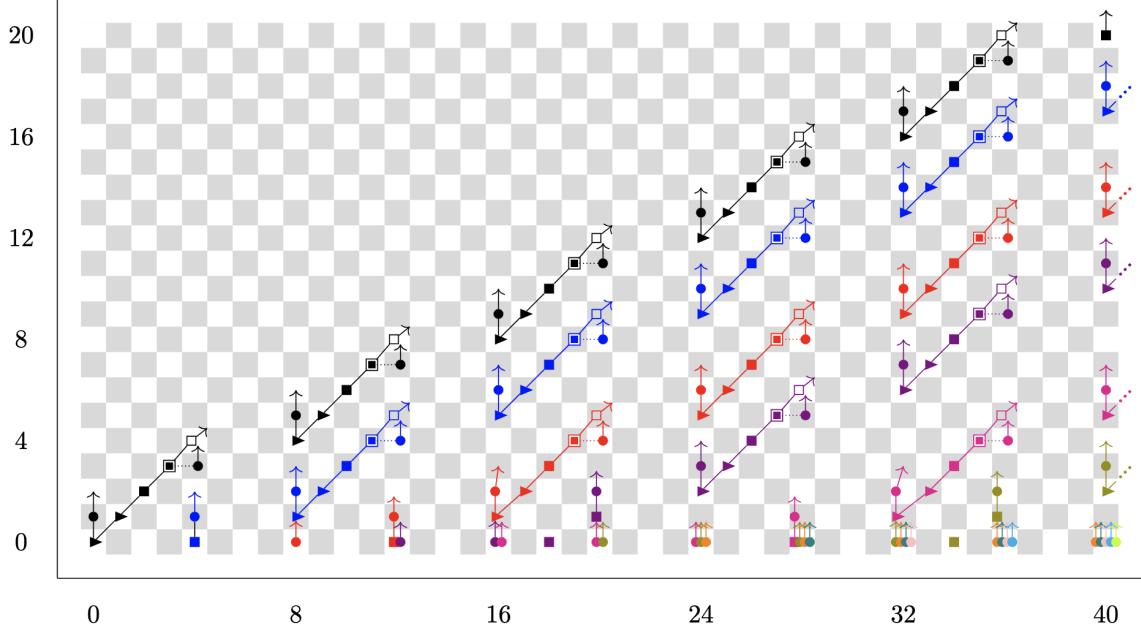


FIGURE 14. The $\mathbf{mASS}^{\mathbb{F}_q}(kq \otimes kq)$ for $q \equiv 3(4)$.

Remark 4.14. We expect that the formula given for $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0(k))$ generalizes well, since the short exact sequence of Brown–Gitler comodules [Proposition 2.6](#) holds over any field with 2 invertible. We also expect that one can produce a spectrum level splitting of $kq \otimes kq$. Such a splitting would need to

incorporate a mix of h_1 -truncations of Adams covers of kq and ksp , and would need to account for the lack of non-nilpotent h_1 -towers in low stem degrees.

Remark 4.15. In future work, and indeed one of our initial reasons for investigating the ring of cooperations over \mathbb{F}_q , we will use the computation of $\pi_{**}^F(kq \otimes kq)$ for $F = \mathbb{C}, \mathbb{R}, \mathbb{F}_3$ to describe $\pi_{**}^{\mathbb{Z}[1/2]}(kq \otimes kq)$ by the methods developed in [BO22].

5. APPLICATION TO THE KQ-RESOLUTION

We conclude by examining the E_1 -page of the kq -resolution. In this section, we do not distinguish between the cases of a trivial or nontrivial Bockstein action on \mathbb{F}_q .

Recall that the kq -resolution for the sphere has signature

$$E_1 = \pi_{s+f,w}^{\mathbb{F}_q}(kq \otimes \overline{kq})^{\otimes f} \implies \pi_{s,w}^{\mathbb{F}_q}\mathbb{S}.$$

We can determine each filtration $f = n$ -line of the E_1 -page by an extension of the techniques used for the computation of $\pi_{**}^{\mathbb{F}_q}(kq \otimes kq)$. For each $n \geq 0$, there is a motivic Adams spectral sequence of the form

$$E_2 = \text{Ext}_{\mathcal{A}^\vee}^{s,f,w}(H_{**}(kq \otimes \overline{kq})^{\otimes n}) \implies \pi_{s,w}^{\mathbb{F}_q}(kq \otimes \overline{kq})^{\otimes n}.$$

We begin by decomposing the E_2 -page.

