Algebraic Topology

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Contents

Introduction

We won't follow the book line by line, but it will be the reference for the course. Read the first three pages of chapter 0, talking about homotopy. We will really get started on chapter 1, section 1.1, about the fundamental group.

1. Introduction

Today will be an introductory account of what algebraic topology actually is. In topology the objects of interest are topological spaces where the natural equivalence relation is a homeomorphism, i.e. a bijection $f : X \to Y$ such that f and f^{-1} are continuous. Somehow the goal is classifying topological spaces up to homeomorphism, so the basic question is somehow:

QUESTION 1. Given topological spaces X and Y, is $X \cong Y$.

In these terms, algebraic topology is somehow a way of translating this into an algebraic question. More specifically, *algebraic topology* is the construction and study of functors from **Top** to some categories of algebraic objects (e.g. groups [0.1], abelian groups, vector spaces, rings, modules, ...). Recall this means we have a map from topological spaces $X \to A(X)$ for some algebraic object $A(X)$. In addition, for every $f : X \to Y$ we get a morphism $f_* : A(X) \to A(Y)$. Then these have to satisfy the conditions that

$$(gf)_* = g_* f_* , \qquad\qquad (\mathrm{id})_* = \mathrm{id} .$$

EXERCISE 1.1. Show that $X \cong Y$ implies that $A(X) \cong A(Y)$.

EXAMPLE 0.1 (Fundamental group). Let X be a topological space. We will construct a group $\pi_1(X)$.

EXAMPLE 0.2 (Higher homotopy groups). There is also something, $\pi_n(X)$, called the nth homotopy group. As it turns out for $n \geq 2$ this is abelian.

EXAMPLE 0.3 (Singular homology). We will define abelian groups $H_n(X)$ (for $n \geq 0$) called the nth singular homology group.

We will also define real vector spaces $H_n(X; \mathbb{R})$ for $n \geq 0$ which are the nth singular homology with coefficients in $\mathbb{R}$.

EXAMPLE 0.4 (Cohomology). We will also have the nth (singular) cohomology rings $H^*(X)$. These is actually a graded ring.

[0.1]Once Professor Gordon was giving a job talk about knot cobordisms. As it turns out these form a semigroup rather than a group. But if you add some sort of 4 dimensional equiv relation you get an honest group. So he was going on about how semigroups aren't so useful. After the talk he found out the chairman of the department worked on semigroups.

WARNING 0.1. Above we actually should have said we're dealign with what are *covariant* functors, but in this case we are actually dealing with a *contravariant* functor. This just means we have:

$$f : X \to Y \rightsquigarrow f^* : H^*(Y) \to H^*(X) \ .$$

REMARK 0.1. The point here is that problems about topological spaces and maps are "continuous" and "hard". But on the algebraic side these problems become somehow "discrete" and "easy".

2. A bit more specific

Recall in $\mathbb{R}^n$ we define:

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \qquad S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

EXAMPLE 0.5. Two examples of surfaces are S^2 and $T^2 = S^1 \times S^1$. They clearly aren't homeomorphic, but how are we supposed to prove such a fact? We will see that $\pi_1(S^2) = 1$ whereas $\pi^1(T^2) = \mathbb{Z} \times \mathbb{Z}$. Since these are not isomorphic, the spaces cannot be homeomorphic.

2.1. Retraction. Let $A \subset X$ be a space and a subspace.

DEFINITION 0.1. A *retraction* from X to A is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$, i.e. the following diagram commutes:

$$\begin{array}{ccc} & X & \\ {}^{i}\nearrow & & \searrow^{r} \\ A & \xrightarrow[\mathrm{id}]{} & A \end{array} \ .$$

Note that r is certainly surjective since id is.

EXAMPLE 0.6. If X is any nonempty space, $x_0 \in X$, define $r : X \to \{x_0\}$ as $r(x) = x_0$. So every nonempty space always retract onto a point.

EXAMPLE 0.7. Think of $A \subset A \times B$ by fixing some $b_0 \in B$ and sending

$$A \longleftrightarrow B$$

$$a \longmapsto (a, b_0)$$

Then $r : A \to B \to$ defined by $r(a, b) = a$ is a retraction.

Recall that for $f : X \to Y$ for X path connected, then $f(X)$ is also path connected. Recall that $D^1 = [-1, 1] \subset \mathbb{R}$ is path connected, whereas $S^0 = \{\pm 1\}$ is not. Therefore there cannot be a retraction $D^1 \to S^0$. This is a basic fact, but it motivates a more general statement which is not so clear.

Suppose there exists a retraction $r : D^1 \to S^0$. Then this means the following diagram commutes:

$$\begin{array}{ccc} & D^1 & \\ \nearrow & & \searrow^{r} \\ S^0 & \xrightarrow[\mathrm{id}]{} & S^0 \end{array} \ .$$

If we apply the functor H_0, we will see that

$$H_0(X) \cong \bigoplus_{\text{path components of } X} \mathbb{Z} \ .$$

So if we apply H_0 to the diagram we get:

$$H_0\left(D^1\right)$$

$$H_0\left(S^0\right) \xrightarrow{\text{id}} H_0\left(S^0\right) \qquad = \qquad \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}\oplus\mathbb{Z}$$

with i_*, r_* and $\mathbb{Z}$, i_*, r_* labeling the arrows.

but this is clearly impossible.

In the same way we will see the much harder fact:

FACT 1 (Brouwer). *There does not exist a retraction $D^n \to S^{n-1}$ (for $n \geq 2$).*

We will see this by applying H_{n-1}. The idea is that

$$H_{n-1}\left(D^n\right) = 0 \qquad\qquad H_{n-1}\left(S^{n-1}\right) = \mathbb{Z}$$

which means we would have the diagram:

$$0$$
$$\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}$$

which is a contradiction.

This turns out to imply the famous:

THEOREM 0.1 (Brouwer fixed point theorem). *Every map $f : D^n \to D^n$ $(n \geq 1)$ has a fixed point (i.e. a point $x \in D^n$ such that $f(x) = x$). In this case one says that D^n has the* fixed point property *(FPP).*

PROOF. Suppose there exists an $f : D^n \to D^n$ such that $\forall x \in D^n$ $f(x) \neq x$. Now draw a straight line from x to $f(x)$ and continue it to the boundary S^{n-1}. Call this point $g(x)$. Then this defines a map $g : D^n \to S^{n-1}$. g is continuous and $g|_{S^{n-1}} = \text{id}$. Therefore g is a retraction $D^n \to S^{n-1}$ which we saw cannot exist. $\square$

2.2. Dimension. We know $\mathbb{R}^n$ somehow has dimension n. But what does this really mean? The intuition is that $\mathbb{R}^2$ somehow has more points than $\mathbb{R}$. But then in 1877 Cantor proved that there is in fact a bijection $\mathbb{R} \to \mathbb{R}^2$. But this is highly non-continuous, so this tells us continuity should have something to do with it. But then in 1890 Peano showed that there exists a continuous suggestion $\mathbb{R} \to \mathbb{R}^2$ as well. In 1910, using homology, Brouwer proved:[0.2]

THEOREM 0.2. *For $m < n$ $\nexists$ continuous injection $\mathbb{R}^n \to \mathbb{R}^m$.*

We will prove this. A corollary of this is the famous invariance of dimension. I.e. $\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$. The proof uses separation properties of $n-1$-sphere in $\mathbb{R}^n$, thus in turn uses H_*.

EXERCISE 2.1. Find an easy proof for $m = 1$.

THEOREM 0.3 (Jordan curve theorem). *For a subset $C \subset \mathbb{R}^2$ such that $C \simeq S^1$ then $\mathbb{R}^2 \setminus C$ has exactly 2 components A and B. In addition $C = Fr(A) = Fr(B)$. (Recall the* frontier *is defined as $Fr(X) = \overline{X} \cap \overline{(Y \setminus X)}$ for any $X \subset Y$.)*

[0.2] When he proved this Lebesgue contacted him saying that he could prove it too. So he sent him his proof, and Brouwer saw some errors. So over many years he eventually corrected it. In the end Brouwer summarized it by saying that it was really just his own proof.

THEOREM 0.4 (Schönflies). *Let A, B, and C be as above and assume A is a bounded component of $\mathbb{R}^2 \setminus X$. Then $\overline{A} \cong D^2$.*

The higher dimensional analog of JCT is true Brouwer). The proof will use H_*. The generalization is that for $\Sigma \subset \mathbb{R}^n$ then $\Sigma \cong S^{n-1}$ As it turns out the higher dimensional analog of the Schönflies theorem is false. The counterexample is the famous Alexander horned space. I.e. there exists $\Sigma \subset \mathbb{R}^3$, $\Sigma \cong S^2$; A and bounded component of $\mathbb{R}^2 \setminus \Sigma$ and $\overline{A} \not\cong D^3$.
    Recall

THEOREM 0.5 (Heine-Borel). *$X \subset \mathbb{R}^n$ compact is equivalent to X being closed and bounded.*

So consider $\emptyset \neq X \subset \mathbb{R}$ compact and connected. This is equivalent to being an interval $X = [a, b]$ ($a \leq b$). In $\mathbb{R}^2$ things get much worse. In particular there exists compact, connected subset such that $\mathbb{R}^2 \setminus X$ has exactly three components A, B, and C, and in particular every neighborhood of every point in X meets all three components. This is known as the 'lakes of Wada'. We start with an island with two lakes, and then start digging canals which somehow get closer and closer to the lakes. In the end every point of what is left is arbitrarily close to all three lakes.

QUESTION 2 (Open question). Let X be a compact connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus X$ is connected. Does X have the fixed point property?

CHAPTER 1

Homotopy

Let $I = [0, 1]$.

