Worksheet 3

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1. Consider the function:

$$f(x) = \begin{cases} e^x & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

   a. Sketch this function.

   b. Is this function continuous? Why or why not?

   This function just looks like the exponential we know so well for negative x, and x^2 for positive x. As we can see in fig. 1, this is not continuous at 0, since the limit at 0 from one side disagrees with the limit at 0 from the other side.

2. Come up with a piecewise function which is linear.

   Consider the function from the previous example. If we alter it by shifting one of the pieces to get the new function:

$$g(x) = \begin{cases} e^x & x < 0 \\ x^2 + 1 & x \geq 0 \end{cases}$$

   then g is continuous.

3. For what value of a is the following function continuous:

$$f(x) = \begin{cases} x^3 + a & x < 0 \\ \cos x & x \geq 0 \end{cases}$$

   For $a = 0$ this function is plotted in fig. 2. First note that $\cos x$ and x^3 are continuous everywhere, so the only potential problem is at $x = 0$. This makes it clear that a value of $a = 1$ is sufficient to make this function continuous. We can also do this algebraically by plugging in 0 to both parts. For the positive part this gives us 1, and for the negative part this gives us a. For the limits from both sides at 0 to agree, we again just need $a = 1$.

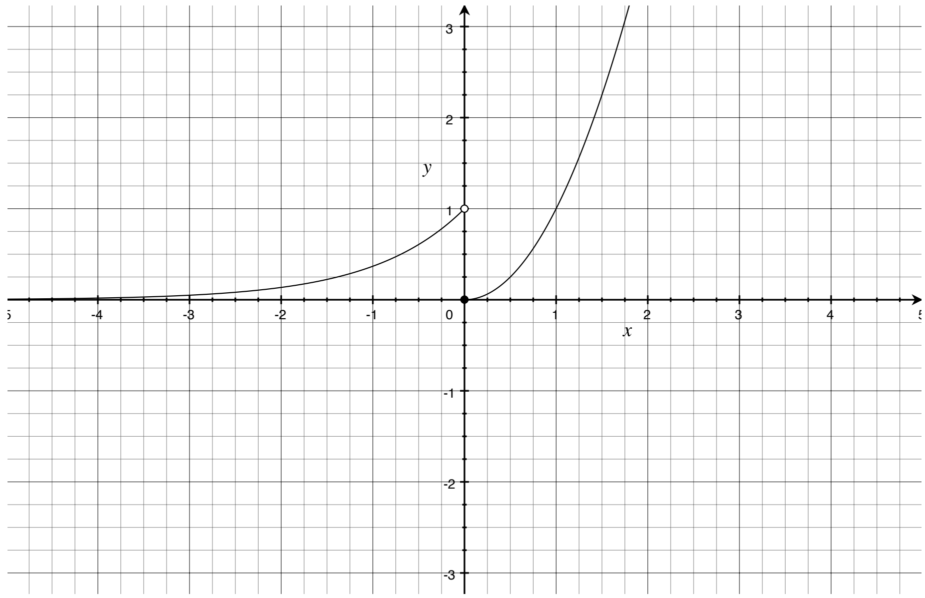

Figure 1: The plot of the function $f(x)$ from the first question.

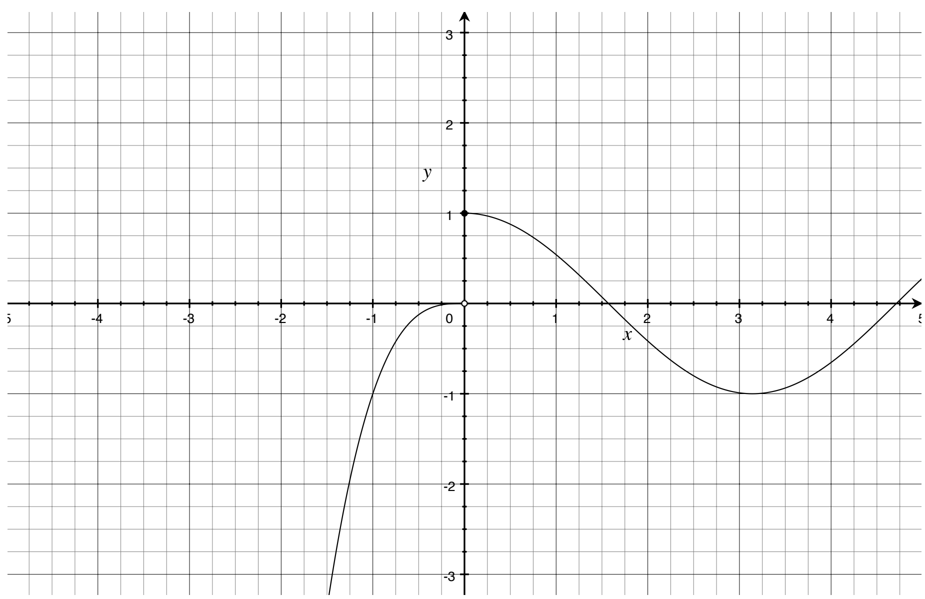

Figure 2: The piecewise plot from item 3.

