Worksheet 6

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1. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$x^2 + y^2 = \left(2x^2 + 2y^2 - x\right)^2 \qquad \left(0, \frac{1}{2}\right)$$

Write the equation with $y \rightsquigarrow y(x)$ to remind us that y depends on x.

$$\frac{d}{dx}\left(x^2 + y(x)^2\right) = 2x + 2y(x)\,y'(x)$$

$$\frac{d}{dx}\left(\left(2x^2 + 2y(x)^2 - x\right)^2\right) = 2\left(2x^2 + 2y(x)^2 - x\right)\left(4x + 4y(x)\,y'(x) - 1\right)$$

So now we can drop the dependency on x, and simplify:

$$\begin{aligned} 2x + 2yy' &= 2\left(2x^2 + 2y^2 - x\right)\left(4x + 4yy' - 1\right) \\ x + yy' &= \left(2x^2 + 2y^2 - x\right)\left(4x + 4yy' - 1\right) \\ yy' &= 8x^3 + 8x^2yy' - 2x^2 + 8xy^2 + 8y^3y' - 2y^2 - 4x^2 - 4xyy' \end{aligned}$$

Now we isolate y':

$$\begin{aligned} y'\left(y - 8x^2y - 8y^3 + 4xy\right) &= 8x^3 + -2x^2 + 8xy^2 - 2y^2 - 4x^2 \\ y' &= \frac{8x^3 + -6x^2 + 8xy^2 - 2y^2}{y - 8x^2y - 8y^3 + 4xy} \end{aligned}$$

Now plugging in the point we get:

$$y'(0, 1/2) = \frac{-1/2}{1/2 - 1} = 1$$

which means the equation of the tangent line at this point is:

$$y = x + \frac{1}{2}$$

2. Find the local extrema, intervals of increasing and decreasing, the inflection points, and the intervals of concavity for each f.

   a. $f(x) = x^2$
   b. $f(x) = x^3 - 300x$
   c. $f(x) = \sin x + \cos x$

   a. We first find the critical values of the function by calculating $f'(x) = 2x$. This is zero only when $x = 0$, and it is defined everywhere on $\mathbb{R}$, so this is the only critical value. Since the derivative changes from negative to positive at 0, we know this is a local minimum. The function is decreasing for $(-\infty, 0)$ and increasing on $(0, \infty)$. Now to find the inflection points and intervals over which the function is concave up/concave down, we calculate $f''(x) = 2$, which is always positive, so the function is always concave up and there are no inflection points.

   b. We first find the critical values by differentiating to get $f'(x) = 3x^2 - 300$. This is zero precisely when $x^2 - 100 = (x-10)(x+10) = 0$ which means $x = \pm 10$. By plugging in points, we can see that $f'(x)$ is positive before -10, since $f'(-100) = 10^4 - 300 > 0$ and after 10 since $f'(100) = 10^4 - 300 > 0$. It is also negative between $x = \pm 10$ since $f'(0) = -300 < 0$. Therefore the function is increasing on $(-\infty, -10) \cup (10, \infty)$, and decreasing on $(-10, 10)$.

      To find the inflection points we take a second derivative to get $f''(x) = 6x$. This is zero precisely when $x = 0$, and is defined everywhere. Therefore there is only one inflection point at 0. Since $f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$, we see that the graph is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

   c. To find the critical values we differentiate $f'(x) = \cos x - \sin x$. This is defined everywhere, and zero for x such that $\cos x = \sin x$ which is equivalent to $\pi/4 + n\pi$ for any $n \in \mathbb{Z}$. If n is even then between $\pi/4 + (n-1)\pi$ and $\pi/4 + n\pi$ (before the point) the derivative is positive, since $f'(n\pi) = 1 > 0$.

      Then between the point $\pi/4 + n\pi$ and $\pi/4 + (n+1)\pi$ (after the point) the derivative is negative since $f'(n+1) = -1 < 0$. Therefore for every even n the point $x = \pi/4 + n\pi$ is a local maximum, and similarly for odd n the point $x = \pi/4 + n\pi$ is a local minimum and the regions between then alternate between increasing and decreasing.

      To find the inflection points we calculate $f''(x) = -\sin x - \cos x$ which is 0 when $\sin x = -\cos x$. This is only the case when $x = 3\pi/4 + n\pi$ for $n \in \mathbb{Z}$ so these are the inflection points. Between $x = 3\pi/4 + n\pi$ and the next inflection point, the second derivative is positive since $f''((n+1)\pi) = 1$. Between $x = 3\pi + n\pi$ and the previous inflection point, the second derivative is negative since

$f''(n\pi) = -1$. This means the regions between these inflection points alternate between concave up and concave down.

In conclusion:

$$
\begin{aligned}
f \text{ is increasing on } &\left(\frac{\pi}{4} + n\pi, \frac{\pi}{4} + (n+1)\pi\right) && n \text{ even} \\
f \text{ is decreasing on } &\left(\frac{\pi}{4} + n\pi, \frac{\pi}{4} + (n+1)\pi\right) && n \text{ odd} \\
f \text{ is concave up on } &\left(\frac{3\pi}{4} + n\pi, \frac{3\pi}{4} + (n+1)\pi\right) && n \text{ even} \\
f \text{ is concave down on } &\left(\frac{3\pi}{4} + n\pi, \frac{3\pi}{4} + (n+1)\pi\right) && n \text{ odd}
\end{aligned}
$$

3. A 15 m ladder is resting against the wall. The bottom is initially 10 m away from the wall and is being pushed towards the wall at a rate of 0.25 m per second. How fast is the top of the ladder moving up the wall when the bottom is 1 m away from the wall?

   Write $x(t)$ for the distance from the bottom of the ladder to the wall, and $y(t)$ for the distance from the top of the ladder to the ground. From the Pythagorean theorem we have $x^2(t) + y^2(t) = 15^2 = 225$. Now we can differentiate both sides (implicitly) to get

   $$2x(t)\,x'(t) + 2y(t)\,y'(t) = 0$$

   We know $x'(t) = -0.25$ for all t, so $-x(t)/2 + 2y(t)\,y'(t) = 0$ which can be solved:

   $$y'(t) = \frac{-x(t)\,x'(t)}{y(t)}$$

   When the bottom is 1 m away from the wall, $x(t) = 1$ and $y(t) = \sqrt{225 - 1} \sim 14.966$, so

   $$y'(t) = \frac{0.25}{14.966} = 0.0167$$