Dial-Jackson-homework2

Jackson Dial

2023-01-15

Q1

A

Task 3

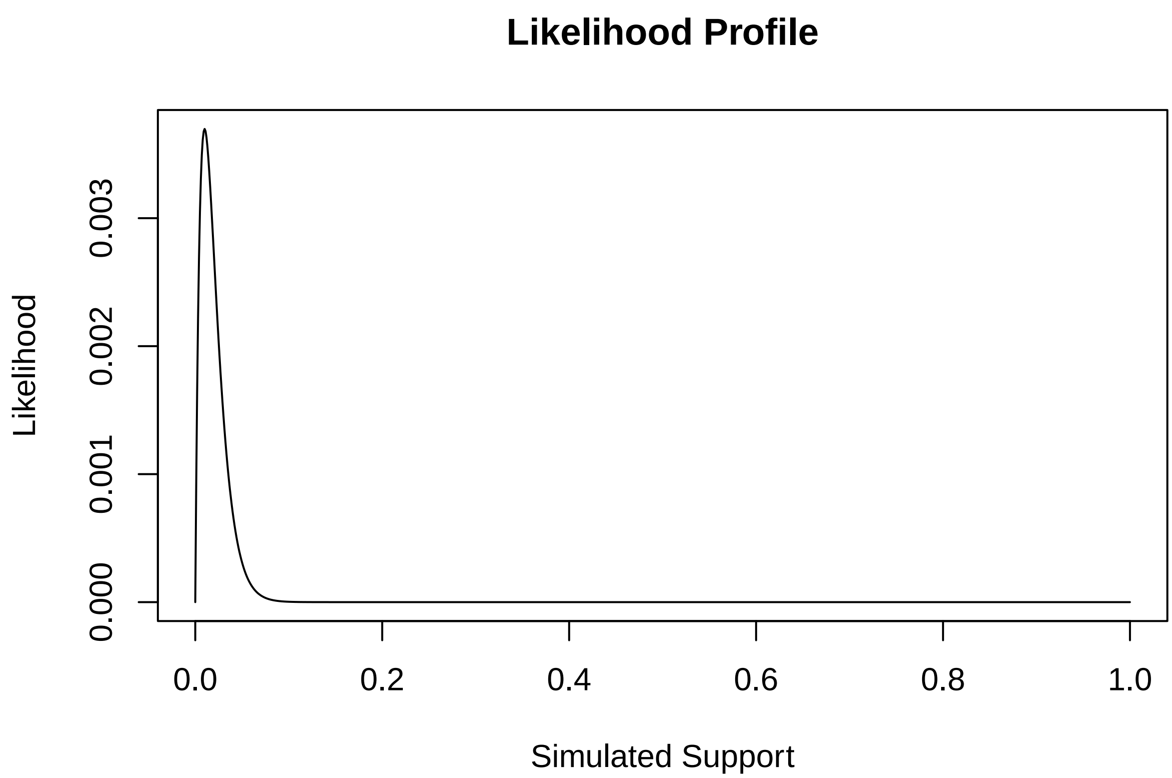
```
obs_data <- rbinom(n = 100, size = 1, prob = .01)

myBernLH <- function(obs_data, theta){
  N <- length(obs_data)
  x <- sum(obs_data)
  LH <- (theta^x)* ((1-theta)^(N-x))
  return(LH)
}

#create grid
theta_sim <- seq(0, 1, length=1000)
#store the LH values
sim_lh <- myBernLH(obs_data, theta_sim)
#create the plot
head(theta_sim)
```

```
## [1] 0.000000000 0.001001001 0.002002002 0.003003003 0.004004004 0.005005005
```

```
tail(sim_lh)
```

```
## [1] 1.733277e-228 4.417875e-238 1.891101e-250 6.984188e-268 1.103016e-297
## [6]  0.000000e+00
```

```
plot(theta_sim, sim_lh, type = "l", main = "Likelihood Profile", xlab = "Simulated Support", ylab = "Li
```



B

Task 4

```r
set.seed(1234)

obs_data <- rbinom(n = 100, size = 1, prob = .01)

BetaBernMod <- function(a,b,data){
  sum_data <- sum(data)
  n <- length(data)
  a1 <- sum_data + a
  b1 <- n - sum_data + b
  return (c(a1, b1))
}

BetaBernMod(1,1,obs_data)
```

```
## [1]   2 100
```