4. If an equation of the tangent line to the curve $y = f(x)$ at the point where $x = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

   The tangent line to the curve $f(x)$ at $x = 2$ must match the point $(2, f(2))$. Since at $x = 2$ the line is at $4 \cdot 2 - 5 = 3$, this means $f(2) = 3$. We also know that the line must have slope matching the derivative of f at 2, $f'(2)$. The slope of the line is 4, which means $f'(2) = 4$.

5. Consider the function $f(x) = 2x$.

   a. Sketch this function.

   b. Below this function, sketch the derivative of this function based on the notion that the instantaneous rate of change of f is given by the derivative.

   c. What might you guess the derivative of this function is based on your sketches?

   d. How does this all change if we instead do the problem for $f(x) = 2x + 5$?

   The graph of this function is of course just the line with slope 2 intersecting the y-axis at $(0, 0)$. Now the slope of this is 2 everywhere, so the derivative is just the constant function $f(x) = 2$. If we instead do this for the function $f(x) = 2x + 5$ this extra shift doesn't change the slope, so of course it doesn't change the derivative.

6. Consider the function $f(x) = e^x$.

   a. Sketch this function.

   b. Below this function, sketch the derivative of this function based on the notion that the instantaneous rate of change of f is given by the derivative.

   c. What might you guess the derivative of this function is based on your sketches?

   d. How would this all change if we instead do the problem for $f(x) = e^x + 100$?

   We sketch e^x as usual in fig. 3. First we check if the graph has any portions which are flat, since these would correspond to zeros of the derivative. But the exponential function never gets flat. But we can notice that around $x = 0$, e^x seems to have a slope of $+1$. So we plot a positive 1 in the derivative. In addition to this, for negative x, we have that the slope gradually increases from a very low positive slop to the slope of 1 at 0. Then in positive x we get that the slope begins to rapidly increase. The sketch of the exponential looks an awful lot like the exponential itself. And indeed, the derivative of the exponential is just the exponential. Again, adding a constant doesn't change the slope of the function at any point, so the derivative is again just the exponential function.

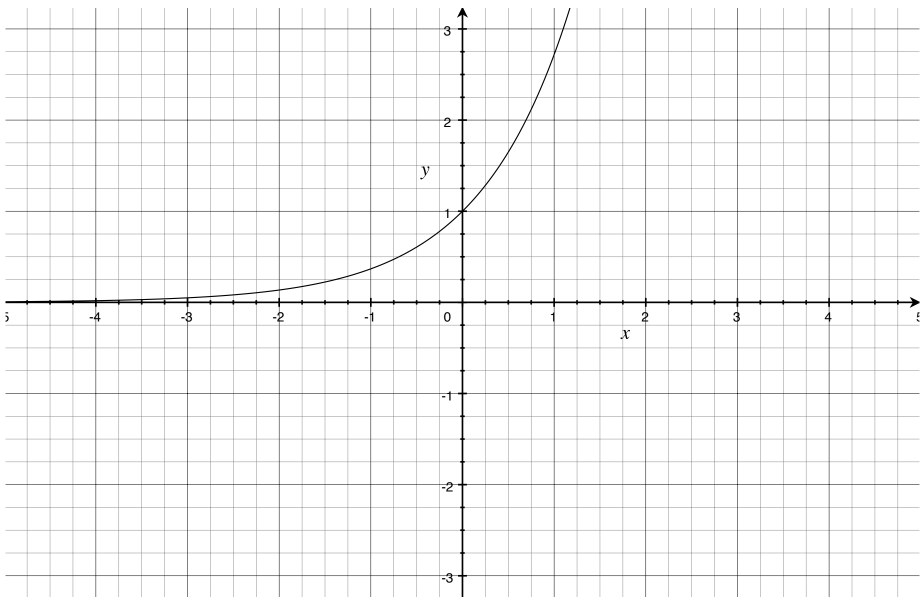

Figure 3: The function $f(x) = e^x$.