DEFINITION 1.1. Let $f, g : X \to Y$. Say f is homotopic to g (and write $f \simeq g$) iff there exists a map $F : X \times I \to Y$ such that for all $x \in X$

$$F(x, 0) = f(x) \qquad\qquad F(x, 1) = g(x) \ .$$

Define $F_t : X \to Y$ by $F_t(x) = F(x, t)$. Then F_t is a continuous 1-parameter family of maps $X \to Y$ such that $F_0 = f$ and $F_1 = g$.
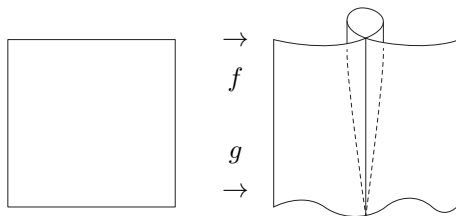

FIGURE 1.

EXAMPLE 1.1. Let $F : S^{n-1} \times I \to \mathbb{R}^n$ be defined by $F(x, t) = (1 - t) x$. Then F_0 is the inclusion inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$ and F_1 is the constant map $S^{n-1} \to$ origin.

Lecture 2; September 3, 2019

LEMMA 1.1. *Homotopy is an equivalence relation on the set of all maps $X \to Y$.*

PROOF. (i) $f \simeq f$: Let F be the constant homotopy, i.e. $F(x, y) = f(x)$ for all $t \in I$ and for all $x \in X$.

(ii) $f \simeq g \implies g \simeq f$: Suppose $f \simeq_F g$. Let $\bar{F}$ be the reverse homotopy $\bar{F}(x, t) = F(x, 1 - t)$. Then $g \simeq_{\bar{F}} f$.

(iii) $f \simeq g, g \simeq h \implies f \simeq h$: Define $H : X \times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq 1/2 \\ G(x, 2t - 1) & 1/2 \leq t \leq 1 \end{cases} .$$

Then by exercise 2 on homework 1 H is continuous and therefore $f \simeq_H h$.
□

LEMMA 1.2 (compositions of homotopic maps are homotopic). *If $f \simeq f' : X \to Y$, $g \simeq g' : Y \to Z$, then $gf \simeq g'f' : X \to Z$.*

PROOF. Suppose $f \simeq_F f'$, $g \simeq_G g'$. Then the composition

$$X \times I \xrightarrow{F} Y \xrightarrow{g} Z$$

is a homotopy from gf to gf'. Then the composition

$$X \times I \xrightarrow{f' \times \mathrm{id}} Y \times I \xrightarrow{G} Z$$

is a homotopy from gf' to $g'f'$. By transitivity of equivalence relations from lemma 1.1. □

Let $[X, Y]$ be the set of homotopy classes of maps from $X \to Y$.

REMARK 1.1. We should probably assume $X \neq \emptyset \neq Y$.

Lemma 1.2 then tells us that composition defines a function

$$[X, Y] \times [Y, Z] \to [X, Z] \ .$$

0.1. Homotopy equivalence.

DEFINITION 1.2. X is *homotopy equivalent* to Y (or *of the same homotopy type as Y*) written $X \simeq Y$ if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. Then we say f is a homotopy equivalence and g is a homotopy inverse of f.

EXAMPLE 1.2. A homeomorphism is a homotopy equivalence, but in general this is much weaker.

LEMMA 1.3. *Homotopy equivalence is an equivalence relation.*

PROOF.   (i) $X \simeq X$: $f = g = \mathrm{id}_X$, lemma 1.1.
(ii) $X \simeq Y \implies Y \simeq X$: by definition.
(iii) $X \simeq Y$, $Y \simeq Z \implies X \simeq Z$: So we have

$$X \overset{f}{\underset{f'}{\rightleftarrows}} Y$$

where $f'f \simeq \mathrm{id}_X$, $ff' \simeq \mathrm{id}_Y$ and similarly

$$Y \overset{g}{\underset{g'}{\rightleftarrows}} Z$$

where $g'g \simeq \mathrm{id}_X$, $gg' \simeq \mathrm{id}_Y$. Then we compose:

$$X \overset{gf}{\underset{f'g'}{\rightleftarrows}} Y$$

and we have that

$$(f'g')\,gf = f'\,(g'g)\,f \simeq f'\,\mathrm{id}_Y\,f = f'f \simeq \mathrm{id}_X$$

(from lemma 1.2) and similarly

$$(gf)\,(f'g') \simeq \mathrm{id}_X$$

so $gf : X \to Z$ is a homotopy equivalence.

□

LEMMA 1.4. $X \simeq X'$, $Y \simeq Y' \implies X \times Y \simeq X' \times Y'$.

PROOF. (exercise)   □

REMARK 1.2. Many functors in algebraic topology (e.g. π_1, H_n, ...) have the property that $f \simeq g \implies f_* = g_*$. In other words they factor through the homotopy category where the objects are topological spaces and morphisms are just homotopy classes of maps:

$$\mathbf{Top}/\simeq$$

$$\mathbf{Top} \longrightarrow \{\text{Algebraic objects, morphisms}\} \quad \cdot$$

$$f : X \to Y \longrightarrow f_* : A(X) \to A(Y)$$

DEFINITION 1.3. Let $A \subset X$, $f, g : X \to Y$ such that $f|_A = g|_A$. Then $f \simeq g$ (rel. A) iff there exists $f \simeq_F g$ such that

$$F_t|_A = f|_A \left(= g|_A\right)$$

for all $t \in I$.

Homotopy equivalence rel. A is an equivalence relation on the set

$$\{f : X \to Y \mid f|_A \text{ is a fixed map.}\}$$

Let $i : A \to X$ be the inclusion. Then i being a homotopy equivalence means there exists some $f : X \to A$ such that
  (i) $if \simeq \mathrm{id}_X$, and
  (ii) $fi \simeq \mathrm{id}_A$.
Now we can strengthen this in multiple ways.

DEFINITION 1.4. If we strengthen (ii) to say $fi = \mathrm{id}_A$ (i.e. f is a retraction) then $\mathrm{id}_X \simeq_F if$ is a *deformation retraction of X onto A*. In other words we have

$$F : X \times I \to X$$

such that $F_0 = \mathrm{id}_X$, $F_1(x) \in A$ for all $x \in A$, and $F_1|_A = A$ (since $F_1 = if$).

DEFINITION 1.5. If in addition we strengthen (i) to say that $\mathrm{id}_X \simeq_F if$ (rel A) then F is a *strong deformation retraction* (of X onto A). In other words we have $F : X \times I \to X$ such that

$$F_0 = \mathrm{id}_X \qquad \forall x \in X, F_1(x) \in A \qquad \forall t \in I, \forall a \in A, F_t(a) = a \ .$$

The idea here is that X strong deformation retracts to A implies X deformation retracts to A which implies $i : A \hookrightarrow X$ is a homotopy equivalence. As it will turn out, both of these implications are strict.

EXAMPLE 1.3. $X \times \{0\}$ is a strong deformation retract of $X \times I$:

$$F : (X \times I) \times I \to X \times I$$

where $F((x, s), t) = (x, (1-t)s)$.

EXAMPLE 1.4. $\mathbb{R}^n \setminus \{0\}$ strong deformation retracts to S^{n-1}.

EXAMPLE 1.5. $A = S^1 \times [-1, 1]$ strong deformation retracts to $S^1 \times \{0\}$. A Möbius band B also strong deformation retracts to S^1. Therefore $A \simeq B$.

EXAMPLE 1.6. Let X be a twice punctured disk. Then it is sort of clear that this strong deformation retracts to

(i) the boundary along with one arc passing between the punctures,
(ii) the wedge of two circles, and
(iii) two circles connected by an interval.

This means these three are all homotopy equivalent.

EXAMPLE 1.7. Consider a once-punctured torus $X = T^2 \setminus \text{int}\left(D^2\right)$. This strong deformation retracts to the wedge of two circles. This tells us that the once punctured torus is actually homotopy equivalent to the disk with two punctures.

These examples show that homotopy equivalence does not imply homeomorphism, even for surfaces with boundary. We can immediately see these examples are homeomorphic because there's no way for the boundaries to be mapped to one another.

0.2. Manifolds. Topological spaces can be very wild, but manifolds are usually quite nice.

DEFINITION 1.6. An *n-dimensional manifold* is a Hausdorff, second countable space M such that every $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

EXAMPLE 1.8. $\mathbb{R}^n$, S^n, $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$.

DEFINITION 1.7. A manifold is *closed* iff it is compact.

DEFINITION 1.8. A surface is a closed 2-manifold.

EXAMPLE 1.9. We have all of the two-sided or orientable surfaces (S^2, T^2, ...) and then non orientable ones like $\mathbb{P}^2$ and the Klein bottle. As it turns out this is a complete list up to homeomorphism.

For any two spaces $X \cong Y$ implies $X \simeq Y$. As we have seen, the converse isn't even true for surfaces with boundary. But for closed manifolds, there are many interesting cases where it is true:

(1) For M, M' closed surfaces, $M \simeq M' \implies M \cong M'$.
(2) For M an n-manifold, $M \simeq S^n \implies M \cong S^n$. (Generalized Poincaré conjecture)

For $n = 0, 1, 2$ this is not so bad. For $n \geq 5$, Connell and Newman independently proved it. The next step was proving this for $n = 4$. This was proved by Freedman. Finally for $n = 3$ Perelman proved this. The smooth 4-dimensional version is still open. I.e. the statement for diffeomorphism.

REMARK 1.3. Poincaré originally posed this conjecture as saying that having the same homology as S^n was sufficient. But he discovered a counterexample, now called the Poincaré homology sphere. This shares homology with S^3 but has different fundamental group. This is in fact why he invented the fundamental group.

DEFINITION 1.9. $f : X \to Y$ is a *constant map* if there is some $y_0 \in Y$ such that for all $x \in X$ $f(x) = y_0$. We write $f = c_{y_0}$.

DEFINITION 1.10. A map $f : X \to Y$ is *null-homotopic* iff $f \simeq$ a constant map.

