Algebraic Topology

Lectures: Professor Cameron Gordon
Notes: Jackson Van Dyke

Contents

Introduction

We won't follow the book line by line, but it will be the reference for the course. Read the first three pages of chapter 0, talking about homotopy. We will really get started on chapter 1, section 1.1, about the fundamental group.

Lecture 1;
August 29, 2019

1. Introduction

Today will be an introductory account of what algebraic topology actually is. In topology the objects of interest are topological spaces where the natural equivalence relation is a homeomorphism, i.e. a bijection $f : X \to Y$ such that f and f^{-1} are continuous. Somehow the goal is classifying topological spaces up to homeomorphism, so the basic question is somehow:

QUESTION 1. Given topological spaces X and Y, is $X \cong Y$.

In these terms, algebraic topology is somehow a way of translating this into an algebraic question. More specifically, *algebraic topology* is the construction and study of functors from **Top** to some categories of algebraic objects (e.g. groups [0.1], abelian groups, vector spaces, rings, modules, ...). Recall this means we have a map from topological spaces $X \to A(X)$ for some algebraic object $A(X)$. In addition, for every $f : X \to Y$ we get a morphism $f_* : A(X) \to A(Y)$. Then these have to satisfy the conditions that

$$(gf)_* = g_* f_* \ , \qquad\qquad (\mathrm{id})_* = \mathrm{id} \ .$$

EXERCISE 1.1. Show that $X \cong Y$ implies that $A(X) \cong A(Y)$.

EXAMPLE 0.1 (Fundamental group). Let X be a topological space. We will construct a group $\pi_1(X)$.

[0.1] Once Professor Gordon was giving a job talk about knot cobordisms. As it turns out these form a semigroup rather than a group. But if you add some sort of 4 dimensional equiv relation you get an honest group. So he was going on about how semigroups aren't so useful. After the talk he found out the chairman of the department worked on semigroups.

Example 0.2 (Higher homotopy groups). There is also something, $\pi_n(X)$, called the nth homotopy group. As it turns out for $n \geq 2$ this is abelian.

Example 0.3 (Singular homology). We will define abelian groups $H_n(X)$ (for $n \geq 0$) called the nth singular homology group.

We will also define real vector spaces $H_n(X;\mathbb{R})$ for $n \geq 0$ which are the nth singular homology with coefficients in $\mathbb{R}$.

Example 0.4 (Cohomology). We will also have the nth (singular) cohomology rings $H^*(X)$. These is actually a graded ring.

Warning 0.1. Above we actually should have said we're dealign with what are *covariant* functors, but in this case we are actually dealing with a *contravariant* functor. This just means we have:

$$f : X \to Y \rightsquigarrow f^* : H^*(Y) \to H^*(X) \ .$$

Remark 0.1. The point here is that problems about topological spaces and maps are "continuous" and "hard". But on the algebraic side these problems become somehow "discrete" and "easy".

2. A bit more specific

Recall in $\mathbb{R}^n$ we define:

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \qquad S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

Example 0.5. Two examples of surfaces are S^2 and $T^2 = S^1 \times S^1$. They clearly aren't homeomorphic, but how are we supposed to prove such a fact? We will see that $\pi_1(S^2) = 1$ whereas $\pi^1(T^2) = \mathbb{Z} \times \mathbb{Z}$. Since these are not isomorphic, the spaces cannot be homeomorphic.

2.1. Retraction. Let $A \subset X$ be a space and a subspace.

Definition 0.1. A *retraction* from X to A is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$, i.e. the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ A \arrow[ur, hook, "i"] \arrow[rr, "\mathrm{id}"] & & A \end{tikzcd} \ .$$

Note that r is certainly surjective since id is.

Example 0.6. If X is any nonempty space, $x_0 \in X$, define $r : X \to \{x_0\}$ as $r(x) = x_0$. So every nonempty space always retract onto a point.

Example 0.7. Think of $A \subset A \times B$ by fixing some $b_0 \in B$ and sending

$$\begin{aligned} A &\hookrightarrow B \\ &. \\ a &\longmapsto (a, b_0) \end{aligned}$$

Then $r : A \to B \to$ defined by $r(a, b) = a$ is a retraction.

Recall that for $f : X \to Y$ for X path connected, then $f(X)$ is also path connected. Recall that $D^1 = [-1, 1] \subset \mathbb{R}$ is path connected, whereas $S^0 = \{\pm 1\}$ is not. Therefore there cannot be a retraction $D^1 \to S^0$. This is a basic fact, but it motivates a more general statement which is not so clear.

Suppose there exists a retraction $r : D^1 \to S^0$. Then this means the following diagram commutes:

$$\begin{tikzcd} & D^1 \arrow[dr, "r"] & \\ S^0 \arrow[ur, hook] \arrow[rr, "\mathrm{id}"] & & S^0 \end{tikzcd} .$$

If we apply the functor H_0, we will see that

$$H_0(X) \cong \bigoplus_{\text{path components of } X} \mathbb{Z} .$$

So if we apply H_0 to the diagram we get:

$$\begin{tikzcd} & H_0(D^1) \arrow[dr, "r_*"] & \\ H_0(S^0) \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & H_0(S^0) \end{tikzcd} = \begin{tikzcd} & \mathbb{Z} \arrow[dr, "r_*"] & \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd}$$

but this is clearly impossible.

