Algebraic Topology

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Contents

Introduction

Hatcher is the reference for the course. We won't follow too closely.

Lecture 1; August 29, 2019

1. Introduction

Today will be an introductory account of what algebraic topology actually is. In topology the objects of interest are topological spaces where the natural equivalence relation is a homeomorphism, i.e. a bijection $f : X \to Y$ such that f and f^{-1} are continuous. Somehow the goal is classifying topological spaces up to homeomorphism, so the basic question is somehow:

Question 1. Given topological spaces X and Y, is $X \cong Y$?

In these terms, algebraic topology is somehow a way of translating this into an algebraic question. More specifically, *algebraic topology* is the construction and study of functors from **Top** to some categories of algebraic objects (e.g. groups [0.1], abelian groups, vector spaces, rings, modules, ...). Recall this means we have a map from topological spaces $X \to A(X)$ for some algebraic object $A(X)$. In addition, for every $f : X \to Y$ we get a morphism $f_* : A(X) \to A(Y)$. Then these have to satisfy the conditions that

$$(gf)_* = g_* f_* \ , \qquad (\mathrm{id})_* = \mathrm{id} \ . \tag{0.1}$$

Exercise 1.1. Show that $X \cong Y$ implies that $A(X) \cong A(Y)$.

Example 0.1 (Fundamental group). Let X be a topological space. We will construct a group $\pi_1(X)$.

Example 0.2 (Higher homotopy groups). There is also something, $\pi_n(X)$, called the nth homotopy group. As it turns out for $n \geq 2$ this is abelian.

Example 0.3 (Singular homology). We will define abelian groups $H_n(X)$ (for $n \geq 0$) called the nth singular homology group.

We will also define real vector spaces $H_n(X; \mathbb{R})$ for $n \geq 0$ which are the nth singular homology with coefficients in $\mathbb{R}$.

Example 0.4 (Cohomology). We will also have the nth (singular) cohomology rings $H^*(X)$. These is actually a graded ring.

Warning 0.1. Above we actually should have said we're dealing with what are *covariant* functors, but in this case we are actually dealing with a *contravariant* functor. This just means we have:

$$f : X \to Y \rightsquigarrow f^* : H^*(Y) \to H^*(X) \ . \tag{0.2}$$

[0.1] Once Professor Gordon was giving a job talk about knot cobordisms. As it turns out these form a semigroup rather than a group. But if you add some sort of 4 dimensional equiv relation you get an honest group. So he was going on about how semigroups aren't so useful. After the talk he found out the chairman of the department worked on semigroups.

REMARK 0.1. The point here is that problems about topological spaces and maps are "continuous" and "hard". But on the algebraic side these problems become somehow "discrete" and "easy".

2. A bit more specific

Recall in $\mathbb{R}^n$ we define:

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \qquad S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ . \tag{0.3}$$

EXAMPLE 0.5. Two examples of surfaces are S^2 and $T^2 = S^1 \times S^1$. They clearly aren't homeomorphic, but how are we supposed to prove such a fact? We will see that $\pi_1\left(S^2\right) = 1$ whereas $\pi^1\left(T^2\right) = \mathbb{Z} \times \mathbb{Z}$. Since these are not isomorphic, the spaces cannot be homeomorphic.

2.1. Retraction.

Let $A \subset X$ be a space and a subspace.

DEFINITION 0.1. A *retraction* from X to A is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$, i.e. the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ A \arrow[ur, hook, "i"] \arrow[rr, "\mathrm{id}"] & & A \end{tikzcd} \ . \tag{0.4}$$

Note that r is certainly surjective since id is.

EXAMPLE 0.6. If X is any nonempty space, $x_0 \in X$, define $r : X \to \{x_0\}$ as $r(x) = x_0$. So every nonempty space always retract onto a point.

EXAMPLE 0.7. Think of $A \subset A \times B$ by fixing some $b_0 \in B$ and sending

$$\begin{tikzcd} A \arrow[r, hook] & B \\ a \arrow[r, mapsto] & (a, b_0) \end{tikzcd} \ . \tag{0.5}$$

Then $r : A \to B \to$ defined by $r(a, b) = a$ is a retraction.

Recall that for $f : X \to Y$ for X path connected, then $f(X)$ is also path connected. Recall that $D^1 = [-1, 1] \subset \mathbb{R}$ is path connected, whereas $S^0 = \{\pm 1\}$ is not. Therefore there cannot be a retraction $D^1 \to S^0$. This is a basic fact, but it motivates a more general statement which is not so clear.

Suppose there exists a retraction $r : D^1 \to S^0$. Then this means the following diagram commutes:

$$\begin{tikzcd} & D^1 \arrow[dr, "r"] & \\ S^0 \arrow[ur, hook] \arrow[rr, "\mathrm{id}"] & & S^0 \end{tikzcd} \ . \tag{0.6}$$

If we apply the functor H_0, we will see that

$$H_0(X) \cong \bigoplus_{\text{path components of } X} \mathbb{Z} \ . \tag{0.7}$$

So if we apply H_0 to the diagram we get:

$$\begin{tikzcd} & H_0\left(D^1\right) \arrow[dr, "r_*"] & \\ H_0\left(S^0\right) \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & H_0\left(S^0\right) \end{tikzcd} = \begin{tikzcd} & \mathbb{Z} \arrow[dr, "r_*"] & \\ \mathbb{Z}\oplus\mathbb{Z} \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & \mathbb{Z}\oplus\mathbb{Z} \end{tikzcd} \tag{0.8}$$

but this is clearly impossible.

In the same way we will see the much harder fact:

Fact 1 (Brouwer). *There does not exist a retraction $D^n \to S^{n-1}$ (for $n \geq 2$).*

We will see this by applying H_{n-1}. The idea is that

$$H_{n-1}\left(D^n\right) = 0 \qquad\qquad H_{n-1}\left(S^{n-1}\right) = \mathbb{Z} \tag{0.9}$$

which means we would have the diagram:

$$\begin{tikzcd} & 0 \arrow[dr] & \\ \mathbb{Z} \arrow[ur] \arrow[rr, "\mathrm{id}"] & & \mathbb{Z} \end{tikzcd} \tag{0.10}$$

which is a contradiction.

This turns out to imply the famous:

Theorem 0.1 (Brouwer fixed point theorem). *Every map $f : D^n \to D^n$ ($n \geq 1$) has a fixed point (i.e. a point $x \in D^n$ such that $f(x) = x$). In this case one says that D^n has the* fixed point property *(FPP).*

Proof. Suppose there exists an $f : D^n \to D^n$ such that $\forall x \in D^n\ f(x) \neq x$. Now draw a straight line from x to $f(x)$ and continue it to the boundary S^{n-1}. Call this point $g(x)$. Then this defines a map $g : D^n \to S^{n-1}$. g is continuous and $g|_{S^{n-1}} = \mathrm{id}$. Therefore g is a retraction $D^n \to S^{n-1}$ which we saw cannot exist. ☐

2.2. Dimension. We know $\mathbb{R}^n$ somehow has dimension n. But what does this really mean? The intuition is that $\mathbb{R}^2$ somehow has more points than $\mathbb{R}$. But then in 1877 Cantor proved that there is in fact a bijection $\mathbb{R} \to \mathbb{R}^2$. But this is highly non-continuous, so this tells us continuity should have something to do with it. But then in 1890 Peano showed that there exists a continuous surjection $\mathbb{R} \to \mathbb{R}^2$ as well. In 1910, using homology, Brouwer proved:[0.2]

Theorem 0.2. *For $m < n$ $\nexists$ continuous injection $\mathbb{R}^n \to \mathbb{R}^m$.*

We will prove this. A corollary of this is the famous invariance of dimension. I.e. $\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$. The proof uses separation properties of the $n-1$-sphere in $\mathbb{R}^n$, thus in turn it uses H_*.

Exercise 2.1. Find an easy proof for $m = 1$.

Theorem 0.3 (Jordan curve theorem). *For a subset $C \subset \mathbb{R}^2$ such that $C \simeq S^1$ then $\mathbb{R}^2 \setminus C$ has exactly 2 components A and B. In addition $C = Fr(A) = Fr(B)$. (Recall the* frontier *is defined as $Fr(X) = \overline{X} \cap \overline{(Y \setminus X)}$ for any $X \subset Y$.)*

[0.2] When he proved this Lebesgue contacted him saying that he could prove it too. So he sent him his proof, and Brouwer saw some errors. So over many years he eventually corrected it. In the end Brouwer summarized it by saying that it was really just his own proof.

Theorem 0.4 (Schönflies). *Let A, B, and C be as above and assume A is a bounded component of* $\mathbb{R}^2 \setminus X$*. Then* $\overline{A} \cong D^2$.

The higher dimensional analog of JCT is true (Brouwer). The proof will use H_*. The generalization is that for $\Sigma \subset \mathbb{R}^n$ then $\Sigma \cong S^{n-1}$. As it turns out the higher dimensional analog of the Schönflies theorem is false. The counterexample is the famous Alexander horned space. I.e. there exists $\Sigma \subset \mathbb{R}^3$, $\Sigma \cong S^2$; A and bounded component of $\mathbb{R}^2 \setminus \Sigma$ and $\overline{A} \ncong D^3$.

Recall

Theorem 0.5 (Heine-Borel). *$X \subset \mathbb{R}^n$ compact is equivalent to X being closed and bounded.*

So consider $\emptyset \neq X \subset \mathbb{R}$ compact and connected. This is equivalent to being an interval $X = [a, b]$ $(a \leq b)$. In $\mathbb{R}^2$ things get much worse. In particular there exists a compact, connected subset such X that $\mathbb{R}^2 \setminus X$ has exactly three components A, B, and C, and in particular every neighborhood of every point in X meets all three components. This is known as the 'lakes of Wada'. We start with an island with two lakes, and then start digging canals which somehow get closer and closer to the lakes. In the end every point of what is left is arbitrarily close to all three lakes.

Question 2 (Open question). Let X be a compact connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus X$ is connected. Does X have the fixed point property?

CHAPTER 1

Homotopy

Let $I = [0,1]$.

Definition 1.1. Let $f, g : X \to Y$. Say f is homotopic to g (and write $f \simeq g$) iff there exists a map $F : X \times I \to Y$ such that for all $x \in X$

$$F(x,0) = f(x) \qquad\qquad F(x,1) = g(x) \ . \tag{1.1}$$

Define $F_t : X \to Y$ by $F_t(x) = F(x,t)$. Then F_t is a continuous 1-parameter family of maps $X \to Y$ such that $F_0 = f$ and $F_1 = g$.

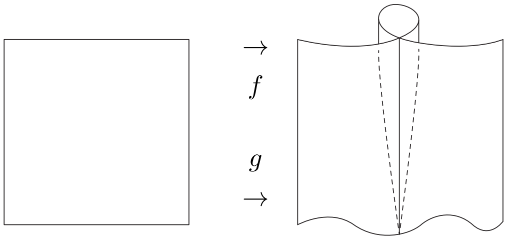

Figure 1.

Example 1.1. Let $F : S^{n-1} \times I \to \mathbb{R}^n$ be defined by $F(x,t) = (1-t)x$. Then F_0 is the inclusion inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$ and F_1 is the constant map $S^{n-1} \to$ origin.

Lecture 2;
September 3, 2019

Lemma 1.1. *Homotopy is an equivalence relation on the set of all maps $X \to Y$.*

Proof.
(i) $f \simeq f$: Let F be the constant homotopy, i.e. $F(x,y) = f(x)$ for all $t \in I$ and for all $x \in X$.
(ii) $f \simeq g \implies g \simeq f$: Suppose $f \simeq_F g$. Let $\bar{F}$ be the reverse homotopy $\bar{F}(x,t) = F(x,1-t)$. Then $g \simeq_{\bar{F}} f$.
(iii) $f \simeq g$, $g \simeq h \implies f \simeq h$: Define $H : X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases} \ . \tag{1.2}$$

Then by exercise 2 on homework 1 H is continuous and therefore $f \simeq_H h$.

□

Lemma 1.2 (compositions of homotopic maps are homotopic)**.** *If $f \simeq f' : X \to Y$, $g \simeq g' : Y \to Z$, then $gf \simeq g'f' : X \to Z$.*

PROOF. Suppose $f \simeq_F f'$, $g \simeq_G g'$. Then the composition

$$X \times I \xrightarrow{F} Y \xrightarrow{g} Z \tag{1.3}$$

is a homotopy from gf to gf'. Then the composition

$$X \times I \xrightarrow{f' \times \mathrm{id}} Y \times I \xrightarrow{G} Z \tag{1.4}$$

is a homotopy from gf' to $g'f'$. By transitivity of equivalence relations from Lemma 1.1. □

Let $[X, Y]$ be the set of homotopy classes of maps from $X \to Y$.

REMARK 1.1. We should probably assume $X \neq \emptyset \neq Y$.

Lemma 1.2 then tells us that composition defines a function

$$[X, Y] \times [Y, Z] \to [X, Z] \ . \tag{1.5}$$

0.1. Homotopy equivalence.

DEFINITION 1.2. X is *homotopy equivalent* to Y (or *of the same homotopy type* as Y) written $X \simeq Y$ if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. Then we say f is a homotopy equivalence and g is a homotopy inverse of f.

EXAMPLE 1.2. A homeomorphism is a homotopy equivalence, but in general this is much weaker.

Lemma 1.3. *Homotopy equivalence is an equivalence relation.*

PROOF. (i) $X \simeq X$: $f = g = \mathrm{id}_X$, Lemma 1.1.

(ii) $X \simeq Y \implies Y \simeq X$: by definition.

(iii) $X \simeq Y$, $Y \simeq Z \implies X \simeq Z$: So we have

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f'"] \end{tikzcd} \tag{1.6}$$

where $f'f \simeq \mathrm{id}_X$, $ff' \simeq \mathrm{id}_Y$ and similarly

$$\begin{tikzcd} Y \arrow[r, bend left, "g"] & Z \arrow[l, bend left, "g'"] \end{tikzcd} \tag{1.7}$$

where $g'g \simeq \mathrm{id}_X$, $gg' \simeq \mathrm{id}_Y$. Then we compose:

$$\begin{tikzcd} X \arrow[r, bend left, "gf"] & Y \arrow[l, bend left, "f'g'"] \end{tikzcd} \tag{1.8}$$

and we have that

$$(f'g')\, gf = f'\, (g'g)\, f \simeq f'\, \mathrm{id}_Y\, f = f'f \simeq \mathrm{id}_X \tag{1.9}$$

(from Lemma 1.2) and similarly

$$(gf)\,(f'g') \simeq \mathrm{id}_X \tag{1.10}$$

so $gf : X \to Z$ is a homotopy equivalence.

□

Lemma 1.4. $X \simeq X'$, $Y \simeq Y' \implies X \times Y \simeq X' \times Y'$.

PROOF. (exercise) □

Remark 1.2. Many functors in algebraic topology (e.g. π_1, H_n, ...) have the property that $f \simeq g \implies f_* = g_*$. In other words they factor through the homotopy category where the objects are topological spaces and morphisms are just homotopy classes of maps:

$$\begin{tikzcd} & \mathbf{Top}/\simeq \arrow[dr, dashed] & \\ \mathbf{Top} \arrow[ur] \arrow[rr] & & \{\text{Algebraic objects, morphisms}\} \end{tikzcd} \cdot \tag{1.11}$$

$$f : X \to Y \longrightarrow f_* : A\left(X\right) \to A\left(Y\right)$$

Definition 1.3. Let $A \subset X$, $f, g : X \to Y$ such that $f|_A = g|_A$. Then $f \simeq g$ (rel. A) iff there exists $f \simeq_F g$ such that

$$F_t|_A = f|_A \left(= g|_A\right) \tag{1.12}$$

for all $t \in I$.

Homotopy equivalence rel. A is an equivalence relation on the set

$$\{f : X \to Y \mid f|_A \text{ is a fixed map.}\} \tag{1.13}$$

Let $i : A \to X$ be the inclusion. Then i being a homotopy equivalence means there exists some $f : X \to A$ such that

(i) $if \simeq \mathrm{id}_X$, and
(ii) $fi \simeq \mathrm{id}_A$.

Now we can strengthen this in multiple ways.

Definition 1.4. If we strengthen (ii) to say $fi = \mathrm{id}_A$ (i.e. f is a retraction) then $\mathrm{id}_X \simeq_F if$ is a *deformation retraction of* X *onto* A. In other words we have

$$F : X \times I \to X \tag{1.14}$$

such that $F_0 = \mathrm{id}_X$, $F_1\left(x\right) \in A$ for all $x \in A$, and $F_1|_A = \mathrm{id}_A$ (since $F_1 = if$).

Definition 1.5. If in addition we strengthen (i) to say that $\mathrm{id}_X \simeq_F if\,(\mathrm{rel}\, A)$ then F is a *strong deformation retraction* (of X onto A). In other words we have $F : X \times I \to X$ such that

$$F_0 = \mathrm{id}_X \qquad \forall x \in X, F_1\left(x\right) \in A \qquad \forall t \in I, \forall a \in A, F_t\left(a\right) = a\ . \tag{1.15}$$

The idea here is that X strong deformation retracts to A implies X deformation retracts to A which implies $i : A \hookrightarrow X$ is a homotopy equivalence. As it will turn out, both of these implications are strict.

Example 1.3. $X \times \{0\}$ is a strong deformation retract of $X \times I$:

$$F : \left(X \times I\right) \times I \to X \times I \tag{1.16}$$

where $F\left(\left(x, s\right), t\right) = \left(x, \left(1 - t\right) s\right)$.

Example 1.4. $\mathbb{R}^n \setminus \{0\}$ strong deformation retracts to S^{n-1}.

Example 1.5. $A = S^1 \times [-1, 1]$ strong deformation retracts to $S^1 \times \{0\}$. A Möbius band B also strong deformation retracts to S^1. Therefore $A \simeq B$.

Example 1.6. Let X be a twice punctured disk. Then it is sort of clear that this strong deformation retracts to

(i) the boundary along with one arc passing between the punctures,

(ii) the wedge of two circles, and
(iii) two circles connected by an interval.

This means these three are all homotopy equivalent.

Example 1.7. Consider a once-punctured torus $X = T^2 \setminus \operatorname{int}\left(D^2\right)$. This strong deformation retracts to the wedge of two circles. This tells us that the once punctured torus is actually homotopy equivalent to the disk with two punctures.

These examples show that homotopy equivalence does not imply homeomorphism, even for surfaces with boundary. We can immediately see these examples are not homeomorphic because there's no way for the boundaries to be mapped to one another.

0.2. Manifolds. Topological spaces can be very wild, but manifolds are usually quite nice.

Definition 1.6. An *n-dimensional manifold* is a Hausdorff, second countable space M such that every $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Example 1.8. $\mathbb{R}^n$, S^n, $T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$.

Definition 1.7. A manifold is *closed* iff it is compact.

Definition 1.8. A surface is a closed 2-manifold.

Example 1.9. We have all of the two-sided or orientable surfaces (S^2, T^2, ...) and then non orientable ones like $\mathbb{P}^2$ and the Klein bottle. As it turns out this is a complete list up to homeomorphism.

For any two spaces $X \cong Y$ implies $X \simeq Y$. As we have seen, the converse isn't even true for surfaces with boundary. But for closed manifolds, there are many interesting cases where it is true:

(1) For M, M' closed surfaces, $M \simeq M' \implies M \cong M'$.
(2) For M an n-manifold, $M \simeq S^n \implies M \cong S^n$. (Generalized Poincaré conjecture)

For $n = 0, 1, 2$ this is not so bad. For $n \geq 5$, Connell and Newman independently proved it. The next step was proving this for $n = 4$. This was proved by Freedman. Finally for $n = 3$ Perelman proved this. The smooth 4-dimensional version is still open. I.e. the statement for diffeomorphism.

Remark 1.3. Poincaré originally posed this conjecture as saying that having the same homology as S^n was sufficient. But he discovered a counterexample, now called the Poincaré homology sphere. This shares homology with S^3 but has different fundamental group. This is in fact why he invented the fundamental group.

Definition 1.9. $f : X \to Y$ is a *constant map* if there is some $y_0 \in Y$ such that for all $x \in X$ $f(x) = y_0$. We write $f = c_{y_0}$.

Definition 1.10. A map $f : X \to Y$ is *null-homotopic* iff $f \simeq$ a constant map.

Definition 1.11. X is *contractible* if id_X is null-homotopic. I.e. there exists some $x_0 \in X$ such that X deformation retracts to x_0.

Example 1.10. (1) Any nonempty convex[1.1] subspace of $\mathbb{R}^n$ is contractible. Choose an arbitrary point $x_0 \in X$. Define $F : X \times I \to X$ by

$$F(x,t) = (1-t)x + tx_0 \ . \tag{1.17}$$

[1.1] Recall $X \subset \mathbb{R}^n$ is convex if $x, y \in X$ implies $tx + (1-t)y \in X$ for all $t \in I$.

Then $\mathrm{id}_X \simeq_F c_{x_0}$. In fact this is a strong deformation retraction. Sometimes you can do this for some point but not all, and for some you can't do it for any points.

(2) S^1 is not contractible.

Lecture 3; September 5, 2019

Lemma 1.5. *For a topological space X TFAE:*

1. *X is contractible,*
2. *$\forall x_0 \in X$, X deformation retracts to $\{x_0\}$,*
3. *$X \simeq \{pt\}$,*
4. *$\forall Y$, any two maps $Y \to X$ are homotopic.*
5. *$\forall Y$, any map $X \to Y$ is null-homotopic.*

Proof. (1) $\implies$ (3): (1) is equivalent to saying that X deformation retracts to a point, so the inclusion map is certainly a homotopy equivalence.

(3) $\implies$ (4): Let $f : X \to \{z\}$ be a homotopy equivalence. By homework 1 exercise 3, we get an induced function:

$$f_* : [Y, X] \to [Y, \{z\}] \tag{1.18}$$

but there is only one map in the target set, so clearly there is only one homotopy class of maps $Y \to \{z\}$.

(4) $\implies$ (2): Take $Y = X$, and take any $x_0 \in X$. This means $\mathrm{id}_X \simeq c_{x_0}$, but this is exactly saying that X deformation retracts to x_0.

(2) $\implies$ (5): Let $f : X \to Y$ and $x_0 \in X$. Then (2) implies $\mathrm{id}_X \simeq_F c_{x_0}$. Then

$$f \circ \mathrm{id}_X \simeq_{f\circ F} f \circ c_{x_0} \tag{1.19}$$

i.e. f is nullhomotopic.

(5) $\implies$ (1): Take $Y = X$. ☐

Corollary 1.6. *For X, Y contractible, then*

1. *$X \simeq Y$,*
2. *any map $X \to Y$ is a homotopy equivalence.*

Proof.
1. If $X, Y \simeq \{\mathrm{pt}\}$ then $X \simeq Y$.
2. Given $f : X \to Y$, let $g : Y \to X$ be any map. $gf : X \to X$, but X is contractible, so $gf \simeq \mathrm{id}_X$ by Lemma 1.5.

☐

Now we will give an example of a deformation retraction which is not a strong deformation retraction. Recall X strong deformation retracts to A implies X deformation retracts to A which implies $i : A \hookrightarrow X$ is a homotopy equivalence, but none of these implications are reversible.

Example 1.11 (Comb space). Define the comb space $C \subset I \times I \subset \mathbb{R}^2$ to be:

$$C = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y = 0, 0 \le x \le 1; 0 \le y \le 1, x = 0, 1/n\,(n = 1, 2, \ldots)\right\} . \tag{1.20}$$

This should be pictured as a bunch of vertical intervals.[1.2] The first thing to note is that C strong deformation retracts to $(0, 0)$. Therefore C is contractible. C also deformation retracts to $(0, 1)$. [More generally: if X deformation retracts to some $x_0 \in X$ and X is path connected, then X deformation retracts to any $x \in X$].

[1.2] Which is supposed to look like a comb.

Claim 1.1. But it does not strong deformation retract to $(0,1)$.

Proof. Let $F : C \times I \to C$ be such a strong deformation retraction. Let U be some open disc of radius $1/2$ centered at $(0,1)$. $F^{-1}(U) \subset X \times I$ contains $(0,1) \times I$. Therefore for all $t \in I$ there exists some neighborhood V_t of $(0,1) \times \{t\}$ such that $V_t \subset F^{-1}(U)$. But $V_t = W_t \times Z_t$ for W_t some neighborhood of $(0,1)$ in C and Z_t some neighborhood of t in I. I is compact which means $\exists t_1, \ldots, t_m$ such that

$$\bigcup_{i=1}^{m} Z_{t_i} = I \ . \tag{1.21}$$

Let

$$W = \bigcap_{i=1}^{m} W_{t_i} \ . \tag{1.22}$$

This is a neighborhood of $(0,1)$ in C, and $W \times I \subset F^{-1}(U)$. (This is sometimes called the tube lemma). Pick n such that $(1/n, 1) \in W$. Then $F((1/n,1), t)$, $0 \le t \le 1$, is a path in U from $(1/n,1)$ to $(0,1)$ but there clearly isn't such a path since these two points are in different path components. ☐

Corollary 1.7. *Let $C \subset I^2 \subset \mathbb{R}^2$ where C, I^2 are both contractible. Then the inclusion $i : C \to I^2$ is a homotopy equivalence. But there does not exist a deformation retraction $I^2 \to C$. In fact there is no retraction at all.*

Remark 1.4. There exists a space X such that X is contractible (therefore $\{x\} \hookrightarrow X$ is a homotopy equivalence for all $x \in X$) but there does not exist a strong deformation retraction from X to any $x \in X$. (e.g. Hatcher chapter 0, exercise 6(b)).

0.3. Fixed point property. A space X has the fixed point property (FPP) iff $\forall f : X \to X$, $\exists x \in X$ such that $f(x) = x$. X being contractible does not imply X has the FPP (e.g. $\mathbb{R}^1$).

Question 3 (Borsuk). If X is compact and contractible does contractible imply FPP?

CHAPTER 2

The fundamental group

Definition 2.1. A *path* from x_0 to x is a map $\sigma : I \to X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x$.

Definition 2.2. Let σ be a path in X from x_0 to x_1, and τ a path in X from x_1 to x_2. Their *concatenation* $\sigma * \tau$ is a path from x_0 to x_2 given by:

$$(\sigma * \tau)(s) = \begin{cases} \sigma(2s) & 0 \le s \le 1/2 \\ \tau(2s-1) & 1/2 \le s \le 1 \end{cases} . \tag{2.1}$$

Definition 2.3. The *homotopy class* of σ is

$$[\sigma] = \{\sigma' \mid \sigma' \simeq \sigma\,(\mathrm{rel}\,\partial I)\} . \tag{2.2}$$

Lemma 2.1. *If $[\sigma] = [\sigma']$ and $[\tau] = [\tau']$ where $\sigma(1) = \tau(0)$ then $[\sigma * \tau] = [\sigma' * \tau']$.*

Proof. If $\sigma \simeq_{F_t} \sigma'$ and $\tau \simeq_{G_t} \tau'$ then $\sigma * \tau \simeq_{F_t * G_t} \sigma' * \tau'\,(\mathrm{rel}\,\partial I)$. ☐

This means we can define the product of two homotopy classes to the be the homotopy class of the concatenation. This is well defined by the lemma.

Lemma 2.2 (Reparameterization)**.** *Let $u : I \to I$ be a map such that $u|_{\partial I} = \mathrm{id}$. Then $u \simeq \mathrm{id}_I\,(\mathrm{rel}\,\partial I)$.*

Proof. Define $F : I \times I \to I$ by $F(s,t) = ts + (1-t)\,u(s)$. $F_0 = u$, $F_1 = \mathrm{id}_I$, $F_t|_{\partial I} = \mathrm{id}$ for all $t \in I$. ☐

Lemma 2.3 (Associativity)**.** *Let ρ, σ, τ be paths in X such that $\rho(1) = \sigma(0)$, $\sigma(1) = \tau(0)$. Then*

$$([\rho]\,[\sigma])\,[\tau] = [\rho]\,([\sigma]\,[\tau]) . \tag{2.3}$$

Proof. Define $u : I \to I$ by

$$u(s) = \begin{cases} 2s & 0 \le s \le 1/4 \\ s + 1/4 & 1/4 \le s \le 1/2 \\ (s+1)/2 & 1/2 \le s \le 1 \end{cases} . \tag{2.4}$$

Then

$$(\rho * (\sigma * \tau))\,u = (\rho * \sigma) * \tau . \tag{2.5}$$

but $u \simeq \mathrm{id}_I\,(\mathrm{rel}\,\partial I)$ so $(\rho * (\sigma * \tau)) = (\rho * \sigma) * \tau\,(\mathrm{rel}\,\partial I)$ ☐

Let $c_{x_0} : I \to X$ be the constant path given by $c_{x_0} = x_0$ for all $s \in I$.

Lemma 2.4. *For σ a path in X from x_0 to x_1 then*

$$[\sigma] = [\sigma]\,[c_{x_1}] = [c_{x_0}]\,[\sigma] \quad . \tag{2.6}$$

Proof. Let $u : I \to I$ be

$$u(s) = \begin{cases} 2s & 0 \le s \le 1/2 \\ 1 & 1/2 \le s \le 1 \end{cases} \quad . \tag{2.7}$$

Then $\sigma * c_{x_1} = \sigma * u$

$$[\sigma] = [\sigma]\,[c_{x_1}] \tag{2.8}$$

by Lemma 2.2. The proof is the same for the other part. ☐

If σ is a path from x_0 to x_1, the reverse of σ is the path $\bar{\sigma}$ from x_1 to x_0 given by

$$\bar{\sigma}(s) = \sigma(1-s) \quad . \tag{2.9}$$

Note that immediately we have $\overline{(\bar{\sigma})} = \sigma$.

Lemma 2.5. $[\sigma]\,[\bar{\sigma}] = [c_{x_0}]$.

