Worksheet 4

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1. Calculate derivatives of the following functions, making a note of which differentiation laws you use at each step:

   a. $f(x) = x^2 + \sqrt{x}$
   b. $f(x) = 10e^3$
   c. $f(x) = 3e^x + 4/\sqrt[3]{x}$
   d. $f(x) = (x/2)^5$

   a. We know the derivative of the sum is the sum of the derivatives, so we can write
   $$\frac{df}{dx} = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sqrt{x}\right)$$
   Now we can use the exponent differentiation law to get that $f'(x) = 2x + 1/\left(2\sqrt{X}\right)$.

   b. This is just a constant, so the derivative is 0.

   c. Again we can split the sum up, so we get
   $$\begin{aligned} f'(x) &= \frac{d}{dx}\left(3e^x\right) + \frac{d}{dx}\left(\frac{4}{x^{1/3}}\right) \\ &= 3e^x + \frac{d}{dx}\left(4x^{-1/3}\right) \\ &= 3e^x - \frac{4}{3}x^{-4/3} \end{aligned}$$

   d. First we need to pull the constant out front, then we can apply the power rule:
   $$\frac{df}{dx}(x) = 2^{-5}\frac{d}{dx}\left(x^5\right) = \frac{5}{2^5}x^4$$

2. Find the equation of the tangent line to the curve at the given point.

   a. $f(x) = x^4 + 2x^2 - x$ at $(1, 2)$
   b. $f(x) = \sin(x)$ at $(\pi/2, 1)$

a. First we take the derivative to get $f'(x) = 4x^3 + 4x - 1$. Then we plug the x-value of the point into the derivative to get the instantaneous rate of change at that point, $f'(1) = 4 + 4 - 1 = 7$. Now this means the tangent line at the point $(1, 2)$ is the line with slope 7 which goes through this point. Therefore we can write the equation of the line as: $y - 2 = 7(x - 1)$ or $y = 7x - 5$.

b. First take the derivative to get $f'(x) = \cos(x)$. This tells us that the slope of f at $x = \pi/2$ is $f'(\pi/2) = 0$. Therefore the tangent line to f at this point is the line of slope 0 which goes through $(\pi/2, 1)$. This has equation $y - 1 = 0(x - 2)$ or just $y = 1$.

3. Consider the function $f(x) = x^2 e^x$.

   a. Find the derivative of this function.

   b. What is the equation of the tangent line to f at $(1, e)$?

   a. We first use the product rule to calculate the derivative:

$$f'(x) = \frac{d}{dx}\left(x^2\right)e^x + x^2\frac{d}{dx}\left(e^x\right) = 2xe^x + x^2e^x = e^x\left(2x + x^2\right)$$

   b. Now we can plug in the point $x = 1$ to get the slope of f at this point:

$$f'(1) = e(2 + 1) = 3e$$

   This means the tangent line to f at this point is the line of slope $3e$ going through $(1, e)$. This has the equation $y - e = 3e(x - 1)$, or $y = 3ex - 2e$.

4. Consider the function:

$$f(x) = \frac{2x}{x + 1}$$

   a. Find the derivative of this function.

   b. What is the equation of the tangent line to f at $(1, 1)$?

   a. To take the derivative of this, we use the quotient rule:

$$f'(x) = \frac{2(x + 1) - 2x}{(x + 1)^2}$$

   b. To get the slope of f at the point $(1, 1)$ we plug $x = 1$ into the derivative to get $f'(1) = 2/4 = 1/2$. Therefore the tangent line is the line of slope $1/2$ which goes through the point $(1, 1)$. This has the equation $y - 1 = 1/2(x - 1)$, or equivalently

$$y = \frac{x}{2} + \frac{1}{2}$$

5. Prove the quotient rule using the product rule, and without using the chain rule.

   The quotient rule requires $g(x) \neq 0$, so we can write:

$$f'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}g(x)\right) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)g(x) + g'(x)\frac{f(x)}{g(x)}$$

   where we have used the product rule. Now we can simplify this expression to give us:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g(x)} - g'(x)\frac{f(x)}{(g(x))^2} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$$