One Eight

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Contents

1 Key Points

- Determine when a function is a linear transformatoin
- Compute the standard matrix for a linear transformation
- Create rotation, reflection and projection matrix transformations

2 Definition

A **function** is a rule that associates with each element of a set A exactly one element is a set B.

$$f : A \to B$$

The set A is called the **domain** of f and the set B is the **codomain** of f. The **range** of f is the subset of B that consists of all images of elements from the domain.

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

When n = m, the transformation is called an **operator** on $\mathbb{R}^n$

3 Definition

A tranformation T: $\mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if the following 2 properties hold for all vectors **u** and **v** in $\mathbb{R}^n$ and for every scalar k:

1. $T(u+v) = T(u) + T(v)$
2. $T(ku) = kT(u)$

T can be defined by matrix multiplication

3.1 Test 2 Example Problem

You will be asked to determine if a given transformation is linear or non-linear

Determine if the transformation is linear or non-linear

1. $T : R \to R$
   - $x \to 2x^2$
2. $T : R \to R$
   - $x \to x^2 + 1$

3.1.1 How to get to Solution

1. WTS (i)(ii). Prove that this holds for any input (proof in general), let u & v be arbitrary numbers

$T(u) = 2u^2$ $T(v) = 2v^2$ $T(u+v) = 2(u+v)^2 = 2u^2+4uv+2v^2 \neq 2u^2+2v^2 = T(u) + T(v)$

Non-linear

1. WTS part b from above

$T(u) = 2u$ $T(v) = 2v$ $T(u+v) = 2(u+v) = 2u+2v = 2u+2v = T(u)+T(v)$ Property 1 is satisfied Let k be any constant $T(ku) = 2(ku) = k(2u) = kT(u)$ Property 2 is satisifed

Linear

4 Examples

T can be defined by matrix multiplicaiton

1. Let T: $\mathbb{R}^3 \to \mathbb{R}^3$ be defined by multiplication $\begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 5 \\ 7 & 5 & 6 \end{bmatrix}$

   Find the image, T(X) = A(x) under this transformation for each

1. $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, T(x) = A(x) = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 5 \\ 7 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
2. $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, T(x) = A(x) = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 5 \\ 7 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 22 \\ 35 \end{bmatrix}$
3. $x = \begin{bmatrix} z \\ y \\ z \end{bmatrix}$
4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by the multiplication of A $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & 4 \end{bmatrix}$
   - $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y - z \\ 3x + 7y + 4z \end{bmatrix} = (x+2y-z, 3x+7y+4z) =$ $T(x^{\rightarrow} = T(x, y, z)$
   - $T(e_1) = T(1, 0, 0) = (1, 3)$
   - $T(e_2) = T(0, 1, 0) = (2, 7)$
   - $T(e_3) = T(0, 0, 1) = (-1, 4)$
   - **Very Important**, we note that, if we collect T(e_1), T(e_2), and T(e_3) and write them in colum form we will have matrix A. Thus, if T($x^{\rightarrow}$) is given (not A(x)), we could find matrix A by the following formula. $A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$

- Note: We define $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ ... \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ ...0 \end{bmatrix}$ $e_n = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ Where the 1 is in row n

5. Describe in words the linar transformation given by each matrix

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$
      - Negates the y, reflecting the vector below the x-axis

   (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$
      - Eliminates the y component of the vector, making it parallel to the x axis
      - Projection of the vector onto the x-axis

   (c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
      - Increases magnitude of the vector

   (d) $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$
      - This rotates the vector counterclockwise by an angle of θ

5 Theorem

For every matrix A the matrix transformatoin $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **linear transformation** of $\mathbb{R}^n \rightarrow \mathbb{R}^m$ and has the following properties for all vectors $\mathrm{v}^{\rightarrow}$ and $\mathrm{v}^{\rightarrow}$ and for any scalar k $\in \mathbb{R}$

1. $\mathrm{T_A(0) = 0}$
2. $\mathrm{T_A(ku) = kT_A(u)}$
   - Linear transformation of a product
3. $\mathrm{T_A(u+v) = T_A(u) + T_A(v)}$
   - Linear transformation of a sum
4. $\mathrm{T_A(u\text{-}v) = T_A(u)\ \text{-}\ T_A(v)}$
   - Linear transformation of a difference

5.1 Example

$$T_A(k_1u_1 + k_2u_2 + ...) = k_1T_A(u_1) + k_2T_A(u_2) + ...$$

The linear combination of the images of each vector under the transformation

when doing linear transformation it must be in a row

1. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by T(x,y) = (x,3x,2x+y)
   - $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
   - $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
   - T(e_1) = T(1,0) = (1,3,2)
   - T(e_2) = T(0,1) = (0,0,1)
   - So A, = $\begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}$
2. Given that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that T(1,0,0) = (3,2,5), T(0,1,0) = (1,4,7) and T(0,0,1) = (2,0,6). Find T(10,25,21)
   - Method 1: $\begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & 0 \\ 5 & 7 & 6 \end{bmatrix} \Rightarrow T(10, 25, 21) = A \begin{bmatrix} 10 \\ 25 \\ 21 \end{bmatrix}$
     - find the image of T(10,25,21) so after we find A we just attach A to the vector and do a matrix multiplication
     - $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & 0 \\ 5 & 7 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ 25 \\ 21 \end{bmatrix} = \begin{bmatrix} 97 \\ 120 \\ 351 \end{bmatrix}$
   - Method 2: $\begin{bmatrix} 10 \\ 25 \\ 21 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 25 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 21 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 10e_1 + 25e_2 + 21e_3$
     - So T(10,25,21) = 10T(e_1)+25T(e_2)+21T(e_3) = same as above
3. Create the standard matrix that represents each transformation

- $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$
- Rotate a vector(x,y) clockwise (-45)
  - $\$[cos45 \& sin45 \backslash \text{sin45\&cos45}] = \left[\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\right.$
  - Project a vector(x,y,z) onto the z-axis
    * $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
    * $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$

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[3,1,2;2,4,0;5,7,6] * [10;25;21]
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