In the same way we will see the much harder fact:

Fact 1 (Brouwer). *There does not exist a retraction $D^n \to S^{n-1}$ (for $n \geq 2$).*

We will see this by applying H_{n-1}. The idea is that

$$H_{n-1}(D^n) = 0 \qquad H_{n-1}\left(S^{n-1}\right) = \mathbb{Z}$$

which means we would have the diagram:

$$\begin{tikzcd} & 0 \arrow[dr] & \\ \mathbb{Z} \arrow[ur] \arrow[rr, "\mathrm{id}"] & & \mathbb{Z} \end{tikzcd}$$

which is a contradiction.

This turns out to imply the famous:

THEOREM 0.1 (Brouwer fixed point theorem). *Every map $f : D^n \to D^n$ $(n \geq 1)$ has a fixed point (i.e. a point $x \in D^n$ such that $f(x) = x$). In this case one says that D^n has the* fixed point property *(FPP).*

PROOF. Suppose there exists an $f : D^n \to D^n$ such that $\forall x \in D^n$ $f(x) \neq x$. Now draw a straight line from x to $f(x)$ and continue it to the boundary S^{n-1}. Call this point $g(x)$. Then this defines a map $g : D^n \to S^{n-1}$. g is continuous and $g|_{S^{n-1}} = \mathrm{id}$. Therefore g is a retraction $D^n \to S^{n-1}$ which we saw cannot exist. □

2.2. Dimension. We know $\mathbb{R}^n$ somehow has dimension n. But what does this really mean? The intuition is that $\mathbb{R}^2$ somehow has more points than $\mathbb{R}$. But then in 1877 Cantor proved that there is in fact a bijection $\mathbb{R} \to \mathbb{R}^2$. But this is highly non-continuous, so this tells us continuity should have something to do with it. But then in 1890 Peano showed that there exists a continuous suggestion $\mathbb{R} \to \mathbb{R}^2$ as well. In 1910, using homology, Brouwer proved:[0.2]

THEOREM 0.2. *For $m < n$ $\nexists$ continuous injection $\mathbb{R}^n \to \mathbb{R}^m$.*

We will prove this. A corollary of this is the famous invariance of dimension. I.e. $\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$. The proof uses separation properties of $n-1$-sphere in $\mathbb{R}^n$, thus in turn uses H_*.

EXERCISE 2.1. Find an easy proof for $m = 1$.

THEOREM 0.3 (Jordan curve theorem). *For a subset $C \subset \mathbb{R}^2$ such that $C \simeq S^1$ then $\mathbb{R}^2 \setminus C$ has exactly 2 components A and B. In addition $C = Fr(A) = Fr(B)$. (Recall the* frontier *is defined as $Fr(X) = \overline{X} \cap \overline{(Y \setminus X)}$ for any $X \subset Y$.)*

THEOREM 0.4 (Schönflies). *Let A, B, and C be as above and assume A is a bounded component of $\mathbb{R}^2 \setminus X$. Then $\overline{A} \cong D^2$.*

The higher dimensional analog of JCT is true Brouwer). The proof will use H_*. The generalization is that for $\Sigma \subset \mathbb{R}^n$ then $\Sigma \cong S^{n-1}$ As it turns out the higher dimensional analog of the Schönflies theorem is false. The counterexample is the famous Alexander horned space. I.e. there exists $\Sigma \subset \mathbb{R}^3$, $\Sigma \cong S^2$; A and bounded component of $\mathbb{R}^2 \setminus \Sigma$ and $\overline{A} \ncong D^3$.

Recall

[0.2] When he proved this Lebesgue contacted him saying that he could prove it too. So he sent him his proof, and Brouwer saw some errors. So over many years he eventually corrected it. In the end Brouwer summarized it by saying that it was really just his own proof.

THEOREM 0.5 (Heine-Borel). *$X \subset \mathbb{R}^n$ compact is equivalent to X being closed and bounded.*

So consider $\emptyset \neq X \subset \mathbb{R}$ compact and connected. This is equivalent to being an interval $X = [a, b]$ $(a \leq b)$. In $\mathbb{R}^2$ things get much worse. In particular there exists compact, connected subset such that $\mathbb{R}^2 \setminus X$ has exactly three components A, B, and C, and in particular every neighborhood of every point in X meets all three components. This is known as the 'lakes of Wada'. We start with an island with two lakes, and then start digging canals which somehow get closer and closer to the lakes. In the end every point of what is left is arbitrarily close to all three lakes.

QUESTION 2 (Open question). Let X be a compact connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus X$ is connected. Does X have the fixed point property?

CHAPTER 1

Homotopy

Let $I = [0, 1]$.

Definition 1.1. Let $f, g : X \to Y$. Say f is homotopic to g (and write $f \simeq g$) iff there exists a map $F : X \times I \to Y$ such that for all $x \in X$

$$F(x, 0) = f(x) \qquad\qquad F(x, 1) = g(x) \ .$$

Define $F_t : X \to Y$ by $F_t(x) = F(x, t)$. Then F_t is a continuous 1-parameter family of maps $X \to Y$ such that $F_0 = f$ and $F_1 = g$.

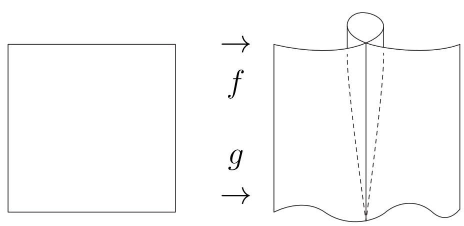

Figure 1.