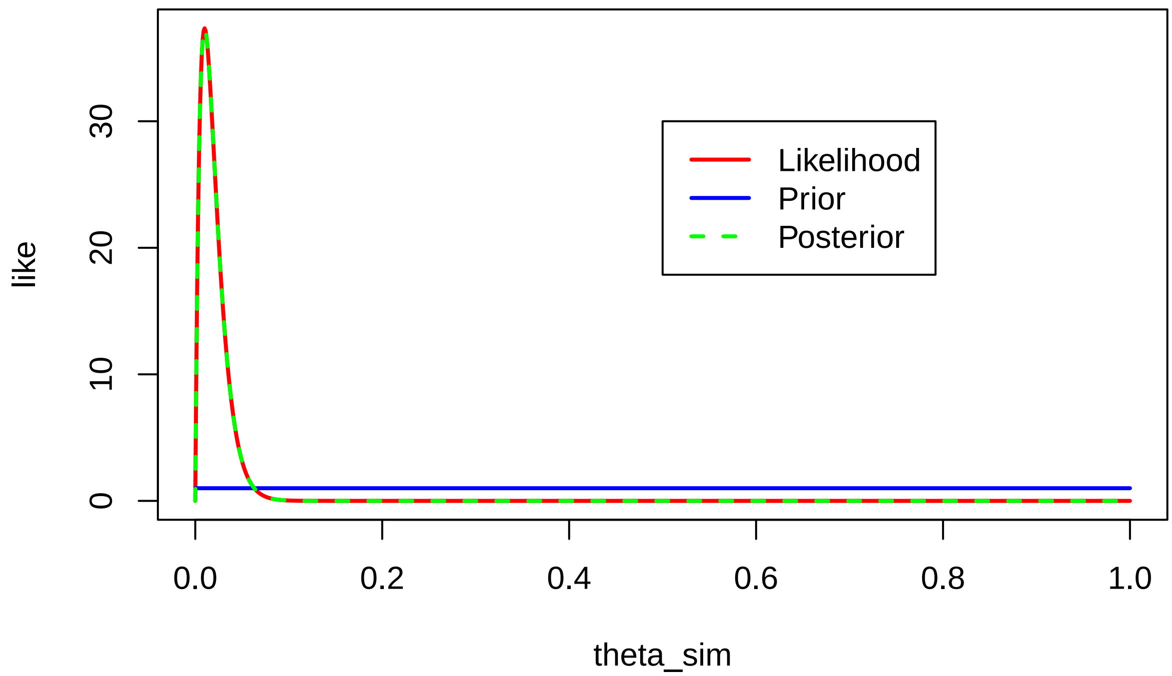
```r
BetaBernMod(3,1,obs_data)
```

```
## [1]   4 100
```

