Worksheet 12

GSI: Jackson Van Dyke

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1 Standard

1. Compute the following integrals using integration by parts:

   a.
   $$\int xe^{x/2}\,dx$$

   b.
   $$\int x\cos 5x\,dx$$

   a. Set $u = x$ and $dv = e^{x/2}$. Then we can rewrite the integral as:

   $$\int xe^{x/2}\,dx = 2xe^{x/2} - 2\int e^{x/2} = 2xe^{x/2} - 4e^{x/2} + c = \boxed{2e^{x/2}(x-2)+c}$$

   b. Set $u = x$ and $dv = \cos 5x$. Then we can rewrite the integral as:

   $$\int x\cos 5x\,dx = \frac{x\sin 5x}{5} - \frac{1}{5}\int \sin 5x = \boxed{\frac{x\sin 5x}{5} + \frac{\cos 5x}{25} + c}$$

2. **True or false:** It doesn't matter what you choose for u and dv when you use integration by parts since the formula holds either way.

   This is **false**. Though it is true that the formula holds no matter what u and dv you use, some choices of u and dv will yield a more difficult problem.

3. Evaluate the following integral:

   $$\int \frac{x-9}{(x+5)(x-2)}\,dx$$

First we split the expression up using partial fraction decomposition:

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$
$$x-9 = (x-2)A + (x+5)B$$

Now setting $x = 2$ yields $B = -1$, and setting $x = -5$ yields $A = 2$. Therefore we have split the integral up as follows:

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int \frac{2}{x+5}\,dx - \int \frac{1}{x-2}\,dx$$

which we can anti-differentiate to yield:

$$\boxed{2\ln|x+5| - \ln|x-2| + c}$$

2 Challenge

1. Calculate the following integral:

$$I \int x^3 e^{x^2}\,dx$$

First set $u = x^2$, which means $du = 2x\,dx$, so we can rewrite the integral as:

$$I = \int x^3 e^{x^2}\,dx = \frac{1}{2}\int u e^u\,du$$

now we integrate by parts! Set $u = u$ and $dv = e^u$. This means:

$$I = \frac{1}{2}ue^u - \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u(u-1) = \boxed{\frac{1}{2}e^{x^2}\left(x^2-1\right)}$$

2. Integration by parts is only useful when we are trying to take the integral of a product of two functions of x. For example, the integral of $x \arctan x$ might be solved using integration by parts, whereas the integral of $\arctan x$ cannot.

This is **false**. Integration by parts can be used to show:

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) + c$$

3. Calculate the following integral:

$$I = \int \frac{x}{(x-a)(x-b)}\,dx$$

First we break up the expression using partial fraction decomposition:

$$\frac{x}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

is equivalent to

$$x = (x-b)\,A + (x-a)\,B$$

so setting $x = b$ we get $b = (b-a)\,B$, or

$$\boxed{B = -\frac{b}{a-b}}$$

and similarly, setting $x = a$ gives us $a = (a-b)\,A$, so

$$\boxed{A = \frac{a}{a-b}}$$

which means

$$\begin{aligned}
I &= \int \frac{a}{(x-a)(a-b)}\,dx - \int \frac{b}{(a-b)(x-b)}\,dx \\
&= \frac{1}{a-b}\left(\int \frac{a}{x-a}\,dx - \int \frac{b}{x-b}\,dx\right) \\
&= \frac{1}{a-b}\left(a\ln|x-a| - b\ln|x-b|\right) + c \\
&= \boxed{\frac{a\ln|x-a| - b\ln|x-b|}{a-b} + c}
\end{aligned}$$