Lorentz Invariance and the Gravitational Field

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An argument leading from the Lorentz invariance of the Lagrangian to the introduction of the gravitational field is presented. Utiyama's discussion is extended by considering the 10-parameter group of inhomogeneous Lorentz transformations, involving variation of the coordinates as well as the field variables. It is then unnecessary to introduce *a priori* curvilinear coordinates or a Riemannian metric, and the new field variables introduced as a consequence of the argument include the vierbein components $h_k{}^\mu$ as well as the "local affine connection" $A^i{}_{j\mu}$. The extended transformations for which the 10 parameters become arbitrary functions of position may be interpreted as general coordinate transformations and rotations of the vierbein system. The free Lagrangian for the new fields is shown to be a function of two covariant quantities analogous to $F_{\mu\nu}$ for the electromagnetic field, and the simplest possible form is just the usual curvature scalar density expressed in terms of $h_k{}^\mu$ and $A^i{}_{j\mu}$. This Lagrangian is of first order in the derivatives, and is the analog for the vierbein formalism of Palatini's Lagrangian. In the absence of matter, it yields the familiar equations $R_{\mu\nu}=0$ for empty space, but when matter is present there is a difference from the usual theory (first pointed out by Weyl) which arises from the fact that $A^i{}_{j\mu}$ appears in the matter field Lagrangian, so that the equation of motion relating $A^i{}_{j\mu}$ to $h_k{}^\mu$ is changed. In particular, this means that, although the covariant derivative of the metric vanishes, the affine connection $\Gamma^\lambda{}_{\mu\nu}$ is nonsymmetric. The theory may be reexpressed in terms of the Christoffel connection, and in that case additional terms quadratic in the "spin density" $S^k{}_{ij}$ appear in the Lagrangian. These terms are almost certainly too small to make any experimentally detectable difference to the predictions of the usual metric theory.

1. INTRODUCTION

IT has long been realized that the existence of certain fields, notably the electromagnetic field, can be related to invariance properties of the Lagrangian.[1] Thus, if the Lagrangian is invariant under phase transformations $\psi \to e^{i\epsilon\lambda}\psi$, and if we wish to make it invariant under the general gauge transformations for which λ is a function of x, then it is necessary to introduce a new field A_μ which transforms according to $A_\mu \to A_\mu - \partial_\mu\lambda$, and to replace $\partial_\mu\psi$ in the Lagrangian by a "covariant derivative" $(\partial_\mu+ieA_\mu)\psi$. A similar argument has been applied by Yang and Mills[2] to isotopic spin rotations, and in that case yields a triplet of vector fields. It is thus an attractive idea to relate the existence of the gravitational field to the Lorentz invariance of the Lagrangian. Utiyama[3] has proposed a method which leads to the introduction of 24 new field variables $A^{ij}{}_\mu$ by considering the homogeneous Lorentz transformations specified by six parameters ϵ^{ij}. However, in order to do this it was necessary to introduce *a priori* curvilinear coordinates and a set of 16 parameters $h_k{}^\mu$. Initially, the $h_k{}^\mu$ were treated as given functions of x, but at a later stage they were regarded as field variables and interpreted as the components of a vierbein system in a Riemannian space. This is a rather unsatisfactory procedure since it is the purpose of the discussion to supply an argument for introducing the gravitational field variables, which include the metric as well as the affine connection. The new field variables $A^{ij}{}_\mu$ were subsequently related to the Christoffel connection $\Gamma^\lambda{}_{\mu\nu}$ in the Riemannian space, but this could only be done uniquely by making the *ad hoc* assumption

that the quantity $\Gamma'^\lambda{}_{\mu\nu}$ calculated from $A^{ij}{}_\mu$ was symmetric.

It is the purpose of this paper to show that the vierbein components $h_k{}^\mu$, as well as the "local affine connection" $A^{ij}{}_\mu$, can be introduced as new field variables analogous to A_μ if one considers the full 10-parameter group of inhomogeneous Lorentz transformations in place of the restricted six-parameter group. This implies that one must consider transformations of the coordinates as well as the field variables, which will necessitate some changes in the argument, but it also means that only one system of coordinates is required, and that a Riemannian metric need not be introduced *a priori*. The interpretation of the theory in terms of a Riemannian space may be made later if desired. The starting point of the discussion is the ordinary formulation of Lorentz invariance (including translational invariance) in terms of rectangular coordinates in flat space. We shall follow the analogy with gauge transformations as far as possible, and for purposes of comparison we give in Sec. 2 a brief discussion of linear transformations of the field variables. This is essentially a summary of Utiyama's argument, though the emphasis is rather different, particularly with regard to the covariant and noncovariant conservation laws.

In Sec. 3 we discuss the invariance under Lorentz transformations, and in Sec. 4 we extend the discussion to the corresponding group in which the ten parameters become arbitrary functions of position. We show that to maintain invariance of the Lagrangian, it is necessary to introduce 40 new variables so that a suitable covariant derivative may be constructed. To make the action integral invariant, one actually requires the Lagrangian to be an invariant density rather than an invariant, and one must, therefore, multiply the invariant by a suitable (and uniquely determined) function of the new fields. In Sec. 5 we consider the possible forms of the free Lagrangian for the new fields. As in the case of the

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[1] See, for example, H. Weyl, *Gruppentheorie und Quantenmechanik* (S. Hirzel, Leipzig, 1931), 2nd ed., Chap. 2, p. 89; and earlier references cited there.
[2] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[3] Ryoyu Utiyama, Phys. Rev. **101**, 1597 (1956).

electromagnetic field, we choose the Lagrangian of lowest degree which satisfies the invariance requirements.

The geometrical interpretation in terms of a Riemannian space is discussed in Sec. 6, where we show that the free Lagrangian we have obtained is just the usual curvature scalar density, though expressed in terms of an affine connection $\Gamma^\lambda_{\mu\nu}$ which is not necessarily symmetric. In fact, when no matter is present it is symmetric as a consequence of the equations of motion, but otherwise it has an antisymmetric part expressible in terms of the "spin density" $\mathfrak{S}^\mu{}_{ij}$. Thus there is a difference between this theory and the usual metric theory of gravitation. This difference was first pointed out by Weyl,[4] and has more recently been discussed by Sciama.[5] It arises from the fact that our free Lagrangian is of first order in the derivatives, with the $h_k{}^\mu$ and $A^{ij}{}_\mu$ as independent variables. It is possible to re-express the theory in terms of the Christoffel connection $^0\Gamma^\lambda_{\mu\nu}$ or its local analog $^0A^{ij}{}_\mu$, and this is done in Sec. 7. In that case, additional terms quadratic in $\mathfrak{S}^\mu{}_{ij}$, and multiplied by the gravitational constant, appear in the Lagrangian.

2. LINEAR TRANSFORMATIONS

We consider a set of field variables $\chi_A(x)$, which we regard as the elements of a column matrix $\chi(x)$, with the Lagrangian

$$L(x) \equiv L\{\chi(x), \chi_{,\mu}(x)\},$$

where $\chi_{,\mu} = \partial_\mu \chi$. We also consider linear transformations of the form

$$\delta\chi = \epsilon^a T_a \chi, \tag{2.1}$$

where the ϵ^a are n constant infinitesimal parameters, and the T_a are n given matrices satisfying commutation rules appropriate to the generators of a Lie group,

$$[T_a, T_b] = f_a{}^c{}_b T_c.$$

The Lagrangian is invariant under these transformations if the n identities

$$(\partial L/\partial \chi) T_a \chi + (\partial L/\partial \chi_{,\mu}) T_a \chi_{,\mu} \equiv 0, \tag{2.2}$$

are satisfied, and we shall assume that this is so. Note that $\partial/\partial\chi$ must be regarded as a row matrix. The equations of motion imply n conservation laws

$$J^\mu{}_{a,\mu} = 0,$$

where the "currents" are defined by[6]

$$J^\mu{}_a \equiv -(\partial L/\partial \chi_{,\mu}) T_a \chi. \tag{2.3}$$

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[4] H. Weyl, Phys. Rev. **77**, 699 (1950).

[5] D. W. Sciama, *Festschrift for Infeld* (Pergamon Press, New York), to be published.