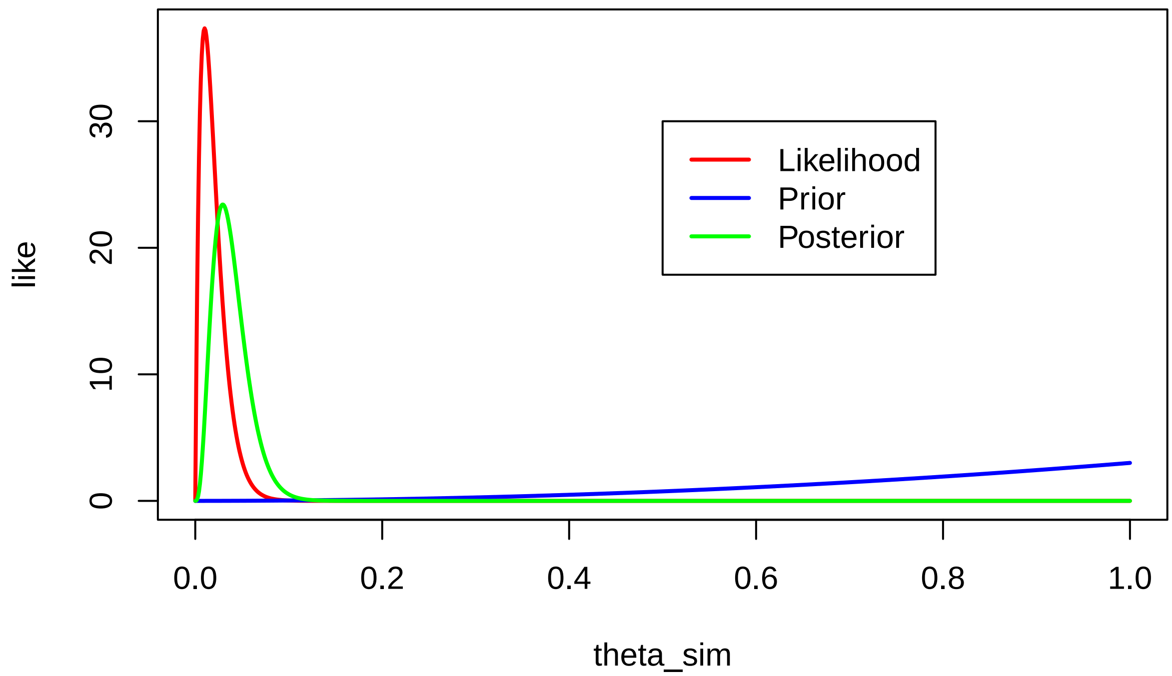
C

Task 5

```r
set.seed(1234)
obs_data <- rbinom(n = 100, size = 1, prob = .01)
theta_sim <- seq(0, 1, length=1000)

n <- length(obs_data)
x <- sum(obs_data)
like <- dbeta(theta_sim, x +1, n-x+1)

#make the non informative distributions
parameters_non <- BetaBernMod(1,1,obs_data)
non_prior <- dbeta(theta_sim, 1,1)
non_post <- dbeta(theta_sim, parameters_non[1], parameters_non[2])

#make the informative distributions
parameters_info <- BetaBernMod(3,1,obs_data)
info_prior <- dbeta(theta_sim, 3,1)
info_post <- dbeta(theta_sim, parameters_info[1], parameters_info[2])

#plot all 3 with the non informative prior
plot(theta_sim, like, type = "l", col = "red", lwd = 2)
lines(theta_sim, non_prior, type = "l", lwd = 2, col = "blue")
lines(theta_sim, non_post, type = "l", lty = 2, lwd = 2, col = "green")
legend(.5, 30, legend = c("Likelihood", "Prior", "Posterior"), col = c("red", "blue", "green"), lty = c
```




```
#plot them all together with the informative prior
plot(theta_sim, like, type = "l", col = "red", lwd = 2)
lines(theta_sim, info_prior, type = "l", lwd = 2, col = "blue")
lines(theta_sim, info_post, type = "l", lwd = 2, col = "green")
legend(.5, 30, legend = c("Likelihood", "Prior", "Posterior"), col = c("red", "blue", "green"), lty = c
```


Likelihood
Prior
Posterior
like
theta_sim
0
10
20
30
0.0
0.2
0.4
0.6
0.8
1.0

Q2

A

9:18 PM Thu Jan 19 ... 72%

Q2 A

prior:

$$p(\theta) = \text{Gamma}(\theta \mid A, b) = \frac{b^A}{\Gamma(A)} \theta^{A-1} \exp(-b\theta)$$

data:

$$p(x \mid \theta) = \text{Exp}(x \mid \theta) = \theta \exp(-\theta x)$$

$$P(\theta \mid x_{1:N}) \underset{\theta}{\propto} \underbrace{P(x_{1:N} \mid \theta)}_{\text{likelihood}} \underbrace{P(\theta)}_{\text{prior}}$$

goal: Find posterior $P(\theta \mid x_{1:N})$

$$l(\theta) = \prod_{i=1}^{N} \theta \exp(-\theta x_i) = P(x_{1:N} \mid \theta)$$

$$= \theta^N \exp\left(-\theta \sum_{i=1}^{N} x_i\right) = \text{likelihood}$$

prior

$$\left[\theta^N \exp\left(-\theta \sum_{i=1}^{N} x_i\right)\right] \cdot \left[\frac{b^A}{\Gamma(A)} \theta^{A-1} \exp(-b\theta)\right]$$

$$\rightarrow \theta^{N+A-1} \cdot \exp\left(-\theta \sum_{i=1}^{N} x_i - b\theta\right) \cdot \frac{b^A}{\Gamma(A)}$$

$$\rightarrow \theta^{N+A-1} \cdot \exp\left(\theta\left(\sum_{i=1}^{N} x_i - b\right)\right) \cdot \frac{b^A}{\Gamma(A)}$$

$$p(\theta \mid x_{1:N}) = \theta^{N+A-1} \cdot \exp\left(\theta\left(\sum_{i=1}^{N} x_i - b\right)\right) \cdot \frac{b^A}{\Gamma(A)}$$