DEFINITION 1.11. X is *contractible* if id_X is null-homotopic. I.e. there exists some $x_0 \in X$ such that X deformation retracts to x_0.

EXAMPLE 1.10.   (1) Any nonempty convex[1.1] subspace of $\mathbb{R}^n$ is contractible. Choose an arbitrary point $x_0 \in X$. Define $F : X \times I \to X$ by

$$F(x, t) = (1 - t) x + t x_0 .$$

Then $\mathrm{id}_X \simeq_F c_{x_0}$. In fact this is a strong deformation retraction. Sometimes you can do this for some point but not all, and for some you can't do it for any points.

   (2) S^1 is not contractible.

LEMMA 1.5. *For a topological space X TFAE:*

   (1) X is contractible,
   (2) $\forall x_0 \in X$, X deformation retracts to $\{x_0\}$,
   (3) $X \simeq \{pt\}$,
   (4) $\forall Y$, any two maps $Y \to X$ are homotopic.
   (5) $\forall Y$, any map $X \to Y$ is null-homotopic.

PROOF. (1) $\implies$ (3): (1) is equivalent to saying that X deformation retracts to a point, so the inclusion map is certainly a homotopy equivalence.

   (3) $\implies$ (4): Let $f : X \to \{z\}$ be a homotopy equivalence. By homework 1 exercise 3, we get an induced function:

$$f_* : [Y, X] \to [Y, \{z\}]$$

but there is only one map in the target set, so clearly there is only one homotopy class of maps $Y \to \{z\}$.

   (4) $\implies$ (2): Take $Y = X$, and take any $x_0 \in X$. This means $\mathrm{id}_X \simeq c_{x_0}$, but this is exactly saying that X deformation retracts to x_0.

   (2) $\implies$ (5): Let $f : X \to Y$ and $x_0 \in X$. Then (2) implies $\mathrm{id}_X \simeq_F c_{x_0}$. Then

$$f \circ \mathrm{id}_X \simeq_{f \circ F} f \circ c_{x_0}$$

i.e. f is nullhomotopic.

   (5) $\implies$ (1): Take $Y = X$. $\square$

COROLLARY 1.1. *For X, Y contractible, then*

   (1) $X \simeq Y$,
   (2) any map $X \to Y$ is a homotopy equivalence.

PROOF.   (1) If $X, Y \simeq \{pt\}$ then $X \simeq Y$.
   (2) Given $f : X \to Y$, let $g : Y \to X$ be any map. $gf : X \to X$, but X is contractible, so $gf \simeq \mathrm{id}_X$ by lemma 1.5.

                                                                      $\square$

Now we will give an example of a deformation retraction which is not a strong deformation retraction. Recall X strong deformation retracts to A implies X deformation retracts to A which implies $i : A \hookrightarrow X$ is a homotopy equivalence, but none of these implications are reversible.

[1.1] Recall $X \subset \mathbb{R}^n$ is convex if $x, y \in X$ implies $tx + (1 - t) y \in X$ for all $t \in I$.

EXAMPLE 1.11 (Comb space). Define the comb space $C \subset I \times I \subset \mathbb{R}^2$ to be:

$$C = \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, y = 0, 0 \le x \le 1; 0 \le y \le 1, x = 0, 1/n \, (n = 1, 2, \ldots) \right\} \ .$$

This should be pictured as a bunch of vertical intervals.[1.2] The first thing to note is that C strong deformation retracts to $(0, 0)$. Therefore C is contractible. C also deformation retracts to $(0, 1)$. [More generally: if X deformation retracts to some $x_0 \in X$ and X is path connected, then X deformation retracts to any $x \in X$].

CLAIM 1.1. But it does not strong deformation retract to $(0, 1)$.

PROOF. Let $F : C \times I \to C$ be such a strong deformation retraction. Let U be some open disc of radius $1/2$ centered at $(0, 1)$. $F^{-1}(U) \subset X \times I$ contains $(0, 1) \times I$. Therefore for all $t \in I$ there exists some neighborhood V_t of $(0, 1) \times \{t\}$ such that $V_t \subset F^{-1}(U)$. But $V_t = W_t \times Z_t$ for W_t some neighborhood of $(0, 1)$ in C and Z_t some neighborhood of t in I. I is compact which means $\exists t_1, \ldots, t_m$ such that

$$\bigcup_{i=1}^{m} Z_{t_i} = I \ .$$

Let

$$W = \bigcap_{i=1}^{m} W_{t_i} \ .$$

This is a neighborhood of $(0, 1)$ in C, and $W \times I \subset F^{-1}(U)$. (This is sometimes called the tube lemma). Pick n such that $(1/n, 1) \in W$. Then $F((1/n, 1), t)$, $0 \le t \le 1$, is a path in U from $(1/n, 1)$ to $(0, 1)$ but there clearly isn't such a path since these two points are in different path components. $\square$

COROLLARY 1.2. *Let* $X \subset I^2 \subset \mathbb{R}^2$ *where* C, I^2 *are both contractible. Then the inclusion* $i : C \to I^2$ *is a homotopy equivalence. But there does not exist a deformation retraction* $I^2 \to C$. *In fact there is no retraction at all.*

REMARK 1.4. There exists a space X such that X is contractible (therefore $\{x\} \hookrightarrow X$ is a homotopy equivalence for all $x \in X$) but there does not exist a deformation retraction from X to any $x \in X$. (e.g. Hatcher chapter 0, 6(b)).

0.3. Fixed point property. A space X has the fixed point property (FPP) iff $\forall f : X \to x$, $\exists x \in X$ such that $f(x) = x$. X being contractible does not imply X has the FPP (e.g. $\mathbb{R}^1$).

QUESTION 3 (Borsuk). If X is compact and contractible does contractible imply FPP?

[1.2]Which is supposed to look like a comb.

The fundamental group

DEFINITION 2.1. A *path* from x_0 to x is a map $\sigma : I \to X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x$.

DEFINITION 2.2. Let σ be a path in X from x_0 to x_1, and τ a path in X from x_1 to x_2. Their *concatenation* $\sigma * \tau$ is a path from x_0 to x_2 given by:

$$(\sigma * \tau)(s) = \begin{cases} \sigma(2s) & 0 \le s \le 1/2 \\ \tau(2s - 1) & 1/2 \le s \le 1 \end{cases}.$$

DEFINITION 2.3. The *homotopy class* of σ is

$$[\sigma] = \{\sigma' \,|\, \sigma' \simeq \sigma \,(\mathrm{rel}\,\partial I)\} .$$

LEMMA 2.1. *If* $[\sigma] = [\sigma']$ *and* $[\tau] = [\tau']$ *where* $\sigma(1) = \tau(0)$ *then* $[\sigma * \tau] = [\sigma' * \tau']$.

PROOF. If $\sigma \simeq_{F_t} \sigma'$ and $\tau \simeq_{G_t} \tau'$ then $\sigma * \tau \simeq_{F_t * G_t} \sigma' * \tau' \,(\mathrm{rel}\,\partial I)$. $\qquad\square$

This means we can define the product of two homotopy classes to the be the homotopy class of the concatenation. This is well defined by the lemma.

LEMMA 2.2 (Reparameterization). *Let* $u : I \to I$ *be a map such that* $u|_{\partial I} = \mathrm{id}$. *Then* $u \simeq \mathrm{id}_I \,(\mathrm{rel}\,\partial I)$.

PROOF. Define $F : I \times I \to I$ by $F(s, t) = ts + (1 - t)u(s)$. $F_0 = u$, $F_1 = \mathrm{id}_I$, $F_t|_{\partial I} = \mathrm{id}$ for all $t \in I$. $\qquad\square$

LEMMA 2.3 (Associativity). *Let* ρ, σ, τ *be paths in* X *such that* $\rho(1) = \sigma(0)$, $\sigma(1) = \tau(0)$. *Then*

$$([\rho][\sigma])[\tau] = [\rho]([\sigma][\tau]) .$$

PROOF. Define $u : I \to I$ by

$$u(s) = \begin{cases} 2s & 0 \le s \le 1/4 \\ s + 1/4 & 1/4 \le s \le 1/2 \\ (s + 1)/2 & 1/2 \le s \le 1 \end{cases}.$$

Then

$$(\rho * (\sigma * \tau))\,u = (\rho * \sigma) * \tau .$$

but $u \simeq \mathrm{id}_I \,(\mathrm{rel}\,\partial I)$ so $(\rho * (\sigma * \tau)) = (\rho * \sigma) * \tau \,(\mathrm{rel}\,\partial I)$ $\qquad\square$

Let $c_{x_0} : I \to X$ be the constant path given by $c_{x_0} = x_0$ for all $s \in I$.

LEMMA 2.4. *For σ a path in X from x_0 to x_1 then*

$$[\sigma] = [\sigma]\,[c_{x_1}] = [c_{x_0}]\,[\sigma] \ .$$

PROOF. Let $u : I \to I$ be

$$u\,(s) = \begin{cases} 2s & 0 \le s \le 1/2 \\ 1 & 1/2 \le s \le 1 \end{cases} .$$

Then $\sigma * c_{x_1} = \sigma * u$

$$[\sigma] = [\sigma]\,[c_{x_1}]$$

by lemma 2.2. The proof is the same for the other part. □

If σ is a path from x_0 to x_1, the reverse of σ is the path $\bar{\sigma}$ from x_1 to x_0 given by

$$\bar{\sigma}\,(s) = \sigma\,(1 - s) \ .$$

Note that immediately we have $\overline{(\bar{\sigma})} = \sigma$.

