LECTURE 2

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1. Homotopy

Recall:

Definition 1. Let $f, g : X \to Y$. Say f is homotopic to g iff there exists a map $F : X \times I \to Y$ such that for all $x \in X$

$$F(x, 0) = f(x) \qquad\qquad F(x, 1) = g(x) \ .$$

In this case we say F is a homotopy from f to g and write $f \simeq_F g$.

Lemma 1. *Homotopy is an equivalence relation on the set of all maps $X \to Y$.*

Proof. (i) $f \simeq f$: Let F be the constant homotopy, i.e. $F(x, y) = f(x)$ for all $t \in I$ and for all $x \in X$.

(ii) $f \simeq g \implies g \simeq f$: Suppose $f \simeq_F g$. Let $\bar{F}$ be the reverse homotopy $\bar{F}(x, t) = F(x, 1 - t)$. Then $g \simeq_{\bar{F}} f$.

(iii) $f \simeq g$, $g \simeq h \implies f \simeq h$: Define $H : X \times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq 1/2 \\ G(x, 2t - 1) & 1/2 \leq t \leq 1 \end{cases} \ .$$

Then by exercise 2 on homework 1 H is continuous and therefore $f \simeq_H h$. □

Lemma 2 (compositions of homotopic maps are homotopic)**.** *If $f \simeq f' : X \to Y$, $g \simeq g' : Y \to Z$, then $gf \simeq g'f' : X \to Z$.*

Proof. Suppose $f \simeq_F f'$, $g \simeq_G g'$. Then the composition

$$X \times I \xrightarrow{F} Y \xrightarrow{g} Z$$

is a homotopy from gf to gf'. Then the composition

$$X \times I \xrightarrow{f' \times \mathrm{id}} Y \times I \xrightarrow{G} Z$$

is a homotopy from gf' to $g'f'$. By transitivity of equivalence relations from lemma 1. □

Let $[X, Y]$ be the set of homotopy classes of maps from $X \to Y$.

Remark 1. We should probably assume $X \neq \emptyset \neq Y$.

Lemma 2 then tells us that composition defines a function

$$[X, Y] \times [Y, Z] \to [X, Z] \ .$$

Date: September 3, 2019.

1.1. Homotopy equivalence.

Definition 2. X is *homotopy equivalent* to Y (or *of the same homotopy type* as Y) written $X \simeq Y$ if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. Then we say f is a homotopy equivalence and g is a homotopy inverse of f.

Example 1. A homeomorphism is a homotopy equivalence, but in general this is much weaker.

Lemma 3. *Homotopy equivalence is an equivalence relation.*

Proof.
- (i) $X \simeq X$: $f = g = \mathrm{id}_X$, lemma 1.
- (ii) $X \simeq Y \implies Y \simeq X$: by definition.
- (iii) $X \simeq Y$, $Y \simeq Z \implies X \simeq Z$: So we have

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f'"] \end{tikzcd}$$

where $f'f \simeq \mathrm{id}_X$, $ff' \simeq \mathrm{id}_Y$ and similarly

$$\begin{tikzcd} Y \arrow[r, bend left, "g"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

where $g'g \simeq \mathrm{id}_X$, $gg' \simeq \mathrm{id}_Y$. Then we compose:

$$\begin{tikzcd} X \arrow[r, bend left, "gf"] & Y \arrow[l, bend left, "f'g'"] \end{tikzcd}$$

and we have that

$$(f'g')\, gf = f'\,(g'g)\, f \simeq f'\,\mathrm{id}_Y\, f = f'f \simeq \mathrm{id}_X$$

(from lemma 2) and similarly

$$(gf)\,(f'g') \simeq \mathrm{id}_X$$

so $gf : X \to Z$ is a homotopy equivalence.

□

Lemma 4. $X \simeq X'$, $Y \simeq Y' \implies X \times Y \simeq X' \times Y'$.

Proof. (exercise) □

Remark 2. Many functors in algebraic topology (e.g. π_1, H_n, ...) have the property that $f \simeq g \implies f_* = g_*$. In other words they factor through the homotopy category where the objects are topological spaces and morphisms are just homotopy classes of maps:

$$\begin{tikzcd} & \mathbf{Top}/\simeq \arrow[dr, dashed] & \\ \mathbf{Top} \arrow[ur] \arrow[rr] & & \{\text{Algebraic objects, morphisms}\} \end{tikzcd} .$$

$$f : X \to Y \longmapsto f_* : A(X) \to A(Y)$$

Definition 3. Let $A \subset X$, $f, g : X \to Y$ such that $f|_A = g|_A$. Then $f \simeq g$ (rel. A) iff there exists $f \simeq_F g$ such that

$$F_t|_A = f|_A \left(= g|_A\right)$$

for all $t \in I$.

Homotopy equivalence rel. A is an equivalence relation on the set

$$\{f : X \to Y \mid f|_A \text{ is a fixed map.}\}$$

Let $i : A \to X$ be the inclusion. Then i being a homotopy equivalence means there exists some $f : X \to A$ such that

(i) $if \simeq \mathrm{id}_X$, and
(ii) $fi \simeq \mathrm{id}_A$.

