GRAVITY IS THE GAUGE THEORY OF THE PARALLEL - TRANSPORT MODIFICATION OF THE POINCARE GROUP

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Abstract

We prove that only the Dynamically - Restricted Anholonomized General Coordinate Transformation Group reproduces Einstein's theory of Gravitation directly when gauged. This amounts to a Modified Poincaré group where translations are replaced by Parallel transport. We also explain the role of GL(4R) and explore the Modified Affine Group. Using the Ogievetsky theorem, we present several No-Go theorems restricting the joint application of Conformal and Affine Symmetries.

1. Introduction: Gauge Theories

The first local gauge invariance principle (LGIP, or just "gauge") to be suggested [Weyl, 1919] dealt with dilations, and was introduced as an addition to Einstein's Gravity. H. Weyl was looking for a geometrical derivation of Electromagratism, which would thereby also "unify" it with Gravitation. His first theory invoked dilation invariance, and failed at the time since macroscopic evidence appeared to be clearly in disagreement with such a postulate. This particular theory has recently been revived at the quantum level as a gauge invariance with "spontaneous breakdown" [Englert et. al., 1975]. The geometrical derivation itself was revived after the advent of quantum mechanics as a U(1) gauge [Weyl, 1929] i.e. a locally dependent phase for complex charged matter fields instead of scale invariance. We would now render it as a Principal Bundle B $(\mathcal{M}^4$, G) with Minkowski Space-Time $\mathcal{M}^4$ as base space, and G = U(1) as structure group. The gauge transformations are given by the set of Bundle automorphisms whose action on $\mathcal{M}^4$ is the identity, i.e. leaving a point $x \in \mathcal{M}^4$ invariant. They thus act only in the fiber above that point, and can be written as $g(x)$, $g \in G$. They belong to the "stability group" of translations in $\mathcal{M}^4$.

This abstract "internal" gauge invariance was H. Weyl's second definition, and it won wide acceptance. Three decades later, it served as a model for the (G = SU(2)) local Non-Abelian internal gauge of C.N. Yang and R.L. Mills [1954; see also Shaw 1954]. The method was further generalized [Ne'eman 1961, Gell-Mann 1962, Salam and

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Ward 1961] to SU(3) and in principle to any Semi-simple group [Gell-Mann and Glashow 1961, Ionides 1961]. In recent years, this SU(3) universal (and therefore gauge-like) coupling which is indeed observed in the coupling of hadrons to massive vector-mesons (the ρ, ω, ϕ, ψ, Υ, with $J^{PC} = 1^{--}$) has to be regarded as a pole-dominance approximation for phenomenological vector fields [Gell-Mann, 1962]. On the other hand, an $SU(2)_{Left} \times U(1)$ LGIP involving a subgroup of that SU(3) but acting on leptons and on SU(3) invariant quarks as well is favored as a Weak and Electromagnetic Unified Gauge [Weinberg 1967, Salam 1968] (though other groups are still possible), and an $SU(3)_{color}$ LGIP is believed to represent the quark-glueing [Nambu 1965, Fritzsch and Gell-Mann 1972, Weinberg 1973] (and confining?) part of the Strong Interactions. Those applications have become serious candidate dynamical theories since the achievement of G. 't Hooft and M. Veltman ['t Hooft 1971, 't Hooft and Veltman 1972, Lee and Zinn-Justin 1972] in completing the renormalization of the Yang-Mills interaction [Feynmann 1963; De Witt 1964, 1967; Faddeev and Popov 1967; Fradkin and Tyutin 1970; Veltman 1970], including the case of "spontaneous breakdown" [Higgs 1964a, 1964b, 1966; Englert and Brout 1964; Guralnik et al 1964; Kibble 1967] of the local gauge coupled with a Goldstone- Nambu realization of the global symmetry [Goldstone 1961, Nambu and Jona - Lasinio 1961]. For the Strong Interactions, renormalization has also led to the discovery of Asymptotic Freedom [Politzer 1973, Gross and Wilczek 1973] which seems particularly fitting for short range quark interactions, and appears to support the $SU(3)_{color}$ gauge idea. We review the highlights of a Yang-Mills type gauge.

The dynamical variables in a $B(\mathcal{M}^4, G)$ gauge theory may include matter fields (quarks) $q^a(x)$ which are generally represented as sections of a vector bundle E associated to B,

$$E = B \times_G \Lambda(G)$$

where $\Lambda(G)$ is the (3 x 3 for quarks) appropriate representation of G on q^a:

$$(b(g_o(x)),\ q(x) \circ g = (b(g_o(x)g(x)), \quad \Lambda(g^{-1})q(x))$$

The covariant derivative in E involves matrix connections (potentials)

$$\rho(x) = \rho_\mu{}^A(x)\ X_A\ dx^\mu \tag{1.1}$$

where X_A is the Lie-algebra of G in the $\Lambda(g)$ representation. The covariant derivative in E is then

$$(D\ q)^a = d\ q^a - (\rho)^a{}_b\ q^b \tag{1.2}$$

and the dynamical theory is derived by the replacement

$$(\delta_b{}^a\ \partial_\mu) \rightarrow (D_\mu)_b{}^a \tag{1.3}$$

known as a "minimal" or "universal" coupling. Indeed, with a free Lagrangian

$$\mathcal{L}_o = - \overline{q}_a\ (\gamma^M\ \partial_\mu + m)\ q^a \tag{1.4}$$

the unwanted contribution due to $\partial_\mu\ g \neq 0$

$$- \overline{q}_a\ \gamma^\mu\ (g^{-1}\ \partial_\mu\ g)_b{}^a\ q^b$$

is cancelled by

$$\rho \rightarrow (g\ \rho\ g^{-1} + g^{-1}\ d\ g) \tag{1.5}$$

For an infinitesimal transformation $(\Lambda(g))_b{}^a = \delta_b^a + (i\alpha^A\ X_A)_b{}^a$, the unwanted $\partial_\mu\alpha^A$ term arises in

$$\partial_\mu\alpha^A\ \frac{\partial}{\partial(\partial_\mu q^a)}\ (X_A)_b{}^a\ q^b = \partial_\mu\alpha^A\ J_A{}^\mu$$

where $J_A{}^\mu$ is the Noether current, satisfying a covariant conservation law

$$D\ J_A = 0 \tag{1.6}$$

The curvature

$$R\ = (d\ \rho - \rho \wedge \rho)\ = (d\rho - \frac{1}{2}\ [\rho,\ \rho])\ = (R^A\ X_A) \tag{1.7}$$

similarly satisfies the Bianchi identity

$$(DR)\ = 0 \tag{1.8}$$

The equations of motion are

$$(D\ {*R}\ )\ = {*J}^A \tag{1.9}$$

where * stands for the duals

$$*R_{\mu\nu}{}^{A} = \frac{1}{2}\,\varepsilon_{\sigma\tau\mu\nu}\,R_{\sigma\tau}{}^{A} \tag{1.10a}$$

$$*J_{\mu\sigma\tau}{}^{A} = \frac{1}{6}\,\varepsilon_{\nu\mu\sigma\tau}\,J_{\nu}{}^{A} \tag{1.10b}$$

The equations of motion can be used to turn (1.6) into a non-G-covariant conservation law for a new current

$$d\,J'_A = 0 \tag{1.11}$$

where J'_A will include contributions from the ρ^A potentials themselves. This will be more problematic in Gravitation.

Connections, covariant derivatives and curvatures can also be introduced in B itself, where they will regulate their own gauge invariance (no "sources"). The matrices X_A will now belong to the adjoint representations,

$$(X_A)_B{}^C = -\,C_{AB}{}^C$$

The definitions are

$$R^A = d\,\rho^A - \tfrac{1}{2}\,\rho^B \wedge \rho^C\,C_{BC}{}^A = \tfrac{1}{2}\rho^B \wedge \rho^C\,R_{BC}{}^A \tag{1.12}$$

$$(D\rho)^A = d\,\rho^A - \rho^B \wedge \rho^C\,C_{BC}{}^A \tag{1.13}$$

$$(DR)^A = 0 \tag{1.14}$$

using contractions with vector-fields D_A,

$$D_A = \Delta_A{}^{\mu}\,\partial_{\mu} \qquad\qquad \rho^A\,(\Delta_B) = \delta^A{}_B \tag{1.15}$$

and with the resulting commutator (from double contraction of (1.12)),

$$[D_A, D_B] = (C^C{}_{AB} + R^C{}_{AB})\,D_C \tag{1.16}$$

where $C^C{}_{AB} = 0$ in $\mathcal{M}^4$ but not in "Superspace" $\mathfrak{R}^{4/4N}$ as we shall later see.

