Worksheet 18

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1 IVPs, BVPs, complex numbers

1. a. Write down Euler's formula.
   b. Use it to evaluate $e^{i\pi}$.

   a. Euler's formula is:
   $$e^{ix} = \cos x + i \sin x$$
   b. Set $x = \pi$, then
   $$e^{i\pi} = \cos \pi + i \sin \pi = -1$$

2. Consider the following ODE:
   $$y'' + 4y = 0$$
   a. Find the general solution.
   b. Find the particular solution if we have initial conditions $y(0) = 0$ and $y'(0) = 2$.
   c. Now return to your general solution. Does the boundary value problem with the conditions $y(0) = 0$, $y(\pi) = 1$ have a unique solution like we had for the IVP? If it does not, can you find a different set of boundary values which does give us a unique solution?

   a. To find the general solution we write down the characteristic polynomial: $r^2 + 4r = 0$. This has roots $\pm 2i = \alpha \pm \beta i$ for $\alpha = 0$ and $\beta = 2$. Therefore the general solution is:
   $$\boxed{y = C_1 \cos 2t + C_2 \sin 2t}$$
   b. Using the first conditions we get
   $$0 = C_1 \cos 0 + C_2 \sin 0 = C_1$$
   so now the solution is:
   $$y = C_2 \sin 2t$$

and since the derivative of this is:

$$y' = 2C_2 \cos 2t$$

second condition gives us:

$$2 = 2C_2 \cos 0 = 2C_2$$

so $C_2 = 1$. Therefore the unique solution is:

$$\boxed{y = \sin 2t}$$

c. Returning to our answer to the first part, we now insist on these boundary conditions. The first one still gives us $C_1 = 0$, so the solution becomes $y = C_2 \sin 2t$. Now if we try to insist on $y(\pi) = 1$, we run into problems, because $y(\pi) = C_1 \sin 2\pi = 0 \neq 1$. Therefore this BVP has no solutions. If we switch the conditions to be $y(\pi) = 0$ instead, then we get infinitely many solutions. Alternatively we could change the second condition to be $y(\pi/2) = 1$. This would give us $C_2 = 1$ and we would have a unique solution.

3. **True or false:** The Taylor polynomial for e^x is given by:

$$e^x = 1 + \frac{x}{1} + \frac{x}{2} + \frac{x}{3} + \cdots$$

This is **false**. The denominators should be factorials.

2 Equilibrium solutions and predator-prey

1. Find the equilibrium solutions to the following predator-prey system of ODEs:

$$\frac{dx}{dt} = -0.05x + 0.001xy \qquad \frac{dy}{dt} = 0.1y - 0.005xy$$

Recall an equilibrium solution is when the derivative of the independent variable is 0. These to be 0 to get

$$0 = x(-0.01 + 0.001y) \qquad 0 = y(0.1 - 0.005x)$$

so $x = y = 0$ is a trivial equilibrium solution. The other solution is:

$$x = 0.1/0.005 = 20 \qquad y = 0.01/0.001 = 10$$

so when the populations are 20 and 10, neither population is changing.

2. Describe what type of system is being described by the following system of ODEs:

$$\frac{dx}{dt} = 0.1x - 0.001x^2 - 0.001xy \qquad \frac{dy}{dt} = -0.1y + 0.001xy$$

We have two quantities x and y. We first describe their independent behaviour. As x gets bigger, its rate of growth also gets bigger, but once it reaches some critical point it will begin to flatten out. As y gets bigger its rate of growth gets smaller.

As x gets bigger, the rate of growth of y gets larger, and as y gets bigger, the rate of growth of x gets smaller. This is a predator-prey situation where x also has some environmental limitations which give the solutions of x in the absence of y a logistic behaviour.