COUNTEREXAMPLES TO HKR IN POSITIVE CHARACTERISTIC

ROK GREGORIC

1. Hochschild homology

Fix a field k (not necessarily of characteristic 0) and R a k-algebra (automatically flat for fields). Define:

$$\mathrm{HH}(R/k) \coloneqq R \overset{L}{\otimes}_{R\otimes_k R} R \tag{1}$$

Let $M \in \mathbf{Mod}\text{-}A$ and $N \in A\text{-}\mathbf{Mod}$. Recall the ordinary tensor product is defined as the co-equalizer of:

$$M \otimes A \otimes N \rightrightarrows M \otimes N \tag{2}$$

where

$$\begin{gathered} x \otimes a \otimes y \overset{d_0}{\mapsto} xa \otimes y \\ x \otimes a \otimes y \overset{d_1}{\mapsto} x \otimes ay \end{gathered} . \tag{3}$$

This is equivalent to

$$M \otimes_A N = \operatorname{coker}\left(M \otimes A \otimes N \xrightarrow{d_0 - d_1} M \otimes N\right) . \tag{4}$$

Now we can define the derived tensor product in the analogous way:

$$M \overset{L}{\otimes}_R N \simeq \left(\dots \overset{d}{\to} M \otimes R \otimes R \to N \overset{d}{\to} M \otimes R \otimes N \overset{d}{\to} M \otimes N\right) \tag{5}$$

where

$$x \otimes a \otimes b \otimes y \mapsto xa \otimes b \otimes y - x \otimes ab \otimes y + x \otimes a \otimes by . \tag{6}$$

Note that

$$H_i\left(M \overset{L}{\otimes} N\right) = \mathrm{Tor}_i^R(M, N) . \tag{7}$$

To get to HH, set $M = N = R$, $A = R \otimes R$. Then the Hochschild complex looks like:

$$\dots \to R \otimes_R (R \otimes R) \otimes_R (R \otimes R) \otimes_R R \to R \otimes_R (R \otimes R) \otimes_R R \to R \otimes_R R \tag{8}$$

Date: February 21, 2020.
Notes by: Jackson Van Dyke, all errors introduced are my own.

which can be rewritten as:

$$
\begin{array}{ccccc}
\ldots \longrightarrow R\otimes R\otimes R & \longrightarrow & R\otimes R & \xrightarrow{\quad 0 \quad} & R \\
x\otimes y\otimes z \longmapsto xy\otimes z - x\otimes yz + zx\otimes y & & & &
\end{array} \tag{9}
$$

$$x\otimes y \mapsto xy - yx = 0$$

Now we can read off that

$$\mathrm{HH}_0\left(R\right)\simeq R \tag{10}$$

$$\mathrm{HH}_1\left(R\right)\simeq R\otimes R/\left(xy\otimes z - x\otimes yz + zx\otimes y\right)\ . \tag{11}$$

So this relation says that:

$$x\otimes yz = xy\otimes z + xz\otimes y\ . \tag{12}$$

Now recall Kähler differentials are given by:

$$\Omega^1_{Rk}\simeq\left\{x\,dy\mid x\,d\left(yz\right)\ = xy\,dz\ + xz\,dy\right\}\ . \tag{13}$$

Now these look awfully familiar, since they are the same. We get an isomorphism by sending $x\otimes y\mapsto x\,dy$.

This required no assumptions on the ring, or on the characteristic. In fact if R/k is smooth:

$$\mathrm{HH}_i\left(R/k\right)\cong\Omega^i_{R/k}\ . \tag{14}$$

2. Derived Hochschild homology