HW 8

① $Y_{ij} = \beta_1 + \beta_2 t_{ij} + b_{1i} + b_{2i} t_{ij} + \epsilon_{ij}$

$i = 1 \dots N \qquad j = 1 \dots n_i$

$\epsilon_{ij} \sim N(0, \sigma^2_\epsilon)$

Var-Cov Mat.

$Var(b_{1i})$

$$\begin{bmatrix} b_{1i} \\ b_{2i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \right)$$

$Cov(b_{1i}, b_{2i}) \qquad Var(b_{2i})$

A) $Var(Y_{ij}) = ?$

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

$$Var\left[\beta_1 + \beta_2 t_{ij} + b_{1i} + b_{2i} t_{ij} + \epsilon_{ij}\right]$$

$$= \cancelto{0}{Var[\beta_1]} + \cancelto{0}{Var[\beta_2 t_{ij}]} + Var[b_{1i} + b_{2i} t_{ij}] + Var[\epsilon_{ij}]$$

$$= Var[b_{1i} + b_{2i}] = Var[b_{1i}] + Var[b_{2i} t_{ij}] + 2\,Cov[b_{1i}, b_{2i} t_{ij}] + \sigma^2_\epsilon$$

$$= g_{11} + g_{22} t_{ij}^2 + 2 t_{ij} g_{12} + \sigma^2_\epsilon$$

B) Expression for $Cov[Y_{ij}, Y_{ik}]$

$$Cov(Y_{ik}, Y_{im}) = E[(Y_{ik} - E[Y_{ik}])(Y_{im} - E[Y_{im}])]$$

$$\rightarrow E[(b_{1i} + b_{2i}t_{ij} + \epsilon_{ij})(b_{1i} + b_{2i}t_{ik} + \epsilon_{ik})]$$

$$\rightarrow E[b_{1i}^2 + b_{1i}b_{2i}t_{ik} + \cancel{b_{1i}\epsilon_{ik}} + b_{1i}b_{2i}t_{ij} + b_{2i}^2 t_{ij}t_{ik} + \cancel{b_{2i}t_{ij}\epsilon_{ik}} +$$

$$\cancel{b_{1i}\epsilon_{ij}} + \cancel{b_{2i}t_{ik}\epsilon_{ij}} + \cancel{\epsilon_{ij}}\epsilon_{ik}]$$ because $E[\epsilon] = \emptyset$

$E(XY) = Cov(X, Y) = g_{12}$

$E(X^2) = Var(X) + [E(X)]^2$

$$\rightarrow E[b_{1i}^2 + b_{1i}b_{2i}t_{ik} + b_{1i}b_{2i}t_{ij} + b_{2i}^2 t_{ij}t_{ik}]$$

$$\rightarrow E[b_{1i}^2] + t_{ik}\underbrace{E[b_{1i}b_{2i}]} + t_{ij}\underbrace{E[b_{1i}b_{2i}]} + t_{ij}t_{ik}E[b_{2i}^2]$$

$$Cov(b_1, b_2) = E[b_1 b_2] - \cancel{E(b_1)E(b_2)}$$

0

$$Cov(b_1, b_2) = E[b_1 b_2]$$

$$g_{12} = E[b_{1i} b_{2i}]$$

$$\rightarrow E[b_{1i}^2] + t_{ik}g_{12} + t_{ij}g_{12} + t_{ij}t_{ik}E[b_{2i}^2]$$

b/c $E[b]=0 \rightarrow E[b_{1i}^2] = Var(b_{1i})$

$\rightarrow Var(b_{1i}) + Var(b_{2i}) + t_{ik} g_{12} + t_{ij} g_{12}$

$\rightarrow g_{11} + t_{ij} t_{ik} g_{22} + g_{12} [t_{ik} + t_{ij}]$

c)

(c) If

$$\begin{bmatrix} b_{1i} \\ b_{2i} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right)$$

what is the value of σ_e^2 if correlation between responses at time t=0 and time t=2 is equal to 0.5, i.e. $Corr(Y_{ij}, Y_{ik}) = 0.5$ for $t_{ij} = 0$ and $t_{ik} = 2$.

$$Corr(Y_{ij}, Y_{ik}) = \frac{Cov(Y_{ij}, Y_{ik})}{\sigma_{ij} \cdot \sigma_{ik}}$$

$$\rightarrow \frac{g_{11} + t_{ij} t_{ik} g_{22} + g_{12}[t_{ik} + t_{ij}]}{\sqrt{g_{11} + g_{22} t_{ij}^2 + 2t_{ij} g_{12} + \sigma_e^2} \cdot \sqrt{g_{11} + g_{22} t_{ik}^2 + 2t_{ik} g_{12} + \sigma_e^2}}$$

$t_{ij} = 0$
$t_{ik} = 2$
$Corr = .5$

$g_{11} = 1$
$g_{22} = 1$
$g_{12} = .5$

$$0.5 = \frac{g_{11} + 0 \cdot 2 \cdot g_{22} + g_{12}[2+0]}{\sqrt{g_{11} + g_{22} 0^2 + 2 \cdot 0 \cdot g_{12} + \sigma_e^2} \cdot \sqrt{g_{11} + g_{22} 2^2 + 2 \cdot 2 \cdot g_{12} + \sigma_e^2}}$$

$$0.5 = \frac{g_{11} + 2g_{12}}{\ldots}$$

$$\sqrt{g_{11}+\sigma^2_\epsilon} \cdot \sqrt{g_{11}+4g_{22}+4g_{12}+\sigma^2_\epsilon}$$

$$0.5 = \frac{1+2\cdot .5}{\sqrt{1+\sigma^2_\epsilon} \cdot \sqrt{1+4+2+\sigma^2_\epsilon}} \rightarrow \frac{2}{\sqrt{1+\sigma^2_\epsilon} \cdot \sqrt{7+\sigma^2_\epsilon}}$$

$$0.5 = \frac{2}{\sqrt{7+8\sigma^2_\epsilon+(\sigma^2_\epsilon)^2}}$$

$\sigma^2_\epsilon = \phi$

$$0.5 = \frac{2}{\sqrt{\phi^2+8\phi+7}}$$

$$\rightarrow \sqrt{\phi^2+8\phi+7} = 4 \rightarrow \phi^2+8\phi+7=16$$

$$\rightarrow \phi^2+8\phi-9=0 \qquad \frac{-8\pm\sqrt{64-4\cdot 1\cdot(-9)}}{2}$$

$$\frac{-8\pm\sqrt{64+36}}{2} = \frac{-8\pm 10}{2}$$

$$= 1, -9$$

$\sigma^2_\epsilon = 1$