Notice that in the adjoint representation, (1.13) can also be written as $D\rho = d\rho - [\rho, \rho]$ and is not equivalent to R. This is due to the antisymmetry of $C_{BC}{}^A$ or $(-X_B)_C{}^A$ in the (B,C) indices, as against $(\Lambda_B)_b{}^a$ in (1.2) for $(\rho^B\,\lambda_B)_b{}^a\,q^b$ where there is no such link between B and b. The antisymmetry implies a factor 2 in contracting with $(\frac{1}{2}\,dx^{\mu} \wedge dx^{\mu})$ as against the curl $d\rho$.

2. The First Step: Gauging the (intrinsic) Lorentz Group

We first return to Gravity when R. Utiyama [1956] attempts to derive that theory from a Gauge Principle. Since not much was known at the time about the renormalizability of Yang-Mills LGIP theories, this was in the main an aesthetic urge. Utiyama gauged the (homogeneous) Lorentz group $G = SL(2,C) =: L$ using the equivalent of connection one-forms

$$\rho^{ij} = \rho_\mu{}^{ij}\, dx^\mu \quad (i,j = 0,1\ldots 3 \text{ in a local frame; } \mu = 0,1\ldots 3 \text{ holonomic} \tag{2.1}$$

However, to reproduce Einstein's theory it appeared that he had to introduce a-priori curvilinear coordinates, and a set of 16 "parameters" $\Delta_k{}^\mu(x)$. These were initially treated as given functions of x and later became field variables, to be identified with orthonormal vector fields Δ_k reciprocal to a vierbein frame,

$$\rho^i\,(\Delta_k) = \delta^i_k \tag{2.2a}$$

the $\rho_\mu{}^i$ thus arising as vierbein fields, with (η_{ij} in the Minkowski metric)

$$\eta_{ij}\,\rho_\mu{}^i\,\rho_\nu{}^j = g_{\mu\nu}(x) \tag{2.2b}$$

Still, the relationship of $\rho_\mu{}^{ij}$, to the Christoffel connection $\Gamma_{\mu\nu}{}^\lambda$ was incomplete, since the formula he derived was forced by an arbitrary assumption to select only ($\mu\ \nu$) symmetric contributions to $\Gamma_{\mu\nu}{}^\lambda$. As we shall see, this role of the Connection ("Affinity" in holonomic - "world tensor" - language) as a Gauge Potential has since been perfected. However, it contrasted sharply with the physical intuition of workers in Gravitation [e.g. Thirring 1977] who regard the metric (or vierbein) as the Gravitational Potential, and consider the Connection as the analogue to the Field Strength in Electrodynamics.

Sciama [1962] and Kibble [1961] continued Utiyama's project. Although they were aiming at a full Poincaré gauge ($G = ISL(2,C) := P$), their main achievement consisted in clarifying the Lorentz gauge. They showed that this consisted only in the stability group over $\mathcal{M}^4$, i.e. the "internal" action of $H = SL(2,C) =: L$, which we generally describe as the Spin of the Matter fields (though it does not include contributions to physical spin due to the holonomic - "Greek" - indices of gauge fields, curvatures etc., i.e. in particular, the photon or the Yang-Mills' fields own spins). This "Latin" or "anholonomic" spin $s_{ij}{}^\mu$ gives rise to a new interaction term, in which it is minimally coupled to the connection $\rho_\mu{}^{ij}$.

$$A_S = \int d^\mu x \quad e \, \rho_\mu{}^{ij} \, S_{ij}{}^\mu = \int \rho^{ij} \wedge {*S_{ij}} \tag{2.3}$$

where $*S_{ij}$ is a dual three-form

$$*S_{ij} = \frac{1}{6} \varepsilon_{\tau\nu\rho\sigma} \quad S_{ij}{}^\tau \, dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{2.4}$$

Indeed, this arises when we perform the replacement

$$\delta_i{}^\mu \, \partial_\mu \longmapsto \Delta_i{}^\mu \, D_\mu{}^{(H)} \quad , \qquad D_\mu{}^{(H)} = \partial_\mu + \tfrac{1}{2}\rho_\mu{}^{ij} \, f_{ij} \tag{2.5}$$

where $\frac{1}{2}f_{ij}$ is a representation of the Lorentz generators, appropriate for action on the ψ matter field in $\mathcal{L}_M(\psi, \partial_\mu \psi)$

$$\frac{\partial \mathcal{L}_M}{\partial \rho_\mu{}^{ij}} = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \, f_{ij} \, \psi =: e \, s_{ij}{}^\mu \tag{2.6}$$

The factor $e = \det \rho_\mu{}^i$ arises in the replacement $d^u x \longmapsto e \, d^u x$ of the matter action measure. However, the variation of the action by $\delta \rho^{ij}$ also receives a contribution from the Einstein free action ($\lambda := 8\pi c^{-3} G$, G being Newton's constant; $[K] = [L^2]$ in "natural" units)

$$A = \frac{1}{8} \int \frac{1}{k} R^{ij} \wedge \rho^k \wedge \rho^\ell \, \varepsilon_{ijk\ell} = \frac{1}{8k} \int R^{ij} \wedge \zeta_{ij} \tag{2.7}$$

so that one has a new equation of motion (besides Einstein's) involving three-forms

$$R^k \wedge \rho^t \, \varepsilon_{ijk\ell} = -\frac{1}{2}\lambda \, {*s_{ij}} \tag{2.8}$$

In these equations, R^{ij} and R^i are the curvature two-forms, with

$$R^{ij} = dx^\mu \wedge dx^\nu \, R_{\mu\nu}{}^{ij} = d\rho^{ij} + \rho^i{}_k \wedge \rho^k{}_\ell = \tfrac{1}{2}\rho^k \wedge \rho^\iota \, R_{k\iota}{}^{ij} \tag{2.9}$$

$$R^i = dx^\mu \wedge dx^\nu \, R_{\mu\nu}{}^i = d\rho^i + \rho^i{}_k \wedge \rho^k = D^{(L)}\rho^i = \tfrac{1}{2}\rho^k \wedge \rho^\iota \, R_{k\iota}{}^i \tag{2.10}$$

In conventional nomenclature, R^{ij} is the Riemannian curvature and R^i the Cartan "tors ion". For empty space, (2.8) becomes $R^i = 0$ and solving (2.10) for $\rho^i{}_k$ (in $D^{(L)} \rho^i$) then produces the Christoffel symbol formula. However, when (Latin) spinning matter is present, solving for $\rho^i{}_k$ will produce in addition an antisymmetric contribution to the Christoffel connection.

Contracting R^k with two vector-fields we find,

$$R_{ij}^{\cdot\cdot}{}^{k} = \Delta_i{}^{\mu} \Delta_j{}^{\nu} R_{\mu\nu}{}^{k} = (\Delta_i, \Delta_j, R^k) \tag{2.11}$$

$$= (\Delta_i, \Delta_j, (d\rho^k + \rho^k{}_{\iota} \wedge \rho^{\iota}))$$

$$= \Omega_{ij}^{\cdot\cdot}{}^{k} + \tfrac{1}{2}(\rho_i{}^{k}{}_{\iota}\, \delta_j{}^{\iota} - \rho_j{}^{k}{}_{\iota}\, \delta_i{}^{\iota})$$

Thus,

$$R_{ij}{}^{k} = \Omega_{ij}^{\cdot\cdot}{}^{k} + \tfrac{1}{2}\,(\rho_i{}^{k}{}_{j} - \rho_j{}^{k}{}_{i}) \tag{2.12}$$

The doubly contracted exterior derivative $\Omega_{ij}^{\cdot\cdot}{}^{k}$ has been called "the object of Anholonomity", [Schouten 1954; Hehl et. al 1976a]. Using the Minkowski metric (in the tangent space) we can lower the ^{k} index, and remembering that the antisymmetry of the Lorentz generators imposes

$$\rho_{i\,k\,j} = -\rho_{i\,j\,k} \tag{2.13}$$

we can extract $\rho_{i\,j\,k}$,

$$\rho_{i\,j\,k} =: \Omega_{i\,j\,k} - \Omega_{j\,k\,i} + \Omega_{k\,i\,j} + R_{j\,k\,i} - R_{k\,i\,j} - R_{i\,j\,k} \tag{2.14}$$

The last three terms, making up together the "Contortion tensor" $K_{i\,j\,k}$, vanish for $R^k = 0$ and represent the contribution of ("Latin") spinning matter when present.

