LECTURE 5
MATH 242

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1. Moser's trick

Recall that last lecture we saw that if M is a compact smooth manifold, and $\{\omega_t\}_{t\in[0,1]}$ is a smooth family of symplectic forms on M in the same cohomology class, then there is an isotopy

$$\{\varphi_t \in \mathrm{Diff}\,(M) \mid t \in [0,1]\}$$

with $\varphi_0 = \mathrm{id}_M$ and $\varphi_t^* \omega_t = \omega_0$. I.e. nothing changes up to isotopy if we continuously deform the symplectic form without changing the cohomology class. This is proved using Moser's trick. There is also a relative version of this.

Theorem 1 (Relative Moser trick)**.** *Let M be a smooth manifold, and let $X \subseteq M$ be a compact submanifold. Let ω_0 and ω_1 be symplectic forms on M such that for each point $p \in X$, $\omega_0|_{T_pM} = \omega_1|_{T_pM}$. Then there are neighborhoods $X \subset U_0, U_1 \subset M$ and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi|_X = \mathrm{id}_X$ and $\varphi^*\omega_1 = \omega_0$.*

Proof. The following is a standard fact in differential topology: we can choose a tubular neighborhood U of X and a diffeomorphism $U \simeq N$, where N is the normal bundle, which sends X to the 0-section in the obvious way. Now we want to apply Moser's argument in (part of) N. The first thing we need to know is that ω_0 and ω_1 are in the same cohomology class. This is true because they do the same thing on X.

Lemma 1. *There is a 1-form α on N such that $d\alpha = \omega_1 - \omega_0$ and $\alpha|_{T_pN} = 0$ for all $p \in X$.*

Proof. We define a chain homotopy $K : \Omega^i(N) \to \Omega^{i-1}(N)$. Recall this means

$$K\beta = \int_0^1 (\psi_t^* \iota_{V_t}\beta)\ dt$$

where for $t \in [0,1]$ we have a map $\psi_t : N \to N$ which just multiplies vectors by t, and V_t is the vector field given by the derivative of ψ_t. Now we can calculate the

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following:

$$\begin{aligned}
dK\beta &= \int_0^1 d\left(\psi_t^* \iota_{V_t} \beta\right) dt \; = \int_0^1 \left(\psi_t^* d\iota_{V_t} \beta\right) dt \\
K\, d\beta \; &= \int_0^1 \left(\psi_t^* \iota_{V_t}\, d\beta\,\right) dt \\
\left(dK + Kd\right)\beta &= \int_0^1 \psi_t^* \left(d\iota_{V_t}\beta + \iota_{V_t}\, d\beta\,\right) dt \; = \int_0^1 \psi_t^* \left(\mathcal{L}_{V_t}\beta\right) dt
\end{aligned}$$

but since $d/dt\left(\psi_t^*\left(-\right)\right) = \psi_t^*\left(\mathcal{L}_{V_t}\left(-\right)\right)$, we can write this as

$$\begin{aligned}
\left(dK + Kd\right)\beta &= \int_0^1 \left(\frac{d}{dt}\psi_t^*\beta\right) dt \\
&= \psi_1^*\beta - \psi_0^*\beta = \boxed{\beta - \pi^* i^* \beta}
\end{aligned}$$

where $i : X \hookrightarrow N$. To prove the lemma, let $\alpha = K\left(\omega_1 - \omega_0\right)$. Then the chain homotopy equation says that

$$dK\left(\omega_1 - \omega_0\right) + \cancel{Kd\left(\omega_1 - \omega_0\right)} = \omega_1 - \omega_0 = \, d\alpha$$

since $\pi^* i^*\left(\omega_1 - \omega_0\right) = 0$, and since these are symplectic forms.

Now for $p \in X$, we have $\alpha|_{T_p N} = 0$ because $V_t = 0$ on X. □

Now for $t \in [0,1]$, let $\omega_t = (1-t)\,\omega_0 + t\omega_1$ be a 1-parameter family of closed 2-forms on M. Note these are certainly closed, but not necessarily non-degenerate, so they may not be symplectic. However, because $\omega_0 = \omega_1$ on the zero section X, it follows that for some neighborhood of X in N, the ω_t are symplectic. For this part of the argument we really need them to be the same on all of TM. (Before we only needed them to agree on X.) Now we will do the Moser trick to find an isotopy $\{\varphi_t\}_{t\in[0,1]}$ where φ_t is a diffeomorphism between two neighborhoods of X, $\varphi_0 = \mathrm{id}$, and $\varphi_t^*\omega_t = \omega_0$. Moreover, we want $\varphi|_X = \mathrm{id}_X$. Now we want

$$\begin{aligned}
0 = \frac{d}{dt}\varphi_t^*\omega_t &= \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}\right) \\
&= \varphi_t^*\left(d\iota_{X_t}\omega_t + \cancel{\iota_{X_t} d\omega_t} + \left(\omega_1 - \omega_0\right)\right) \\
&= \varphi_t^*\left(d\iota_{X_t}\omega_t + \, d\alpha\,\right)
\end{aligned}$$

where $X_t = d\varphi_t/dt$. To get this, it is enough to get $\iota_{X_t}\omega_t + \alpha = 0$. But there is a unique X_t satisfying this, because ω_t is nondegenerate as long as we're in a sufficiently small neighborhood of the zero section. We also know $X_t = 0$ on the zero section, so if we choose our neighborhood small enough, this will generate an isotopy which doesn't move the zero section at all, so $\varphi = \varphi_1$ is the required isotopy. ■

2. Darboux's theorem

Now that we have proved this, we can prove Darboux's theorem.