LEMMA 2.5. $[\sigma]\,[\bar{\sigma}] = [c_{x_0}] \ .$

PROOF. Define $F : I \times I \to X$ by

$$F\,(s, t) = \begin{cases} \sigma\,(2st) & 0 \le s \le 1/2 \\ \sigma\,(2\,(1 - s)\,t) & 1/2 \le 2 \le 1 \end{cases} .$$

Note that $F_0 = c_{x_0}$ and $F_1 = \sigma * \bar{\sigma}$ so we are done. □

A *loop* in a space X based at a point $x_0 \in X$ is a path in X from x_0 to x_0. If   Lecture 4; σ and τ are loops at x_0 then this implies $\sigma * \tau$ is a loop at x_0. Let   September 10, 2019

$$\pi_1\,(X, x_0) = \{[\sigma]\,|\,\sigma \text{ loop in } X \text{ based at } x_0\} \ .$$

THEOREM 2.1. *$\pi\,(X, x_0)$ is a group with respect to the operation $[\sigma]\,[\tau]$; the fundamental group of X with basepoint x_0.*

PROOF. We proved associativity last time in lemma 2.3 the identity is the constant map $[c_{x_0}]$ as shown in lemma 2.4, and inverses are given by $[\sigma]^{-1} = [\bar{\sigma}]$ as shown in lemma 2.5. □

EXAMPLE 2.1. Suppose X strong deformation retracts to some point x_0. This means there exists some homotopy $F : X \times I \to X$ such that $F_0 = \mathrm{id}_X$ and $F_1\,(x) = x_0$ for all $x \in X$. Since it is a strong deformation retraction for all $t \in I$ we have $F_t\,(x_0) = x_0$. Now let $\sigma : I \to X$ be a loop based at x_0. Then $\sigma \simeq_{F_t \sigma} c_{x_0}$ (rel ∂I). Therefore $[\sigma] = 1$ and $\pi_1\,(X, x_0) = 1$.

REMARK 2.1. We will actually prove a more general fact later, when we discuss change of basepoint.

Now we will finally have an example with nontrivial fundamental group. For all we know every space is contractible.[2.1]

[2.1]Professor Gordon says our ignorance is extensive.

FIGURE 1. p mapping $\mathbb{R}$ to S^1. Note the preimage of a $1 \in S^1$ looks like $\mathbb{Z}$.

1. Fundamental group of S^1

A key example is $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$.

THEOREM 2.2. $\pi_1\left(S^1, 1\right) \simeq \mathbb{Z}$.

REMARK 2.2. The idea of the proof is to somehow unwrap the circle. This proof led to the idea of a covering space.

Let $p : \mathbb{R} \to S^1$ be the map defined by $p(x) = e^{2\pi i x}$. The picture is as in fig. 1. Note that $p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$.

LEMMA 2.6 (path-lifting). *Let $\sigma : I \to S^1$ be a path with $\sigma(0) = 1$. Then there exists a unique path $\tilde{\sigma} : I \to \mathbb{R}$ such that $\tilde{\sigma}(0) = 0$ and $p\tilde{\sigma} = \sigma$, i.e. $\tilde{\sigma}$ is a lift of σ so the following diagram commutes:*

$$
\begin{array}{ccc}
 & & \mathbb{R} \\
 & \overset{\tilde{\sigma}}{\nearrow} & \downarrow p \\
I & \overset{\sigma}{\longrightarrow} & S^1
\end{array} \quad .
$$

PROOF. Let

$$U = \left\{ e^{i\theta} \,\middle|\, -\frac{3\pi}{4} < \theta < \frac{3\pi}{4} \right\}$$

$$V = \left\{ e^{i\theta} \,\middle|\, \frac{\pi}{4} < \theta < \frac{7\pi}{4} \right\} \quad .$$

This looks as in fig. 2. Then $\{U, V\}$ is an open cover of S^1. First notice that the distance between the points $\pi/4$ and $3\pi/4$ is $\sqrt{2}$. This means that for any $z, z' \in S^1$ we have that $d(z, z') < \sqrt{2}$ implies either $z, z' \in U$ and $z, z' \in V$.

So we have some $\sigma : I \to S^1$, $\sigma(0) = 1$. Since I is compact this implies $\exists \delta > 0$ such that $d(s, s') < \delta$ implies $d(\sigma(s), \sigma(s')) < \sqrt{2}$.

Now we decompose the interval. Let $0 = s_0 < s_1 < \ldots < s_m = 1$ such that $|s_i - s_{i-1}| < \delta$ for all i. Then $\sigma([s_{i-1}, s_i]) \subset U$ or V for all i.

Now we look at the inverse image in $\mathbb{R}$ and we get these overlapping intervals covering $\mathbb{R}$ as in fig. 3.
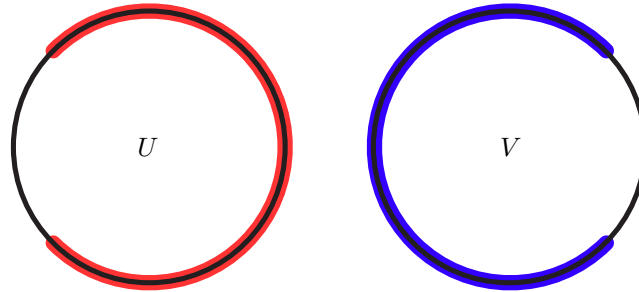
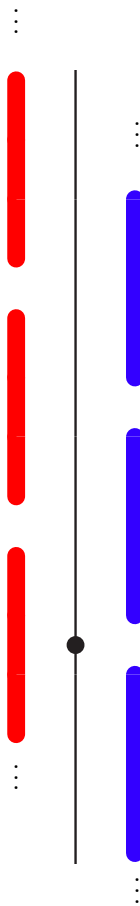FIGURE 2. An open cover for the circle is given by the two sets U and V.

FIGURE 3. The preimage of the cover U and V in $\mathbb{R}$ under the map p.

In particular

$$p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} \tilde{U}_n \qquad\qquad \tilde{U}_n = \left(n - \frac{3}{8}, n + \frac{3}{8}\right)$$

$$p^{-1}(V) = \bigcup_{n \in \mathbb{Z}} \tilde{V}_n \qquad\qquad \tilde{V}_n = \left(n + \frac{1}{8}, n + \frac{7}{8}\right) .$$

The restrictions

$$p|_{\tilde{U}_n} : \tilde{U}_n \to U \qquad\qquad p|_{\tilde{V}_n} : \tilde{V}_n \to V$$

are homeomorphisms for all n. This means if we try to lift the path from s_0 to s_1 we have no choice in the lift since it stays in U in S^1. So we get a unique lift. In detail, we define $\tilde{\sigma}$ inductively on $[0, s_k]$, $0 \le k \le m$. For $k = 0$, $\sigma(0) = 0$. So now suppose $\tilde{\sigma}$ is defined on $[0, s_{k-1}]$ for some k such that $1 \le k \le m$. Then WLOG

$$\sigma([s_{k-1}, s_k]) \subset U$$

(rather than V). This means $\tilde{\sigma}(s_{k-1}) \in \tilde{U}_r$ for some $r \in \mathbb{Z}$. Since $p|_{\tilde{U}_r}$ is a homeomorphism let $q_r : U \to \tilde{U}_r$ be $\left(p|_{\tilde{U}_r}\right)^{-1}$. Define

$$\tilde{\sigma}|_{[s_{k-1}, s_k]} = q_r \sigma|_{[s_{k-1}, s_k]}$$

and note this is the only choice. Therefore by induction $\tilde{\sigma}$ exists and is unique. $\square$

LEMMA 2.7 (Homotopy lifting). *Let $\sigma, \tau : I \to S^1$ be paths with $\sigma(0) = \tau(0) = 1$ and let $F : I \times I \to S^1$ be a homotopy from σ to τ (rel ∂I). (So $\sigma(1) = \tau(1)$). Then there exists a unique $\tilde{F} : I \times I \to \mathbb{R}$, a homotopy from $\tilde{\sigma}$ to $\tilde{\tau}$ (rel ∂I), i.e. the following diagram commutes:*

$$
\begin{array}{ccc}
 & & \mathbb{R} \\
 & \nearrow & \downarrow p \\
I \times I & \xrightarrow{\ F\ } & S^1
\end{array}
\quad .
$$
$\tilde{F}$

PROOF. This proof is very similar to the proof of lemma 2.6. $I \times I$ is compact so there exists some $0 < s_0 < s_1 < \ldots < s_m = 1$ and $0 = t_0 < t_1 < \ldots < t_n = 1$ such that if

$$R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$$

(for $1 \le i \le m$, $1 \le j \le n$) then $F(R_{ij}) \subset U$ or V. Order the $R_{i,j}$ and relabel them as R_k. Then let

$$S_k = \bigcup_{i \le k} R_i .$$

Now we define $\tilde{F}$ inductively on S_k. First $\tilde{F}(S_0) = \tilde{F}(0,0) = 0$. Now suppose $\tilde{F}$ is defined on S_{k-1}. Then $S_k = S_{k-1} \cup R_k$ and $F(R_k) \subset U$ or V; say U. Then $S_{k-1} \cap R_k$ is nonempty and connected and in U, and therefore

$$\tilde{F}(S_{k-1} \cap R_k) \subset \tilde{U}_r$$

for some $r \in \mathbb{Z}$. So we can (and must) define

$$\tilde{F}\Big|_{R_k} = q_r\, F|_{R_k} .$$

Therefore $\tilde{F}$ exists and is unique. $\tilde{F}_0$ is a lift of $F_0 = \sigma$, starting at 0, $\tilde{F}_1$ is a lift of $F_1 = \tau$, starting at 0, therefore by the uniqueness of lemma 2.6, $\tilde{F}_0 = \tilde{\sigma}$ and $\tilde{F}_1 = \tilde{\tau}$. Finally $p^{-1}(1) = \mathbb{Z}$ is discrete in $\mathbb{R}$, so $\tilde{F}_t(0) = \tilde{\sigma}(0) = 0$ and $\tilde{F}_t(1) = \tilde{\sigma}(1)$ for all $t \in I$, so indeed $\tilde{\sigma} \simeq_{F_t} \tilde{\tau} \,(\mathrm{rel}\,\partial I)$. $\qquad\square$

