Riddler 1-7-22

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Riddler Classic

Problem. Amare the ant is traveling within triangle △ABC, shown in fig 1. Angle ∠A measures 15 degrees, and sides AB and AC both have length 1.

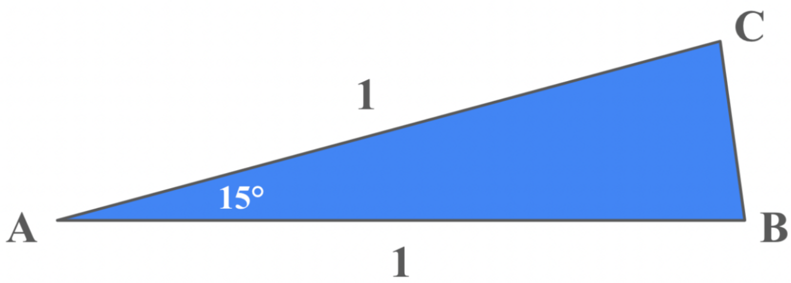

Figure 1. Amare the ant will travel across triangle △ABC.

Amare starts at point B and wants to arrive on side AC, but the colony's queen requires that Amare's path follow several constraints:

(1) Start at point B.
(2) Touch any point on side AC.
(3) Touch any point on side AB.
(4) Reach any point on side AC.

What is the shortest distance Amare can travel while following the queen's path?

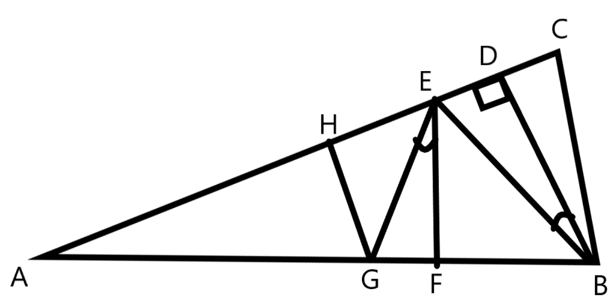

Figure 2. Amare the ant will travel across triangle △ABC on a path from points B to E to G to H.

Solution. We will add several points to triangle △ABC to describe the path that Amare will take.

Amare could take any straight line path to side AC, but going towards point C takes Amare farther from side AB. Traveling directly to side AC would minimize that portion of the trip, but Amare can shorten later sections by traveling to a point E that lies closer to A. Therefore, we define the segment perpendicular to AC and passing through B with the point D. Amare travels to the point E on AD, and we define angle $\angle\text{DBE} = \theta$.

Similar logic applies to suggest that Amare could travel directly to side AB from point E, but should aim for a point G closer to A. As in the previous step, we can define the point F on AB via the segment perpendicular to AB that intersects E, and we define angle $\angle\text{FEG} = \phi$.

Finally, Amare should travel to AC directly, and so we define the point H on AC via the segment perpendicular to AC that intersects G.

With the previous construction, shown in Fig. 2, Amare will take a partwise straight-line path from points B to E to G to H. We need to analyze the situation to find the values of θ and ϕ that will minimize Amare's distance traveled.

The distance traveled is given by $\overline{BE} + \overline{EG} + \overline{GH}$. We will develop expressions for each length in turn.

We begin by looking at the right triangle $\triangle\text{ABD}$. Since $\angle\text{CAB} = 15°$ and $\overline{AB} = 1$, we have $\overline{BD} = \sin(15°)$. We can look at the right triangle $\triangle\text{BDE}$,

$$\begin{aligned}\overline{BE} &= \frac{\overline{BD}}{\cos\theta} \\ &= \frac{\sin(15°)}{\cos\theta}.\end{aligned}$$

We can then look at the right triangle $\triangle\text{BFE}$. Since $\angle\text{ABD} = 75°$, we have $\angle\text{EBF} = 75° - \theta$. By the right angle, $\overline{EF} = \overline{BE}\sin(\angle\text{EBF})$. Turning to the right triangle $\triangle\text{FEG}$, we can determine $\overline{EG}$,

$$\begin{aligned}\overline{EG} &= \frac{\overline{EF}}{\cos\phi} \\ &= \frac{\sin(75° - \theta)}{\cos\phi}\frac{\sin 15°}{\cos\theta}.\end{aligned}$$

We note that $\angle\text{AEF} = 75°$, implying $\angle\text{HEG} = \angle\text{AEG} = 75° - \phi$. Looking at the right triangle $\triangle\text{EHG}$, we can express $\overline{GH}$,

$$\begin{aligned}\overline{GH} &= \overline{EG}\sin(\angle\text{GEH}) \\ &= \sin(75° - \phi)\frac{\sin(75° - \theta)}{\cos\phi}\frac{\sin 15°}{\cos\theta}.\end{aligned}$$

Putting everything together, and noting $\sin(75° - x) = \cos(15° + x)$, we can write the total distance traveled by Amare as

$$d(\theta,\phi) = \frac{\cos(15° + \phi)}{\cos\phi}\frac{\cos(15° + \theta)}{\cos\theta}\sin 15° + \frac{\sin(15°)}{\cos\phi}\frac{\cos(15° + \theta)}{\cos\theta} + \frac{\sin(15°)}{\cos\theta}.$$

We recall from their definition that the appropriate domain for θ and ϕ are $[0, 75°]$. At the extremes, we have $d(75°, \phi) = 1$ for any ϕ, and $d(0, 0) = \sin(15°)(1 + \cos(15°) + \cos^2(15°))$.

Using some calculus, we could find the minimum of the distance function over the domain. WolframAlpha gives a local minimum at $(\theta, \phi) = (\pi/6, \pi/12) = (30°, 15°)$ with minimum distance $1/\sqrt{2}$.

This minimum happens to be the side length of an isosceles right triangle with hypotenuse 1. This value is not a coincidence, as there is a geometric proof involving such a triangle that simplifies the solution for this problem.