Proof. Define $F : I \times I \to X$ by

$$F(s,t) = \begin{cases} \sigma(2st) & 0 \le s \le 1/2 \\ \sigma(2(1-s)t) & 1/2 \le 2 \le 1 \end{cases} \quad . \tag{2.10}$$

Note that $F_0 = c_{x_0}$ and $F_1 = \sigma * \bar{\sigma}$ so we are done. ☐

Lecture 4; September 10, 2019

A *loop* in a space X based at a point $x_0 \in X$ is a path in X from x_0 to x_0. If σ and τ are loops at x_0 then this implies $\sigma * \tau$ is a loop at x_0. Let

$$\pi_1(X, x_0) = \{[\sigma] \mid \sigma \text{ loop in } X \text{ based at } x_0\} \quad . \tag{2.11}$$

Theorem 2.6. *$\pi(X, x_0)$ is a group with respect to the operation $[\sigma]\,[\tau]$; the fundamental group of X with basepoint x_0.*

Proof. We proved associativity last time in Lemma 2.3 the identity is the constant map $[c_{x_0}]$ as shown in Lemma 2.4, and inverses are given by $[\sigma]^{-1} = [\bar{\sigma}]$ as shown in Lemma 2.5. ☐

Example 2.1. Suppose X strong deformation retracts to some point x_0. This means there exists some homotopy $F : X \times I \to X$ such that $F_0 = \mathrm{id}_X$ and $F_1(x) = x_0$ for all $x \in X$. Since it is a strong deformation retraction for all $t \in I$ we have $F_t(x_0) = x_0$. Now let $\sigma : I \to X$ be a loop based at x_0. Then $\sigma \simeq_{F_t \sigma} c_{x_0} \,(\mathrm{rel}\, \partial I)$. Therefore $[\sigma] = 1$ and $\pi_1(X, x_0) = 1$.

Remark 2.1. We will actually prove a more general fact later, when we discuss change of basepoint.

Now we will finally have an example with nontrivial fundamental group. For all we know every space is contractible.[2.1]

[2.1] Professor Gordon says our ignorance is extensive.

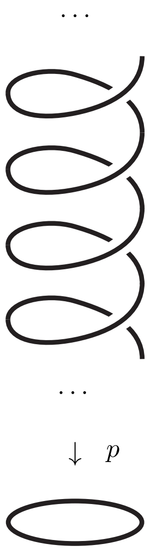

FIGURE 1. p mapping $\mathbb{R}$ to S^1. Note the preimage of a $1 \in S^1$ looks like $\mathbb{Z}$.

1. Fundamental group of S^1

A key example is $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

THEOREM 2.7. $\pi_1\left(S^1, 1\right) \simeq \mathbb{Z}$.

REMARK 2.2. The idea of the proof is to somehow unwrap the circle. This proof led to the idea of a covering space.

Let $p : \mathbb{R} \to S^1$ be the map defined by $p(x) = e^{2\pi i x}$. The picture is as in Fig. 1.

Note that $p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$.

Lemma 2.8 (path-lifting)**.** *Let $\sigma : I \to S^1$ be a path with $\sigma(0) = 1$. Then there exists a unique path $\tilde{\sigma} : I \to \mathbb{R}$ such that $\tilde{\sigma}(0) = 0$ and $p\tilde{\sigma} = \sigma$, i.e. $\tilde{\sigma}$ is a lift of σ so the following diagram commutes:*

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{\sigma}"] \arrow[r, "\sigma"] & S^1 \end{tikzcd} . \tag{2.12}$$

PROOF. Let

$$U = \left\{ e^{i\theta} \,\middle|\, -\frac{3\pi}{4} < \theta < \frac{3\pi}{4} \right\} \tag{2.13}$$

$$V = \left\{ e^{i\theta} \,\middle|\, \frac{\pi}{4} < \theta < \frac{7\pi}{4} \right\} . \tag{2.14}$$

This looks as in Fig. 2. Then $\{U, V\}$ is an open cover of S^1. First notice that the distance between the points $\pi/4$ and $3\pi/4$ is $\sqrt{2}$. This means that for any $z, z' \in S^1$ we have that $d(z, z') < \sqrt{2}$ implies either $z, z' \in U$ and $z, z' \in V$.

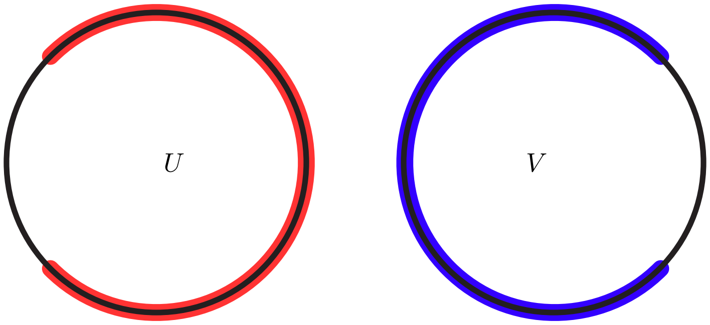

FIGURE 2. An open cover for the circle is given by the two sets U and V.

So we have some $\sigma : I \to S^1$, $\sigma(0) = 1$. Since I is compact this implies $\exists \delta > 0$ such that $d(s, s') < \delta$ implies $d(\sigma(s), \sigma(s')) < \sqrt{2}$.

Now we decompose the interval. Let $0 = s_0 < s_1 < \ldots < s_m = 1$ such that $|s_i - s_{i-1}| < \delta$ for all i. Then $\sigma([s_{i-1}, s_i]) \subset U$ or V for all i.

Now we look at the inverse image in $\mathbb{R}$ and we get these overlapping intervals covering $\mathbb{R}$ as in Fig. 3.

In particular

$$p^{-1}(U) = \bigcup_{n\in\mathbb{Z}} \tilde{U}_n \qquad \tilde{U}_n = \left(n - \frac{3}{8}, n + \frac{3}{8}\right) \tag{2.15}$$

$$p^{-1}(V) = \bigcup_{n\in\mathbb{Z}} \tilde{V}_n \qquad \tilde{V}_n = \left(n + \frac{1}{8}, n + \frac{7}{8}\right) \,. \tag{2.16}$$

The restrictions

$$p|_{\tilde{U}_n} : \tilde{U}_n \to U \qquad p|_{\tilde{V}_n} : \tilde{V}_n \to V \tag{2.17}$$

are homeomorphisms for all n. This means if we try to lift the path from s_0 to s_1 we have no choice in the lift since it stays in U in S^1. So we get a unique lift. In detail, we define $\tilde{\sigma}$ inductively on $[0, s_k]$, $0 \leq k \leq m$. For $k = 0$, $\sigma(0) = 0$. So now suppose $\tilde{\sigma}$ is defined on $[0, s_{k-1}]$ for some k such that $1 \leq k \leq m$. Then WLOG

$$\sigma([s_{k-1}, s_k]) \subset U \tag{2.18}$$

(rather than V). This means $\tilde{\sigma}(s_{k-1}) \in \tilde{U}_r$ for some $r \in \mathbb{Z}$. Since $p|_{\tilde{U}_r}$ is a homeomorphism let $q_r : U \to \tilde{U}_r$ be $\left(p|_{\tilde{U}_r}\right)^{-1}$. Define

$$\tilde{\sigma}|_{[s_{k-1}, s_k]} = q_r \sigma|_{[s_{k-1}, s_k]} \tag{2.19}$$

and note this is the only choice. Therefore by induction $\tilde{\sigma}$ exists and is unique. □

Lemma 2.9 (Homotopy lifting)**.** *Let $\sigma, \tau : I \to S^1$ be paths with $\sigma(0) = \tau(0) = 1$ and let $F : I \times I \to S^1$ be a homotopy from σ to τ $(\operatorname{rel} \partial I)$. (So $\sigma(1) = \tau(1)$). Then there*

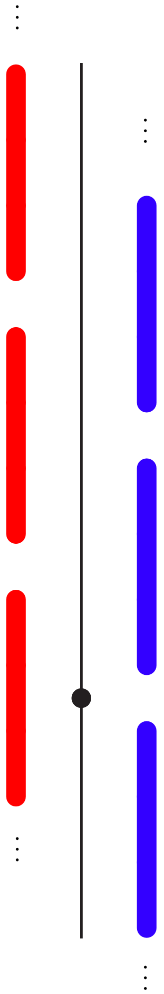

FIGURE 3. The preimage of the cover U and V in $\mathbb{R}$ under the map p.

exists a unique $\tilde{F} : I \times I \to \mathbb{R}$*, a homotopy from* $\tilde{\sigma}$ *to* $\tilde{\tau}$ $(\operatorname{rel} \partial I)$*, i.e. the following diagram commutes:*

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \times I \arrow[ur, dashed, "\tilde{F}"] \arrow[r, "F"] & S^1 \end{tikzcd} . \tag{2.20}$$

PROOF. This proof is very similar to the proof of Lemma 2.8. $I \times I$ is compact so there exists some $0 < s_0 < s_1 < \ldots < s_m = 1$ and $0 = t_0 < t_1 < \ldots < t_n = 1$ such that if

$$R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j] \tag{2.21}$$

(for $1 \leq i \leq m$, $1 \leq j \leq n$) then $F(R_{ij}) \subset U$ or V. Order the $R_{i,j}$ and relabel them as R_k. Then let

$$S_k = \bigcup_{i \leq k} R_i \ . \tag{2.22}$$

Now we define $\tilde{F}$ inductively on S_k. First $\tilde{F}(S_0) = \tilde{F}(0,0) = 0$. Now suppose $\tilde{F}$ is defined on S_{k-1}. Then $S_k = S_{k-1} \cup R_k$ and $F(R_k) \subset U$ or V; say U. Then $S_{k-1} \cap R_k$ is nonempty and connected and in U, and therefore

$$\tilde{F}(S_{k-1} \cap R_k) \subset \tilde{U}_r \tag{2.23}$$

for some $r \in \mathbb{Z}$. So we can (and must) define

$$\left.\tilde{F}\right|_{R_k} = q_r \left.F\right|_{R_k} \ . \tag{2.24}$$

Therefore $\tilde{F}$ exists and is unique. $\tilde{F}_0$ is a lift of $F_0 = \sigma$, starting at 0, $\tilde{F}_1$ is a lift of $F_1 = \tau$, starting at 0, therefore by the uniqueness of Lemma 2.8, $\tilde{F}_0 = \tilde{\sigma}$ and $\tilde{F}_1 = \tilde{\tau}$. Finally $p^{-1}(1) = \mathbb{Z}$ is discrete in $\mathbb{R}$, so $\tilde{F}_t(0) = \tilde{\sigma}(0) = 0$ and $\tilde{F}_t(1) = \tilde{\sigma}(1)$ for all $t \in I$, so indeed $\tilde{\sigma} \simeq_{F_t} \tilde{\tau} \, (\mathrm{rel}\, \partial I)$. □

Proof of Theorem 2.7. Let σ be a loop in S^1 at 1. Define $\varphi : \pi_1(S^1, 1) \to \mathbb{Z}$ by $\varphi([\sigma]) = \tilde{\sigma}(1)$. Then we claim:

1. φ is well defined: This follows from Lemmata 2.8 and 2.9.
2. φ is onto: Let $n \in \mathbb{Z}$; define $\tilde{\sigma} : I \to \mathbb{R}$ by $\tilde{\sigma}(s) = ns$. Let $\sigma = p\tilde{\sigma}$. Then $\varphi([\sigma]) = \tilde{\sigma}(1) = n$.
3. φ is a homomorphism: Let $[\sigma], [\tau] \in \pi_1(S^1, 1)$ with $\varphi([\sigma]) = m$, $\varphi([\tau]) = n$ and

$$[\sigma][\tau] = [\sigma * \tau] \ . \tag{2.25}$$

Then

$$\widetilde{\sigma * \tau} = \tilde{\sigma} * \hat{\tau} \tag{2.26}$$

where $\hat{\tau}(s) = \tilde{s} + m$ and therefore

$$\varphi([\sigma][\tau]) = \varphi([\sigma * \tau]) = \left(\widetilde{\sigma * \tau}\right)(1) = (\tilde{\sigma} * \hat{\tau})(1) = m + n = \varphi([\sigma]) + \varphi([\tau]) \tag{2.27}$$

as desired.

4. φ is one-to-one: Suppose $\varphi([\sigma]) = 0$, i.e. $\tilde{\sigma}$ is a loop in $\mathbb{R}$ at 0. $\mathbb{R}$ strong deformation retracts to 0, so $\tilde{\sigma} \simeq x_0 \, (\mathrm{rel}\, \partial I)$. Therefore $[\sigma] = 1$.

□

2. Induced homomorphisms

So consider some map $f : (X, x_0) \to (Y, y_0)$. Then the claim is that we can define a group homomorphism

$$f_* : \pi(X, x_0) \to \pi_1(Y, y_0) \tag{2.28}$$

by $f_*([\sigma]) = [f\sigma]$.

Theorem 2.10. *(1) f_* is a well-defined homomorphism.*
(2) $(gf)_* = g_* f_*$, $\mathrm{id}_* = \mathrm{id}$.
(3) $f \simeq g \, (\mathrm{rel}\, x_0)$ implies $f_* = g_*$.

PROOF. (1) Suppose we have two loops $\sigma \simeq_{F_t} \sigma' \, (\operatorname{rel} \partial I)$. Then this means

$$f\sigma \simeq_{fF_t} f\sigma' \, (\operatorname{rel} \partial I) \tag{2.29}$$

which means f_* is well-defined.

(2)

$$f_*([\sigma][\tau]) = f_*([\sigma * \tau]) = [f(\sigma * \tau)] \tag{2.30}$$

$$= [f(\sigma) * f(\tau)] = [f\sigma][f\tau] \tag{2.31}$$

$$= f_*([\sigma]) f_*([\tau]) \ . \tag{2.32}$$

□

Lecture 5; September 12, 2019

EXAMPLE 2.2. Let $\mu_n(z) = z^n$ for $n \in \mathbb{Z}$. Then

$$\mu_{n*} : \pi_1(S^1, 1) \to \pi_1(S^1, 1) \tag{2.33}$$

is multiplication by n, i.e.

$$\begin{tikzcd} \pi_1(S^1,1) \arrow[r, "\mu_{n*}"] \arrow[d, "\varphi"] & \pi_1(S^1,1) \arrow[d, "\varphi"] \\ \mathbb{Z} \arrow[r, "\times n"] & \mathbb{Z} \end{tikzcd} \tag{2.34}$$

PROOF. The loop $\sigma : I \to S^1$ with $\sigma(s) = e^{2\pi i s}$ represents $1 \in \mathbb{Z} \simeq \pi_1(S^1, 1)$, i.e. $\varphi([\sigma]) = 1$. Then

$$\mu_{n*}([\sigma]) = [\mu_n \sigma] \tag{2.35}$$

where $(\mu_n \sigma)(s) = e^{2\pi i n s}$. Therefore $\varphi([\mu_n \sigma]) = n$. □

The following is an application of Theorem 2.7.

THEOREM 2.11. *There is no retraction $D^2 \to S^1$.*

PROOF. Let $r : D^2 \to S^1$ be a retraction. Then

$$\begin{tikzcd} & D^2 \arrow[dr, "r"] & \\ S^1 \arrow[ur, hook, "i"] \arrow[rr, "\mathrm{id}"] & & S^1 \end{tikzcd} \tag{2.36}$$

commutes. Then we apply the functor π_1 to get:

$$\begin{tikzcd} & \pi_1(D^2,1) \arrow[dr, "r_*"] & \\ \pi_1(S^1,1) \arrow[ur, hook, "i_*"] \arrow[rr, "\mathrm{id}_*"] & & \pi_1(S^1,1) \end{tikzcd} = \begin{tikzcd} & 0 \arrow[dr] & \\ \mathbb{Z} \arrow[ur] \arrow[rr] & & \mathbb{Z} \end{tikzcd} \tag{2.37}$$

which is a contradiction. □

COROLLARY 2.12. *Every map $D^2 \to D^2$ has a fixed point.*

3. Dependence of π_1 on the basepoint

Let $\alpha : I \to X$ be a path from x_0 to x_1. Define $\alpha_{\#} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\alpha_{\#}[\sigma] = [\bar{\alpha}][\sigma][\alpha] = [\bar{\alpha} * \sigma * \alpha] \ . \tag{2.38}$$

The point is that we take a loop at one basepoint and attach this path on both ends (with one reversed) to get a loop at the other basepoint.

Theorem 2.13. *(1) $\alpha_{\#}$ is an isomorphism.*
(2) $\alpha \simeq \beta \,(\mathrm{rel}\,\partial I)$ implies $\alpha_{\#} = \beta_{\#}$.
(3) $(\alpha * \beta)_{\#} = \beta_{\#}\alpha_{\#}$.
(4) If $f : X \to Y$ is a map then

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\alpha_{\#}"] & \pi_1(Y, f(x_0)) \arrow[d, "(f\alpha)_{\#}"] \\ \pi_1(X, x_1) \arrow[r, "f_*"] & \pi_1(Y, f(x_1)) \end{tikzcd} \tag{2.39}$$

commutes.

Remark 2.3. Professor Cameron learned algebraic topology from Hilton and Wylie. He is surprised he learned anything at all because they decided that since homology is a covariant functor and cohomology is a contravariant functor homology should be cohomology and cohomology should be "contrahomology". They also wrote all of the maps on the right.

Proof. (1) We can directly verify $\alpha_{\#}$ is a homomorphism:

$$\alpha_{\#}([\sigma][\tau]) = [\bar{\alpha}][\sigma][\tau][\alpha] = [\bar{\alpha}][\sigma][c_{x_0}][\tau][\alpha] \tag{2.40}$$

$$= [\bar{\alpha}][\sigma]([\alpha][\bar{\alpha}])[\tau][\alpha] \tag{2.41}$$

$$= \alpha_{\#}([\sigma])\,\alpha_{\#}([\tau]) \ . \tag{2.42}$$

$\alpha_{\#}$ is a bijection because $(\bar{\alpha})_{\#}$ is an inverse of $\alpha_{\#}$.
(2) Obvious.
(3) Exercise.
(4) Exercise.

□

Remark 2.4. If $x_1 = x_0$ then α is a loop at x_0 so

$$\alpha_{\#}([\sigma]) = [\bar{\alpha}][\sigma][\alpha] = [\alpha]^{-1}[\sigma][\alpha] \tag{2.43}$$

i.e. $\alpha_{\#}$ is conjugation by α.

Corollary 2.14. *If X is path-connected then $\pi_1(X, x_0)$ is independent of x_0 (up to isomorphism).*

Definition 2.4. X is *simply connected* if X is path-connected and $\pi_1(X) = 1$.

Lemma 2.15. *Let $f, g : X \to Y$. Let F be a homotopy from f to g. Let α be the path $\alpha(t) = F(x_0, t)$ in Y from $f(x_0)$ to $g(x_0)$. Then the diagram*

$$\begin{tikzcd} & \pi_1(Y, f(x_0)) \arrow[dd, "\alpha_{\#}", "\cong"'] \\ \pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\ & \pi_1(Y, g(x_0)) \end{tikzcd} \tag{2.44}$$

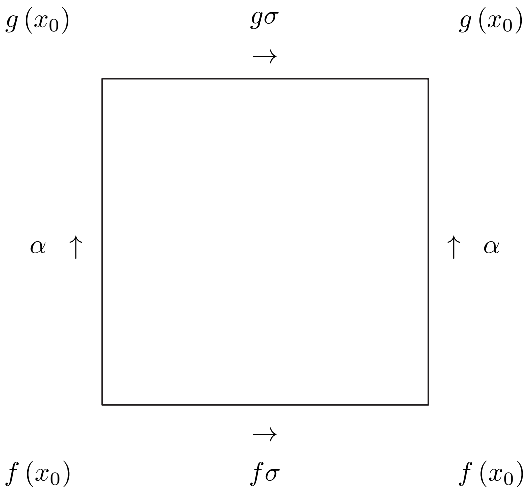

FIGURE 4. $I \times I$ labelled as in Lemma 2.15.

commutes.

PROOF. Let σ be a loop in X at x_0. Let $H = F(\sigma \times \mathrm{id}) : I \times I \to Y$. Pictorially we have: $H|_{\partial(I\times I)}$ (suitably reparameterized) is a loop in Y at $g(x_0)$. Reading Fig. 4 counterclockwise we get

$$\left[H|_{\partial(I\times I)}\right] = [\bar{\alpha}]\,[f\sigma]\,[\alpha]\,[g\sigma]^{-1} \ . \tag{2.45}$$

But this extends over $I \times I$ so it is just $1 \in \pi_1(Y, g(x_0))$, i.e. $\alpha_{\#} f_*([\sigma]) = g_*([\sigma])$ which implies $\alpha_{\#} f_* = g_*$. ☐

THEOREM 2.16. *Let $f : X \to Y$ be a homotopy equivalence. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism for all $x_0 \in X$.*

PROOF. Let $g : Y \to X$ be a homotopy inverse of f. So we get maps

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y_* f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0)) \tag{2.46}$$

but we have $gf \simeq \mathrm{id}_X$ which means $(gf)_* = g_* f_*$ is an isomorphism. by Lemma 2.15 and Theorem 2.13 (1) so g_* is onto.

Then we have

$$\pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, gf(x_0)) \xrightarrow{f_*} \pi_1(Y, fgf(x_0)) \tag{2.47}$$

and $fg \simeq \mathrm{id}_Y$ so $f_* g_*$ is an isomorphism so g_* is one-to-one. Therefore g_* is an isomorphism and $g_* f_*$ is an isomorphism which implies f_* is an isomorphism. ☐

COROLLARY 2.17. *If X is contractible then $\pi_1(X, x_0) = 1$ for all $x_0 \in X$.*

PROOF. X being contractible is equivalent to $i : \{x_0\} \hookrightarrow X$ being a homotopy equivalence. ☐

4. Fundamental theorem of algebra

THEOREM 2.18 (Fundamental theorem of algebra). *Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

REMARK 2.5. Professor Cameron says this topological proof of an algebraic statement should provide topologists some comfort in light of the Poincaré conjecture receiving some sort of proof via PDEs.

PROOF. Let $f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ for all $a_i \in \mathbb{C}$ for $n \geq 1$. Suppose $f(z) \neq 0$ for all $z \in \mathbb{C}$; so $a_0 \neq 0$. So f is a map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$.

The idea of the proof is as follows. As $|z| \to \infty$ somehow $f \rightsquigarrow z^n$, i.e. $f \rightsquigarrow \mu_n$ and similarly as $|z| \to 0$ somehow $f \rightsquigarrow a_0$, i.e. $f \rightsquigarrow c_{a_0}$ but this implies $\mu_n \simeq c_{a_0}$ which is a contradiction. Now we make this precise.

Let $R \in \mathbb{R}$, $R \geq 0$. Define $F : S^1 \times I \to S^1$ by

$$F(z,t) = F_t(z) = \frac{f(tRz)}{|f(tRz)|} \ . \tag{2.48}$$

Note that $F_0(z) = 1$ so $F_0 = c_1$.

Let

$$g(z) = a_{n-1}z^{n-1} + \ldots + a_0 \tag{2.49}$$

so $f(z) = z^n + g(z)$. Choose

$$R > \max\left\{1, \sum_{k=0}^{n-1} |a_k|\right\} \ . \tag{2.50}$$

Then if $|z| = R$

$$|g(z)| \leq \sum_{k=0}^{n-1} \left|a_k z^k\right| = \sum_{k=0}^{n-1} |a_k| R^k \leq^{2.2} \left(\sum_{k=0}^{n-1} |a_k|\right) R^{n-1} < R^n = |z|^n \ . \tag{2.51}$$

Therefore for all $t \in I$

$$|z^m + tg(z)| \geq |z|^n - t|g(z)| \geq |z|^n - |g(z)| > 0 \ . \tag{2.52}$$

Let $h_t(z) = z^n + tg(z)$. So $h_0(z) = z^n$ and $h_1(z) = f(z)$.

So define $H : S^1 \times I \to S^1$ by

$$H(z,t) = \frac{h_t(Rz)}{h_t(Rz)} \tag{2.53}$$

and

$$H_1(z) = \frac{f(Rz)}{|f(Rz)|} = F_1(z) \qquad\qquad H_0(z) = \frac{(Rz)^n}{|Rz|^n} = z^n \tag{2.54}$$

and therefore $H_0 = \mu_n$.

Now we can put it all together. We have

$$c_1 = F_0 \simeq F_1 = H_1 \simeq H_0 = \mu_n \tag{2.55}$$

[2.2] Since $R > 1$

so $c_1 \simeq \mu_n$. Therefore by Lemma 2.15

$$\begin{tikzcd} & \pi_1\left(S^1,1\right) \arrow[dd, "\alpha_\#"] \\ \pi_1\left(S^1,1\right) \arrow[ur, "\mu_{n*}"] \arrow[dr, "c_{1*}"'] & \\ & \pi_1\left(S^1,1\right) \end{tikzcd} = \begin{tikzcd} & \mathbb{Z} \arrow[dd, "\simeq"] \\ \mathbb{Z} \arrow[ur, "\times n"] \arrow[dr, "0"'] & \\ & \mathbb{Z} \end{tikzcd} \tag{2.56}$$

which is a contradiction. Note $\alpha_\#$ is conjugation by $[\alpha]$, but $\mathbb{Z}$ is abelian so this is just the identity. ☐

5. Cartesian products

If $f : Z \to X$ and $g : Z \to Y$ are maps, let $(f,g) : Z \to X \times Y$ be the map given by

$$(f,g)(z) = (f(z), g(z)) \ . \tag{2.57}$$

We will use the same notation for group homomorphisms. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projections.

THEOREM 2.19. $(p_*, q_*) : \pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

PROOF. Exercise. ☐

EXAMPLE 2.3. $\pi_1(T^n) \simeq \mathbb{Z}^n$.

6. Retractions

Consider some $i : A \hookrightarrow X$ and some $a_0 \in A$. Suppose there exists a retraction $r : X \to A$. Then we get a commutative diagram

$$\begin{tikzcd} & \pi_1(X, a_0) \arrow[dr, "r_*"] & \\ \pi_1(A, a_0) \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & \pi_1(A, a_0) \end{tikzcd} \tag{2.58}$$

i.e. $r_* i_* = \mathrm{id}$ so therefore i_* is one-to-one and r_* is onto.

CHAPTER 3

Van Kampen's theorem

The idea is to calculate the fundamental group of $X = X_1 \cup X_2$ from the data of the groups $\pi_1(X_1)$, $\pi_1(X_2)$, $\pi_1(X_1 \cap X_2)$ and the associated inclusion maps. These always fit into the diagram:

$$\begin{tikzcd}
\pi_1(X_1 \cap X_2) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"] & \pi_1(X_1) \arrow[d, "j_{1*}"] \\
\pi_1(X_2) \arrow[r, "j_{2*}"] & \pi_1(X)
\end{tikzcd} . \tag{3.1}$$

Lecture 6;
September 24, 2019

1. Basic combinatorial group theory

1.1. Free products. Let A and B be groups. The free product is a group D and homomorphisms $i_A : A \to D$ and $i_B : B \to D$ such that given a group G and homomorphisms $\alpha : A \to G$ and $\beta : B \to G$, there exists a unique homomorphism $\varphi : D \to G$ such that $\varphi i_A = \alpha$ and $\varphi i_B = \beta$; i.e. the following push-out diagram commutes:

$$\begin{tikzcd}
 & A \arrow[d, "i_A"] \arrow[ddr, bend left] & \\
B \arrow[r, "i_B"] \arrow[drr, bend right, "\beta"'] & D \arrow[dr, dashed, "\varphi"] & \\
 & & G
\end{tikzcd} . \tag{3.2}$$

Remark 3.1. This is the coproduct in the category **Grp**.

Lemma 3.1. *If D exists then it is unique (up to unique isomorphism).*

Proof. If D, D' are free products of A and B there exist unique φ, ψ as shown:

$$\begin{tikzcd}
D' \arrow[dr, shift left, "\psi"] & A \arrow[l, "i'_A"'] \arrow[d, "i_A"] \\
B \arrow[u, "i'_B"] \arrow[r, "i_B"'] & D \arrow[ul, shift left, "\varphi"]
\end{tikzcd} \tag{3.3}$$

□

Now we can write *the* free product of A and B as $A * B$, and say A and B are the *factors* of $A * B$.

Theorem 3.2. *Free products exist.*

PROOF. Define $A * B$ to be the set of all sequences $x = (x_1x_2 \dots x_m)$ for $m \geq 0$ where $x_i \in A$ or B; $x_i \neq 1$; and x_i and x_{i+1} are in different factors. m is the *length* of x, 1 is the empty sequence.

Define the product of two sequence by

$$(x_1 \dots x_m)(y_1 \dots y_n) = \begin{cases} (x_1 \dots x_m y_1 \dots y_n) & x_m, y_1 \in \text{ different fac.} \\ (x_1 \dots x_{m-1}(x_m y_1) y_2 \dots y_m) & x_m, y_1 \in \text{ same fac.}, x_m y_1 \neq 1 \\ (x_1 \dots x_{m-1})(y_2 \dots y_n) & x_m, y_1 \in \text{ same fac.}, x_m y_1 = 1 \end{cases} . \tag{3.4}$$

Then $x1 = 1x = x$ for all $x \in A * B$. Define the inverse $x^{-1} = \left(x_m^{-1} \dots x_1^{-1}\right)$, then $xx^{-1} = 1 = x^{-1}x$ for all $x \in A * B$.

So we just have to check associativity. $x(yz) = (xy)z$. We induct on n (where n is the length of y). $n = 0$ is trivial. For $n = 1$, WLOG $y = (a)$ for $a \in A$, $a \neq 1$. Then there are several cases, e.g.

- $x_m \in B$, $z_1 \in B$;
- $x_m \in B$, $z_1 \in A$;
- $\dots$

For $n > 1$, $y = y'y''$ where the length of y' and y'' are less than n. Then by induction

$$(xy)z = (x(y'y''))z = ((xy')y'')z = (xy')(y''z) . \tag{3.5}$$

On the other hand

$$x(yz) = x((y'y'')z) = x(y'(y''z)) = (xy'(y''z)) \tag{3.6}$$

so $A * B$ is a group.

Now we see it has the universal properties. Define $i_A : A \to A * B$ by $i_A(a) = (a)$ and similarly for B. Given $\alpha : A \to G$ and $\beta : B \to G$ then define $\varphi : A * B \to G$ by

$$\varphi(x_1 \dots x_m) = \gamma(x_1) \dots \gamma(x_m) \tag{3.7}$$

where $\gamma_i = \alpha$ if $x_i \in A$ and $\gamma_i = \beta$ if $x_i \in B$. Therefore φ is unique as a homomorphism and it has the desired properties:

$$\varphi i_A = \alpha \qquad\qquad \varphi i_B = \beta . \tag{3.8}$$

□

REMARK 3.2. i_A and i_B are actually injective in this case. For general pushouts they won't be. This means we can identify A and B with their images in $A * B$. In other words we can drop the parentheses. Each element in $A * B$ has a unique expression as $x_1 \dots x_m$ for $x_i \in A$ or B; $x_i \neq 1$; x_i, x_{i+1} in different factors. This is called a *reduced word* in A and B. We say $A * B$ is a non-trivial free product if $A \neq \text{id} \neq B$.

