Worksheet 13

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1 Standard

1. Calculate the following integral:

$$I = \int \frac{x-2}{x^2+2x+1}\,dx$$

First factor the denominator, so

$$I = \int \frac{x-2}{(x+1)^2}\,dx$$

Now we will calculate the partial fraction decomposition of this:

$$\frac{x-2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$
$$x-2 = (x+1)A + B$$

Now setting $x = -1$ we get $B = -3$ and setting $x = 0$ we get $A = -2 - B = 1$. Therefore we can rewrite the integral as:

$$I = \int \frac{dx}{x+1} + \int -\frac{3}{(x+1)^2}\,dx = \ln|x+1| + c - 3\int \frac{1}{(x+1)^2}\,dx$$

Now to compute the second integral we can make the u-substitution $u = x+1$ to get $du = dx$, so

$$\int \frac{1}{(x+1)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + c = -\frac{1}{x+1} + c$$

so finally our integral comes out to:

$$I = \boxed{\ln|x+1| + 3\frac{1}{x+1} + c}$$

2. **True or false**: Since

$$\int_0^\infty f(x)\ dx$$

is the area under the curve all the way to ∞, unless $f(x)$ is eventually equal to 0 this is always infinite.

This is **false**. For example

$$\int_0^\infty \frac{1}{(x+1)^2}$$

converges to 1.

3. Determine if this integral converges or diverges. If it converges, evaluate it.

$$I = \int_1^\infty \frac{x+1}{x^2+2x}\, dx \qquad\qquad I = \int_{-\infty}^\infty xe^{-x^2}\, dx$$

a. First we compute the indefinite integral of this:

$$\begin{aligned} I &= \int \frac{1}{x+2}\, dx + \int \frac{1}{x^2+2x}\, dx \\ &= \ln|x+2| + c + \int \frac{1}{x(x+2)}\, dx \end{aligned}$$

For the second integral we can do a partial fraction decomposition:

$$\begin{aligned} \frac{1}{x(x+2)} &= \frac{A}{x} + \frac{B}{x+2} \\ 1 &= A(x+2) + Bx \end{aligned}$$

so setting $x = -2$ we get $B = -1/2$ and setting $x = 0$ we get $A = 1/2$. Therefore we can rewrite the second integral as:

$$\begin{aligned} \int \frac{1}{x(x+2)}\, dx &= \int \frac{1}{2x}\, dx - \frac{1}{2}\int \frac{1}{(x+2)}\, dx \\ &= \frac{1}{2}\ln x - \frac{1}{2}\ln|x+2| \end{aligned}$$

so all together,

$$I = \ln|x+2| + \frac{1}{2}\ln x - \frac{1}{2}\ln|x+2| = \frac{1}{2}(\ln x + \ln|x+2|)$$

Now taking the integral to infinity is just taking the limit of the definite integral as the upper bound goes to infinity. So we can write:

$$\begin{aligned}\int_1^\infty \frac{x+1}{x^2+2x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{x+2}{x^2+2x}\,dx \\ &= \frac{1}{2}\lim_{b\to\infty}\left[\ln x + \ln|x+2|\right]_1^b \\ &= \frac{1}{2}\lim_{b\to\infty}\left(\ln b + \ln|b+2| - \cancel{\ln 1} - \ln 2\right) \\ &= \frac{1}{2}\lim_{b\to\infty}\left(\ln b + \ln|b+2| - \ln 2\right)\end{aligned}$$

but this limit $\boxed{\text{diverges}}$.

b. First we compute the indefinite integral of this. Setting $u = x^2$ we get $du = 2x\,dx$, so

$$\begin{aligned}I &= \int xe^{-x^2}\,dx \\ &= \frac{1}{2}\int e^{-u}\,du \\ &= -\frac{1}{2}e^{-u} + c \\ &= -\frac{1}{2}e^{-x^2} + c\end{aligned}$$

Now when we take an infinite limit in both directions we split the integral up at 0, and calculate them separately. Then if they both exist, we will add the result. First we calculate:

$$\begin{aligned}\int_0^\infty xe^{-x^2}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx \\ &= -\frac{1}{2}\lim_{b\to\infty}\left[e^{-x^2} + c\right]_0^b \\ &= -\frac{1}{2}\lim_{b\to\infty}\left(e^{-b^2} - 1\right) \\ &= -\frac{1}{2}(-1) = \frac{1}{2}\end{aligned}$$

where we have used the fact that:

$$\lim_{b\to\infty} e^{-b^2} = 0$$

Now we make a similar calculation in the other direction:

$$\begin{aligned}\int_{-\infty}^{0} xe^{-x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 xe^{-x^2} \\ &= \lim_{a\to-\infty} -\frac{1}{2}\left[e^{-x^2}+c\right]_a^0 \\ &= -\frac{1}{2}\lim_{a\to-\infty} 1-e^{-a^2} \\ &= -\frac{1}{2}\end{aligned}$$

where we have again used the limit from above. Therefore the whole integral is:

$$I = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx = \frac{1}{2}-\frac{1}{2} = \boxed{0}$$

