One Six

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Contents

1 Objectives

- Expanding the Equivalency Theorem
- Given a singular matrix A, find all matricies b such that Ax = b is consistent

2 Theorem

If A is an n x n matrix, then the following statements are equivalent

1. A is invertible
2. Ax = 0 has only the trivial solution
3. The reduced row-echelon form of A is I_n
4. A is expressible as a product of elementary matricies
5. Ax = b is consistent for each n x 1 matrix b

6. Ax = b has exactly one solution for each n x 1 matrix b

3 Theorem

For A and B square matricies of the same size, if AB is invertible, then A and B must also be invertible

3.1 Proof

4 Problem

If A is either not square or not invertible, find all b such that Ax = b is consistent

If a is invertible then it has 1 solution If a is non-invertible then it either has infinite solutions or no solutions

4.1 Examples

1. Find all b such that $x + 3y = b_1$ $2x + 6y = b_2$ has a solution

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ r_2 - $2r_1 \to r_2$ $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

The last row tells us that $0x_1 + 0x_2 = b_2 - 2b_1$, which can only be true if $b_2 = 2b_1$

So, if $b_2 - 2b_1 \neq 0$, then we have no solutions. Thus, we would have solutions when $b_2 - 2b_1 = 0$

How to write a single column $b^{\to} = \begin{bmatrix} b_1 \\ 2b_1 \end{bmatrix}$

1. Find all b such that $x+3y+2z = b_1$ $2x+7y+4z = b_2$ $4x+5y+8z = b_3$

$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 4 & 5 & 8 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 4 & 5 & 8 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 4b_1 + 7(b_2 - 2b_1 \end{array}\right]$$

No solution unless $b_3 - 4b_1 + 7(b_2 - 2b_1) = 0$

Let $b_1 = t, b_2 = s$

$b_3 - 18t + 7s = 0$ $b_3 = 18t - 7s$

Thus, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ 18t - 7s \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 18t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ -7s \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 18 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix}$

When you add 2 vectors you add component wise, in the above column vector we have 2 variables, so we write in 2 separate column vectors. First entry we have t, so pull it out