PHYS 510 - General Relativity - 2018

Assignments 2 and 3 - Due October 25, 2018

Part A consists of problems that help you complete missing steps in the lectures but it is also aimed at understanding notation and learning analytical techniques. Part B consists of comprehensive problems that make use of the entire machinery developed in the lectures.

I. PART A

1. This is a problem in two dimensions and shows the power of covariant derivatives. In polar coordinates the velocity field of a fluid is given by

$$v^r = a\cos\theta, \quad v^\theta = b\sin\theta/r,$$

where a and b are constants.

   (a) Find the Cartesian components (v^x, v^y) of the velocity field. [4]

   (b) Compute the covariant derivative of the velocity field $\nabla_\beta v^\alpha$ both in Cartesian and polar coordinate systems (two derivatives of each of the two components). [6]

   (c) For what (non-zero) values of a and b are all the covariant derivatives zero in the two coordinate systems? Are the values of a and b, which make the covariant derivative in the two coordinate systems, the same? [5]

2. This question is about projection operator and its properties. Consider a timelike vector u^a of unit norm and the tensor P_{ab} defined by:

$$P_{ab} = \eta_{ab} + u_a u_b.$$

(In a curved space one replaces η by g.) Show that P_{ab} is a projection operator that projects an arbitrary vector v^a into one orthogonal to u^a. That is, show that $v^a_\perp$ defined by $v^a_\perp = P^a_b v^b$ is (i) orthogonal to u^a and (ii) unaffected by P_{ab} (i.e., show that $P^2 = P$). [10]

3. In a hurricane air has a density $\rho = 1$ kg per m^3, pressure p of 1 atmosphere (i.e., 10^6 dyne per cm^2) and a velocity of $v^x = 10^5$ m per hour, $v^y = v^z = 0$. Treat the air as a perfect fluid for which the energy-momentum tensor is:

$$T^{ab} = (\rho + p)u^a u^b + p\eta^{ab}.$$

   (a) Work out the non-zero components of the energy-momentum tensor of the air. Make sure that you express both pressure and density in the same units using appropriate factors of the speed of light. [5]

   (b) Next assume that all the velocity components of the fluid are non-zero but also non-relativistic (i.e. $u^a \approx (1, \vec{v})$, $|\vec{v}| \ll 1$). For such a fluid, show that

$$T^{00} \approx \rho, \quad T^{0j} \approx \rho v^j, \quad T^{ij} \approx (\rho + p)v^i v^j + p\delta^{ij},$$

   where the symbol $\approx$ stands for "approximately equal to" and $i, j = x, y, z$. [5]

(c) The conservation law says $\partial_a T^{ab} = 0$. The time-component of this equation gives:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (v^i \rho)}{\partial x^i} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v}\rho) = 0,$$

which is the equation of continuity. Show that the rest of the conservation equations, namely $\partial_\mu T^{i\mu} = 0$, give the Euler equation for the fluid flow. [5]

II. PART B

4. In this problem you will work out the geodesic deviation equation on a sphere. The metric on the sphere is:

$$d\ell^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2),$$

where a is a constant that gives the scale of the sphere and (θ, ϕ) are the polar coordinates. (Note that a is the radius of the sphere only when you think of embedding the sphere in R^3, more generally, it is just a length scale.)

(a) Compute all the nonzero Christoffel symbols on the sphere. [4]

(b) Compute all the nonzero components of the Riemann tensor (there is really only one nontrivial component, the rest are obtained by symmetry property of the Riemann tensor). [6]

(c) Compute the Ricci tensor and Ricci scalar. [4]

(d) Is the Weyl tensor nonzero, explain your answer? [4]

(e) Consider two neighboring geodesics (great circles) on the sphere, one the equator and the other a geodesic slightly displaced from the equator by $\delta\theta = b$ and parallel to it at $\phi = 0$. Let $\vec{\xi}$ be the separation vector between the two geodesics. At $\phi = 0$, $\vec{\xi} = b\partial/\partial\theta$. Let ℓ be the proper distance along the equatorial geodesic, so $d/d\ell = \vec{u}$ is its tangent vector.

i. Show that $\ell = a\phi$ along the equatorial geodesic. [4]

ii. Show that the equation of geodesic deviation for the pair of geodesics discussed above reduces to:

$$\frac{d^2\xi^\theta}{d\phi^2} = -\xi^\theta, \quad \frac{d^2\xi^\phi}{d\phi^2} = 0.$$

[6]

iii. Solve this, subject to appropriate initial conditions, to obtain

$$\xi^\theta = b\cos\phi, \quad \xi^\phi = 0.$$

[2]

Nonzero Christoffel symbols:

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\,\cos\theta, \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta.$$

Nonzero Riemann tensor components:

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} = a^2\,\sin^2\theta.$$