Worksheet 1

Jackson Van Dyke

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1. What is the range of the following functions:
   (a) $f(x) = x^2$
   (b) $f(x) = \sqrt{x}$
   (c) $f(x) = \ln(x)$
   (d) $f(x) = \sin(x)$
   (e) $f(x) = \arctan(x)$
   (f) $f(x) = \sqrt{2-x}$

   (a) The range is $[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$.
   (b) The range is $[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$.
   (c) The range is $\mathbb{R}$.
   (d) The range is $[-1, 1] = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$.
   (e) The range[1] is $(-\pi/2, \pi/2) = \{x \in \mathbb{R} \mid \pi/2 \lneq x \lneq \pi/2\}$.
   (f) The range is $[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$.

2. Evaluate the difference quotient:
$$\frac{f(3+h) - f(3)}{h}$$
   where
$$f(x) = 4 + 3x - x^2$$

   First calculate:
$$\begin{aligned} f(3+h) &= 4 + 3(3+h) - (3+h)^2 \\ &= 4 + 9 + 3h - 9 - h^2 - 6h \\ &= -h^2 - 3h + 4 \end{aligned}$$

[1] The symbol $\lneq$ means strictly less than.

and

$$f(3) = 4 + 9 - 9 = 4$$

so we can write:

$$\frac{f(3+h) - f(3)}{h} = \frac{-h^2 - 3h + 4 - 4}{h} = \boxed{-h - 3}$$

3. (a) We know the vertical line test determines if a curve in the plane determines a function or not. What might be a "horizontal line test" which could be satisfied by a curve?

   (b) Now consider a curve which satisfies the horizontal and vertical line tests. What does this mean about the function corresponding to this curve?

   (a) The horizontal line test might be whether or not there exists a horizontal line which crosses the curve more than once. This isn't really part of the course, just something that I thought might help you get comfortable thinking about functions.

   (b) We can think of the function as some sort of machine which takes some x value and gives us a y value. Then the vertical line test tells us that functions must, by definition, only associate a single y to every x. If the function satisfies this alternative test, it would mean that for any choice of y, if the function associates any value of x to it, it only does it once. And indeed it might not associate a single x value to this particular y value. Consider for example $f(x) = x^2$. The value $y = -1$ is not in the range of x^2. However x^2 actually brings 2 values of x to $y = 1$, namely $f(1) = f(-1) = 1$. Since x^2 turns around, it somehow hits all of the positive values twice, and none of the negative values. $f(x) = x^3$ hits all of the values, both negative and positive, and doesn't bring any two values of x to the same y value.

   Note that there are functions which hit every possible y value, and still do not pass the so-called "horizontal line test."

4. Sketch the graph of the function $f(x) = 2x + x^2$.

   Rewrite f as
   $$f(x) = 2x + x^2 = (x+1)^2 - 1$$
   which tells us that it is the parabola x^2 shifted one unit down, and one unit to the left as in fig. 1. Another option is plotting points and figuring out what happens in between.

5. (a) If the point $(4, 2)$ is on the graph of an even function, what other point must be on the graph?

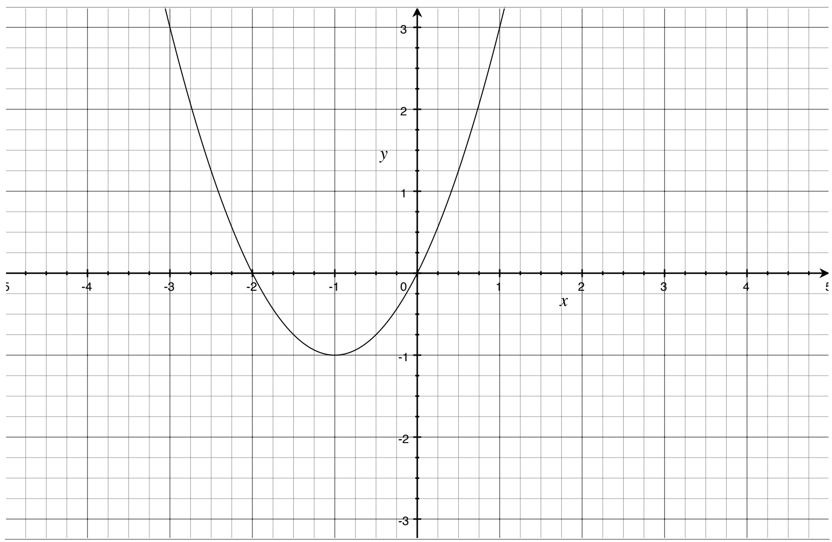

Figure 1: The graph of $f(x) = (x+1)^2 - 1$.

(b) If the point $(5, 2)$ is on the graph of an odd function, what other point must also be on the graph?

(a) Since even functions are symmetric about the y axis, the point $(-4, 2)$ would also be on the graph.

(b) Since off functions are symmetric with respect to the origin. Said differently, $f(-x) = -f(x)$, so $(-5, -2)$ is also on the graph.

6. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides. What is the domain of this function?

Call one side length x, and the other side length y. Since the whole perimeter is 20 m, we can write $20 = 2x + 2y$. This can be rewritten as $y = 10 - x$. We also know that the area is $A = xy$, which means

$$A = xy = x(10 - x) = 10x - x^2$$

Since the area is strictly positive, the domain of this function is $(0, 10)$.

7. Explain how to get the graph of the resulting function from the graph of $f(x)$:

(a) $f(2x + 5)$

(b) $2f(x + 5)$

(c) $f(|x|)$

(a) Rewrite the argument of f to get $f(2x+5) = f(2(x+2.5))$. This shows us that we take the graph of f, we shift it to the left by 2.5 units, and then shrink its width by 1/2.

(b) Shift to the left by 5, and twice as tall.

(c) The portion of the graph for $x > 0$ is reflected over the y axis to replace the portion of f for negative x.

8. Express the following function in the form $f \circ g$:

$$F(x) = \frac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$$

This is the case for:

$$f = \frac{x}{1+x} \qquad\qquad g = \sqrt[3]{x}$$