Worksheet 14

GSI: Jackson Van Dyke

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1 Standard

1. Calculate the area between the curves x^2 and x^3 between 0 and 1.

   Recall the area between two curves in an interval is just the integral of their difference over that interval. Therefore this area is:

   $$A = \int_0^1 x^2 - x^3 \, dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

2. We can never use the same integration techniques we have been using all along to find a volume, since volumes are three dimensional quantities, and the functions we have considered in this class are only two-dimensional.

   This is **false**. We can cleverly set up some integrals to calculate a volume such that it is an integral we can solve with the methods we know.

3. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ around the x-axis.

   To calculate the volume bounded by this "surface of revolution" we have to calculate the following integral:

   $$V = \int_a^b \pi \left(f(x) \right)^2 \, dx$$

   where $f(x) = \sqrt{x-1}$, $a = 1$, and $b = 5$. We get a and b by thinking about the region bounded by these three curved in the x, y plane. This means we can write:

$$\begin{aligned}
V &= \int_0^5 \pi \left(\sqrt{x-1}\right)^2 \, dx \\
&= \int_0^5 \pi \left(x-1\right) \, dx \\
&= \pi \left[\frac{1}{2}x^2 - x\right]_1^5 \\
&= \pi \left(\frac{25}{2} - 5 - \frac{1}{2} + 1\right) \\
&= \boxed{8\pi}
\end{aligned}$$

4. Check that $y = \cos t$ is a solution to the following differential equation:

$$\frac{d^2y}{dt^2} + y = 0$$

Can you think of any other solutions?

Take the second derivative?

$$y'' = -\cos t$$

so indeed $y'' + y = 0$. $\sin t$ is also a solution as well as any linear combination of $\sin t$ and $\cos t$.

2 Challenge

1. The region bounded by the graphs of $y = x^2$ and $y = 6 - |x|$ is revolved around the y-axis. What is the volume of the generated solid?

If we imagine the plots of these functions and calculate the intersection points between the graphs, we can split this volume up into two integrals:

$$\begin{aligned}
V &= \int_0^4 \pi \left(\sqrt{y}\right)^2 \, dy + \int_4^6 (6-y)^2 \, dy \\
&= \pi \int_0^4 y \, dy + \pi \int_2^0 u^2 \, (-\, dy\,) \\
&= \left[\frac{y^2}{2}\right]_0^4 - \pi \left[\frac{1}{3}u^2\right]_2^0 \\
&= 8\pi + \frac{8}{3}\pi = \boxed{\frac{32}{3}\pi}
\end{aligned}$$

where in the second integral I have used a u-substitution of $u = 6 - y$.

2. Let a and b be positive numbers. The region in the second quadrant bounded by the graphs of $y = ax^2$ and $y = -bx$ is revolved around the x-axis. Which of the following relationships between a and b would imply that the volume of this solid of revolution is a constant, independent of a and b?

   The region bounded by these curves in the plane is between $x = -b/a$ and $x = 0$. Therefore the volume is given by:

$$
\begin{aligned}
V &= \pi \int_{-b/a}^{0} \left[(-ba)^2 - \left(ax^2\right)^2 \right] dx \\
&= \pi \int_{-b/a}^{0} b^2a^2 - a^2x^4 \, dx \\
&= \pi \left[\frac{b^2}{3}x^3 - \frac{a^2}{5}x^5 \right]_{-b/a}^{0} \\
&= \frac{\pi b^5}{a^3}\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{2\pi b^5}{15a^3}}
\end{aligned}
$$

   Now the problem is basically done. The hard work was doing this integral. But for this to be independent of a, b just means that $b^5 = ca^3$ for any constant c.

3. Consider the following differential equation:

$$y' = xy^2$$

   a. What will a solution of this equation look like near $x = 0$?
   b. Is the following a solution?

$$y = \frac{1}{x^2 + c}$$

   c. Is the following a solution?

$$y = \frac{1}{c - x^2/2}$$

   d. Can you find the solution to this yourself? [I don't know if you did this in class so the answer might be no.]

   a. It will have a slope of 0 since y' is the slope of the solution, and y' is zero at $x = 0$.
   b. Check this by explicitly differentiating y and seeing if it satisfies the equation $y' = xy^2$. It does not satisfy it.

c. We can check this by differentiating y to get:

$$y' = \frac{-1}{\left(c - x^2/2\right)^2}\,(-x) = \frac{x}{(c - x^2/2)}$$

but this last expression is just xy^2 by the definition of y, so this is a solution to the given differential equation.