Example 1.1. Let $F : S^{n-1} \times I \to \mathbb{R}^n$ be defined by $F(x, t) = (1 - t)x$. Then F_0 is the inclusion inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$ and F_1 is the constant map $S^{n-1} \to$ origin.

Lecture 2; September 3, 2019

Lemma 1.1. *Homotopy is an equivalence relation on the set of all maps $X \to Y$.*

Proof.
- (i) $f \simeq f$: Let F be the constant homotopy, i.e. $F(x, y) = f(x)$ for all $t \in I$ and for all $x \in X$.
- (ii) $f \simeq g \implies g \simeq f$: Suppose $f \simeq_F g$. Let $\bar{F}$ be the reverse homotopy $\bar{F}(x, t) = F(x, 1 - t)$. Then $g \simeq_{\bar{F}} f$.
- (iii) $f \simeq g$, $g \simeq h \implies f \simeq h$: Define $H : X \times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \le t \le 1/2 \\ G(x, 2t - 1) & 1/2 \le t \le 1 \end{cases} .$$

Then by exercise 2 on homework 1 H is continuous and therefore $f \simeq_H h$.

$\square$

LEMMA 1.2 (compositions of homotopic maps are homotopic). *If* $f \simeq f' : X \to Y$, $g \simeq g' : Y \to Z$, *then* $gf \simeq g'f' : X \to Z$.

PROOF. Suppose $f \simeq_F f'$, $g \simeq_G g'$. Then the composition

$$X \times I \xrightarrow{F} Y \xrightarrow{g} Z$$

is a homotopy from gf to gf'. Then the composition

$$X \times I \xrightarrow{f' \times \mathrm{id}} Y \times I \xrightarrow{G} Z$$

is a homotopy from gf' to $g'f'$. By transitivity of equivalence relations from lemma 1.1. $\square$

Let $[X, Y]$ be the set of homotopy classes of maps from $X \to Y$.

REMARK 1.1. We should probably assume $X \neq \emptyset \neq Y$.

Lemma 1.2 then tells us that composition defines a function

$$[X, Y] \times [Y, Z] \to [X, Z] \ .$$

0.1. Homotopy equivalence.

DEFINITION 1.2. X is *homotopy equivalent* to Y (or *of the same homotopy type* as Y) written $X \simeq Y$ if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. Then we say f is a homotopy equivalence and g is a homotopy inverse of f.

EXAMPLE 1.2. A homeomorphism is a homotopy equivalence, but in general this is much weaker.

LEMMA 1.3. *Homotopy equivalence is an equivalence relation.*

PROOF. (i) $X \simeq X$: $f = g = \mathrm{id}_X$, lemma 1.1.
(ii) $X \simeq Y \implies Y \simeq X$: by definition.
(iii) $X \simeq Y, Y \simeq Z \implies X \simeq Z$: So we have

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f'"] \end{tikzcd}$$

where $f'f \simeq \mathrm{id}_X$, $ff' \simeq \mathrm{id}_Y$ and similarly

$$\begin{tikzcd} Y \arrow[r, bend left, "g"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

where $g'g \simeq \mathrm{id}_X$, $gg' \simeq \mathrm{id}_Y$. Then we compose:

$$\begin{tikzcd} X \arrow[r, bend left, "gf"] & Y \arrow[l, bend left, "f'g'"] \end{tikzcd}$$

and we have that

$$(f'g')\, gf = f'\,(g'g)\, f \simeq f' \,\mathrm{id}_Y\, f = f'f \simeq \mathrm{id}_X$$

(from lemma 1.2) and similarly

$$(gf)\,(f'g') \simeq \mathrm{id}_X$$

so $gf : X \to Z$ is a homotopy equivalence.

□

LEMMA 1.4. $X \simeq X'$, $Y \simeq Y' \implies X \times Y \simeq X' \times Y'$.

PROOF. (exercise) □

REMARK 1.2. Many functors in algebraic topology (e.g. π_1, H_n, ...) have the property that $f \simeq g \implies f_* = g_*$. In other words they factor through the homotopy category where the objects are topological spaces and morphisms are just homotopy classes of maps:

$$\begin{tikzcd} & \mathbf{Top}/\simeq \arrow[dr, dashed] & \\ \mathbf{Top} \arrow[ur] \arrow[rr] & & \{\text{Algebraic objects, morphisms}\} \end{tikzcd} .$$

$$f : X \to Y \longrightarrow f_* : A\,(X) \to A\,(Y)$$

DEFINITION 1.3. Let $A \subset X$, $f, g : X \to Y$ such that $f|_A = g|_A$. Then $f \simeq g$ (rel. A) iff there exists $f \simeq_F g$ such that

$$F_t|_A = f|_A\,(= g|_A)$$

for all $t \in I$.

Homotopy equivalence rel. A is an equivalence relation on the set

$$\{f : X \to Y \mid f|_A \text{ is a fixed map.}\}$$

Let $i : A \to X$ be the inclusion. Then i being a homotopy equivalence means there exists some $f : X \to A$ such that

(i) $if \simeq \mathrm{id}_X$, and
(ii) $fi \simeq \mathrm{id}_A$.

Now we can strengthen this in multiple ways.

Definition 1.4. If we strengthen (ii) to say $fi = \mathrm{id}_A$ (i.e. f is a retraction) then $\mathrm{id}_X \simeq_F if$ is a *deformation retraction of X onto A*. In other words we have

$$F : X \times I \to X$$

such that $F_0 = \mathrm{id}_X$, $F_1(x) \in A$ for all $x \in A$, and $F_1|_A = A$ (since $F_1 = if$).

