Derivatives of inverse trig functions

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Seldom taught in high-school calculus, is the surprisingly simple[1] differentiation law:

$$\frac{d}{dx}\left(f^{-1}(y)\right) = \frac{1}{f'(x)}$$

This is of special interest to anyone who doesn't like memorizing long tables, since, with a little practice, this rule renders the infamous table of derivatives of inverse trig functions useless.

We will do one example carefully, and then the rest in less detail. **I strongly suggest attempting these on your own before reading this, and only consulting this for hints when you get stuck**.

Let's try to calculate $(\arcsin(y))'$. So our $f(x) = \sin(x)$, and we want to calculate the derivative of its inverse. Well we know $f'(x) = \cos(x)$, which means we can write

$$\frac{d}{dx}\left(f^{-1}(y)\right) = (\arcsin(y))' = \frac{1}{\cos(x)}$$

but we want to calculate the derivative of this as a function of y, not as a function of x. Since $x = \arcsin(y)$, if we take cos of both sides, we get $\cos(x) = \cos(\arcsin(y))$, so now we can express the derivative as a function of y, but it appears that we have made the situation much more messy. In fact, we can rewrite this expression in a much more simple way. Imagine a right triangle with hypotenuse length 1, and side lengths x and y as in fig. 1. Call the marked angle θ. Then $\arcsin y = \theta$, which means $\cos(\arcsin y) = \cos(\theta) = x$. But by

[1] As an exercise, try to prove this law from the chain rule. [Hint: the composition of f with its inverse is just x, now take the derivative of both sides.]

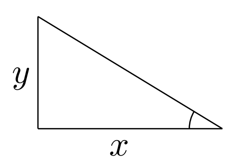

Figure 1: A right triangle with hypotenuse length 1.

the Pythagorean theorem, $x = \sqrt{1-y^2}$, which means

$$\boxed{\frac{d}{dx}\left(\arcsin\left(y\right)\right) = \frac{1}{\sqrt{1-y^2}}}$$

which is indeed what is on[2] your "cheat sheet" for the derivative of $\arcsin(y)$.

Now let's set $f(x) = y = \cos(x)$. Therefore $f^{-1}(y) = x = \arccos(y)$. As before,

$$\frac{d}{dx}\arccos\left(y\right) = \frac{1}{-\sin\left(x\right)} = \frac{1}{-\sin\left(\arccos\left(y\right)\right)}$$

Now take the triangle from fig. 1 and let θ be the non-marked angle. This gives us that $\arccos(y) = \theta$, so

$$\sin\left(\arccos\left(y\right)\right) = \sin\left(\theta\right) = x = \sqrt{1-y^2}$$

so this time,

$$\boxed{\frac{d}{dx}\arccos\left(y\right) = -\frac{1}{\sqrt{1-y^2}}}$$

Now set $f(x) = \tan(x)$, so we calculate:

$$\frac{d}{dx}f^{-1}\left(x\right) = \frac{d}{dx}\arctan\left(y\right) = \frac{1}{\cos^{-2}\left(x\right)} = \cos^2\left(x\right) = \cos^2\left(\arctan\left(y\right)\right)$$

since the derivative of $\tan(x) = \cos^{-2}(x)$. Now take the triangle in fig. 1, and do the following:

- Let the marked angle be θ.
- Let the vertical side be y.
- Let the horizontal side (labelled x) have length 1.

This means the hypotenuse has length $\sqrt{1+y^2}$. Now we calculate

$$\cos^2\left(\theta\right) = \cos^2\left(\arctan\left(y\right)\right) = \frac{1}{1+y^2}$$

which means

$$\boxed{\frac{d}{dx}\left(\arctan\left(y\right)\right) = \frac{1}{1+y^2}}$$

[2] Unless your cheat sheet is wrong of course.