[6] We have defined $J^\mu{}_a$ with the opposite sign to that used by Utiyama.[3] This is because with this choice of sign the analogous quantity for translations is $T^\mu{}_\nu$ rather than $-T^\mu{}_\nu$. The change may be considered as a change of sign of ϵ^a and T_a, and there is a corresponding change of sign in (2.6). This convention has the additional advantage that the "local affine connection" $A^{ij}{}_\mu$ defined in Sec. 4 specifies covariant derivatives according to the same rule as $\Gamma^\lambda_{\mu\nu}$.

Now, under the more general transformations of the form (2.1), but in which the parameters ϵ^a become arbitrary functions of position, the Lagrangian is no longer invariant, because the derivatives transform according to

$$\delta\chi_{,\mu} = \epsilon^a T_a \chi_{,\mu} + \epsilon^a{}_{,\mu} T_a \chi, \tag{2.4}$$

and the terms in $\epsilon^a{}_{,\mu}$ do not cancel. In fact, one finds

$$\delta L = -\epsilon^a{}_{,\mu} J^\mu{}_a.$$

However, one can obtain a modified Lagrangian which is invariant by replacing $\chi_{,\mu}$ in L by a quantity $\chi_{;\mu}$ which transforms according to

$$\delta\chi_{;\mu} = \epsilon^a T_a \chi_{;\mu}. \tag{2.5}$$

To do this[7] it is necessary to introduce $4n$ new field variables $A^a{}_\mu$ whose transformation properties involve $\epsilon^a{}_{,\mu}$. In fact, if one takes

$$\chi_{;\mu} \equiv \chi_{,\mu} + A^a{}_\mu T_a \chi, \tag{2.6}$$

then the condition (2.5) determines the transformation properties of the new fields uniquely. They are

$$\delta A^a{}_\mu = \epsilon^b f_b{}^a{}_c A^c{}_\mu - \epsilon^a{}_{,\mu}. \tag{2.7}$$

In this way one obtains the invariant Lagrangian

$$L'\{\chi, \chi_{,\mu}, A^a{}_\mu\} \equiv L\{\chi, \chi_{;\mu}\}.$$

The expression $\chi_{;\mu}$ may be called the covariant derivative of χ with respect to the transformations (2.1). One may define covariant currents by

$$J'^\mu{}_a \equiv -(\partial L'/\partial A^a{}_\mu) \equiv -(\partial L/\partial \chi_{;\mu}) T_a \chi, \tag{2.8}$$

where L is regarded as a function of χ and $\chi_{;\mu}$. They transform linearly according to

$$\delta J'^\mu{}_a = -\epsilon^b f_b{}^c{}_a J'^\mu{}_c,$$

and their covariant divergences vanish in virtue of the equations of motion and the identities (2.2):

$$J'^\mu{}_{a;\mu} \equiv J'^\mu{}_{a,\mu} - A^b{}_\mu f_b{}^c{}_a J'^\mu{}_c$$
$$= 0.$$

Two covariant differentiations do not in general commute. From (2.6) one finds

$$\chi_{;\mu\nu} - \chi_{;\nu\mu} = F^a{}_{\mu\nu} T_a \chi,$$

where

$$F^a{}_{\mu\nu} \equiv A^a{}_{\mu,\nu} - A^a{}_{\nu,\mu} - f_b{}^a{}_c A^b{}_\mu A^c{}_\nu. \tag{2.9}$$

Unlike $A^a{}_\mu$, the expression $F^a{}_{\mu\nu}$ is a covariant quantity transforming according to

$$\delta F^a{}_{\mu\nu} = \epsilon^b f_b{}^a{}_c F^c{}_{\mu\nu},$$

and one may, therefore, define its covariant derivative in an obvious manner. It satisfies the cyclic identity

$$F^a{}_{\mu\nu;\rho} + F^a{}_{\nu\rho;\mu} + F^a{}_{\rho\mu;\nu} \equiv 0.$$

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[7] For a full discussion, see footnote 3.

It remains to find a free Lagrangian L_0 for the new fields. Clearly L_0 must be separately invariant, and it is easy to see[8] that this implies that it must contain $A^a{}_\mu$ only through the covariant combination $F^a{}_{\mu\nu}$. The simplest such Lagrangian is[8]

$$L_0 = -\tfrac{1}{4}F^a{}_{\mu\nu}F_a{}^{\mu\nu}, \qquad (2.10)$$

where the tensor indices are raised with the flat-space metric $\eta^{\mu\nu}$ with diagonal elements $(1, -1, -1, -1)$, and the index a is lowered with the metric[8a]

$$g_{ab} \equiv f_a{}^c{}_d f_c{}^d{}_b$$

associated with the Lie group (except of course for a one-parameter group). It is clear that this Lagrangian is not unique. All that is required is that it should be a scalar both in coordinate space and in the Lie-group space, and one could add to it terms of higher degree in $F^a{}_{\mu\nu}$. However, it seems reasonable to choose the Lagrangian of lowest degree which satisfies the invariance requirements.

With the choice (2.10) of L_0, the equations of motion for the new fields are

$$F_a{}^{\mu\nu}{}_{;\nu} = J'^\mu{}_a.$$

Because of the antisymmetry of $F_a{}^{\mu\nu}$ one can define another current which is conserved in the strict sense:

$$(J'^\mu{}_a + j^\mu{}_a)_{,\mu} = 0, \qquad (2.11)$$

where

$$j^\mu{}_a \equiv A^b{}_\nu f_b{}^c{}_a F_c{}^{\mu\nu}.$$

This extra current $j^\mu{}_a$ may be regarded as the current of the new field $A^a{}_\mu$ itself, since it is expressible in the form

$$j^\mu{}_a \equiv -(\partial L_0/\partial A^a{}_\mu) \equiv -(\partial L_0/\partial A^b{}_{\nu,\mu})f_a{}^b{}_c A^c{}_\nu, \quad (2.12)$$

which should be compared with (2.8). Note, however, that it is not a covariant quantity. To obtain a strict conservation law one must sacrifice the covariance of the current.

3. LORENTZ TRANSFORMATIONS

We now wish to consider infinitesimal variations of both the coordinates and the field variables,

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu + \delta x^\mu, \\ \chi(x) &\to \chi'(x') = \chi(x) + \delta\chi(x). \end{aligned} \qquad (3.1)$$

It will be convenient to allow for the possibility that the Lagrangian may depend on x explicitly. Then, under a variation (3.1), the change in L is

$$\delta L \equiv (\partial L/\partial \chi)\delta\chi + (\partial L/\partial \chi_{,\mu})\delta\chi_{,\mu} + (\partial L/\partial x^\mu)\delta x^\mu,$$

where $\partial L/\partial x^\mu$ denotes the partial derivative with fixed χ. It is sometimes useful to consider also the variation at a fixed value of x,

$$\delta_0 \chi = \chi'(x) - \chi(x) = \delta\chi - \delta x^\mu \chi_{,\mu}. \qquad (3.2)$$

In particular, it is obvious that δ_0 commutes with ∂_μ, whence

$$\delta\chi_{,\mu} = (\delta\chi)_{,\mu} - (\delta x^\nu)_{,\mu}\chi_{,\nu}. \qquad (3.3)$$

The action integral

$$I(\Omega) \equiv \int_\Omega L(x)d_4x$$

over a space-time region Ω is transformed under (3.1) into

$$I'(\Omega) \equiv \int_\Omega L'(x')\|\partial_\nu x'^\mu\|d_4x.$$

Thus the action integral over an arbitrary region is invariant if[9]

$$\delta L + L(\delta x^\mu)_{,\mu} \equiv \delta_0 L + (L\delta x^\mu)_{,\mu} \equiv 0. \qquad (3.4)$$

This is of course the typical transformation law of an invariant density.

