MT1 Review
Section 201

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1. Consider the function:

$$f(x) = \begin{cases} x^2 e^x & x < 0 \\ \ln(x+1) & x \geq 0 \end{cases}$$

a. State the definition of continuous.

b. Is f continuous at 0? Show directly why or why not from the definition you have stated above.

a. The definition of continuous is the following: A function f is continuous at $x = a$ if and only if the limit of f as x approaches a from the left, agrees with the limit of f as x approaches a from the right, and also that both of these limit agree with the value $f(a)$.

b. f is continuous at 0, since we can calculate

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(x^2 e^x\right) = \lim_{x \to 0^-} \left(x^2\right) \lim_{x \to 0^-} \left(e^x\right) = 0 \cdot 1 = 0$$

and similarly:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln(x+1) = \ln(1) = 0$$

and of course $f(0) = \ln(0+1) = 0$, so indeed these three quantities all agree.

The grading scheme for this question was as follows: For the first part I awarded one point for mentioning the limit from the right, one point for mentioning the limit from the left (or two points for insisting that the limit exists), one point for mentioning $f(a)$, and two points for insisting they are all equal.

For the second part, I awarded two points for calculating the right limit, two points for calculating the left limit, and one point for evaluating the function at 0 and checking they are all equal.

2. Find the global max and min of the function

$$f(x) = x^3 - 12x$$

in the interval $[-3, 3]$. First, since $x^3 - 12x$ is a polynomial, it is continuous, so we can use the closed interval method from the text. Plug in the endpoints to get:

$$f(3) = 3 \cdot 9 - 4 \cdot 9 = -9$$
$$f(-3) = -3 \cdot 9 + 4 \cdot 9 = 9$$

Now take the derivative to get $f'(x) = 3x^2 - 12$ where we have used the power rule. Setting this equal to zero, gives us $x^2 - 4 = 0$, which we can factor to give us $(x-2)(x+2) = 0$, so the critical points are $x = \pm 2$. But remember, if the derivative is zero, this doesn't necessarily mean the function has a max or min. So we need to plug in these critical values:

$$f(2) = 2^3 - 3 \cdot 2^3 = -2^4 = -16 \qquad f(-2) = -2^3 + 3 \cdot 2^3 = 2^4 = 16$$

Now since $16 > 9$, we see that at $x = 2$ the function has a global min, and at $x = -2$ the function has a global max.

Grading scheme: I awarded one point for realizing you have to plug in the endpoints, and one point for plugging in the endpoints correctly. Then I awarded one point for knowing to take the derivative, one point for taking it properly, one point for finding the critical points, one point for mentioning that it's defined everywhere so there are no other critical points. Then I awarded two points for finding whether these critical points correspond to extrema. This could be done by noting the closed interval method, or by the first derivative test. Finally, I awarded one point for knowing to find the critical values, and two for finding them.

3. Compute the limit:

$$\lim_{x \to \infty} \frac{e^{\sqrt{x}}}{x}$$

Write the limit as L. We get an indeterminate form which is fractional, and the derivative of the bottom is not zero near ∞, so we can apply l'Hospital and get

$$L = \lim_{x \to \infty} \frac{e^{\sqrt{x}} / (2\sqrt{x})}{1} = \lim_{x \to \infty} \frac{e^{\sqrt{x}}}{2\sqrt{x}} = \lim_{x \to \infty} \frac{e^{\sqrt{x}} x}{2x^{3/2}}$$

This is again a fractional indeterminate, and the derivative of the bottom is not zero near ∞. Therefore we can again apply l'Hospital's rule to get:

$$L = \lim_{x \to \infty} \frac{e^{\sqrt{x}} + \sqrt{x} e^{\sqrt{x}}/2}{3\sqrt{x}} = \lim_{x \to \infty} \frac{e^{\sqrt{x}}}{3\sqrt{x}} + \lim_{x \to \infty} \frac{e^{\sqrt{x}}}{6} = \frac{2}{3} L + \lim_{x \to \infty} \frac{e^{\sqrt{x}}}{6}$$

which we can rewrite as

$$L = 3 \lim_{x\to\infty} \frac{e^{\sqrt{x}}}{6} = \boxed{\infty}$$

4. Compute the limit:

$$\lim_{x\to\infty} \left(\sqrt{x} - \ln x\right)$$

We get an indeterminate form of the form $\infty - \infty$. To remedy this, we rewrite this as:

$$L = \lim_{x\to\infty} \left(\sqrt{x} - \ln x\right) = \ln\left(\lim_{x\to\infty} \frac{e^{\sqrt{x}}}{e^{\ln x}}\right) = \ln\left(\lim_{x\to\infty} \frac{e^{\sqrt{x}}}{x}\right)$$

But this is just the limit from the first question, which was infinite, meaning this limit is infinite as well.

Alternate solution: Rewrite this as:

$$\sqrt{x} - \ln x = \frac{\left(\sqrt{x} - \ln x\right)\left(\sqrt{x} + \ln x\right)}{\sqrt{x} + \ln x} = \frac{x - \ln^2(x)}{\sqrt{x} + \ln x}$$

now divide by the leading term of the denominator to get

$$\frac{x - \ln^2(x)}{\sqrt{x} + \ln x} = \frac{\sqrt{x} - \ln^2 x/\sqrt{x}}{\ln x/\sqrt{x} + 1}$$

Now we have

$$\lim_{x\to\infty} -\frac{\ln^2 x}{\sqrt{x}} = 0 \qquad\qquad \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$$

which means we are left with

$$\lim_{x\to\infty} \frac{\sqrt{x} - \cancel{\ln^2 x/\sqrt{x}}}{\cancel{\ln x/\sqrt{x}} + 1} = \lim_{x\to\infty} \sqrt{x} = \boxed{\infty}$$

5. Find the Taylor polynomial of degree 4 at the point $x = 0$ of the function $f(x) = \cos x$. [If you remember the formula for the Taylor expansion of $\cos x$ from the "hall of fame" in lecture, write the formula down and show how you use it to get your answer. If you don't remember the formula, use the definition of the Taylor polynomial to get it directly.] If you remembered the formula, you should have written something like:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} .$$

This tells us that the constant term should be 1, then the linear term is 0, the x^2 term should be $-x^2/2$, the x^3 should be 0, and the x^4 term should be $x^4/4!$ so the polynomial is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

Alternatively, you could start from the definition of the Taylor polynomial of degree 4 at a:

$$P_4(x) = f(a) + f'(a)\,x\frac{f''(a)\,x^2}{2} + \frac{f^{(3)}(a)\,x^3}{3!} + \frac{f^{(4)}(a)}{4!}$$

and here $a = 0$, so we can explicitly calculate the values:

$$\begin{aligned} f(0) &= 1 \qquad & f'(0) &= -\sin(0) = 0 \qquad & f''(0) &= -\cos(0) = -1 \\ & & f^{(3)}(0) &= \sin(0) = 0 & f^{(4)}(0) &= \cos(0) = 1 \end{aligned}$$

which means the polynomial is:

$$1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

as we had above.