LECTURE 4
MATH 242

LECTURE: PROFESSOR MICHAEL HUTCHINGS
NOTES: JACKSON VAN DYKE

1. Exact Lagrangians

Recall a Lagrangian submanifold in a symplectic manifold $\left(M^{2n},\omega\right)$ is an n-dimensional submanifold $L \subset M$ such that $\omega|_L = 0$. There is a related notion of an exact (with respect to λ) Lagrangian which is that $\lambda|_L = df$ for some $f : L \to \mathbb{R}$.

Conjecture 1 (Nearby Lagrangian conjecture)**.** *If X is a compact smooth n-manifold then every compact exact Lagrangian in T^*X is Hamiltonian isotopic to the 0-section.*

There is a related theorem of Gromov:

Theorem 1 (Gromov)**.** *There does not exist a compact exact Lagrangian in $\mathbb{R}^{2n}$.*

Since $H_1\left(\mathbb{R}^{2n}\right) = 0$, the definition of "exact Lagrangian" is independent of the choice of the primitive λ.

If $L \subseteq \mathbb{R}^{2n}$ is Lagrangian, we can define a map $\alpha : H_1(L) \to \mathbb{R}$ as follows. If γ is an oriented loop in L, then

$$\alpha(\gamma) = \int_{D^2} \varphi^*\omega$$

where $\varphi : D^2 \to \mathbb{R}^{2n}$ with $\varphi|_{\partial D} = \gamma$. Then L is exact iff $\alpha = 0$.

Example 1. If $\gamma_1, \cdots, \gamma_n$ are embedded loops in $\mathbb{C}$, then $L = \gamma_1 \times \cdots \times \gamma_n \subseteq \mathbb{C}^n$ is Lagrangian but not exact, because α sends each of the γ_i to the area that it encloses in $\mathbb{C}$.

2. Intersections of Lagrangians

Definition 1. A Lagrangian $L \subset M$ is displaceable if there is a Hamiltonian isotopy from L to L' such that $L \cap L' = \emptyset$.

Example 2. Suppose $M = S^2$. A Lagrangian in this manifold is given by a simple closed curve. A Hamiltonian isotopy induces a symplectomorphism, which has to conserve area. So the curve is displaceable iff it divides S^2 into two components of unequal area.

Example 3. Consider a Lagrangian in $M = T^2$, i.e. a simple closed curve. If the homology class of L is trivial, and cuts the torus into pieces of unequal area, then it is displaceable. If the homology class $[L] \neq 0 \in H_1\left(T^2\right)$ then L is not displaceable. The reason for this is as follows.

Claim 1. If $\{\varphi_t\}$ is a Hamiltonian isotopy from id to φ, then the area swept out by $\{\varphi_t(L)\}$ is zero i.e.

$$\int_{L\times[0,1]} \Phi^*\omega = 0$$

for $\Phi : L \times [0,1] \to T^2$ sending $(x,t) \mapsto \varphi_t(x)$.

Proof. Exercise. □

What this means, is that if L were displaceable, then the region between L and the displaced Lagrangian L', $\mathrm{Area}(U) \in \mathbb{Z}$, but $\mathrm{Area}(T^2) = 1$ so we have a contradiction.[1]

Remark 1. Later we will see that there is a much more systematic way to understand the question of when Lagrangians intersect using Lagrangian Floer homology, which is an invariant of a pair of Lagrangians up to Hamiltonian isotopy.

Example 4. If $\varphi : (M,\omega) \to (M,\omega)$ then the graph

$$\Gamma(\varphi) = \{(x,\varphi(x))\} \subset (M \times M, (-\omega,\omega))$$

is a Lagrangian. An element of $T_{(p,\varphi(p))}\Gamma$ has the form $(v, \varphi^* v)$ for $v \in T_p M$. Then we have

$$(-\omega,\omega)((v,\varphi_* v),(w,\varphi_* w)) = -\omega(v,w) + \omega(\varphi_* v, \varphi_* w) = 0$$

since $\varphi^*\omega = \omega$. Note that the fixed points are in bijection with $\Gamma(\varphi) \cap \Delta$ where $\Delta = \{(x,x)\}$ is the diagonal.

Remark 2. If $L \subset M$ is a Lagrangian submanifold (or more generally an immersion) then ω defines a canonical isomorphism of vector bundles from the normal bundle of L to the cotangent bundle of L:

$$NL = TM|_L / TL \xrightarrow{\simeq} T^*L \,.$$

This gives some restrictions on what sort of Lagrangians are allowed.

Example 5. Let $L \subset \mathbb{R}^4$ be a compact connected Lagrangian. Then $\chi(L) = 0$ because of the following. More generally, if L is a compact Lagrangian surface in some compact oriented symplectic 4-manifold (M^4,ω), then the self-intersection number is $L \cdot L = -\chi(L)$. This is because of the following. We have this isomorphism $\omega : NL \to T^*L$, and then the Euler number of NL is $L \cdot L$ and the Euler number of T^*L is $\chi(L)$. The map $NL \to T * L$ is orientation reversing.[2]

Example 6. For $M = \mathbb{R}^4$ and L the x_1, x_2 plane, then $NL = y_1, y_2$ plane.

