Worksheet 7

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1. Find the positive numbers whose product makes 100 and whose sum is minimum.

   If x and y are two numbers which have product equal to 100, then $xy = 100$. If we want their sum to be minimum, then we want $x + y$ to be as small as possible. Since $y = 100/x$, this just means $x + 100/x$ needs to be as small as possible. Set $f(x) = x + 100/x$, and then we are just trying to find the global minimum of this function.

   Take the derivative to get

   $$f'(x) = 1 - \frac{100}{x^2}$$

   Now setting this equal to zero we get $x^2 = 100$, so $x = \pm 10$, but we are only allowing for positive values, so $x = 10$, which means $y = 10$ as well.

2. Find the point on the line $y = 2x + 3$ that is closest to the origin.

   The distance from a point (x, y) to the origin is given by $\sqrt{x^2 + y^2}$. Now if the point is on the above line, this means we can substitute the expression $2x + 3$ for y and instead write the distance as a function

   $$d(x) = \sqrt{x^2 + (2x + 3)^2}$$

   Now we take a derivative to get:

   $$d'(x) = \frac{2x + 4(2x + 3)}{x^2 + (2x + 3)^2}$$

   where we have used the chain rule twice. Now if we set this to be 0, we get

   $$10x + 12 = 0$$

   since the denominator doesn't make a difference. This means $x = -6/5$, so $y = -12/5 + 3 = 3/5$ so the point is $(-6/5, 3/5)$.

3. Calculate the following limits. If l'Hospital's rule applies, you can use it, and it not, explain why.

a.

$$\lim_{x\to(\pi/2)^+} \frac{\cos x}{1-\sin x}$$

b.

$$\lim_{x\to 0} \frac{e^t-1}{t^3}$$

c.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

d.

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$$

a. If we attempt to plug $\pi/2$ into the expression we get $0/0$, which is an indeterminate form. We attempt to use l'Hospital's rule, and indeed we can, since the derivative of the denominator, $-\cos x$ is nonzero near (but not at) $\pi/2$. This means

$$\lim_{x\to(\pi/2)^+} \frac{\cos x}{1-\sin x} = \lim_{x\to(\pi/2)^+} \frac{\sin x}{\cos x} = -\infty$$

b. If we attempt to plug in 0 to the expression, we get the indeterminate form $0/0$. We attempt to use l'Hospital's rule, and indeed we can, since the derivative of the denominator, $3t^2$ is nonzero near (but not at) 0. This means

$$\lim_{x\to 0} \frac{e^t-1}{t^3} = \lim_{x\to 0} \frac{e^t}{3t^2} = 1/0 = \infty$$

c. If we attempt to plug in ∞ to the expression, we get the indeterminate form ∞/∞. We attempt to use l'Hospital's rule, but we need to check that the derivative of the denominator is not zero near ∞. But the derivative of $\sqrt{x}$ is $1/\left(2\sqrt{x}\right)$ which is 0 near infinity. We can instead calculate

$$\lim_{x\to\infty} \frac{\ln x\sqrt{x}}{x}$$

Again we get ∞/∞, and now the derivative of the bottom is 1. So we get

$$\lim_{x\to\infty} \frac{\ln x\sqrt{x}}{x} = \lim_{x\to\infty} \frac{\sqrt{x}/x + \ln x/\left(2\sqrt{x}\right)}{1} = \lim_{x\to\infty} \frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}$$

where we have used the product rule. Now subtracting

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$$

from both sides, we can write:

$$\frac{1}{2}\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$$

so the original limit is 0.

d. If we plug 0 into the expression we get 0/0. We attempt to use l'Hospital's rule, and indeed we can, since the derivative of the denominator, is nonzero everywhere. Therefore we can calculate:

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} = \lim_{x\to 0}\frac{2}{2\sqrt{1+2x}} - \frac{-4}{2\sqrt{1-4x}} = 3$$