Midterm 2 Review
Section 201

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1. A definite integral is a number whereas an indefinite integral is a function.

   a. True
   b. False

   This is **true**.

2. $$\int_0^1 x = 1 + c$$

   a. False
   b. True

   This is **false**. Definite integrals don't require you to add a constant. Even still, this integral is equal to 1/2.

3. Calculate the following integral:

   $$\int_0^1 \left(2 + \sqrt{1 - x^2}\right)$$

   This is a semicircle of radius 1 shifted above the x-axis by 2-units as in fig. 1. Therefore the area under this curve is $2 + \pi 1^2/4 = \boxed{2 + \pi/4}$. You could also do it explicitly, but that is much harder.

4. If $u = 2\sqrt{x}$ then $\,du = \,dx\,\sqrt{x}$.

   a. True
   b. False

   This is **false**. This would be true if it said $\,du = \,dx\,/\sqrt{x}$.

5. We learned FTC II to replace FTC I.

   a. True

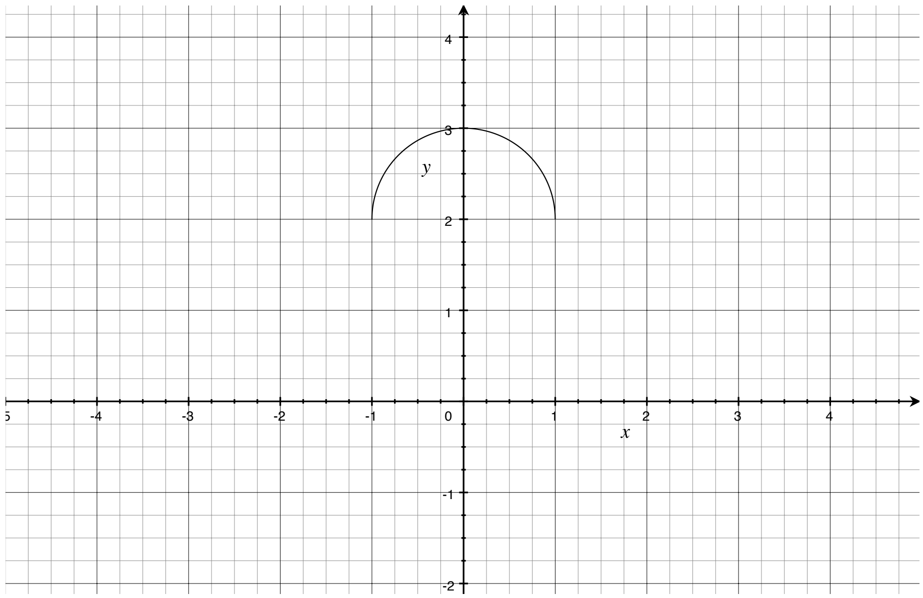

Figure 1: The graph of $f(x) = 2 + \sqrt{1 - x^2}$.

b. False

This is **false**. These are two different, but related, results.

6. Evaluate the following definite integral:

$$\int_1^2 x^2 \ln x \, dx$$

We first calculate the definite integral using integration by parts. Assign $u = \ln x$ and $dv = x^2 \, dx$, the integral becomes:

$$\begin{aligned} \int x^2 \ln x \, dx &= \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^2 \\ &= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c \\ &= \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) + c \end{aligned}$$

Now we can just evaluate at the endpoints to get:

$$\frac{2^3}{3} \left(\ln 2 - \frac{1}{3} \right) - \frac{1}{3} \left(0 - \frac{1}{3} \right) = \frac{8}{3} \ln 2 - \frac{8}{9} + \frac{1}{9} = \boxed{\frac{8}{3} \ln 2 - \frac{7}{9}}$$

7. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x) \leq \int_a^b g(x)$$

   a. True

   b. False

   This is **true**.

8. The equation for the volume of a sphere can be calculated using integrals.

   a. True

   b. False

   This is **true**.

9. Determine whether the integral converges, and if it converges, evaluate it.

$$I = \int_1^\infty \frac{x}{x^2 + 2x}$$

   First do the improper integral:

$$\int \frac{x}{x^2 + 2x} = \int \frac{1}{x+2} = \ln|x+2|$$

   Now evaluating at the endpoints gives us:

$$I = \lim_{b \to \infty} \ln(b+2) - \cancel{\ln(1)} = \lim_{b \to \infty} \ln(b+2)$$

   which $\boxed{\text{diverges}}$.

10. The only way to solve separable ODEs is using integrating factors.

    a. True

    b. False

    This is **false**. If a differential equations is separable, you should typically not resort to integrating factors.

11. The ODE $y' + y^2 = x$ is separable since we can get the x terms and the y terms one opposite sides of the equation.

    a. True

    b. False

    This is **false**. The definition of separable says that it can be written as $y' = f(t)\, g(y)$. So they can be separated by multiplication, not addition.

12. Consider the following ODE:

$$ty' + 3y = \frac{e^t}{t^2}$$

a. Is it pure time? Is it autonomous? Is it separable? Is it linear?

b. Solve the ODE.

a. This is not pure time since y' is not only a function of t, it is not autonomous since y' is not only a function of y, and it is not separable since it cannot be written as a product of functions which are themselves only functors of t and y. It is linear though, since there is never an instance of y or a derivative of y with power greater than 1.

b. We solve this using an integrating factor. Divide by x to get:

$$y' + \frac{3}{t}y = \frac{e^t}{t^3}$$

comparing this with the usual PQ form, we see that $P(t) = 3/t$. Therefore our integrating factor is:

$$I(t) = e^{\int P(t)\,dt} = e^{3\int 1/t\,dt} = e^{3\ln t} = e^{\ln t^3} = \pm t^3$$

Now multiplying both sides by this we get:

$$t^3y' + 3t^2y = e^t$$

now undoing the product rule we get. $\left(t^3y\right)' = e^t$. Integrating this gives us: $t^3y = e^t + C$, or solving for y,

$$\boxed{y = e^t t^{-3} + Ct^{-3}}$$

for any C.

13. Boundary value problems always have 1 unique solution.

a. True

b. False

This is **false**. BVPs can have 0, 1, or infinitely many solutions.

14. If an ODE has a term with t^2, it is not linear. (Where t is the independent variable.)

a. True

b. False

This is **false**. The coefficients involving t can have higher powers. An ODE is linear if it is linear in y.

15. Use Euler's method with step size 0.1 to estimate $y(0.2)$ where $y(0) = 0$ and y satisfies the ODE:

$$y' = 2x + y + 3$$

We can fill in the x_n column immediately. At the $n = 0$ step we have $y_0 = 0$, and $y'(x_0, y_0) = y'(0, 0) = 3$. This means

$$y_1 = y_0 + hy'(0, 0) = 0 + 0.1 \cdot 3 = 0.3$$

At the $n = 1$ step we have $y'(x_1, y_1) = y'(0.1, 0.3) = 0.2 + 0.3 + 3 = 3.5$. This means

$$y_2 = y_1 + hy'(0.1, 0.3) = 0.3 + 0.1 \cdot 3.5 = \boxed{0.65}$$

so this is our estimation. This is all summarized in the table below.

n	x_n	y_n	$y'(x_n, y_n)$
0	0	0	3
1	0.1	0.3	3.5
2	0.2	0.65	–