PROOF OF THEOREM 2.1. Let σ be a loop in S^1 at 1. Define $\varphi : \pi_1\left(S^1, 1\right) \to \mathbb{Z}$ by $\varphi\left([\sigma]\right) = \tilde{\sigma}(1)$. Then we claim:

(1) φ is well defined: This follows from lemmata 2.6 and 2.7.
(2) φ is onto: Let $n \in \mathbb{Z}$; define $\tilde{\sigma} : I \to \mathbb{R}$ by $\tilde{\sigma}(s) = ns$. Let $\sigma = p\tilde{\sigma}$. Then $\varphi\left([\sigma]\right) = \tilde{\sigma}(1) = n$.
(3) φ is a homomorphism: Let $[\sigma], [\tau] \in \pi_1\left(S^1, 1\right)$ with $\varphi\left([\sigma]\right) = m$, $\varphi\left([\tau]\right) = n$ and

$$[\sigma][\tau] = [\sigma * \tau] \ .$$

Then

$$\widetilde{\sigma * \tau} = \tilde{\sigma} * \hat{\tau}$$

where $\hat{\tau}(s) = \tilde{s} + m$ and therefore

$$\varphi\left([\sigma][\tau]\right) = \varphi\left([\sigma * \tau]\right) = \left(\widetilde{\sigma * \tau}\right)(1) = (\tilde{\sigma} * \hat{\tau})(1) = m + n = \varphi\left([\sigma]\right) + \varphi\left([\tau]\right)$$

as desired.

(4) φ is one-to-one: Suppose $\varphi\left([\sigma]\right) = 0$, i.e. $\tilde{\sigma}$ is a loop in $\mathbb{R}$ at 0. $\mathbb{R}$ strong deformation retracts to 0, so $\tilde{\sigma} \simeq x_0 \,(\mathrm{rel}\,\partial I)$. Therefore $[\sigma] = 1$.

$\qquad\square$

2. Induced homomorphisms

So consider some map $f : (X, x_0) \to (Y, y_0)$. Then the claim is that we can define a group homomorphism

$$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$$

by $f_*\left([\sigma]\right) = [f\sigma]$.

THEOREM 2.3.   (1) f_* is a well-defined homomorphism.
(2) $(gf)_* = g_* f_*$, $\mathrm{id}_* = \mathrm{id}$.
(3) $f \simeq g\,(\mathrm{rel}\,x_0)$ implies $f_* = g_*$.

PROOF.   (1) Suppose we have two loops $\sigma \simeq_{F_t} \sigma'\,(\mathrm{rel}\,\partial I)$. Then this means

$$f\sigma \simeq_{fF_t} f\sigma'\,(\mathrm{rel}\,\partial I)$$

which means f_* is well-defined.

(2)

$$\begin{aligned}
f_*\left([\sigma][\tau]\right) = f_*\left([\sigma * \tau]\right) &= [f(\sigma * \tau)] \\
&= [f(\sigma) * f(\tau)] = [f\sigma][f\tau] \\
&= f_*\left([\sigma]\right) f_*\left([\tau]\right) \ .
\end{aligned}$$

$\qquad\square$

EXAMPLE 2.2. Let $\mu_n(z) = z^n$ for $n \in \mathbb{Z}$. Then

$$\mu_{n*} : \pi_1\left(S^1, 1\right) \to \pi_1\left(S^1, 1\right)$$

is multiplication by n, i.e.

$$
\begin{array}{ccc}
\pi_1\left(S^1, 1\right) & \xrightarrow{\mu_{n*}} & \pi_1\left(S^1, 1\right) \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
\mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z}
\end{array}
$$

PROOF. The loop $\sigma : I \to S^1$ with $\sigma(s) = e^{2\pi i s}$ represents $1 \in \mathbb{Z} \simeq \pi_1\left(S^1, 1\right)$, i.e. $\varphi\left([\sigma]\right) = 1$. Then

$$\mu_{n*}\left([\sigma]\right) = [\mu_n \sigma]$$

where $(\mu_n \sigma)(s) = e^{2\pi i n s}$. Therefore $\varphi\left([\mu_n \sigma]\right) = n$. $\square$

The following is an application of Theorem 2.1.

THEOREM 2.4. *There is no retraction $D^2 \to S^1$.*

PROOF. Let $r : D^2 \to S^1$ be a retraction. Then

$$
\begin{array}{ccc}
 & D^2 & \\
{\scriptstyle i}\nearrow & & \searrow{\scriptstyle r} \\
S^1 & \xrightarrow{\;\;\mathrm{id}\;\;} & S^1
\end{array}
$$

commutes. Then we apply the functor π_1 to get:

$$
\begin{array}{ccccccc}
 & \pi_1\left(D^2, 1\right) & & & & 0 & \\
{\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} & & & & \searrow \\
\pi_1\left(S^1, 1\right) & \xrightarrow{\;\;\mathrm{id}_*\;\;} & \pi_1\left(S^1, 1\right) & & = & \mathbb{Z} \longrightarrow & \mathbb{Z}
\end{array}
$$

which is a contradiction. $\square$

COROLLARY 2.1. *Every map $D^2 \to D^2$ has a fixed point.*

3. Dependence of π_1 on the basepoint

Let $\alpha : I \to X$ be a path from x_0 to x_1. Define $\alpha_\# : \pi_1\left(X, x_0\right) \to \pi_1\left(X, x_1\right)$ by

$$\alpha_\#[\sigma] = [\bar{\alpha}][\sigma][\alpha] = [\bar{\alpha} * \sigma * \alpha] \ .$$

The point is that we take a loop at one basepoint and attach this path on both ends (with one reversed) to get a loop at the other basepoint.

THEOREM 2.5.   *(1) $\alpha_\#$ is an isomorphism.*
(2) $\alpha \simeq \beta \,(\mathrm{rel}\,\partial I)$ implies $\alpha_\# = \beta_\#$.
(3) $(\alpha * \beta)_\# = \beta_\# \alpha_\#$.
(4) If $f : X \to Y$ is a map then

$$
\begin{array}{ccc}
\pi_1\left(X, x_0\right) & \xrightarrow{\;f_*\;} & \pi_1\left(Y, f\left(x_0\right)\right) \\
\downarrow{\alpha_\#} & & \downarrow{(f\alpha)_\#} \\
\pi_1\left(X, x_1\right) & \xrightarrow{\;f_*\;} & \pi_1\left(Y, f\left(x_1\right)\right)
\end{array}
$$

commutes.

REMARK 2.3. Professor Cameron learned algebraic topology from Hilton and Wylie. He is surprised he learned anything at all because they decided that since homology is a covariant functor and cohomology is a contravariant functor homology should be cohomology and cohomology should be ''contrahomology''. They also wrote all of the maps on the right.

PROOF. (1) We can directly verify $\alpha_\#$ is a homomorphism:

$$\alpha_\# \left([\sigma] [\tau] \right) = [\bar\alpha] [\sigma] [\tau] [\alpha] = [\bar\alpha] [\sigma] [c_{x_0}] [\tau] [\alpha]$$
$$= [\bar\alpha] [\sigma] \left([\alpha] [\bar\alpha] \right) [\tau] [\alpha]$$
$$= \alpha_\# \left([\sigma] \right) \alpha_\# \left([\tau] \right) \ .$$

$\alpha_\#$ is a bijection because $(\bar\alpha)_\#$ is an inverse of $\alpha_\#$.
(2) Obvious.
(3) Exercise.
(4) Exercise.

$\square$

REMARK 2.4. If $x_1 = x_0$ then α is a loop at x_0 so

$$\alpha_\# \left([\sigma] \right) = [\bar\alpha] [\sigma] [\alpha] = [\alpha]^{-1} [\sigma] [\alpha]$$

i.e. $\alpha_\#$ is conjugation by α.

COROLLARY 2.2. *If X is path-connected then $\pi_1 (X, x_0)$ is independent of x_0 (up to isomorphism).*

DEFINITION 2.4. *X is simply connected if X is path-connected and $\pi_1 (X) = 1$.*

LEMMA 2.8. *Let $f, g : X \to Y$. Let F be a homotopy from f to g. Let α be the path $\alpha (t) = F (x_0, t)$ in Y from $f (x_0)$ to $g (x_0)$. Then the diagram*

$$
\begin{array}{ccc}
 & & \pi_1 (Y, f (x_0)) \\
 & \nearrow^{f_*} & \\
\pi_1 (X, x_0) & & \cong \Big\downarrow \alpha_\# \\
 & \searrow_{g_*} & \\
 & & \pi_1 (Y, g (x_0))
\end{array}
$$

commutes.