Lemma 5.1. *There is a Künneth isomorphism*

$$H_{**}(kq \otimes \overline{kq})^{\otimes n} \cong H_{**}(kq) \otimes H_{**}(\overline{kq})^{\otimes n}.$$

Proof. We induct on n . The Künneth spectral sequence takes the form

$$E_2 = \text{Tor}_{\mathcal{M}_2^{\mathbb{F}_q}}^{M_2^{\mathbb{F}_q}}(H_{**}(kq), H_{**}(\overline{kq})) \implies H_{**}(kq \otimes \overline{kq}).$$

Since $H_{**}(kq) = (\mathcal{A} // \mathcal{A}(1))^\vee$ is the $\mathcal{M}_2^{\mathbb{F}_q}$ -linear dual of a free $\mathcal{M}_2^{\mathbb{F}_q}$ -module, it is also free. This implies that the spectral sequence collapses. Now, suppose the result is true for all $i < n$. We again have a Künneth spectral sequence which takes the form

$$E_2 = \text{Tor}_{\mathcal{M}_2^{\mathbb{F}_q}}^{M_2^{\mathbb{F}_q}}(H_{**}(kq \otimes \overline{kq})^{\otimes n-1}, H_{**}(\overline{kq})) \implies H_{**}(kq \otimes \overline{kq})^{\otimes n}.$$

By induction, we have a Künneth isomorphism on the left hand factor. Now, note that $H_{**}(\overline{kq})$ is also free over $\mathcal{M}_2^{\mathbb{F}_q}$, which one can see by the long exact sequence in homology associated to the cofiber sequence

$$\mathbb{S} \rightarrow kq \rightarrow \overline{kq}.$$

Thus the spectral sequence collapses, giving the result. \square

Note that there is an isomorphism of $\mathcal{A}(1)^\vee$ -comodules [Mor25b, Proposition 2.20]

$$(5.2) \quad H_{**}(\overline{kq}) \cong \bigoplus_{k \geq 1} \Sigma^{4k, 2k} B_0^{\mathbb{F}_q}(k).$$

We can now deduce the E_2 -page.

Proposition 5.3. *The E_2 -page of the $\text{mASS}^{\mathbb{F}_q}(kq \otimes \overline{kq})^{\otimes n}$ may be rewritten as*

$$E_2^{s,f,w} \cong \bigoplus_{K \in \mathcal{K}_n} \Sigma^{|K|, 2|K|} \text{Ext}_{\mathcal{A}(1)^\vee}^{s,f,w}(B_0^{\mathbb{F}_q}(K)),$$

where $\mathcal{K}_n = \{K = (k_1, \dots, k_n) : k_j \geq 1 \text{ for all } j\}$, $|K| = \sum_{j=1}^n k_j$, and $B_0^{\mathbb{F}_q}(K) = \bigotimes_{j=1}^n B_0^{\mathbb{F}_q}(k_j)$.

Proof. By Lemma 5.1 and the change of rings isomorphism, we may rewrite the E_2 -page as

$$\mathrm{Ext}_{\mathcal{A}^\vee}^{***}(H_{**}(\mathrm{kq}) \otimes H_{**}(\overline{\mathrm{kq}})^{\otimes n}) \cong \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(H_{**}(\overline{\mathrm{kq}})^{\otimes n}).$$

The identification of Equation (5.2) allows us to rewrite the first factor of $H_{**}(\overline{\mathrm{kq}})$, leaving us with

$$\bigoplus_{k_1 \geq 1} \Sigma^{4k_1, 2k_1} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(k_1) \otimes H_{**}(\mathrm{kq})^{\otimes n-1}).$$

Rewriting the next factor of $H_{**}(\overline{\mathrm{kq}})$ gives us

$$\bigoplus_{k_1 \geq 1} \Sigma^{4k_1, 2k_1} \left(\bigoplus_{k_2 \geq 1} \Sigma^{4k_2, 2k_2} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(k_1) \otimes B_0^{\mathbb{F}_q}(k_2) \otimes H_{**}(\overline{\mathrm{kq}})^{\otimes n-2}) \right),$$

which we may rewrite as

$$\bigoplus_{k_1, k_2 \geq 1} \Sigma^{4(k_1+k_2), 2(k_1+k_2)} \mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(k_1) \otimes B_0^{\mathbb{F}_q}(k_2) \otimes H_{**}(\overline{\mathrm{kq}})^{\otimes n-2}).$$

The result follows by extending these techniques and rewriting all factors of $H_{**}(\overline{\mathrm{kq}})$ in terms of Brown–Gitler comodules. \square

