DRP WEEK 2: GROUP GENERATION

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Let G be a group.

Question 1. What is the free group with n generators? Write this as F_n.

Question 2. Show that F_1 is the same as the group $\mathbb{Z}$ under addition. [Hint: to show a group is isomorphic to another group, we have to provide a group homomorphism in both directions. What is a group homomorphism $F_1 \to \mathbb{Z}$? What is a group homomorphism $\mathbb{Z} \to F_1$?]

Question 3. What is a group presentation?

Question 4. Write down a group presentation for $\mathbb{Z}$. [Hint: This is basically the answer to Question 2.]

Question 5. Write down a group presentation for $\mathbb{Z}^2$.

Question 6. Consider some group elements $g_i \in G$ for every $i \in \{1, \dots, n\}$. What does it mean for the set of elements $\{g_i\}$ to *generate* G? When n is finite we say G is *finitely generated*.

Question 7. Is $\mathbb{Z}$ under addition finitely generated? What about $\mathbb{Q}$ under addition?

Question 8. Is F_n finitely generated for finite n? [This is kind of a trick question.]

Question 9. Are subgroups of finitely generated groups necessarily finitely generated? [Hint: Consider the free group on two elements F_2. Now consider the group generated by the (infinite) collection of elements

$$\left\{ y^n x y^{-n} \,\middle|\, x, y \in F_2, n \geq 1 \right\} \ . \tag{1}$$

Is this subgroup finitely generated?]

Remark 1. The last question is very hard. There is probably some fiddly way to do it manually, but the only way I know off the top of my head uses machinery which is unnecessary[1]. The point of this question is to get you to realize that, even though it seems counter-intuitive at first, finitely generated groups can in fact have infinitely generated subgroups!

Date: February 12, 2020.

[1] In particular, it uses algebraic topology! Which is very neat if you ask me.