$$K_{i\,j\,k} = R_{j\,k\,i} - R_{k\,i\,j} - R_{i\,j\,k}$$

Inserting the last expression for R^i in (2.10) into (2.8), and replacing the holonomic index in $S_{ij}{}^{\mu}$

$$S_{ij}^{\cdot\cdot}{}^{k} = \rho_{\mu}^{\cdot}{}^{k}\, S_{ij}{}^{\mu} \tag{2.15}$$

we get the equation of motion,

$$R_{i\,j}{}^{k} - \delta_i^k\, R_{\iota\,j}{}^{\iota} - \delta_j^k\, R_{i\,\iota}{}^{\iota} =: T_{i\,j}{}^{k} = k\, S_{i\,j}{}^{k} \tag{2.16}$$

$T_{ij}^{\cdot\cdot}{}^{k}$ is sometimes named the "Modified Torsion". We can also contract the upper (naming) index of the torsion tensor $R_{\mu\nu}{}^{k}$ in (2.11)

$$R_{\mu\nu}{}^{\rho} = \Delta_k{}^{\rho} R_{\mu\nu}{}^{k} \doteq \Delta_k{}^{\rho} \rho_\mu{}^{i} \rho_\nu{}^{j} R_{ij}{}^{k}$$

If we now insert (2.12) we find

$$R_{\mu\nu}{}^{\rho} = -\tfrac{1}{2} (\rho_{\mu\nu}{}^{\rho} - \rho_{\nu\mu}{}^{\rho}) = -\tfrac{1}{2} (K_{\mu\nu}{}^{\rho} - K_{\nu\mu}{}^{\rho}) \qquad (2.17)$$

Holonomically, torsion thus corresponds to the antisymmetric part of the connection. Note that these are not the indices which are antisymmetric in the anholonomic connection due to the Lorentz gauge. Returning to the equation of motion we derived, we note that Eq. (2.16) being algebraic (due to 2.17) rather than differential (due to the particular choice of the Einstein Lagrangian which is linear in the canonical momenta), the connection potential $\rho_\mu{}^{ij}$ does not propagate. Instead, like a gauge connection in a current-field identity [Lee et al 1967] it is replaced by the spin-current itself, so that (2.3) becomes a spin-spin term with very weak coupling k^2, a contact term. Sciama and Kibble thus rediscovered Cartan's modification [1922-25] of Einstein's Relativity. At the same time, this can be regarded as a "first order" or Palatini [1969] formalism for that theory (independent variations for ρ^i and ρ^{ij}). It then differs from it by that $k^2 s^i s_i$ term only [Weyl 1950]. This theory, further analyzed by Hehl [1970] and by Trautman [1972] is known as the Einstein - Cartan - Sciama - Kibble theory (or U_4 theory), and is thus indeed derivable <u>in its spin-torsion parts</u> from a Lorentz gauge.

3. Difficulties in Gauging Translations; Pseudo-Invariance

The attempt to reproduce Gravity had of course to come to grips with the main part of the theory - the universal coupling of the Energy-Momentum tensor-current to the gravitational potential (i.e. to the metric $g_{\mu\nu}$ or in a vierbein formalism, to the $\rho_\mu{}^i$ of (2.2)). Indeed, varying ρ^i in (2.7) yields Einstein's equation for empty space,

$$R^{ij} \wedge \rho^k \varepsilon_{ijk\ell} = 0 \tag{3.1}$$

which becomes, in holonomic language, after some manipulations,

$$R^\mu{}_\nu - \tfrac{1}{2} R\, \delta^\mu{}_\nu = 0 \tag{3.2}$$

In the presence of matter we have

$$R^{ij} \wedge \rho^k \varepsilon_{ijk\ell} = k\, {*t_\ell} \tag{3.3}$$

where $*t_\ell$ is the energy-momentum current 3-form

$$*t_\ell = \frac{1}{6} \varepsilon_{ijkm}\, \rho^j \wedge \rho^k \wedge \rho^m\, t^i{}_\ell \tag{3.4}$$

for the density $t^i{}_\ell$.

The Sciama-Kibble approach fell short of attaining this goal by a gauge principle. Kibble noted that the Lorentz-gauge invariance having been ensured by the covariant derivative (2.5), the remaining unwanted gradient term corresponding to translations is a homogeneous term, in contradistinction to the Yang-Mills case,

$$\delta(D_\mu{}^{(L)}\psi) = \tfrac{1}{2}\varepsilon^{ij}\, f_{ij}\, (D_\mu{}^{(L)}\psi) - \partial_\mu \xi^\nu\, D_\nu{}^{(L)}\psi$$

its removal is achieved by a multiplicative application rather than by the usual additive construction. Indeed, taking

$$D_k = \Delta_k{}^\mu D_\mu{}^{(L)} \quad , \qquad \delta\Delta_k{}^\mu = -\, \varepsilon^i{}_k\, \Delta_i{}^\mu + \partial_\nu \xi^\mu \Delta_k{}^\nu \tag{3.5}$$

yields

$$\delta D_k\psi = \tfrac{1}{2}\, \varepsilon^{ij}\, f_{ij}\, D_k\psi - \varepsilon^i{}_k\, D_i\psi \tag{3.6}$$

Kibble thus attributed to (the vector field) $\Delta_k{}^\mu$ the role of a translation gauge field, with ξ^μ as the translation parameter. This fitted an analysis of the action of the Poincaré group on fields, in which the intrinsic Lorentz action was given anholomic indices, but where all the rest (both orbital angular momentum action and translations) was incorporated in the General Coordinate Transformation and represented holonomically,

$$\delta\psi = \tfrac{1}{2}\,\varepsilon^{ij}\, f_{ij}\,\psi \qquad\qquad \delta x^\mu = \xi^\mu = \varepsilon^\mu{}_\nu x^\nu + \varepsilon^\mu \tag{3.7}$$

with in addition

$$\delta_o\,\psi = -\xi^\mu\,\partial_\mu\psi + \tfrac{1}{2}\,\varepsilon^{ij}\, f_{ij}\,\psi \tag{3.8}$$

The separation (3.7) in which the "orbital" action of ε^{ij} appeared as $\varepsilon^{\mu\nu}$ and was incorporated in ξ^μ corresponds indeed to the Fiber Bundle picture, in which the gauged group is the stability subgroup of P, the Poincaré group. However, the assignment of ξ^μ to coordinate transformations precluded any form of gauging for trans lations. A variation $\delta_o\psi$ had been introduced so as to reproduce

$$\delta_o\psi = \psi'(x) - \psi(x) = S\psi - \xi^\mu\,\partial_\mu\psi - \psi(x) \tag{3.9a}$$

since

$$S\psi(x) = \psi'(x') \tag{3.9b}$$

i.e. a resetting of the value of the argument to its original value x, after the action of a Lorentz transformation, in view of the latter's simultaneous effect on the coordinates (its orbital action).

Sometime using $\delta_o\psi$ in his interpretation, Kibble remarked that one could also regard (3.5) as involving a translation-gauge field $(\Delta_k{}^\mu - \delta_k{}^\mu)$

$$D_k = \delta_k{}^\mu\,\partial_\mu + (\Delta_k{}^\mu - \delta_k{}^\mu)\partial_\mu + \delta_k^\mu\;\tfrac{1}{2}\,\rho_\mu{}^{ij}\, f_{ij} \tag{3.10}$$

where the second term could correspond to ∂_μ as the $\delta_o\psi$ algebraic generator in (3.8) multiplied by its gauge field. A recent attempt to pursue this idea [Cho 1976] appears to have failed due to the difficulty of expressing $e = \det \rho^k$ in that interpretation.