Definition 1.5. If in addition we strengthen (i) to say that $\mathrm{id}_X \simeq_F if\,(\mathrm{rel}\,A)$ then F is a *strong deformation retraction* (of X onto A). In other words we have $F : X \times I \to X$ such that

$$F_0 = \mathrm{id}_X \quad \forall x \in X, F_1(x) \in A \quad \forall t \in I, \forall a \in A, F_t(a) = a \ .$$

The idea here is that X strong deformation retracts to A implies X deformation retracts to A which implies $i : A \hookrightarrow X$ is a homotopy equivalence. As it will turn out, both of these implications are strict.

Example 1.3. $X \times \{0\}$ is a strong deformation retract of $X \times I$:

$$F : (X \times I) \times I \to X \times I$$

where $F((x,s),t) = (x,(1-t)s)$.

Example 1.4. $\mathbb{R}^n \setminus \{0\}$ strong deformation retracts to S^{n-1}.

Example 1.5. $A = S^1 \times [-1,1]$ strong deformation retracts to $S^1 \times \{0\}$. A Möbius band B also strong deformation retracts to S^1. Therefore $A \simeq B$.

Example 1.6. Let X be a twice punctured disk. Then it is sort of clear that this strong deformation retracts to

(i) the boundary along with one arc passing between the punctures,
(ii) the wedge of two circles, and
(iii) two circles connected by an interval.

This means these three are all homotopy equivalent.

Example 1.7. Consider a once-punctured torus $X = T^2 \setminus \mathrm{int}(D^2)$. This strong deformation retracts to the wedge of two circles. This tells us that the once punctured torus is actually homotopy equivalent to the disk with two punctures.

These examples show that homotopy equivalence does not imply homeomorphism, even for surfaces with boundary. We can immediately see these examples are homeomorphic because there's no way for the boundaries to be mapped to one another.

0.2. Manifolds. Topological spaces can be very wild, but manifolds are usually quite nice.

DEFINITION 1.6. An *n-dimensional manifold* is a Hausdorff, second countable space M such that every $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

EXAMPLE 1.8. $\mathbb{R}^n$, S^n, $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$.

DEFINITION 1.7. A manifold is *closed* iff it is compact.

DEFINITION 1.8. A surface is a closed 2-manifold.

EXAMPLE 1.9. We have all of the two-sided or orientable surfaces (S^2, T^2, ...) and then non orientable ones like $\mathbb{P}^2$ and the Klein bottle. As it turns out this is a complete list up to homeomorphism.

For any two spaces $X \cong Y$ implies $X \simeq Y$. As we have seen, the converse isn't even true for surfaces with boundary. But for closed manifolds, there are many interesting cases where it is true:

(1) For M, M' closed surfaces, $M \simeq M' \implies M \cong M'$.
(2) For M an n-manifold, $M \simeq S^n \implies M \cong S^n$. (Generalized Poincaré conjecture)

For $n = 0, 1, 2$ this is not so bad. For $n \geq 5$, Connell and Newman independently proved it. The next step was proving this for $n = 4$. This was proved by Freedman. Finally for $n = 3$ Perelman proved this. The smooth 4-dimensional version is still open. I.e. the statement for diffeomorphism.

REMARK 1.3. Poincaré originally posed this conjecture as saying that having the same homology as S^n was sufficient. But he discovered a counterexample, now called the Poincaré homology sphere. This shares homology with S^3 but has different fundamental group. This is in fact why he invented the fundamental group.

DEFINITION 1.9. $f : X \to Y$ is a *constant map* if there is some $y_0 \in Y$ such that for all $x \in X$ $f(x) = y_0$. We write $f = c_{y_0}$.

DEFINITION 1.10. A map $f : X \to Y$ is *null-homotopic* iff $f \simeq$ a constant map.

DEFINITION 1.11. X is *contractible* if id_X is null-homotopic. I.e. there exists some $x_0 \in X$ such that X deformation retracts to x_0.

EXAMPLE 1.10.
(1) Any nonempty convex[1.1] subspace of $\mathbb{R}^n$ is contractible. Choose an arbitrary point $x_0 \in X$. Define $F : X \times I \to X$ by
$$F(x,t) = (1-t)\,x + tx_0 \ .$$
Then $\mathrm{id}_X \simeq_F c_{x_0}$. In fact this is a strong deformation retraction. Sometimes you can do this for some point but not all, and for some you can't do it for any points.
(2) S^1 is not contractible.

Lecture 3; September 5, 2019

LEMMA 1.5. *For a topological space X TFAE:*
(1) X is contractible,
(2) $\forall x_0 \in X$, X deformation retracts to $\{x_0\}$,
(3) $X \simeq \{pt\}$,
(4) $\forall Y$, any two maps $Y \to X$ are homotopic.
(5) $\forall Y$, any map $X \to Y$ is null-homotopic.

PROOF. (1) $\implies$ (3): (1) is equivalent to saying that X deformation retracts to a point, so the inclusion map is certainly a homotopy equivalence.

(3) $\implies$ (4): Let $f : X \to \{z\}$ be a homotopy equivalence. By homework 1 exercise 3, we get an induced function:
$$f_* : [Y, X] \to [Y, \{z\}]$$
but there is only one map in the target set, so clearly there is only one homotopy class of maps $Y \to \{z\}$.

