② $X=x \mid \theta \sim \text{Uniform}(0,\theta)$ data

$\theta \sim \text{Pareto}(\alpha, \beta)$ prior

$p(\theta) = \frac{\alpha\beta^{\alpha}}{\theta^{\alpha+1}} I_{(\beta,\infty)}(\theta)$. Write out likelihood $p(X=x \mid \theta)$. Then calculate the posterior distribution of $\theta \mid X=x$.

$$\ell(\theta) = \frac{1}{\theta} I(0 < x_{1:N}) \cdot I(x_{N:N} < \theta)$$

$$p(\theta \mid x) \propto \frac{1}{\theta} I_{(0,\theta)}(x) \cdot \frac{\alpha\beta^{\alpha}}{\theta^{\alpha+1}} I_{(\beta,\infty)}(\theta)$$

$$\rightarrow p(\theta \mid x) \propto \frac{\alpha\beta^{\alpha}}{\theta^{\alpha+2}} I_{(0,\theta)}(x) \cdot I_{(\beta,\infty)}(\theta)$$

$$\rightarrow p(\theta \mid x) \propto \frac{\alpha\beta^{\alpha}}{\theta^{\alpha+2}} I_{(x,\infty)}(\theta) \cdot I_{(\beta,\infty)}(\theta)$$

$$I_{(0,\theta)}(x) = \begin{cases} 1 & 0<x<\theta \\ 0 & \text{o/w} \end{cases}$$

$\downarrow$

$$I_{(x,\infty)}(\theta) = \begin{cases} 1 & x<\theta<\infty \\ 0 & \text{o/w} \end{cases}$$

$$\rightarrow \theta \mid X=x \sim \text{pareto}(\alpha+1, \max(x,\beta))$$

③ Ⓐ Find Bayes estimator when the

Loss function is:

$$L(\theta, \delta(x)) = c(\theta - \delta(x))^2$$, where $c > 0$ is a constant

$$p(\theta, \delta(x)) = E[L(\theta, \delta(x)) \mid X_{1:N}]$$

$$= E[(c(\theta - \delta(x))^2) \mid X]$$

$$= E[c(\theta^2 - 2\theta\delta(x) + \delta^2(x)) \mid X]$$

X is known

$$\Rightarrow E[c\theta^2 - 2c\theta\delta(x) + c\delta^2(x) \mid X]$$

$$\Rightarrow c E_\theta(\theta^2 \mid x) - E_\theta(\theta \mid x) \cdot 2c\delta(x) + c\delta^2(x)$$

Now minimize with respect to $\delta(x)$

$$\frac{\partial[p(\theta, \delta(x))]}{\partial \delta(x)}$$

$$= \frac{\partial[E(\theta^2 \mid x) - E(\theta \mid x) \cdot 2c\delta(x) + c\delta^2(x)]}{\partial \delta(x)}$$

$\rightarrow \quad 0 - E(\theta|x)\cdot 2c + 2c\delta(x)$

$\rightarrow -2cE(\theta|x) = -2c\delta(x)$

$E(\theta|x) = \hat{\delta}(x)$

To show convexity:

$\frac{\partial}{\partial\delta(x)} -E(\theta|x)\cdot 2c + 2c\delta(x)$

$\rightarrow \quad 2c \qquad\qquad 2c \geq 0$

(B) Derive the Bayes estimator when:

$L(\theta, \delta(x)) = w(\theta)(g(\theta) - \delta(x))^2$

w/o integrals

$\rho(\theta, \delta(x))$

$\rho(\theta, \delta(x)) = E[\ell(\theta, \delta(x)) | X_{1:N}]$

$$E[w(\theta)(g(\theta) - \delta(x))^2 \mid x]$$

$$E[w(\theta)[g^2(\theta) - 2g(\theta)\delta(x) + \delta^2(x)] \mid x]$$

$$E[w(\theta) \mid x]\{E[g^2(\theta) \mid x] - E[g(\theta) \mid x] \cdot \delta(x)] + \delta^2(x)\}$$

Now minimize with respect to $\delta(x)$

$$\frac{\partial[\rho(\theta, \delta(x))]}{\partial \delta(x)} =$$

$$\frac{\partial[E[w(\theta) \mid x]\{E[g^2(\theta) \mid x] - 2E[g(\theta) \mid x] \cdot \delta(x)] + \delta^2(x)\}]}{\partial \delta(x)}$$

$$\frac{\partial[\cancelto{0}{E[w(\theta)g^2(\theta) \mid x]} - 2E[w(\theta)g(\theta) \mid x \ \delta(x)] + E[w(\theta) \mid x \cdot \delta^2(x)]]}{\partial \delta(x)}$$

$$\rightarrow -2E[w(\theta)g(\theta)|x] + 2E[w(\theta)|x]\,\delta(x)$$

☆ Solve for $\delta(x)$

$$\cancel{2}E[w(\theta)g(\theta)|x] = \cancel{2}E[w(\theta)|x]\,\delta(x)$$

$$\delta(x) = \frac{E[w(\theta)g(\theta)|x]}{E[w(\theta)|x]}$$

☆ For convexity ☆

$$\frac{\partial}{\partial \delta(x)}\, 2E[w(\theta)|x]\,\delta(x) - 2E[w(\theta)g(\theta)|x]$$

$$= 2\,w(\theta)$$ is positive

Yay!