We now consider the specific case of Lorentz transformations,

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu + \epsilon^\mu, \qquad \delta\chi = \tfrac{1}{2}\epsilon^{\mu\nu}S_{\mu\nu}\chi, \qquad (3.5)$$

where ϵ^μ and $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ are 10 real infinitesimal parameters, and the $S_{\mu\nu}$ are matrices satisfying

$$S_{\mu\nu} + S_{\nu\mu} = 0,$$
$$[S_{\mu\nu}, S_{\rho\sigma}] = \eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\nu\sigma}S_{\mu\rho} - \eta_{\mu\rho}S_{\nu\sigma} \equiv \tfrac{1}{2}f_{\mu\nu}{}^{\kappa\lambda}{}_{\rho\sigma}S_{\kappa\lambda}.$$

From (3.3) one has

$$\delta\chi_{,\mu} = \tfrac{1}{2}\epsilon^{\rho\sigma}S_{\rho\sigma}\chi_{,\mu} - \epsilon^\rho{}_\mu \chi_{,\rho}. \qquad (3.6)$$

Moreover, since $(\delta x^\mu)_{,\mu} = \epsilon^\mu{}_\mu = 0$, the condition (3.4) for invariance of the action integral again reduces to $\delta L \equiv 0$, and yields the 10 identities[10]

$$\partial L/\partial x^\rho \equiv L_{,\rho} - (\partial L/\partial \chi)\chi_{,\rho} - (\partial L/\partial \chi_{,\mu})\chi_{,\mu\rho} \equiv 0, \qquad (3.7)$$

$$(\partial L/\partial \chi)S_{\rho\sigma}\chi + (\partial L/\partial \chi_{,\mu})(S_{\rho\sigma}\chi_{,\mu} + \eta_{\mu\rho}\chi_{,\sigma} - \eta_{\mu\sigma}\chi_{,\rho}) \equiv 0. \quad (3.8)$$

These are evidently the analogs of the identities (2.2), and we shall assume that they are satisfied. Note that (3.7), which express the conditions for translational invariance, are equivalent to the requirement that L be explicitly independent of x, as might be expected.

As before, the equations of motion may be used to obtain 10 conservation laws which follow from these identities, namely,

$$T^\mu{}_{\rho,\mu} = 0, \qquad (S^\mu{}_{\rho\sigma} - x_\rho T^\mu{}_\sigma + x_\sigma T^\mu{}_\rho)_{,\mu} = 0,$$

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[8] There could of course be a constant factor multiplying (2.10), but this can be absorbed by a trivial change of definition of $A^a{}_\mu$ and T_a.

[8a] The discussion here applies only to semisimple groups since otherwise g_{ab} is singular. (I am indebted to the referee for this remark.)

[9] See L. Rosenfeld, Ann. Physik 5, 113 (1930).

[10] Compare L. Rosenfeld, Ann. inst. Henri Poincaré 2, 25 (1931).

where

$$T^\mu{}_\rho \equiv (\partial L/\partial X_{,\mu})X_{,\rho} - \delta^\mu{}_\rho L, \quad S^\mu{}_{\rho\sigma} \equiv -(\partial L/\partial X_{,\mu})S_{\rho\sigma}X.$$

These are the conservation laws of energy, momentum, and angular momentum.

It is instructive to examine these transformations in terms of the variation $\delta_0 X$ also, which in this case is

$$\delta_0 X = -\epsilon^\rho \partial_\rho X + \tfrac{1}{2}\epsilon^{\rho\sigma}(S_{\rho\sigma} + x_\rho \partial_\sigma - x_\sigma \partial_\rho)X.$$

On comparing this with (2.1), one sees that the role of the matrices T_a is played by the differential operators $-\partial_\mu$ and $S_{\rho\sigma} + x_\rho\partial_\sigma - x_\sigma\partial_\rho$. Thus, by analogy with the definition (2.3) of the currents $J^\mu{}_a$, one might expect the currents in this case to be

$$J^\mu{}_\rho \equiv (\partial L/\partial X_{,\mu})X_{,\rho}, \quad J^\mu{}_{\rho\sigma} \equiv S^\mu{}_{\rho\sigma} - x_\rho J^\mu{}_\sigma + x_\sigma J^\mu{}_\rho,$$

corresponding to the parameters ϵ^ρ, $\epsilon^{\rho\sigma}$, respectively. However, in terms of δ_0, the condition for invariance (3.4) is not simply $\delta_0 L \equiv 0$, and the additional term $\delta x^\rho L_{,\rho}$ is responsible for the appearance of the term $L_{,\rho}$ in the identities (3.7), and hence for the term $\delta^\mu{}_\rho L$ in $T^\mu{}_\rho$.

4. GENERALIZED LORENTZ TRANSFORMATIONS

We now turn to a consideration of the generalized transformations (3.5) in which the parameters ϵ^μ and $\epsilon^{\mu\nu}$ become arbitrary functions of position. It is more convenient, and clearly equivalent, to regard as independent functions $\epsilon^{\mu\nu}$ and

$$\xi^\mu \equiv \epsilon^\mu{}_\nu x^\nu + \epsilon^\nu,$$

since this avoids the explicit appearance of x. Moreover, one could consider generalized transformations with $\xi^\mu = 0$ but nonzero $\epsilon^{\mu\nu}$, so that the coordinate and field transformations can be completely separated. In view of this fact, it is convenient to use Latin indices for ϵ^{ij} (and for the matrices S_{ij}), retaining the Greek ones for ξ^μ and x^μ. Thus the transformations under consideration are

$$\delta x^\mu = \xi^\mu, \quad \delta X = \tfrac{1}{2}\epsilon^{ij}S_{ij}X \qquad (4.1)$$

or

$$\delta_0 X = -\xi^\mu X_{,\mu} + \tfrac{1}{2}\epsilon^{ij}S_{ij}X. \qquad (4.2)$$

This notation emphasizes the similarity of the ϵ^{ij} transformations to the linear transformations discussed in Sec. 2. These transformations alone were considered by Utiyama.[3] Evidently, the four functions ξ^μ specify a general coordinate transformation. The geometrical significance of the ϵ^{ij} will be discussed in Sec. 6.

According to our convention, the differential operator ∂_μ must have a Greek index. However, in the Lagrangian function L it would be inconvenient to have two kinds of indices, and we shall, therefore, regard L as a given function of X and X_k (no comma),[11] satisfying the identities (3.7) and (3.8). The original Lagrangian is then

obtained by setting

$$X_k = \delta_k{}^\mu X_{,\mu}.$$

It is of course not invariant under the generalized transformations (4.1), but we shall later obtain an invariant expression by replacing X_k by a suitable quantity $X_{;k}$.

The transformation of $X_{,\mu}$ is given by

$$\delta X_{,\mu} = \tfrac{1}{2}\epsilon^{ij}S_{ij}X_{,\mu} + \tfrac{1}{2}\epsilon^{ij}{}_{,\mu}S_{ij}X - \xi^\nu{}_{,\mu}X_{,\nu}, \qquad (4.3)$$

and so the original Lagrangian transforms according to

$$\delta L \equiv -\xi^\rho{}_{,\mu}J^\mu{}_\rho - \tfrac{1}{2}\epsilon^{ij}{}_{,\mu}S^\mu{}_{ij}.$$

Note that it is $J^\mu{}_\rho$ rather than $T^\mu{}_\rho$ which appears here. The reason for this is that we have not included the extra term $L(\delta x^\mu)_{,\mu}$ in (3.4). The left-hand side of (3.4) actually has the value

$$\delta L + L(\delta x^\mu)_{,\mu} \equiv -\xi^\rho{}_{,\mu}T^\mu{}_\rho - \tfrac{1}{2}\epsilon^{ij}{}_{,\mu}S^\mu{}_{ij}.$$

We now look for a modified Lagrangian which makes the action integral invariant. The additional term just mentioned is of a different kind to those previously encountered, in that it involves L and not $\partial L/\partial X_k$. In particular, it includes contributions from terms in L which do not contain derivatives. Thus it is clear that we cannot remove it by replacing the derivative by a suitable covariant derivative. For this reason, we shall consider the problem in two stages. We first eliminate the noninvariance arising from the fact that $X_{,\mu}$ is not a covariant quantity, and thus obtain an expression L' satisfying

$$\delta L' \equiv 0. \qquad (4.4)$$

Then, because the condition (3.4) for invariance of the action integral requires the Lagrangian to be an invariant density rather than an invariant, we make a further modification, replacing L' by $\mathfrak{L}'$, which satisfies

$$\delta\mathfrak{L}' + \xi^\mu{}_{,\mu}\mathfrak{L}' \equiv 0. \qquad (4.5)$$

The first part of this program can be accomplished by replacing X_k in L by a "covariant derivative" $X_{;k}$ which transforms according to

$$\delta X_{;k} = \tfrac{1}{2}\epsilon^{ij}S_{ij}X_{;k} - \epsilon^i{}_k X_{;i}. \qquad (4.6)$$

The condition (4.4) then follows from the identities (3.8). To do this it is necessary to introduce forty new field variables. We consider first the ϵ^{ij} transformations, and eliminate the $\epsilon^{ij}{}_{,\mu}$ term in (4.3) by setting[12]

$$X_{|\mu} \equiv X_{,\mu} + \tfrac{1}{2}A^{ij}{}_\mu S_{ij}X, \qquad (4.7)$$

where the $A^{ij}{}_\mu = -A^{ji}{}_\mu$ are 24 new field variables. We can then impose the condition

$$\delta X_{|\mu} = \tfrac{1}{2}\epsilon^{ij}S_{ij}X_{|\mu} - \xi^\nu{}_{,\mu}X_{|\nu} \qquad (4.8)$$

which determines the transformation properties of $A^{ij}{}_\mu$

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[11] Note that since we are using Latin indices for S_{ij} the various tensor components of X must also have Latin indices, and for spinor components the Dirac matrices must be γ^k.