★ looks like Gamma ★

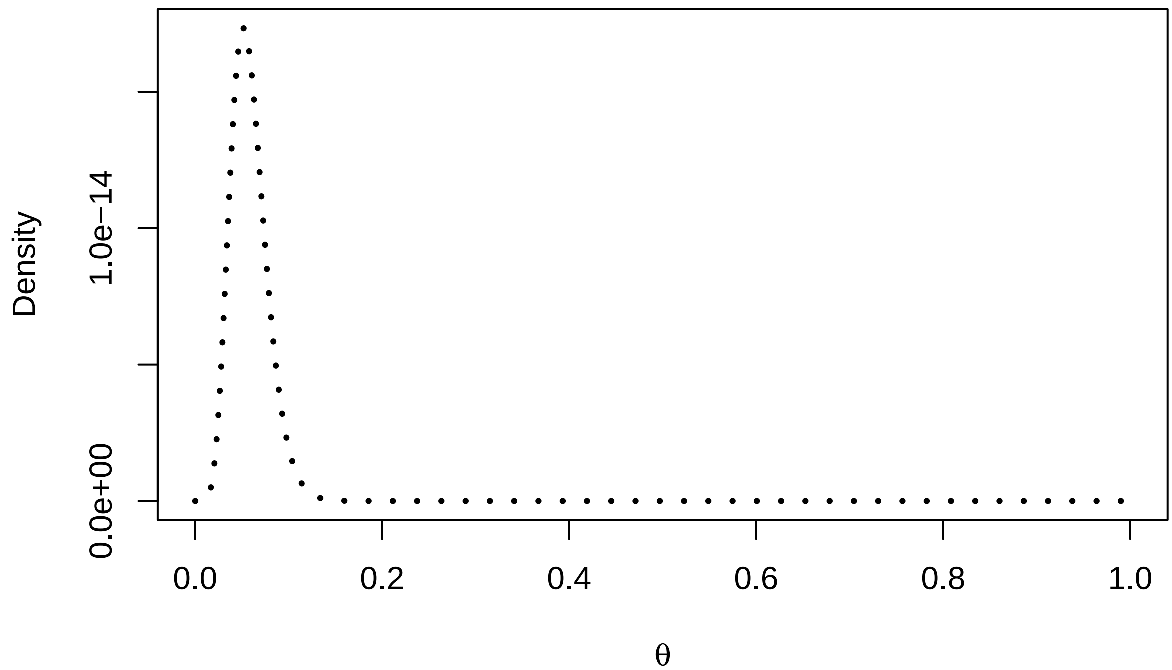
$$\theta \mid x_{1:N} \sim \text{Gamma}\left(\theta \mid N+A, \sum_{i=1}^{N} x_i - b\right)$$

B

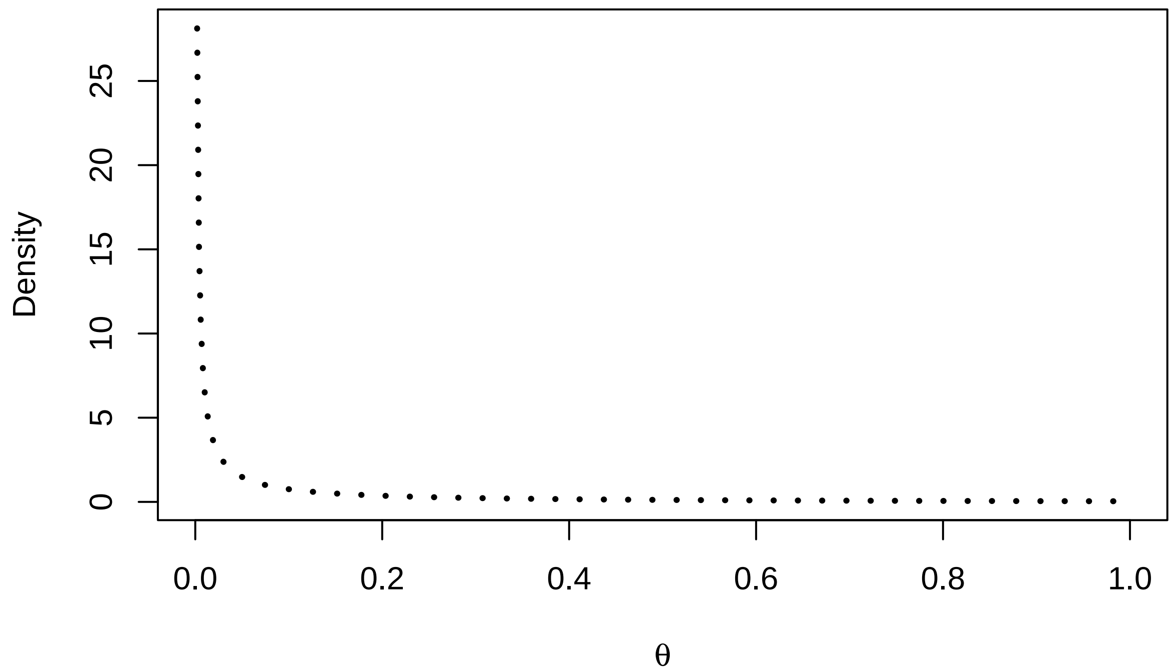
The posterior is a proper pdf because it has the form of a known distribution, the Gamma distribution.

