DRP WEEK 4: ONE BIG EXAMPLE

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Let G be a group and X and Y be sets. Consider the set of functions $f : X \to Y$. Write this set as F:

$$F = \{f : X \to Y\} \ . \tag{1}$$

Assume we have a group action of G on X, written $G \circlearrowright X$. This means we have a function:

$$a : G \times X \to X \ .$$

This means for every $g \in G$, we get a function $\varphi_g : X \to X$ given by

$$\varphi_g(x) = a(g, x) \ .$$

Remark 1. The idea of restricting to an element of G to get a function on X is a very subtle, confusing, and important one. So take some time to meditate on it.

Exercise 1. Show that G acts on F too. [Hint: we need a rule for eating a group element $g \in G$ and a function $f : X \to Y$, and producing a new function $X \to Y$. From the action of G on X, g defines a function $\varphi_g : X \to X$. So now we can rephrase the question: how do we turn f and φ_g into a new function $X \to Y$?]

Exercise 2. Let X have n elements and Y have m elements. How many elements does F have? Your answer should be in terms of m and n.

Exercise 3. Define a group homomorphism from G to a symmetric group (of some order). [Hint: We know G has an action on F. Does a symmetric group act on F?] [Another hint: The order of the correct symmetric group is the size of F...]

So we have a homomorphism $\psi : G \to S_N$ from G to a symmetric group. Now imagine I hand you a collection of elements $g_1, \ldots, g_n \in G$ and I claim they generate G. You don't believe me, so you come up with a very complicated element $\tilde{g} \in G$ which you believe is outside of the subgroup generated by the elements $\{g_1, \ldots, g_n\}$. Note $\psi(g_1), \ldots, \psi(g_n)$ are elements of S_N. Generate a subgroup using these elements. Call this subgroup P. So now we can ask if $\psi(\tilde{g}) \in P$ or not.

Exercise 4. If $\psi(\tilde{g})$ is in P, is $\tilde{g}$ necessarily in the subgroup of G generated by $g_1, \ldots, g_n$?

Exercise 5. If $\psi(\tilde{g})$ is *NOT* in P, is $\tilde{g}$ necessarily outside of the subgroup of G generated by $g_1, \ldots, g_n$?

Date: February 26, 2020.