PROOF. Let σ be a loop in X at x_0. Let $H = F (\sigma \times \mathrm{id}) : I \times I \to Y$. Pictorially we have: $H|_{\partial (I \times I)}$ (suitably reparameterized) is a loop in Y at $g (x_0)$. Reading fig. 4 counterclockwise we get
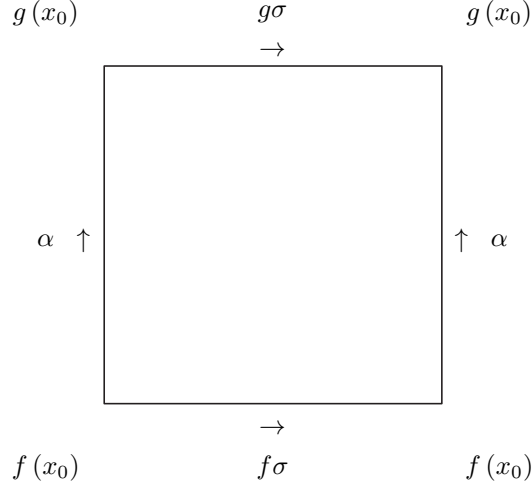
$$\left[H|_{\partial (I \times I)} \right] = [\bar\alpha] [f\sigma] [\alpha] [g\sigma]^{-1} \ .$$

But this extends over $I \times I$ so it is just $1 \in \pi_1 (Y, g (x_0))$, i.e. $\alpha_\# f_* \left([\sigma] \right) = g_* \left([\sigma] \right)$ which implies $\alpha_\# f_* = g_*$. $\square$

THEOREM 2.6. *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : \pi_1 (X, x_0) \to \pi_1 (Y, f (x_0))$ is an isomorphism for all $x_0 \in X$.*

PROOF. Let $g : Y \to X$ be a homotopy inverse of f. So we get maps

$$\pi_1 (X, x_0) \xrightarrow{\ f_*\ } \pi_1 (Y_* f (x_0)) \xrightarrow{\ g_*\ } \pi_1 (X, gf (x_0))$$

$$g\left(x_0\right) \qquad\qquad g\sigma \qquad\qquad g\left(x_0\right)$$

$$\rightarrow$$

$$\alpha \;\uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow\; \alpha$$

$$\rightarrow$$

$$f\left(x_0\right) \qquad\qquad f\sigma \qquad\qquad f\left(x_0\right)$$

FIGURE 4. $I \times I$ labelled as in lemma 2.8.

but we have $gf \simeq \mathrm{id}_X$ which means $(gf)_* = g_* f_*$ is an isomorphism. by lemma 2.8 and theorem 2.5 (1) so g_* is onto.

Then we have

$$\pi_1\left(Y, f\left(x_0\right)\right) \xrightarrow{\;g_*\;} \pi_1\left(X, gf\left(x_0\right)\right) \xrightarrow{\;f_*\;} \pi_1\left(Y, fgf\left(x_0\right)\right)$$

and $fg \simeq \mathrm{id}_Y$ so $f_* g_*$ is an isomorphism so g_* is one-to-one. Therefore g_* is an isomorphism and $g_* f_*$ is an isomorphism which implies f_* is an isomorphism. $\qquad\square$

COROLLARY 2.3. *IF X is contractible then $\pi_1\left(X, x_0\right) = 1$ for all $x_0 \in X$.*

PROOF. X being contractible is equivalent to $i : \{x_0\} \hookrightarrow X$ being a homotopy equivalence. $\qquad\square$

4. Fundamental theorem of algebra

THEOREM 2.7 (Fundamental theorem of algebra). *Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

REMARK 2.5. Professor Cameron says this topological proof of an algebraic statement should provide topologists some comfort in light of the Poincaré conjecture receiving some sort of proof via PDEs.

PROOF. Let $f(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_0$ for all $a_i \in \mathbb{C}$ for $n \geq 1$. Suppose $f(z) \neq 0$ for all $z \in \mathbb{C}$; so $a_0 \neq 0$. So f is a map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$.

The idea of the proof is as follows. As $|z| \to \infty$ somehow $f \rightsquigarrow z^n$, i.e. $f \rightsquigarrow \mu_n$ and similarly as $|z| \to 0$ somehow $f \rightsquigarrow a_0$, i.e. $f \rightsquigarrow c_{a_0}$ but this implies $\mu_n \simeq c_{a_0}$ which is a contradiction. Now we make this precise.

Let $R \in \mathbb{R}$, $R \geq 0$. Define $F : S^1 \times I \to S^1$ by

$$F(z, t) = F_t(z) = \frac{f(tRz)}{|f(tRz)|} \; .$$

Note that $F_0(z) = 1$ so $F_0 = c_1$.

Let
$$g(z) = a_{n-1}z^{n-1} + \ldots + a_0$$
so $f(z) = z^n + g(z)$. Choose
$$R > \max\left\{1, \sum_{k=0}^{n-1} |a_k|\right\} .$$
Then if $|z| = R$
$$|g(z)| \leq \sum_{k=0}^{n-1} \left|a_k z^k\right| = \sum_{k=0}^{n-1} |a_k| R^k \leq^{2.2} \left(\sum_{k=0}^{n-1} |a_k|\right) R^{n-1} < R^n = |z|^n .$$
Therefore for all $t \in I$
$$|z^m + tg(z)| \geq |z|^n - t|g(z)| \geq |z|^n - |g(z)| > 0 .$$
Let $h_t(z) = z^n + tg(z)$. So $h_0(z) = z^n$ and $h_1(z) = f(z)$.

So define $H : S^1 \times I \to S^1$ by
$$H(z,t) = \frac{h_t(Rz)}{h_t(Rz)}$$
and
$$H_1(z) = \frac{f(Rz)}{|f(Rz)|} = F_1(z) \qquad\qquad H_0(z) = \frac{(Rz)^n}{|Rz|^n} = z^n$$
and therefore $H_0 = \mu_n$.

Now we can put it all together. We have
$$c_1 = F_0 \simeq F_1 = H_1 \simeq H_0 = \mu_n$$
so $c_1 \simeq \mu_n$. Therefore by lemma 2.8

$$
\begin{array}{ccc}
& \pi_1\left(S^1, 1\right) & \\
\mu_{n*}\nearrow & \downarrow \alpha_\# & \times n \nearrow \quad \mathbb{Z} \\
\pi_1\left(S^1, 1\right) & = \quad \mathbb{Z} & \downarrow \simeq \\
c_{1*}\searrow & \downarrow & 0 \searrow \\
& \pi_1\left(S^1, 1\right) & \mathbb{Z}
\end{array}
$$

which is a contradiction. Note $\alpha_\#$ is conjugation by $[\alpha]$, but $\mathbb{Z}$ is abelian so this is just the identity. $\qquad\square$

5. Cartesian products

If $f : Z \to X$ and $g : Z \to Y$ are maps, let $(f,g) : Z \to X \times Y$ be the map given by
$$(f,g)(z) = (f(z), g(z)) .$$
We will use the same notation for group homomorphisms. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projections.

THEOREM 2.8. $(p_*, q_*) : \pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

PROOF. Exercise. $\qquad\square$

EXAMPLE 2.3. $\pi_1(T^n) \simeq \mathbb{Z}^n$.

[2.2]Since $R > 1$

6. Retractions

Consider some $i : A \hookrightarrow X$ and some $a_0 \in A$. Suppose there exists a retraction $r : X \to A$. Then we get a commutative diagram

$$
\begin{array}{ccc}
 & \pi_1\left(X, a_0\right) & \\
 & & \\
\pi_1\left(A, a_0\right) & \xrightarrow{\quad \text{id} \quad} & \pi_1\left(A, a_0\right)
\end{array}
$$

with i_* and r_*.

i.e. $r_* i_* = \text{id}$ so therefore i_* is one-to-one and r_* is onto.

CHAPTER 3

Van Kampen's theorem

The idea is to calculate the fundamental group of $X = X_1 \cup X_2$ from the data of the groups $\pi_1(X_1)$, $\pi_1(X_2)$, $\pi_1(X_1 \cap X_2)$ and the associated inclusion maps. These always fit into the diagram:

$$
\begin{array}{ccc}
\pi_1(X_1 \cap X_2) & \xrightarrow{\ i_{1*}\ } & \pi_1(X_1) \\
\Big\downarrow{\scriptstyle i_{2*}} & & \Big\downarrow{\scriptstyle j_{1*}} \\
\pi_1(X_2) & \xrightarrow{\ j_{2*}\ } & \pi_1(X)
\end{array}
\quad .
$$

1. Basic combinatorial group theory

1.1. Free products. Let A and B be groups. The free product is a group D and homomorphisms $i_A : A \to D$ and $i_B : B \to D$ such that given a group G and homomorphisms $\alpha : A \to G$ and $\beta : B \to G$, there exists a unique homomorphism $\varphi : D \to G$ such that $\varphi i_A = \alpha$ and $\varphi i_B = \beta$; i.e. the following push-out diagram commutes:

REMARK 3.1. This is the coproduct in the category **Grp**.

LEMMA 3.1. *If D exists then it is unique (up to unique isomorphism).*

PROOF. If D, D' are free products of A and B there exist unique φ, ψ as shown:

$\square$

Now we can write *the* free product of A and B as $A * B$, and say A and B are the *factors* of $A * B$.

THEOREM 3.1. *Free products exist.*

PROOF. Define $A * B$ to be the set of all sequences $x = (x_1 x_2 \ldots x_m)$ for $m \geq 0$ where $x_i \in A$ or B; $x_i \neq 1$; and x_i and x_{i+1} are in different factors. m is the *length* of x, 1 is the empty sequence.

Define the product of two sequence by

$$(x_1 \ldots x_m)(y_1 \ldots y_n) = \begin{cases} (x_1 \ldots x_m y_1 \ldots y_n) & x_m, y_1 \in \text{ different fac.} \\ (x_1 \ldots x_{m-1}(x_m y_1) y_2 \ldots y_m) & x_m, y_1 \in \text{ same fac.}, x_m y_1 \neq 1 \\ (x_1 \ldots x_{m-1})(y_2 \ldots y_n) & x_m, y_1 \in \text{ same fac.}, x_m y_1 = 1 \end{cases} .$$

Then $x1 = 1x = x$ for all $x \in A * B$. Define the inverse $x^{-1} = (x_m^{-1} \ldots x_1^{-1})$, then $xx^{-1} = 1 = x^{-1}x$ for all $x \in A * B$.

So we just have to check associativity. $x(yz) = (xy)z$. We induct on n (where n is the length of y). $n = 0$ is trivial. For $n = 1$, WLOG $y = (a)$ for $a \in A$, $a \neq 1$. Then there are several cases, e.g.

