Worksheet 11

GSI: Jackson Van Dyke

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1. Calculate the following integrals:

   a.
   $$\int \frac{(\ln x)^2}{x}\, dx$$

   b.
   $$\int \frac{dx}{5-3x}$$

   c.
   $$\int e^x\sqrt{1+e^x}\, dx$$

   d.
   $$\int \frac{\cos x}{\sin^2 x}\, dx$$

   a. Set $u = \ln x$. This means $du = dx/x$, so we can rewrite the integral as:
   $$\int \frac{(\ln x)^2}{x}\, dx = \int u^2\, du = \frac{u^3}{3} + c = \frac{(\ln x)^3}{3} + c$$

   b. Set $u = 5 - 3x$ to get $du = -3\, dx$, so we can rewrite the integral as:
   $$\int \frac{dx}{5-3x} = \int -\frac{du}{3u} = -\frac{\ln|u|}{3} + c = -\frac{\ln|5-3x|}{3} + c$$

   c. Set $u = 1 + e^x$ so $du = e^x\, dx$, so we can rewrite the integral as:
   $$\int e^x\sqrt{1+e^x}\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + c = \frac{2}{3}(1+e^x)^{3/2} + c$$

   d. Set $u = \sin x$ so $du = \cos x\, dx$, so we can rewrite the integral as:
   $$\int \frac{\cos x}{\sin^2 x}\, dx = \int \frac{du}{u^2} = -u^{-1} + c = -\frac{1}{\sin x} + c$$

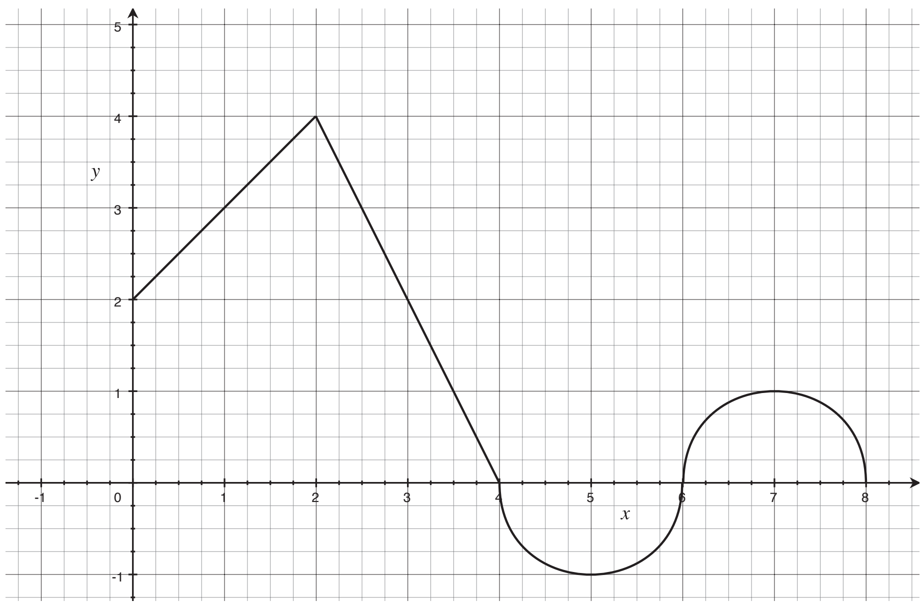

Figure 1: The function $f(x)$.

2. Let $f(x)$ be the function in fig. 1. Define the function $g(x)$ to be:

$$g(x) = \int_0^x f(t)\, dt$$

Evaluate $g(2)$, $g(4)$, and $g(8)$. Once you get your head around what is going on here, this question is just asking you to count the area under the curve. In particular, **evaluating $g(a)$ is just counting the area under f between 0 and $x = a$.** Therefore the answers are:

$$g(2) = 6 \qquad g(4) = 10 \qquad g(8) = 10$$

3. **True or False:** Integrals of odd functions don't require any tricks or ILs to evaluate since they can all be simplified using the fact that:

$$\int_{-a}^{a} f(x) = 0$$

This is **false**. This only simplifies definite integrals where the two bounds add up to 0. Beyond this, these integrals can be just as hard or easy as integrals of functions which are not odd.

4. **True or False:** The following is an IL:

$$\int f(x)\, g(x)\, dx = \int f(x)\, dx \int g(x)\, dx$$

This is **false**! This would make life much easier.

Challenge: Calculate the following integral:

$$I = \int \frac{x^2\,dx}{\sqrt{1-x^2}}$$

Hint: substitute $u = \arcsin x$. This is the same as substituting $x = \sin u$, which means $dx = \cos\theta\,du$. This lets us rewrite the integral:

$$\begin{aligned} I &= \int \frac{x^2\,dx}{\sqrt{1-x^2}} = \int \frac{\sin^2 u \cos u\,du}{\sqrt{1-\sin^2 u}} \\ &= \int \frac{\sin^2 u \cos u\,du}{\sqrt{\cos u}} = \int \frac{\sin^2 u \cos u\,du}{\cos u} \\ &= \int \sin^2 u\,du \end{aligned}$$

Now we need to use the trig identity:

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u)$$

which lets us write:

$$I = \frac{1}{2}\int (1-\cos 2u)\,du = \frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right) + c = \frac{1}{2}(u - \cos u \sin u)$$

where we have used the trig identity $\sin 2u = \cos u \sin u$. Now to get this in terms of x, we can to recall that $\cos \arcsin x = \sqrt{1-x^2}$, so

$$I = \frac{1}{2}\left(\arcsin x - x\sqrt{1-x^2}\right) + c$$