EXAMPLE 3.1. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \{1, a, b, ab, ba, aba, bab, abab, \dots\}$.

REMARK 3.3. A non-trivial free product is infinite: $a \in A \setminus \{1\}$, $b \in B \setminus \{1\}$, then ab has infinite order in $A * B$.

REMARK 3.4. We can define free product $* A_\lambda$ for any collection $\{A_\lambda\}$ of groups. As a special case we can take $A_\lambda = \mathbb{Z}$ for all λ, then write x_λ for the generator of A_λ. Then $* A_\lambda$ is the *free group on the set* $X = \{x_\lambda\}$, written $F(X)$.

Any function $\{x_{\lambda_0}\} \to G$ extends to a unique homomorphism $\mathbb{Z} = \mathbb{Z}(x_{\lambda_0}) \to G$. So $F(X)$ is characterized by the following. For any group G and any function $f : X \to G$, there exists unique homomorphism $\varphi : F(X) \to G$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & G \\ F(X) \arrow[ur, dashed, "\varphi"] & \end{tikzcd} \tag{3.9}$$

commutes.

Note that every element of $F(X)$ has a unique expression as a reduced word

$$x_{\lambda_1}^{n_1} x_{\lambda_2}^{n_2} \dots x_{\lambda_m}^{n_m} \tag{3.10}$$

where $\lambda_i \neq \lambda_{i+1}$ and $n_i \in \mathbb{Z}$, $n_i \neq 0$.

Definition 3.1. $(X : R)$ is a *presentation* of the group G if $R \subset F(X)$ and there exists an epimorphism $\varphi : F(X) \to G$ such that $\ker \varphi = N(R)$ is the normal closure of R in $F(X)$, the smallest normal subgroup of $F(X)$ that contains R, i.e. the set of all products of conjugates of elements in R and their inverses:

$$\left\{ \prod_{i=1}^{k} u_i^{-1} r_i^{\epsilon_i} u_i \,\middle|\, u_i \in F(X), r_i \in R, \epsilon_i = \pm 1 \right\} . \tag{3.11}$$

Theorem 3.3. *Every group has a presentation.*

Proof. Let $Y = \{y_\lambda\}$ be any set of generators of G. (Worst case we could take $Y = G$.) Let $X = \{x_\lambda\}$ and define a function $f : X \to G$ by mapping $f(x_\lambda) = y_\lambda$. f extends to a homomorphism $\varphi : F(X) \to G$ which is onto (since Y generates G). (Worst case take $R = \ker \varphi$.) ☐

Definition 3.2. G is *finitely generated* iff there exists a finite set of generators for G, iff G has a presentation $(X : R)$ with X finite.

G is *finitely presentable* iff there exists a presentation $(X : R)$ of G with X and R finite.

If $(X : R)$ is a presentation of G we say R is the set of relators, and X is the set of generators. We often suppress φ, i.e. we regard $X \subset G$, and write $r = 1$ $(r \in R)$ to indicate that $\varphi(r) = 1 \in G$. We also write $G = \langle X : R \rangle$. Write $(\{x_1, \dots\} : \{r_1, \dots\})$ as $(x_1, \dots : r_1, \dots)$.

Example 3.2.
1. $\langle X \,|\, \emptyset \rangle \cong F(X)$,
2. $\langle \{x\} \,|\, x^n = 1 \rangle \cong \mathbb{Z}_{|n|}$,
3. $\langle \{x, y\} \,|\, xy = yx \rangle \cong \mathbb{Z} \times \mathbb{Z}$,
4. $\langle a, b \,|\, a^n = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle$

Theorem 3.4. *If $X \cap Y = \emptyset$ then $\langle X \,|\, R \rangle * \langle Y \,|\, S \rangle \cong \langle X \cup Y \,|\, R \cup S \rangle$.*

Example 3.3. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle a, b \,|\, a^2 = b^2 = 1 \rangle$.

Exercise 1.1.
1. Show $\langle x, y \,|\, x^{-1}yx = y^2, y^{-1}xy = x^2 \rangle$ is trivial.
2. Is $\langle x, y, z \,|\, x^{-1}yx = y^2, y^{-1}zy = z^2, z^{-1}xz = x^2 \rangle$ trivial?
3. Is $\langle x, y, z, w \,|\, x^{-1}yx = y^2, y^{-1}zy = z^2, z^{-1}wz = w^2, w^{-1}xw = x^2 \rangle$ trivial?
4. Show $\langle x, y \,|\, xyx = yxy, x^n = y^{n+1} \rangle$ is trivial for $n \geq 0$.

These are called balanced presentations of the trivial group because the set of relators is the same size as the set of generators.

Conjecture 1. *Every balanced presentation can be taken to the trivial balanced presentation under some particular "moves".*

Remark 3.5. Dealing with groups via their finite presentations is notoriously difficult. It is an unsolvable problem to find whether a group is trivial based only on a presentation. There is not algorithm which determines whether a group is trivial from its finite presentation.

Remark 3.6. Novikov and Boone proved this unsolvability independently.[3.1]

1.2. Push-outs. Let A, B, and C be groups with homomorphisms $j_A : C \to A$, $j_B : C \to B$. Then a push-out of this diagram is a group D and homomorphisms $i_A : A \to D$ and $i_B : B \to D$ such that $i_A j_A = i_B j_B$, and given a group G and homomorphisms $\alpha : A \to G$, $\beta : B \to G$ such that $\alpha j_A = \beta j_B$ there exists a unique homomorphism $\varphi : D \to G$ such that $\varphi i_A = \alpha$ and $\varphi i_B = \beta$. In other words, the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "j_A"] \arrow[d, "j_B"] & A \arrow[d, "i_A"] \arrow[ddr, bend left, "\alpha"] & \\
B \arrow[r, "i_B"] \arrow[drr, bend right, "\beta"] & D \arrow[dr, dashed, "\varphi"] & \\
 & & G
\end{tikzcd} \ . \tag{3.12}$$

Theorem 3.5. *Push-outs exist.*

Proof. Consider $A * B$ (regard $A, B \subset A * B$). Let

$$N = N\left(\left\{ j_A\left(c\right) j_b\left(c\right)^{-1} \,\middle|\, c \in C \right\}\right) \lhd A * B \ . \tag{3.13}$$

Let $q : A * B \to A * B / N$ be quotient homomorphisms. Define $i_A = q\big|_A$, $i_B = q\big|_B$. Then $A * B / N$ is the pushout. $\square$

Remark 3.7. In the case $A * B$ (i.e. $C = 1$), i_A and i_B are injective. This is not true in an arbitrary pushout.

Counterexample 1. Consider $\mathbb{Z}\left(y\right) \xleftarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}\left(x\right)$. Then the pushout is $\langle x, y \mid x = 1 \rangle \simeq \mathbb{Z}$:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\text{id}"] \arrow[d, "0"] & \mathbb{Z}\left(x\right) \arrow[d, "0"] \\
\mathbb{Z}\left(y\right) \arrow[r, "\text{id}"] & \mathbb{Z}
\end{tikzcd} \tag{3.14}$$

so i_A is not injective.

If j_A and j_B are injective, then i_A and i_B are injective. The pushout is called the free product of A and B amalgamated along C, $A *_C B$.

[3.1] Boone was at UIUC at the time. He had only one copy of his manuscript and when he biked home in the snow, he lost it. So he summoned his graduate students to find it.

Example 3.4. Take $\mathbb{Z} \xleftarrow{q} \mathbb{Z} \xrightarrow{p} \mathbb{Z}$ for $p, q \geq 2$. Then the pushout is:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times p"] \arrow[d, "\times q"] & \mathbb{Z}\left(x\right) \arrow[d] \\
\mathbb{Z}\left(y\right) \arrow[r] & \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}
\end{tikzcd} \tag{3.15}$$

where $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle x, y \,|\, x^p = y^q \rangle$.

Lecture 7;
September 26, 2019

Theorem 3.6. *Push-outs exist.*

Proof. So we need to show that given $B \leftarrow C \rightarrow A$ we have:

$$\begin{tikzcd}
C \arrow[r, "j_A"] \arrow[d, "j_B"] & A \arrow[d, "i_A"] \arrow[ddr, bend left, "\alpha"] & \\
B \arrow[r, "i_B"] \arrow[drr, bend right, "\beta"'] & D \arrow[dr, dashed, "\varphi"] & \\
 & & G
\end{tikzcd} . \tag{3.16}$$

Define

$$N = N\left(\left\{j_A\left(c\right) j_b\left(c\right)^{-1} \forall c \in C\right\}\right) \lhd A * B \ . \tag{3.17}$$

where we regard $A, B < A * B$. Let $q : A * B \rightarrow A * B/N$ be the quotient homomorphism and let $i_A = q|_A$ and $i_B = q|_B$. Then the diagram certainly commutes.

Claim 3.1. This is the push-out.

Proof. Given $\alpha : A \rightarrow G$ and $\beta : B \rightarrow G$ such that $\alpha j_A = \beta j_B : C \rightarrow G$ there exists $\psi : A * B \rightarrow G$ such that $\psi|_A = \alpha$ and $\psi|_B = \beta$ by Theorem 3.2.

Now $\alpha j_A = \beta j_B$ implies that $\psi j_A = \psi j_B$, which means $\psi\left(j_A\left(c\right) j_B\left(c\right)^{-1}\right) = 1$ for all $c \in C$, which means $\psi\left(N\right) = 1$. This means ψ induces $\varphi : A * B/N \rightarrow G$, i.e.

$$\begin{tikzcd}
A * B \arrow[rr, "q"] \arrow[dr, "\psi"'] & & A * B/N \arrow[dl, "\varphi"] \\
 & G &
\end{tikzcd} \tag{3.18}$$

commutes. I.e. $\varphi\, q|_A = \alpha$ and $\varphi q|_B = \beta$, i.e. $\varphi i_A = \alpha$ and $\varphi i_B = \beta$. Then $A \cup B$ generates $A * B$, and therefore $q\left(A\right) \cup q\left(B\right)$ generates $A * B/N$. Then φ is determined by $\varphi|_{q(A)}$ and $\varphi|_{q(B)}$. But we have to define $\varphi q|_A = \alpha$ and $\varphi q|_A = \beta$ and therefore φ is unique. □

■

Remark 3.8. It follows from Theorem 3.4 that if $A = \langle X \,|\, R \rangle$ and $B = \langle Y \,|\, S \rangle$ then the pushout it

$$\langle X \cup Y \,|\, R \cup S, j_A\left(c\right) = j_B\left(c\right), \forall c \in C \rangle \ . \tag{3.19}$$

We could also replace C by any set generating C.

Remark 3.9. Push-outs are defined and exist for $\{A_\lambda, j_\lambda : C \rightarrow A_\lambda\}$.

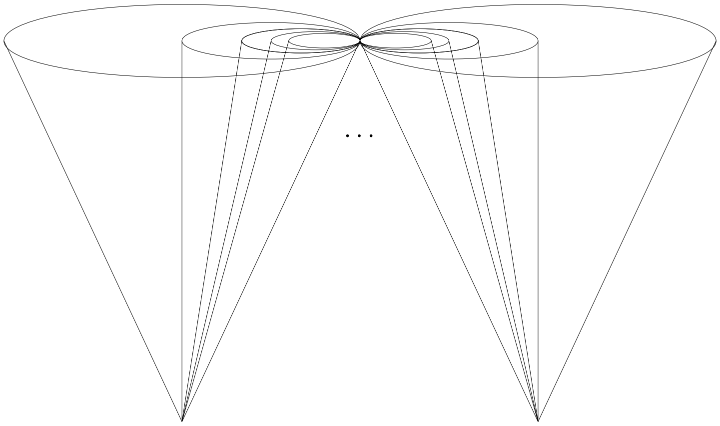

Figure 1. The cone of two Hawaiian earrings joined at a point. This has nontrivial fundamental group even though it is the union of two things with trivial fundamental group.

2. Statement of the theorem

Theorem 3.7 (van Kampen's theorem). *Suppose $X = X_1 \cup X_2$, with X_1, X_2, $X_1 \cap X_2$ open and path-connected. Let $x_0 \in X_1 \cap X_2$ Then $\pi_1(X, x_0)$ is the push-out:*

$$\begin{tikzcd}
\pi_1(X_1 \cap X_2, x_0) \arrow[r] \arrow[d] & \pi_1(X_1, x_0) \arrow[d] \\
\pi_1(X_2, x_0) \arrow[r] & \pi_1(X, x_0)
\end{tikzcd} \tag{3.20}$$

where all homomorphisms are induced by inclusions.

Remark 3.10. (1) There exists an analogous statement when $X = \cup_\lambda X_\lambda$.

(2) In practice, one often applies Theorem 3.7 when X_1 and X_2 are closed in X but where X_i has an open subsets $U_i \subset X$ such that there exists a strong deformation retractions of pairs:

$$(U_1, U_1 \cap X_2) \to (X_1, X_1 \cap X_2) \qquad (U_2, U_2 \cap X_1) \to (X_2, X_1 \cap X_2)\ . \tag{3.21}$$

(3) Theorem 3.7 is false without the openness assumption. The counterexample is the double cone of the wedge of two Hawaiian earrings as in Fig. 1. If we take each cone of an earring to be one of the X_i then the intersection is a point, so it satisfies all of the assumptions except the openness. But notice that each of the X_i is contractible since cones are always contractible. However $\pi_1(X_1 \cup X_2) \neq 1$ because the two earrings themselves form a nontrivial loop.

Corollary 3.8. *Let X_1, X_2 be as in Theorem 3.7. Then $\pi_1(X_1) = \pi_1(X_2) = 1$ which implies $\pi_1(X) = 1$.*

Corollary 3.9. *If $n \geq 2$ then $\pi_1(S^n) = 1$.*

Proof. We can write $S^n = D^n_+ \cup D^n_-$ where $D_+ \cap D_- \cong S^{n-1}$. For $n \geq 2$ S^{n-1} is path-connected and D^n is contractible, so $\pi_1(D_+) = \pi_1(D_-) = 1$. □

Corollary 3.10. *If* $\pi_1(X_1 \cap X_2) = 1$ *then* $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2)$.

3. Cell complexes

Let Y be a topological space and $f : S^{n-1} \to Y$ a map. Then the quotient space

$$X = (D^n \amalg Y) / \left(x \sim f(x), \forall x \in S^{n-1}\right) \tag{3.22}$$

is *obtained from Y by attaching an n cell*, and f is the *attaching map*.

Note that X has the quotient topology, i.e. if $q : D^n \amalg Y \to X$ is the quotient map, then $U \subset X$ is open iff $q^{-1}(U) \subset D^n \amalg Y$ is open. Also note that one can similarly add multiple n-cells to Y via maps $f_\lambda : S^{n-1}_\lambda \to Y$.

An *n-complex* (or n-dimensional CW complex) is defined inductively by:

- a -1-complex is $\emptyset$,
- an n-complex is a space obtained by attached a collection of n-cells to an $n-1$-complex so a 0-complex is a discrete set of points.

An n-complex is finite if there exists only finitely many cells. Note that we can define an inf-dimensional CW-complex by setting

$$X = \cup_{n=0}^{\infty} X_n \tag{3.23}$$

for X_n an n-complex such that $X_i \subset X_{i+1}$ for all i. Note that $U \subset X$ is open iff $U \cap X_n$ is open for all n.

For X a cell-complex define the *n-skeleton of X*, $X^{(n)}$, to be the union of all cells of dimension $\leq n$.

Example 3.5. The same space might have multiple cell decompositions. Consider S^2. We can think of this as a cell-complex as being one 0-cell, one 1-cell, and two 2-cells (hemispheres). A more simple way of seeing this is one 0-cell and a 2-cell with the entire boundary mapping to the point.

3.1. Effect of adding a cell on the fundamental group.

Example 3.6. Consider $X = Y \cup$ some 2-cell, i.e. $X = Y \cup_f D^2$ where f is the attaching map. Let $u \in D^2$. Then define $X_1 = Y \cup_f \left(D^2 \setminus \{0\}\right)$ and $X_2 = \operatorname{int}\left(D^2\right) \subset X$. This means X_1 and X_2 are open in X, $X = X_1 \cup X_2$, and

$$X_1 \cap X_2 = \operatorname{int}\left(D^2\right) \setminus \{0\} \simeq S^1 \times \mathbb{R} \simeq S^1 \ . \tag{3.24}$$

By van Kampen $\pi_1(X)$ is the push-out of

$$\begin{tikzcd}
\mathbb{Z} \simeq \pi_1(X_1 \cap X_2) \arrow[r, "j_{2*}"] & \pi_1(X_1) \\
\pi_1(X_2) = 1 &
\end{tikzcd} . \tag{3.25}$$

Let $w \in \pi_1(Y) = \pi_1(Y, f(1))$ be the element represented by $f : S^1 \to Y$. We have a strong deformation retraction $D^2 \setminus \{0\} \to S^1$, which induces a strong deformation retraction $X_1 \to Y$ so $j_{1*}(z) = w$ and

$$\boxed{\pi_1(X) / N(w)} \ . \tag{3.26}$$

Similarly, if we attach a collection of 2-cells we get $\pi_1(X) / N(\{w_\lambda\})$.

Remark 3.11. The same argument shows, using the fact that $\pi_1\left(S^{n-1}\right) = 1$ $(n \geq 3)$ that if $X = Y \cup$ some n-cells then the inclusion map $\pi_1(Y) \to \pi_1(X)$ is an isomorphism.

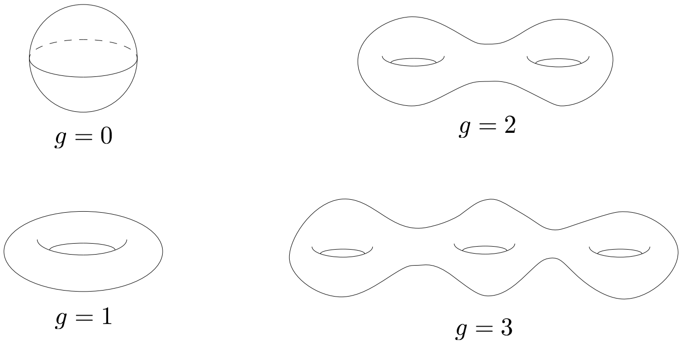

Figure 2. The first few closed orientable surfaces.

Example 3.7. Let

$$X = \bigvee_{\lambda} S_\lambda \ . \tag{3.27}$$

We can consider the open set around the common point:

$$U = \bigcup_{\lambda} I_\lambda \subset X \tag{3.28}$$

where each I_λ is an open interval in the S_λ around the common point. So attach this U to each circle:

$$X_\lambda = S^1_\lambda \cup U \tag{3.29}$$

and then apply van Kampen's theorem to get:

$$\pi_1(X) = \mathop{\ast}_{\lambda} \pi_1(X_\lambda) = \mathop{\ast}_{\lambda} \mathbb{Z} = F(\{x_\lambda\}) \ . \tag{3.30}$$

Lecture 8; October 1, 2019

Recall a closed n-dimensional manifold M is a compact metric space such that every $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$. Then a *surface* is a connected, closed 2-manifold.

If S_1 and S_2 are two surfaces, let $D_i \subset S_i$ be a disk, define $S_1 \# S_2$ (the *connect sum* of S_1 and S_2) to be

$$\mathrm{Cl}\left((S_1 \setminus D_1)\right) \cup_f \mathrm{Cl}\left((S_2 \setminus D_2)\right) \tag{3.31}$$

where f is any homeomorphism $f : \partial D_1 \to \partial D_2$. One can show that this is well-defined, i.e. the homeomorphism type of $S_1 \# S_2$ is independent of all choices.

Example 3.8. S^2 and T^2 are surfaces. Then we can take connect sums to get the closed orientable genus 2 surface $T^2 \# T^2$, and similarly we get the closed orientable genus 3 surface by taking $T^2 \# T^2 \# T^2$. These look as in Fig. 2.

The projective plane is given by:

$$\mathbb{P}^2 = S^2 / \left\{ x \sim -x \forall x \in S^2 \right\} \ . \tag{3.32}$$

This can be viewed as a Möbius band M with a disk D attached. We could also connect some these together to get what are call the *non-orientable surfaces* (or one-sided surfaces).

THEOREM 3.11. *Any surface is homeomorphic to one of the following:*

$$\#_g T^2, (g \geq 0) \qquad \#_k \mathbb{P}^2, (k \geq 1) \ . \tag{3.33}$$

EXAMPLE 3.9 (Surfaces). Recall $\pi_1\left(S^2\right) = 1$ and $\pi_1\left(T^2\right) = \mathbb{Z}^2$.

If we consider the wedge of two circles (a and b) we can also build a Torus by attaching a disk, so we get

$$\pi_1\left(T^2\right) = \left\langle a, b \,\middle|\, aba^{-1}b^{-1} = 1\right\rangle \cong \mathbb{Z}^2 \ . \tag{3.34}$$

Now we want to calculate the fundamental group of a closed orientable surface of any genus. First consider the $2g$ curves in Fig. 3. Call a neighborhood of these loops N. N itself is shown in the bottom of Fig. 3. Notice that N strong deformation retracts to the wedge of $2g$ circles, so we get:

$$\pi_1(N) = F_{2g} = F\left(a_1, b_1, \ldots, a_g, b_g\right) \ . \tag{3.35}$$

For $f : S^1 \to \partial N \subset N$ the attaching map we have

$$f_*(\text{gen.}) = \prod_{i=1}^{g} \left[a_i, b_i\right] \in \pi_1(N) = F\left(a_1, b_1, \ldots, a_g, b_g\right) \tag{3.36}$$

where $[g, h] = ghg^{-1}h^{-1}$. So by van Kampen we get:

$$\pi_1\left(\#_g T^2\right) = \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} \left[a_i, b_i\right] = 1 \right\rangle \ . \tag{3.37}$$

Note also that $\#_g T^2$ can also be viewed as in Fig. 4.

This tells us that it is a cell complex with one 0-cell, $2g$ 1-cells, and one 2-cell.

EXAMPLE 3.10. Recall $\mathbb{P}^2 = S^2/(x \sim -x)$. We can view S^2 as being two disks glued together along $S^1 \times I$. Then we can view $\mathbb{P}^2$ as a Möbius band B with a disk glued in. Recall $\pi_1(B) \simeq \mathbb{Z}$ so this disk kills certain classes and we get

$$\pi_1\left(\mathbb{P}^2\right) = \left\langle x \,\middle|\, x^2 = 1\right\rangle \simeq \mathbb{Z}/2\mathbb{Z} \ . \tag{3.38}$$

EXAMPLE 3.11. The Klein bottle[3.2] is glued out of two copies of $\mathbb{P}^2$. Recall it was the quotient of a square as in Fig. 5, where it is also shown as the union of two Möbius bands. We know:

$$\pi_1(KB) = \left\langle a, b \,\middle|\, aba^{-1}b = 1\right\rangle \tag{3.39}$$

but we also know:

$$\pi_1\left(\mathbb{P}^2 \# \mathbb{P}^2\right) = \left\langle x, y \,\middle|\, x^2 = y^2\right\rangle \ . \tag{3.40}$$

EXERCISE 3.1. Show these groups are isomorphic.

Now we can connect sum an arbitrary number of projective planes to get:

$$\#_k \mathbb{P}^2 \cong N \cup D^2 \tag{3.41}$$

where N is pictured in Fig. 6.

[3.2] The name of the Klein bottle is actually a joke in German based on the German words for surfaces and bottles.

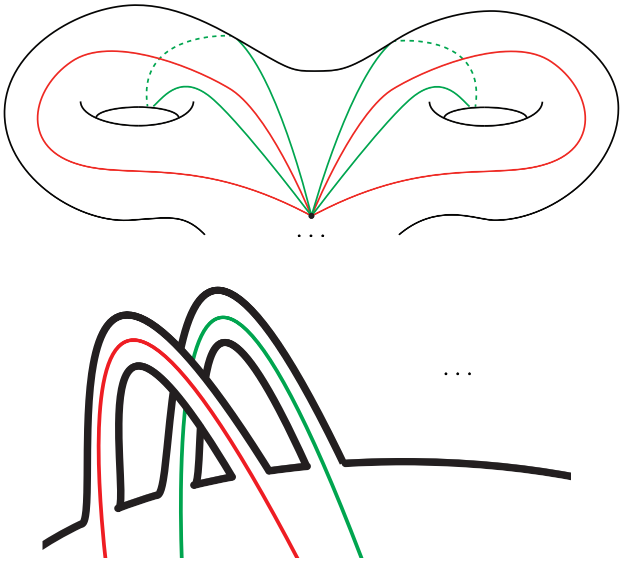

Figure 3. Top: Two “handles” of a closed surface. If we take a neighborhood of the two loops in each handle, the complement is just a disk. Bottom: A better picture of the neighborhood of these loops. Note this space strong deformation retracts to the wedge of $2g$ copies of S^1.

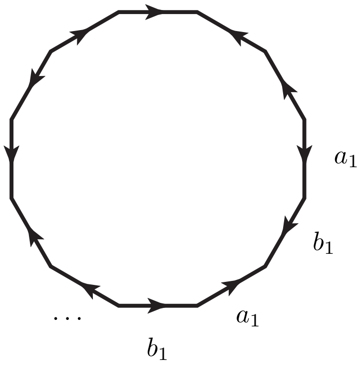

Figure 4. We can quotient this polygon to get a closed orientable genus g surface.

Then by van Kampen we get:

$$\pi_1\left(\#_k\mathbb{P}^2\right) = \left\langle a_1, \ldots, a_k \,\middle|\, \prod_{i=1}^{k} a_i^2 = 1 \right\rangle . \tag{3.42}$$

We can also view this as a quotient space as in Fig. 7.

This tells us this has a cell-composition with one 0-cell, k 1-cells, and one 2-cell.

Let G be a group. Then the *commutator subgroup*, $[G, G]$, is the subgroup generated by $\{[g, h] \mid g, h \in G\}$. Note that $[G, G] \lhd G$ and $G/[G, G] = G_{\text{ab}}$ is abelian. Call this the *abelianization* of G. Let A be an abelian group and α be a group homomorphism. Then

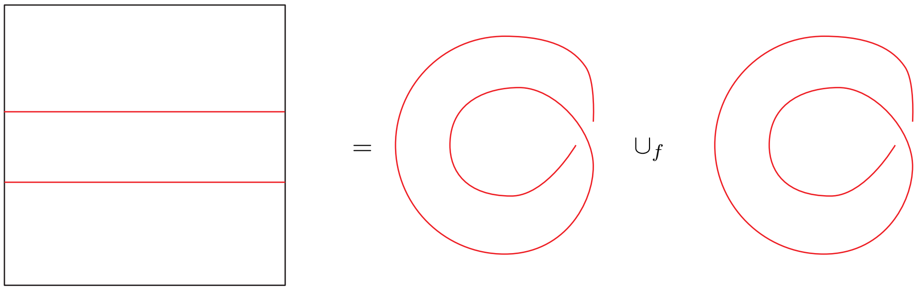

Figure 5. The Klein bottle viewed as a quotient of the square, and as the union of two Möbius bands.

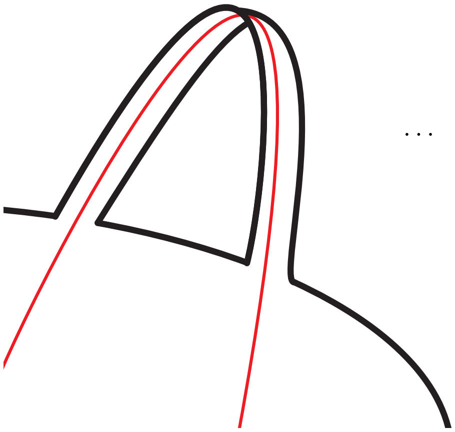

Figure 6. We can view the connect sum of k copies of $\mathbb{P}^2$ as being a disk glued to this space N.

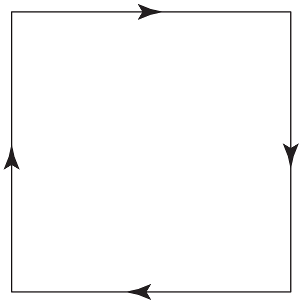

Figure 7. We can quotient this square as indicated to get the projective plane. In fact we can view all connect sums of copies of $\mathbb{P}^2$ as quotients of polygons. This is the non-orientable analogue to Fig. 4.

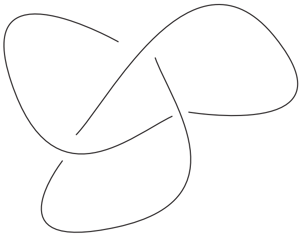

FIGURE 8. (Left) The trefoil knot.

clearly G_{ab} satisfies

$$\begin{tikzcd}
 & G_{\text{ab}} \arrow[dr, dashed, "\varphi"] & \\
G \arrow[ur, "\pi"] \arrow[rr, "\alpha"'] & & A
\end{tikzcd} \tag{3.43}$$

,

i.e. there exists a unique $\varphi : G_{\text{ab}} \to A$ such that $\alpha = \varphi\pi$.

Note that $[G, G]$ is also a characteristic subgroup, which means $G \simeq H$ implies $G_{\text{ab}} \simeq H_{\text{ab}}$.

THEOREM 3.12. *The surfaces listed are pairwise non-homeomorphic.*

PROOF. $\left(\pi_1\left(\#_g T^2\right)\right)_{\text{ab}}$ is the free abelian group on $2g$ generators, i.e. $\bigoplus_{2g} \mathbb{Z}$. On the other hand, $\left(\pi_1\left(\#_k \mathbb{P}^2\right)\right)_{\text{ab}}$ is an abelian group with generators $a_1, \dots, a_k$ with the relation

$$2\left(a_1 + a_2 + \ldots + a_k\right) = 0 \; . \tag{3.44}$$

Then $\{a_1, a_2, \dots, a_1 + a_2 + \dots a_k\}$ is a $\mathbb{Z}$ basis for the free abelian group on $\{a_1, \dots, a_k\}$ which means

$$\boxed{\left(\pi_1\left(\#_k \mathbb{P}^2\right)\right)_{\text{ab}} \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}} \; . \tag{3.45}$$

Therefore the surfaces listed have pairwise non-isomorphic abelianized fundamental groups and are therefore not homeomorphic. ☐

4. Knot groups

A *knot*[3.3] is a smooth subset $K \subset S^3$ such that $K \cong S^1$. See Fig. 8 for the example of the trefoil knot. We say $K_1 \simeq K_2$ if there exists a homeomorphism $h : S^3 \to S^3$ such that $h\left(K_1\right) = K_2$.

