MT1 Review
Section 206

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1. Consider the function $f(x) = x^3 + e^x$.

   a. Find the derivative $f'(x)$ using differentiation laws and **explain** every step.

   b. Find the equation of the tangent line at $(0, 1)$, and **explain** every step.

   a. First we split this up as a sum of derivatives: $f'(x) = \left(x^3\right)' + (e^x)'$. Then we can use the power rule for the first term, and the fact that the derivative of e^x is itself for the second term to get $f'(x) = 3x^2 + e^x$.

   b. To find the slope of f at $x = 0$, we plug this into the derivative $f'(0) = 3 \cdot 0 + e^0 = 1$. This means that the tangent line is the line with slope 1 which goes through the point $(0, 1)$ which has equation $y - 1 = 1(x - 0)$ or equivalently $y = x + 1$.

   The grading scheme for this problem is the following: For the first part, I awarded one point for mentioning that the derivative of the sum is the sum of the derivatives, one point for knowing the power rule, one point for knowing the derivative of e^x, and two points for using them correctly on the actual given function.

   For the second part, I awarded one point for knowing what a tangent line is, one point for knowing the derivative gives you the slope, one point for correctly plugging it in, one point for knowing the equation of a line, one point for plugging it in correctly.

2. Find y' using implicit differentiation where

   $$e^y \cos x = 1 + \sin(xy)$$

   First write $y(x)$ for y to remind us that y depends on x. Then we can differentiate both sides with respect to x to get:

   $$\begin{aligned} e^y y' \cos x - e^y \sin x &= \cos(xy)\left(y + xy'\right) \\ &= y \cos(xy) + xy' \cos(xy) \end{aligned}$$

where we have use the chain rule and the product rule on both sides. Then we can rearrange:

$$\begin{aligned} y'\left(e^y \cos x - x\cos(xy)\right) &= y\cos(xy) + e^y \sin x \\ y' &= \frac{y\cos(xy) + e^y \sin x}{e^y \cos x - x\cos(xy)} \end{aligned}$$

Grading scheme: I awarded four points for each side of the differentiation, two for recognizing each rule, and two for properly using each rule. Then I awarded one point for recognizing you have to rearrange, and one point for properly rearranging.

3. Compute the following limit:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

First notice that the limit evaluates to the indeterminate form $0/0$, and since the derivative of the denominator, $2x$, is nonzero near (but not at) $x = 0$, we can apply l'Hospital's rule. This means the limit we are looking for is:

$$\lim_{x\to 0} \frac{e^x - 1}{2x}$$

This is again the indeterminate form $0/0$, and since the derivative of the denominator, 2, is nonzero everywhere, we can apply l'Hospital's rule again to write the limit as:

$$\lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$$

4. Compute the limit:

$$\lim_{x\to\infty} \frac{e^{\sqrt{x}}}{x}$$

Write the limit as L. We get an indeterminate form which is fractional, and the derivative of the bottom is not zero near ∞, so we can apply l'Hospital and get

$$L = \lim_{x\to\infty} \frac{e^{\sqrt{x}}/\left(2\sqrt{x}\right)}{1} = \lim_{x\to\infty} \frac{e^{\sqrt{x}}}{2\sqrt{x}} = \lim_{x\to\infty} \frac{e^{\sqrt{x}}x}{2x^{3/2}}$$

This is again a fractional indeterminate, and the derivative of the bottom is not zero near ∞. Therefore we can again apply l'Hospital's rule to get:

$$L = \lim_{x\to\infty} \frac{e^{\sqrt{x}} + \sqrt{x}e^{\sqrt{x}}/2}{3\sqrt{x}} = \lim_{x\to\infty} \frac{e^{\sqrt{x}}}{3\sqrt{x}} + \lim_{x\to\infty} \frac{e^{\sqrt{x}}}{6} = \frac{2}{3}L + \lim_{x\to\infty} \frac{e^{\sqrt{x}}}{6}$$

which we can rewrite as

$$L = 3\lim_{x\to\infty} \frac{e^{\sqrt{x}}}{6} = \boxed{\infty}$$

5. Compute the limit:

$$\lim_{x\to\infty}\left(\sqrt{x}-\ln x\right)$$

We get an indeterminate form of the form $\infty - \infty$. To remedy this, we rewrite this as:

$$L=\lim_{x\to\infty}\left(\sqrt{x}-\ln x\right)=\ln\left(\lim_{x\to\infty}\frac{e^{\sqrt{x}}}{e^{\ln x}}\right)=\ln\left(\lim_{x\to\infty}\frac{e^{\sqrt{x}}}{x}\right)$$

But this is just the limit from the first question, which was infinite, meaning this limit is infinite as well.

Alternate solution: Rewrite this as:

$$\sqrt{x}-\ln x=\frac{\left(\sqrt{x}-\ln x\right)\left(\sqrt{x}+\ln x\right)}{\sqrt{x}+\ln x}=\frac{x-\ln^2(x)}{\sqrt{x}+\ln x}$$

now divide by the leading term of the denominator to get

$$\frac{x-\ln^2(x)}{\sqrt{x}+\ln x}=\frac{\sqrt{x}-\ln^2 x/\sqrt{x}}{\ln x/\sqrt{x}+1}$$

Now we have

$$\lim_{x\to\infty}-\frac{\ln^2 x}{\sqrt{x}}=0 \qquad\qquad \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=0$$

which means we are left with

$$\lim_{x\to\infty}\frac{\sqrt{x}-\cancel{\ln^2 x/\sqrt{x}}}{\cancel{\ln x/\sqrt{x}}+1}=\lim_{x\to\infty}\sqrt{x}=\boxed{\infty}$$