Worksheet 2

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1 Based off HW 2

1. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
   (b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of the family.
   (c) Which function belongs to both families?

   a. The family of linear functions with slope 2 consists of the functions of the form $f(x) = 2x + a$ for any constant a. Some examples can be seen in fig. 1.
   b. The family of linear functions such that $f(2) = 1$ consists of the functions of the form $f(x) = bx + 1 - 2b$ for some constant b. Some examples can be see in fig. 2.
   c. The function which belongs to both families is $f(x) = 2x - 3$.

2. Find the cubic function such that $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

   All cubic functions are of the form

$$f(x) = ax^3 + bx^2 + cx + d$$

   for some constants a, b, c, d. The conditions given in the problem imply the following:

$$6 = a + b + c + d \qquad 0 = -a + b - c + d$$
$$0 = d \qquad 0 = 8a + 4b + 2c + d$$

   this means we have already solved $\boxed{\mathrm{d} = 0}$. This simplifies the problem to have only three constraints:

$$6 = a + b + c \qquad 0 = -a + b - c \qquad 0 = 4a + 2b + c \tag{1}$$

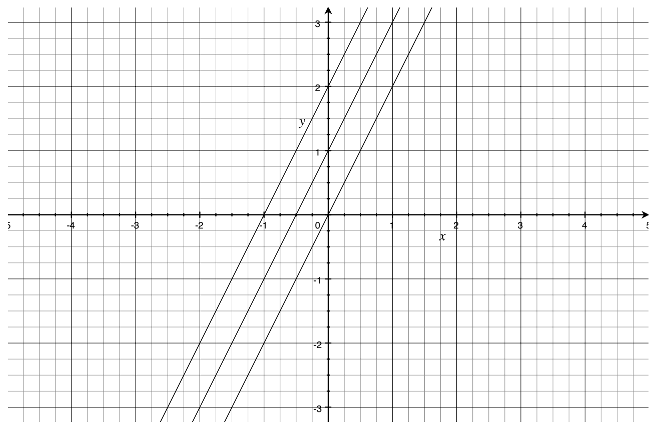

Figure 1: The functions $f(x) = 2x, 2x+1, 2x+2$.

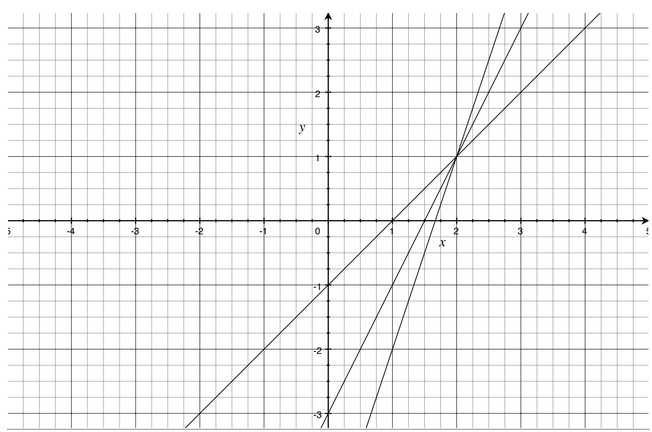

Figure 2: Plots of the functions $f(x) = x-1, 2x-3, 3x-5$.

The middle equation implies $a = b - c$, which we can substitute into the first equation to get $6 = b - c + b + c = 2b$, so $\boxed{\text{b = 3}}$. This simplifies the constrains further to:

$$3 = a + c \qquad\qquad 4a + c = -6$$

which can be solved to get $\boxed{\text{c = 6}}$, and $\boxed{\text{a = -3}}$. Therefore we can write the final function as:

$$\boxed{f(x) = -3x^3 + 3x^2 + 6x}$$

3. Simplify the following:

   (a) $a^8 (2a)^5$

   (b) $\left(6a^3\right)^4 / \left(2a^5\right)$

   a. $2^5 a^{13} = 32a^{13}$

   b. $6^4 a^7 / 2 = 648a^7$

4. Is the function $x^2 - 2x$ one-to-one?

   No, this function fails the horizontal line test for every value of $y > -1$.