The *group of* K is $\pi\left(K\right) = \pi_1\left(S^3 \setminus K\right)$. Then $K_1 \sim K_2$ implies $S^3 \setminus K_1 \cong S^3 \setminus K_2$, which implies $\pi\left(K_1\right) \cong \pi\left(K_2\right)$.

[3.3] Once Professor Gordon was giving a big important talk at a conference about knots. He prepared a joke for the beginning of his talk, which was that it's hard not to give knot puns in your talk. So a nice woman comes up to introduce him and says something along the lines of: ''He's talking about knots, not talking about not knots". So he gets up and says the line anyway, for it was just too late. And as he was saying it, he was thinking to himself how bad of an idea it was.

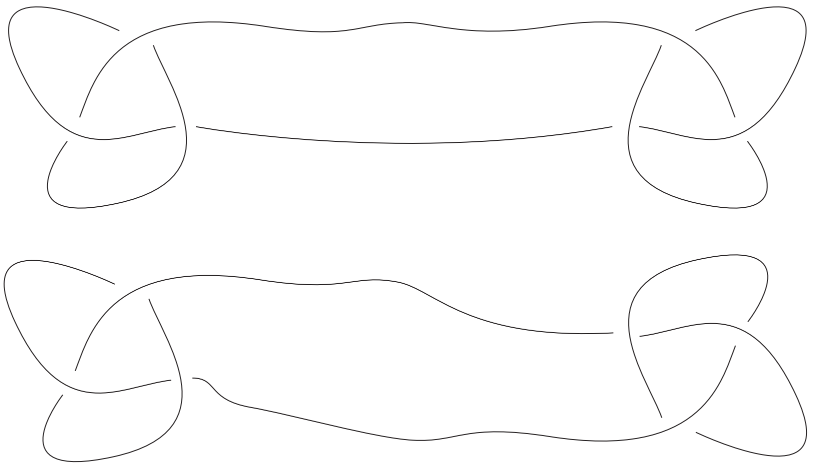

FIGURE 9. The square knot (top) and the granny knot (bottom). They are both connect sums of the trefoil.

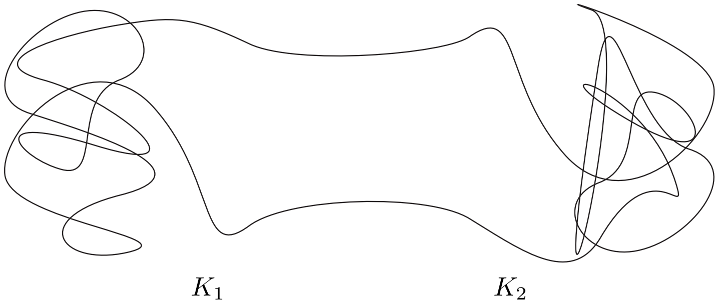

FIGURE 10. The connect sum of two knots K_1 and K_2.

EXAMPLE 3.12. For K the unknot, we have $\pi(K) \cong \mathbb{Z}$. Essentially using van Kampen's theorem one can then show that the group of the trefoil is

$$\langle a, b \,|\, aba = bab \rangle \tag{3.46}$$

which is not $\mathbb{Z}$, so it is not the unknot.

In fact we also have:

Lemma 3.13 (Dehn)**.** *K is trivial iff $\pi(K) \cong \mathbb{Z}$.*

One might wonder if $\pi(K) \cong \pi(K')$ implies $K \sim K'$. This is not true.

COUNTEREXAMPLE 2. Consider the square knot and granny knot in Fig. 9. These have the same group but $K \nsim K'$.

We can also connect sum knots. The idea is to cut out a small piece out of each and then past the gaps together as in Fig. 10.

We say K is prime if $K = K_1 \# K_2$ implies either K_1 or K_2 is trivial.

FIGURE 11. The 5, 2 torus knot. (Photo from the knot atlas.)

THEOREM 3.14. *Let K be prime. Then $\pi(K) \cong \pi(K')$ implies $K \sim K'$.*

EXAMPLE 3.13 (Torus knots). Let $p, q \geq 1$ relatively prime. Then identify the ends of a solid cylinder $I \times D^2$ by $2\pi p/q$. This gives a solid torus for any p,q. Now take q arcs in $S^1 \times I \hookrightarrow D^2 \times I$ given by

$$\left\{\frac{2\pi k}{q}\right\} \times I \tag{3.47}$$

for $k = 0, 1, \ldots, q-1$. In the quotient space $D^2 \times S^1$, this gives a circle $K_{p,q} \subset \partial\left(D^2 \times S^1\right)$. This is called the p, q torus knot $T_{p,q} \subset S^3$. For example the trefoil is $T_{2,3}$. another example is in Fig. 11.

We can decompose $S^3 = V \cup W$ where V, W are solid tori $D^1 \times S^1$. Write

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$$A = T \setminus K_{p,q} \cong S^1 \times \mathbb{R}\ . \tag{3.48}$$

Then

$$S^3 \setminus K_{p,q} = (V \setminus K_{p,q}) \cup_A (W \setminus K_{p,q}) \tag{3.49}$$

so we can apply van Kampen. In other words $\pi(K_{p,q})$ is the push-out:

$$\begin{tikzcd}
\pi_1(A) = \mathbb{Z} \arrow[r, "\times p"] \arrow[d, "\times q"] & \pi_1(V) = \mathbb{Z} \arrow[d] \\
\pi_1(W) = \mathbb{Z} \arrow[r] & \pi_1\left(S^3 \setminus K_{p,q}\right)
\end{tikzcd} \tag{3.50}$$

which means

$$\pi_1(K_{p,q}) = G_{p,q} \cong \langle x, y \,|\, x^p = y^q \rangle\ . \tag{3.51}$$

REMARK 3.12. If p or $q = 1$ $K_{p,q}$ is the unknot.

THEOREM 3.15. *(1) If $p, q > 1$ then $K_{p,q}$ is non-trivial.*
(2) $K_{p,q} \sim K_{p',q'}$ iff $\{p, q\} = \{p', q'\}$.

PROOF. (1) π (trivial knot) $= \mathbb{Z}$. Suppose $p, q > 1$. Then

$$G_{p,q} = \langle x, y \,|\, x^p = y^q \rangle \tag{3.52}$$

has quotient

$$\langle x, y \,|\, x^p = y^q = 1\rangle = \langle x, y \,|\, x^p = 1, y^q = 1\rangle \cong \langle x \,|\, x^p = 1\rangle * \langle y \,|\, y^q = 1\rangle \cong \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} \tag{3.53}$$

which is non-abelian, so this cannot be $\mathbb{Z}$, and therefore K cannot be trivial.

(2) Assume $p, q > 1$. In $G_{p,q}$, let $z = x^p = y^q$. Clearly z commutes with $x + y$. Therefore $z \in Z(G_{p,q})$, so $\langle\langle z \rangle\rangle = (z)$, and

$$G_{p,q}/Z = \langle x, y \,|\, x^p = x^q = 1\rangle \cong \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}\ . \tag{3.54}$$

But $Z(\mathbb{Z}_p * \mathbb{Z}_q) = 1$, so $(z) = Z(G_{p,q})$. In other words:

$$G_{p,q}/Z(G_{p,q}) \cong \mathbb{Z}_p * \mathbb{Z}_q \tag{3.55}$$

which means $G_{p,q} \cong G_{p',q'}$ which implies $\mathbb{Z}_p * \mathbb{Z}_q \cong \mathbb{Z}_{p'} * \mathbb{Z}_{q'}$. Considering elements of finite order (Hatcher §1.2, exercise 1) we get that $\{p, q\} = \{p', q'\}$.

□

REMARK 3.13. These are the only knots such that the group has nontrivial center.

THEOREM 3.16. *For any group G, there exists a path-connected 2-complex X with $\pi_1(X) \cong G$.*

PROOF. G has a presentation $\langle \{x_\lambda\} \,|\, \{r_\mu\}\rangle$ for $r_\mu \in F(\{x_\lambda\})$. Then let

$$W = \bigvee_\lambda S^1_\lambda\ . \tag{3.56}$$

This means $\pi_1(W) \cong F(\{x_\lambda\})$. Attach 2-cells $\{D_\mu\}$, with attaching maps

$$f_\mu : \left(S^1, 1\right) \to (W, x_0) \tag{3.57}$$

such that $f_{\mu_*}(\text{gen}) = r_\mu \in \pi_1(W)$. Now let

$$X = W \cup_{f_\mu} \{\text{2-cells } D_\mu\}\ . \tag{3.58}$$

This is a path-connected 2-complex, $\pi_1(X) \cong G$ by van Kampen. □

THEOREM 3.17. *Let $f, g : S^{n-1} \to Y$ be maps such that $f \simeq g$. Then*

$$Y \cup_f D^n \simeq Y \cup_g D^n\ . \tag{3.59}$$

PROOF. Let $F : S^{n-1} \times I \to Y$ be a homotopy with $F_0 = f$, $F_1 = g$. Let $W = Y \cup_F (D^n \times I)$. Then we have:

$$D^n \cup_F Y = Y \cup_{F_0} (D^n \times \{0\}) \subset W \supset Y \cup_{F_1} (D^n \times \{1\}) = Y \cup_g D^n\ . \tag{3.60}$$

There exists a strong deformation retraction

$$D^n \times I \to D^n \times \{0\} \cup S^{n-1} \times I\ . \tag{3.61}$$

Similarly, there exists a strong deformation retraction

$$D^n \times I \to D^n \times \{1\} \cup S^{n-1} \times I\ . \tag{3.62}$$

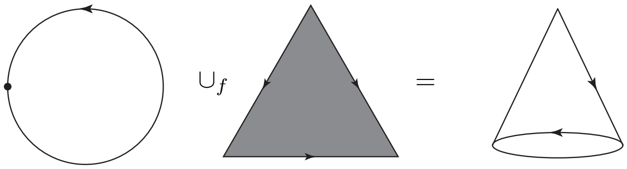

FIGURE 12. All three edges get identified with the 1-cell on the left with the indicated orientations. If we identify the bottom edge of the triangle with the 1-cell on the left, we get the figure on the right of the equality. This is why it is called the dunce hat. At this stage it is obviously contractible, but after making the final identification (indicated with the arrows) it becomes more difficult to see this.

These induce a strong deformation retractions:

$$\begin{tikzcd} & Y \cup_f D^n \\ W \arrow[ur] \arrow[dr] & \\ & Y \cup_g D^n \end{tikzcd} . \tag{3.63}$$

Therefore

$$Y \cup_f D^n \simeq W \simeq Y \cup_d D^n \ . \tag{3.64}$$

□

EXAMPLE 3.14 (Dunce hat). Consider the space in Fig. 12. The attaching map is homotopic to $\mathrm{id} : S^1 \to S^1$. Therefore

$$X = S^1 \cup_f D^2 \simeq S^1 \cup_{\mathrm{id}} D^2 = D^2 \ . \tag{3.65}$$

Therefore X is contractible.

CHAPTER 4

Covering spaces

Let X be a topological space.

Definition 4.1. A map $p : \tilde{X} \to X$ is a *covering projection* iff for all $x \in X$, there exists an open neighborhood U of x such that U is *evenly covered* by p, i.e. $p^{-1}(U)$ is a non-empty disjoint union of open subsets if $\tilde{X}$, each of which is mapped by p homeomorphically onto U. We call these the *sheets over U*. We call $\tilde{X}$ the *covering space of X*.

Remark 4.1. For all $x \in X$, $p^{-1}(x)$, the *fiber over x* is a discrete subspace of $\tilde{X}$.

Example 4.1.

(1) $\mathrm{id} : X \to X$

(2) $p : \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi i x}$ is a covering projection. This is the map we used to calculate $\pi_1(S^1)$. See Fig. 1.

Exercise 0.1. Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be covering projections. Then $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$ is a covering projection.

Exercise 0.2. X is the quotient:

$$\tilde{X} / (\tilde{x}_1 \sim \tilde{x}_2 \iff p(\tilde{x}_1) = p(\tilde{x}_2)) \tag{4.1}$$

with the quotient topology.

Example 4.2. If we map

$$\begin{aligned} S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ (\theta, \varphi) &\longmapsto (m\varphi, n\varphi) \end{aligned} \tag{4.2}$$

(for $m, n \geq 1$) we get a covering projection.

Similarly

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\longrightarrow S^1 \times S^1 \\ (x, y) &\longmapsto \left(e^{2\pi i x}, e^{2\pi i y}\right) \end{aligned} \tag{4.3}$$

is a covering projection. See Fig. 1.

Example 4.3. Consider the connect sum $\#_4 T^2$. The connect sum $\#_7 T^2$ actually forms a covering space of this as in Fig. 2.

Example 4.4. $\mathbb{R}^2$ is also a covering space for the Klein bottle in a similar way that we constructed a cover for T^2. In fact we get a 2-sheet cover $T^2 \to KB$.

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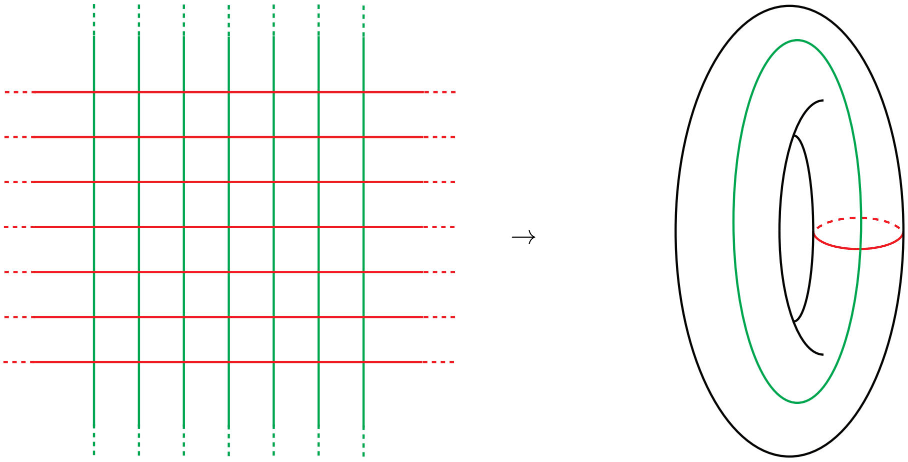


FIGURE 1. $\mathbb{R}^2$ mapping to T^2 is a covering projection. Each I^2 gets quotiented as usual.

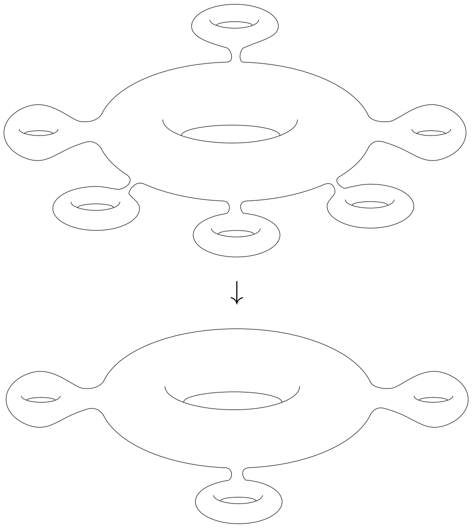


FIGURE 2. The connect sum of 7 tori forms a covering space over the connect sum of 4 tori. In general the connect sum of $2g-1$ tori is a covering space over the connect sum of g tori.

EXAMPLE 4.5. $q : S^n \to S^n/\left(x \sim -x\right) = \mathbb{RP}^n$ is a covering projection.

EXAMPLE 4.6. The map indicated in Fig. 3 is a covering projection. This is what is called a regular covering space. The group $\mathbb{Z}/3\mathbb{Z}$ is acting on $\tilde{X}$.

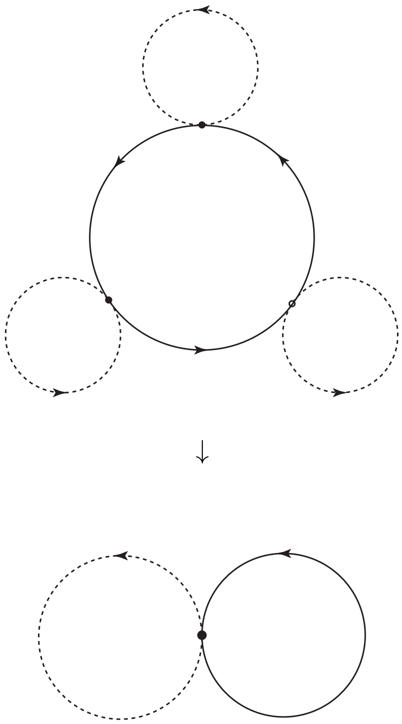


Figure 3. The map identifying solid with solid and dashed with dashed (with the indicated orientations) is a covering projection.

Example 4.7. Consider a different three-fold cover (of the same base as the previous example) which is pictured in Fig. 4. We will eventually call this an irregular covering space.

Example 4.8 (TV antenna). Now consider the covering projection in Fig. 5 of the same base as the previous examples. This has an obvious action of the free group on it. This example is called the TV antenna.[4.1]

Slogan: Taking covering spaces of X corresponds to unwrapping $\pi_1(X)$.

For a covering projection $p : \tilde{X} \to X$, $f : Y \to \tilde{X}$ is a *lift* of a map $f : Y \to X$ iff the following diagram commutes:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd} , \tag{4.4}$$

i.e. $p\tilde{f} = f$.

Example 4.9. Map $p : \mathbb{R} \to S^1$ where $p(x) = e^{2\pi i x}$. Then define $f : I \to S^1$ by $f(t) = e^{2\pi i t}$ $(t \in I)$. Then for any $n \in \mathbb{Z}$, $\tilde{f}(t) = t + n$ is a life of f.

[4.1] “When TV first arrived it was going to be this great tool for education. Right. . . ”

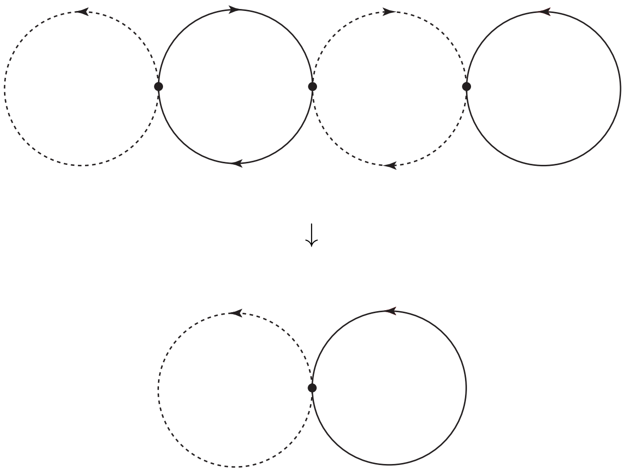


FIGURE 4. Another example of a three-fold covering space.

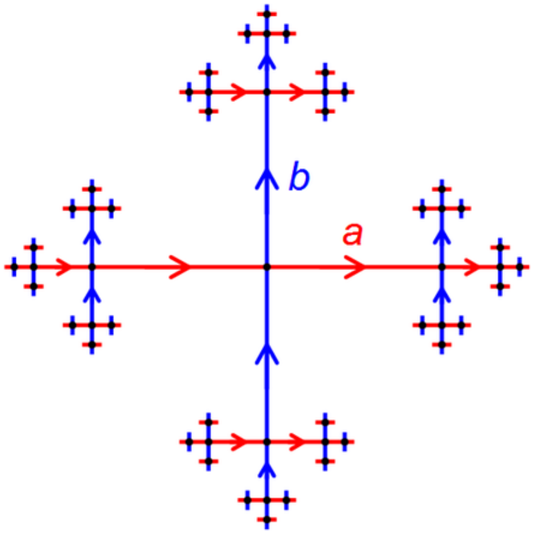


FIGURE 5. The Cayley graph of the free group on two generators. This can be viewed as a covering space for the same base as above. Figure from wikipedia.

Lemma 4.1 (Unique lifting)**.** *Let* $p : \tilde{X} \to X$ *be a covering projection and* $f : Y \to X$ *be a map, where* Y *is connected. Let* $\tilde{f}, \tilde{g} : Y \to \tilde{X}$ *be lifts of* f *such that* $\tilde{f}(y_0) = \tilde{g}(y_0)$ *for some* $y_0 \in Y$*. Then* $\tilde{f} = \tilde{g}$*.*

PROOF. Let

$$A = \left\{ y \in Y \,\middle|\, \tilde{f}(Y) = \tilde{g}(y) \right\} \qquad D = \left\{ y \in Y \,\middle|\, \tilde{f}(y) \neq \tilde{g}(y) \right\} \ . \tag{4.5}$$

Then $Y = A \amalg D$ and $A \neq \emptyset$ by hypothesis. We will show that A and D are both open. Therefore $A = Y$ since Y is connected.

(i) A is open: Let $y \in A$, U be an evenly covered open neighborhood of $f(y) \in X$. Then $\tilde f(y) = \tilde g(y)$ are in some sheet $\tilde U$. Then

$$\tilde f^{-1}\left(\tilde U\right) \cap \tilde g^{-1}\left(\tilde U\right) \tag{4.6}$$

is an open neighborhood of $Y \subset A$.

(ii) D is open: the picture here is the same. But now $\tilde f(y)$ and $\tilde g(y)$ are in different sheets, since otherwise they would agree. Then $\tilde f^{-1}\left(\tilde U\right) \cap \tilde g^{-1}\left(\tilde V\right)$ is an open neighborhood of $y \in Y$, contained in D.

□

Theorem 4.2 (Homotopy lifting). *Let $p : \tilde X \to X$ be a covering projection and $f : Y \to X$ a map with lift $\tilde f : Y \to \tilde X$ and $F : Y \times I \to X$ a homotopy such that $F_0 = f$. Then F has a unique lift $\tilde F : Y \times I \to \tilde X$ such that $\tilde F_0 = \tilde f$, i.e. there exists a unique $\tilde F$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & & \tilde X \arrow[d, "p"] \\
Y \times \{0\} \arrow[urr, bend left, "\tilde f"] \arrow[r, hook] \arrow[rr, bend right, "f"'] & Y \times I \arrow[ur, dashed, "\tilde F"] \arrow[r, "F"] & X
\end{tikzcd} . \tag{4.7}$$

Proof. **Uniqueness:** $\{y\} \times I$ is connected so $\tilde F\Big|_{\{y\}\times I}$ is unique for all $y \in Y$ (from Lemma 4.1). Therefore $\tilde F$ is unique.

Existence: This is similar to the proof of Lemma 2.9. Let $y \in Y$. By compactness of I, we can write it as a union $I = I_1 \cup \ldots \cup I_n$ such that $F(\{y\} \times I_i)$ is contained in some evenly covered $U_i \subset X$.

Then we can define $\tilde F$ on $\{y\} \times I$ such that $\tilde F(\{y\} \times I) \subset \tilde U_i$ for some sheet over U_i. Then I_i is compact, which means there exists a neighborhood N_i of y in Y such that $\tilde F(N_i \times I_i) \subset \tilde U_i$. Let

$$N = N_y = \bigcap_{i=1}^{n} N_i \ . \tag{4.8}$$

Then N is a neighborhood of $y \in Y$ such that $F(N \times I_i) \subset U_i$ $(1 \leq i \leq n)$. So now we can define $\tilde F(N \times I_i)$ (contained in $\tilde U_i$) for all i. Therefore $\tilde F_y$ is defined on $N_y \times I$.

Claim 4.1. The $\tilde F_y$s fit together to give $\tilde F : Y \times I \to \tilde X$.

Let $z \in N_y \cap N_{y'}$. Then $\tilde F_y \times \tilde F_{y'}$ are defined on $\{z\} \times I$ and

$$\tilde F_Y(z, 0) = \tilde f(z) = \tilde F_{y'}(z, 0) \ . \tag{4.9}$$

$\{z\} \times I$ is connected, and therefore by Lemma 4.1 $\tilde F_y$ and $F_{y'}$ agree on $\{z\} \times I$, and therefore they agree on $N_y \cap N_{y'}$, so setting $\tilde F(y, t) = \tilde F_y(y, t)$ gives a well-defined map $\tilde F : Y \times I \to \tilde X$ and $\tilde F_0 = \tilde f$. □

Corollary 4.3 (Path lifting). *Let $p : \left(\tilde X, \tilde x_0\right) \to (X, x_0)$ be a covering projection.*

1. *Let* $\sigma : I \to X$ *be a path such that* $\sigma(0) = x_0$*. Then* σ *has a unique lift* $\tilde{\sigma} : I \to \tilde{X}$ *such that* $\tilde{\sigma}(0) = \tilde{x}_0$*.*
2. *If* $\sigma, \tau : I \to X$ *are paths such that* $\sigma(0) = \tau(0) = x_0$ *and* $\sigma \simeq \tau \, (\mathrm{rel}\, \partial I)$ *then* $\tilde{\sigma}(1) = \tilde{\tau}(1)$ *and* $\tilde{\sigma} \simeq \tilde{\tau} \, (\mathrm{rel}\, \partial I)$*.*

PROOF. (1) Apply Theorem 4.2 with $Y = \{\mathrm{pt}\}$.
(2) Apply Theorem 4.2 with $Y = I$.

□

Corollary 4.4. *Let* $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ *be a covering projection. Then* $p_* : \pi_1\left(\tilde{X}, \tilde{x}_0\right) \to \pi_1(X, x_0)$ *is injective.*

PROOF. Let $\hat{\sigma}$ be a loop in $\tilde{X}$ at $\tilde{x}_0$ such that $p_*[\hat{\sigma}] = 1 \in \pi_1(X, x_0)$. By hypothesis $\hat{\simeq} c_{x_0} (\mathrm{rel}\, \partial I)$. $\hat{\sigma}$ is the lift $\tilde{\sigma}$ of σ in Corollary 4.3. Therefore by Corollary 4.3 $\hat{\sigma} \simeq \tilde{c}_{x_0} = c_{\tilde{x}_0} (\mathrm{rel}\, \partial I)$. Therefore $[\hat{\sigma}] = 1 \in \pi_1\left(\tilde{X}, \tilde{x}_0\right)$. □

Lecture 11; October 10, 2019

THEOREM 4.5 (Lifting criterion for loops). *Let* $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ *be a covering projection. Then a loop* σ *in* X *at* x_0 *lifts to a loop in* $\tilde{X}$ *at* $\tilde{x}_0$ *iff*

$$[\sigma] \in p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right) (< \pi_1(X, x_0)) \ . \tag{4.10}$$

DEFINITION 4.2. A space Y is *locally path-connected* (lpc) iff for all $y \in Y$ and for all neighborhoods U of $y \in Y$, there exists some neighborhood V of y such that $V \subset U$ and V is path-connected.

EXAMPLE 4.10. The comb space is not lpc.

REMARK 4.2. (1) Path connected does not imply locally-path-connected (e.g. the comb space).
(2) Theorem 4.6 is *false* if we omit the lpc hypothesis.

THEOREM 4.6 (Lifting criterion). *Let* $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ *be a covering projection. Let* Y *be connected and lpc and let* $f : (Y, y_0) \to (X, x_0)$ *be a map. Then* f *has a lift* $\tilde{f} : (Y, y_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ *iff*

$$f_*(\pi_1(Y, y_0)) < p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right) (< \pi_1(X, x_0)) \ . \tag{4.11}$$

PROOF. ($\implies$): This direction is clear:

$$\begin{tikzcd} & \pi_1\left(\tilde{X}, \tilde{x}_0\right) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd} \ . \tag{4.12}$$

($\Longleftarrow$): Define $\tilde{f}$ as follows. Given $y \in Y$, Y lps, let α be a path in Y form y_0 to y. Then $f\alpha$ is a path in X from x_0 to $f(y)$. Let $\widetilde{f\alpha}$ be the lift of $f\alpha$ such that

$$\widetilde{f\alpha}(0) = \tilde{x}_0 \ . \tag{4.13}$$

Then define

$$\tilde{f}\left(y\right)=\widetilde{f\alpha}\left(1\right)\ . \tag{4.14}$$

First we show this is well-define. For α and β paths in Y from y_0 to y we have

$$\alpha\simeq\left(\alpha*\bar{\beta}*\beta\right)=\underbrace{\sigma}_{\alpha*\bar{\beta}}*\beta\left(\operatorname{rel}\partial I\right) \tag{4.15}$$

which means

$$f\alpha\simeq f\sigma*f\beta\left(\operatorname{rel}\partial I\right)\ . \tag{4.16}$$

But

$$\left[f\sigma\right]=f_*\left[\sigma\right]\in p_*\pi_1\left(\tilde{X},\tilde{x}_0\right) \tag{4.17}$$

by hypothesis. Therefore by Theorem 4.5 $f\sigma$ lifts to a *loop* $\widetilde{f\sigma}$ in $\tilde{X}$ at $\tilde{x}_0$. Therefore by Corollary 4.3 we have that $\widetilde{f\alpha}\simeq\widetilde{f\sigma}*\widetilde{f\beta}\left(\operatorname{rel}\partial I\right)$, so $\widetilde{f\alpha}\left(1\right)=\widetilde{f\beta}\left(1\right)$.