(4) $\implies$ (2): Take $Y = X$, and take any $x_0 \in X$. This means $\mathrm{id}_X \simeq c_{x_0}$, but this is exactly saying that X deformation retracts to x_0.

(2) $\implies$ (5): Let $f : X \to Y$ and $x_0 \in X$. Then (2) implies $\mathrm{id}_X \simeq_F c_{x_0}$. Then
$$f \circ \mathrm{id}_X \simeq_{f\circ F} f \circ c_{x_0}$$
i.e. f is nullhomotopic.

(5) $\implies$ (1): Take $Y = X$. □

COROLLARY 1.1. *For X, Y contractible, then*
(1) $X \simeq Y$,
(2) any map $X \to Y$ is a homotopy equivalence.

PROOF.
(1) If $X, Y \simeq \{\mathrm{pt}\}$ then $X \simeq Y$.

[1.1] Recall $X \subset \mathbb{R}^n$ is convex if $x, y \in X$ implies $tx + (1-t)\,y \in X$ for all $t \in I$.

(2) Given $f : X \to Y$, let $g : Y \to X$ be any map. $gf : X \to X$, but X is contractible, so $gf \simeq \mathrm{id}_X$ by lemma 1.5.

□

Now we will give an example of a deformation retraction which is not a strong deformation retraction. Recall X strong deformation retracts to A implies X deformation retracts to A which implies $i : A \hookrightarrow X$ is a homotopy equivalence, but none of these implications are reversible.

EXAMPLE 1.11 (Comb space). Define the comb space $C \subset I \times I \subset \mathbb{R}^2$ to be:

$$C = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y = 0, 0 \leq x \leq 1; 0 \leq y \leq 1, x = 0, 1/n\, (n = 1, 2, \ldots)\right\} .$$

This should be pictured as a bunch of vertical intervals.[1.2] The first thing to note is that C strong deformation retracts to $(0, 0)$. Therefore C is contractible. C also deformation retracts to $(0, 1)$. [More generally: if X deformation retracts to some $x_0 \in X$ and X is path connected, then X deformation retracts to any $x \in X$].

CLAIM 1.1. But it does not strong deformation retract to $(0, 1)$.

PROOF. Let $F : C \times I \to C$ be such a strong deformation retraction. Let U be some open disc of radius $1/2$ centered at $(0, 1)$. $F^{-1}(U) \subset X \times I$ contains $(0, 1) \times I$. Therefore for all $t \in I$ there exists some neighborhood V_t of $(0, 1) \times \{t\}$ such that $V_t \subset F^{-1}(U)$. But $V_t = W_t \times Z_t$ for W_t some neighborhood of $(0, 1)$ in C and Z_t some neighborhood of t in I. I is compact which means $\exists t_1, \ldots, t_m$ such that

$$\bigcup_{i=1}^{m} Z_{t_i} = I .$$

Let

$$W = \bigcap_{i=1}^{m} W_{t_i} .$$

This is a neighborhood of $(0, 1)$ in C, and $W \times I \subset F^{-1}(U)$. (This is sometimes called the tube lemma). Pick n such that $(1/n, 1) \in W$. Then $F((1/n, 1), t)$, $0 \leq t \leq 1$, is a path in U from $(1/n, 1)$ to $(0, 1)$ but there clearly isn't such a path since these two points are in different path components. □

COROLLARY 1.2. *Let $X \subset I^2 \subset \mathbb{R}^2$ where C, I^2 are both contractible. Then the inclusion $i : C \to I^2$ is a homotopy equivalence.*

[1.2] Which is supposed to look like a comb.

But there does not exist a deformation retraction $I^2 \to C$. In fact there is no retraction at all.

REMARK 1.4. There exists a space X such that X is contractible (therefore $\{x\} \hookrightarrow X$ is a homotopy equivalence for all $x \in X$) but there does not exist a deformation retraction from X to any $x \in X$. (e.g. Hatcher chapter 0, 6(b)).

0.3. Fixed point property. A space X has the fixed point property (FPP) iff $\forall f : X \to x$, $\exists x \in X$ such that $f(x) = x$. X being contractible does not imply X has the FPP (e.g. $\mathbb{R}^1$).

QUESTION 3 (Borsuk). If X is compact and contractible does contractible imply FPP?

CHAPTER 2

The fundamental group

DEFINITION 2.1. A *path* from x_0 to x is a map $\sigma : I \to X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x$.

DEFINITION 2.2. Let σ be a path in X from x_0 to x_1, and τ a path in X from x_1 to x_2. Their *concatenation* $\sigma * \tau$ is a path from x_0 to x_2 given by:

$$(\sigma * \tau)(s) = \begin{cases} \sigma(2s) & 0 \le s \le 1/2 \\ \tau(2s-1) & 1/2 \le s \le 1 \end{cases} .$$

DEFINITION 2.3. The *homotopy class* of σ is

$$[\sigma] = \{\sigma' \mid \sigma' \simeq \sigma \,(\mathrm{rel}\, \partial I)\} .$$

LEMMA 2.1. *If* $[\sigma] = [\sigma']$ *and* $[\tau] = [\tau']$ *where* $\sigma(1) = \tau(0)$ *then* $[\sigma * \tau] = [\sigma' * \tau']$.