[12] Our $A^{ij}{}_\mu$ differs in sign from that of Utiyama.[3] Compare footnote 6.

uniquely. They are

$$\delta A^{ij}{}_{\mu} = \epsilon^i{}_k A^{kj}{}_{\mu} + \epsilon^j{}_k A^{ik}{}_{\mu} - \xi^v{}_{,\mu} A^{ij}{}_{v} - \epsilon^{ij}{}_{,\mu}. \qquad (4.9)$$

The position with regard to the last term in (4.3) is rather different. The term involving $\epsilon^{ij}{}_{,\mu}$ is inhomogeneous in the sense that it contains χ rather than $\chi_{,\mu}$, just like the second term of (2.4), but this is not true of the last term.[13] Correspondingly, the transformation law (4.8) of $\chi_{|\mu}$ is already homogeneous. This means that to obtain an expression $\chi_{;k}$ transforming according to (4.6) we should add to $\chi_{|\mu}$ not a term in χ but rather a term in $\chi_{|\mu}$ itself. In other words, we can merely multiply by a new field:

$$\chi_{;k} \equiv h_k{}^{\mu} \chi_{|\mu}. \qquad (4.10)$$

Here the $h_k{}^{\mu}$ are 16 new field variables with transformation properties determined by (4.6) to be

$$\delta h_k{}^{\mu} = \xi^{\mu}{}_{,v} h_k{}^{v} - \epsilon^i{}_k h_i{}^{\mu}. \qquad (4.11)$$

It should be noted that the fields $h_k{}^{\mu}$ and $A^{ij}{}_{\mu}$ are quite independent and unrelated at this stage, though of course they will be related by equations of motion.

We have now found an invariant L'. We can easily obtain an invariant density $\mathfrak{L}'$ by multiplying by a suitable function of the fields already introduced:

$$\mathfrak{L}' \equiv \mathfrak{H} L'.$$

Then (4.5) is satisfied provided that $\mathfrak{H}$ is itself an invariant density,

$$\delta \mathfrak{H} + \xi^{\mu}{}_{,\mu} \mathfrak{H} \equiv 0.$$

It is easy to see that the only function of the new fields which obeys this transformation law, and does not involve derivatives, is

$$\mathfrak{H} = [\det(h_k{}^{\mu})]^{-1},$$

where the arbitrary constant factor has been chosen so that $\mathfrak{H}$ reduces to 1 when $h_k{}^{\mu}$ is set equal to $\delta_k{}^{\mu}$.[14]

The final form of our modified Lagrangian is

$$\mathfrak{L}\{\chi, \chi_{,\mu}, h_k{}^{\mu}, A^{ij}{}_{\mu}\} \equiv \mathfrak{H} L\{\chi, \chi_{;k}\}.$$

(We can drop the prime without risk of confusion.) It may be asked whether this Lagrangian is unique in the same sense as the modified Lagrangian L' of Sec. 2, and in fact it is easy to see that it is not. The reason for this is that if one starts with two Lagrangians L_1 and L_2 which differ by an explicit divergence, and are therefore

equivalent, then the modified Lagrangians $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are not necessarily equivalent. Consider for example the Lagrangian for a real scalar field written in its first-order form

$$L_1 = \pi^k \varphi_{,k} - \tfrac{1}{2} \pi^k \pi_k - \tfrac{1}{2} m^2 \varphi^2. \qquad (4.12)$$

This is equivalent to

$$L_2 = -\pi^k{}_{,k} \varphi - \tfrac{1}{2} \pi^k \pi_k - \tfrac{1}{2} m^2 \varphi^2, \qquad (4.13)$$

but the corresponding modified Lagrangians differ by

$$\mathfrak{L}_1 - \mathfrak{L}_2 \equiv \mathfrak{H}(\pi^k \varphi)_{;k}$$
$$\equiv \mathfrak{H} h_k{}^{\mu} [(\pi^k \varphi)_{,\mu} + A^k{}_{i\mu} \pi^i \varphi] \qquad (4.14)$$

which is not an explicit divergence. Thus in order to define the modified Lagrangian $\mathfrak{L}$ completely it would be necessary to specify which of the possible equivalent forms of the original Lagrangian is to be chosen. The reasons for this situation and the problem of choosing the correct form are discussed in the Appendix.

As in Sec. 2, one may define modified "currents" in terms of $L \equiv L\{\chi, \chi_{;k}\}$ by

$$\mathfrak{T}^k{}_{\mu} \equiv \partial \mathfrak{L}/\partial h_k{}^{\mu} \equiv \mathfrak{H} b^i{}_{\mu} \{ (\partial L/\partial \chi_{;k}) \chi_{;i} - \delta^k{}_i L \}, \qquad (4.15)$$

$$\mathfrak{S}^{\mu}{}_{ij} \equiv -2(\partial \mathfrak{L}/\partial A^{ij}{}_{\mu}) \equiv -\mathfrak{H} h_k{}^{\mu} (\partial L/\partial \chi_{;k}) S_{ij} \chi, \qquad (4.16)$$

where $b^i{}_{\mu}$ is the inverse of $h_i{}^{\mu}$, satisfying

$$b^i{}_{\mu} h_i{}^{v} = \delta_{\mu}{}^{v}, \quad b^i{}_{\mu} h_j{}^{\mu} = \delta^i{}_j.$$

To express the "conservation laws" which these currents satisfy in a simple form, it is convenient to extend the definition of the covariant derivative $\chi_{|\mu}$ (not $\chi_{;k}$). Originally, it is defined for χ and, therefore, by a trivial extension for any other quantity which is invariant under ξ^{μ} transformations, and transforms linearly under ϵ^{ij} transformations. We wish to extend it to any quantity which transforms linearly under ϵ^{ij} transformations, by simply ignoring the ξ^{μ} transformation properties altogether. Thus, for example, we would have

$$h_i{}^{\mu}{}_{|v} \equiv h_i{}^{\mu}{}_{,v} - A^k{}_{iv} h_k{}^{\mu}, \qquad (4.17)$$

according to the ϵ^{ij} transformation law of $h_i{}^{\mu}$. We shall call this the ϵ covariant derivative. Later we shall define another covariant derivative which takes account of ξ^{μ} transformations also.

One can easily calculate the commutator of two ϵ covariant differentiations.[15] This gives

$$\chi_{|\mu v} - \chi_{|v\mu} = \tfrac{1}{2} R^{ij}{}_{\mu v} S_{ij} \chi, \qquad (4.18)$$

where

$$R^i{}_{j\mu v} \equiv A^i{}_{j\mu,v} - A^i{}_{jv,\mu} - A^i{}_{k\mu} A^k{}_{jv} + A^i{}_{kv} A^k{}_{j\mu}.$$

This quantity is covariant under ϵ^{ij} transformations, and satisfies the cyclic identity

$$R^i{}_{j\mu v|\rho} + R^i{}_{jv\rho|\mu} + R^i{}_{j\rho\mu|v} \equiv 0.$$

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[13] The reason for this may be seen in terms of the variation $\delta_0 \chi$ given by (4.2). The analogs of the matrices T_a are clearly $-\partial_{\mu}$ and S_{ij}, so that the presence of the derivative $\chi_{,\mu}$ in the last term of (4.3) is to be expected. By analogy with (2.6) we should expect the covariant derivative to have the form

$$\chi_{;k} = \delta_k{}^{\mu} \chi_{,\mu} + \tfrac{1}{2} A^{ij}{}_k S_{ij} \chi - A^{\mu}{}_k \partial_{\mu} \chi.$$

Because of the appearance of derivatives, the first and last terms can be combined in the form $h_k{}^{\mu} \chi_{,\mu}$, where $h_k{}^{\mu} = \delta_k{}^{\mu} - A^{\mu}{}_k$. If we then set $A^{ij}{}_k = h_k{}^{\mu} A^{ij}{}_{\mu}$, we arrive at the same form for $\chi_{;k}$ as that obtained in the text.