Now we show $\tilde{f}$ is continuous. Let $y\in Y$. Any neighborhood of $\tilde{f}\left(y\right)$ contains a neighborhood $\tilde{U}$, a sheet over an evenly covered neighborhood U of $f\left(y\right)$. f continuous implies that there exists a neighborhood V of $y\in Y$ such that $f\left(V\right)\subset U$. Then Y lpc implies there exists a path connected neighborhood W of y, $W\subset V$. Therefore for all $y'\in W$ there exists a path β in W from y to y'. Then $f\beta$ is a path in U from $f\left(y\right)$ to $f\left(y'\right)$. But $p|_{\tilde{U}}:\tilde{U}\to U$ is a homeomorphism, so $f\beta$ lifts to a path $\widetilde{f\beta}$ in $\tilde{U}$ from $\tilde{f}\left(y\right)$ to $\tilde{f}\left(y'\right)$. Therefore $\tilde{f}\left(W\right)\subset\tilde{U}$, so $\tilde{f}$ is continuous. ☐

Corollary 4.7. *If Y is lpc and simply-connected then any map $f:Y\to X$ lifts.*

Lemma 4.8. *Let $p:\left(\tilde{X},\tilde{x}_0\right)\to\left(X,x_0\right)$ be a covering projection, $\tilde{X}$ path-connected. Let $\tilde{x}_1\in p^{-1}\left(x_0\right)$. Then $p_*\pi_1\left(\tilde{X},\tilde{x}_0\right)$ and $p_*\pi_1\left(\tilde{X},\tilde{x}_1\right)$ are conjugate in $\pi_1\left(X,x_0\right)$.*[4.2]

Proof. For $\tilde{X}$ pc, let α be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. This gives us

$$\alpha_\#:\pi_1\left(\tilde{X},\tilde{x}_0\right)\xrightarrow{\cong}\pi_1\left(\tilde{X},\tilde{x}_1\right) \tag{4.18}$$

where

$$\alpha_\#\left(\left[\sigma\right]\right)=\left[\bar{\alpha}*\sigma*\alpha\right] \tag{4.19}$$

which means

$$p_*\left(\alpha_\#\left[\sigma\right]\right)=\left[\overline{p\alpha}*p\sigma*p\alpha\right]\ .\alpha^{-1}p_*\left(\left[\sigma\right]\right)a\ . \tag{4.20}$$

where $p\alpha$ is a loop in X at x_0, so we can write $\left[p\alpha\right]=a\in\pi_1\left(X,x_0\right)$ and actually

$$p_*\left(\alpha_\#\left[\sigma\right]\right)=\left[\overline{p\alpha}*p\sigma*p\alpha\right]\ .\alpha^{-1}p_*\left(\left[\sigma\right]\right)a\ . \tag{4.21}$$

☐

[4.2] Professor Gordon accidentally misnumbered this lemma. He then reminded us that there are three kinds of mathematicians.

1. Covering transformations

An *isomorphism of covering spaces* $\left(\tilde{X}, p_1\right) \xrightarrow{\cong} \left(\tilde{X}_2, p_2\right)$ is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that

(4.22)
$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"] & & \tilde{X}_2 \arrow[ll, bend right, "g"'] \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

commutes.

Let $g = f^{-1}$. Then f is said to be a lift of p_1 and f is said to be a lift of p_2. Therefore Theorem 4.6 gives

Theorem 4.9. *For $\left(\tilde{X}_i, p_i\right)$ a covering space of X, $\tilde{X}_i$ connected and lpc, then $\left(\tilde{X}_1, p_1\right) \cong \left(\tilde{X}_2, p_2\right)$ by a homeomorphism taking $\tilde{x}_1$ to $\tilde{x}_2$ iff*

$$p_{1*}\left(\pi_1\left(\tilde{X}_1, \tilde{x}_1\right)\right) = p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{x}_2\right)\right) \quad . \tag{4.23}$$

This and Lemma 4.8 gives us the forward implication of the following: The other implication is left as an exercise.

Theorem 4.10. *Let $\left(\tilde{X}_i, p_i\right)$ be as above. Then $\left(\tilde{X}_1, p_1\right) \cong \left(\tilde{X}_2, p_2\right)$ iff for all $\tilde{x}_i \in \tilde{X}_i$ such that $p_1\left(\tilde{x}_1\right) = p_2\left(\tilde{x}_2\right)$ $(= x_0 \in X,$ say)*

$$p_{1*}\left(\pi_1\left(\tilde{X}_1, \tilde{x}_1\right)\right) \qquad \textit{and} \qquad p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{x}_2\right)\right) \tag{4.24}$$

are conjugate.

Definition 4.3. A *covering transformation* of $\left(\tilde{X}, p\right)$ is an automorphism of $\left(\tilde{X}, p\right)$, i.e. a homeomorphism $f : \tilde{X} \to \tilde{X}$ such that $pf = p$:

(4.25)
$$\begin{tikzcd}
\tilde{X} \arrow[rr, "f"] \arrow[dr, "p"] & & \tilde{X} \arrow[dl, "p"] \\
& X &
\end{tikzcd} \quad .$$

Remark 4.3. (1) A covering transformation is a lift of p.
(2) The covering transformations of $\left(\tilde{X}, p\right)$ form a group under composition.

Digression 1. Now we learn some basics of group actions. Let Y be a space and G a group. A *left action* of G on Y is a map

$$\begin{aligned} G \times Y &\longrightarrow Y \\ (g, y) &\longmapsto gy \end{aligned} \tag{4.26}$$

such that

(1) $1y = y$

(2) $\forall y \in Y, g(h(y)) = (gh)(y)$
(3) $\forall g \in G$, $y \mapsto gy$ is continuous.

It follows from this that for all $g \in G$, $y \mapsto gy$ is a homeomorphism $Y \to Y$. So a left action of G on Y is equivalent to a homeomorphism $G \to \mathrm{Hom}(Y, Y)$. The action is transitive if for all $y_1, y_2 \in Y$ there exists $g \in G$ such that $g(y_1) = y_2$. Define a equivalence relation on Y by $y_1 \sim y_2$ iff there exists some $g \in G$ such that $g(y_1) = y_2$. This gives us a quotient space:

$$Y/G = Y/(y_1 \sim y_2) \tag{4.27}$$

with the quotient topology. Note that if G is transitive, $Y/G = \mathrm{pt}$.

Now we return to covering transformations. Since these form a group $G\left(\tilde{X}, p\right)$, we have a left group action of this on $\tilde{X}$. By unique lifting, if $\tilde{X}$ is connected and $g_1, g_2 \in G\left(\tilde{X}\right)$ such that $g_1(\tilde{x}) = g_2(\tilde{x})$ for some $\tilde{x} \in X$, then $g_1 = g_2$. Equivalently, for all $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$, there is at most one covering transformation taking $\tilde{x}_1 \to \tilde{x}_2$.

Example 4.11. Consider the covering space $p : \mathbb{R} \to S^1$. For any $n \in \mathbb{Z}$ we can map $\mathbb{R} \to \mathbb{R}$ by sending $t \mapsto t + n$. This is a covering transformation. But there is at most one such transformation, as we just noticed, which means this is all of them:

$$G(\mathbb{R}) = \{g_n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}\ . \tag{4.28}$$

In addition, the quotient is $\mathbb{R}/G(\mathbb{R}) \cong S^1$.

Example 4.12. Recall Example 4.6. There is a $\mathbb{Z}/3\mathbb{Z}$ action on $\tilde{X}$, so $G\left(\tilde{X}\right) \cong \mathbb{Z}/3\mathbb{Z}$, and

$$\tilde{X}/(\mathbb{Z}/3\mathbb{Z}) \cong X\ . \tag{4.29}$$

Example 4.13. Recall Example 4.7. In this case $G\left(\tilde{X}\right) = \{1\}$.

Theorem 4.9 immediately gives us:

Theorem 4.11. *Let $p : \tilde{X} \to X$ be a covering projection. Suppose $\tilde{X}$ is connected and lpc, $x_0 \in X$, and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$. Then there exists a covering transformation of $\tilde{X}$ taking $\tilde{x}_1$ to $\tilde{x}_2$ iff $p_*\pi_1\left(\tilde{X}, \tilde{x}_1\right) = p_*\pi_1\left(\tilde{X}, \tilde{x}_2\right)$.*

Example 4.14. Consider Example 4.7 again. In this case $H_i = p_*\pi_1\left(\tilde{X}, \tilde{x}_i\right)$ for $i = 1, 2, 3$ are all distinct. We know these are contained in $F(a, b)$. Then we can explicitly write them as the subgroups generated by:

$$H_1 = \left\langle b, a^2, ab^2a^{-1}, (ab)\,a\left(b^{-1}a^{-1}\right)\right\rangle \tag{4.30}$$

$$H_2 = \left\langle a^2, b^2, aba^{-1}, bab^{-1}\right\rangle \tag{4.31}$$

$$H_3 = \left\langle a, b^2, ba^1b^{-1}, (ba)\,b\left(a^{-1}b^{-1}\right)\right\rangle\ . \tag{4.32}$$

So these are all distinct, but conjugate in $\pi_1\left(\tilde{X}\right)$.

A covering projection $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ is *regular* iff $p_*\pi_1\left(\tilde{X}, \tilde{x}_0\right)$ is normal in $\pi_1(X, x_0)$.

Lecture 12; October 15, 2019

REMARK 4.4. If $\tilde{X}$ is path-connected then this is independent of x_0 and $\tilde{x}_0$. The reason is as follows. Suppose have some $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ with $x_0 = p(\tilde{x}_0)$, $x_1 = p(\tilde{x}_1)$. Then $\tilde{X}$ path-connected implies there exists a path α in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$ so we get a commutative diagram

$$\begin{tikzcd} \pi_1\left(\tilde{X}, \tilde{x}_0\right) \arrow[r, "\alpha_\#", "\cong"'] \arrow[d, "p_*"] & \pi_1\left(\tilde{X}, \tilde{x}_1\right) \arrow[d, "p_*"] \\ \pi_1(X, x_0) \arrow[r, "(p\alpha)_\#", "\cong"'] & \pi_1(X, x_1) \end{tikzcd} \tag{4.33}$$

where these are isomorphisms from Corollary 2.14.

Lemma 4.8 and Theorem 4.11 immediately give:

THEOREM 4.12. *Let $p : \tilde{X} \to X$ be a covering projection; $\tilde{X}$ connected, lpc; $x_0 \in X$. Then p is regular iff $G\left(\tilde{X}, p\right)$ acts transitively on $p^{-1}(x_0)$.*

Recall for G a group and $H < G$ a subgroup, the normalizer of H in G is

$$N(H) = \left\{ g \in G \,\middle|\, g^{-1} H g = H \right\} \ . \tag{4.34}$$

Note that H is clearly always normal in its normalizer, and H is a normal subgroup if its normalizer is all of G.

THEOREM 4.13. *Let $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ be a covering projection, $\tilde{X}$ connected and lpc. Let $H = p_* \pi_1\left(\tilde{X}, \tilde{x}_0\right) < \pi_1(X, x_0)$. Then*

$$G\left(\tilde{X}, p\right) \cong N(H)/H \ . \tag{4.35}$$

Corollary 4.14. *Let p be as above. If the covering is regular, then*

$$G\left(\tilde{X}, p\right) \cong \pi_1(X, x_0)/p_* \pi_1\left(\tilde{X}, \tilde{x}_0\right) \ . \tag{4.36}$$

In particular if $\pi_1\left(\tilde{X}, \tilde{x}_0\right) = 1$, then

$$G\left(\tilde{X}, p\right) \cong \pi_1(X, x_0) \ . \tag{4.37}$$

EXAMPLE 4.15. Consider the usual covering $p : \mathbb{R} \to S^1$. Then we have

$$G(\mathbb{R}, p) \cong \mathbb{Z} \cong \pi_1\left(S^1, 1\right) \ . \tag{4.38}$$

EXAMPLE 4.16. Recall Example 4.7. For each possible basepoint in $\tilde{X}$ we had three subgroups $H_i = p_* \pi_1\left(\tilde{X}, \tilde{x}_i\right)$. Recall $\pi_1(X, x_0) \cong F_2$. Now we have $G\left(\tilde{X}, p\right) = \{1\}$ and therefore from Theorem 4.13 we have

$$N(H_i) = H_i \ . \tag{4.39}$$

PROOF OF THEOREM 4.13. Let $[\sigma] \in N(H)$, σ a loop in X at x_0. Let $\tilde{\sigma}$ be the lift of σ such that $\tilde{\sigma}(0) = \tilde{x}_0$; let $\tilde{x}_1 = \tilde{\sigma}(1)$.

By the proof of Lemma 4.8:

$$p_* \pi_1\left(\tilde{X}, \tilde{x}_1\right) = [\sigma]^{-1} H [\sigma] = H \tag{4.40}$$

by assumption. Therefore by Theorem 4.11 there exists a covering transformation $g \in G\left(\tilde{X}, p\right)$ such that $g(\tilde{x}_0) = \tilde{x}_1$. g is unique by Remark 4.4.

Define $\varphi : N(H) \to G\left(\tilde{X}, p\right)$ by sending $\varphi([\sigma]) = g$. Now we need to check that φ is well-defined. Suppose we have two representatives of the same class $[\sigma] = [\tau]$. This means $\sigma \simeq \tau\,(\mathrm{rel}\,\partial I)$, which means $\tilde{\sigma} \simeq \tilde{\tau}\,(\partial I)$, which means $\tilde{\sigma}(1) = \tilde{\tau}(1)$. Now we need to check that φ is a homomorphism. Consider $[\sigma], [\tau] \in N(H)$. These will correspond to $\varphi([\sigma]) = g$ and $\varphi([\tau]) = h$. This means

$$g(\tilde{x}_0) = \tilde{x}_1 \qquad\qquad h(\tilde{x}_0) = \tilde{x}_2\ . \tag{4.41}$$

Then we have $\varphi([\sigma][\tau]) = \varphi([\sigma * \tau])$. But we know

$$\widetilde{\sigma * \tau} = \tilde{\sigma} * g(\tilde{\tau}) \tag{4.42}$$

which means

$$\widetilde{\sigma * \tau}(1) = g(\tilde{x}_2) = gh(\tilde{x}_0) \tag{4.43}$$

so finally

$$\varphi([\sigma][\tau]) = gh = \varphi([\sigma])\,\varphi([\tau]) \tag{4.44}$$

as desired.

Now we want to show φ is onto. Let $g \in G\left(\tilde{X}, p\right)$ and $\tilde{\sigma}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $g(\tilde{x}_0)$. Then $p\tilde{\sigma} = \sigma$ is a loop in X at x_0. By Theorem 4.11 and Lemma 4.8 we have

$$H = p_*\pi_1\left(\tilde{X}, \tilde{x}_0\right) = p_*\pi_1\left(\tilde{X}, g(\tilde{x}_0)\right) = [\sigma]^{-1} H [\sigma]\ . \tag{4.45}$$

Therefore $[\sigma] \in N(H)$, $\tilde{\sigma}(1) = g(\tilde{x}_0)$. Therefore $\varphi([\sigma]) = g$.

Now we show $\ker\varphi = H$. $\varphi([\sigma]) = \mathrm{id}$ iff $\tilde{\sigma}(1) = \tilde{x}_0$ iff $\tilde{\sigma}$ is a loop in $\tilde{X}$ at $\tilde{x}_0$. But this is equivalent to:

$$[\sigma] = [p\tilde{\sigma}] = p_*([\tilde{\sigma}]) \in p_*\pi_1\left(\tilde{X}, \tilde{x}_0\right) = H\ . \tag{4.46}$$

□

THEOREM 4.15. *Let $p : \tilde{X} \to X$ be a regular covering projection with $\tilde{X}$ connected and lpc. Write $G = G\left(\tilde{X}, p\right)$ for the group of covering transformations. Then there is a homeomorphism*

$$\tilde{X}/G \xrightarrow{\cong} X \tag{4.47}$$

such that

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \tilde{X}/G \arrow[rr, "\cong"] & & X \end{tikzcd}\ . \tag{4.48}$$

PROOF. By Theorem 4.12, if $\tilde{x}, \tilde{y} \in \tilde{X}$ then $p(\tilde{x}) = p(\tilde{y})$ iff there is a group element $g \in G$ such that $g(\tilde{x}) = \tilde{y}$. By definition, this is the case iff $q(\tilde{x}) = q(\tilde{y})$. So we just need to check that p is a quotient map, i.e. that for V open in $\tilde{X}$ we have $p(V)$ open in X. This is left as an exercise. □

For "nice" group actions there is a sort of converse to Theorem 4.15. A (left) action of G on Y is a *covering space action* (CSA) iff for all $y \in Y$ there is some neighborhood U of Y such that if $U \cap gU \neq \emptyset$ for some $g \in G$ then $g = \mathrm{id}$. Note that this certainly implies that the action has no fixed points. It actually turns out to be stronger.

Remark 4.5. (1) This condition is equivalent to $g_1U \cap g_2U_2 \neq \emptyset$ iff $g_1 = g_2$.

(2) Let $\tilde{X} \to X$ be covering projection and $\tilde{X}$ connected and lpc. Then the action of $G\left(\tilde{X}, p\right)$ on $\tilde{X}$ is a CSA.

Theorem 4.16. *Let $G \times Y \to Y$ be a CSA. Then*

(1) The quotient map $p : Y \to Y/G$ is a regular covering projection,
(2) $G \cong G(Y, p)$; hence
(3) $G \simeq \pi_1(Y/G)/p_*\pi_1(Y)$.

Example 4.17. Consider the antipodal map $a : S^n \to S^n$ defined by $a(x) = -x$. Since $a^2 = \mathrm{id}$ a generated a $\mathbb{Z}/2\mathbb{Z}$ action on S^n. By definition $\mathbb{RP}^n = S^n/(\mathbb{Z}/2)$. Clearly the action is a CSA, and therefore the quotient map

$$p : S^n \to \mathbb{RP}^n \tag{4.49}$$

is a covering projection with group of covering transformations isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

For $n = 0$ this is just a map from two points to one point. For $n = 1$ this is a rotation, i.e. it sends $p(\theta) = 2\theta$. The induced map sends $\pi_1\left(S^1\right) = \mathbb{Z} \xrightarrow{p_*} \pi_1\left(S^1\right) = \mathbb{Z}$ by multiplying by 2. Then of course the quotient is $\mathbb{Z}/2\mathbb{Z}$.

For $n > 1$ we have $\pi_1(S^n) = 1$, which means $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 4.18. Let G be the group of homeomorphisms of $\mathbb{R}^2$ generated by

$$h(x, y) = (x + 1, y) \qquad v(x, y) = (-x, y + 1) \ . \tag{4.50}$$

It is easy to see that this is a CSA. Let

$$R = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, -\frac{1}{2} \leq x \leq \frac{1}{2}, 0 \leq y \leq 1\right\} \ . \tag{4.51}$$

Every points in $\mathbb{R}^2$ can be obtained by some point of R after sufficiently many applications of h and v. In particular, $\mathbb{R}^2/G$ is some kind of quotient of R. In particular we quotient the edges exactly as in the description of the Klein bottle.

Notice that

$$h^m v^n (0, 0) = (m, n) \tag{4.52}$$

so certainly we have a bijection $\mathbb{Z} \times \mathbb{Z} \leftrightarrow G$. Now notice that

$$vhv^{-1} = h^{-1} \ , \tag{4.53}$$

i.e. as a set:

$$G = \{h^m v^n \mid m, n \in \mathbb{Z}\} \ , \tag{4.54}$$

and to find the multiplication we use the rule in (4.53) to get:

$$(h^{m_1}v^{n_1})(h^{m_2}v^{n_2}) = \begin{cases} h^{m_1+m_2}v^{n_1+n_2} & n_1 \text{ even} \\ h^{m_1-m_2}v^{n_1+n_2} & n_1 \text{ odd} \end{cases} \ . \tag{4.55}$$

In other words:

$$\pi_1(KB) = \left\langle h, v \,\middle|\, vhv^{-1} = h^{-1}\right\rangle \ . \tag{4.56}$$

Exercise 1.1. Recall we saw a different presentation for this group by a more direct argument. Show that

$$\left\langle a,b \,\middle|\, a^2b^2=1\right\rangle = \left\langle h,v \,\middle|\, vhv^{-1}=h^{-1}\right\rangle \ . \tag{4.57}$$

Lecture 13; October 17, 2019

An n-manifold is a second countable Hausdorff space M such that every $x \in M$ has a neighborhood $\cong \mathbb{R}^n$. M is *closed* if it is compact.

Let $\tilde{M} \to M$ be a covering projection (such that $\tilde{M}$ is connected). Then we have the following:

1. If M is an n-manifold, then $\tilde{M}$ is an n-manifold.[4.3]
2. Suppose $\tilde{M}$ is an n-manifold. Then M is locally euclidean and second countable, but M may not be Hausdorff.

Example 4.19. Let $Y = \mathbb{R}^2 \setminus \{(x,0) \,|\, x \leq 0\}$.

Exercise 1.2. Show this is homeomorphic to $\mathbb{R}^2$.

Define a homeomorphism $h : Y \to Y$ by

$$h\left(x,y\right) = \left(2x, y/2\right) \ . \tag{4.58}$$

This generates an action of $\mathbb{Z}$ on Y.

Exercise 1.3. Show this is a CSA.

Therefore the quotient map $p : Y \to Y/\mathbb{Z} =: X$ is a regular covering projection. We know $Y \cong \mathbb{R}^2$ so $\pi_1\left(Y\right) = 1$, so $\pi_1\left(Y/\mathbb{Z}\right) \cong \mathbb{Z}$. X is locally euclidean, but as it turns out, X is not Hausdorff.[4.4] The reason is that $p\left((1,0)\right)$ and $p\left((0,1)\right)$ cannot be separated. If U is a neighborhood of $(1,0)$, $h^{-n}\left(U\right)$ approaches the y-axis as $n \to \infty$. Similarly, for V a neighborhood of $(0,1)$, then $h^{n}\left(V\right)$ approaches the x-axis as $n \to \infty$. Then the point is that for all U, V there exists n such that $h^{-n}\left(U\right) \cap h^{n}\left(V\right) \neq \emptyset$.

An action of G on Y is *free* if $g\left(y\right) = y$ for some $Y \in Y$ iff $g = \mathrm{id}$.

Exercise 1.4. Show that if G is finite, acts freely on Y, and Y is Hausdorff, then Y/G is Hausdorff.

The upshot is that for M an n-manifold, G finite acting freely on M we have M/G is an n-manifold.

Example 4.20. Consider

$$S^3 = \left\{(z,w) \,\middle|\, z,w \in \mathbb{C}, \left|z^2\right| + \left|w^2\right| = 1\right\} \subset \mathbb{C}^2 \ . \tag{4.59}$$

Let $p,q \in \mathbb{Z}$, $p \geq 2$, $(p,q) = 1$. Then define $h : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$h\left(z,w\right) = \left(e^{2\pi i/p}z, e^{2\pi iq/p}w\right) \ . \tag{4.60}$$

This fixes the origin, but otherwise this action is free. In particular we can restrict this to the unit sphere to get a free $\mathbb{Z}/p\mathbb{Z}$ action on S^3. Therefore the quotient is a three-manifold. In particular

$$S^3 \to S^3/\left(\mathbb{Z}/p\mathbb{Z}\right) \tag{4.61}$$

[4.3] It is clear that it is locally euclidean and Hausdorff. It is not so clear that it is second countable, but it is.

[4.4] Even if you don't want non-Hausdorff spaces, they might want you.

is a regular $\mathbb{Z}/p\mathbb{Z}$-covering projection. $S^3/(\mathbb{Z}/p\mathbb{Z})$ is a (closed) three-manifold, $L(p,q)$, called *Lens space*. Since $\pi_1\left(S^3\right) = 1$, we get that

$$\pi_1\left(L\left(p,q\right)\right) \cong \mathbb{Z}/p\mathbb{Z}\ . \tag{4.62}$$

Note that $L(2,1) = \mathbb{RP}^3$.

FACT 2 (Reidemeister). *$L(p,q) \cong L(p',q')$ iff $p = p'$ and either:*

- $q \equiv \pm q' \pmod p$*, or*
- $qq' \equiv \pm 1 \pmod p$

The difficult direction of this is the forward implication.

EXAMPLE 4.21. $L(5,1) \ncong L(5,2)$, and in fact $L(5,1) \not\simeq L(5,2)$. $L(7,1) \simeq L(7,2)$, but $L(7,1) \ncong L(7,2)$.

FACT 3 (Perelman). *Let M and M' be closed prime three-manifolds, with $\pi_1(M) \cong \pi_1(M')$. Then either $M \cong M'$ or $M = L(p,q)$, $M' = L(p,q')$.*

The moral is to not be disheartened when you find counterexamples to what you're trying to prove.

2. Existence of covering spaces

Let $p : \left(\tilde{X}, x_0\right) \to (X, x_0)$ be a covering projection. We know

$$\pi_1\left(\tilde{X}, x_0\right) \cong p_*\pi_1\left(\tilde{X}, \tilde{x}_0\right) < \pi_1\left(X, x_0\right)\ . \tag{4.63}$$

Now we might wonder if, given a subgroup of $\pi_1(X, x_0)$, there is some covering space $\tilde{X}$ such that $p_*\pi_1\left(\tilde{X}, \tilde{x}_0\right) = H$. This is **false**.

Say a space X is *semi-locally simply-connected* (slsc) iff for all $x \in X$ there exists a neighborhood U of x such that

$$i_* : \pi_1(U, x) \to \pi_1(X, x) \tag{4.64}$$

is trivial.

Lemma 4.17. *Let $p : \tilde{X} \to X$ be a covering projection, $\tilde{X}$ simply-connected. Then X is slsc.*

PROOF. Let $x \in X$, U an evenly covered neighborhood of x, and $\tilde{U}$ a sheet over U. Let $\tilde{x} \in \tilde{U}$ such that $p(\tilde{x}) = x$. Then we have

$$\begin{tikzcd}
\pi_1\left(\tilde{U}, \tilde{x}\right) \arrow[r, "\tilde{i}_*"] \arrow[d, "\left(p|_{\tilde{U}}\right)_*", "\cong"'] & \pi_1\left(\tilde{X}, \tilde{x}\right) = 1 \arrow[d, "p_*"] \\
\pi_1(U, x) \arrow[r, "i_*"] & \pi_1(X, x)
\end{tikzcd} \tag{4.65}$$

so indeed X is slsc. □

EXAMPLE 4.22. Consider the Hawaiian earring. This is not slsc at the origin (common point of all the circles). Therefore X has no simply-connected covering space.

REMARK 4.6. Cell complexes are lpc and slsc.

THEOREM 4.18. *Let X be connected, lpc, and slsc. Let $x_0 \in X$ and $H < \pi_1(X, x_0)$. Then there exists a covering projection $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$, $\tilde{X}$ pc such that $p_* \pi_1\left(\tilde{X}, \tilde{x}_0\right) = H$.*

PROOF SKETCH.

(1) Assume $H = 1$: The idea is as follows. If we did have $p : \tilde{X} \to X$, $\tilde{X}$ simply connected, then for $\tilde{x} \in \tilde{X}$, any two paths from $\tilde{x}_0$ to $\tilde{x}$ are $\simeq$ (rel ∂I). Therefore $\tilde{X}$ is in bijection with the set:

$$\left\{[\tilde{\alpha}] \,\middle|\, \tilde{\alpha} \text{ path in } \tilde{X} \text{ s.t. } \tilde{\alpha}(0) = \tilde{x}_0\right\} . \tag{4.66}$$

by path-lifting (Corollary 4.3) we have that this is in bijection with

$$S = \{[\alpha] \mid \alpha \text{ in } X \text{ s.t. } \alpha(0) = x_0\} . \tag{4.67}$$

Define $p : S \to X$ by $p([\alpha]) = \alpha(1)$.

(a) Topologize S: Let $[\alpha] \in S$. $\alpha(1)$ has a neighborhood U in X, where U is pc, and $\pi_1(U) \to \pi_1(X)$ is trivial. Define

$$[\alpha, U] = \{[\alpha * \beta] \mid \beta \text{ a path in } U \text{ s.t. } \beta(0) = \alpha(1)\} \subset S . \tag{4.68}$$

Then

$$p|_{[\alpha, U]} : [\alpha, U] \to U \tag{4.69}$$

is a bijection.

EXERCISE 2.1. Show that the set of $[\alpha, U]$s is a basis for a topology on S.

(b)

EXERCISE 2.2. Show p is continuous.

(c)

EXERCISE 2.3. S is path-connected.

(d)

EXERCISE 2.4. Show $\pi_1(S) = 1$.

(2) Assume $H < \pi_1(X, x_0)$: Let $\tilde{X}$ be the S as above. $p : \tilde{X} \to X$ is a covering projection, $\tilde{X}$ is simply-connected. $G = \pi_1(X, x_0)$ acts on $\tilde{X}$ as the group of covering transformations Corollary 4.14. By Theorem 4.15 we have $\tilde{X} \xrightarrow{p} \tilde{X}/G = X$. Let $H < G$. Then we have:

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[rr, bend right, "p"'] & \tilde{X}/H \arrow[r, "p_H"] & \tilde{X}/G = X \end{tikzcd} . \tag{4.70}$$

EXERCISE 2.5. Show p_H is a covering projection.

$\pi_1\left(\tilde{X}/H\right) \cong H$ so we are done.

□

Combining Theorem 4.10 and Theorem 4.18 we get:

THEOREM 4.19 (Galois correspondence). *Let X be connected, lpc, slsc. Then there exists a bijection between isomorphism classes of connected covering spaces of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

We have seen that for any group G there exists a connected 2-complex X such that $\pi_1(X) = G$ so we have seen basically this entire correspondence between algebra and topology.

Lecture 14; October 22, 2019

Definition 4.4. Let X be pc, lpc. A *universal covering space* of X is a covering projection $p : \tilde{X} \to X$ such that $\tilde{X}$ is simply connected, i.e. pc and $\pi_1\left(\tilde{X}\right) = 1$.

By Theorem 4.10 if $\left(\tilde{X}, p\right)$ exists, then it is unique (up to isomorphism of covering spaces). By Lemma 4.17 an Theorem 4.18 it exists iff X is slsc.

Counterexample 3. The Hawaiian earring has no universal covering.

This is universal in the sense that if $q : \hat{X} \to X$ is a covering projection ($\hat{X}$ connected) then there exists a covering projection $\hat{p} : \tilde{X} \to \hat{X}$ such that $p = q\hat{p}$, i.e. that the following diagram commutes:

$$\begin{tikzcd}
\tilde{X} \arrow[dd, "p"] \arrow[dr, "\hat{p}"] & \\
& \hat{X} \arrow[dl, "q"] \\
X &
\end{tikzcd} \tag{4.71}$$

See the proof of Theorem 4.18.

Let $p : \tilde{X} \to X$ be a covering projection and $x \in X$. $\left|p^{-1}(x)\right|$ is locally constant. Therefore if $\tilde{X}$ is connected, $\left|p^{-1}(x)\right|$ is constant. This is the *index* of the covering. This is the same as the number of sheets.

Lemma 4.20. *If $\tilde{X}$ is pc, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$, then $\left|p^{-1}(x_0)\right|$ is the index of $p_*\pi_1\left(\tilde{X}, \tilde{x}_0\right)$ in $\pi_1(X, x_0)$.*

Proof. Let $H = p_*\pi_1\left(\tilde{X}, \tilde{x}_0\right) < \pi_1(X, x_0)$. $[\sigma] \in \pi_1(X, x_0)$, let $\tilde{\sigma}$ be a lift of σ in $\tilde{X}$ with $\tilde{\sigma}(0) = \tilde{x}_0$.