- $x_m \in B$, $z_1 \in B$;
- $x_m \in B$, $z_1 \in A$;
- $\ldots$

For $n > 1$, $y = y'y''$ where the length of y' and y'' are less than n. Then by induction

$$(xy)z = (x(y'y''))z = ((xy')y'')z = (xy')(y''z) .$$

On the other hand

$$x(yz) = x((y'y'')z) = x(y'(y''z)) = (xy'(y''z))$$

so $A * B$ is a group.

Now we see it has the universal properties. Define $i_A : A \to A*B$ by $i_A(a) = (a)$ and similarly for B. Given $\alpha : A \to G$ and $\beta : B \to G$ then define $\varphi : A * B \to G$ by

$$\varphi(x_1 \ldots x_m) = \gamma(x_1) \ldots \gamma(x_m)$$

where $\gamma_i = \alpha$ if $x_i \in A$ and $\gamma_i = \beta$ if $x_i \in B$. Therefore φ is unique as a homomorphism and it has the desired properties:

$$\varphi i_A = \alpha \qquad\qquad\qquad \varphi i_B = \beta .$$

$\square$

REMARK 3.2. i_A and i_B are actually injective in this case. For general pushouts they won't be. This means we can identify A and B with their images in $A * B$. In other words we can drop the parentheses. Each element in $A * B$ has a unique expression as $x_1 \ldots x_m$ for $x_i \in A$ or B; $x_i \neq 1$; x_i, x_{i+1} in different factors. This is called a *reduced word* in A and B. We say $A * B$ is a non-trivial free product if $A \neq \text{id} \neq B$.

EXAMPLE 3.1. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \{1, a, b, ab, ba, aba, bab, abab, \ldots\}$.

REMARK 3.3. A non-trivial free product is infinite: $a \in A \setminus \{1\}$, $b \in B \setminus \{1\}$, then ab has infinite order in $A * B$.

REMARK 3.4. We can define free product $* A_\lambda$ for any collection $\{A_\lambda\}$ of groups. As a special case we can take $A_\lambda = \mathbb{Z}$ for all λ, then write x_λ for the generator of A_λ. Then $* A_\lambda$ is the *free group on the set* $X = \{x_\lambda\}$, written $F(X)$.

Any function $\{x_{\lambda_0}\} \to G$ extends to a unique homomorphism $\mathbb{Z} = \mathbb{Z}(x_{\lambda_0}) \to G$. So $F(X)$ is characterized by the following. For any group G and any function $f : X \to G$, there exists unique homomorphism $\varphi : F(X) \to G$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & G \\
\downarrow{\scriptstyle i} & \nearrow{\scriptstyle \varphi} & \\
F(X) & &
\end{array}
$$

commutes.

Note that every element of $F(X)$ has a unique expression as a reduced word

$$x_{\lambda_1}^{n_1} x_{\lambda_2}^{n_2} \ldots x_{\lambda_m}^{n_m}$$

where $\lambda_i \neq \lambda_{i+1}$ and $n_i \in \mathbb{Z}$, $n_i \neq 0$.

DEFINITION 3.1. $(X : R)$ is a *presentation* of the group G if $R \subset F(X)$ and there exists an epimorphism $\varphi : F(X) \to G$ such that $\ker \varphi = N(R)$ is the normal closure of R in $F(X)$, the smallest normal subgroup of $F(X)$ that contains R, i.e. the set of all products of conjugates of elements in R and their inverses:

$$
\left\{ \prod_{i=1}^{k} u_i^{-1} r_i^{\epsilon_i} u_i \ \middle|\ u_i \in F(X), r_i \in R, \epsilon_i = \pm 1 \right\}.
$$

THEOREM 3.2. *Every group has a presentation.*

PROOF. Let $Y = \{y_\lambda\}$ be any set of generators of G. (Worst case we could take $Y = G$.) Let $X = \{x_\lambda\}$ and define a function $f : X \to G$ by mapping $f(x_\lambda) = y_\lambda$. f extends to a homomorphism $\varphi : F(X) \to G$ which is onto (since Y generates G). (Worst case take $R = \ker \varphi$.) $\square$

DEFINITION 3.2. G is *finitely generated* iff there exists a finite set of generators for G, iff G has a presentation $(X : R)$ with X finite.

G is *finitely presentable* iff there exists a presentation $(X : R)$ of G with X and R finite.

If $(X : R)$ is a presentation of G we say R is the set of relators, and X is the set of generators. We often suppress φ, i.e. we regard $X \subset G$, and write $r = 1$ ($r \in R$) to indicate that $\varphi(r) = 1 \in G$. We also write $G = \langle X : R \rangle$. Write $(\{x_1, \ldots\} : \{r_1, \ldots\})$ as $(x_1, \ldots : r_1, \ldots)$.

EXAMPLE 3.2. (1) $\langle X \mid \emptyset \rangle \cong F(X)$,
(2) $\langle \{x\} \mid x^n = 1 \rangle \cong \mathbb{Z}_{|n|}$,
(3) $\langle \{x, y\} \mid xy = yx \rangle \cong \mathbb{Z} \times \mathbb{Z}$,
(4) $\langle a, b \mid a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$

THEOREM 3.3. *If $X \cap Y = \emptyset$ then $\langle X \mid R \rangle * \langle Y \mid S \rangle \cong \langle X \cup Y \mid R \cup S \rangle$.*

EXAMPLE 3.3. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle a, b \mid a^2 = b^2 = 1 \rangle$.

EXERCISE 1.1. (1) Show $\langle x, y \mid x^{-1}yx = y^2, y^{-1}xy = x^2 \rangle$ is trivial.
(2) Is $\langle x, y, z \mid x^{-1}yx = y^2, y^{-1}zy = z^2, z^{-1}xz = x^2 \rangle$ trivial?
(3) Is $\langle x, y, z, w \mid x^{-1}yx = y^2, y^{-1}zy = z^2, z^{-1}wz = w^2, w^{-1}xw = x^2 \rangle$ trivial?
(4) Show $\langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle$ is trivial for $n \geq 0$.

These are called balanced presentations of the trivial group because the set of relators is the same size as the set of generators.

CONJECTURE 1. *Every balanced presentation can be taken to the trivial balanced presentation under some particular "moves".*

REMARK 3.5. Dealing with groups via their finite presentations is notoriously difficult. It is an unsolvable problem to find whether a group is trivial based only on a presentation. There is not algorithm which determines whether a group is trivial from its finite presentation.

REMARK 3.6. Novikov and Boone proved this unsolvability independently. Boone was at UIUC at the time. He had only one copy of his manuscript and when he biked home in the snow, he lost it. So he summoned his graduate students to find it.

1.2. Push-outs. Let A, B, and C be groups with homomorphisms $j_A : C \to A$, $j_B : C \to B$. Then a push-out of this diagram is a group D and homomorphisms $i_A : A \to D$ and $i_B : B \to D$ such that $i_A j_A = i_B j_B$, and given a group G and homomorphisms $\alpha : A \to G$, $\beta : B \to G$ such that $\alpha j_A = \beta j_B$ there exists a unique homomorphism $\varphi : D \to G$ such that $\varphi i_A = \alpha$ and $\varphi i_B = \beta$. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{j_A} & A \\
\downarrow{j_B} & & \downarrow{i_A} \\
B & \xrightarrow{i_B} & D
\end{array}
\quad
\begin{array}{c}
\alpha \\
\\
\varphi \\
\beta \\
G
\end{array}
\;.
$$

THEOREM 3.4. *Push-outs exist.*

PROOF. Consider $A * B$ (regard $A, B \subset A * B$). Let

$$
N = N \left(\left\{ j_A (c) \, j_b (c)^{-1} \,\middle|\, c \in C \right\} \right) \lhd A * B \;.
$$

Let $q : A * B \to A * B / N$ be quotient homomorphisms. Define $i_A = q|_A$, $i_B = q|_B$. Then $A * B / N$ is the pushout. □

REMARK 3.7. In the case $A * B$ (i.e. $C = 1$), i_A and i_B are injective. This is not true in an arbitrary pushout.

COUNTEREXAMPLE 1. Consider $\mathbb{Z}(y) \xleftarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}(x)$. Then the pushout is $\langle x, y \mid x = 1 \rangle \simeq \mathbb{Z}$:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}(x) \\
\downarrow{0} & & \downarrow{0} \\
\mathbb{Z}(y) & \xrightarrow{\mathrm{id}} & \mathbb{Z}
\end{array}
$$

so i_A is not injective.

If j_A and j_B are injective, then i_A and i_B are injective. The pushout is called the free product of A and B amalgamated along C, $A *_C B$.

EXAMPLE 3.4. Take $\mathbb{Z} \xleftarrow{q} \mathbb{Z} \xrightarrow{p} \mathbb{Z}$ for $p, q \geq 2$. Then the pushout is:

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\times p} & \mathbb{Z}(x) \\
\downarrow{\scriptstyle \times q} & & \downarrow \\
\mathbb{Z}(y) & \longrightarrow & \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}
\end{array}
$$

where $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle x, y \mid x^p = y^q \rangle$.

THEOREM 3.5. *Push-outs exist.*

PROOF. So we need to show that given $B \leftarrow C \rightarrow A$ we have:

$$
\begin{array}{ccc}
C & \xrightarrow{j_A} & A \\
\downarrow{\scriptstyle j_B} & & \downarrow{\scriptstyle i_A} \\
B & \xrightarrow{i_B} & D
\end{array}
$$

with maps α, β to G and dashed $\varphi : D \to G$.

Define

$$
N = N\left(\left\{ j_A(c) j_b(c)^{-1} \, \forall c \in C \right\} \right) \triangleleft A * B \ .
$$

where we regard $A, B < A * B$. Let $q : A * B \to A * B/N$ be the quotient homomorphism and let $i_A = q|_A$ and $i_B = q|_B$. Then the diagram certainly commutes.

CLAIM 3.1. This is the push-out.