One may easily alter the charts given in Figure 7 and Figure 14 to illustrate the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \overline{\mathrm{kq}})$, i.e. the 1-line of the kq -resolution, by omitting the black $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$ summand originating in stem $s = 0$. For higher Adams filtration, one may compute $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(K))$ by an analysis akin to the one we give in this paper, using a combination of the short exact sequences of Brown–Gitler comodules and algebraic Atiyah–Hirzebruch spectral sequences. In particular, while the Ext group is substantially larger, we can still decompose it, modulo v_1 -torsion, into:

- shifts of h_1 -truncations $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathrm{kq}^{\langle n \rangle})$ and $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathrm{ksp}^{\langle n \rangle})$,
- $\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathrm{HZ})$ -summands, and
- summands of the form $(\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathrm{HZ}))[h_1]/(h_0, h_1^2)$.

Note that the 0-line of the kq -resolution is $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq})$, and so we have described the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ as a module over the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$.

Theorem 5.4. *The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$.*

Proof. In the same way that we were able to determine differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ by using the structure of $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(K))$ as a module over $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$, the map

$$\mathrm{kq} \rightarrow \mathrm{kq} \otimes \mathrm{kq}^{\otimes n} \rightarrow \mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}$$

allows us to determine the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ by using the structure of $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(K))$ as a module over $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$. The result follows. \square

This gives an additive description of the n -line E_1 -page of the kq -resolution, modulo v_1 -torsion, in terms of h_1 -truncated Adams covers of kq and ksp , summands of HZ , and stunted η -towers. Moreover, we have described the module structure of the n -line over the 0-line. We expect these observations to be useful in future work on the kq -resolution.

Remark 5.5. These results are not particular to $\mathrm{SH}(\mathbb{F}_q)$, and they also hold in $\mathrm{SH}(\mathbb{R})$. We will use this in our forthcoming study of the kq -resolution over \mathbb{R} and \mathbb{F}_q [Mor25a]. In another direction, it would be interesting to investigate the the ring of cooperations for kq in $\mathrm{SH}(\mathbb{Q}_p)$.

Remark 5.6. An interesting phenomenon encountered in this paper is that any differential in a motivic Adams spectral sequence we have considered has been determined by a differential in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$. Similar observations were made by the author and Li–Petersen–Tatum in [Li+25] for the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{BPGL}\langle 1 \rangle)$ and the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{BPGL}\langle 1 \rangle \otimes \mathrm{BPGL}\langle 1 \rangle)$, by Ormsby in [Orm11] for the $\mathbf{mASS}^{\mathbb{Q}_p}(\mathrm{BPGL}\langle n \rangle)$, and by Ormsby–Østvær in [OØ13] for the $\mathbf{mASS}^{\mathbb{Q}}(\mathrm{BPGL}\langle n \rangle)$.

Say that a 2-complete motivic spectrum E is fp if its homology $H_{**}(E)$ is finitely-presented as an \mathcal{A}^\vee -comodule. This is a motivic analogue of the notion of fp-spectra introduced by Mahowald–Rezk [MR99]. Examples of motivic fp-spectra include HF_2 , $\mathrm{H}\mathbb{Z}$, $\mathrm{BPGL}\langle n \rangle$, kq , and mmf , the motivic modular forms spectrum. Our observations motivate the following question.

Question 5.7. Let $E \in \mathrm{SH}(F)$ be an fp-spectrum, where F is any field. To what extent are the differentials in the $\mathbf{mASS}^F(E)$ determined by the differentials in the $\mathbf{mASS}^F(\mathrm{H}\mathbb{Z})$?

A similar question was studied by Ormsby–Østvær in the context of the slice spectral sequence [OØ14], where they determined that over fields of low cohomological dimension, there is an isomorphism

$$\pi_{**}^F(\mathrm{BPGL}\langle n \rangle) \cong (\pi_{**}^F(\mathrm{H}\mathbb{Z}))[v_1, \dots, v_n].$$

In particular, this implies that the differentials in the $\mathbf{mASS}^F(\mathrm{BPGL}\langle n \rangle)$ are completely determined by v_n -linearity and the differentials in the $\mathbf{mASS}^F(\mathrm{H}\mathbb{Z})$.

Notice that if an fp-spectrum E is additionally a ring spectrum, then an affirmative answer to the above question implies that there is a large degree of control over the E -based motivic Adams spectral sequence. As a particular topic of investigation, we give the following.

