OVERVIEW

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The seminar website is here. We will follow Lurie's notes.

The idea is to start with some collection of spaces, and attach some kind of algebraic invariant to them. Then we might hope that not too much information is lost so we can understand these spaces via the easier-to-understand algebraic counterparts. One example of this is a cohomology theory, which is a functor

$$E^* : \mathbf{Spaces}^{\mathrm{op}} \to \mathbf{Ab} \tag{1}$$

satisfying the Eilenberg-Steenrod axioms. To a cohomology theory, we can consider $E^*(\mathrm{pt})$, the *coefficients* of E.

Theorem 1 (Eilenberg-Steenrod). *If*

$$E^i(pt) = \begin{cases} A & i = 0 \\ 0 & i \neq 0 \end{cases} \tag{2}$$

then E is ordinary *cohomology:*

$$E^*(X) \cong H^*(X; A) \ . \tag{3}$$

Otherwise we say E^* is *extraordinary*.

Example 1.

$$KU^0(X) = \{\mathbb{C}\text{-vector bundles on } X\} / \tag{4}$$

which canonically extends to a cohomology theory KU, called *complex K-theory*, with coefficients:

$$KU^*(\mathrm{pt}) = \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases} , \tag{5}$$

so this is an extraordinary cohomology theory.

As it turns out, cohomology theories are equivalent to spectra, which can be thought of as "jazzed up" spaces.

We will assume E is *complex-oriented*, i.e.

$$E^*(\mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[\![t]\!] \tag{6}$$

where t has degree $|t| = 2$.

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Example 2. Ordinary cohomology, which as a spectrum we will denote $E = H\mathbb{Z}$, is complex oriented since:

$$H^*\left(\mathbb{CP}^\infty; \mathbb{Z}\right) \simeq \mathbb{Z}[\![t]\!] \ . \tag{7}$$

One reason to assume this is that this allows us to develop a theory of characteristic classes. The idea is that for $\mathcal{L}$ a line bundle on X, we would like to attach a class $c_1(\mathcal{L}) \in H^2(X; \mathbb{Z})$. A line is characterized by a function $f\colon X \to \mathbb{CP}^\infty$. Explicitly

$$\mathcal{L} \cong f^*\mathcal{O}(1) \ . \tag{8}$$

Using (6), we can define

$$c_1(\mathcal{L}) \coloneqq f^*(t) \in H^2(X; \mathbb{Z}) \ . \tag{9}$$

So for general complex oriented E we can define

$$c_1^E(\mathcal{L}) \in E^2(X) \ . \tag{10}$$

In the integral case, we have

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}') \ , \tag{11}$$

but this fails for the generalized classes c_1^E. There is, however, a weaker version:

$$c_1^E(\mathcal{L} \otimes \mathcal{L}') = f\left(c_1^E(\mathcal{L}), c_1^E(\mathcal{L}')\right) \tag{12}$$

for some power series:

$$f \in E^*(\mathrm{pt})[\![u, v]\!] \cong E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \ . \tag{13}$$

f satisfies some properties, which ultimately come from properties which line bundles satisfy. One thing which line bundles satisfy the property that

$$\mathcal{L} \otimes \text{trivial} \simeq \mathcal{L} \simeq \text{trivial} \otimes \mathcal{L} \tag{14}$$

which implies

$$f(u, 0) = u = f(0, u) \ . \tag{15}$$

It also satisfies

$$\mathcal{L} \otimes \mathcal{L}' \simeq \mathcal{L}' \otimes \mathcal{L} \tag{16}$$

which implies

$$f(u, v) = f(v, u) \ . \tag{17}$$

Finally it satisfies

$$(\mathcal{L} \otimes \mathcal{L}') \otimes \mathcal{L}'' \simeq \mathcal{L} \otimes (\mathcal{L}' \otimes \mathcal{L}'') \tag{18}$$

which implies

$$f(f(u, v), w) = f(u, f(v, w)) \ . \tag{19}$$

When f satisfies properties (15), (17), and (19) we say that f is a *formal group law*. This defines a group operation on