C

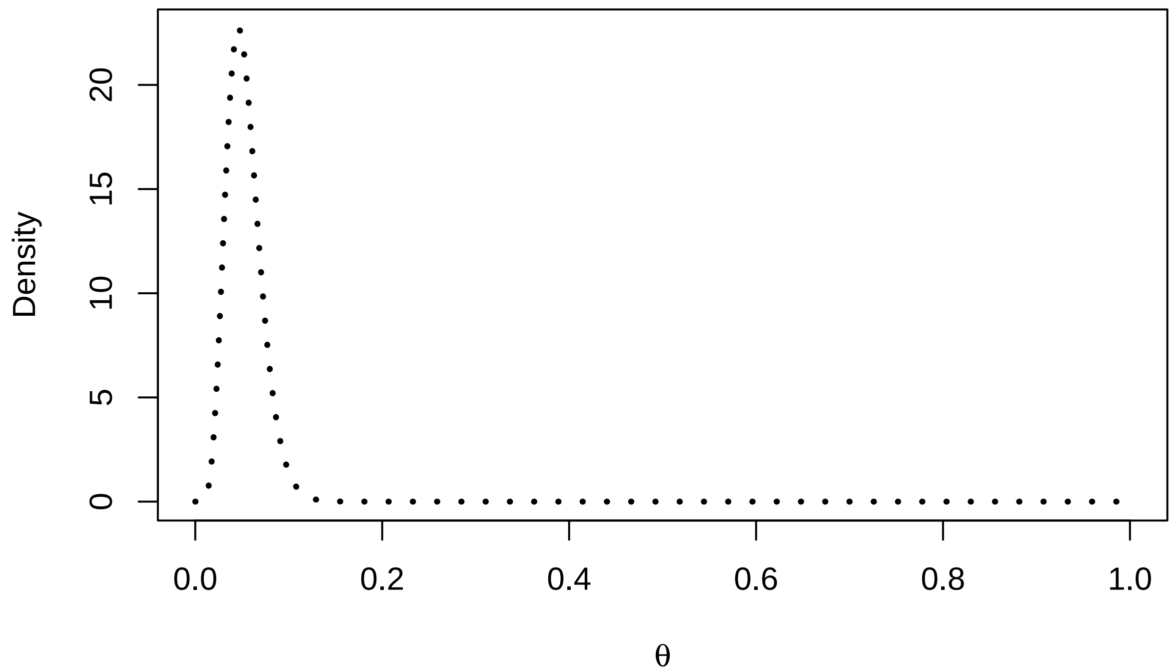
```
#sum of all data = 154.5
n <- 8

a <- .1
b <- 1
th <- seq(0, 1, length = 500)

x <- 154.5 #not sure if this is actually correct

like <- (th ^ n) * exp(-th * x)
prior <- dgamma(th, a, b)
post <- dgamma(th, n + a, x - b)

#plot the likelihood function
plot(
  th,
  like,
  type = "l",
  ylab = "Density",
  lty = 3,
  lwd = 3,
  xlab = expression(theta)
)
```




```r
#plot the prior distribution
plot(
  th,
  prior,
  type = "l",
  ylab = "Density",
  lty = 3,
  lwd = 3,
  xlab = expression(theta)
)
```

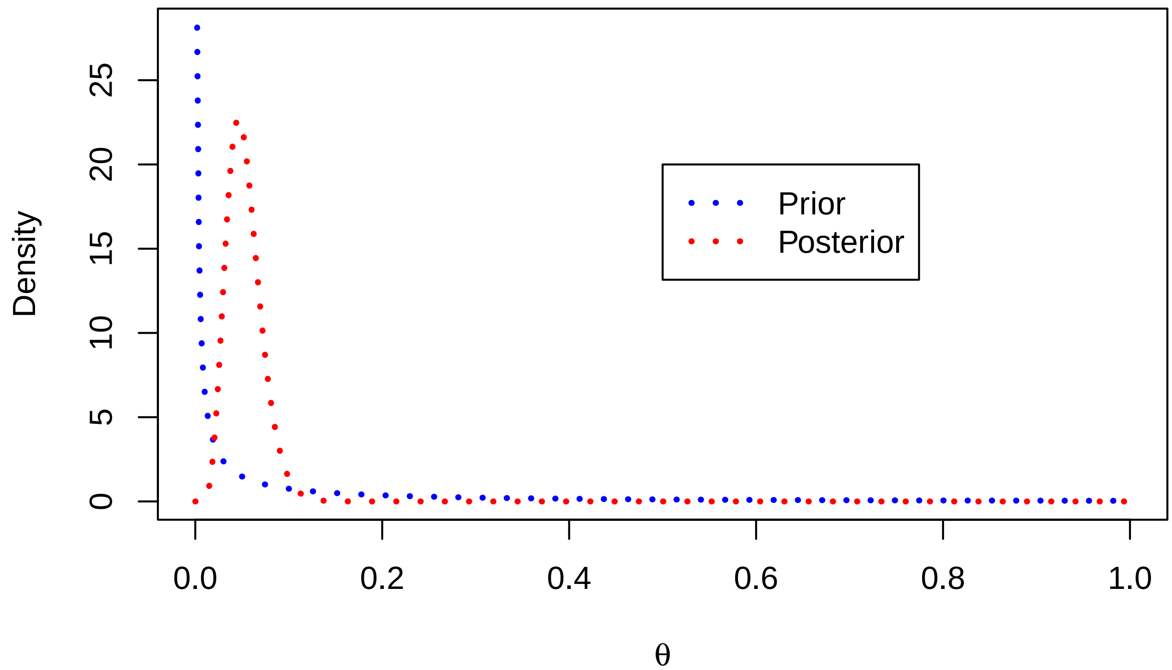



```r
#plot the posterior distribution
plot(
  th,
  post,
  type = "l",
  ylab = "Density",
  lty = 3,
  lwd = 3,
  xlab = expression(theta)
)
```




```r
#plot the prior and posterior together
plot(
  th,
  prior,
  col = "blue",
  type = "l",
  ylab = "Density",
  lty = 3,
  lwd = 3,
  xlab = expression(theta)
)
lines(
  th,
  post,
  col = "red",
  type = "l",
  lty = 3,
  lwd = 3
)
legend(
  0.5,
  20,
  legend = c("Prior", "Posterior"),
  col = c("blue", "red"),
  lwd = c(3, 3),
  lty = c(3, 3)
```

```
)
```