[14] Multiplication of the entire Lagrangian by a constant factor is of course unimportant.

[15] Note that this could not be done without extending the definition, since one must know how to treat the index on $\chi_{1\mu}$. Here, as in Sec. 2, we simply ignore it.

It is thus closely analogous to $F^a{}_{\mu\nu}$. Note that $R^{ij}{}_{\mu\nu}$ is antisymmetric in both pairs of indices.

In terms of the ϵ covariant derivative, the "conservation laws" can be expressed in the form[5]

$$(\mathfrak{T}^k{}_{\mu}h_k{}^{\mu})_{|\mu} + \mathfrak{T}^k{}_{\mu}h_k{}^{\mu}{}_{|\nu} = \tfrac{1}{2}\mathfrak{S}^{\mu}{}_{ij}R^{ij}{}_{\mu\nu}, \qquad (4.19)$$

$$\mathfrak{S}^{\mu}{}_{ij|\mu} = \mathfrak{T}_i{}_{\mu}h_j{}^{\mu} - \mathfrak{T}_{j\mu}h_i{}^{\mu}. \qquad (4.20)$$

5. FREE GRAVITATIONAL LAGRANGIAN

We now wish to examine the quantity $\chi_{;k}$, rather than $\chi_{|\mu}$. As before, the covariant derivative of any quantity which transforms in a similar way to χ may be defined analogously. Now in particular $\chi_{;k}$ itself (unlike $\chi_{|\mu}$) is such a quantity, and therefore without extending the definition of covariant derivative one can evaluate the commutator $\chi_{;kl} - \chi_{;lk}$. However, this quantity is not simply obtained by multiplying $\chi_{|\mu\nu} - \chi_{|\nu\mu}$ by $h_k{}^{\mu}h_l{}^{\nu}$, as one might expect. The reason for this is that in evaluating $\chi_{;kl}$ one differentiates the $h_k{}^{\mu}$ in $\chi_{;k}$, and moreover adds an extra $A^i{}_{k\mu}$ term on account of the index k. Thus one finds

$$\chi_{;kl} - \chi_{;lk} = \tfrac{1}{2}R^{ij}{}_{kl}S_{ij}\chi - C^i{}_{kl}\chi_{;i}, \qquad (5.1)$$

where

$$R^{ij}{}_{kl} \equiv h_k{}^{\mu}h_l{}^{\nu}R^{ij}{}_{\mu\nu}, \qquad (5.2)$$

$$C^i{}_{kl} \equiv (h_k{}^{\mu}h_l{}^{\nu} - h_l{}^{\mu}h_k{}^{\nu})b^i{}_{\mu|\nu}. \qquad (5.3)$$

Note that (5.1) is not simply proportional to χ, but involves $\chi_{;i}$ also.[16]

We now look for a free Lagrangian $\mathfrak{L}_0$ for the new fields. Clearly $\mathfrak{L}_0$ must be an invariant density, and if we set

$$\mathfrak{L}_0 \equiv \mathfrak{H}L_0,$$

then it is easy to see, as in the case of linear transformations, that the invariant L_0 must be a function only of the covariant quantities $R^{ij}{}_{kl}$ and $C^i{}_{kl}$. As before, there are many possible forms for $\mathfrak{L}_0$, but there is a difference between this case and the previous one in that all the indices on these expressions are of the same type (unlike $F^a{}_{\mu\nu}$), and one can, therefore, contract the upper indices with the lower. In fact, the condition that L_0 be a scalar in two separate spaces is now reduced to the condition that it be a scalar in one space. In particular, this means that there exists a linear invariant which has no analog in the previous case, namely,

$$R \equiv R^{ij}{}_{ij}.$$

There are in addition several quadratic invariants. However, if we again choose for L_0 the form of lowest possible degree, then we are led to the free Lagrangian[17]

$$\mathfrak{L}_0 = \tfrac{1}{2}\mathfrak{H}R \qquad (5.4)$$

which differs from (2.10) in being only linear in the derivatives.

With this choice of Lagrangian, the equations of motion for the new fields are

$$\mathfrak{H}(R^{ik}{}_{jk} - \tfrac{1}{2}\delta^i{}_j R) = -\mathfrak{T}^i{}_{\mu}h_j{}^{\mu}, \qquad (5.5)$$

$$-[\mathfrak{H}(h_i{}^{\mu}h_j{}^{\nu} - h_j{}^{\mu}h_i{}^{\nu})]_{|\nu}$$
$$\equiv \mathfrak{H}(h_k{}^{\mu}C^k{}_{ij} - h_j{}^{\mu}C^k{}_{ik} - h_i{}^{\mu}C^k{}_{kj}) = \mathfrak{S}^{\mu}{}_{ij}. \qquad (5.6)$$

From Eq. (5.6) one can immediately obtain a strict conservation law

$$(\mathfrak{S}^{\mu}{}_{ij} + \mathfrak{s}^{\mu}{}_{ij})_{,\mu} = 0, \qquad (5.7)$$

where

$$\mathfrak{s}^{\mu}{}_{ij} \equiv \mathfrak{H}A^k{}_{i\nu}(h_j{}^{\mu}h_k{}^{\nu} - h_k{}^{\mu}h_j{}^{\nu}) - \mathfrak{H}A^k{}_{j\nu}(h_i{}^{\mu}h_k{}^{\nu} - h_k{}^{\mu}h_i{}^{\nu}).$$

This quantity is expressible in the form

$$\mathfrak{s}^{\mu}{}_{ij} \equiv -2(\partial\mathfrak{L}_0/\partial A^{ij}{}_{\mu}) \equiv -\tfrac{1}{2}(\partial\mathfrak{L}_0/\partial A^{mn}{}_{\nu,\mu})f_{ij}{}^{mn}{}_{kl}A^{kl}{}_{\nu},$$

which is closely analogous to (2.12), and should be compared with (4.16). Equation (5.7) is a rather surprising result, since $\mathfrak{S}^{\mu}{}_{ij}$ may very reasonably be interpreted as the spin density of the matter field,[18] so that it appears to be a law of conservation of spin with no reference to the orbital angular momentum. In fact, however, the orbital angular momentum appears in the corresponding "covariant conservation law" (4.20), and therefore part of the "spin" of the gravitational field, $\mathfrak{s}^{\mu}{}_{ij}$, may be regarded as arising from this source. Nevertheless, Eq. (5.7) differs from other statements of angular momentum conservation in that the coordinates do not appear explicitly.

It would also be possible to deduce from Eq. (5.5) a strict conservation law

$$[h_k{}^{\nu}(\mathfrak{T}^k{}_{\mu} + t^k{}_{\mu})]_{,\nu} = 0, \qquad (5.8)$$

but there is a considerable amount of freedom in choosing $t^k{}_{\mu}$. The most natural definition, by analogy with (4.15) would be

$$t^k{}_{\mu} \equiv \partial\mathfrak{L}_0/\partial h_k{}^{\mu},$$

and this quantity does indeed satisfy (5.8). However, in this case the expression within the parentheses itself vanishes, so that (5.8) is rather trivial. We shall not discuss the question of the correct choice of $t^k{}_{\mu}$ further, as this lies beyond the scope of the present paper.[19]

It should be noted that Eq. (5.6) can be solved, at least in principle, for $A^{ij}{}_{\mu}$. In the simple case when $\mathfrak{S}^{\mu}{}_{ij}$ vanishes, one finds[20]

$$A_{ij\mu} = {}^0A_{ij\mu} \equiv \tfrac{1}{2}b^k{}_{\mu}(c_{kij} - c_{ijk} - c_{jki}),$$
$$c^k{}_{ij} \equiv (h_i{}^{\mu}h_j{}^{\nu} - h_j{}^{\mu}h_i{}^{\nu})b^k{}_{\mu,\nu}. \qquad (5.9)$$

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[16] This is another example of the fact that for ξ^{μ} transformations derivatives play the role of the matrices T_a. Compare footnote 13.

[17] We choose units in which $\kappa = 1$ (as well as $c = \hbar = 1$).

[18] See H. J. Belinfante, Physica 6, 887 (1939), and footnote 5.