2 Challenge

1. Rewrite the following integral using partial fraction decomposition:

$$\int \frac{3x^2+6x+30}{(x^2+2)(x-5)}\,dx$$

Since the term on the bottom, $\left(x^2+2\right)$ is irreducible over the real numbers, we can just directly find the partial fraction decomposition. However we have to be careful and allow for the numerator which will correspond to the term with a quadratic denominator (i.e. the term with a squared on the bottom) must be linear (i.e. must have an x on top). In other words:

$$\frac{3x^2+6x+30}{(x^2+2)(x-5)} = \frac{Ax+B}{x^2+2}+\frac{C}{x-5}$$

Now we can solve this as usual:

$$3x^2+6x+30 = (Ax+B)(x-5)+C\left(x^2+2\right)$$

First setting $x=5$ we get

$$3\cdot 25+6\cdot 5+30 = 0+27C$$

which means $\boxed{C = 135/27 = 5}$. Therefore, setting $x=0$ we get

$$30 = -5B+2C = -5B+10$$

which means $\boxed{B=-4}$. Now we can rewrite the expression as:

$$3x^2+6x+30 = (Ax-4)(x-5)+5\left(x^2+2\right)$$

Now setting $x = 1$ we get

$$3 + 6 + 30 = 39 = -4\,(A - 4) + 5 \cdot 3$$

or $-24/4 + 4 = \boxed{-2 = A}$ Finally we can rewrite the integral as:

$$\begin{aligned} I &= -2 \int \frac{x+2}{x^2+2}\,dx + \int \frac{5}{x-5}\,dx \\ &= -2\left(\int \frac{x}{x^2+2}\,dx + \int \frac{2}{x^2+2}\,dx\right) + \int \frac{5}{x-5}\,dx \end{aligned}$$

We will now do these integrals one by one. First make the substitution $u = x^2 + 2$ to get $du = 2x\,dx$, so

$$\int \frac{x}{x^2+x}\,dx = \int \frac{du}{2u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln\left(x^2+1\right) + c$$

Now make the substitution $u = x/\sqrt{2}$ to get $du = dx/\sqrt{2}$, so

$$\begin{aligned} \int \frac{2}{x^2+2}\,dx &= 2\sqrt{2}\int \frac{1}{2\,(x^2/2+1)}\,du \\ &= \sqrt{2}\int \frac{1}{(u^2+1)}\,du \\ &= \sqrt{3}\arctan u + c \\ &= \sqrt{3}\arctan x/\sqrt{2} + c \end{aligned}$$

and finally,

$$\int \frac{5}{x-5}\,dx = 5\ln|x-5|$$

putting this all together we get:

$$\boxed{I = -\ln\left(x^2+1\right) + 2\sqrt{2}\arctan\frac{x}{\sqrt{2}} + 5\ln|x-5|}$$

2. **True or false**: There are no irreducible polynomials of degree 3.

   This is **true**. We can convince ourselves of this because if we take the limit at ∞ and $-\infty$ these will always have a different sign, meaning the graph of the polynomial must cross the x-axis at some point a, which means we can factor $(x - a)$ out of the polynomial.

3. Let n be a number for which the improper integral:

$$\int_e^{\infty} \frac{dx}{x\,(\ln x)^n}$$

   converges. Determine the value of this integral.

Make the substitution $u = \ln x$, then $du = dx/x$, so

$$I = \int \frac{dx}{x \ln^n x} = \int \frac{du}{u^n}$$
$$= \begin{cases} \frac{1}{1-n} u^{-n+1} & n \neq -1 \\ \ln u & n = -1 \end{cases} = \begin{cases} \frac{1}{1-n} (\ln x)^{1-n} & n \neq -1 \\ \ln \ln x & n = -1 \end{cases}$$

Now we can notice that since:

$$\lim_{x \to \infty} \ln x = \infty$$

we need a $\ln x$ to be in the denominator, but in this is equivalent to $n \gneq 1$. Therefore the indefinite integral is just:

$$I = \frac{1}{1-n} \frac{1}{\ln^{n-1} x} + c$$

Now we can write:

$$\int_e^\infty \frac{dx}{x(\ln x)^n} = \lim_{b \to \infty} \int_e^b \frac{dx}{x(\ln x)^n}$$
$$= \lim_{b \to \infty} \left[\frac{1}{1-n} \frac{1}{\ln^{n-1} x} \right]_e^b$$
$$= \frac{1}{1-n} \lim_{b \to \infty} \left(\frac{1}{\ln^{n-1} b} - 1 \right) = \boxed{\frac{1}{n-1}}$$

where we have used the fact that:

$$\lim_{b \to \infty} \frac{1}{\ln^{n-1} b} = 0$$