A tetrad field is defined by erecting at every point x a frame of vectors $r_x{}^i(x)$

$$\rho_\mu{}^i\,(X) := \left(\frac{\partial r_x{}^i(x)}{\partial x^\mu}\right)_{x = X} \tag{3.11}$$

and its variation under a coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ is given by

$$\delta \rho_\mu{}^i (x) = - \partial_\mu \xi^\nu \rho_\nu{}^i \tag{3.12}$$

which is indeed the inverse of the ξ^μ variation of $\Delta_k{}^\mu$ in (3.5). However, one can replace the linear connection ρ^{ij} corresponding to gauging the Lorentz group, by a Cartan connection [Kobayashi 1956, 1972, Trautman 1973], in which the Bundle Structure Group G is the Affine (or Poincaré) Group. For the Poincaré group this means having in the Bundle "intrinsic" translations and altogether 10 connections. By choosing the origin in that fiber, one can make the translation-connection coincide with the frame, except that we now have an anholonomic translation gauge, with variation,

$$\delta \rho_\mu{}^i (x) = \partial_\mu \varepsilon^i + \rho_\mu{}^i{}_j \varepsilon^j - \varepsilon^i{}_j \rho_\mu{}^j = D_\mu^{(P)} \varepsilon^i \tag{3.13a}$$

or for the forms

$$\delta \rho^i = d\varepsilon^i + \rho^{ij} \wedge \varepsilon^j - \rho^j \wedge \varepsilon^{ij} = D^{(P)} \varepsilon^i \tag{3.13b}$$

Such a translation-gauge was indeed suggested by Trautman [1973] and by Petti [1976]. It yields the universal coupling of eq. (3.3) through Noether's theorem or the Bianchi identities. The difficulty is that the Einstein Lagrangian itself is not Poincaré-gauge invariant [e.g. Ne'eman and Regge 1978a]. Under the translation gauge (3.13) we find terms in $D^{(P)} \varepsilon^i$ arising from ζ_{ij} in (2.7). Integration by parts then makes the action produce a variation proportional to the torsion R^i. (The Action is of course trivially translation-invariant.)

One way out of this dilemma is to abandon the concept of invariance for a weaker "pseudo-invariance", holding only after the application of the equations of motion. This was done somewhat half-heartedly in Supergravity [Freedman et al 1976; Deser and Zumimo 1976; Freedman and van Nieuwenhuizen 1976] so emphasized by C. Teitelboim [1977], and generalized to gravity by J. Thierry-Mieg [1978]. Indeed, applying (2.8) for empty space (i.e. $R^i = 0$) after the variation makes Einstein's free Lagrangian invariant under (3.13). However this interpretation does not guarantee the possibility of exponentiation to a finite gauge, i.e. group action. In addition, $\delta \rho^{ij} = 0$ under the translation gauge, which has to be modified so as to fit $R^i = 0$, with no gauge mechanism to provide for the new $\delta \rho^{ij}$. Moreover, the interpretation fails when spinning matter is present.

4. The Parallel Transport Gauge (AGCT)

The next step in solving the mystery of the translation gauge is due to von der Heyde [1976; see also Hehl et al 1976a]. Returning to Kibble's $\delta_o\psi$ concept (eq. 3.9a - 3.9b) he noticed that with space being already "curved" due to the Lorentz gauge, the transport term had to involve parallel transport, i.e. the covariant derivative $D_\mu^{(L)}\psi$ rather than $\partial_\mu\psi$. Moreover, to preserve the Poincaré group appartenance of the translation generators, the operator D_k of (3.5) should be used, with the anholonomic ("Latin") indices covering P rather than just L. This solution thus combines the idea of 10 connections, including the vierbeins ρ^i, with that of parallel transport.

$$\overline{\delta}_o\, \psi^m(x) = \tfrac{1}{2}\, (\varepsilon^{ij}\, f_{ij})^m_n\, \psi^n - (\varepsilon^k\, D_k)^m_n\, \psi^n \tag{4.1}$$

We have seen that $\varepsilon^{ij} = \delta^i_\mu\, \delta^j_\nu\, \varepsilon^{\mu\nu}$, but for ε^k

$$\varepsilon^k = \varepsilon^\mu\, \rho_\mu{}^k \tag{4.2}$$

This is due to the "flatness" of the fiber (parameters $\varepsilon^{ij} = \varepsilon^{\mu\nu}$) as against the curvature induced by the Lorentz gauge in $\mathcal{M}^4$. Equation (4.1) can be interpreted as an active Lorentz transformation followed by a passive resetting of the coordinate frame to the original value of x. We can convince ourselves of the role of $-\varepsilon^k D_k$ as a translation-gauge by noting that the entire $\overline{\delta}_o\psi$ transformation amounts to a trivial action on the base space, a conclusion which would still be true in the principal bundle when taking the $\rho^A(x)$ for $\psi(x)$, except for the gauge term. Thus as the homogeneous part of an infinitesimal Poincaré transformation in the extended bundle with G = P, it should be considered as a gauge transformation. The interaction Lagrangian is still produced by the replacement

$$\delta_k{}^\mu\, \partial_\mu \longmapsto D_k \tag{4.3}$$

and the Equivalence Principle to maintained, independently of the existence of microscopic torsion. Indeed, Special-relativistic matter in a non-inertial frame is always locally equivalent to the same matter in a gravitational field [von der Heyde 1975]. Note also that in this derivation, the appearance of curvature is natural, due to our improved understanding of geometry: Utiyama and Kibble had to make a jump to curvilinear coordinates, whereas the Fiber Bundle picture tells us that curvature is nothing but the base-space effect of gauging a group in the Fiber. Indeed, even the electromagnetic U(1) or the modern SU(3) gauges induce curvature in space time (the $F_{\mu\nu}{}^a$).

We have recently generalized this approach [Ne'eman and Regge 1978a,b] , showing that the Supersymmetric ("local") transformations of Supergravity correspond to a similar parallel-transport action in Superspace with a further restriction to $\mathcal{M}^4$.

We now analyze the geometric and algebraic structure of the parallel-transport gauges.

To understand these gauges and indeed to analyze the entire problem of gauging a "non-Internal" group, i.e. a group with some action on space-time, we revert to a new manifold. Noting that in gauging P, the Fiber was L with 16 dimensions, and the base-space $\mathcal{M}^4$ had 4 dimensions, we observe that the Bundle dimensionality was 10, the same as that of the Poincaré group. In Supergravity, with a 14-dimensional group, workers in Superspace $\mathfrak{R}^{4/4}$, an 8-dimensional manifold, found that they had to restrict the pure gauge group to L = SO(3.1). Adding, we find again 8 + 6 = 14, the group manifold dimensionality.

In the formalism we recently developed with T. Regge [1978a,b] for the gauging of non-internal groups, we work in the Group Manifold. Generalized curvatures R^A ($A = i$, $[ij]$ in P) appear as the non-vanishing right-hand side of the Cartan-Maurer equations for Left invariant forms ω^A, when such forms are replaced by a "perturbed" set ρ^A (a ten-bein) (see (1.12))

$$d\rho^A - \tfrac{1}{2}\,\rho^B \wedge \rho^E\, C_{BE}{}^A = R^A = \tfrac{1}{2}\,\rho^B \wedge \rho^E\, R_{BE}{}^A \tag{4.4a}$$

or

$$d\rho^A - \tfrac{1}{2}\,\rho^B \wedge \rho^E\, (C_{BE}{}^A + R_{BE}{}^A) = 0 \tag{4.4b}$$

For an orthonormal basis of vector fields D_B orthogonal to the ρ^A,

$$\rho^A\,(D_B) = \delta^A_B \qquad \omega^A\,(D^{L.I.}_B) = \delta^A_B \qquad R^A \xrightarrow[\rho\to\omega]{} 0 \tag{4.5}$$

$$[D_B,\, D_E] = (C_{BE}{}^A + R_{BE}{}^A)\, D_A \tag{4.6}$$

$$[D^{L.I.}_B,\, D^{L.I.}_E] = C_{BE}{}^A\, D_A \tag{4.7}$$

(4.7) is the Left-invariant generator algebra. In (4.8) we have an algebra with "structure functions" instead of constants.

We can now also calculate the variation of D_E:

$$(\delta D)_E = [\varepsilon^B D_B,\, D_E] = \varepsilon^B\,(C_{BE}{}^A + R_{BE}{}^A)\, D_A \tag{4.8a}$$

$$\frac{(\delta D)_E}{\delta\varepsilon^B} = (C_{BE}{}^A + R_{BE}{}^A)\, D_A = (\delta D)_{EB} \tag{4.8b}$$

and since the product $\rho^E D_E$ is invariant, we can derive the variations of the adjoint representation ρ^E from those of the co-adjoint D_E (the difference is important when the group is not semi-simple, which is the case for P, GP, Extended GP, GA(4R) etc. but not for the Conformal SU(2,2), Graded-Conformal SU(2,2/1) or Extended G. Conformal SU(2,2/N). The factor $(-1)^{be}$ takes care of the grading in case of a Graded (or Super)Group.