PROOF. If $\sigma \simeq_{F_t} \sigma'$ and $\tau \simeq_{G_t} \tau'$ then $\sigma * \tau \simeq_{F_t * G_t} \sigma' * \tau' \,(\mathrm{rel}\, \partial I)$. $\square$

This means we can define the product of two homotopy classes to the be the homotopy class of the concatenation. This is well defined by the lemma.

LEMMA 2.2 (Reparameterization). *Let* $u : I \to I$ *be a map such that* $u|_{\partial I} = \mathrm{id}$. *Then* $u \simeq \mathrm{id}_I \,(\mathrm{rel}\, \partial I)$.

PROOF. Define $F : I \times I \to I$ by $F(s,t) = ts + (1-t)\,u(s)$. $F_0 = u$, $F_1 = \mathrm{id}_I$, $F_t|_{\partial I} = \mathrm{id}$ for all $t \in I$. $\square$

LEMMA 2.3 (Associativity). *Let* ρ, σ, τ *be paths in* X *such that* $\rho(1) = \sigma(0)$, $\sigma(1) = \tau(0)$. *Then*

$$([\rho][\sigma])[\tau] = [\rho]([\sigma][\tau]) .$$

PROOF. Define $u : I \to I$ by

$$u(s) = \begin{cases} 2s & 0 \le s \le 1/4 \\ s + 1/4 & 1/4 \le s \le 1/2 \\ (s+1)/2 & 1/2 \le s \le 1 \end{cases} .$$

Then

$$(\rho * (\sigma * \tau))\, u = (\rho * \sigma) * \tau \ .$$

but $u \simeq \mathrm{id}_I\,(\mathrm{rel}\,\partial I)$ so $(\rho * (\sigma * \tau)) = (\rho * \sigma) * \tau\,(\mathrm{rel}\,\partial I)$ $\square$

Let $c_{x_0} : I \to X$ be the constant path given by $c_{x_0} = x_0$ for all $s \in I$.

LEMMA 2.4. *For σ a path in X from x_0 to x_1 then*

$$[\sigma] = [\sigma]\,[c_{x_1}] = [c_{x_0}]\,[\sigma] \ .$$

PROOF. Let $u : I \to I$ be

$$u\,(s) = \begin{cases} 2s & 0 \leq s \leq 1/2 \\ 1 & 1/2 \leq s \leq 1 \end{cases} .$$

Then $\sigma * c_{x_1} = \sigma * u$

$$[\sigma] = [\sigma]\,[c_{x_1}]$$

by lemma 2.2. The proof is the same for the other part. $\square$

If σ is a path from x_0 to x_1, the reverse of σ is the path $\bar{\sigma}$ from x_1 to x_0 given by

$$\bar{\sigma}\,(s) = \sigma\,(1 - s) \ .$$

Note that immediately we have $\overline{(\bar{\sigma})} = \sigma$.

LEMMA 2.5. $[\sigma]\,[\bar{\sigma}] = [c_{x_0}]$.

PROOF. Define $F : I \times I \to X$ by

$$F\,(s, t) = \begin{cases} \sigma\,(2st) & 0 \leq s \leq 1/2 \\ \sigma\,(2\,(1 - s)\,t) & 1/2 \leq 2 \leq 1 \end{cases} .$$

Note that $F_0 = c_{x_0}$ and $F_1 = \sigma * \bar{\sigma}$ so we are done. $\square$

A *loop* in a space X based at a point $x_0 \in X$ is a path in X from x_0 to x_0. If σ and τ are loops at x_0 then this implies $\sigma * \tau$ is a loop at x_0. Let Lecture 4; September 10, 2019

$$\pi_1\,(X, x_0) = \{[\sigma] \mid \sigma \text{ loop in } X \text{ based at } x_0\} \ .$$

THEOREM 2.1. *$\pi\,(X, x_0)$ is a group with respect to the operation $[\sigma]\,[\tau]$; the fundamental group of X with basepoint x_0.*

PROOF. We proved associativity last time in lemma 2.3 the identity is the constant map $[c_{x_0}]$ as shown in lemma 2.4, and inverses are given by $[\sigma]^{-1} = [\bar{\sigma}]$ as shown in lemma 2.5. $\square$

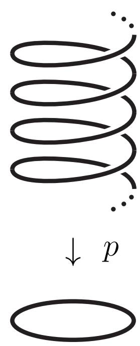

Figure 1. p mapping $\mathbb{R}$ to S^1. Note the preimage of a $1 \in S^1$ looks like $\mathbb{Z}$.

Example 2.1. Suppose X strong deformation retracts to some point x_0. This means there exists some homotopy $F : X \times I \to X$ such that $F_0 = \mathrm{id}_X$ and $F_1(x) = x_0$ for all $x \in X$. Since it is a strong deformation retraction for all $t \in I$ we have $F_t(x_0) = x_0$. Now let $\sigma : I \to X$ be a loop based at x_0. Then $\sigma \simeq_{F_t\sigma} c_{x_0}\,(\mathrm{rel}\,\partial I)$. Therefore $[\sigma] = 1$ and $\pi_1(X, x_0) = 1$.

Remark 2.1. We will actually prove a more general fact later, when we discuss change of basepoint.

Now we will finally have an example with nontrivial fundamental group. For all we know every space is contractible.[2.1]

1. Fundamental group of S^1

A key example is $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Theorem 2.2. *$\pi_1(S^1, 1) \simeq \mathbb{Z}$.*

Remark 2.2. The idea of the proof is to somehow unwrap the circle. This proof led to the idea of a covering space.