Define a function φ from the right cosets of H in $\pi_1(X, x_0)$ to $p^{-1}(x_0)$ by $\varphi(H[\sigma]) = \tilde{\sigma}(1)$. Consider two loops $[\sigma], [\tau] \in \pi_1(X, x_0)$. Then we have:

$$\tilde{\sigma}(1) = \tilde{\tau}(1) \iff \left(\widetilde{\sigma * \bar{\tau}}\right)(1) = \tilde{x}_0 \tag{4.72}$$

$$\iff [\sigma * \bar{\tau}] = [\sigma][\tau]^{-1} \in H \tag{4.73}$$

$$\iff H[\sigma] = H\tau\ . \tag{4.74}$$

This shows φ is well-defined and injective. φ is onto since $\tilde{X}$ is pc.

Exercise 2.6. Work out the details of this.

□

3. Coverings of cell complexes

Let $X = Y \cup_a D^n$ $a : S^{n-1} \to Y$ the attaching map. Then we have the diagram:

$$\begin{tikzcd} D^n \arrow[r, hook] \arrow[rr, bend right, "f"'] & D^n \amalg Y \arrow[r, "q"] & X & Y \arrow[l, hook', "i"] \end{tikzcd} \tag{4.75}$$

where $f|_{S^{n-1}} = ia$.

Now let $p : \tilde{X} \to X$ be a covering projection. Let $\tilde{Y} = p^{-1}(Y)$. Then $p|_{\tilde{Y}} : \tilde{Y} \to Y$ is a covering projection (Hatcher §1.3, Q1). Let $0 \in D^n$, $x = f(0) \in X$, $\tilde{x} \in p^{-1}(x)$. We know D^n is lpc and simply connected. Therefore by Theorem 4.6 and Lemma 4.1 f has a unique lift

$$f_{\tilde{x}} : (D^n, 0) \to \left(\tilde{X}, \tilde{x}\right) \tag{4.76}$$

and in particular

$$f_{\tilde{x}}\left(S^{n-1}\right) \subset \tilde{Y} \overset{\tilde{i}}{\hookrightarrow} \tilde{X} \tag{4.77}$$

and

$$f_{\tilde{x}} = \tilde{i} a_{\tilde{x}} \tag{4.78}$$

for $a_{\tilde{x}} : S^{n-1} \to \tilde{Y}$. Then

$$\left\{a_{\tilde{x}} \,\middle|\, \tilde{x} \in p^{-1}(x)\right\} \tag{4.79}$$

defined a map

$$\tilde{a} : \coprod_{\tilde{x} \in p^{-1}(x)} S^{n-1}_{\tilde{x}} \to \tilde{Y} \tag{4.80}$$

Lemma 4.21.

$$\tilde{X} = \tilde{Y} \cup_{\tilde{a}} \left(\coprod_{\tilde{x} \in p^{-1}(x)} D^n_{\tilde{x}} \right) . \tag{4.81}$$

Sketch of proof. We have a commutative diagram

$$\begin{tikzcd} \coprod_{\tilde{x} \in p^{-1}(x)} D^n_{\tilde{x}} \arrow[d, "s"] & \amalg & \tilde{Y} \arrow[r, "\tilde{q}"] \arrow[d, "p|_{\tilde{Y}}"] & \tilde{X} \arrow[d, "p"] \\ D^n & \amalg & Y \arrow[r, "q"] & X \end{tikzcd} \tag{4.82}$$

.

Exercise 3.1. Show that $\tilde{q}$ is the quotient map wrt the given topology on $\tilde{X}$.

☐

Similarly, for attaching a collection of n cells to Y.

Corollary 4.22. *A covering $\tilde{X}$ of a cell complex X is a cell complex. If X is finite dimensional, then $\tilde{X}$ is finite-dimensional and* $\dim X = \dim \tilde{X}$.

Proof. Induct over n-skeleta $X^{(n)}$ of X. ☐

If X is a finite complex, and $\tilde{X} \to X$ is an n-sheeted covering, then $\tilde{X}$ is also finite complex. In particular, if we have N_k k-cells in X, we have $n \cdot N_k$ k-cells in $\tilde{X}$. Recall the Euler characteristic is

$$\chi(X) = \sum (-1)^k N_k \ . \tag{4.83}$$

Remark 4.7. We will see later (using homology) that $\chi(X)$ depends only on the homeomorphism type[4.5] of X. In particular, this means it is independent of the particular cell structure.

Corollary 4.23.

$$\chi\left(\tilde{X}\right) = n\chi(X) \ . \tag{4.84}$$

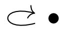
4. Free groups and 1-complexes

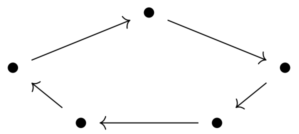
A 1-complex is also called a graph, where the 0-cells are vertices and the 1-cells are the edges.

A graph Γ is *connected* iff Γ is path connected, iff any 2 vertices can be joined by a reduced edge. A *path* is a concatenation of edges. A graph is a *tree*[4.6] iff it is connected and with has no closed nontrivial reduced edge paths.

Example 4.23. The following are not trees:

(4.85)





Therefore γ is a tree iff any 2 vertices can be joined by a unique reduced edge path.

Lemma 4.24. *Let T be a tree, v_0 be a vertex of T. Then there exists a strong deformation retraction $T \to v_0$. In particular T is contractible.*

Proof. For each vertex v of T, there exists a unique reduced edge path P_v from v to v_0. If P_v has n edges, identify P_v with $[0, n]$ in the obvious way. Now define a homotopy $F_t(v) : P_v \to P_v$ by

$$F_t^{(v)}(x) = (1-t)\,x \tag{4.86}$$

so $F_0^{(v)} = \mathrm{id}$, $F_1^{(v)}(P_v) = v_0$, and $F_t^{(v)}(v_0) = v_0$ for all $t \in I$.

Note:

(i) $P_v \cap P_{v'} = P_{v''}$ for some v'' (possibly $v'' = v_0$).
(ii) $T = \bigcup_v P_v$.

So we have

$$F_t^{(v)}\Big|_{P_{v''}} = F_t^{(v')}\Big|_{P_{v''}} \tag{4.87}$$

[4.5] In fact, only the homotopy type.

[4.6] If you look outside you see why these are called trees. Same reason pigs are called pigs and shrimp are called shrimp.

which means the $F_t^{(s)}$s induce

$$F_t : T \times I \to T \tag{4.88}$$

such that $F_0 = \mathrm{id}$, $F_1(t) = v_0$, and $F_t(v_0) = v_0$ for all $t \in I$. ☐

Lemma 4.25. *Let Γ be a graph. Then:*

1. *Γ contains a* maximal *subtree (wrt inclusion).*
2. *If Γ is connected, a subtree of Γ is maximal iff it contains all of the vertices.*

Proof. The set of subtrees of Γ is partially ordered by inclusion. Let $\mathfrak{T}$ be the set of subtrees of Γ that is totally ordered. I.e. for all T and $T' \in \mathfrak{T}$ we have either $T \subset T'$ or $T' \subset T$. Then

$$\bigcup_{T \in \mathfrak{T}} T \tag{4.89}$$

is a subtree of Γ.

To see this, let $e_1, \ldots, e_k$ be a closed reduced edge path in $\cup_T T$. Then $e_i \subset T_i$ for some $T_i \in \mathfrak{T}$. $\mathfrak{T}$ is totally ordered, so for all i, $1 \leq i \leq k$, $e_i \subset T_j$ for some j. But this is a contradiction since T_j is a tree.

Therefore there exists a maximal subtree of Γ by Zorn's lemma.

Exercise 4.1. Prove the second part.

☐

A pair of spaces (X, Y) has the *homotopy extension proper* (HEP) iff given a space Z and maps $f : X \to Z$, $G : Y \times I \to Z$, such that $G(y, 0) = f(y)$ for all $y \in Y$ there exists $F : X \times I \to Z$ such that $F|_{Y \times I} = G$ and $F(x, 0) = f(x)$ for all $x \in X$.

Lemma 4.26. *Let X be an n-complex and Y a subcomplex. Then (X, Y) has the HEP.*

Proof. Note if there exists a retraction $r : X \times I \to X \times 0 \cup Y \times I$, then (X, Y) has the HEP. Given $f \cup G : X \times 0 \cup Y \times I \to Z$, define $F : X \times I \to Z$ by $(f \cup G) \circ r$.

If $X = Y \cup \{n\text{-cells}\}$ then we have a retraction $D^n \times I \to D^n \times 0 \cup S^{n-1} \times I$. This induces a retraction $X \times I \to X \times 0 \cup Y \times I$. ☐

Exercise 4.2. Show that for X a cell complex and Y a subcomplex, then XY is a cell complex.

Lemma 4.27. *Let (X, Y) be a pair of m-complexes with Y contractible. Then the quotient map $X \to X/Y$ is a homotopy equivalence.*

Proof. Y is contractible, so we have $G : Y \times I \to X$ such that $G_0 : Y \hookrightarrow X$, $G_1(Y) = y_0 \in Y$, $G_t(Y) \subset Y$ for all $t \in I$. We know (X, Y) has the HEP, so there exists $F : X \times I \to X$ such that $F_0 = \mathrm{id}$, $F|_{Y \times I} = G$.

The we have $F_1(Y) = G_1(Y) = y_0$ which means F_1 induces a well-defined map $f : X/Y \to X$. So then we have a commuting diagram

(4.90)
$$\begin{tikzcd} X \arrow[rr, "F_1"] \arrow[dr, "q"'] & & X \\ & XY \arrow[ur, "f"'] & \end{tikzcd}$$

which means $fq = F_1 \simeq F_0 = \mathrm{id}$.

Then for all $t \in I$

$$F_t(Y) = G_t(Y) \subset Y \tag{4.91}$$

which means F_t induces $F_t' : X/Y \to X/Y$ and we have the diagram:

$$\begin{tikzcd} X \arrow[r, "F_1"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/Y \arrow[r, "F_1'"] \arrow[ur, "f"] & X/Y \end{tikzcd} \tag{4.92}$$

The square commutes, and the upper triangle commutes, which implies the lower triangle commutes (since q is onto) and therefore $qf = F_1' \simeq F_0' = \mathrm{id}$ which means $fg \simeq \mathrm{id} : X \to X$ so q is a homotopy equivalence. ☐

Lecture 15; October 24, 2019

Corollary 4.28. *Let X be a connected n-complex and T a maximal tree in $X^{(1)}$. Then $X \simeq X/T$ an n-complex with a single 0 cell.*

Corollary 4.29. *Let Γ be a connected graph. Let T be a maximal tree in Γ and let $\{e_\lambda\}$ be the set of edges of Γ that are not in T. Then $\pi_1(T)$ is the free group $F(\{e_\lambda\})$.*

PROOF. $\Gamma \simeq \Gamma/T$, which is a wedge of circles, one for each edge e_λ not in T. ☐

THEOREM 4.30. *(1) (Nielsen-Schreier)[4.7] A subgroup of a free group is free.*
(2) If F is a free group of rank k and $H < F$ of index n, then H is a free group of rank $n(k-1)+1$.

PROOF. (1) Let F be a free group and $H < F$. We know $F \simeq \pi_1(X)$ for X a wedge of circles.

FACT 4 (Hatcher). *Cell complexes are locally contractible. In particular they are slsc and lpc.*

Then there exists a connected covering space $\tilde{X} \to X$ with $\pi_1\left(\tilde{X}\right) \cong H$ (Theorem 4.18). By Corollary 4.22 $\tilde{X}$ is also a 1-complex. Therefore $\pi_1\left(\tilde{X}\right)$ is free by Corollary 4.29.

(2) Let Γ be a finite graph with vertices $V(\gamma)$ and edges $E(\gamma)$. Then any maximal tree T in Γ has $V(\Gamma)-1$ edges. Therefore Γ/T is the wedge of

$$E[\Gamma] - (V(\Gamma)-1) = 1 - V(\Gamma) + E(\gamma) = 1 - \chi(\Gamma) \tag{4.93}$$

circles. Let F be a free group of rank k. Then $F \cong \pi_1(X)$ where X is the wedge of k circles. For $H < F$ of index n there exists an n-sheeted covering $\tilde{X} \to X$ with $\pi_1\left(\tilde{X}\right) \cong H$. By Corollary 4.23 we get

$$\chi\left(\tilde{X}\right) = n\chi(X) = n(1-k) \ . \tag{4.94}$$

Therefore by Corollary 4.29 $\pi_1\left(\tilde{X}\right)$ is free of rank

$$1 - \chi\left(\tilde{X}\right) = 1 - n(1-k) = n(k-1)+1 \ . \tag{4.95}$$

☐

[4.7] These two wrote a book once, but since they didn't have Xerox machines, they only had one copy. They left it in a car in Berlin and it was stolen. Well the briefcase it was in was probably stolen.

5. Surfaces

Lemma 4.31. *Let X and Y be finite cell complexes such that $X \cap Y$ is a subcomplex of X and Y. Then ($X \cup Y$ is a finite cell complex) and*

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y) \tag{4.96}$$

Proof. For any k, the number of k-cells in $X \cup Y$ is given by the number of k-cells in X, plus the number of k-cells in Y, minus the number of k-cells in $X \cap Y$. □

Let S be a surface, $D \subset S$ a disk. Let $S_0 = S \setminus D$. Then $S = S_0 \cup D$, $S_0 \cap D = \partial D \simeq S^1$. Then we have

$$\chi(S) = \chi(S_0) + \underbrace{\chi(D)}_{=1} - \underbrace{\chi(\partial D)}_{=0} = \chi(S) - 1 \tag{4.97}$$

Recall we saw that when we removed a disk from the connect sum of g copies of T^2, we got something homotopy equivalent to the wedge of $2g$ circles. This tells us that, as expected, the Euler characteristic is:

$$\chi\left(\#_g T^2\right) = 2 - 2g\ . \tag{4.98}$$

Similarly, we saw that when we removed a disk from the connect sum of k copies of $\mathbb{P}^2$, we got something homotopy equivalent to the wedge of k circles. This tells us that, as expected, the Euler characteristic is:

$$\chi\left(\#_k \mathbb{P}^2\right) = 2 - k\ . \tag{4.99}$$

Define $\omega(S)$ to be $+$ if S is orientable and $-$ if S is nonorientable. The classification of closed surfaces shows the following.

Theorem 4.32. *Let S and S' be two compact surfaces with (possibly empty) boundary. Then $S \simeq S'$ iff*

1. $\chi(S) = \chi(S')$
2. $|\partial S| = |\partial S'|$
3. $\omega(S) = \omega(S')$.

Note that if $\partial S \neq \emptyset$, then S strong deformation retracts to a wedge of circles, so $\pi_1(S)$ is free.

Suppose $\tilde{S} \to S$ is an n-sheeted covering, where S is a compact, connected surface (possibly with boundary). Then $\tilde{S}$ is also a compact, connected surface. In addition, $\omega(S) = +$ implies $\omega\left(\tilde{S}\right) = +$ and $\chi\left(\tilde{S}\right) = n\chi(S)$.

Example 4.24. Let S be the connect sum of 3 copies of $\mathbb{P}^2$. Let $\tilde{S}$ be a 4-sheeted (connected) covering of S. We know $\chi(S) = 2 - 3 = -1$ which means $\chi\left(\tilde{S}\right) = -4$. This has even Euler characteristic, so there are two possibilities. The connect sum of 3 tori or 6 projective planes. In fact these are both realized. Figure 6 shows the latter.

Example 4.25. If S is compact, connected, and $\tilde{S} \to S$ is a connected n-sheeted covering. We know:

1. $\chi\left(\tilde{S}\right) = n\chi(S)$.
2. $|\partial S| \leq \left|\partial \tilde{S}\right| \leq n\,|\partial S|$.

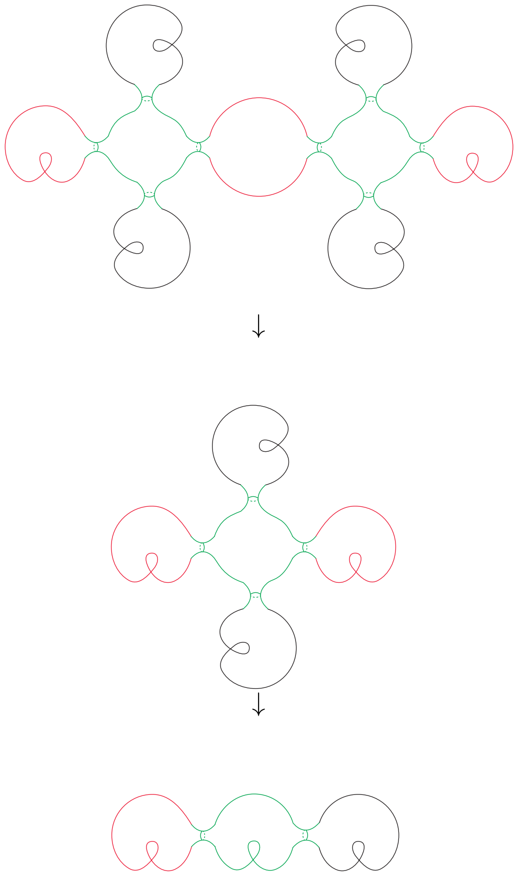


Figure 6. The composition of these maps gives us a covering of the connect sum of three copies of $\mathbb{P}^2$ by the connect sum of 6 copies of $\mathbb{P}^2$. The bottom map is involution on the green sphere, and the top is involution on the red sphere.

Now we might try to realize when these are saturated inequalities, but in fact we have:

Fact 5. *If S is orientable and (1) and (2) hold, then $\tilde{S}$ is an n-sheeted covering of S.*

This is false if S is non-orientable.

There is a generalization of Nielsen-Schreier which tells us that:

Theorem 4.33 (Kurosh subgroup theorem). *Let H be a subgroup of $A * B$. Then $H = (*_\lambda H_\lambda) * F$, where each H_λ is a subgroup of a conjugate of A or B (in $A * B$) and F is a free group. (And similarly for arbitrary free products $*_\lambda A_\lambda$.)*

Proof. Let X_A and X_B be connected 2-complexes with $\pi_1(X_A) \cong A$ and $\pi_1(X_B) \cong B$. We can take X_A and X_B to have a zero cell x_A and x_B. Let X be obtained from $X_A \amalg X_B$ by attaching a 1-cell e which connects these two 0-cells. Take x_0 to be the midpoint of e.

Then by van Kampen we have

$$\pi_1(X, x_0) \cong \pi_1(X_A, x_A) * \pi_1(X_B, x_B) \cong A * B\ . \tag{4.100}$$

So for $H < A{*}B$ we have a connected covering $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ such that $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right) = H$. Each component of $p^{-1}(X_A)$ is a covering space of X_A, each component of $p^{-1}(X_B)$ is a covering space of X_B, and each component of $p^{-1}(e)$ is a covering space of a 1-cell. Let $\tilde{x}_A$ and $\tilde{x}_B$ be the endpoints of the 1-cell in $\tilde{X}$ containing $\tilde{x}_0$. Let X_λ be a component of $p^{-1}(X_A)$ or $p^{-1}(X_B)$ (say $p^{-1}(X_A)$). Let $x_\lambda \in p^{-1}(x_A) \in X_\lambda$. Let γ_λ be a path in $\tilde{X}$ from x_λ to $\tilde{x}_A$. Then $p\gamma_\lambda$ is a loop in X at x_A which means $g_\lambda \in \pi_1(X, x_A) \cong \pi_1(X, x_0)$.

Then

$$\pi_1(X_\lambda, x_\lambda) \to \pi_1\left(\tilde{X}, x_\lambda\right) \xrightarrow{\gamma_{\lambda\#}} \pi_1\left(\tilde{X}, \tilde{x}_A\right) \tag{4.101}$$

and

$$p_*(\pi_1(X_\lambda, x_\lambda)) \subset g_\lambda^{-1}\pi_1(X_A, x_A)\, g_\lambda = g_\lambda^{-1} A g_\lambda\ . \tag{4.102}$$

Now take a maximal tree in $X_\lambda^{(1)}$, and retract this to a point to get $X'_\lambda \simeq X_\lambda$, where X'_λ has a single 0-cell x'_λ. Then $\tilde{X} \simeq \tilde{X}'$ is a graph Γ with X'_λs attached to the vertices. Now identify a maximal tree in Γ to a point. Now by van Kampen

$$\pi_1\left(\tilde{X}, \tilde{x}_0\right) \cong \pi_1\left(\tilde{X}', \tilde{x}'_0\right) = \mathop{*}_{\lambda} \underbrace{\pi_1(X'_\lambda, x'_\lambda)}_{*_\lambda \pi_1(X_\lambda, x_\lambda)} * \pi_1(\Gamma) \tag{4.103}$$

so in the end we get

$$H = p_* \pi_1\left(\tilde{X}, \tilde{x}_0\right) \tag{4.104}$$

and

$$p_*(\pi_1(X_\lambda, x_\lambda)) \subset g_\lambda^{-1}\{A \text{ or } B\}\, g_\lambda \tag{4.105}$$

and we are done. ☐

CHAPTER 5

Homology

A subset $A \subseteq \mathbb{R}^m$ is *convex* if $a_1, a_2 \in A$ implies $ta_1 + (1-t)\, a_2 \in A$ for all $t \in I$. For $A, B \subseteq \mathbb{R}^m$ are convex, then a map $f : A \to B$ is *affine* if

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$$f\,(ta_1 + (1-t)\, a_2) = tf\,(a-1) + (1-t)\, f\,(a_2) \tag{5.1}$$

for all $a_1, a_2 \in A$ and for all $t \in I$.

Let R be a commutative ring. A sequence of R-modules[5.1]

$$\ldots \to A_{q+1} \xrightarrow{\alpha_{q+1}} A_q \xrightarrow{\alpha_q} A_{q-1} \xrightarrow{\alpha_{q-1}} \ldots \tag{5.2}$$

is *exact* at A_q if $\operatorname{im}(\alpha_{q+1}) = \ker(\alpha_q)$. A sequence of exact if it is exact at A_q for all q. A *short exact sequence* (SES) is an exact sequence of the form

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \ , \tag{5.3}$$

i.e. α is injective, β is surjective, and $\operatorname{im}(\alpha) = \ker(\beta)$.

Let $v_0, v_1, \ldots v_q \in \mathbb{R}^q$ be the points

$$v_0 = (0, \ldots, 0) \tag{5.4}$$

$$v_1 = (1, 0, \ldots, 0) \tag{5.5}$$

$$v_2 = (0, 1, 0, \ldots, 0) \tag{5.6}$$

$$\ldots \tag{5.7}$$

$$v_q = (0, \ldots, 1) \ . \tag{5.8}$$

Let Δ^q be the *convex hull* of $\{v_0, \ldots, v_q\}$:

$$\Delta^q \coloneqq \left\{ \sum_{i=0}^{q} \lambda_i v_i \,\middle|\, \lambda_i \geq 0, \sum_{i=0}^{q} \lambda_i = 1 \right\} \ . \tag{5.9}$$

For X a topological space, a (singular) q-simplex in X is a map $\sigma : \Delta^q \to X$. Let $\Sigma_q\,(X)$ be the set of all singular q-simplices in X. Now we make this into an abelian group by taking the free abelian group on $\Sigma_q\,(X)$. We should thin of this as the set of all formal linear combinations of simplices:

$$S_q\,(X) = \left\{ \sum_{\sigma \in \Sigma(X)} n_\sigma \sigma \,\middle|\, n_\sigma \in \mathbb{Z}, n_\sigma = 0 \ \forall \text{ but fin'tly many } \sigma \in \Sigma_q\,(X) \right\} \ . \tag{5.10}$$

We call this group *the group of (singular) q-chains in X*. If $q < 0$ define $S_q\,(X) = 0$.

If $f : X \to Y$, then we can define $f_\# : S_q\,(X) \to S_q\,(Y)$ by

$$f_\#\,(\sigma) = f\sigma \tag{5.11}$$

[5.1] E.g. for $R = \mathbb{Z}$ these are abelian groups, for $R = \mathbb{R}$ these are vector spaces.

then we just extend this linearly. Then $\mathrm{id}_{\#} = \mathrm{id}$ and $(gf)_{\#} = g_{\#} f_{\#}$. So even at this point we have functoriality.

Now we define the boundary homomorphisms. First we form the affine idea, and just apply it to the singular chains. Consider the standard 1-simplex

$$v_0 \longrightarrow v_1 \quad . \tag{5.12}$$

We want to think of the boundary of this as $\partial (v_0 v_1) = v_1 - v_0$. Now consider the standard 2-simplex. We think of orienting it as:

$$\begin{tikzcd}
v_2 \arrow[d] & \\
v_0 \arrow[r] & v_1 \arrow[lu]
\end{tikzcd} \quad . \tag{5.13}$$

Then we want to think of the boundary of this as:

$$\partial (v_0 v_1 v_2) = (v_1 v_2) - (v_0 v_2) + (v_0 v_1) \quad . \tag{5.14}$$

Note that

$$\partial^2 (v_0 v_1 v_2) = \partial (v_1 v_2) - \partial (v_0 v_2) + \partial (v_0 v_1) \tag{5.15}$$

$$= (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) \tag{5.16}$$

$$= 0 \; . \tag{5.17}$$

Now we want to express this idea in terms of maps. Let $A \subset \mathbb{R}^m$ be a convex set. If $a_0, \ldots, a_q \in A$, let $[a_0 \ldots a_q]$ denote the singular q-simplex given by the affine map $\Delta^q \to A$ given by

$$v_i \mapsto a_i \tag{5.18}$$

for $0 \leq i \leq q$. So $[a_0 \ldots a_q] \in \Sigma_q (A)$. In particular,

$$\Sigma_q (\Delta^q) \ni [v_0 \ldots v_q] = \mathrm{id}^q : \Delta^q \to \Delta^q \; . \tag{5.19}$$

For $0 \leq i \leq q$, define the *ith face of* id^q by

$$\mathrm{id}^q_{(i)} = [v_0 \ldots \hat{v}_i \ldots v_q] : \Delta^{q-1} \to \Delta^q \tag{5.20}$$

where $\hat{v}_i$ indicates that we omit v_i. For arbitrary $\sigma \in \Sigma_q (X)$ define the *ith face of* σ to be

$$\sigma_{(i)} = \sigma \, \mathrm{id}^q_{(i)} = \sigma_{\#} \left(\mathrm{id}^q_{(i)} \right) \; . \tag{5.21}$$

The q-dimensional boundary homomorphism $\partial_q : S_q (X) \to S_{q-1} (X)$ is defined to be

$$\partial_q (\sigma) = \sum_{i=0}^{q} (-1)^i \sigma_{(i)} \tag{5.22}$$

and then extended linearly.

Remark 5.1. (1) $\partial_q (\sigma) = \sigma_{\#} (\partial_q (\mathrm{id}^q))$

(2) For $f : X \to Y$ a map,

$$\begin{tikzcd}
S_q (X) \arrow[r, "f_{\#}"] \arrow[d, "\partial_q"] & S_q (Y) \arrow[d, "\partial_q"] \\
S_{q-1} (X) \arrow[r, "f_{\#}"] & S_{q-1} (Y)
\end{tikzcd} \tag{5.23}$$

commutes.

Lemma 5.1. $\partial_{q-1}\partial_q = 0$.

Proof. It is sufficient to show that $\partial_{q-1}\partial_q(\sigma) = 0$ for all $\sigma \in \Sigma_q(X)$. But this is just

$$\sigma_\#\left(\partial_{q-1}\partial_q\left(\mathrm{id}^q\right)\right) \tag{5.24}$$

so it it sufficient to show:

$$\partial_{q-1}\partial_q\left(\mathrm{id}^q\right) = \partial_{q-1}\left(\sum_{i=0}^{q}(-1)^i\left[v_0 \ldots \hat{v}_i \ldots v_q\right]\right) \tag{5.25}$$

$$= \sum_{i=0}^{q}\sum_{j=0}^{i-1}(-1)^i(-1)^j\left[v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_q\right] \tag{5.26}$$

$$+\sum_{i=0}^{q}\sum_{j=i+1}^{q}(-1)^i(-1)^{j-1}\left[v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_q\right] \tag{5.27}$$

$$= 0\ . \tag{5.28}$$

☐

A *chain complex* is an indexed set of abelian groups $C = \{C_q\}_{q\in\mathbb{Z}}$ and a *boundary homomorphism* $\partial = \{\partial_q\}$ where $\partial_q : C_q \to C_{q-1}$ (we say ∂ has degree -1) such that $\partial_{q-1}\partial_q = 0$. We write $\partial : C \to C$ and say $\partial^2 = 0$. We call C_q the *q-chains* of C, we write

$$Z_q(C) = \ker\partial_q \subset C_q \tag{5.29}$$

for the *q-cycles*, and we write

$$B_q(C) = \operatorname{im}\partial_{q+1} \subset C_q \tag{5.30}$$

for the *q-boundaries.* Note that since $\partial^2 = 0$,

$$B_q(C) \subset Z_q(C) \subset C_q\ . \tag{5.31}$$

Now define the *q-th homology group of C* to be

$$H_q(C) = Z_q(C)/B_q(C)\ . \tag{5.32}$$

We write $H(C) = \{H_q(C)\}_{q\in\mathbb{Z}}$.

Write $S(X) = \{S_q(X)\}$ for the singular chain complex of X. Then $H_q(X)$ is the q-dimensional singular homology of X. By definition, $H_q(X) = 0$ for $q < 0$.

Suppose we have two chain complexes C and D. A chain map $\varphi : C \to D$ is an indexed set of homomorphisms $\{\varphi_q : C_q \to D_q\}$ (we say φ has degree 0) such that

$$\begin{tikzcd} C_q \arrow[r, "\varphi_q"] \arrow[d, "\partial_q"] & D_q \arrow[d, "\partial_q"] \\ C_{q-1} \arrow[r, "\varphi_{q-1}"] & D_{q-1} \end{tikzcd} \tag{5.33}$$

commutes, i.e. $\partial\varphi = \varphi\partial$.

Example 5.1. If $f : X \to Y$ is a map then we get a chain map $f_\# : S(X) \to S(Y)$.