PROOF. Given $\alpha : A \to G$ and $\beta : B \to G$ such that $\alpha j_A = \beta j_B : C \to G$ there exists $\psi : A * B \to G$ such that $\psi|_A = \alpha$ and $\psi|_B = \beta$ by theorem 3.1.

Now $\alpha j_A = \beta j_B$ implies that $\psi j_A = \psi j_B$, which means $\psi\left(j_A(c) j_B(c)^{-1} \right) = 1$ for all $c \in C$, which means $\psi(N) = 1$. This means ψ induces $\varphi : A * B/N \to G$, i.e.

$$
\begin{array}{ccc}
A * B & \xrightarrow{\quad q \quad} & A * B/N \\
& {\scriptstyle \psi} \searrow \quad \swarrow {\scriptstyle \varphi} & \\
& G &
\end{array}
$$

commutes. I.e. $\varphi \, q|_A = \alpha$ and $\varphi q|_B = \beta$, i.e. $\varphi i_A = \alpha$ and $\varphi i_B = \beta$. Then $A \cup B$ generates $A * B$, and therefore $q(A) \cup q(B)$ generates $A * B/N$. Then φ is determined by $\varphi|_{q(A)}$ and $\varphi|_{q(B)}$. But we have to define $\varphi q|_A = \alpha$ and $\varphi q|_A = \beta$ and therefore φ is unique. $\qquad \square$

$\blacksquare$

REMARK 3.8. It follows from theorem 3.3 that if $A = \langle X \mid R \rangle$ and $B = \langle Y \mid S \rangle$ then the pushout it

$$
\langle X \cup Y \mid R \cup S, j_A(c) = j_B(c), \forall c \in C \rangle \ .
$$

We could also replace C by any set generating C.

REMARK 3.9. Push-outs are defined and exist for $\{A_\lambda, j_\lambda : C \to A_\lambda\}$.
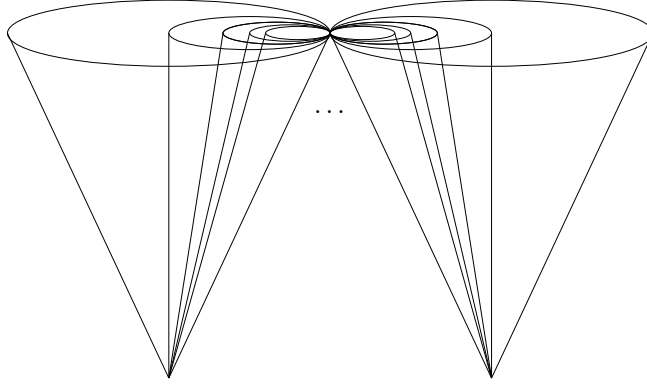
FIGURE 1. The cone of two Hawaiian earrings joined at a point. This has nontrivial fundamental group even though it is the union of two things with trivial fundamental group.

2. Statement of the theorem

THEOREM 3.6 (Van Kampen's theorem). *Suppose $X = X_1 \cup X_2$, with X_1, X_2, $X_1 \cap X_2$ open and path-connected. Let $x_0 \in X_1 \cap X_2$ Then $\pi_1(X, x_0)$ is the push-out:*

$$
\begin{array}{ccc}
\pi_1(X_1 \cap X_2, x_0) & \longrightarrow & \pi_1(X_1, x_0) \\
\downarrow & & \downarrow \\
\pi_1(X_2, x_0) & \longrightarrow & \pi_1(X, x_0)
\end{array}
$$

where all homomorphisms are induced by inclusions.

REMARK 3.10.   (1) There exists an analogous statement when $X = \cup_\lambda X_\lambda$.
  (2) In practice, one often applies theorem 3.6 when X_1 and X_2 are closed in X but where X_i has an open subsets $U_i \subset X$ such that there exists a strong deformation retractions of pairs:

$$(U_1, U_1 \cap X_2) \to (X_1, X_1 \cap X_2) \qquad (U_2, U_2 \cap X_1) \to (X_2, X_1 \cap X_2) \ .$$

  (3) Theorem 3.6 is false without the openness assumption. The counterexample is the double cone of the wedge of two Hawaiian earrings as in fig. 1. If we take each cone of an earring to be one of the X_i then the intersection is a point, so it satisfies all of the assumptions except the openness. But notice that each of the X_i is contractible since cones are always contractible. However $\pi_1(X_1 \cup X_2) \neq 1$ because the two earrings themselves form a nontrivial loop.

COROLLARY 3.1. *Let X_1, X_2 be as in theorem 3.6. Then $\pi_1(X_1) = \pi_1(X_2) = 1$ which implies $\pi_1(X) = 1$.*

COROLLARY 3.2. *If $n \geq 2$ then $\pi_1(S^n) = 1$.*

PROOF. We can write $S^n = D_+^n \cup D_-^n$ where $D_+ \cap D_- \cong S^{n-1}$. For $n \geq 2$ S^{n-1} is path-connected and D^n is contractible, so $\pi_1(D_+) = \pi_1(D_-) = 1$.   □

COROLLARY 3.3. *If $\pi_1(X_1 \cap X_2) = 1$ then $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2)$.*

3. Cell complexes

Let Y be a topological space and $f : S^{n-1} \to Y$ a map. Then the quotient space
$$X = (D^n \amalg Y) / \left(x \mathrm{im} f(x), \forall x \in S^{n-1}\right)$$
is *obtained from Y by attaching an n cell*, and f is the *attaching map*.

Note that X has the quotient topology, i.e. if $q : D^n \amalg Y \to X$ is the quotient map, then $U \subset X$ is open iff $q^{-1}(U) \subset D^n \amalg Y$ is open. Also note that one can similarly add multiple n-cells to Y via maps $f_\lambda : S^{n-1}_\lambda \to Y$.

An *n-complex* (or n-dimensional CW complex) is defined inductively by:

- a -1-complex is $\emptyset$,
- an n-complex is a space obtained by attached a collection of n-cells to an $n-1$-complex so a 0-complex is a discrete set of points.

An n-complex is finite if there exists only finitely many cells. Note that we can define an inf-dimensional CW-complex by setting
$$X = \cup_{n=0}^{\infty} X_n$$
for X_n an n-complex such that $X_i \subset X_{i+1}$ for all i. Note that $U \subset X$ is open iff $U \cap X_n$ is open for all n.

For X a cell-complex define the *n-skeleton of X*, $X^{(n)}$, to be the union of all cells of dimension $\leq n$.

EXAMPLE 3.5. The same space might have multiple cell decompositions. Consider S^2. We can think of this as a cell-complex as being one 0-cell, one 1-cell, and two 2-cells (hemispheres). A more simple way of seeing this is one 0-cell and a 2-cell with the entire boundary mapping to the point.

3.1. Effect of adding a cell on the fundamental group.

EXAMPLE 3.6. Consider $X = Y \cup$ some 2-cell, i.e. $X = Y \cup_f D^2$ where f is the attaching map. Let $u \in D^2$. Then define $X_1 = Y \cup_f \left(D^2 \setminus \{0\}\right)$ and $X_2 = \mathrm{int}\left(D^2\right) \subset X$. This means X_1 and X_2 are open in X, $X = X_1 \cup X_2$, and
$$X_1 \cap X_2 = \mathrm{int}\left(D^2\right) \setminus \{0\} \simeq S^1 \times \mathbb{R} \simeq S^1 .$$
By Van Kampen $\pi_1(X)$ is the push-out of
$$\mathbb{Z} \simeq \pi_1(X_1 \cap X_2) \xrightarrow{j_{2*}} \pi_1(X_1)$$

.

$$\pi_1(X_2) = 1$$

Let $w \in \pi_1(Y) = \pi_1(Y, f(1))$ be the element represented by $f : S^1 \to Y$. We have a strong deformation retraction $D^2 \setminus \{0\} \to S^1$, which induces a strong deformation retraction $X_1 \to Y$ so $j_{1*}(z) = w$ and
$$\boxed{\pi_1(X) / N(w)} .$$

Similarly, if we attach a collection of 2-cells we get $\pi_1(X) / N(\{w_\lambda\})$.

REMARK 3.11. The same argument shows, using the fact that $\pi_1\left(S^{n-1}\right) = 1$ ($n \geq 3$) that if $X = Y \cup$ some n-cells then the inclusion map $\pi_1(Y) \to \pi_1(X)$ is an isomorphism.

EXAMPLE 3.7. Let
$$X = \bigvee_\lambda S_\lambda \ .$$
We can consider the open set around the common point:
$$U = \bigcup_\lambda I_\lambda \subset X$$
where each I_λ is an open interval in the S_λ around the common point. So attach this U to each circle:
$$X_\lambda = S_\lambda^1 \cup U$$
and then apply Van Kampen's theorem to get:
$$\pi_1(X) = \underset{\lambda}{*}\, \pi_1(X_\lambda) == \underset{\lambda}{*}\, \mathbb{Z} = F(\{x_\lambda\}) \ .$$

EXAMPLE 3.8 (Surfaces). $\pi_1\left(S^2 = 1\right)$. Let
$$P^2 = S^2 / \left\{x \sim -x \forall x \in S^2\right\} \ .$$
This is a Möbius band M with a disk D attached. Recall $\pi_1(B) \simeq \mathbb{Z}$ so this disk somehow kills classes such that
$$\pi_1\left(P^2\right) = \left\langle x \,\middle|\, x^2 = 1\right\rangle \simeq \mathbb{Z}/2\mathbb{Z} \ .$$

Consider the wedge of two circles (a and b). We can build a Torus by attaching a disk, so we get
$$\pi_1\left(T^2\right) = \left\langle a, b \,\middle|\, aba^{-1}b^{-1} = 1\right\rangle \cong \mathbb{Z}^2 \ .$$
For the Klein bottle[3.1] we get $\left\langle a, b \,\middle|\, aba^{-1}b = 1\right\rangle$.

[3.1] The name of the Klein bottle is actually a joke in German based on the German words for surfaces and bottles.