D

An exponential model is typically used to model the expected time until an event occurs. This event could be the time from now until the next hurricane strikes Florida.

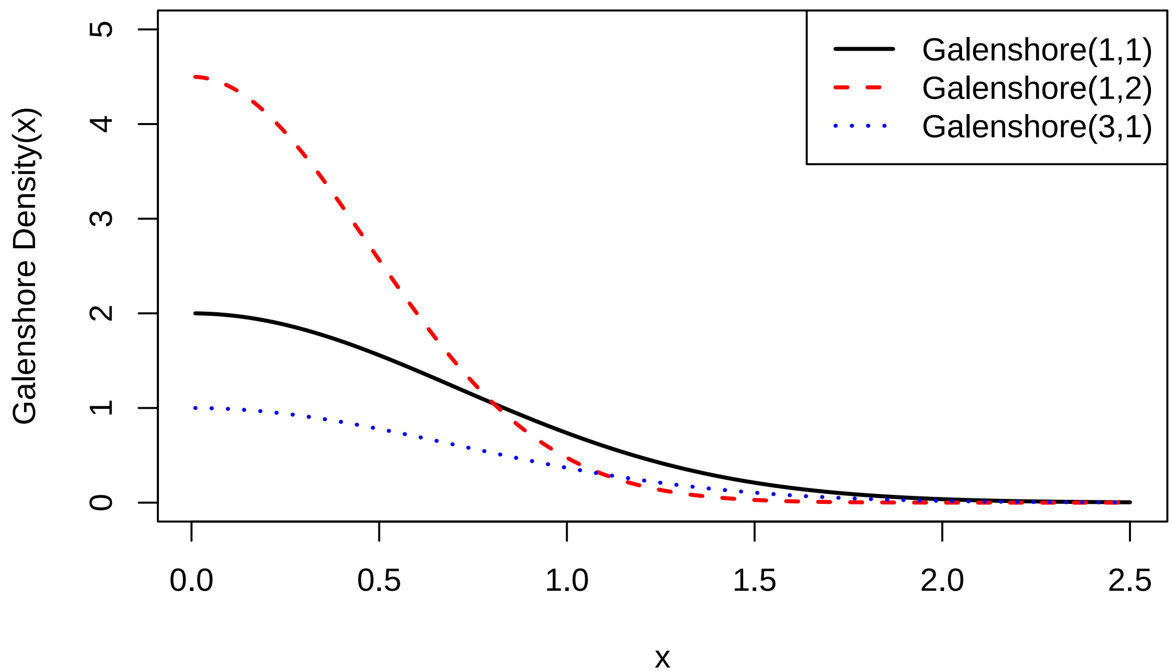
An exponential model would not be a good fit for something like the number of times that an individual will experience a hurricane in his/her/their lifetime.