Let $p : \mathbb{R} \to S^1$ be the map defined by $p(x) = e^{2\pi i x}$. The picture is as in fig. 1.

Note that $p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$.

Lemma 2.6 (path-lifting). *Let $\sigma : I \to S^1$ be a path with $\sigma(0) = 1$. Then there exists a unique path $\tilde{\sigma} : I \to \mathbb{R}$ such that $\tilde{\sigma}(0) = 0$ and*

[2.1]Professor Gordon says our ignorance is extensive.

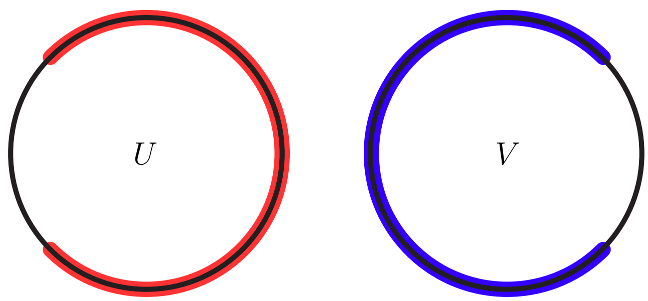

FIGURE 2. An open cover for the circle is given by the two sets U and V.

$p\tilde{\sigma} = \sigma$, i.e. $\tilde{\sigma}$ is a lift of σ so the following diagram commutes:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{\sigma}"] \arrow[r, "\sigma"] & S^1 \end{tikzcd} .$$

PROOF. Let

$$U = \left\{ e^{i\theta} \middle| -\frac{3\pi}{4} < \theta < \frac{3\pi}{4} \right\}$$
$$V = \left\{ e^{i\theta} \middle| \frac{\pi}{4} < \theta < \frac{7\pi}{4} \right\} .$$

This looks as in fig. 2. Then $\{U, V\}$ is an open cover of S^1. First notice that the distance between the points $\pi/4$ and $3\pi/4$ is $\sqrt{2}$. This means that for any $z, z' \in S^1$ we have that $d(z, z') < \sqrt{2}$ implies either $z, z' \in U$ and $z, z' \in V$.

So we have some $\sigma : I \to S^1$, $\sigma(0) = 1$. Since I is compact this implies $\exists \delta > 0$ such that $d(s, s') < \delta$ implies $d(\sigma(s), \sigma(s')) < \sqrt{2}$.

Now we decompose the interval. Let $0 = s_0 < s_1 < \ldots < s_m = 1$ such that $|s_i - s_{i-1}| < \delta$ for all i. Then $\sigma([s_{i-1}, s_i]) \subset U$ or V for all i.

Now we look at the inverse image in $\mathbb{R}$ and we get these overlapping intervals covering $\mathbb{R}$ as in fig. 3.

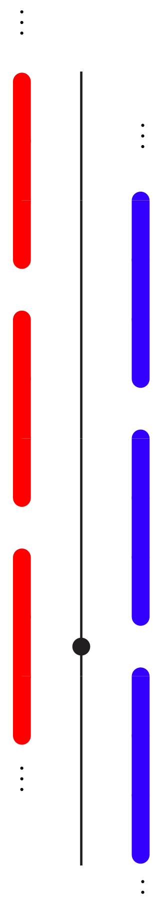

FIGURE 3. The preimage of the cover U and V in $\mathbb{R}$ under the map p.

In particular

$$p^{-1}(U) = \bigcup_{n\in\mathbb{Z}} \tilde{U}_n \qquad \tilde{U}_n = \left(n - \frac{3}{8}, n + \frac{3}{8}\right)$$

$$p^{-1}(V) = \bigcup_{n\in\mathbb{Z}} \tilde{V}_n \qquad \tilde{V}_n = \left(n + \frac{1}{8}, n + \frac{7}{8}\right) .$$

The restrictions

$$p|_{\tilde{U}_n} : \tilde{U}_n \to U \qquad p|_{\tilde{V}_n} : \tilde{V}_n \to V$$

are homeomorphisms for all n. This means if we try to lift the path from s_0 to s_1 we have no choice in the lift since it stays in U in S^1. So we get a unique lift. In detail, we define $\tilde{\sigma}$ inductively on $[0, s_k]$,

$0 \leq k \leq m$. For $k = 0$, $\sigma(0) = 0$. So now suppose $\tilde{\sigma}$ is defined on $[0, s_{k-1}]$ for some k such that $1 \leq k \leq m$. Then WLOG

$$\sigma([s_{k-1}, s_k]) \subset U$$

(rather than V). This means $\tilde{\sigma}(s_{k-1}) \in \tilde{U}_r$ for some $r \in \mathbb{Z}$. Since $p|_{\tilde{U}_r}$ is a homeomorphism let $q_r : U \to \tilde{U}_r$ be $\left(p|_{\tilde{U}_r}\right)^{-1}$. Define

$$\tilde{\sigma}|_{[s_{k-1}, s_k]} = q_r \sigma|_{[s_{k-1}, s_k]}$$

and note this is the only choice. Therefore by induction $\tilde{\sigma}$ exists and is unique. □

Lemma 2.7 (Homotopy lifting). *Let $\sigma, \tau : I \to S^1$ be paths with $\sigma(0) = \tau(0) = 1$ and let $F : I \times I \to S^1$ be a homotopy from σ to τ $(\operatorname{rel} \partial I)$. (So $\sigma(1) = \tau(1)$). Then there exists a unique $\tilde{F} : I \times I \to \mathbb{R}$, a homotopy from $\tilde{\sigma}$ to $\tilde{\tau}$ $(\operatorname{rel} \partial I)$, i.e. the following diagram commutes:*

