Worksheet 21

1. Solve the following system of linear equations by putting them into augmented matrix form, using row operations to get the system into upper triangular form, and then using back substitution:

$$\begin{cases} 2x + 3y - z = 0 \\ x + 2y + z = 3 \\ x + 3y + 3z = 7 \end{cases}$$

Do the following operations:

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 3 & 7 \end{array}\right) \xrightarrow{\text{II}-\text{I}/2,\text{III}-\text{I}/2} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 3/2 & 7/2 & 7 \end{array}\right)$$

$$\xrightarrow{\text{III}-3\text{II}} \begin{pmatrix} 2 & 3 & -1 & 0 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

Therefore $z = 2$ is a solution, and $y/2 + 3z/2 = 3$, or $y/2 + 3 = 3$, so $y = 0$. Then $2x + 3y - z = 0$, so $x = 1$, and the solutions is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

2. Use the row operations we learned in class to convert the following augmented matrices $\left(A|\vec{b}\right)$ into the form $(U|\vec{c})$ where U is upper triangular. Using your results, solve the corresponding original systems of linear equations:

a.

$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 1 & 2 & 1 & 1 \\ 4 & 5 & 0 & 2 \end{array}\right)$$

b.

$$\left(\begin{array}{ccc|c} 2 & 1 & 8 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & 5 & 4 & 1 \end{array}\right)$$

c.

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 2 \\ 2 & 4 & -8 & 5 \\ -3 & -6 & 12 & -6 \end{array}\right)$$

Now find the determinants of the 3×3 (non-augmented) matrices A in each of the systems above. Which determinants are 0 and which are not? Which are invertible and which are not? Does this correspond to the number of solutions to the systems that you found? Explain.

a. First switch the third and first row so the top left is nonzero. Then:

$$\left(\begin{array}{ccc|c} 4 & 5 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right) \xrightarrow{\text{II}-\text{I}/4} \begin{pmatrix} 4 & 5 & 0 & 2 \\ 0 & 3/4 & 1 & 1/2 \\ 0 & 1 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{III}-4\text{II}/3} \left(\begin{array}{ccc|c} 4 & 5 & 0 & 2 \\ 0 & 3/4 & 1 & 1/2 \\ 0 & 0 & 5/3 & -2/3 \end{array}\right)$$

Therefore $5z = -2$, so $z = -2/5$, then $3y/4+z = 1/2$, so $3/4y-2/5 = 1/2$, so $y = 4 \cdot 9/30 = 6/5$. Finally $4x + 5y = 2$, so $4x + 6 = 2$, so $x = -1$. Therefore the solution is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6/5 \\ -2/5 \end{pmatrix}$$

The determinant is $0 + 5 \cdot 3 + 4 - 4 \cdot 2 \cdot 3 - 0 - 0 = -5 \neq 0$ so it is invertible, which is what we expect because we have exactly one solution.

b.

$$\left(\begin{array}{ccc|c} 2 & 1 & 8 & 1 \\ -1 & 1 & -1 & 0 \\ -2 & 5 & 4 & 1 \end{array}\right) \xrightarrow{\text{II}+\text{I}/2,\text{III}+\text{I}} \begin{pmatrix} 2 & 1 & 8 & 1 \\ 0 & 3/2 & 3 & 1/2 \\ 0 & 6 & 12 & 2 \end{pmatrix}$$

$$\xrightarrow{\text{III}-4\text{II}} \begin{pmatrix} 2 & 1 & 8 & 1 \\ 0 & 3/2 & 3 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The bottom line means the equations don't constrain z, so it can be anything. It does however tell us that $3y/2 + 3z = 1/2$, so $y =$

$1/2-2z$. Similarly, $2x+y+8z=1$, so $x = 1//3-3z$. Therefore there are infinitely many solutions. They are all of the following form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 - 3z \\ 1/3 - 2z \\ z \end{pmatrix}$$

We can calculate the determinant to be $2 \cdot 1 \cdot 4 + 2 \cdot (-1) \cdot (-2) + 8 \cdot (-1) \cdot 5 - 2\,(-1) \cdot 5 - 1\,(-1) \cdot 4 - 8\,(-2) = 0$. Therefore the matrix is not invertible, but this is exactly what we expect because we have infinitely many solutions.

c.

