Three Three

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Contents

- Determine whether 2 vectors are parallel, orthogonal or neither
- Compute the projection of one vector onto another
- Decompose one vector into a sum of orthogonal and parallel parts
- Compute distances between points and lines or points and planes
- Compute a unit vector orthogonal to two vectors in R^3
- Compute a unit vector orthogonal to any number of vectors in R^n

1 Definition

Two nonzero vectors u and v in $\mathrm{R^n}$ are orthogonal (or perpendicular) if $\mathrm{u \cdot v = 0}$

- $cos\theta = \frac{u \cdot v}{||u||||v||} = 0$

The zero vector in $\mathrm{R^n}$ is orthogonal to every vector in $\mathrm{R^n}$ A nonempty set of vectors is called an orthogonal set if all pairs of distinct vectors in the set are orhtogonal An orthogonal set of unit vectors is called an orthogonal set

- $u_1, u_2, ..., u_n$ is an orthogonal set
  - Then $\mathrm{u_i \cdot u_j = 0 \; \forall \; i,j \in \{1,\ n\}}$

$\mathrm{R^3 = i,\ j,\ k \Rightarrow \{i,\ j,\ k\}}$

1.1 Example

The standard unit vectors in any space

- $R^n = \{e_1, e_2, ..., e_n\} =$ orthogonal set
- $R^2 = \{(1, 0), (0, 1)\}$
- $R^3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

1.2 Make it Orthogonal

- $(1, 2, 3) \cdot (-3, 0, 1) = 0$
- $(1, 2, 3) \cdot (1, -5, 3) = 0$
- $(-3, 0, 1) \cdot (1, -5, 3) = 0$

It is already orthogonal

2 Point-Normal Equations of Lines and Planes

One application of orthogonality is to define a line or place by using a nonzero vector n, called a normal, that is orthogonal to the line or plane

For example $n \cdot P_0P_1 = 0$, where $P_0P_1 = (x - x_0, y - y_0)$ or $P_0P_1 = (x - x_0, y - y_0, z - z_0)$

Find the equation of the plane the goes through the origin and has normal (10, 3, 13)

- Equation of a plane through a point (x_0, y_0, z_0) and has normal (a, b, c)
- $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
- $10x + 3y + 13z = 0$

Find the equation of the plane that goes through the point (1,2,3) and has normal (10,3,13)

- $10(x - 1) + 3(y - 2) + 13(z - 3) = 0$

3 Orthogonal Projections

If u and a are vectors in R^{n} and if a $\neq$ 0, then u = w_1 + w_2, hwere w_1 is a scalar multiple of a and w_2 is orthogonal to a

- w_1 is the vector component of u along a
  - $w_1 = proj_u(u) = \frac{u \cdot a}{a \cdot a} a$
- and w_2 is the vector component of u orthogonal to a
  - $w_2 = u - proj_a(u) = u - \frac{u \cdot a}{a \cdot a} a$

3.1 Example

u = (1,9,0,5), a = (3,-2,4,1)

- $w_1 = proj_a u = \frac{u \cdot a}{a \cdot a} a = \frac{1(3)+9(-2)+0(-4)+5(1)}{3^2+-2^2+4^2-1^2} \times (3, -2, 4, 1) = \frac{-10}{30}(3, -2, 4, 1) = (-1, \frac{2}{3}, \frac{-4}{3}, \frac{-1}{3})$
- $w_2 = u - proj_a u = (1, 9, 0, 5) - (-1, \frac{2}{3}, \frac{-4}{3}, \frac{-1}{3}) = (2, \frac{25}{3}, \frac{4}{3}, \frac{16}{3})$

3.2 Another one

u=(2,1,-4,6) and a=(1,-2,2,1)

- $w_1 = \frac{2(1)+1(-2)+(-4)(2)+6(1)}{1^2+(-2)^2+2^2+1^2} \times (1, -2, 2, 1) = \frac{-2}{10}(1, -2, 2, 1) = (\frac{-1}{5}, \frac{2}{5}, \frac{-2}{5}, \frac{-1}{5})$
- $w_2 = (2, 1, -4, 6) - (\frac{-1}{5}, \frac{2}{5}, \frac{-2}{5}, \frac{-1}{5}) = (\frac{11}{5}, \frac{3}{5}, \frac{-18}{5}, \frac{31}{5})$

3.3 Checks

3.3.1 Problem 1

$\mathrm{w}_2 \cdot \mathrm{a}$ = ? This should be orthogonal component

3.3.2 Problem 2

$w_2 \cdot a$ = ? This should be orthogonal component

3.4 Finding the Norm

- $\sqrt{1^2 + \frac{2}{3}^2 +}$

4 Pythag

in R^n: If u and v are orthogonal vectors in R^n with the Euclidean inner product, then $||u+v||^2 = ||u||^2 + ||v||^2$

We know that $||a||^2 = a \cdot a$

- $||u + v||^2 = (v + v) \cdot (u + u) = u{\cdot}u + u{\cdot}v + v{\cdot}v + v{\cdot}u$

4.1 find the distance between a point ($\mathbf{x_0}$,$\mathbf{y_0}$) and a line ax+by+c = 0 in $\mathbf{R^2}$

- $D = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$
- $D = \frac{|5(2)-11-9|}{\sqrt{5^2+(-1)^2}} = \frac{10}{\sqrt{26}}$

4.2 find a distance between a point and a plane in $\mathbf{R^3}$

- $D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$
- $D = \frac{|2(3)+3(5)-10+4|}{\sqrt{2^2+3^2+(-1^2)}} = \frac{15}{\sqrt{14}}$