5. Let $f(x) = 4 + x + e^x$. Find $f^{-1}(5)$.

   First we notice that this function is one-to-one, and the inverse is well defined at 5. The answer to this question must be a number a such that $f(a) = 5$. But if this is true, then it must be the case that $4 + a + e^a = 5$, which means $a + e^a = 1$, and from this it is clear that $a = 0$, since $0 + e^0 = 1$.

2 Based off HW 3

1. Consider the following table of data:

x	2	4	6	8	10	12
y	0.08	0.12	0.18	0.26	0.35	0.53

   (a) Draw a scatter plot of the data points.

   (b) Make a semilog and log-log plot of the data.

   (c) Is a linear, power, or exponential function appropriate as a model?

   a. See fig. 3.

   b. See fig. 4.

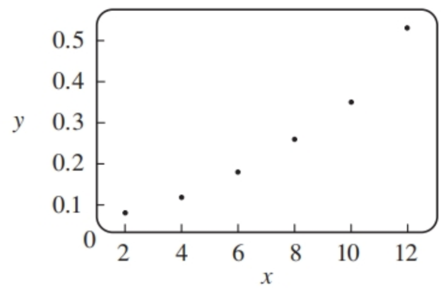

Figure 3: Plot of the data with linear scales on both axes.

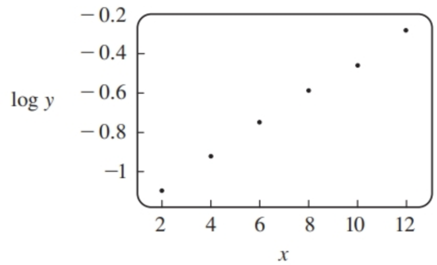

Figure 4: Semilog plot of the data.

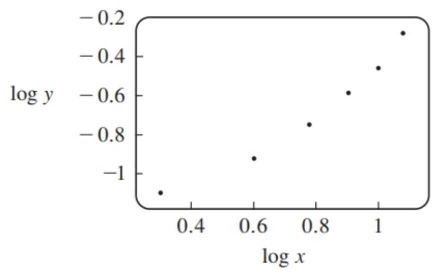

Figure 5: log-log plot of the data.

c. See fig. 5.

2. Let $a_n = 1 - (0.1)^n$. Does this sequence converge? If so, what does it converge to, and if not, why not?

   This sequence converges to 1.

3. Evaluate the following limits:

   (a) $\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{2x-x^2}$

   (b) $\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$

   (c) $\lim_{x\to\infty} \arctan x$

   (d) $\lim_{x\to\pi/2^-} \tan x$

   (e) $\lim_{x\to 1} \frac{3-x}{(x-1)^2}$

   a. -1

   b. $-1/2$

   c. $\pi/2$

   d. ∞

   e. ∞

4. State the squeeze theorem.

   Theorem 1. *If $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly at a) and*

   $$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

   then

   $$\lim_{x\to a} g(x) = L \tag{2}$$

5. Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$. [Hint: You cannot use the fact that the limit of a product is the product of the limits (why?). Instead, you must use the squeeze theorem...]

   We cannot just split this limit as a product, because the limit as $x \to 0$ of $\sin(1/x)$ does not exist. Because of this, we have to take a different route.

   Proof. As suggested by the previous question, we use the squeeze theorem. Now we know $\sin(1/x)$ takes values between -1 and 1. Because of this, we can write:

   $$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

   which means we can apply the squeeze theorem for $f(x) = -x^2$ and $h(x) = x^2$. Since

   $$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$$

we have

$$\boxed{\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0}$$

as desired. $\square$

6. Does the following limit exist?

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{|x|}\right)$$

If so, what is it, if not why does it not exist?

No, this limit does not exist, because the limit from the left is $-\infty$, and the limit from the right is 0, and the two-sided limit at 0 exists iff the two one-sided limits agree.