$$\delta(\rho^E D_E)_B = (\delta\rho)_B{}^E D_E + (-1)^{be} \rho^E (\delta D)_{EB} = 0$$

$$(\delta\rho)_B{}^A D_A = -(-1)^{be} \rho^E (C_{BE}{}^A + R_{BE}{}^A) D_A$$

$$\delta_B \rho^A = -(-1)^{be} \rho^E (C_{BE}{}^A + R_{BE}{}^A) \tag{4.8c}$$

If we treat $\varepsilon^B(Z)$ as a local gauge (Z is 10-dimensional for P), we have to add the necessary gradient term. Summing over the B index we get, (we leave out the gradings for simplicity, $D^{(G)}$ is the covariant derivative defined over the group G)

$$\delta\rho^A = d\varepsilon^A - \rho^E \wedge \varepsilon^B (C_{EB}{}^A + R_{EB}{}^A) = D^{(G)}\varepsilon^A - \rho^E \varepsilon^B R_{EB}{}^A \tag{4.9}$$

or also (see definition following (4.17))

$$\delta\rho^A = D^{(G)} \varepsilon^A - 2 (\varepsilon, R^A) \tag{4.10}$$

$$(D^{(G)}\eta)^A = d\eta^A - \rho^B \wedge \eta^E C_{BE} \tag{4.11}$$

We have shown that all this is unchanged when a subgroup H (the Lorentz group L for both P and GP) is factorized out in the group manifold (so that we are left with $\mathcal{M}^4$ for P and $\mathfrak{R}^{4/4}$, i.e. "Superspace", for GP, as base spaces M). This also corresponds to H being gauged, as in section 2. In that case, denoting by E, F the indices in the range of G/H, and by A, B $\in$ H, ρ^A contains only dx differentials ($x \in M = G/H$) and ω^A itself, ρ^F only dx differentials,

$$D_A = D_A{}^{L.I.} \tag{4.12a}$$

since

$$\rho^A(D_B{}^{L.I.}) = \omega^A (D_B{}^{L.I.}) = \delta^A{}_B, \quad \rho^F(D_B{}^{L.I.}) = 0 \tag{4.12b}$$

which also implies

$$R_{BJ}{}^K = 0 \ ; \quad I,J,K \in G \ ; \ A,B \in H \ ; \ E,F \in G/H \tag{4.13}$$

Similarly, for holonomic indices, since the only "perturbed" forms are constructed of M differentials,

$$R_{QU}{}^{K} = 0 \quad ; \quad Q,R \in H \quad ; \quad V,U \in G \quad ; \quad Y,Z \in G/H \tag{4.14}$$

To further our understanding of these parallel transport gauges, we analyze the effect of a general coordinate transformation (either in the G-manifold, or after factorization in G/H) on our one forms ρ^K

$$\delta x^U = \varepsilon^U \tag{4.15}$$

$$\delta\rho^K = \delta(dx^U \rho_U{}^K) = Dx^V \frac{\partial\varepsilon^U}{\partial x^V} \rho_U{}^K + dx^U \varepsilon^V \frac{\partial}{\partial x^V} \rho^U$$

$$= dx^V \{\frac{\partial\varepsilon^K}{\partial x^V} + \varepsilon^U \frac{\partial}{\partial x^U} \rho_V{}^K - \varepsilon^U \frac{\partial}{\partial x^V} \rho_U{}^K\}$$

where we have defined (see (4.2))

$$\varepsilon^K = \varepsilon^U \rho_U{}^K \tag{4.16}$$

Since

$$d\rho^K = - \tfrac{1}{2} (dx^V \wedge dx^U) (\frac{\partial}{\partial x^U} \rho_V{}^K - \frac{\partial}{\partial x^V} \rho_U{}^K)$$

we can regroup the terms in $\delta\rho^K$,

$$\delta\rho^K = d\varepsilon^K - 2 (\varepsilon, d\rho^K) \tag{4.17}$$

where the scalar product parenthesis represents contraction with the second factor in the two-form. Also,'

$$\delta\rho^K = D^{(G)} \varepsilon^K + \rho^I \wedge \varepsilon^U \rho_U{}^J C_{IJ}{}^K - 2 (\varepsilon, d\rho^K) =$$

$$= D^{(G)} \varepsilon^K + (\varepsilon, - 2 d\rho^K + \rho^I \wedge \rho^J C_{IJ}{}^K) =$$

$$= D^{(G)} \varepsilon^K - 2(\varepsilon, R^K) \tag{4.18}$$

$$= D^{(G)} \varepsilon^K - \rho^I \varepsilon^J R_{IJ}{}^K \tag{4.19}$$

The algebra of parallel transport operators in G is thus in fact an algebra generating "anholonomized" (see (4.16)) General Coordinate Transformations (AGCT) on the G manifold. That gauge invariance is thus guaranteed by the General Covariance of the Lagrangian. Indeed, it is this gauge which reproduces the General Covariance Group, rather than GL(4R)-gauging, as commonly believed.

5. Dynamically Restricted AGCT gauges

We can construct the D_E for the factorized case. These correspond to translations in the quotient space G/H. From (4.5), (4.12b)

$$\rho^B (D_E) = 0 \quad , \quad \rho^F (D_E) = \delta_E^F \tag{5.1}$$

we find (still using the indices as in (4.13) - (4.14))

$$D_E = \Delta_E{}^Y D_Y{}^{(H)} \tag{5.2}$$

$$D_Y{}^{(H)} = \frac{\partial}{\partial x^Y} - \tfrac{1}{2} \sum_{A \in \{H\}} \rho_Y{}^A S_A \tag{5.3}$$

where $D^{(H)}$ is the H-covariant derivative, ρ^A is the post-factorization form on M itself and S_A is the Right-Invariant algebra (of left-translation), which commutes with the $D_A^{L.T.}$ and has structure constants $- C_{BE}{}^A$. For the Poincare group with Ξ^{ab} the SO(3.1) variable to be factorized,

$$\hat{\rho}^{ij} (\Xi,x) = (\Xi^{-1} d\Xi)^{ij} + \rho^{kl}(x) \; \Xi^{lj} \; \Xi^{ki} \tag{5.4}$$

$$\hat{\rho}^i (\Xi,x) = \Xi^{ki} \rho^k (x) \tag{5.5}$$

The $\rho^{ij}(x)$ are the connection potentials we introduced in (2.1) and used in Section 2, while we have used $\hat{\rho}^{ij}$ and $\hat{\rho}^i$ in these last equations to denote the pre-factorization one-forms. Our previous discussion of the parallel-transport or AGCT gauges holds for either set.

For the parallel-transport modified translation gauge we thus get the variations,

$$\delta\rho^{ij} = D^{(P)} \varepsilon^{ij} - \rho^k \varepsilon^l \; R_{kl}{}^{ij} \tag{5.6}$$

$$\delta\rho^i = D^{(P)} \varepsilon^i - \rho^k \varepsilon^l \; R_{kl}{}^i \tag{5.7}$$

where

$$D^{(P)} \eta^{ij} = d\eta^{ij} + \rho^{ik} \wedge \eta^{kj} - \rho^{kj} \wedge \eta^{ik} \tag{5.8}$$

$$D^{(P)}\ \eta^i = d\eta^i + \rho^{ik} \wedge \eta^k - \rho^k \wedge \eta^{ik} \tag{5.8b}$$

Compare (5.7) with (3.13b) and with (3.12)!

The parallel-transport gauges((5.6)-(5.7)) introduced by Von der Heyde [1976; see also Hehl et al 1976a] (in space-time; here generalized to the Group manifold), are still "semi-trivial", since they only reproduce General Covariance. Note that one very important point is guaranteed: we realize that AGCT form a group and can be exponentiated, since they are just a subset of the Group of Diffeomorphisms.

