Midterm 2 Review
Section 206

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1. If a function takes only positive nonzero values everywhere on its domain, then any definite integral of this function must be positive.

   a. True
   b. False

   This is **false**. Here are two counterexamples:

   $$\int_a^a f(x)\,dx = 0 \qquad\qquad \int_1^0 2x\,dx = -1$$

2. A finite Riemann sum is always an overestimate.

   a. True
   b. False

   This is **false**. It can be an overestimate, an underestimate, or even an exact match.

3. Find the function f such that

   $$f'(x) = 1 - x + 6x^2$$

   and $f(0) = 10$.

   We first compute the anti-derivative:

   $$F(x) = x - \frac{x}{2} + 2x^3 + c$$

   then plugging in the initial condition, we can calculate:

   $$F(0) = 0 - 0 + 0 + c = 10$$

   which means the function is:

   $$f(x) = x - \frac{x}{2} + 2x^3 + c$$

Grading scheme: I awarded one point for realizing that we need to anti-differentiate, and four points for properly finding the anti-derivative. I awarded one point for adding the constant, one point for plugging in 0, one point for plugging it in properly, and one point for presenting the final function on its own.

4. Integration by parts says that for f and g differentiable functions,

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int g(x)\, f'(x)\, dx$$

   a. True
   b. False

   This is **true**.

5. Integration by parts cannot be used to evaluate definite integrals.

   a. True
   b. False

   This is **false**. It can be used to evaluate definite integrals.

6. Evaluate the following definite integral:

$$\int_{-1}^{3} \frac{dx}{3+2x}$$

First we do the indefinite integral by making the substitution $u = 3 + 2x$. This means $du = 2\, dx$, so we can rewrite the integral as:

$$\int \frac{du}{2u} = \frac{1}{2} \int \frac{du}{u} = \frac{\ln u}{2} + c = \frac{\ln(3+2x)}{2} + c$$

Now we can plug in the bounds to get:

$$\frac{\ln 9}{2} - \frac{\ln 1}{2} = \boxed{\ln 3}$$

Important: If you choose to change your bounds by plugging them into u, you must not change back to x in the end. Please ask me if you have more questions about this.

Grading scheme: One point for recognizing u-sub, two points for choosing the correct u, two points for properly calculating du, one point for substituting the u correctly, two points for taking the anti-derivative properly, and two points for properly handling the bounds.

7. If the limit as x goes to infinity of $f(x)$ is zero, then the following improper integral converges:

$$\int_0^\infty f(x)\ dx$$

   a. True

   b. False

   This is **false**. Consider $f(x) = 1/x$ as an example.

8. If $f(x) = g(x)$,

$$A = \int_a^b [f(x) - g(x)]\ dx\ = 0$$

   a. True

   b. False

   This is **true**.

9. Find the area between $\sin x$ and $\cos x$ between $a = -3\pi/4$ and $b = \pi/4$.

   Recall the area between two functions in an interval is given by the integral of their difference over this interval. Therefore the area is given by:

$$\begin{aligned}
\int_{-3\pi/4}^{\pi/4} \cos x - \sin x\, dx &= [\sin x + \cos x]_{-3\pi/4}^{\pi/4} \\
&= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \left(\sin\left(-\frac{3\pi}{4}\right) + \cos\left(-\frac{3\pi}{4}\right)\right) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \\
&= \boxed{2\sqrt{2}}
\end{aligned}$$

   where I put $\cos x$ first since it is larger than $\sin x$ at every point in the interval.

   Grading scheme: One point for knowing the area between the curves is the integral. Four points for setting up the integral correctly, two points for integrating correctly, two points for evaluating correctly at the endpoints, and one point for simplifying.

10. Every pure-time ODE is separable.

   a. True

   b. False

   This is **true**.

11. The differential equation $y' = t^2 y$ is autonomous since it has y on the right hand side.

a. True

b. False

This is **false**. It needs to only have y on the RHS.

12. Consider the following ODE:

$$\left(t^2-1\right)y' = ty$$

 a. Is this pure time? Is this autonomous? Is this separable? Is this linear?

 b. Solve this differential equation.

13. This is not pure time or autonomous because there are factors of x and y on both sides of the equation. It is separable, since we can write it as a product of two functions, one only of t and one only of y. In particular, $y' = f(t)\,g(y)$ for $f(t) = t/\left(t^2-1\right)$ and $g(y) = y$. It is also linear since we never have a power of y (or some derivative of y) greater than 1.

14. Since this is separable, assuming $y \neq 0$ (if it is, that gives us a solution $\boxed{y=0}$) we can just divide both sides by $\left(t^2-1\right)y$ to get:

$$\frac{y'}{y} = \frac{t}{t^2-1}$$

and now we can integrate to get:

$$\ln|y| = \int \frac{t}{t^2-1}\,dt = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + C = \ln\sqrt{t^2-1} + C$$

where I have set $u = t^2-1$. Solving for y we get:

$$y = C\sqrt{t^2-1}$$

for $C \neq 0$, but $y = 0$ is a solution so this is really for any C.

Grading scheme: one point for each of the four questions in part a, two points for recognizing and recalling the integrating factors technique, two points for properly completing the IF technique, and two points for solving for y and dealing with C correctly.

15. The sum of two solutions to a homogeneous ODE is also a solution to the same ODE.

 a. True

 b. False

This is **true**. This follows from the definition of a homogeneous ODE and the linearity of the derivative (DL+).

16. A quadratic polynomial always has two real roots.

    a. True

    b. False

    This is **false**. This is why we need the third case when solving homogeneous second-order ODEs with constant coefficients.

17. Solve the following ODE:

$$y'' - 2y' + 4y = 0$$

This is a homogeneous second-order ODE with constant coefficients. Therefore we first write down the characteristic polynomial:

$$r^2 - 2r + 4 = 0$$

From the quadratic equation we see that the roots are:

$$1 \pm \sqrt{3}i$$

which means our α is 1 and our β is $\sqrt{3}$, so the general solution is:

$$y = e^t \left(C_1 \cos \sqrt{3}t + C_2 \sin \sqrt{3}t \right)$$

for all $C_1, C_2 \in \mathbb{R}$.