Lemma 5.2. *A chain map $\varphi : C \to D$ induces a homomorphism at the level of homology $\varphi_* : H(C) \to H(D)$ (of degree 0).*

PROOF. Suppose we have $c \in C_{q+1}$. Then $\varphi(\partial c) = \partial(\varphi c)$ which means

$$\varphi(B_q(C)) \subset B_q(D) \ . \tag{5.34}$$

Also, $z \in Z_q(c)$ implies $\partial z = 0$, which implies $\varphi(\partial z) = 0$, so $\partial(\varphi(z)) = 0$, so $\varphi(z) \in Z_q(D)$. Therefore

$$\varphi(Z_q(C)) \subset Z_q(D) \ . \tag{5.35}$$

Therefore φ induces a homomorphism

$$Z_q(C)/B_q(C) = H_q(C) \to H_q(D) = Z_q(D)/B_q(D) \ . \tag{5.36}$$

□

REMARK 5.2. $\mathrm{id}_* = \mathrm{id}$ and $(\psi\varphi)_* = \psi_*\varphi_*$.

For $f : X \to Y$ this induces a chain map $f_\# : S(X) \to S(Y)$, which in turn induces $f_* : H(X) \to H(Y)$.

Elementary computations

THEOREM 5.3. *Suppose we have a space X with path components $\{X_\lambda\}$. Then*

$$H(X) = \bigoplus_\lambda H(X_\lambda) \ . \tag{5.37}$$

PROOF. Δ_q is path-connected, so

$$\Sigma_q(X) = \bigoplus_\lambda \Sigma_q(X_\lambda) \tag{5.38}$$

hence

$$S_q(X) = \bigoplus_\lambda S_q(X_\lambda) \tag{5.39}$$

as abelian groups. Then we have that

$$\partial_q(S_q(X_\lambda)) \subset S_{q-1}(X_\lambda) \tag{5.40}$$

so as chain complexes

$$S(X) \cong \bigoplus_\lambda S(X_\lambda) \tag{5.41}$$

which yields the result. □

THEOREM 5.4 (Homology of a point). *Let $X = \{x_0\}$ be a one-point space. Then*

$$H_q(X) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & o/w \end{cases} \ . \tag{5.42}$$

PROOF. For every q there is a unique elements $\sigma_q \in \Sigma_q(X)$. Therefore for $q \geq 0$, $S_q(X) \cong \mathbb{Z}$. For $q > 0$

$$\partial\sigma_q = \sum_{i=0}^{q} (-1)^i \sigma_{q-1} = \begin{cases} \sigma_{q-1} & q \text{ even} \\ 0 & q \text{ odd} \end{cases} \tag{5.43}$$

so the chain complex $S\left(X\right)$ is

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} 0 \to 0 \to \cdots \ . \tag{5.44}$$

So for $q > 0$

$$Z_q\left(X\right) = B_q\left(X\right) = \begin{cases} 0 & q \text{ even} \\ \mathbb{Z} & q \text{ odd} \end{cases} \tag{5.45}$$

so we get exactly what we wanted. □

Suppose X is nonempty. Then there exists a unique map $e : X \to \{x_0\}$. Define the *augmentation map*

$$\epsilon = e_\# : S_0\left(X\right) \to S_0\left(\{x_0\}\right) \cong \mathbb{Z} \ . \tag{5.46}$$

Clearly ϵ is onto. Also,

$$\epsilon\partial_1 = e_\#\partial_1 = \partial_1 e_\# = 0 \tag{5.47}$$

since $\partial_1 : S_1\left(\{x_0\}\right) \to S_0\left(\{x_0\}\right)$ is 0 from the proof of Theorem 5.4. This means we get an *augmented chain complex*

$$\ldots \to S_q\left(X\right) \to S_{q-1}\left(X\right) \to \ldots \to S_1\left(X\right) \xrightarrow{\partial_1} S_0\left(X\right) \xrightarrow{\epsilon} \mathbb{Z} \to 0 \to 0 \ldots \ . \tag{5.48}$$

Remark 5.3. If $c \in S_0\left(X\right)$, $c = \sum_{x\in X} n_x x$. (Note we are identifying a singular 0-simplex in X with its image, i.e. a point in X). Then explicitly the augmentation homomorphism is

$$\epsilon\left(c\right) = \sum n_x \ . \tag{5.49}$$

Theorem 5.5. *For $\{X_\lambda\}$ the set of path-components of X, then*

$$H_0\left(X\right) \cong \bigoplus_\lambda \mathbb{Z} \ . \tag{5.50}$$

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Proof. Theorem 5.3 tells us that it is sufficient to show that if X is nonempty and pc, then $H_0\left(X\right) \cong \mathbb{Z}$. Since $\epsilon\partial_1 = 0$, we know $\operatorname{im}\partial_1 \subset \ker\epsilon$. So we want to show the converse. Let $c_0 = \sum_{x\in X} n_x x \in \ker\epsilon$, i.e. $\sum n_x = 0$. Let $x_0 \in X$. For $x \in X$ let $\alpha_x : \Delta^1 \to X$ be such that

$$\alpha_x\left(v_0\right) = x_0 \qquad\qquad \alpha\left(v_1\right) = x \tag{5.51}$$

(where we are using that X is pc). Then $\partial\alpha_x = x - x_0$. Let

$$c_1 = \sum n_x \alpha_x \in S_1\left(X\right) \ . \tag{5.52}$$

Then

$$\partial_1 c_1 = \sum n_x\left(x - x_0\right) = c_0 - \left(\underbrace{\sum n_x}_{0}\right) x_0 = c_0 \ . \tag{5.53}$$

Therefore $\ker\epsilon \subset \operatorname{im}\partial_1$. Therefore $\ker\epsilon = \operatorname{im}\partial_1 = B_0\left(X\right)$. This means

$$H_0\left(X\right) \cong Z_0\left(X\right)/B_0\left(X\right) = S_0\left(X\right)/B_0\left(X\right) = S_0\left(X\right)/\ker\epsilon \cong \operatorname{im}\alpha \cong \mathbb{Z} \tag{5.54}$$

as desired. □

Write $\tilde{S}(X)$ for the augmented chain complex in (5.48). Recall that by definition $\tilde{S}_q(X) = S_q(X)$ for $q \neq -1$ and $\tilde{S}_{-1}(X) = \mathbb{Z}$. Then define *reduced (singular) homology of* X to be

$$H\left(\tilde{S}(X)\right) = \tilde{H}(X) \ . \tag{5.55}$$

By construction

$$\tilde{H}_q(X) \cong H_q(X) \tag{5.56}$$

for $q \neq 0$, and in fact we have

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z} \ . \tag{5.57}$$

To see this, first notice we have a SES

$$\begin{tikzcd} 0 \arrow[r] & \ker\epsilon \arrow[r] \arrow[d, equal] & S_0(X) \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] & 0 \\ & \tilde{Z}_0(X) & Z_0(X) & & \end{tikzcd} \tag{5.58}$$

and since $\mathbb{Z}$ is free abelian, this splits, i.e.

$$Z_0(X) \cong \tilde{Z}_0(X) \oplus \mathbb{Z} \ . \tag{5.59}$$

Exercise 0.1. Show that for any SES

$$0 \to A \to B \to \mathbb{Z} \to 0 \tag{5.60}$$

(or more generally if $\mathbb{Z}$ is replaced by something free abelian) then $B \cong A \oplus \mathbb{Z}$.

This means

$$B_0(X) = \tilde{B}_0(X) \subset \tilde{Z}_0(X) \tag{5.61}$$

which means

$$H_0(X) \cong Z_0(X)/B_0(X) \cong \frac{\tilde{Z}_0(X) \oplus \mathbb{Z}}{\tilde{B}_0(X)} \cong \tilde{H}_0(X) \oplus \mathbb{Z} \ . \tag{5.62}$$

1. Homology exact sequence of a pair

Let C be a chain complex, and $D \subset C$ a subcomplex. Then we get a quotient complex C/D.

Example 5.2. Let (X, A) be a pair of topological spaces. Then $S(A) \subset S(X)$ is a subcomplex.

We define the *(singular) chain complex of* (X, A) to be $S(X, A) = S(X)/S(A)$.

Remark 5.4.
1. $S_q(X, A)$ is the free abelian group on $\Sigma_q(X) \setminus \Sigma_q(A)$.
2. $S(X, \emptyset) = S(X)$.
3. For $f : (X, A) \to (Y, B)$ a map of pairs, we get an induced chain map

$$f_{\#} : S(X) \to S(Y) \tag{5.63}$$

such that $f_{\#}(S(A)) \subset S(B)$ so it really induces a chain map

$$f_{\#} : S(X, A) \otimes S(Y, B) \ . \tag{5.64}$$

(4) Let $i : A \hookrightarrow X$, $j : (A, \emptyset) \to (X, A)$ be the inclusion maps. Then we get a SES of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & S(A) \arrow[r, "i_\#"] & S(X) \arrow[r, "j_\#"] \arrow[d, equal] & S(X, A) \arrow[r] & 0 \\ & & S(X, \emptyset) & S(X) / S(A) & \end{tikzcd} \; . \tag{5.65}$$

(5) $H(S(X, A)) = H(X, A)$ is the *singular homology* of (X, A). Note $H(X, \emptyset) = H(X)$. For $f : (X, A) \to (Y, B)$ we get a (graded) homomorphism

$$f_* : H(X, A) \to H(Y, B) \; . \tag{5.66}$$

Now the goal is to relate $H(A)$, $H(X)$, and $H(X, A)$ by means of an exact sequence.

THEOREM 5.6. *Let*

$$0 \to C \xrightarrow{\varphi} D \xrightarrow{\psi} E \to 0 \tag{5.67}$$

be a SES of chain complexes and chain maps. Then there is a (long) exact sequence

$$\ldots \to H_q(C) \xrightarrow{\varphi_*} H_q(D) \xrightarrow{\psi_*} H_q(E) \xrightarrow{\Delta} H_{q-1}(C) \to \ldots \tag{5.68}$$

PROOF. (1) First we prove exactness at $H_q(D)$.

- (i) ($\operatorname{im}\varphi_* \subset \ker\psi_*$): By assumption, $\psi\varphi = 0$, which means $(\psi\varphi)_* = \psi_*\varphi_* = 0$. So $\operatorname{im}\varphi_* \subset \ker\psi_*$.
- (ii) ($\ker\psi_* \subset \operatorname{im}\varphi_*$): Suppose $u \in H_q(D)$ is in $\ker\psi_*$. Let $u = [z]$, for some $z \in Z_q(D)$. Then

$$[\psi(z)] = \psi_*(u) = 0 \in H_q(E) \; . \tag{5.69}$$

Therefore $\psi(z) \in B_q(E)$, so there exists $y \in E_{q+1}$ such that $\partial y = \psi(z)$. Since ψ is surjective, there exists $x \in D_{q+1}$ such that $\psi(x) = y$. The diagram is:

$$\begin{tikzcd} & & x \arrow[r, mapsto] & y & \\ & & D_{q+1} \arrow[r, "\psi"] \arrow[d, "\partial"] & E_{q+1} \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & C_q \arrow[r, "\varphi"] & D_q \arrow[r, "\psi"] & E_q \arrow[r] & 0 \\ & & z - \partial x \arrow[r, mapsto] & 0 & \end{tikzcd} \; . \tag{5.70}$$

Then we can ask where this goes in D_q, and it would be nice if it was z, but we can at least look at their difference $z - \partial x$, and then

$$\psi(z - \partial x) = \psi(z) - \psi(\partial x) = \partial y - \partial\psi(x) = \partial y - \partial y = 0 \; . \tag{5.71}$$

Therefore $z - \partial x \in \ker\psi = \operatorname{im}\varphi$, so there is $w \in C_q$ such that $\varphi(w) = z - \partial x$. Therefore

$$\varphi(\partial w) = \partial(z - \partial x) = \partial z - \partial^2 x = 0 \; . \tag{5.72}$$

Then φ is injective, so $\partial w = 0$, i.e. $w \in Z_q(C)$. Let $v = [w] \in H_q(C)$. Then $\varphi_*(v) = [\varphi(w)] = [z - \partial x] = [z] = u$, so $u \in \operatorname{im}\varphi_*$ so $\ker\psi_* \subset \operatorname{im}\varphi_*$.

(2) (Definition of Δ): This is often called the connecting homomorphism. Recall this is a homomorphism $\Delta : H_q(E) \to H_{q-1}(C)$. Let $[z] \in H_q(E)$, $z \in Z_q(E)$. ψ is surjective, so there exists $d \in D_q$ such that $\psi(d) = z$. The diagram is:

$$d \longmapsto z$$

$$\begin{tikzcd}
 & & D_q \arrow[r, "\psi"] \arrow[d, "\partial"] & E_q \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{q-1} \arrow[r, "\varphi"] & D_{q-1} \arrow[r, "\psi"] & E_{q-1} \arrow[r] & 0
\end{tikzcd} . \tag{5.73}$$

$$\partial d$$

Then we have

$$\psi(\partial d) = \partial(\psi(d)) = \partial z = 0 \tag{5.74}$$

which means

$$\partial d \in \ker\psi = \operatorname{im}\varphi \ . \tag{5.75}$$

Therefore there exists $c \in C_{q-1}$ such that $\varphi(c) = \partial d$. Then $\varphi(\partial c) = \partial\varphi(c) = \partial^2 d = 0$. Now φ is injective which means $\partial c = 0$, i.e. $c \in Z_{q-1}(C)$. Define

$$\Delta([z]) = [c] \in H_{q-1}(C) \ . \tag{5.76}$$

Remark 5.5. A trick to remember this definition is to think of Δ as $\varphi^{-1}\partial\psi^{-1}z$. Don't take this literally, but it is the idea.

Exercise 1.1. Show Δ is well-defined and a homomorphism.

(3) (Exactness at $H_q(E)$):

Exercise 1.2. Show $\operatorname{im}\psi_* \subset \ker\Delta$, then $\ker\Delta \subset \operatorname{im}\psi_*$.

(4) (Exactness at $H_q(C)$):

Exercise 1.3. Show $\operatorname{im}\Delta \subset \ker\varphi_*$ and $\ker\varphi_* \subset \operatorname{im}\Delta$.

□

Corollary 5.7. *If (X, A) is a pair of topological spaces, then there exists an exact sequence*

$$\cdots \longrightarrow H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\Delta} H_{q-1}(A) \longrightarrow \cdots \tag{5.77}$$

called the homology exact sequence of the pair (X, A).

Proof. We get a SES of chain complexes $0 \to S(A) \to S(X) \to S(X, A) \to 0$ and then applying Theorem 5.6 we get the desired long exact sequence. □

REMARK 5.6.
1. Δ in Corollary 5.7 is functorial, i.e. for $f : (X,A) \to (Y,B)$ then

   (5.78)
   $$\begin{tikzcd} H_q(X,A) \arrow[r, "\Delta"] \arrow[d, "\varphi_*"] & H_{q-1}(A) \arrow[d, "(f|_A)_*"] \\ H_q(Y,B) \arrow[r, "\Delta"] & H_{q-1}(B) \end{tikzcd}$$

   commutes. Therefore the entire exact sequence in Corollary 5.7 is functorial.
2. In the setting of $H(X,A)$, Δ has a nice topological interpretation. Define the relative q cycles in (X,A) to be

   $$Z_q^A(X) = \{c \in S_q(X) \mid \partial c \in S_{q-1}(A)\} \ . \tag{5.79}$$

   Similarly we can define the relative q boundaries as:

   $$B_q^A(X) = \{c \in S_q(X) \mid \exists d \in S_{q+1}(X) \text{ s.t. } \partial d = c + a\,(a \in S_q(a))\} \ . \tag{5.80}$$

   We can see that actually

   $$H_q(X,A) = \frac{Z_q(X,A)}{B_q(X,A)} \cong \frac{Z_q^A(X)/S(A)}{B_q^A(X)/S(A)} \cong Z_q^A(X)/B_q^A(X) \ . \tag{5.81}$$

   Then it turns out that for $[c] \in Z_q^A(X)$

   $$\Delta([c]) = [\partial c] \in H_{q-1}(A) \ . \tag{5.82}$$

   We can think of c as being some chain in X which doesn't have zero boundary, but the boundary is in $A \subset X$.

Lecture 18; November 12, 2019

Recall for nonempty X we can define the augmented singular chain complex $\tilde{S}(X)$ For nonempty A, we get a SES of chain complexes

$$0 \to \tilde{S}(A) \to \tilde{S}(X) \to \tilde{S}(X,A) \to 0 \ . \tag{5.83}$$

But now notice that

$$\tilde{S}(X,A) = \tilde{S}(X)/\tilde{S}(A) \cong S(X)/S(A) \tag{5.84}$$

since $\tilde{S}_{-1}(A) \cong \tilde{S}_{-1}(X) \cong \mathbb{Z}$ which implies $\tilde{S}_{-1}(X,A) = 0 = S_{-1}(X,A)$. This induces a long exact sequence in $\tilde{H}$:

(5.85)
$$\begin{tikzcd} \cdots \arrow[r] & \tilde{H}_q(A) \arrow[r] & \tilde{H}_q(X) \arrow[r] & \tilde{H}_q(X,A) \arrow[r] & \cdots \arrow[r] & \tilde{H}_0(A) \arrow[r] & \tilde{H}_0(X) \arrow[r] & \tilde{H}_0(X,A) \arrow[r] \arrow[d, equal] & \cdots \\ & & & & & & & H_0(X,A) & \end{tikzcd}$$

DEFINITION 5.1. We say chain maps $\varphi, \psi : C \to D$ are *chain homotopic*, $\varphi \overset{c}{\cong} \psi$ iff there exists a homomorphism $T : C \to D$ of degree $+1$ such that $\partial T + T\partial = \varphi - \psi$. We say T is a *chain homotopy* from φ to ψ.

REMARK 5.7. Anything of the form $\partial T + T\partial$ is automatically a chain map because

$$\partial(\partial T + T\partial) = \partial T \partial = (\partial T + T\partial)\partial \ . \tag{5.86}$$

DEFINITION 5.2. We say $\varphi : C \to D$ is a *chain homotopy equivalence* iff there exists $\psi : D \to C$ such that $\psi\varphi \overset{c}{\cong} \mathrm{id}_C$ and $\varphi\psi \overset{c}{\cong} \mathrm{id}_D$. We say C and D are *chain homotopy equivalent*.

Lemma 5.8. *Let* $\varphi, \psi : C \to D$ *be chain maps. Then* $\varphi \overset{c}{\cong} \psi$ *implies* $\varphi_* = \psi_* : H(C) \to H(D)$.

Proof. Let $z \in Z_q(C)$. Then

$$(\varphi_* - \psi_*)([z]) = [\varphi(z) - \psi(z)] = [(\varphi - \psi)(z)] \tag{5.87}$$

$$= [(\partial T + T\partial)(z)] = [\partial T(z)] = 0 \in H_q(D)\ . \tag{5.88}$$

□

Corollary 5.9. $C \overset{c}{\cong} D$ *implies* $H(C) \cong H(D)$.

The definition of $\overset{c}{\cong}$ is motivated by topology. In particular:

Theorem 5.10. $f \simeq g : (X, A) \to (Y, B)$ *implies* $f_\# \overset{c}{\cong} g_\# : S(X, A) \to S(Y, B)$.

This gives us the homotopy invariance of singular homology.

Corollary 5.11. $f \simeq g : (X, A) \to (Y, B)$ *implies* $f_* \simeq g_* : H(X, A) \to H(Y, B)$.

Corollary 5.12. $(X, A) \simeq (Y, B)$ *implies* $H(X, A) \cong H(Y, B)$.

To prove Theorem 5.10 it is sufficient to construct

$$T : S_q(X) \to S_{q+1}(X \times I) \tag{5.89}$$

such that

$$\partial T + T\partial = i_{1\#} - i_{0\#}\ , \tag{5.90}$$

where $i_\epsilon : X \to X \times I$ is inclusion $i_\epsilon(x) = (x, \epsilon)$ where $\epsilon \in \{0, 1\}$.

To this, it is enough to take $X = \Delta^q$ and define

$$T(\mathrm{id}^q) \in S_{q+1}(\Delta^q \times I)\ . \tag{5.91}$$

Then define $T : S_q(X) \to S_{q+1}(X \times I)$ by

$$\boxed{T(\sigma) = (\sigma \times \mathrm{id})_\# T(\mathrm{id}^q)} \tag{5.92}$$

for $\sigma \in \Sigma_q(X)$.

Note that T is automatically fuctorial, i.e. if $f : X \to Y$ then

$$\begin{tikzcd} S_q(X) \arrow[r, "T"] \arrow[d, "g_\#"] & S_{q+1}(X \times I) \arrow[d, "(f \times \mathrm{id})_\#"] \\ S_q(Y) \arrow[r, "T"] & S_{q+1}(Y \times I) \end{tikzcd}\ . \tag{5.93}$$

The idea of (5.91) is that $T(\mathrm{id}^q)$ is a linear combination of affine $(q+1)$-simplices in $\Delta^q \times I$, whose union is $\Delta^q \times I$, with coefficients ± 1 chosen so that ∂s cancel, except for

$$\partial(\Delta^q \times I) = \Delta^q \times \{1\} \cup \Delta^1 \times \{0\} \cup \partial\Delta^q \times I\ . \tag{5.94}$$

Let $a_i = (v_i, 0), b_i = (v_i, 1) \in \Delta^q \times I$ of $0 \le i \le q$. Then define

$$T(\mathrm{id}^q) = \sum_{i=0}^{q} (-1)^i [a_0 \dots a_i b_i \dots b_q]\ . \tag{5.95}$$

Proposition 5.13. $\partial T(\mathrm{id}^q) = [b_0 \dots b_q] - [a_0 \dots a_q] - T(\partial(\mathrm{id}^q))$.

The proof of this is just an explicit computation. See Hatcher.

EXAMPLE 5.3. For $q = 0$, $\Delta^0 \times I$ is just an interval running from a_0 to b_0:

$$a_0 \longrightarrow b_0 \quad . \tag{5.96}$$

Here we have

$$T\left(\mathrm{id}^0\right) = [a_0 b_0] \tag{5.97}$$

$$\partial T\left(\mathrm{id}^0\right) = [b_0] - [a_0] \quad . \tag{5.98}$$

EXAMPLE 5.4. For $q = 1$ we have a square which we decompose as two simplices:

$$\begin{tikzcd} b_0 \arrow[r] & b_1 \arrow[d] \\ a_0 \arrow[u] \arrow[ur, dash, "\circlearrowright", "\circlearrowright"'] & a_1 \arrow[l] \end{tikzcd} \quad . \tag{5.99}$$

The idea is that

$$\partial = \begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \uparrow & & \downarrow \\ \cdot & \longleftarrow & \cdot \end{array} = \left(\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ & & \\ \cdot & \longleftarrow & \cdot \end{array}\right) - \left(\begin{array}{ccc} \cdot & & \cdot \\ \downarrow & & \uparrow \\ \cdot & & \cdot \end{array}\right) \quad . \tag{5.100}$$

Notice the inner edge is oriented opposite ways when we look at each individual simplex. As the algebra will show us, this is such that it will cancel. In particular

$$T\left(\mathrm{id}^1\right) = [a_0 b_0 b_1] - [a_0 a_1 b_1] \tag{5.101}$$

and

$$\partial T\left(\mathrm{id}^1\right) = ([b_0 b_1] - [a_0 b_1] + a_0 b_0) - ([a_1 b_1] - [a_0 b_1] + [a_0 a_1]) \tag{5.102}$$

$$= [b_0 b_1] - [a_0 a_1] - ([a_1 b_1] - [a_0 b_0]) \quad . \tag{5.103}$$

Then we have

$$T([v_0]) = T\left(\mathrm{id}^0\right) = [a_0 b_0] \tag{5.104}$$

and

$$T([v_1]) = ([v_1] \times \mathrm{id})_{\#}\, T\left(\mathrm{id}^0\right) \tag{5.105}$$

$$= ([v_1] \times \mathrm{id})\,[a_0 b_0] = [a_1 b_1] \tag{5.106}$$

which means

$$[a_1 b_1] - [a_0 b_0] = T([v_1]) - T([v_0]) = T([v_1] - [v_0]) = \partial T\left(\mathrm{id}^1\right) \quad . \tag{5.107}$$

Now consider $T : S_q(X) \to S_{q+1}(X \times I)$.

Lemma 5.14. $\partial T + T\partial = i_{1\#} - i_{0\#}$.

PROOF. It is enough to consider $\sigma \in \Sigma_q(X)$. Then we have

$$(\partial T + T\partial)(\sigma) = \partial\,(\sigma \times \mathrm{id})_{\#}\, T(\mathrm{id}^q) + T\sigma_{\#}\partial(\mathrm{id}^q) \tag{5.108}$$

$$= (\sigma \times \mathrm{id})_{\#}\, \partial T(\mathrm{id}^q) + (\sigma \times \mathrm{id})_{\#}\, T\partial(\mathrm{id}^q) \tag{5.109}$$

$$= (\sigma \times \mathrm{id})_{\#}\, (\partial T + T\partial)(\mathrm{id}^q) \tag{5.110}$$

$$= (\sigma \times \mathrm{id})_{\#}\, ([b_0 \ldots b_q] - [a_0 \ldots a_q])(\mathrm{id}^q) \tag{5.111}$$

$$= i_q \circ \sigma - i_0 \sigma = i_{1\#}(\sigma) - i_{0\#}(\sigma) \quad . \tag{5.112}$$

□

Proof of Theorem 5.10. Let $F : X \times I \to Y$ be a homotopy from f to g. F induces $F_\# : S_{q+1}(X \times I) \to S_{q+1}(Y)$. Let

$$T' = F_\# T : S_q(X) \to S_{q+1}(Y) \ . \tag{5.113}$$

Now we check that

$$\partial T' + T'\partial = \partial F_\# T + F_\# T\partial = F_\# (\partial T + T\partial) = F_\# \left(i_{1\#} - i_{0\#}\right) \tag{5.114}$$

$$= (F_{i_1})_\# - (F_{i_0})_\# = g_\# - f_\# \ . \tag{5.115}$$

So t' is a chain homotopy from $f_\#$ to $g_\#$.

Also $T(S(a)) \subset S(A \times I)$ and $F_\#(S(A \times I)) \subset S(B)$. Therefore T' is a chain homotopy of maps of pairs $(S(X), S(A)) \to (S(Y), S(B))$. Therefore this induces a chain homotopy equivalence from $f_\#$ to $g_\# : S(X, A) \otimes S(Y, B)$. ☐

Remark 5.8. $f \simeq g : X \to Y$ implies $f_* : \tilde{H}(X) \to \tilde{H}(Y)$ is an $\cong$.

Corollary 5.15. *If X is contractible then $\tilde{H}(X) = 0$.*

Proof. Recall $\tilde{H}(\text{pt})$ from Theorem 5.4. Then X being contractible implies $x \simeq$ pt. ☐

Lecture 19;
November 14, 2019

2. Excision

This is the property that makes homology easy to compute.

Theorem 5.16. *If $Z \subset A \subset X$ is such that $\mathrm{Cl}(Z) \subset \mathrm{int}(A)$ then the inclusion induces an isomorphism*

$$H(X \setminus Z, A \setminus Z) \to H(X, A) \ . \tag{5.116}$$

In this situation we say Z can be excised.

We will actually prove a more general result. Let $\mathcal{C}$ be a collection of subsets of X. Let $S_q^{\mathcal{C}}(X)$ be the free abelian group on the set of q-simplices such that the image is contained in some $U \in \mathcal{C}$. Call this the group of $\mathcal{C}$-small q-chains in X. So we have a subcomplex

$$S^{\mathcal{C}}(X) = \sum_{U \in \mathcal{C}} S(U) \subset S(X) \ . \tag{5.117}$$

Theorem 5.17. *If $\{\mathrm{int}(U) \mid U \in \mathcal{C}\}$ covers X, then the inclusion $S^{\mathcal{C}}(X) \to S(X)$ is a chain homotopy equivalence.*

Moreover, if $A \in \mathcal{C}$, then the inclusion $\left(S^{\mathcal{C}}(X), S(A)\right) \to (S(X), S(A))$ is a chain homotopy equivalence of pairs.

Proof that Theorem 5.17 $\implies$ Theorem 5.16. Let $B = X \setminus Z$. Then $\mathrm{int}(B) = X \setminus \mathrm{Cl}(Z) \supset X \setminus \mathrm{int}(A)$. Therefore $X = (\mathrm{int}(A)) \cup (\mathrm{int}(B))$. So by Theorem 5.17, the inclusion

$$\left(\underbrace{S^{\{A,B\}}(X)}_{=S(A)+S(B)}, S(A) \right) \to (S(X), S(A)) \tag{5.118}$$

is a chain homotopy equivalence of pairs. Therefore it induces a chain homotopy equivalence

$$\frac{S(A) + S(B)}{S(A)} \to \frac{S(X)}{S(A)} \ , \tag{5.119}$$

but

$$\frac{S(A)+S(B)}{S(A)} \cong \frac{S(B)}{S(A)\cap S(B)} = \frac{S(B)}{S(A\cap B)}\ . \tag{5.120}$$

This induces an isomorphism

$$H(X\setminus Z, A\setminus Z) = H(B, A\cap B) \to H(X,A)\ . \tag{5.121}$$

□

A useful variant of excision is the following.

THEOREM 5.18 (Mayer-Vietoris). *Suppose* $X = (\mathrm{int}(X_1))\cup(\mathrm{int}(X_2))$. *Then there exists a functorial exact sequence*

(5.122)

$$\cdots \longrightarrow H_q(X_1\cap X_2) \xrightarrow{\alpha_*} H_q(X_1)\oplus H_q(X_2) \xrightarrow{\beta_*} H_q(X) \xrightarrow{\Delta} H_{q-1}(X_1\cap X_2) \longrightarrow \cdots$$

where

$$\alpha_*(u) = (i_{1*}(u), -i_{2*}(u)) \tag{5.123}$$

$$\beta_*(v_1,v_2) = j_{1*}(v_1) + j_{2*}(v_2) \tag{5.124}$$

where the i_1, i_2, j_1, *and* j_2 *are the obvious inclusion maps. We will call this sequence the Mayer-Vietoris (M-V) sequence.*

PROOF. For $A, B \subset C$ abelian groups then there exists a SES

$$\begin{tikzcd} 0 \arrow[r] & A\cap B \arrow[r] & A\oplus B \arrow[r] & A+B \arrow[r] & 0 \\ & c \arrow[r, mapsto] & (c,-c) & & . \\ & & (a,b) \arrow[r, mapsto] & a+b & \end{tikzcd} \tag{5.125}$$

Hence there is a SES of chain complexes

(5.126)

$$\begin{tikzcd} 0 \arrow[r] & S(X_1)\cap S(X_2) \arrow[r, "\alpha"] \arrow[d, equal] & S(X_1)\oplus S(X_2) \arrow[r, "\beta"] & S(X_1)+S(X_2) \arrow[r] \arrow[d, equal] & 0 \\ & S(X_1\cap X_2) & & S^{\{X_1,X_2\}}(X) & \end{tikzcd}$$

where

$$\alpha(c) = (c,-c) \qquad\qquad \beta'(a,b) = a+b\ . \tag{5.127}$$

By Theorem 5.6 this induces an exact sequence as stated; where $\beta = ib'$ where $i : S^{\{X_1,X_2\}} \hookrightarrow S(X)$ is the inclusion. i_* is an isomorphism by Theorem 5.17. □

REMARK 5.9. (1) i_* is an isomorphism (so Theorem 5.18 holds) if $X = X_1\cup X_2$ such that x_1 is closed in X and has open V such that $(V, V\cap X_2)$ deformation retracts to $(X_1, X_2\cap X_2)$.