[19] It is well known in the case of the ordinary metric theory of gravitation that many definitions of the energy pseudotensor are possible. See, for example, P. G. Bergmann, Phys. Rev. 112, 287 (1958).

[20] The ${}^0A^{ij}{}_{\mu}$ are Ricci's coefficients of rotation. See for instance V. Fock, Z. Physik 57, 261 (1929).

In general, if we write

$$\mathfrak{S}^\mu{}_{ij} \equiv \mathfrak{H} h_k{}^\mu S^k{}_{ij},$$

then

$$A_{ij\mu} = {}^0A_{ij\mu} - \tfrac{1}{2}b^k{}_\mu(S_{kij} - S_{ijk} - S_{jki} - \eta_{ki}S^l{}_{lj} - \eta_{kj}S^l{}_{il}). \quad (5.10)$$

If the original Lagrangian L is of first order in the derivatives, then $S^k{}_{ij}$ is independent of $A^{ij}{}_\mu$ so that (5.10) is an explicit solution. Otherwise, however, $A^{ij}{}_\mu$ also appears on the right-hand side of this equation.

We conclude this section with a discussion of the Lagrangian for the fields $A^a{}_\mu$ introduced in Sec. 2 when the "gravitational" fields $h_k{}^\mu$ and $A^{ij}{}_\mu$ are also introduced. The fields $A^a{}_\mu$ should not be regarded merely as components of χ when dealing with Lorentz transformations, since one must preserve the invariance under the linear transformations. To find the correct form of the Lagrangian, one should consider simultaneously Lorentz transformations and these linear transformations. This can be done provided that the matrices T^a commute with the S_{ij}, a condition which is always fulfilled in practice. Then one finds that χ_k in L should be replaced by a derivative which is covariant under both (2.1) and (4.1), namely,

$$\chi_{;k} = h_k{}^\mu(\chi_{,\mu} + \tfrac{1}{2}A^{ij}{}_\mu S_{ij}\chi + A^a{}_\mu T_a\chi).$$

The commutator $\chi_{;kl} - \chi_{;lk}$ then contains the extra term

$$F^a{}_{kl}T_a\chi,$$

where

$$F^a{}_{kl} \equiv h_k{}^\mu h_l{}^\nu F^a{}_{\mu\nu},$$

with $F^a{}_{\mu\nu}$ given by (2.9). It is important to notice that the derivatives of $A^a{}_\mu$ in $F^a{}_{\mu\nu}$ are ordinary derivatives, not covariant ones. (We shall see in the next section that the ordinary and covariant curls are not equal, because the affine connection is in general nonsymmetric.) As before, one can see that any invariant function of $A^a{}_\mu$ must be a function of $F^a{}_{kl}$ only, and the simplest free Lagrangian for $A^a{}_\mu$ is, therefore,

$$-\tfrac{1}{4}\mathfrak{H}F^a{}_{kl}F_a{}^{kl}. \quad (5.11)$$

6. GEOMETRICAL INTERPRETATION

Up to this point, we have not given any geometrical significance to the transformations (4.1), or to the new fields $h_k{}^\mu$ and $A^{ij}{}_\mu$, but it is useful to do so in order to be able to compare the theory with the more familiar metric theory of gravitation.

Now the ξ^μ transformations are general coordinate transformations, and according to (4.11) $h_k{}^\mu$ transforms like a contravariant vector under these transformations, while $b^k{}_\mu$ and $A^{ij}{}_\mu$ transform like covariant vectors. Thus the quantity

$$g_{\mu\nu} \equiv b^k{}_\mu b_{k\nu} \quad (6.1)$$

is a symmetric covariant tensor, and may therefore be interpreted as the metric tensor of a Riemannian space. It is moreover invariant under the ϵ^{ij} transformations. Evidently, the Greek indices may be regarded as world tensor indices, and we must of course abandon for them the convention that all indices are to be raised or lowered with the flat-space metric $\eta_{\mu\nu}$, and use $g_{\mu\nu}$ instead. It is easy to see that the scalar density $\mathfrak{H}$ is equal to $(-g)^{\frac{1}{2}}$, where $g = \det(g_{\mu\nu})$.

Now, in view of the relation (6.1), $h_k{}^\mu$ and $b^k{}_\mu$ are the contravariant and covariant components, respectively, of a vierbein system in the Riemannian space.[21] Thus the ϵ^{ij} transformations should be interpreted as vierbein rotations, and the Latin indices as local tensor indices with respect to this vierbein system. The original field χ may be decomposed into local tensors and spinors,[22] and from the tensors one can form corresponding world tensors by multiplying by $h_k{}^\mu$ or $b^k{}_\mu$. For example, from a local vector v^i one can form

$$v^\mu = h_i{}^\mu v^i, \quad v_\mu = b^i{}_\mu v_i. \quad (6.2)$$

No confusion can be caused by using the same symbol v for the local and world vectors, since they are distinguished by the type of index, and indeed we have already used this convention in (5.2). Note that $v_\mu = g_{\mu\nu}v^\nu$, so that (6.2) is consistent with the choice of metric (6.1). We shall frequently use this convention of associating world tensors with given local tensors without explicit mention on each occasion.

The field $A^i{}_{j\mu}$ may reasonably be called a "local affine connection" with respect to the vierbein system, since it specifies the covariant derivatives of local tensors or spinors.[23] For a local vector, this takes the form

$$v^i{}_{|\nu} = v^i{}_{,\nu} + A^i{}_{j\nu}v^j, \quad v_{j|\nu} = v_{j,\nu} - A^i{}_{j\nu}v_i. \quad (6.3)$$

It may be noticed that the relation (4.10) between $\chi_{|\mu}$ and $\chi_{;k}$ is of the same type as (6.2) and could be written simply as

$$\chi_{;\mu} = \chi_{|\mu} \quad (6.4)$$

according to our convention. However, we shall retain the use of two separate symbols because we wish to extend the definition of covariant derivative in a different way to that of Sec. 4. It seems natural to define the covariant derivative of a world tensor in terms of the covariant derivative of the associated local tensor. Thus, for instance, to define the covariant derivatives of the world vectors (6.2) one would form the world tensors corresponding to (6.3). This gives

$$v^\lambda{}_{;\nu} \equiv h_i{}^\lambda v^i{}_{|\nu} = v^\lambda{}_{,\nu} + \Gamma^\lambda{}_{\mu\nu}v^\mu,$$

$$v_{\mu;\nu} \equiv b^i{}_\mu v_{i|\nu} = v_{\mu,\nu} - \Gamma^\lambda{}_{\mu\nu}v_\lambda,$$

where

$$\Gamma^\lambda{}_{\mu\nu} \equiv h_i{}^\lambda b^i{}_{\mu|\nu} \equiv -b^i{}_\mu h_i{}^\lambda{}_{|\nu}. \quad (6.5)$$

Note that this definition of $\Gamma^\lambda{}_{\mu\nu}$ is equivalent to the

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[21] See for instance H. Weyl, Z. Physik **56**, 330 (1929).
[22] H. J. Belinfante, Physica **7**, 305 (1940).
[23] Compare J. A. Schouten, J. Math. and Phys. **10**, 239 (1931).

requirement that the covariant derivatives of the vierbein components should vanish,

$$h_{i}{}^{\lambda}{}_{;\,v}\equiv 0, \quad b^{i}{}_{\mu;\,v}\equiv 0. \tag{6.6}$$