Q3

A

```
#plot galenshore
dgalenshore <- function(x, a, theta){
  return ((2/gamma(a)) * theta^(2 * a) * exp(-theta^2 * x^2))
}
x <- seq(0.01, 2.5, 0.01)
plot(x, dgalenshore(x, 1, 1), type = "l", ylab = "Galenshore Density(x)", col = "black",
    ylim = c(0, 5), lwd = 2)
lines(x, dgalenshore(x, 1, 1.5), type = "l", col = "red", lwd = 2, lty = 2)
```

```r
lines(x, dgalenshore(x, 3, 1), type = "l", col = "blue", lwd = 2, lty = 3)
legend("topright", c("Galenshore(1,1)", "Galenshore(1,2)", "Galenshore(3,1)"),
    col = c("black", "red", "blue"), lwd = c(2, 2, 2), lty = c(1, 2, 3))
```



9:18 PM Thu Jan 19

Q3

(A) To identify a class of conjugate prior densities, I will try first the Galenshore distribution.

$$Y|\theta \sim \text{Galenshore}(a, \theta)$$

$$p(y|\theta) = \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} e^{-\theta^2 y^2}$$

$$p(\theta, d) \sim \text{Gal}(c, d) = \frac{2}{\Gamma(c)} d^{2c} \theta^{2c-1} e^{-d^2\theta^2}$$

(B) Find posterior distribution of $\theta | y_{1:N}$ using prior from conjugate class

$$\text{prior} \Rightarrow \frac{2}{\Gamma(c)} d^{2c} \theta^{2c-1} e^{-d^2\theta^2}$$

$$F(y|\theta) = \prod_{i=1}^{N} \cancel{\frac{2}{\Gamma(a)}} \theta^{2a} \cancel{y_i^{2a-1}} e^{-\theta^2 y_i^2}$$

$$\propto (\theta^{2a})^N e^{-\theta^2 \Sigma y_i^2} = \text{likelihood}$$

2 / 8

$$f(\theta \mid c,d) = \frac{2}{\Gamma(c)} d^{2c} \theta^{2c-1} e^{-d^2\theta^2}$$

$$\propto \theta^{2c-1} e^{-d^2\theta^2}$$ = prior $= p(\theta) = f(\theta \mid c,d)$

$$(\theta^{2a})^N e^{-\theta^2 \sum y_i^2} \cdot \theta^{2c-1} e^{-d^2\theta^2}$$

$$\to \theta^{2NA+2c-1} e^{-\theta^2 \sum y_i^2 - d^2\theta^2} \qquad -\theta^2 \cdot x$$

$$\to \theta^{2(NA+c)-1} e^{-\theta^2(\sum y_i^2 + d^2)}$$

posterior $= \theta \mid c,d \sim GAL\left((NA+c), \sqrt{\sum y_i^2 + d^2}\right)$

(c) Show:

$$\frac{p(\theta_a \mid y_{1:N})}{p(\theta_b \mid y_{1:N})} = \left(\frac{\theta_a}{\theta_b}\right)^{2(aN+c)-1} e^{(\theta_b^2-\theta_a^2)(d^2+\sum y_i^2)}$$

where $\theta_a, \theta_b \sim$ Galenshore(c,d)

STANDARD GAL

$$\frac{2}{\Gamma(c)} d^{2c} \theta^{2c-1} e^{-d^2\theta^2}$$

☆ posterior w/ updated parameters ☆

$$\frac{\cancel{\frac{2}{\Gamma(Na+c)}\left(\sqrt{\Sigma y_i^2+d^2}\right)^{2(Na+c)}}\left(\theta_a^{2(Na+c)-1}\right)\left(e^{-\sqrt{\Sigma y_i^2+d^2}^2\theta_a^2}\right)}{\cancel{\frac{2}{\Gamma(Na+c)}\left(\sqrt{\Sigma y_i^2+d^2}\right)^{2(Na+c)}}\left(\theta_b^{2(Na+c)-1}\right)\left(e^{-\sqrt{\Sigma y_i^2+d^2}^2\theta_b^2}\right)}$$

$$= \frac{\theta_a^{2(Na+c)-1} \; e^{-\sqrt{\Sigma y_i^2+d^2}^2\theta_a^2}}{\theta_b^{2(Na+c)-1} \; e^{-\sqrt{\Sigma y_i^2+d^2}^2\theta_b^2}}$$

$$= \left(\frac{\theta_a}{\theta_b}\right)^{2(Na+c)-1} e^{(\theta_b^2-\theta_a^2)(d^2+\Sigma y_i^2)}$$

by Factorization Theorem & Regular Exponential Class, A Sufficient statistic is:

$$d^2+\Sigma y_i^2$$

(D) Determine $E[\theta \mid y_{1:N}]$

$$E[Y] = \frac{\Gamma(a+1/2)}{\theta\,\Gamma(a)}$$

$a = Na + c$

$\theta = \sqrt{\Sigma y_i^2 + d^2}$

$$E[\theta \mid y_{1:N}] = \frac{\Gamma\left(Na + c + \frac{1}{2}\right)}{\sqrt{\Sigma y_i^2 + d^2}\,\Gamma(Na + c)}$$

