

## Worksheet 26

1. Find a direct formula for  $a_{n+1} = a_n + 2a_{n-1}$  for  $n \geq 1$ ,  $a_0 = 0, a_1 = 3$ , by encoding the recurrence sequence with linear algebra. Follow the steps:

- (a) Let  $\vec{v}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$  for all  $n \geq 0$ . Write down  $\vec{v}_0, \vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$ .
- (b) Find a  $2 \times 2$  matrix  $A$  such that  $A\vec{v}_{n-1} = \vec{v}_n$  for all  $n \geq 1$ .  
Check your answer by verifying that  $A\vec{v}_0 = \vec{v}_1$ ,  $A\vec{v}_1 = \vec{v}_2$ , and  $A\vec{v}_2 = \vec{v}_3$ .
- (c) Find the eigenvalues  $\lambda_1$  and  $\lambda_2$  and corresponding eigenvectors  $\vec{u}_1$  and  $\vec{u}_2$  of  $A$ .
- (d) Write  $\vec{v}_0$  as a linear combination of the eigenvectors  $\vec{u}_1$  and  $\vec{u}_2$ ; i.e., find constants  $c_1$  and  $c_2$  such that  $\vec{v}_0 = c_1\vec{u}_1 + c_2\vec{u}_2$ . (*Hint*: This boils down to a  $2 \times 2$  system.)
- (e) Calculate  $A\vec{u}_1$ ,  $A^2\vec{u}_1$ ,  $A^3\vec{u}_1$ , and  $A^n\vec{u}_1$  for any  $n \geq 0$ . Repeat for  $\vec{u}_2$  instead of  $\vec{u}_1$ . (*Hint*: There are almost no calculations here. Use the property of eigenvectors.)
- (f) Find  $A^n\vec{v}_0$  for any  $n \geq 0$ . (*Hint*: Use your lin. combo in (d) and the fact that  $A^n(c_1\vec{u}_1 + c_2\vec{u}_2) = c_1A^n\vec{u}_1 + c_2A^n\vec{u}_2$ .)
- (g) Noting that  $\vec{v}_n = A^n\vec{v}_0$ , extract from (f) the formula for  $a_n$ .

a.  $\vec{v}_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 9 \\ 15 \end{pmatrix}$

b.  $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

- c. Setting  $\det(A - \lambda I) = 0$  gives us  $\lambda^2 - \lambda - 2 = 0$ , so  $\lambda = 2$  or  $\lambda = -1$ . These are our two eigenvalues  $\lambda_1 = 2$  and  $\lambda_2 = -1$ . Since

$$A - 2I = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix},$$

we have that  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  solves the matrix equation  $(A - 2I)\vec{x} = \vec{0}$ , and

$A + I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ , so  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  solves the matrix equation  $(A + I)\vec{x} = \vec{0}$ .

Therefore we can take our corresponding eigenvectors to be  $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

and  $\vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

d.  $\vec{v}_0 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{u}_1 - \vec{u}_2$  (so  $c_1 = 1, c_2 = -1$ ).

e.  $A\vec{u}_1 = 2\vec{u}_1, A^2\vec{u}_1 = 4\vec{u}_1, A^3\vec{u}_1 = 8\vec{u}_1, A^n\vec{u}_1 = 2^n\vec{u}_1;$   
 $A\vec{u}_2 = -\vec{u}_2, A^2\vec{u}_2 = 1\vec{u}_2, A^3\vec{u}_2 = -\vec{u}_2, A^n\vec{u}_2 = (-1)^n\vec{u}_2.$

f.  $A^n\vec{v}_0 = A^n\vec{u}_1 - A^n\vec{u}_2 = 2^n\vec{u}_1 - (-1)^n\vec{u}_2$

g. The top components of  $A^n\vec{v}_0, \vec{u}_1$ , and  $\vec{u}_2$  are  $a_n, 1$ , and  $1$  respectively, so  $a_n = 2^n - (-1)^n$ .

2. Solve the IVP  $y'' - y' - 2y = 0, y(0) = 0, y'(0) = 3$  by encoding the DE with linear algebra. Follow the following steps:

- Set  $y_1(t) = y(t)$  and  $y_2(t) = y'(t)$ . What are  $y'_1(t)$  and  $y'_2(t)$ , written only in terms of  $y_1(t)$  and  $y_2(t)$ ?
- Write a homogeneous system of two linear 1st order DE's with unknowns  $y_1(t)$  and  $y_2(t)$  that corresponds to solving the given 2nd order DE. (*Hint:* The  $2 \times 2$  matrix is the same as in Problem 1.) Also encode the initial condition via  $y_1(t)$  and  $y_2(t)$ .
- Solve the DE system, using the algorithm from class.
- Which function from your result in (c) is the desired function  $y(t)$ ?
- Now apply the shortcut algorithm for finding solutions to 2nd order DE's to solve the given IVP.

a.  $y'_1(t) = y'(t) = y_2(t);$   
 $y'_2(t) = y''(t) = 2y(t) + y'(t) = 2y_1(t) + y_2(t).$

b.  $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$  (this has the form  $\vec{y}' = A\vec{y}$ ). The initial conditions are encoded as  $\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$

c. We use the eigenvalues and eigenvectors found in Problem 1, which yield the general solution  $\vec{y} = C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

d. Plugging in the initial conditions gives  $\begin{pmatrix} 0 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix},$  and solving for the coefficients gives us  $C_1 = 1$  and  $C_2 = -1$ . So  $\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$  and the top row of the vector equation gives us  $y(t) = e^{2t} - e^{-t}.$

- e. The characteristic equation will be  $r^2 - r - 2 = 0$ , which has roots 2 and  $-1$ , so the general solution is  $y = C_1 e^{2t} + C_2 e^{-t}$ . Plugging in the initial conditions yields  $C_1 = 1$  and  $C_2 = -1$ , so  $y = e^{2t} - e^{-t}$  as in part (d). In the shortcut algorithm, steps (a) and (b) are skipped, while steps (c) and (d) are slightly modified.
3. (Challenge) Let  $A$  be the (same)  $2 \times 2$  matrix that appears in problems 1 and 2. Find  $A^{2018}$  and show all calculations to justify your answer. (*Hint*:  $A = EDE^{-1}$  for some diagonal  $2 \times 2$  matrix  $D$  related to the eigenvalues of  $A$  and some  $2 \times 2$  matrix  $E$  somehow made up of the eigenvectors of  $A$ .)

$A = EDE^{-1}$  where

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

which means

$$E^{-1} = -\frac{1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$

Then we have:

$$A^{2018} = (EDE^{-1})^{2018} = EDE^{-1}EDE^{-1} \dots EDE^{-1} = ED^{2018}E^{-1}$$

so

$$\begin{aligned} A^{2018} &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2^{2018} & 0 \\ 0 & (-1)^{2018} \end{pmatrix} \frac{-1}{3} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \\ &= -\frac{1}{3} \begin{pmatrix} 2^{2018} & 1 \\ 2^{2019} & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \\ &= -\frac{1}{3} \begin{pmatrix} -2^{2018} - 2 & -2^{2018} + 1 \\ -2^{2019} + 2 & -2^{2019} - 1 \end{pmatrix} \end{aligned}$$