Now once a Lagrangian is introduced, it will yield equations of motion. These equations will restrict the values of the $R_{IJ}{}^{K}$ components in (4.19), (5.6-5.7). For instance, we have seen in (4.13) and (4.14) the results of the Lagrangian being gauge-invariant under a subgroup H (the Lorentz group in P and GP). First, the parallel transport generators in the H direction coincide with the Lie-Algebra L.I. generators, so that H-gauging is "conventional". Secondly, applying the equations of motion produces the cancellations (notation as in (4.13))

$$R_{AB}{}^{J} = 0 \qquad\qquad R_{IJ}{}^{E} = 0 \tag{5.9}$$

which makes the Dynamically Restricted AGCT gauge for translations coincide for ρ^E itself (the vierbein ρ^i in Gravity), with an ordinary translation gauge (but not for ρ^A, the connection ρ^{ij}). In Supergravity, where H is also the Lorentz group, D.R. A.G.C.T. translations thus also look like an ordinary gauge for ρ^i itself, but not when acting on ρ^{ij} or ρ^α, the spinor potential. Supersymmetry D.R. A.G.C.T. also produce a variation involving R^{ij} for both vector and spinor variations [Ne'eman and Regge, 1978a,b]:

$$\delta\rho^{ij} = D^{(GP)}\ \varepsilon^{ij} - \rho^c\ \varepsilon^d\ R_{cd}{}^{ij} - \rho^c\ \varepsilon^{\dot\alpha}\ R_{c\dot\alpha}{}^{ij} \tag{5.10}$$

$$\delta\rho^i = D^{(GP)}\ \varepsilon^i \tag{5.11}$$

$$\delta\rho^\alpha = D^{(GP)}\ \varepsilon^\alpha - \rho^c\ \varepsilon^d\ R_{cd}{}^{\alpha} \tag{5.12}$$

The components $R_{c\dot\alpha}{}^{ij}$ in (5.10) are essential to the "local supersymmetry" transformations of Supergravity [Freedman et al 1976; Deser and Zumino 1976; Freedman and van Nieuwenhuizen 1976]

$$D^{(GP)}\ \eta^{ij} = D^{(P)}\ \eta^{ij} \tag{5.13}$$

$$D^{(GP)}\ \eta^i = D^{(P)}\ \eta^i + \bar\rho\gamma^i\eta \tag{5.14}$$

$$D^{(GP)}\ \eta^\alpha = d\eta^\alpha + \tfrac{1}{2}\ (\rho^{ij}\ \sigma^{ij}) \wedge \eta^\alpha - \tfrac{1}{2}\ (\sigma^{ij}\rho)^\alpha \wedge \eta^{ij} \tag{5.15}$$

The action in supergravity is given by

$$A = \frac{1}{8} \int_{\mathcal{M}^4} (R^{ij} \wedge \zeta_{ij} + R^{\alpha} \wedge \overline{\zeta}_{\alpha}) \tag{5.16}$$

with, on a generic $\mathcal{M}^4$, the equations (anholonomic spinor indices are not explicited)

$$R^i = 0 \tag{5.17}$$

$$R^{ij} \rho^k \varepsilon_{ijk\ell} - 2i \overline{R}\gamma_5 \gamma_\ell \rho = 0 \tag{5.18}$$

$$\gamma^i \rho^i R = 0 \tag{5.19}$$

from which one derives

$$R_{IJ}{}^{i} \equiv 0 \quad \forall I,J; \quad R_{\alpha\beta} = 0; \quad R_{i\alpha} = 0; \quad R_{[ij]K} = 0 \quad \forall K \tag{5.20a}$$

$$R_{\alpha m}{}^{ij} \varepsilon_{ijk\ell} - R_{\alpha k}{}^{ij} \varepsilon_{ijm\ell} = 4i R_{mk} (\gamma_5 \gamma_\ell)_\alpha \tag{5.20b}$$

Equation (5.20b) and the $\overline{\varepsilon}^{\dot{\alpha}}$ variation in (5.10) are essential to supergravity. Indeed, the Supersymmetry transformations $\overline{\varepsilon}^{\dot{\alpha}}$ of Supergravity, which were derived directly, posed the problem of what we now know is a Dynamically Restricted AGCT, before it had ever been raised in Gravity, although the survival of the ε^{ℓ} transformation in (5.6) is completely analogous. In both theories, ρ^{ij} does not propagate and is extracted from $R^i = 0$ in terms of the other potentials, which tended to hide the physical importance of either (5.6) or (5.10).

We still have to discuss one more aspect of these theories. Working in the Group Manifold, how come we only use $\mathcal{M}^4$ for the integration in either (2.7) or (5.16)?

First, the reduction of the base space to the quotient of G (P or GP) by its subgroup H (L in both cases, although other such subgroups exist for GP): if $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{F}$ are the Lie algebras of G, H and G/H, the conditions may be

(a) weak reducibility: $[\mathcal{H}, \mathcal{F}] \subset \mathcal{F}$
(b) a symmetric manifold: $[\mathcal{F}, \mathcal{F}] \subset \mathcal{H}$
(c) an ideal: $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$

For G = P and H = L, all three hold, but for G = GP and H = L, we have (a) and (c); for $H = L \otimes A^2$ (left or right handed supersymmetry) we have (a) and (b); for $H = GP_1$ (supersymmetry with only nilpotent elements in the ring of parameters) we have (a), (b) and (c). Each case induces a different theory, with ordinary Supergravity corresponding to H = L. The MacDowell-Mansouri [1977] version of de Sitter

Gravity follows (a) and (b).

The homogeneous spaces $P/L = \mathcal{M}^4$, $GP/L = \mathfrak{R}^{4/4}$ correspond to the "factorized" theories. We conjecture that if a Lagrangian is H gauge-invariant, then it is H-factorizable as a consequence of the equations of motion. A heuristic proof of this hypothesis exists for solutions infinitesimally close to a factorized one. All such solutions can be reduced to factorized ones by an infinitesimal coordinate transformation on G. However, discrete families of factorized solutions with the same boundary conditions but topologically distinct may exist in the large.

This explains restricting the action integral to $\mathcal{M}^4$ in Gravity. In Supergravity, factorization reduces us to $\mathfrak{R}^{4/4}$. However, physics is seen to be completely determined by what happens on a simple $\mathcal{M}^4$. The transfer of information from any $\mathcal{M}^4$ to any other $\mathcal{M}^4$ in $\mathfrak{R}^{4/4}$ corresponds to our AGCT gauges. Partial $\mathcal{M}^4$ slices correspond to all possible supersymmetry-related conventional Supergravity theories.

6. GL(4R) and Affine Gauges

In trying to reproduce Gravity as a gauge theory, several authors[43-46] gauged GL(4R). This group seemed to fit that role, judging from the fact that in holonomic ("world tensor") coordinates, the covariant derivative is

$$D_\mu \phi = \partial_\mu + \Gamma_{\mu\nu}{}^{\rho} \; G^{\nu}{}_{\rho} \, \phi \qquad (6.1)$$

where $G^{\nu}{}_{\rho}$ is the GL(4R) matrix representation corresponding to the world-tensor field ϕ. Indeed, world tensors are classified by the finite irreducible (non-unitary) representations of GL(4R).

However, as proved by DeWitt [1964a] this has nothing to do with a Yang-Mills type gauge. We are dealing with the General Coordinate Transformation Group, and its structure constants do not correspond to a GL(4R) gauge. Indeed, as we have shown, they correspond to an AGCT translation gauge. However, as a group, the G.C.T.G. is represented over its linear subgroup, which happens to be GL(4R). This is true of any such non-linear group, owing to the role played by the Jacobian determinant. We refer the reader to De Witt's text[47] and to the work of A. Joseph and A. I. Solomon [1970], who, in working out the theory of Global and Infinitesimal Nonlinear Chiral transformations, explained the construction of representations and covariant derivatives for such non-linear (and non gauge-factorizable) groups. (In Chiral symmetry, Isospin is the linear subgroup).

One more general point about GL(4R). It had always been assumed in the folklore of general relativity (and often written in texts) that GL(nR) has no double-valued or spinorial representations. E. Cartan [1938] is referred to for this prevalent belief, in two of his theorems. As can be seen in the text, one theorem refers explicitly to spinors with a finite number of components. The other theorem is an overstatement: "the three Unimodular Groups in two dimensions have no multivalued representations". SU(2) is of course compact and simply connected; it is the covering group $SU(2) = \overline{SO(3)}$ of SO(3), where spinors are bivalued. SL(2C) has SU(2) as compact subgroup, and thus has the same topology. Indeed, $SL(2C) = \overline{SO(1.3)}$ is the covering group of the Lorentz group, and Lorentz bivalued representations become single valued here. Now it is true that $SL(2R) = \overline{SO(1.2)}$ and the bivalued representations of SO(1.2) become single-valued in SL(2R), which may explain the error in Cartan's theorem. However, SL(2R) has like SO(2) an infinite covering, and we can find in Bergmann's analysis [1947] double-valued representations of SL(2R), which become single valued in $\overline{SL(2R)} = \overline{\overline{SO(1.3)}}$, etc.

In a recent study [Ne'eman 1977, 1978], we have proved the existence of double-valued representations of SL(nR), GL(nR) and the G.C.T.G. in $\underline{n}$. These reduce to infinite direct sums of SO(n) or O(n) spinors. They are single valued in $\overline{SL(nR)}$, $\overline{GL(nR)}$ and $\overline{GCTG(n)}$. For such "polyfields", (6.1) can be used, provided the $G^{\nu}{}_{\rho}$ are infinite-dimensional.