(E) Show the form of the posterior predictive density is:

$$p(y_{N+1} \mid y_{1:N}) = \frac{2 y_{N+1}^{2a-1}\,\Gamma(aN + a + c)}{\Gamma(a)\,\Gamma(aN + c)} \frac{(d^2 + \Sigma y_i^2)^{aN+c}}{(d^2 + \Sigma y_i^2 + y_{N+1}^2)^{(aN+a+c)}}$$

$$p(y_{N+1} \mid \theta) = \frac{2}{\Gamma(a)} (\theta^{2a})\, y_{N+1}^{2a-1}\, e^{-\theta^2 y_{N+1}^2} = \text{likelihood of new data point}$$

$$\text{Posterior} = \frac{2}{\Gamma(Na+c)} \left(\sqrt{\Sigma y_i^2 + d^2}\right)^{2(Na+c)} \left(\theta^{2(Na+c)-1}\right)\left(e^{-\sqrt{\Sigma y_i^2 + d^2}^{\,2}\theta^2}\right)$$

$$\int_\theta \left[\frac{2}{\Gamma(a)} (\theta^{2a})\, y_{N+1}^{2a-1}\, e^{-\theta^2 y_{N+1}^2}\right]\left[\frac{2}{\Gamma(Na+c)} \left(\sqrt{\Sigma y_i^2 + d^2}\right)^{2(Na+c)} \left(\theta^{2(Na+c)-1}\right)\left(e^{-\sqrt{\Sigma y_i^2 + d^2}^{\,2}\theta^2}\right)\right] d\theta$$

$$\frac{2}{\Gamma(a)}\, y_{N+1}^{2a-1}\, \frac{2}{\Gamma(Na+c)} \left(\sqrt{\Sigma y_i^2 + d^2}\right)^{2(Na+c)} \int_\theta \theta^{2a}\, e^{-\theta^2 y_{N+1}^2}\, \theta^{2(Na+c)-1}\, e^{-\sqrt{\Sigma y_i^2 + d^2}^{\,2}\theta^2}\, d\theta$$

9:19 PM Thu Jan 19 72%

$$= \int_\theta \theta^{2a+2(Na+c)-1} e^{-\theta^2 y_{N+1}^2 - \sqrt{\Sigma y_i^2 + d^2}\,\theta^2} \, d\theta$$

$$\to \int_\theta \theta^{2(a+Na+c)-1} e^{-\theta^2\left(y_{N+1}^2 + \sqrt{\Sigma y_i^2 d^2}\right)} \, d\theta$$

$$\frac{2}{\Gamma(a)} y_{N+1}^{2a-1} \frac{2}{\Gamma(Na+c)} \left(\sqrt{\Sigma y_i^2 + d^2}^{\,2(Na+c)}\right)$$

Normalizing constant is:

$$\frac{2\Gamma(a+Na+c)}{2\Gamma(a+Na+c)}$$

$$\frac{2}{\Gamma(a)} y_{N+1}^{2a-1} \frac{2}{\Gamma(Na+c)} \left(\sqrt{\Sigma y_i^2 + d^2}^{\,2(Na+c)}\right) \frac{2\Gamma(a+Na+c)}{2\Gamma(a+Na+c)} \int_\theta \theta^{2(a+Na+c)-1} e^{-\theta^2\left(y_{N+1}^2 + \sqrt{\Sigma y_i^2 d^2}\right)} \, d\theta$$

→ Is a Galenshore$\left((a+Na+c), \left(y_{N+1}^2 + \sqrt{\Sigma y_i^2 d^2}\right)\right)$ and thus integrates over θ to 1.

$$\to \frac{\cancel{2}}{\Gamma(a)} y_{N+1}^{2a-1} \frac{2}{\Gamma(Na+c)} \left(\Sigma y_i^2 + d^2\right)^{(aN+c)} \frac{\Gamma(a+aN+c)}{\cancel{2}}$$

$$\to \frac{2 y_{N+1}^{a-1} \Gamma(aN+a+c)}{\Gamma(a)\Gamma(aN+c)}$$

* left out the θ^{2a} part for simplicity...
Adding in now *

$$\frac{2 y_{N+1}^{a-1}\,\Gamma(a_N+a+c)}{\Gamma(a)\,\Gamma(a_N+c)}\left(\left(\Sigma y_i^2+d^2\right)^{(a_N+c)}\right)\left(y_{N+1}^2+\sqrt{\Sigma y_i^2 d^2}\right)^{2(a_N+a+c)}$$

$$\rightarrow \frac{2 y_{N+1}^{a-1}\,\Gamma(a_N+a+c)}{\Gamma(a)\,\Gamma(a_N+c)}\;\frac{\left(d^2+\Sigma y_i^2\right)^{a_N+c}}{\left(d^2+\Sigma y_i^2+y_{N+1}^2\right)^{(a_N+a+c)}}$$