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 2 \\ 2 & 4 & -8 & 5 \\ -3 & -6 & 12 & -6 \end{array}\right) \xrightarrow{\text{II}-2\text{I},\text{II}+3\text{I}} \begin{pmatrix} 1 & 2 & -4 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The second row tells us that $0 \cdot x + 0 \cdot y + 0 \cdot z = 0 = 1$ which clearly can't be the case, so there are no solutions. Again, we can calculate the determinant to be: $4 \cdot 12 + 2\,(-8)\,(-3) + (-4)\,(2)\,(-6) - (-4)\,(4)\,(-3) - (-8)\,(-6)\,(1) - 2 \cdot 2 \cdot 12 = 0$. This is what we expect, since there are no solutions.

3. Find the determinant of the following matrix and verify that it is nonzero. Find its inverse using row operations and also using the formula we learned previously.

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{\text{II}-3\text{I}/2} \begin{pmatrix} 2 & 3 & 1 & 0 \\ 0 & -1/2 & -3/2 & 1 \end{pmatrix}$$
$$\xrightarrow{\text{I}+6\text{II}} \begin{pmatrix} 2 & 0 & -8 & 6 \\ 0 & -1/2 & -3/2 & 1 \end{pmatrix}$$
$$\xrightarrow{\text{I}/2,-2\text{II}} \begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -2 \end{pmatrix}$$

so the inverse is

$$A^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$$

indeed, since the determinant is $4 \cdot 2 - 3 \cdot 3 = -1$. the old formula would give us exactly the same answer.

4. Use row operations to find the inverse of the matrix and then check that your answer is correct by verifying both $AA^{-1} = I_3$ and $A^{-1}A = I_3$.

$$\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -3 & 1 & 2 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{II+3I,III+2I}} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 7 & -1 & 3 & 1 & 0 \\ 0 & 6 & -1 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{I}-2\text{I,III}-6\text{II}/7} \left(\begin{array}{ccc|ccc} 1 & 0 & -5/7 & 1/7 & -2/7 & 0 \\ 0 & 7 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1/7 & -4/7 & -6/7 & 1 \end{array}\right)$$

$$\xrightarrow{\text{I}-5\text{III,II}-7\text{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & -5 \\ 0 & 7 & 0 & 7 & 7 & -7 \\ 0 & 0 & -1/7 & -4/7 & -6/7 & 1 \end{array}\right)$$

$$\xrightarrow{\text{II}/7,(-7)\text{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 4 & -5 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right)$$

so the answer is

$$A^{-1} = \begin{pmatrix} 3 & 4 & -5 \\ 1 & 1 & -1 \\ 4 & 6 & -7 \end{pmatrix}$$

now we explicitly multiply A and A^{-1}, and we can see this is the 3×3 identity matrix.

5. **True or false:**

   a. If a matrix is not invertible, we perform row operations to make it invertible.

   b. Now that we are working with matrices that represent systems of equations, we only have to worry about square matrices.

   c. Since we can use Gaussian elimination to find the inverse of a matrix, we can now forget the equation for the inverse of a 2×2 matrix.

   d. Row operations do not change the determinant of a matrix.

   a. This is false. Row operations will not effect whether a matrix is invertible or not.

   b. This is false. Arguably the most important type of matrix is an $n \times 1$ matrix, also known as a vector, which we care quite a lot about.

   c. This is also false. That is a very handy and quick formula. Using Gaussian elimination is tedious, and should be used only for 3×3 matrices and larger.

   d. This is false. Row operations can change the determinant of a matrix.

Challenge

1. For what value of c will the following system of equations have at least one solution? Find all solutions of the resulting system.

$$\begin{cases} x + 3y - z = 4 \\ 2x - y + 3z = 7 \\ 7y - 5z = c \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & -1 & 3 & 7 \\ 0 & 7 & -5 & c \end{array}\right) \xrightarrow{\text{II}-2\text{I}} \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -7 & 5 & -1 \\ 0 & 7 & -5 & c \end{pmatrix}$$

$$\xrightarrow{\text{III}+\text{II}} \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -7 & 5 & -1 \\ 0 & 0 & 0 & c-1 \end{pmatrix}$$

The last line tells us that if $c \neq 1$, there will be no solutions, so we must have $c = 1$. In this case, we can substitute backwards. In the second row we see that $-7y + 5z = -1$, which means $y = (1 + 5z)/7$. The second row says that $x+3y-z = 4$, so $x = 4-3y+z = 4-3/7-15z/7+z = 25/7-8z/7$. Therefore there are infinitely many solutions, which are all of the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (25 - 8z)/7 \\ (1 + 5z)/7 \\ z \end{pmatrix}$$