For a generic quantity α transforming according to

$$\delta\alpha=\tfrac{1}{2}\epsilon^{ij}S_{ij}\alpha+\xi^{\lambda}{}_{,\mu}\Sigma_{\lambda}{}^{\mu}\alpha, \tag{6.7}$$

the covariant derivative is defined by[21]

$$\alpha_{;\,v}\equiv\alpha_{,\,v}+\tfrac{1}{2}A^{ij}{}_{v}S_{ij}\alpha+\Gamma^{\lambda}{}_{\mu v}\Sigma_{\lambda}{}^{\mu}\alpha, \tag{6.8}$$

whereas the ϵ covariant derivative defined in Sec. 4 is obtained by simply omitting the last term of (6.8). Note that the two derivatives are equal for purely local tensors or spinors, but not otherwise. One easily finds that the commutator of two covariant differentiations is given by

$$\alpha_{;\,\mu v}-\alpha_{;\,v\mu}=\tfrac{1}{2}R^{ij}{}_{\mu v}S_{ij}\alpha+R^{\rho}{}_{\sigma\mu v}\Sigma_{\rho}{}^{\sigma}\alpha-C^{\lambda}{}_{\mu v}\alpha_{;\,\lambda},$$

where $R^{\rho}{}_{\sigma\mu v}$ and $C^{\lambda}{}_{\mu v}$ are defined in the usual way in terms of $R^{i}{}_{j\mu v}$ and $C^{i}{}_{kl}$. They are both world tensors, and can easily be expressed in terms of $\Gamma^{\lambda}{}_{\mu v}$, in the form[24]

$$R^{\rho}{}_{\sigma\mu v}=\Gamma^{\rho}{}_{\sigma\mu,\,v}-\Gamma^{\rho}{}_{\sigma v,\,\mu}-\Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma v}+\Gamma^{\rho}{}_{\lambda v}\Gamma^{\lambda}{}_{\sigma v}, \tag{6.9}$$

$$C^{\lambda}{}_{\mu v}=\Gamma^{\lambda}{}_{\mu v}-\Gamma^{\lambda}{}_{v\mu}. \tag{6.10}$$

Thus one sees that $R^{\rho}{}_{\sigma\mu v}$ is just the Riemann tensor formed from the affine connection $\Gamma^{\lambda}{}_{\mu v}$.

From (6.6) it follows that

$$g_{\mu v;\,\rho}\equiv 0, \tag{6.11}$$

so that it is consistent to interpret $\Gamma^{\lambda}{}_{\mu v}$ as an affine connection in the Riemannian space. However, the definition (6.5) evidently does not guarantee that it is symmetric, so that in general it is not the Christoffel connection. The curvature scalar R has the usual form

$$R\equiv R^{\mu}{}_{\mu}, \quad R_{\mu v}\equiv R^{\lambda}{}_{\mu\lambda v},$$

so that the free gravitational Lagrangian is just the usual one except for the nonsymmetry of $\Gamma^{\lambda}{}_{\mu v}$. It should be remarked that it would be incorrect to treat the 64 components of $\Gamma^{\lambda}{}_{\mu v}$ as independent variables, since there are only 24 components of $A^{ij}{}_{\mu}$. In fact the $\Gamma^{\lambda}{}_{\mu v}$ are restricted by the 40 identities (6.11). Thus there is no contradiction with the well-known fact that the first-order Palatini Lagrangian with nonsymmetric $\Gamma^{\lambda}{}_{\mu v}$ does not yield (6.11) as equations of motion.[25]

The equations of motion (5.5) and (5.6) can be rewritten in the form

$$\mathfrak{H}(R_{\mu v}-\tfrac{1}{2}g_{\mu v}R)=-\mathfrak{T}_{\mu v}, \tag{6.12}$$

$$\mathfrak{H}C^{\lambda}{}_{\mu v}=\mathfrak{S}^{\lambda}{}_{\mu v}-\tfrac{1}{2}\delta^{\lambda}{}_{\mu}\mathfrak{S}^{\rho}{}_{\rho v}-\tfrac{1}{2}\delta^{\lambda}{}_{v}\mathfrak{S}^{\rho}{}_{\mu\rho}. \tag{6.13}$$

From Eqs. (6.10) and (6.13) one sees that in the absence of matter the affine connection $\Gamma^{\lambda}{}_{\mu v}$ is symmetric, and

therefore equal to the Christoffel connection ${}^{0}\Gamma^{\lambda}{}_{\mu v}$. (This is the analog for world tensors of ${}^{0}A^{ij}{}_{\mu}$.) Then $R_{\mu v}$ is symmetric, and Eq. (6.12) yields Einstein's familiar equations for empty space,

$$R_{\mu v}=0.$$

However, when matter is present, $\Gamma^{\lambda}{}_{\mu v}$ is no longer symmetric, and its antisymmetric part is given by (6.13). Then the tensor $R_{\mu v}$ is also nonsymmetric, and correspondingly the energy tensor density $\mathfrak{T}_{\mu v}$ is in general nonsymmetric, because $h_{k}{}^{\mu}$ does not appear in $\mathfrak{L}$ only through the symmetric combination $g^{\mu v}$. Thus the theory differs slightly from the usual one, in a way first noted by Weyl.[4] In the following section, we shall investigate this difference in more detail.[5]

Finally, we can rewrite the covariant conservation laws in terms of world tensors. It is convenient to define the contraction

$$C_{\mu}\equiv C^{\lambda}{}_{\mu\lambda},$$

since the covariant divergence of a vector density $\mathfrak{f}^{\mu}$ is then

$$\mathfrak{f}^{\mu}{}_{;\,\mu}=\mathfrak{f}^{\mu}{}_{,\,\mu}+C_{\mu}\mathfrak{f}^{\mu}. \tag{6.14}$$

The conservation laws become

$$\mathfrak{T}^{v}{}_{\mu;\,v}-C_{v}\mathfrak{T}^{v}{}_{\mu}+C^{\lambda}{}_{\mu v}\mathfrak{T}^{v}{}_{\lambda}=\tfrac{1}{2}R^{\rho\sigma}{}_{\mu v}\mathfrak{S}^{v}{}_{\rho\sigma},$$

$$\mathfrak{S}^{\mu}{}_{\rho\sigma;\,\mu}-C_{\mu}\mathfrak{S}^{\mu}{}_{\rho\sigma}=\mathfrak{T}_{\rho\sigma}-\mathfrak{T}_{\sigma\rho}.$$

It may be noticed that these are slightly more complicated than the expressions in terms of the ϵ covariant derivative.

7. COMPARISON WITH METRIC THEORY

For simplicity, we shall assume in this section that L is only of first order in the derivatives, so that (5.10) is an explicit solution for $A^{ij}{}_{\mu}$. The difference between the theory presented here and the usual one arises because we are using a Lagrangian $\mathfrak{L}_{0}$ of first order, in which $h_{k}{}^{\mu}$ and $A^{ij}{}_{\mu}$ are independent variables. The situation is entirely analogous to that which obtains for any theory with "derivative" interaction. In first-order form, the "momenta" $A^{ij}{}_{\mu}$ are not just equal to derivatives of the "coordinates" $h_{k}{}^{\mu}$, or in other words to ${}^{0}A^{ij}{}_{\mu}$. Thus an interaction which appears simple in first-order form will be more complicated if a second-order Lagrangian is used, and vice versa.

The second-order form of the Lagrangian may be obtained by substituting for $A^{ij}{}_{\mu}$ the expression (5.10). This gives

$$\mathfrak{L}'={}^{0}\mathfrak{L}+{}^{0}\mathfrak{L}_{0}+{}^{1}\mathfrak{L},$$

where ${}^{0}\mathfrak{L}$ and ${}^{0}\mathfrak{L}_{0}$ are obtained from $\mathfrak{L}$ and $\mathfrak{L}_{0}$ by replacing $A^{ij}{}_{\mu}$ by ${}^{0}A^{ij}{}_{\mu}$ (or equivalently $\Gamma^{\lambda}{}_{\mu v}$ by ${}^{0}\Gamma^{\lambda}{}_{\mu v}$), and ${}^{1}\mathfrak{L}$ is an additional term quadratic in $S^{k}{}_{ij}$, namely,

$${}^{1}\mathfrak{L}=\tfrac{1}{8}\mathfrak{H}(2S_{ijk}S^{jki}-S_{ijk}S^{ijk}+2S^{i}{}_{ik}S_{j}{}^{jk}). \tag{7.1}$$

In this Lagrangian, only $h_{k}{}^{\mu}$ and X are treated as inde-

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[24] This is a generalization to nonsymmetric affinities of the result proved in the appendix to footnote 3. See also footnotes 4 and 5.

[25] See for instance E. Schrödinger, *Space-time Structure* (Cambridge University Press, New York, 1950).

pendent variables. The equations of motion are equivalent to those previously obtained if the variables $A^{ij}{}_\mu$ are eliminated from the latter by using (5.10).

The usual metric theory, on the other hand, is given by the Lagrangian

$$\mathfrak{L}'' = {}^0\mathfrak{L} + {}^0\mathfrak{L}_0,$$

without the extra terms (7.1). If this Lagrangian were written in a first-order form by introducing additional independent variables $A^{ij}{}_\mu$, then one would arrive at a form identical to the one given here except for the appearance of extra terms equal to (7.1) with a negative sign.

