Worksheet 22

Geometry of $\mathbb{R}^3$

1. What shapes do the equations $x^2+y^2+z^2=9$ and $x+y+z=0$ describe? How would you describe the set of points satisfying both equations?

   The first is a sphere centered at the origin, and the second is a plane through the origin. Therefore the intersection is a circle.

2. If U and V are planes in $\mathbb{R}^3$, what forms can $U \cap V$ take? Construct explicit examples - for each possible form of $U \cap V$, find two planes whose intersection is of that form. What does this tell you about solutions to systems of 2 linear equations in 3 variables?

   - Plane: If U is the plane defined by $x+y+z=0$, and V is the plane defined by $2x+2y+2z=0$, then they are the same plane, and they intersect to give the same plane.
   - Line: If U is the plane defined by $x+y-z=0$, and V is the plane defined by $x+y+z=0$, the intersection is a line.
   - Empty set: If U is the plane $x+y+z=0$ and V is the plane $x+y+z=1$, they have no points in common.

   This tells us that systems of 2 linear equations in 3 variables either have infinitely many solutions, or none.

3. Describe the part of the plane $x+2y+z=4$ contained within the first octant.

   When $x=0$ and $y=0$ and $z>0$ the only point on this plane is $z=4$, so it has the point $(0,0,4)$. Similarly it has the points $(0,2,0)$, and $(4,0,0)$. Therefore this portion of the plane is just a triangle with these three vertices.

4. Write down a system of inequalities describing a cone with base of radius 2 in the xy-plane centered at the origin and height 2 with top point $(0,0,2)$.

   Think about this one first as writing down an inequality for a triangle in the x, y plane. In this case it would just be that $x \leq 2-y$. In the 3d case this becomes

$$\begin{cases} z \geq 0 \\ z \leq 2 - \sqrt{x^2 + y^2} \end{cases}$$

Vectors and dot products

1. a. What are the lengths of the vectors $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

   b. Calculate the dot products $u \cdot u$ and $v \cdot v$. Notice anything?

   a. The length of v is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$. The length of u is $\sqrt{16 + 1} = \sqrt{17}$.

   b. The dot products are $u \cdot u = 1^2 + 2^2 + 3^2 = 14$ and $v \cdot v = 16 + 1 = 17$. I notice that the dot product is the square of the length.

2. Find a vector that has the same direction as $\begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix}$ of length 2.

   If a vector is in the same direction as another vector, then it must be a scalar multiple of that vector. Since the length of the given vector is $\sqrt{8^2 + (-1)^2 + 16} = 9$, we just need to scale this vector by $2/9$ to get:

$$\frac{2}{9} \begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix}$$

3. Consider the triangle in $\mathbb{R}^3$ with vertices $0, \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \vec{u} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

   a. What are the side lengths of the edges of this triangle?

   b. (Challenge) What is the area of this triangle?

   a. The side lengths are just the distances between the points. Therefore they are:

$$\left|\vec{v} - \vec{0}\right| = \sqrt{1 + 4 + 9} = \sqrt{14}$$
$$\left|\vec{u} - \vec{0}\right| = \sqrt{9 + 4 + 1} = \sqrt{14}$$
$$|\vec{v} - \vec{u}| = \sqrt{2^2 + 0^2 + (-2)^2} = \sqrt{4 + 4} = 2\sqrt{2}$$

b. First use the dot product to find the angle between the two vectors:

$$u \cdot v = 3 + 4 + 3 = 10 = |u|\,|v| \cos\theta = 14 \cos\theta$$

so $\theta = \arccos(10/14) = \arccos(5/7)$. Then the area of the triangle is just given by

$$\frac{1}{2}\,|u|\,|v| \sin\theta = \frac{14}{2} \sin(\arccos 5/7) = 14\sqrt{6}/7 = 2\sqrt{6}$$

4. Determine whether angles between the following pairs of vectors are acute, right, or obtuse:

$$\vec{v_1} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \qquad \vec{v_2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \vec{v_3} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{v_4} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

If the vectors are acute the dot product will be positive, if they are obtuse the dot product will be negative, and if they orthogonal (right angle) the dot product will be zero. Therefore we just calculate:

$$\begin{aligned} \vec{v_1} \cdot \vec{v_2} &= 3 - 8 + 5 = 0 & \qquad \vec{v_1} \cdot \vec{v_3} &= -3 + 4 - 5 = -4 \\ \vec{v_1} \cdot \vec{v_4} &= 6 - 4 + 5 = 7 & \qquad \vec{v_2} \cdot \vec{v_3} &= -1 - 2 - 1 = -4 \\ \vec{v_2} \cdot \vec{v_4} &= 2 + 2 + 1 = 5 & \qquad \vec{v_3} \cdot \vec{v_4} &= -2 - 1 - 1 = -4 \end{aligned}$$

5. Find the equation of the plane that passes through the origin and is perpendicular to the vector $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$.

   The equation of this plane is $x - 2y + 5z = 0$.

6. **True or false:** If the dot product of two vectors is zero then one of the initial vectors was the zero vector.

   This is **false**. As a counterexample consider $\vec{u} = (1, 0)$ and $\vec{v} = (0, 1)$ then $\vec{u} \cdot \vec{v} = 1 \cdot 0 + 0 \cdot 1 = 0$.

7. **True or false:** If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors then

$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

   This is **true**.

8. (Challenge) Use the CSB inequality and the formula for the dot product to prove the triangle inequality for vectors:

$$|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$$

For any two vectors $\vec{u}$ and $\vec{v}$,

$$\begin{aligned}
|\vec{u}+\vec{v}|^2 &= (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) \\
&= |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v} \\
&\leq |\vec{u}|^2 + |\vec{v}|^2 + 2\,|\vec{u} \cdot \vec{v}| \\
&\leq |\vec{u}|^2 + |\vec{v}|^2 + 2\,|\vec{u}|\,|\vec{v}| \\
&\leq (|\vec{u}| + |\vec{v}|)^2
\end{aligned}$$