Conjecture 5.1. Over any field F of low cohomological dimension, the E_1 -page of the $\mathrm{BPGL}\langle 2 \rangle$ -motivic Adams spectral sequence can be computed by appropriate lifts of the differentials in the $\mathbf{mASS}^F(\mathrm{H}\mathbb{Z})$. In particular, for $F = \mathbb{C}$ or \mathbb{R} , the $\mathbf{mASS}^F(\mathrm{BPGL}\langle 2 \rangle^{\otimes n})$ collapses on the E_2 -page.

As a final note, let G be a finite cyclic subgroup of the Morava stabilizer subgroup \mathbb{S}_h , $H \subseteq G$ any subgroup, and $\mathrm{BP}^{((G))}\langle m \rangle$ the G -equivariant truncated Brown–Peterson spectra. In recent work, Carrick–Hill [CH25] showed that the spectra $\mathrm{BP}^{((G))}\langle m \rangle^H$ are fp spectra for all $H \subseteq G$ and all $m \geq 0$. We expect an equivariant-motivic analogue, in the sense of [HKO11b], to be true. We will investigate these ideas in future work.

REFERENCES

- [ARØ20] Alexey Ananyevskiy, Oliver Röndigs, and Paul Arne Østvær. “On very effective hermitian K -theory”. In: *Math. Z.* 294.3-4 (2020), pp. 1021–1034.
- [Ban25] Julie Bannwart. *The real Betti realization of motivic Thom spectra and of very effective Hermitian K -theory*. 2025. arXiv: [2505.07297 \[math.AT\]](#).
- [Bea+20] A. Beaudry, M. Behrens, P. Bhattacharya, D. Culver, and Z. Xu. “On the E_2 -term of the bo-Adams spectral sequence”. In: *J. Topol.* 13.1 (2020), pp. 356–415.
- [Beh+19] M. Behrens, K. Ormsby, N. Stapleton, and V. Stojanoska. “On the ring of cooperations for 2-primary connective topological modular forms”. In: *J. Topol.* 12.2 (2019), pp. 577–657.
- [BI22] Eva Belmont and Daniel C. Isaksen. “ \mathbb{R} -motivic stable stems”. In: *J. Topol.* 15.4 (2022), pp. 1755–1793.
- [BIK24] Eva Belmont, Daniel C. Isaksen, and Hana Jia Kong. “ \mathbb{R} -motivic v_1 -periodic homotopy”. In: *Pacific J. Math.* 330.1 (2024), pp. 43–84.
- [BØ22] Tom Bachmann and Paul Arne Østvær. “Topological models for stable motivic invariants of regular number rings”. In: *Forum Math. Sigma* 10 (2022), Paper No. e1, 27.
- [Bou79] A. K. Bousfield. “The localization of spectra with respect to homology”. In: *Topology* 18.4 (1979), pp. 257–281.
- [CH25] Christian Carrick and Michael A. Hill. *On higher real K -theories and finite spectra*. 2025. arXiv: [2507.07051 \[math.KT\]](#).
- [CHN25] Baptiste Calmès, Yonatan Harpaz, and Denis Nardin. *A motivic spectrum representing hermitian K -theory*. 2025. arXiv: [2402.15136 \[math.KT\]](#).