These band-spinors or bandors are all known for SL(2R) [Bargmann 1947] and SL(3R) [Joseph 1970; Sigacki 1975]. They have now also been listed for SL(4R) [Sigacki 1978]. Note that $\overline{SL(4R)} = \overline{\overline{SO(3,3)}}$ and some of these representations had been included in a study of SO(3.3) by A. Kihlberg [1966].

Gauging GL(4R) [Yang 1974] prior to the introduction of bandors implied that spinor matter fields would not be minimally coupled therein. Note that most of these theories did not really exploit GL(4R) anyhow, and added metric restrictions [Mansouri and Chang 1976] which reduced GL(4R) to SO(1.3) or alternatively reduced GA(4R) - the Affine group in 4 dimensions, i.e. GL(4R) x T_4 - to Poincare SO(1.3) x T_4. However, we shall further discuss one consequence of starting with a larger group which is generally disregarded: the representation structure.

We now study the result of a GL(4R) gauge, in the context of a GA(4R) mixed-gauge (ordinary for GL(4R), D.R.AGCT for the translations).

It is [Hehl et al 1976b, 1977a] in the Metric-Affine theory and in its Spinor version [Hehl et al 1977b, 1978] and gauge [Lord, 1978] that the actual enlargement of the sets of connections, curvatures and currents are used, rather than an immediate restriction to Einstein's theory. The spinor matter field is now a polyfield, i.e. an infinite representation of GL(4R), with physical states given by GL(3R) bandors (this is the little group). One such bandor is $\mathcal{D}$ (½,0) which reduces under the spin to the sum $\frac{1}{2} \oplus \frac{5}{2} \oplus \frac{9}{2} \oplus \frac{13}{2} \oplus \ldots$

The connections now include in addition to those of P, ten β^{ij} symmetric in (i,j). The $D_{(ij)}$ generators in the (flat) group space generate shear (for traceless $D_{(ij)}$) and scaling (for the trace). We thus enlarge the angular momentum current tensor into the <u>hypermomentum</u> tensor, with shear, scale and spin currents in its intrinsic part:

$$h_{ab}{}^{\mu} = s_{ab}{}^{\mu} + \tfrac{1}{4}\, \eta_{ab}\, h^{\mu} + \bar{h}_{ab}{}^{\mu} \tag{6.2}$$

where η_{ab} is the Minkowski metric, h^{μ} is the scale (or dilation) current and $\bar{h}_{ab}{}^{\mu}$ is the shear current. Note that the "orbital" part of hypermomentum can be reduced to the set of time-derivatives of gravitational quadrupole moments [Dothan et al 1965; Hehl et al 1977b].

The Noether currents of the theory are given by

$$t_a{}^{\mu} = e^{-1} \frac{\delta \mathcal{L}}{\delta \rho_{\mu}{}^{a}} \tag{6.3}$$

$$h_{ab}{}^{\mu} = - e^{-1} \frac{\delta \mathcal{L}}{\delta \rho_{\mu}{}^{ab}} \tag{6.4}$$

The field equations are ($\mathcal{L}_o$ is the gravitational field Lagrangian)

$$\frac{\delta \mathcal{L}_o}{\delta \rho_{\mu}{}^{a}} = - 2\kappa \; e \; t_a{}^{\mu} \tag{6.5}$$

$$\frac{\delta \mathcal{L}_o}{\delta \rho_{\mu}{}^{ab}} = 2 \; e \; h_{ab}{}^{\mu} \tag{6.6}$$

Choosing the free action

$$A_o = \frac{1}{8K} \int (R^{[ab]} \wedge \zeta_{[ab]} + \beta \, Q^2) \tag{6.7}$$

where

$$Q_{\mu} = \frac{1}{4} \rho_{\mu\sigma}{}^{\sigma} \tag{6.8}$$

we find that equation (6.6) becomes again algebraic in relating connections to hypermomenta, and the $\rho_{\mu}{}^{(ab)}$ does not propagate. Note that the holonomic $\rho_{\mu(\sigma\tau)}$ corresponds to the Non-metricity tensor,

$$\rho_{\mu(\sigma\tau)} = - D_{\mu} \, g_{\sigma\tau} \tag{6.9}$$

which appears in the identity,

$$\rho_{\mu\nu}{}^{\sigma} \equiv g^{\sigma\tau} \, \Delta^{\sigma\beta\gamma}_{\nu\mu\tau} \left(\tfrac{1}{2} \partial_{\alpha} g_{\beta\gamma} - g_{\gamma\varepsilon} R^{\cdot\cdot\varepsilon}_{\alpha\beta} - \tfrac{1}{2} D_{\alpha} g_{\beta\gamma} \right) \tag{6.10}$$

$$\Delta^{\alpha\beta\gamma}_{\nu\mu\tau} := \delta^{\alpha}_{\nu} \, \delta^{\beta}_{\mu} \, \delta^{\gamma}_{\tau} + \delta^{\alpha}_{\mu} \, \delta^{\beta}_{\tau} \, \delta^{\gamma}_{\nu} - \delta^{\alpha}_{\tau} \, \delta^{\beta}_{\nu} \, \delta^{\gamma}_{\mu} \tag{6.11}$$

When no polyfields are present, there is no non-metricity. In the presence of intrinsic hypermomentum, non-metricity exists but is confined to the region where that matter exists, without propagating over intermediary regions. Again, the linearity of the Einstein Lagrangian in derivatives preserves the Riemannian properties of space-time. Macroscopically, one can always define a local Minkowski metric.

7. Extending the Poincaré Group: No-go Theorems

There are three main ways in which one has extended the Poincaré group:

- Conformally, into the simple group
$$Con(4R) = SU(2.2) = \overline{SO(4.2)}$$
- Linearly into the Affine group GA(4R)
- Spinorially, into GP

In fact, the latter extension can also be performed for SU(2.2), extending it thus further into SU(2.2/1) or SU(2.2/N). We have recently shown [Ne'eman and Sherry, 1978] that GA(nR) can be similarly extended into infinite-dimensional graded Lie Groups g GL(nR). Although we have constructed these graded-Affine groups for n = 2,3 only as yet, it appears plausible that g GL(4R) should also exist.

There is one important point we should note when gauging a group G larger than P. Although we may afterwards introduce constraints which will reduce the theory to Einstein's General Relativity, there are still traces of the larger group $G \supset P$. For example, the matter fields physical states have to fit in unitary representations of G. In our case, these would be Polyfields (with either integer or half-integer spins). In Conformal Relativity resulting from gauging the Conformal group [Englert et al 1975; Harnad and Pettitt 1976, 1977; Kaku et al 1977], these would be Mack's [1977] Unitary representations of the Conformal group.

Ogievetsky [1973] has proved that in a holonomic representation of Con(4R) $\cup$ GL(4R) generator algebras, closure occurs only over the entire analytical General Coordinate Transformations Group A. This is due to the commutators of the Special Conformal Transformation generators K_μ and the Shears $S_{(\mu\nu)}$, which keep generating operators

$$x_1^m \, x_2^n \, x_3^r \, x_4^s \, \partial_\mu$$

with ever-increasing powers (m, n, r, s). In more recent work [Borisov and Ogievetsky, 1974; Cho and Freund, 1975] this theorem has been applied to Gravitational theories. We would like to note the following theorems that can be drawn from Ogievetsky's:

(1) Assuming a theory to be (globally) invariant under Con(4R):= C and GL(4R):=G reduces it to a trivial S-matrix. Indeed, we find that if the Lagrangian $\mathcal{L}$ obeys

$$[\mathcal{L}\, , C\,] = 0 \quad , \quad [\mathcal{L}\, , G\,] = 0 \tag{7.1}$$

then

$$[\mathcal{L}\, , [C,G]\,] = [\mathcal{L}\, , A\,] = 0 \tag{7.2}$$

so that we have an infinite number of active-Symmetry Noether theorems.

(2) Gauging both C and G imposes a trivial S-matrix. This results from (7.2) because a local gauge includes the case of a constant (global) gauge.

These theorems are not modified by spontaneous breakdown via a Goldstone mechanism, since this still yields all global Noether currents.

A Higgs-Kibble mechanism breaks the local gauge group but preserves the global conservation laws. Thus, only a Higgs mechanism breaking the A gauge down to global (or local) P invariance can release the S-matrix from triviality.

It is important to remember that the Ogievetsky algebra is a representation of the Diffeomorphisms, but as such is purely a holonomic construct with no (active) Symmetry connotation. Symmetries and their local extension as Gauges are entirely anholonomic.

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References

Bargmann, V. 1947, Ann. Math. 48, 568-640.

Borisov, A. B. and V. I. Ogievetsky, 1974, Teoret. i. Mat. Fiz. 21, 329-342.