Now we can strengthen this in multiple ways.

Definition 4. If we strengthen (ii) to say $fi = \mathrm{id}_A$ (i.e. f is a retraction) then $\mathrm{id}_X \simeq_F if$ is a *deformation retraction of X onto A*. In other words we have

$$F : X \times I \to X$$

such that $F_0 = \mathrm{id}_X$, $F_1(x) \in A$ for all $x \in A$, and $F_1|_A = A$ (since $F_1 = if$).

Definition 5. If in addition we strengthen (i) to say that $\mathrm{id}_X \simeq_F if$ (rel A) then F is a strong deformation retraction (of X onto A). In other words we have $F : X \times I \to X$ such that

$$F_0 = \mathrm{id}_X \qquad \forall x \in X, F_1(x) \in A \qquad \forall t \in I, \forall a \in A, F_t(a) = a\ .$$

The idea here is that X strong deformation retracts to A implies X deformation retracts to A which implies $iA \hookrightarrow X$ is a homotopy equivalence. As it will turn out, both of these implications are strict.

Example 2. $X \times \{0\}$ is a strong deformation retract of $X \times I$:

$$F : (X \times I) \times I \to X \times I$$

where $F((x, s), t) = (x, (1 - t)s)$.

Example 3. $\mathbb{R}^n \setminus \{0\}$ strong deformation retracts to S^{n-1}.

Example 4. $A = S^1 \times [-1, 1]$ strong deformation retracts to $S^1 \times \{0\}$. A Möbius band B also strong deformation retracts to S^1. Therefore $A \simeq B$.

Example 5. Let X be a twice punctured disk. Then it is sort of clear that this strong deformation retracts to

(i) the boundary along with one arc passing between the punctures,
(ii) the wedge of two circles, and
(iii) two circles connected by an interval.

This means these three are all homotopy equivalent.

Example 6. Consider a once-punctured torus $X = T^2 \setminus \mathrm{int}\left(D^2\right)$. This strong deformation retracts to the wedge of two circles. This tells us that the once punctured torus is actually homotopy equivalent to the disk with two punctures.

These examples show that homotopy equivalence does not imply homeomorphism, even for surfaces with boundary. We can immediately see these examples are homeomorphic because there's no way for the boundaries to be mapped to one another.

1.2. **Manifolds.** Topological spaces can be very wild, but manifolds are usually quite nice.

Definition 6. An *n-dimensional manifold* is a Hausdorff, second countable space M such that every $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Example 7. $\mathbb{R}^n$, S^n, $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$.

Definition 7. A manifold is *closed* iff it is compact.

Definition 8. A surface is a closed 2-manifold.

Example 8. We have all of the two-sided or orientable surfaces (S^2, T^2, ...) and then non orientable ones like $\mathbb{P}^2$ and the Klein bottle. As it turns out this is a complete list up to homeomorphism.

For any two spaces $X \cong$ implies $X \simeq$. As we have seen, the converse isn't even true for surfaces with boundary. But for closed manifolds, there are many interesting cases where it is true:

(1) For M, M' closed surfaces, $M \simeq M' \implies M \cong M'$.
(2) For M an n-manifold, $M \simeq S^n \implies M \cong S^n$. (Generalized Poincaré conjecture)

For $n = 0, 1, 2$ this is not so bad. For $n \geq 5$, Connell and Newman independently proved it. The next step was proving this for $n = 4$. This was proved by Freedman. Finally for $n = 3$ Perelman proved this. The smooth 4-dimensional version is still open. I.e. the statement for diffeomorphism.

Remark 3. Poincaré originally posed this conjecture as saying that having the same homology as S^n was sufficient. But he discovered a counterexample, now called the Poincaré homology sphere. This shares homology with S^3 but has different fundamental group. This is in fact why he invented the fundamental group.

Definition 9. $f : X \to Y$ is a *constant map* if there is some $y_0 \in Y$ such that for all $x \in X$ $f(x) = y_0$. We say $f = x_{y_0}$.

Definition 10. A map $f : X \to Y$ is *null-homotopic* iff $f \simeq$ a constant map.

Definition 11. X is *contractible* if id_X is null-homotopic. I.e. there exists some $x_0 \in X$ such that X deformation retracts to x_0.

Example 9. (1) Any nonempty convex[1] subspace of $\mathbb{R}^n$ is contractible. Choose an arbitrary point $x_0 \in X$. Define $F : X \times I \to X$ by
$$F(x,t) = (1-t)\,x + tx_0 \ .$$
Then $\mathrm{id}_X \simeq_F c_{x_0}$. In fact this is a strong deformation retraction. Sometimes you can do this for some point but not all, and for some you can't do it for any points.
(2) S^1 is not contractible.

[1] Recall $X \subset \mathbb{R}^n$ is convex if $x, y \in X$ implies $tx + (1-t)\,y \in X$ for all $t \in I$.