- [CQ21] Dominic Leon Culver and J. D. Quigley. “ kq -resolutions I”. In: *Trans. Amer. Math. Soc.* 374.7 (2021), pp. 4655–4710.
- [DI10] Daniel Dugger and Daniel C. Isaksen. “The motivic Adams spectral sequence”. In: *Geom. Topol.* 14.2 (2010), pp. 967–1014.
- [Fri76] Eric M. Friedlander. “Computations of K -theories of finite fields”. In: *Topology* 15.1 (1976), pp. 87–109.
- [Gon98] Jesús González. “The regular complex in the $\text{BP} < 1 >$ -Adams spectral sequence”. In: *Trans. Amer. Math. Soc.* 350.7 (1998), pp. 2629–2664.
- [Hil11] Michael A. Hill. “Ext and the motivic Steenrod algebra over \mathbb{R} ”. In: *J. Pure Appl. Algebra* 215.5 (2011), pp. 715–727.
- [HK01] Po Hu and Igor Kriz. “Some remarks on Real and algebraic cobordism”. In: *K-Theory* 22.4 (2001), pp. 335–366.
- [HKO11a] P. Hu, I. Kriz, and K. Ormsby. “Convergence of the motivic Adams spectral sequence”. In: *J. K-Theory* 7.3 (2011), pp. 573–596.
- [HKO11b] P. Hu, I. Kriz, and K. Ormsby. “The homotopy limit problem for Hermitian K-theory, equivariant motivic homotopy theory and motivic Real cobordism”. In: *Adv. Math.* 228.1 (2011), pp. 434–480.
- [HKØ17] Marc Hoyois, Shane Kelly, and Paul Arne Østvær. “The motivic Steenrod algebra in positive characteristic”. In: *J. Eur. Math. Soc. (JEMS)* 19.12 (2017), pp. 3813–3849.
- [Hor05] Jens Hornbostel. “ A^1 -representability of Hermitian K -theory and Witt groups”. In: *Topology* 44.3 (2005), pp. 661–687.
- [KQ24] Hana Jia Kong and J. D. Quigley. “The slice spectral sequence for a motivic analogue of the connective $K(1)$ -local sphere”. In: *Proc. Lond. Math. Soc. (3)* 129.5 (2024), Paper No. e70006, 38.
- [Kyl15] Jonas Irgens Kylling. “Hermitian K-theory of finite fields via the motivic Adams spectral sequence”. MA thesis. University of Oslo, 2015.
- [Lev14] Marc Levine. “A comparison of motivic and classical stable homotopy theories”. In: *J. Topol.* 7.2 (2014), pp. 327–362.
- [Li+25] Guchuan Li, Jackson Morris, Sarah Petersen, and Elizabeth Tatum. *Splittings of the motivic Brown-Peterson spectrum $BPL < 1 >$* . In preparation. 2025.
- [LM87] Wolfgang Lellmann and Mark Mahowald. “The bo-Adams spectral sequence”. In: *Trans. Amer. Math. Soc.* 300.2 (1987), pp. 593–623.
- [Mah81] Mark Mahowald. “bo-resolutions”. In: *Pacific J. Math.* 92.2 (1981), pp. 365–383.
- [Mor04] Fabien Morel. “On the motivic π_0 of the sphere spectrum”. In: *Axiomatic, enriched and motivic homotopy theory*. Vol. 131. NATO Sci. Ser. II Math. Phys. Chem. Kluwer Acad. Publ., Dordrecht, 2004, pp. 219–260.
- [Mor25a] Jackson Morris. *kq -resolutions II*. In preparation. 2025.
- [Mor25b] Jackson Morris. *On the ring of cooperations for real Hermitian K-theory*. 2025. arXiv: [2506.16672 \[math.AT\]](https://arxiv.org/abs/2506.16672).
- [MR99] Mark Mahowald and Charles Rezk. “Brown-Comenetz duality and the Adams spectral sequence”. In: *Amer. J. Math.* 121.6 (1999), pp. 1153–1177.
- [OØ13] Kyle M. Ormsby and Paul Arne Østvær. “Motivic Brown-Peterson invariants of the rationals”. In: *Geom. Topol.* 17.3 (2013), pp. 1671–1706.
- [OØ14] Kyle M. Ormsby and Paul Arne Østvær. “Stable motivic π_1 of low-dimensional fields”. In: *Adv. Math.* 265 (2014), pp. 97–131.
- [Orm11] Kyle M. Ormsby. “Motivic invariants of p -adic fields”. In: *J. K-Theory* 7.3 (2011), pp. 597–618.
- [Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*. Vol. 121. Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, 1986, pp. xx+413.
- [RSØ19] Oliver Röndigs, Markus Spitzweck, and Paul Arne Østvær. “The first stable homotopy groups of motivic spheres”. In: *Ann. of Math. (2)* 189.1 (2019), pp. 1–74.
- [Sou79] C. Soulé. “ K -théorie des anneaux d’entiers de corps de nombres et cohomologie étale”. In: *Invent. Math.* 55.3 (1979), pp. 251–295.
- [Spi18] Markus Spitzweck. “A commutative \mathbb{P}^1 -spectrum representing motivic cohomology over Dedekind domains”. In: *Mém. Soc. Math. Fr. (N.S.)* 157 (2018), p. 110.
- [Voe03] Vladimir Voevodsky. “Motivic cohomology with $\mathbb{Z}/2$ -coefficients”. In: *Publ. Math. Inst. Hautes Études Sci.* 98 (2003), pp. 59–104.
- [Win24] Cameron Winter. *Heavy Metal*. 2024.

- [WØ17] Glen Matthew Wilson and Paul Arne Østvær. “Two-complete stable motivic stems over finite fields”. In: *Algebr. Geom. Topol.* 17.2 (2017), pp. 1059–1104.

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