Worksheet 16

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1 Separable ODEs

1. Consider the following ODE:

$$y' = 3t^2 y$$

   a. Is this pure time? Is this autonomous? Is this separable? Is this linear?
   b. Solve the ODE.

   a. This is not pure time or autonomous because there are factors of x and y on both sides of the equation. It is separable, since we can write it as a product of two functions, one only of t and one only of y. In particular, $y' = f(t)\,g(y)$ for $f(t) = 3t^2$ and $g(y) = y$. It is also linear since we never have a power of y (or some derivative of y) greater than 1.
   b. Divide both sides by y to get:

$$\frac{dy}{dt}\frac{1}{y} = 3t^2$$

   and integrate both sides to get:

$$\ln y = t^3 + c\ .$$

   Now taking the exponential of both sides we get:

$$y = ce^{t^3}$$

2. **True or false:** If y is a solution to the differential equation $y' = -2 - y^6$ then y must be decreasing.

   This is **true**. The sign of y' will always be negative no matter what value y takes since the quantity y^6 will always be positive, so $-2-y^6$ will certainly be negative.

3. **True or false:** Every homogeneous linear differential equation has a solution.

   This is **true**. In particular, $y = 0$ is always a solution to such an ODE.

2 Second order linear ODEs

1. Find the general solution to the following second-order ODEs:

   a. $y'' + 2y' - 3y = 0$;

   b. $6y'' - y' - y = 0$;

   c. $y'' + 5y' = 0$;

   d. $y'' - 9y' + 9y = 0$;

   e. $y'' - 4y' + 3y = 0$;

   f. $y'' + 5y' - 6y = 0$;

   g. $y'' + 2y' + 3y = 0$.

2. (Backtracking Challenge) Find the second-order linear ODE whose general solution is $y = X_1 e^{2t} + C_2 e^{-3t}$.

3. Find the solution of the initial value problem

$$\begin{cases} 2y'' - 3y' + y = 0 \\ y(0) = 2,\ y'(0) = 1/2 \end{cases}$$

4. Solve the initial value problem:

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = \alpha,\ y'(0) = 2 \end{cases}$$

   Find α so that the solution approaches zero as $t \to \infty$. (You will need to review some limits of famous functions when $t \to \infty$.)

5. Solve the initial value problem

$$\begin{cases} y'' + 2ay' + \left(a^2 + 1\right) y = 0 \\ y(0) = 2,\ y'(0) = \alpha \geq 0 \end{cases}$$

   where a is an unknown constant.

6. Consider the initial value problem

$$\begin{cases} y'' + 2y' + 6y = 0 \\ y(0) = 2,\ y'(0) = \alpha \geq 0 \end{cases}$$

   a. Find the solution y of this problem.

   b. Find α so that $y = 0$ when $t = 1$.