Worksheet 23

1. The standard basis vectors of $\mathbb{R}^3$ are:

$$\vec{i} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \vec{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \vec{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

   a. Show that $\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0$

   b. Show that $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$

   a. $\vec{i}\cdot\vec{j} = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$ and the rest are almost identical.

   b. $\vec{i}\cdot\vec{i} = 1\cdot 1 + 0\cdot 0 + 0\cdot 0 = 1$ and the rest are almost identical.

2. Find the angle between the vectors $\vec{a} = (-8, 6)$ and $\vec{b} = \left(\sqrt{7}, 3\right)$.

   Since $\vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta$, we calculate

$$\begin{aligned}\vec{a}\cdot\vec{b} &= -8\cdot\sqrt{7} + 6\cdot 3 = -8\sqrt{7} + 18\\ |\vec{a}| &= \sqrt{64+36} = 10\\ \left|\vec{b}\right| &= \sqrt{7+9} = 4\end{aligned}$$

   and now:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\left|\vec{b}\right|} = \frac{9-4\sqrt{7}}{20}$$

   so $\theta \sim 95°$.

3. Find the scalar and vector projections of **b** onto **a**

   a. $\vec{a} = (1, 2)$, $\vec{b} = (-4, 1)$

   b. $\vec{a} = (1, 1, 1)$, $\vec{b} = (1, -1, 1)$

   a. $|a| = \sqrt{5}$, so

$$\begin{aligned}\operatorname{comp}_a b &= \frac{a\cdot b}{|a|} = \frac{-2}{\sqrt{5}}\\ \operatorname{proj}_a b &= \frac{a\cdot b}{|a|}\frac{a}{|a|} = \frac{-2}{\sqrt{5}}\frac{1}{\sqrt{5}}a = (-2/5, -4/5)\end{aligned}$$

b. $|a| = \sqrt{3}$, so

$$\operatorname{comp}_a b = \frac{a \cdot b}{|a|} = \frac{1}{\sqrt{3}}$$
$$\operatorname{proj}_a b = \frac{a \cdot b}{|a|} \frac{a}{|a|} = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} a = (1/3, 1/3, 1/3)$$

4. If $\vec{a} = (3, 0, -1)$, find a vector $\vec{b}$ such that $\operatorname{comp}_{\vec{a}} \vec{b} = 2$.

Since $\operatorname{comp}_a b = 2$, we have

$$\frac{a \cdot b}{|a|} = 2$$

so $a \cdot b = 2|a| = 2\sqrt{10}$. We need b to have components b_1, b_2, and b_3 such that

$$3b_1 + 0b_2 + (-1)\, b_3 = 2\sqrt{10}$$

One such vector is $b = (0, 0, -2)$.

5. Consider the "L" shape made by the two vectors $\vec{u} = (0, 2)$ and $\vec{v} = (1, 0)$. Describe how multiplication by the given matrices change the two vectors.

a.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

b.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \qquad B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

a. $Au = -u$, and $Av = v$, so the y component changes sign, and the x component is left alone. This is reflection over the x-axis. $Bu = u$ and $Bv = -v$, so the x component changes sign and the y component is left along. This is reflection over the y-axis.

b. $Au = (2/3)\, u$, and $Av = v$. The x component of each vector is untouched, whereas the y components are divided by 3. This is a compression in the y direction. $Bu = (1/2)\, u$ and $Bv = (1/2)\, v$. This shrinks both directions by multiplying by $1/2$. This compresses the shape in both directions.

6. Determine whether or not $\mathbf{x}$ is an eigenvector of A. If it is, determine the associated eigenvalue.

a.

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

b.

$$A = \begin{pmatrix} -3 & -1 & 5 \\ -2 & 1 & 2 \\ -2 & -1 & 4 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

c.

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

d.

$$A = \begin{pmatrix} -9 & 4 & 6 \\ -6 & 3 & 4 \\ -9 & 4 & 6 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

Just do the matrix multiplication to find:

a. $Ax = (1, 2)$, so it is an eigenvector of eigenvalue 1.

b. $Ax = (-2, -1, -1)$, so it is an eigenvector of eigenvalue 1.

c. $Ax = (1, -2)$, so it is not an eigenvector.

d. $Ax = \vec{0}$, so it is an eigenvector of eigenvalue 0.