Example 7. If $L \subset \mathbb{R}^4$ is a compact connected Lagrangian then since $\chi(L) = 0$ either $L \simeq T^2$ or $L \simeq$ the Klein bottle. The latter of which is not possible, which was only proved a few years ago.

[1] Professor Hutchings says that if you're ever at a party, and someone asks you what's new in the math world, you should say that they found a new number between 66 and 67 and they're still trying to figure out what's going on with it.

[2] Professor Hutchings says that he has probably spent half of his career figuring out orientation and signs.

Example 8. For $M = \left(S^2 \times S^2, (-\omega, \omega)\right)$, the diagonal Δ is a Lagrangian, and also a sphere. The class is

$$[\Delta] = (1,1) \in H_2(M) = H_2\left(S^2\right) \oplus H_2\left(S^2\right) \ .$$

Generally we have

$$(a_1, b_1) \cdot (a_2, b_2) = -\left(a_1 b_2 + b_1 a_2\right) \ ,$$

so in this case we have

$$[\Delta] \cdot [\Delta] = (1,1)(1,1) = -2 \ .$$

Remark 3. Let X^n be a compact oriented smooth manifold, $A \in H_i(X)$, and $B \in H_j(X)$, where $i + j = n$. In this case we have $A \cdot B \in \mathbb{Z}$. Then the Poincaré duals $A^* \in H^{n-i}(X;\mathbb{Z})$ and $B^* \in H^{n-j}(X;\mathbb{Z})$ satisfy

$$A \cdot B = \langle A^* \smile B^*, [X] \rangle \ .$$

3. A theorem and some proofs

Theorem 2 (Weinstein's Lagrangian Tubular Neighborhood Theorem (WLTN))**.** *If $L \subset (M,\omega)$ is a compact Lagrangian then there is a neighborhood of $L \subset U \subset M$, a neighborhood $U' \subset T^*L$ of the zero-section, and a symplectomorphism $\varphi : (U,\omega) \to (U', d\lambda)$ which identifies L with the zero-section in a canonical way.*

Now we prove Darboux's theorem and WLTN theorem. We will use Moser's trick to prove these.

Proposition 1. *Let M be a compact $2n$ dimensional manifold. Let $\{\omega_t\}_{t\in[0,1]}$ be a smooth 1-parameter family of symplectic forms on M. Then the class $[\omega_t] \in H^2(M;R)$ is constant (so independent of t) iff there exists an isotopy $\{\varphi_t\}_{t\in[0,1]}$ (so $\varphi_t \in \mathrm{Diff}(M)$) with $\varphi_0 = \mathrm{id}_M$ and $\varphi_t^*\omega_t = \omega_0$.*

Proof. ($\Longleftarrow$): By homotopy invariance of de Rham cohomology, $\omega_t = \left(\varphi_t^{-1}\right)^* \omega_0$.

($\Longrightarrow$): Consider an isotopy $\{\varphi_t\}_{t\in[0,1]}$ generated by a 1-parameter family of vector fields $\{X_t\}$, i.e. $\varphi_0 = \mathrm{id}_M$ and

$$\frac{d}{dt}\varphi_t(x) = X_t(\varphi_t(x)) \ .$$

So we know the result is true for $t = 0$, and all we need to show is that

$$\frac{d}{dt}\varphi_t^*\omega_t = 0 \ .$$

We have that

$$\begin{aligned}\frac{d}{dt}\varphi_t^*\omega_t &= \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right)\\ &= \varphi_t^*\left(d\iota_{X_t}\omega_t + \cancel{\iota_{X_t} d\omega_t} + \frac{d}{dt}\omega_t\right)\end{aligned}$$

where $\mathcal{L}$ is the Lie derivative, and this term goes to zero since ω_t is closed. Since φ_t is a diffeomorphism, what we want is equivalent to

$$\boxed{d\iota_{X_t}\omega_t = \frac{-d}{dt}\omega_t} \ .$$

Now we want to find X_t solving this equation. We know that $\frac{d}{dt}\omega_t$ is exact. So there exists a 1-form α_t with $d\alpha_t = \frac{d}{dt}\omega_t$. Now we have a technical claim which we will prove later:

Claim 2. We can choose the α_t to depend smoothly on t.

Then with this claim, we just need $d\iota_{X_t}\omega_t = -d\alpha_t$, and to show this, it is enough to get $\iota_{X_t}\omega_t = -\alpha_t$. But now there is a unique X_t satisfying this equation because ω_t is nondegenerate. The vector field X_t depends smoothly on t because ω_t and α_t do. $\square$