Theorem 2 (Darboux). *Let (M, ω) be a symplectic manifold. For any $p \in M$, there exist local coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$ in a neighborhood of p in which*

$$\omega = \sum_{i=1}^n dx_i\; dy_i \;\; .$$

Proof. We apply the relative Moser trick for $X = \{p\}$. To set this up, choose local coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$ in a neighborhood of p, such that $\omega_p = \sum_{i=1}^n dx_i\ dy_i$. To do this, we need a linear algebra lemma:

Lemma 2. *Let (V, ω) be a symplectic vector space, i.e. V is a finite dimensional real vector space such that $\omega : V \otimes V \to \mathbb{R}$ is a nondegenerate, antisymmetric pairing. Then there exists a basis $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ for V in which $\omega(e_i, e_j) = 0$, $\omega(f_i, f_j) = 0$, and $\omega(e_i, f_j) = \delta_{ij}$.*

Proof. Let e_1 be any nonzero element of V. By non-degeneracy, there exists f_1 with $\omega(e_1, f_1) = 1$. Continuing by induction on the complement:

$$(\mathrm{Span}\{e_1, f_1\})^\omega = \{v \in V \mid \omega(v, e_1) = \omega(v, f_1) = 0\}$$

gives us such a basis. □

So we can apply this lemma to get a basis $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ as above for $\left(T_pM, \omega|_{T_pM}\right)$. Now we can find local coordinates with:

$$\frac{\partial}{\partial x_i} = e_i\ , \qquad\qquad \frac{\partial}{\partial y_i} = f_i$$

at p. So now we have two symplectic forms ω and $\sum_{i=1}^n dx_i\ dy_i$ which agree at p. Now we can apply the relative Moser trick to get that there exists some neighborhood U_1 of p and a diffeomorphism $\varphi : U_1 \to \{p\}$ such that

$$\varphi^*\left(\sum dx_i\ dy_i\right) = \omega$$

so we are done. ■

3. Lagrangian neighborhood theorem

Recall this says the following:

Theorem 3. *Let (M, ω) be a symplectic manifold, and $L \subset M$ be a compact Lagrangian submanifold. Then there are two neighborhoods $L \subset U_0 \subset M$, and $L \subset U_1 \subset T^*L$, and a diffeomorphism $\varphi : U_0 \xrightarrow{\sim} U_1$ with $\varphi|_L = \mathrm{id}_L$, and*

$$\varphi^*(d\lambda) = \omega\ .$$

Proof. This proof will also involve applying the relative Moser trick. To do so, we need to find neighborhoods U_0 of L in M, and U_1 in T^*L, and a diffeomorphism $\varphi : U_0 \to U_1$ such that $\varphi|_L = \mathrm{id}_L$, and for $p \in L$,

$$\varphi^*\ d\lambda\,|_{T_pM} = \omega|_{T_pM}\ \ .$$

For this purpose, it is enough to find a sub-bundle $E \subset TM|_L$ such that for each $p \in L$, $\omega|_E = 0$, and $T_pL \oplus E_p = T_pM$. In other words, E is the Lagrangian complement of T_pL in T_pM. This is enough, because then there is a unique bundle isomorphism $\psi : TM|_L \xrightarrow{\sim} T(T^*L)|_L$ such that we have both:

$$TM|_L \simeq TL \oplus E \qquad\qquad T(T^*L)|_L = TL \oplus T^*L\ .$$

So $\psi : TL \to TL$ canonically sends $\psi : TE \xrightarrow{\simeq} T^*L$, and $\psi^*\,d\lambda = \omega$. That is, we have a bundle isomorphism from $TM|_L \to T(T^*L)|_L$ which preserves the symplectic forms. There is then a diffeomorphism $\varphi : U_0 \to U_1$ as above, whose derivative along L equals φ.

Given the above, we just need the following linear algebra lemma:

Lemma 3. *Let (V, ω) be a symplectic vector space, and let $L \subset V$ be a Lagrangian subspace. Then there is a canonical retraction from*

$$\{complements\ of\ L\} \to \{Lagrangian\ complements\ of\ L\} \ .$$

I.e. there is a canonical way to turn complements of L into Lagrangian ones.

Since the space of complements of L is contractible, then by this lemma the space of Lagrangian complements is as well. $\square$