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \times I \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "F"'] & S^1 \end{tikzcd} .$$

Proof. This proof is very similar to the proof of lemma 2.6. $I \times I$ is compact so there exists some $0 < s_0 < s_1 < \ldots < s_m = 1$ and $0 = t_0 < t_1 < \ldots < t_n = 1$ such that if

$$R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$$

(for $1 \leq i \leq m$, $1 \leq j \leq n$) then $F(R_{ij}) \subset U$ or V. Order the $R_{i,j}$ and relabel them as R_k. Then let

$$S_k = \bigcup_{i \leq k} R_i \ .$$

Now we define $\tilde{F}$ inductively on S_k. First $\tilde{F}(S_0) = \tilde{F}(0,0) = 0$. Now suppose $\tilde{F}$ is defined on S_{k-1}. Then $S_k = S_{k-1} \cup R_k$ and $F(R_k) \subset U$ or V; say U. Then $S_{k-1} \cap R_k$ is nonempty and connected and in U, and therefore

$$\tilde{F}(S_{k-1} \cap R_k) \subset \tilde{U}_r$$

for some $r \in \mathbb{Z}$. So we can (and must) define

$$\tilde{F}\Big|_{R_k} = q_r \, F|_{R_k} \ .$$

Therefore $\tilde{F}$ exists and is unique. $\tilde{F}_0$ is a lift of $F_0 = \sigma$, starting at 0, $\tilde{F}_1$ is a lift of $F_1 = \tau$, starting at 0, therefore by the uniqueness of lemma 2.6, $\tilde{F}_0 = \tilde{\sigma}$ and $\tilde{F}_1 = \tilde{\tau}$. Finally $p^{-1}(1) = \mathbb{Z}$ is discrete in

$\mathbb{R}$, so $\tilde{F}_t(0) = \tilde{\sigma}(0) = 0$ and $\tilde{F}_t(1) = \tilde{\sigma}(1)$ for all $t \in I$, so indeed $\tilde{\sigma} \simeq_{F_t} \tilde{\tau}\,(\mathrm{rel}\,\partial I)$. $\square$

PROOF OF THEOREM 2.1. Let σ be a loop in S^1 at 1. Define $\varphi : \pi_1(S^1, 1) \to \mathbb{Z}$ by $\varphi([\sigma]) = \tilde{\sigma}(1)$. Then we claim:

(1) φ is well defined: This follows from lemmata 2.6 and 2.7.
(2) φ is onto: Let $n \in \mathbb{Z}$; define $\tilde{\sigma} : I \to \mathbb{R}$ by $\tilde{\sigma}(s) = ns$. Let $\sigma = p\tilde{\sigma}$. Then $\varphi([\sigma]) = \tilde{\sigma}(1) = n$.
(3) φ is a homomorphism: Let $[\sigma], [\tau] \in \pi_1(S^1, 1)$ with $\varphi([\sigma]) = m$, $\varphi([\tau]) = n$ and

$$[\sigma][\tau] = [\sigma * \tau] \ .$$

Then

$$\widetilde{\sigma * \tau} = \tilde{\sigma} * \hat{\tau}$$

where $\hat{\tau}(s) = \tilde{s} + m$ and therefore

$$\varphi([\sigma][\tau]) = \varphi([\sigma * \tau]) = \left(\widetilde{\sigma * \tau}\right)(1) = (\tilde{\sigma} * \hat{\tau})(1) = m+n = \varphi([\sigma])+\varphi([\tau])$$

as desired.
(4) φ is one-to-one: Suppose $\varphi([\sigma]) = 0$, i.e. $\tilde{\sigma}$ is a loop in $\mathbb{R}$ at 0. $\mathbb{R}$ strong deformation retracts to 0, so $\tilde{\sigma} \simeq x_0\,(\mathrm{rel}\,\partial I)$. Therefore $[\sigma] = 1$.

$\square$

2. Induced homomorphisms

So consider some map $f : (X, x_0) \to (Y, y_0)$. Then the claim is that we can define a group homomorphism

$$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$$

by $f_*([\sigma]) = [f\sigma]$.

THEOREM 2.3. *(1) f_* is a well-defined homomorphism.*
(2) $(gf)_* = g_* f_*$, $\mathrm{id}_* = \mathrm{id}$.
(3) $f \simeq g\,(\mathrm{rel}\,x_0)$ implies $f_* = g_*$.

PROOF. (1) Suppose we have two loops $\sigma \simeq_{F_t} \sigma'\,(\mathrm{rel}\,\partial I)$. Then this means

$$f\sigma \simeq_{fF_t} f\sigma'\,(\mathrm{rel}\,\partial I)$$

which means f_* is well-defined.
(2)

$$\begin{aligned} f_*([\sigma][\tau]) &= f_*([\sigma * \tau]) = [f(\sigma * \tau)] \\ &= [f(\sigma) * f(\tau)] = [f\sigma][f\tau] \\ &= f_*([\sigma]) f_*([\tau]) \ . \end{aligned}$$

□

Next time we will talk about switching base points. If you're in a path connected space and you switch base points, then we can choose a path between them and concatenate this path to the old loops (on both ends) to get an isomorphism between the associated fundamental groups.