Thus we see that the only difference between the two theories is the presence or absence of these "direct-interaction" terms. Now if we had not set $\kappa = 1$, then $\mathfrak{L}_0$ would have a factor κ^{-1}, whereas the terms (7.1) would appear with the factor κ. They are, therefore, extremely small in comparison to other interaction terms. In particular, for a Dirac field, they would be proportional to (see Appendix)

$$\kappa \bar{\psi} \gamma_k \gamma_5 \psi \bar{\psi} \gamma^k \gamma_5 \psi.$$

Thus they are similar in form to the Fermi interaction terms, but much smaller in magnitude, so that it seems impossible that they would lead to any observable difference between the predictions of the two theories. Hence we must conclude that for all practical purposes the theory presented here is equivalent to the usual one.

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APPENDIX

In this appendix we shall discuss the remaining ambiguity in the modified Lagrangian. It was pointed out in Sec. 4 that the generally covariant Lagrangians obtained from two equivalent Lagrangians L_1 and L_2 are in general inequivalent. One can now see that in fact they differ by a covariant divergence. Thus (4.14) can be written in the form

$$\mathfrak{L}_1 - \mathfrak{L}_2 = (\mathfrak{H} h_k{}^\mu \pi^k \varphi)_{;\mu},$$

but in view of (6.14) this is not equal to the ordinary divergence. It is clear that quite generally changing L by a divergence must change $\mathfrak{L}$ by the covariant divergence of a quantity which is a vector density under coordinate transformations, and invariant under all other transformations. This is the reason for the difference between this case and that of the linear transformations of Sec. 2.

We now wish to investigate the possibility of choosing a criterion which will select a particular form of L, and thus specify $\mathfrak{L}$ completely. There does not seem to be any really compelling reason for one choice rather than

another, but there are plausible arguments for a particular choice.

The most obvious criterion would be to require that the Lagrangian should be written in the symmetrized first-order form suggested by Schwinger,[26] which in the case of the scalar field discussed in Sec. 4 is

$$L = \tfrac{1}{2}(L_1 + L_2).$$

This corresponds to treating φ and π^k on a symmetrical footing. However, this may not in fact be the correct choice, because for some purposes φ and π^k should not be treated in this way. In fact, the two Lagrangians differ in one important respect: $\mathfrak{L}_1$ is independent of $A^{ij}{}_\mu$, whereas $\mathfrak{L}_2$ is not. Correspondingly, for L_1 the quantity $S^k{}_{ij}$ vanishes, whereas for L_2 one finds

$$S^k{}_{ij} = (\delta^k{}_i \pi_j - \delta^k{}_j \pi_i) \varphi.$$

The conservation laws in the two cases are of course the same, because the quantities $T^k{}_i$ also differ. Now the tensor $S^k{}_{ij}$ has often been interpreted as the spin density,[18] so that the two cases differ with regard to the separation of the total angular momentum into orbital and spin terms. The scalar field is normally regarded as a field of spinless particles, so that one would naturally expect $S^k{}_{ij}$ to vanish. This, therefore, furnishes a possible criterion, which would select L_1 rather than L_2. With this choice, a preferred position is assigned to the "wave function" φ rather than the "momenta" π^k, and the derivatives are written on φ only. In this way one achieves a vanishing spin tensor, because the matrices S_{ij} are zero for the scalar field φ, but not for the vector π^k. It may be noticed that L_1 is automatically selected if one writes the Lagrangian in its second-order form in terms of φ only:

$$L_1' = \tfrac{1}{2} \varphi_{,k} \varphi^{,k} - \tfrac{1}{2} m^2 \varphi^2,$$

which yields the modified Lagrangian

$$\mathfrak{L}_1' = \tfrac{1}{2}\mathfrak{H}(g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - m^2 \varphi^2),$$

equivalent to $\mathfrak{L}_1$.[27] This should be contrasted with the second-order form of $\mathfrak{L}_2$, which is

$$\mathfrak{L}_2' = \tfrac{1}{2}\mathfrak{H}^{-1}(\mathfrak{H} h_i{}^\mu \varphi)_{;\mu}(\mathfrak{H} h^{i\nu} \varphi)_{;\nu} - \tfrac{1}{2}\mathfrak{H} m^2 \varphi^2,$$

and clearly differs from $\mathfrak{L}_1'$ by a covariant divergence.

This seems to be a resonable criterion, but the arguments for it cannot be regarded as conclusive. For, although it is true that the spin tensor obtained from L_2 is nonzero, it is still true that the three space-space components of the total spin

$$\mathcal{S}_{ij} = \int d_3x \, S^0{}_{ij}$$

are zero. Thus L_1 and L_2 differ only in the values of the

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[26] J. Schwinger, Phys. Rev. 91, 713 (1953).
[27] Here $\mathfrak{L}_1$ is a "linearization" of $\mathfrak{L}_1'$ in the sense of T. W. B. Kibble and J. C. Polkinghorne, Nuovo cimento 8, 74 (1958).

spin part of the $(0i)$ components of angular momentum. Indeed, one easily sees that it is true in general that adding a divergence to L will change only the $(0i)$ components of S_{ij}. Since it is not at all clear what significance should be attached to the separation of these components into "orbital" and "spin" terms, it might be questioned whether one should expect the spin terms to vanish even for a spinless particle. Even so, the choice of L_1 seems in this case to be the most reasonable.

For a field of spin 1, the corresponding choice would be

$$L_1 = -\tfrac{1}{2}f^{ij}(a_{i,j}-a_{j,i}) + \tfrac{1}{4}f^{ij}f_{ij} + \tfrac{1}{2}m^2 a_i a^i,$$

which is again equivalent to the choice of the second-order Lagrangian in terms of a_i only. It yields

$$S^k{}_{ij} = a_i f_j{}^k - a_j f_i{}^k,$$

which is a reasonable definition of the spin density.[28] The modified Lagrangian may be expressed in terms of the world vector a_μ as

$$\mathfrak{L} = -\tfrac{1}{4}\mathfrak{H}g^{\mu\rho}g^{\nu\sigma}(a_{\mu;\nu}-a_{\nu;\mu})(a_{\rho;\sigma}-a_{\sigma;\rho}) \\ + \tfrac{1}{2}\mathfrak{H}m^2 g^{\mu\nu}a_\mu a_\nu. \quad (A.1)$$

It should be noticed that the electromagnetic Lagrangian is not obtained simply by putting $m=0$ in (A.1). The difference is that the derivatives in (A.1) are covariant derivatives, and since $\Gamma^\lambda{}_{\mu\nu}$ is nonsymmetric the covariant curl is not equal to the ordinary curl (though both

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[28] Compare footnote 18.

are of course tensors). In fact, (A.1) with $m=0$ would not be gauge invariant. The reason for the difference is that a_i is here treated simply as a component of χ, whereas A_μ is introduced along with the gravitational variables to ensure gauge invariance.[29]

For a spinor field ψ, symmetry between ψ and $\bar{\psi}$ appears to demand that one should choose the symmetrized Lagrangian

$$L = \tfrac{1}{2}(\bar{\psi}i\gamma^k\psi_{,k} - \bar{\psi}_{,k}i\gamma^k\psi) - m\bar{\psi}\psi,$$

which yields the spin density

$$S_{kij} = \tfrac{1}{2}\epsilon_{kijl}\bar{\psi}i\gamma^l\gamma_5\psi.$$

Since the Lagrangian $\mathfrak{L}$ must be Hermitian, one could not write the derivative on ψ alone. There remains, however, another possible choice: We could introduce a distinction between the left- and right-handed components, $\psi_\pm = \tfrac{1}{2}(1\pm i\gamma_5)\psi$, treating one of them line φ and the other like π^k. This gives the Lagrangian

$$L = \tfrac{1}{2}\bar{\psi}i\gamma^k(1+i\gamma_5)\psi_{,k} - \tfrac{1}{2}\bar{\psi}_{,k}i\gamma^k(1-i\gamma_5)\psi - m\bar{\psi}\psi.$$

This form of Lagrangian may seem rather unnatural, but it should be mentioned because there are other grounds for treating ψ_+ and ψ_- on a nonsymmetrical footing.[30]

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[29] This has the rather strange consequence that for the electromagnetic field the "spin" tensor $S^k{}_{ij}$ vanishes, since the Lagrangian is independent of $A^{ij}{}_\mu$.

[30] See R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).