Cartan, E. 1922, Com. R. Ac. Sci. (Paris) 174, 593;
1923, 24, 25, Ann. Ec. Nor. Sup. 40, 325; 41, 1; 42, 17;
1938, Lecons sur la Theorie des Spineurs II (Hermann Editeurs, Paris) articles 85-86, pp. 87-91, 177, pp. 89-91.

Cho, Y. M. 1976, Phys. Rev. D14, 2521-2525.

Cho, Y. M. and P. G. O. Freund, 1975, Phys. Rev. D12, 1711-1720.

Deser, S. and B. Zumino, 1976, Phys. Lett. 62B, 335.

De Witt, B. S., 1964a "Dynamical Theory of Groups and Fields" in Relativity, Groups and Topology (Les Houches 1963 Seminar), ed. C. and B. De Witt, Gordon and Breach (N.Y), pp. 587-826. See in particular article 13, pp. 688-689.
- 1964b, Phys. Rev. Lett. 12, 742.
- 1967, Phys. Rev. 162, 1195; 162, 1239.

Dothan, Y., M. Gell-Mann and Y. Ne'eman 1965, Phys. Lett. 17, 148-151.

Englert, F. and R. Brout, 1964, Phys. Rev. Lett. 13, 321.

Englert, F., E. Gunzig, C. Truffin and P. Windey 1975, Phys. Lett. 57B, 73-77.

Feymman, R. P., 1963, Acta. Phys. Polon. 24, 697.

Faddeev, L. D., and V. N. Popov, 1967, Phys. Lett. 25B, 29.

Fradkin, E. S., and I. V. Tyutin, 1970, Phys. Rev. D2, 2841.

Freedman, D. Z., P. van Nieuwenhuizen and S. Ferrara, 1976, Phys. Rev. D13, 3214.

Freedman, D. Z. and P. van Nieuwenhuizen, 1976, Phys. Rev. D14, 912-916.

Fritzsch, H., and M. Gell-Mann, 1972, in Proc. 16th Intern. Conf. H.E.P., J.D.Jackson and A. Roberts,eds., NAL pub. (Batavia), 2, pp. 135-165.

Gell-Mann, M., 1962, Phys. Rev. 125, 1067-1084.

Gell-Mann, M. and S. L. Glashow, 1961, Ann. Phys. 15, 437.

Goldstone, J., 1961, Nuovo Cim. 19, 154.

Gross, D. and F. Wilczck, 1973, Phys. Rev. Lett. 30, 1343.

Guralnik, G. S., C. R. Hagen and T. W. B. Kibble, 1964, Phys. Rev. Lett. 13, 585.

Harnad, J. P., and R. B. Pettitt, 1976, J. Math. Phys. 17, 1827-1837;
- 1977, in Group Theoretical Methods in Physics (5th Int. Coll.) J. Patera and P. Winternitz, eds., Academic Press (N.Y.), pp. 277-301.

Hehl, F. W., 1970, "Spin und Torsion in der allgemeinen Relativitatstheorie, oder die Riemann-Cartansche Geometrie der Welt", Clausthal Tech. Univ. Thesis.

References - continued:

Hehl, F. W., P. v. d. Heyde, G. D. Kerlick and J. M. Nester, 1976a, Rev. Mod.Phys. 48, 393-416.

Hehl, F. W., G. D. Kerlick and P. v. d. Heyde, 1976b, Phys. Lett. 63B, 446-448.

Hehl, F. W., G. D. Kerlick, E. A. Lord and L. L. Smalley, 1977a, Phys. Lett. 70B, 70-72.

Hehl, F. W., E. A. Lord, and Y. Ne'eman, 1977b, Phys. Lett. 71B, 432-434.

Hehl, F. W., E. A. Lord and Y. Ne'eman, 1978, Phys. Rev. D17, 428-433.

von der Heyde, P., 1975, Lett. al Nuovo Cim. 14, 250-252.

von der Heyde, P., 1976, Phys. Lett. 58A, 141-143.

Higgs, P. W., 1964a, Phys. Lett. 12, 132;
- 1964, Phys. Rev. Lett. 13, 508;
- 1966, Phys. Rev. 145, 1156.

't Hooft, G., 1971, Nucl.Phys. B33, 173 and B35, 167.

't Hooft, G., and M. Veltman, 1972, Nucl. Phys. B50, 318.

Ionides, P. 1962, London University Thesis.

Joseph, A. and A. I. Solomon, 1970, J. Math. Phys., 11, 748-761.

Joseph, D. W. 1970, "Representations of the Algebra of SL(3R) with $\Delta j = 2$", University of Nebraska preprint, unpublished.

Kaku, M., P. K. Townsend and P. van Nieuwenhuizen 1977, Phys. Lett. 69B, 304-308.

Kibble, T. W. B. 1961, J. Math. Phys. 2, 212-221;
1967, Phys. Rev. 155, 1554.

Kihlberg, A. 1966, Arkiv f. Fysik, 32, 241-261.

Kobayashi, S. 1956, Canad. J. of Math. 8, 145-156.

Kobayashi, S. 1972, Transformation Groups in Differential Geometry, Springer Pr. (Berlin).

Lee, B. W. and J. Zinn-Justin, 1972, Phys. Rev. D5, 3121, 3137 and 3155.

Lee, T. D., S. Weinberg and B. Zumino 1967, Phys. Rev. Lett. 18, 1029.

Lord, E. A. 1978, Phys. Lett. 65A, 1-4.

MacDowell, S. W. and F. Mansouri 1977, Phys. Rev. Lett. 38, 739.

Mansouri, F. and L. N. Chang 1976, Phys. Rev. D13, 3192-3200.

Mack, G. 1975, DESY 75/50 report.

Nambu, Y. 1965, in Symmetry Principles at High Energy (Coral Gables 1965), B. Kursunoglu, P. Perlmutter and I. Sakmar, eds., Freeman Pub. (San Francisco) pp. 274-283.

Nambu, Y. and G. Iona-Lasinio, 1961, Phys. Rev. 122, 345.

Ne'eman, Y. 1961, Nucl. Phys. 26, 222-229.

Ne'eman, Y. 1977, Proc. Natl. Acad. Sci. U.S.A. 74, 4157-4159.

Ne'eman, Y. 1978, Ann. Inst. Henri Poincare 28.

Ne'eman, Y. and T. Regge 1978a, Rivista del Nuovo Cim.

References - continued:

Ne'eman, Y. and T. Regge 1978b, Phys. Lett.

Ne'eman, Y. and T. Sherry 1978

Ogievetsky, V. I. 1973, Lett. al Nuovo Cim. 8, 988-990.

Palatini, A. 1919, Rend. Circ. Mat. Palermo 43, 203.

Petti, R. J. 1976, Gen. Rel and Grav. 7, 869-883.

Politzer, H. D. 1973, Phys. Rev. Lett. 30, 1346.

Salam, A. 1968, in Proc. Eighth Nobel Symposium, N. Svartholm ed., Almquist and Wiksell pub. (Stockholm) 367-378.

Salam, A. and J. C. Ward 1961, Nuovo Cim. 20, 419.

Schouten, J. A. 1954, Ricci Calculus, Springer Pub. (Berlin).

Sciama, D. W. 1962, in Recent Developments in General Relativity, Pergamon Press (N.Y.-Warsaw) pp. 415-440.

Shaw, T. 1954, Thesis.

Sijacki, Dj. 1975, J. Math. Phys. 16, 298.

Sijacki, Dj. 1978, to be pub.

Teitelboim, C. 1977, Phys. Rev. Lett. 38, 1106-1110.

Thierry-Mieg, J. 1978, to be pub.

Thirring, W., 1978, in this volume.

Trautman, A. 1972, Bull. Acad. Polon. Sci., Ser. Sci. Mat. Ast. et Phys. 20, 185 and 583.

Trautman, A. 1973, Symposia Mathematica 12, 139-162.

Utiyama, R. 1956, Phys. Rev. 101, 1597.

Veltman, M. 1970, Nucl. Phys. B21, 288.

Weinberg, S. 1967, Phys. Rev. Lett. 19, 1264.

Weinberg, S. 1973, Phys. Rev. Lett. 31, 494.

Weyl, H. 1918, Sitzungsberichte d. Preuss. Akad. d. Wissensch.

Weyl, H. 1929, Zeit. f. Phys. 56, 330.

Weyl, H. 1950, Phys. Rev. 77, 699.

Yang, C. N. 1974, Phys. Rev. Lett. 33, 445.

Yang, C. N. and R. L. Mills 1954, Phys. Rev. 96, 191.