(2) Using the description of Δ in Theorem 5.6, we can describe $\Delta : H_q(X) \to H_{q-1}(X_1 \cap X_2)$ in Theorem 5.18. Recall

$$H_q(X) \cong H_q(S(X_1) + S(X_2)) \ , \tag{5.128}$$

so we can write $u \in H_q(X)$ as $[z] = [c_1 + c_2]$ for $c_i \in S_q(X_i)$ for $i \in \{1, 2\}$. Note that $\partial z = \partial c_1 + \partial c_2 = 0$. Recall that Δ consisted of pulling back under β', acting ∂, then pulling back under α. When we pull back under β' we get (c_1, c_2), the differential is $(\partial c_1, \partial c_2) = (\partial c_1, -\partial c_1)$, so after pulling back under α we get ∂c_1. The picture here is that we have two chains with the same boundary (up to orientation) (sitting in $X_1 \cap X_2$) and then we get a cycle since the boundaries cancel.

Now we will do some hard work.

Proof of Theorem 5.17. We will define what is called a *subdivision*. This will be a chain map $s : S(X) \to S(X)$ such that

(1) $s \overset{c}{\cong} \mathrm{id}$ and
(2) $\forall \sigma \in \Sigma_q(X)$, $\exists m = m(\sigma)$ such that $S^m(\sigma) \in S^{\mathcal{C}}(X)$.

Then we will use s to define a *chain deformation retraction* $\rho : S(X) \to S(X)$, i.e. a chain map such that

(1) $\rho \overset{c}{\cong} \mathrm{id}$ and
(2) $\rho|_{S^{\mathcal{C}}(X)} = i : S^{\mathcal{C}}(X) \to S(X)$.

This implies i is a chain homotopy equivalence.

Let $E \subset \mathbb{R}^n$ be convex. Define $S^a(E)$ to be the complex of affine chains in E, i.e. linear combinations of affine simplices in E. Define $C : S^q_p(\Delta^q) \to S^a_{p+1}(\Delta^q)$ by

$$C([e_0 \dots e_p]) = [be_0 \dots e_p] \tag{5.129}$$

where

$$b = \sum_{i=0}^{q} \frac{v_i}{q+1} \tag{5.130}$$

is the barycenter of Δ^q. C can be regarded as *coning on the barycenter of* Δ^q.

Lemma 5.19. *C is a chain homotopy from* id *to* 0*, i.e.* $\partial C + C\partial = \mathrm{id}$.

Proof.

$$\partial C[e_0 \dots e_p] = \partial [be_0 \dots e_p] \tag{5.131}$$

$$= [e_0 \dots e_p] + \sum_{i=0}^{p} (-1)^{i+1} [be_0 \dots \hat{e}_i \dots e_p] \tag{5.132}$$

$$= [e_0 \dots e_p] - C\left(\sum_{i=0}^{q} (-1)^i [e_0 \dots \hat{e}_i \dots e_p]\right) \tag{5.133}$$

$$= [e_0 \dots e_p] - C\partial [e_0 \dots e_p] \ . \tag{5.134}$$

□

Define $s : S_q(X) \to S_q(X)$ (such that $s\left(S^a_q(E)\right) \subset S^a_q(E)$ for E convex) inductively on q. Take $s = \mathrm{id} : S_0(X) \to S_0(X)$. Now it is enough to define it on the identity:

$$s(\mathrm{id}^q) = C(s(\partial(\mathrm{id}^q))) \tag{5.135}$$

since for general σ we can take $s(\sigma) = \sigma_{\#} s(\mathrm{id}^q)$.

Lemma 5.20. *s is a chain map, i.e. $\partial s = s\partial$.*

Proof. It is enough to show $\partial s(\mathrm{id}^q) = d\partial(\mathrm{id}^q)$, induct on q (exercise). ☐

Now we define $T : S_q(X) \to S_{q+1}(X)$ (such that $T\left(S^a_q(E)\right) \subset S^a_{q+1}(E)$ for E convex) inductively on q. Define $T = 0 : S_0(X) \to S_1(X)$. Then define

$$T(\mathrm{id}^q) = C(s - \mathrm{id} - T\partial)(\mathrm{id}^q) \ . \tag{5.136}$$

Lemma 5.21. *$\partial T + T\partial = s - \mathrm{id}$.*

Proof. Enough to show this for id^q, induct on q (exercise). ☐

Let $c \in S^a_q$ for $E \subset \mathbb{R}^n$ convex. Then

$$c = \sum_{i=1}^{k} n_i \sigma_i \tag{5.137}$$

for $n_i \neq 0$ and $\sigma_i \in \Sigma^a_q(E)$ for $1 \leq i \leq k$. Define

$$\mathrm{mesh}(\sigma) = \max\{\mathrm{diam}\, \sigma_i(\Delta^q) \,|\, 1 \leq i \leq k\} \tag{5.138}$$

Lemma 5.22. *Let $E \subset \mathbb{R}^n$ be convex and $c \in S^a_q(E)$. then*

$$\mathrm{mesh}\, s(c) \leq \frac{q}{q+1} \mathrm{mesh}(c) \ . \tag{5.139}$$

Note that if $\sigma = [e_0 \dots e_q] \in \Sigma^a_q(E)$ then

$$\mathrm{diam}\, \sigma = \max_{i \leq i, j \leq q} \{\|e_i - e_j\|\} \ . \tag{5.140}$$

Proof. It is enough to prove for $c = \sigma = [e_0 \dots e_q]$, induct on q (exercise). ☐

For $\sigma \in \Sigma_q(X)$ we have that

$$\left\{\sigma^{-1}(\mathrm{int}(U)) \,\middle|\, U \in \mathcal{C}\right\} \tag{5.141}$$

is an open cover of Δ^q. But Δ^q is compact, so there is some $\epsilon > 0$ such that for $A \subset \Delta^q$ with $\mathrm{diam}\, A < \epsilon$ then $A \subset \sigma^{-1}(\mathrm{int}(U))$ for some $U \in \mathcal{C}$, which implies $\sigma(A) \subset U$. By Lemma 5.22 there is some integer $m \geq 0$ such that

$$\mathrm{mesh}\, s^m(\mathrm{id}^q) < \epsilon \tag{5.142}$$

and then

$$s^m \sigma = \sigma_{\#}(s^m(\mathrm{id}^q)) \in S^{\mathcal{C}}(X) \ . \tag{5.143}$$

Now let $m(\sigma)$ be the least such $m \geq 0$. Note $m(\sigma) = 0$ iff $\sigma \in S^{\mathcal{C}}(X)$, and $m\left(\sigma_{(i)}\right) \leq m(\sigma)$ for $0 \leq i \leq q$.

Now we define $\theta : S(X) \otimes S^{\mathcal{C}}(X)$ by

$$\theta(\sigma) = s^{m(\sigma)}(\sigma) \ . \tag{5.144}$$

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The trouble is this is not a chain map. So there is a bit more fussing. Define $\varphi : S(X) \to S(X)$ by

$$\varphi(\sigma) = \left(\mathrm{id} + s + \ldots + s^{m(\sigma)-1}\right)(\sigma) \tag{5.145}$$

and 0 if $m(\sigma) = 0$. Then we have

$$\theta - \mathrm{id} = (s - \mathrm{id})\,\varphi\ . \tag{5.146}$$

The point is that $T\varphi$ is a chain homotopy from $\rho = \theta + T(\varphi\partial - \partial\varphi)$ to the identity.

Then we just need to check that $\rho(S(X)) \subset S^{\mathcal{C}}(X)$. We know $\theta(S(X)) \subset S^{\mathcal{C}}(X)$, and that $T\left(S^{\mathcal{C}}(X)\right) \subset S^{\mathcal{C}}(X)$. Therefore it is enough to show that

$$(\varphi\partial - \partial\varphi)(S(X)) \subset S^{\mathcal{C}}(X)\ . \tag{5.147}$$

We can directly verify that:

$$(\varphi\partial - \partial\varphi)(\sigma) = \varphi\left(\sum_{i=1}^{q} (-1)^i \sigma_{(i)}\right) - \partial\left(\mathrm{id} + s + \ldots + s^{m(\sigma)-1}\right)(\sigma) \tag{5.148}$$

$$= \sum_{i=0}^{q} \left(\mathrm{id} + s + \ldots + s^{m\left(\sigma_{(i)}\right)-1}\right)\left(\sigma_{(i)}\right) \tag{5.149}$$

$$- \left(\mathrm{id} + s + \ldots + s^{m(\sigma)-1}\right)\left(\sum_{i=0}^{q} (-1)^i \sigma_{(i)}\right) \tag{5.150}$$

$$\in S^{\mathcal{C}}(X) \tag{5.151}$$

since $m\left(\sigma_{(i)}\right) \le m(\sigma)$.

Finally, note that if $A \in \mathcal{C}$, then $\rho(S(A)) \subset S(A)$. ■

REMARK 5.10. If $Z \neq A$, then Theorem 5.16 holds for $\tilde{H}$. If $X_1 \cap X_2 \neq \emptyset$ then Theorem 5.18 holds for $\tilde{H}$.

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of spaces. Recall the wedge of $\{(X_\lambda, x_\lambda)\}$ (where each $x_\lambda \in X_\lambda$) is

$$\bigvee_\lambda X_\lambda = \left(\coprod_\lambda X_\lambda\right) / \left(x_\lambda \sim x_{\lambda'} \forall \lambda, \lambda' \in \Lambda\right)\ . \tag{5.152}$$

THEOREM 5.23. *Let $\{X_\lambda\}$ be a collection of path-connected cell complexes. Then*

$$\tilde{H}\left(\bigvee_\lambda X_\lambda\right) \cong \bigoplus_\lambda \tilde{H}(X_\lambda)\ . \tag{5.153}$$

PROOF. Let X be the 1-complex obtained by attaching 1-cells to the discrete space $\{x_\lambda\} \cup \{x_0\}$ by the attached maps g_λ where $f_\lambda(-1) = x_0$ and $f_\lambda(1) = x_\lambda$. Let

$$Y = \left(C \cup \coprod_\lambda X_\lambda\right) / (x_\lambda \sim x_\lambda)\ . \tag{5.154}$$

Also let

$$Y_1 = \bigcup_\lambda f_\lambda \left([-1, 1/2)\right) \tag{5.155}$$

$$Y_2 = \coprod_\lambda X_\lambda \cup f_\lambda \left((-1/2, 1]\right) \tag{5.156}$$

so that Y_1 is contractible, $Y = Y_1 \cup Y_2$, and

$$Y_1 \cap Y_2 = \coprod_\lambda f_\lambda \left((-1/2, 1/2)\right) \ . \tag{5.157}$$

The theorem is clearly true for $\tilde{H}_0$. By the M-V sequence we get an isomorphism $H_q\left(Y_2\right) \to H_q\left(Y\right)$. Since C is contractible

$$Y \simeq Y/C \cong \bigvee_\lambda X_\lambda \ . \tag{5.158}$$

□

CHAPTER 6

Computations and applications

Let X be a topological space. The suspension SX of X is

$$X \times [-1,1] \,/\, (\forall x, x' \in X, (x, \pm 1) \sim (x', \pm 1)) \ . \tag{6.1}$$

Note that

$$X \times [0,1] \mapsto X_1 \subset SX \tag{6.2}$$

$$X \times [-1,1] \mapsto X_1 \subset SX \tag{6.3}$$

$$\tag{6.4}$$

where each $X_i \cong CX$.

Theorem 6.1. $\tilde{H}_q (SX) \cong \tilde{H}_{q-1} (X)$.

Proof. We have $SX = X_1 \cup X_2$, $X_1 \cap X_2 = X$, and the X_i are contractible. Therefore the pair $\{X_1, X_2\}$ satisfies the remark after Theorem 5.18, so we get a M-V sequence. We know

$$\tilde{H} (X_i) = 0 \tag{6.5}$$

which means

$$\tilde{H}_q (SX) \xrightarrow{\Delta} \tilde{H}_{q-1} (X) \tag{6.6}$$

is an is isomorphism for all q. ☐

Since $S^n \cong SS^{n-1}$ we get

Corollary 6.2.

$$\tilde{H}_q (S^n) \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases} . \tag{6.7}$$

Proof. this is true for $q = 0$, now induct on n using Theorem 6.1. ☐

Corollary 6.3. *$S^m \simeq S^n$ iff $m = n$.*

Corollary 6.4 (Invariance of dimension)**.** *$\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$.*

Proof. $\mathbb{R}^m \cong \mathbb{R}^n$ implies $S^m \cong S^n$, which implies $m = n$ by Corollary 6.3. ☐

Corollary 6.5. *There is no retraction $D^n \to S^{n-1}$ for $n \geq 1$.*

Proof. We have

$$\begin{tikzcd} & D^n \arrow[dr, "r"] & \\ S^{n-1} \arrow[ur, hook, "i"] \arrow[rr, "\mathrm{id}"] & & S^{n-1} \end{tikzcd} \tag{6.8}$$

which gives us

$$\begin{tikzcd} & \tilde{H}_{n-1}\left(D^n\right)=0 \arrow[dr, "r_*"] & \\ \tilde{H}_{n-1}\left(S^{n-1}\right)\cong\mathbb{Z} \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & \tilde{H}_{n-1}\left(S^{n-1}\right)\cong\mathbb{Z} \end{tikzcd} \tag{6.9}$$

which is a contradiction. ☐

Corollary 6.6 (Brouwer fixed point theorem)**.** *Every map $f : D^n \to D^n$ has a fixed point.*

1. Degree

For a map $f : S^n \to S^n$ $(n \geq 0)$ we can consider the induced map

$$f_* : \underbrace{\tilde{H}_n\left(S^n\right)}_{\cong\mathbb{Z}} \to \underbrace{\tilde{H}_n\left(S^n\right)}_{\cong\mathbb{Z}} \tag{6.10}$$

is multiplication by some $d \in \mathbb{Z}$. $d = \deg f$ is the degree of f.

Remark 6.1. (1) $\deg(\mathrm{id}) = 1$.
(2) the degree of a constant map is 0 since it factors through $\tilde{H}_n(\mathrm{pt}) = 0$.
(3) $\deg(gf) = \deg g \deg f$.
(4) $f \simeq g$ implies $\deg f = \deg g$.
(5) If f a homotopy equivalence then $\deg f = \pm 1$.

Let $f : X \to Y$ be a map. Then

$$f \times \mathrm{id} : X \times [-1,1] \to Y \times [-1,1] \tag{6.11}$$

induces a map

$$Sf : SX \to SY\ . \tag{6.12}$$

In particular, $f : S^n \to S^n$ induces a map $Sf : S^{n+1} \to S^{n+1}$

Lemma 6.7. *Let $f : S^n \to S^n$ $(n \geq 0)$ be a map. Then* $\deg(Sf) = \deg f$.

Proof. By the proof of Theorem 6.1 we have a commutative diagram

$$\begin{tikzcd} \tilde{H}_{n+1}\left(SS^n\right) \arrow[r, "\Delta"] \arrow[d, "(Sf)_*"] & \tilde{H}_n\left(S^n\right) \arrow[d, "f_*"] \\ \tilde{H}_{n+1}\left(SS^n\right) \arrow[r, "\Delta"] & \tilde{H}_n\left(S^n\right) \end{tikzcd} \tag{6.13}$$

where Δ is an isomorphism. ☐

Now we want to compute the degree of the antipodal map. Let $f_n : S^n \to S^n$ be

$$(x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)\ . \tag{6.14}$$

Lemma 6.8. $\deg f_n = -1$.

PROOF. $f_{n+1} = Sf_n$, so by Lemma 6.7 it is enough to do $n = 0$. Recall $S^0 = \{\pm 1\} = \{p_\pm\}$. Then we have

$$\begin{tikzcd} S_1\left(S^0\right) \arrow[r, "\partial_1"] & S_0\left(S^0\right) \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0 \\ & \mathbb{Z} \oplus \mathbb{Z} \arrow[u, "\cong"] & & \end{tikzcd} \tag{6.15}$$

and in fact $\partial_1 = 0$. Therefore

$$\tilde{H}_0\left(S^0\right) = \tilde{Z}_0\left(S^0\right) = \ker \epsilon \cong \mathbb{Z} \tag{6.16}$$

where this is generated by $p_+ - p_-$. We know $f_0\left(p_\pm\right) = p_\mp$ which means

$$f_{0*} : \tilde{H}_0\left(S^0\right) \to \tilde{H}_0\left(S^0\right) \tag{6.17}$$

is multiplication by -1. □

Let $a_n : S^n \to S^n$ be the antipodal map, $a_n(x) = -x$.

Lemma 6.9. $\deg a_n = (-1)^{n+1}$.

PROOF. Let $f_n^{(i)} : S^n \to S^n$ be the map

$$(x_0, x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_0, x_1, \ldots, -x_i, \ldots, x_n)\ . \tag{6.18}$$

Then $f_n^{(i)}$ us conjugate to f_n, which means

$$\deg f_n^{(i)} = \deg f_n = -1\ . \tag{6.19}$$

Then

$$a_n = f_n f_n^{(1)} \ldots f_n^{(n)} \tag{6.20}$$

so $\deg a_n = (-1)^{n+1}$ as desired. □

DEFINITION 6.1. A (tangent) vector field on S^n is a map $v : S^n \otimes \mathbb{R}^{n+1}$ such that $v(x)$ is orthogonal to x for all $x \in S^n$.

THEOREM 6.10. *There exists a nowhere zero vector field on S^n iff n is odd.*

PROOF. ($\Longleftarrow$): Assume $n = 2m+1$ is odd. Then define

$$v\left(x_0, \ldots, x_{2m+1}\right) = \left(-x_1, x_0, \ldots, -x_{2m+1}, x_{2m}\right)\ . \tag{6.21}$$

$\Longrightarrow$: Suppose $v : S^n \to \mathbb{R}^{n+1}$ is a nowhere zero vector field. Define $w : S^n \to S^n$ by

$$w(x) = \frac{v(x)}{\|v(x)\|}\ . \tag{6.22}$$

Now the claim is that we can use w to define a homotopy from the antipodal map to the identity. Define $F : S^n \times I \to S^n$ by

$$F(x,t) = \cos(\pi t) \cdot x + \sin(\pi t)\, w(x)\ . \tag{6.23}$$

Since these are orthogonal,

$$\|F(x,t)\|^2 = \cos^2(\pi t) + \sin^2(\pi t) = 1 \tag{6.24}$$

for all t. But now notice that $F(x,0) = x$ and $F(x,1) = -x$. So the antipodal map is homotopic to the identity, and therefore $\deg \mathrm{id} = \deg a_n$, but we know $\deg(\mathrm{id}) = 1$ and $\deg(a_n) = (-1)^{n+1}$. Therefore n is odd. □

Lemma 6.11. *If $f : S^n \to S^n$ has no fixed point then $f \simeq a_n$.*

PROOF.

$$F_t(x) = \frac{(1-t)\,f(x) + t(-x)}{\|(1-t)\,f(x) + t(-x)\|} \tag{6.25}$$

is a homotopy with $F_0 = f$ and $F_1 = a_n$. □

An action of a group G on a space X is *free* if $g(x) = x$ (for some $x \in X$) implies $g = \mathrm{id}$.

EXAMPLE 6.1. $\mathbb{Z}/p\mathbb{Z}$ acts freely on S^1. $\mathbb{Z}/p\mathbb{Z}$ acts freely on S^3. The quotient space gives us the Lens spaces $L(p,q)$. The antipodal maps define a free $\mathbb{Z}/2\mathbb{Z}$ action on S^n for all n.

THEOREM 6.12. *$G \neq 1$ acts freely on S^{2n} iff $G \simeq \mathbb{Z}/2\mathbb{Z}$.*

PROOF. ($\Longleftarrow$): The antipodal map defines such an action.
($\Longrightarrow$): Let $g \in G$, $g \neq 1$. Then g has no fixed points by hypothesis. Therefore by Lemma 6.11, $g \simeq a_{2n}$, so $\deg g = (-1)^{2n+1} = -1$, so for all $g, h \neq 1$

$$\deg(gh) = (-1)(-1) = 1 \tag{6.26}$$

so $gh = \mathrm{id}$, i.e. $G = \mathbb{Z}/2\mathbb{Z}$. □

2. Homology of cell complexes and the Euler characteristic

THEOREM 6.13. *Let $X = Y \cup$ an n-cell, with attaching map $f : S^{n-1} \to Y$. Then*

$$i_* : \tilde{H}_q(Y) \to \tilde{H}_q(X) \tag{6.27}$$

is an isomorphism for $q \neq n, n-1$ and there exists an exact sequence

$$0 \to \tilde{H}_n(Y) \xrightarrow{i_*} \tilde{H}_n(X) \to \tilde{H}_{n-1}\left(S^{n-1}\right) \xrightarrow{f_*} \tilde{H}_{n-1}(Y) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to 0 \ . \tag{6.28}$$

Corollary 6.14. *(1) $\tilde{H}_{n-1}(X) \cong \operatorname{coker} f_*$.*
(2) $\tilde{H}_n(X) \cong \tilde{H}_n(Y) \oplus \ker f_*$.

Let A and B be abelian groups. For $\varphi : A \to B$ a homomorphism, the *cokernel* of φ is $\operatorname{coker}\varphi = B/\operatorname{im}\varphi$. So

(1) we always have an exact sequence

$$0 \to \ker\varphi \to A \xrightarrow{\varphi} B \to \operatorname{coker}\varphi \to 0 \ , \tag{6.29}$$

and

(2) For

$$\ldots \to A_{q+2} \xrightarrow{\alpha_{q+2}} A_{q+1} \xrightarrow{\alpha_{q+1}} A_q \xrightarrow{\alpha_q} A_{q-1} \xrightarrow{\alpha_{q-1}} \ldots \tag{6.30}$$

exact we have a SES

$$0 \to \ker\alpha_q \to A_q \xrightarrow{\alpha_q} \operatorname{im}\alpha_q \to 0 \tag{6.31}$$

but in fact

$$\ker\alpha_q = \operatorname{im}\alpha_{q+1} \cong A_{q+1}/\ker\alpha_{q+1} = \operatorname{coker}\alpha_{q+2} \tag{6.32}$$

so we get the SES

$$0 \to \operatorname{coker}\alpha_{q+2} \to A_q \to \ker\alpha_{q-1} \to 0 \ . \tag{6.33}$$

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Corollary 6.15. *If X is a cell complex of dimension n, then $H_q(X) = 0$ for $q > n$.*

Proof of Theorem 6.13. Let X_1 be the image of the disk of radius $1/2$, $1/2D^n$, in X. Let $X_2 = Y \cup_f D^n \setminus \operatorname{int}(1/2D^n)$. Then $X = X_1 \cup X_2$ and

$$X_1 \cap X_2 = \partial(1/2D^n) \cong S^{n-1} . \tag{6.34}$$

By the remark after Theorem 5.18 there exists a M-V sequence for $\{X_1, X_2\}$. Now notice that there exists a strong deformation retraction

$$(D^n \setminus \{\operatorname{int}(1/2D^n)\}) \to \partial D^n \cong S^n \tag{6.35}$$

which induces a strong deformation retraction

$$r : X_2 \to Y \tag{6.36}$$

so $X_2 \simeq Y$. Now notice that

$$r|_{X_1 \cap X_2} : X_1 \cap X_2 \cong S^{n-1} \to Y \tag{6.37}$$

is just f. So the M-V sequence is

$$\ldots \longrightarrow \tilde{H}_q\left(S^{n-1}\right) \xrightarrow{f_*} \underbrace{\tilde{H}_q(X_1)}_{=0} \oplus \tilde{H}_q(Y) \longrightarrow \tilde{H}_q(X) \longrightarrow H_{q-1}\left(S^{n-1}\right) \longrightarrow \ldots \tag{6.38}$$

so unless $q = n, n-1$ we get equality, and when they are we get exactly the desired exact sequence. ☐

Example 6.2.
Let Y be path-connected. Consider $X = Y \vee S^1$. We can view this as Y with a 1-cell attached. Theorem 6.13 gives us that for $q \neq 1$,

$$\tilde{H}_q(X) \cong \tilde{H}_q(Y) \tag{6.39}$$

and we have an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}_1(Y) \arrow[r] & \tilde{H}_1(X) \arrow[r] & \tilde{H}_0\left(S^0\right) \arrow[r, "f_*"] & \tilde{H}_0(Y) \arrow[d, equal] \\ & & & \mathbb{Z} \arrow[u, "\cong"] & 0 \end{tikzcd} \tag{6.40}$$

so we get that

$$H_1(X) \cong H_1(Y) \oplus \mathbb{Z} . \tag{6.41}$$

This makes no assumptions on Y. However if if Y is e.g. a cell complex, then $\pi_1(X) \cong \pi_1(Y) * \mathbb{Z}$. Then abelianizing, we get

$$\pi_1(X)^{\mathrm{ab}} \cong \pi_1(Y)^{\mathrm{ab}} \oplus \mathbb{Z} . \tag{6.42}$$

Then

Fact 6. *For any space X, $H_1(X) \cong \pi_1(X)^{ab}$.*

Example 6.3. This tells us that $H_n\left(\bigvee S^1\right) \cong \mathbb{Z}^n$. This also follows from Theorem 5.23.

Example 6.4. Let T_g be the closed orientable surface of genus g. Recall as in the discussion of the fundamental groups, we have the picture in Fig. 3 where we remove a disk to get Y, which consists of these strips coming off of a disk, i.e. it is a wedge of circles. Therefore $\pi_1(Y) \cong F_{2g}$ so

$$H_1(Y) \cong \mathbb{Z}^{2g} . \tag{6.43}$$

Then when we attach the disk, we attach along a map $f : S^1 \to Y$, which induces a map $f_* : \pi_1\left(S^1\right) = \mathbb{Z} \to \pi_1(Y)$ which sends

$$1 \mapsto \prod_{i=1}^{g} [a_i, b_i] \ . \tag{6.44}$$

Therefore

$$f_* : H_1\left(S^1\right) \to H_1(Y) \cong \pi_1(Y)^{\mathrm{ab}} \tag{6.45}$$

is just 0.

Note that $H_0(T_g) \cong \mathbb{Z}$. Also, T_g is a 2-complex, so $H_q(T_g) = 0$ for $q \geq 3$. Then we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_2(Y) \arrow[r] \arrow[d, equal] & H_2(T_g) \arrow[r, "\Delta"] & H_1\left(S^1\right) \arrow[r, "f_*=0"] & H_1(Y) \arrow[r] & H_1(T_g) \arrow[r] & 0 \\ & 0 & & \mathbb{Z} \arrow[u, "\cong"] & \mathbb{Z}^{2g} \arrow[u, "\cong"] & & \end{tikzcd} \ . \tag{6.46}$$

Therefore $\Delta : H_2(T_g) \to H_1\left(S^1\right) \cong \mathbb{Z}$ is an isomorphism, and $H_1(Y) \cong \mathbb{Z}^{2g} \to H_q(T_g)$. In summary:

$$H_q(T_g) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^{2g} & q = 1 \\ \mathbb{Z} & q = 2 \\ 0 & q \geq 3 \end{cases} . \tag{6.47}$$

EXAMPLE 6.5. Let $P_h = \#_h \mathbb{P}^2$. Recall from our calculation of $\pi_1(P_h)$, we had that when we removed a disk, we got a wedge of h circles, so $P_h = Y \cup$ a 2-cell and $\pi_1(Y) = F_h$, so $H_1(Y) \cong \mathbb{Z}^h$. Then the attaching map of the missing disk $f : S^1 \to Y$ induces the map:

$$\begin{tikzcd} \pi_1\left(S^1\right) \arrow[r, "f_*"] & \pi_1(Y) \\ 1 \arrow[r, mapsto] & \prod_{i=1}^{h} a_i^2 \end{tikzcd} \ . \tag{6.48}$$

Therefore

$$f_* : H_1\left(S^1\right) \cong \mathbb{Z} \to H_1(Y) \cong \mathbb{Z}^h \tag{6.49}$$

sends

$$f_*(1) = (2, \ldots, 2) \ . \tag{6.50}$$

Now M-V gives us

$$\begin{tikzcd} 0 \arrow[r] & H_2(P_h) \arrow[r] & H_1\left(S^2\right) \arrow[r, "f_*"] & H_1(Y) \arrow[r] & H_1(P_h) \arrow[r] & 0 \\ & & \mathbb{Z} \arrow[u, "\cong"] & \mathbb{Z}^h \arrow[u, "\cong"] & & \end{tikzcd} \ . \tag{6.51}$$

Now $\mathbb{Z}^h$ has a basis

$$e_1 = (1, \ldots, 1) \tag{6.52}$$

$$e_2 = (0, 1, \ldots, 0) \tag{6.53}$$

$$\ldots \tag{6.54}$$

$$e_h = (0, \ldots, 1) \tag{6.55}$$

and $f_*(1) = 2e_1$. Therefore $\ker f_* = 0$ which implies $H_2(P_h) = 0$. Now we have

$$H_1(P_h) \cong \operatorname{coker} f_* \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} \tag{6.56}$$

which means

$$H_q(P_h) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} & q = 1 \\ 0 & q \geq 2 \end{cases} . \tag{6.57}$$

2.1. Coefficients. Let R be a commutative ring with unit. Instead of forming the free abelian group on the set of simplices, we could have defined $S_q(X, R)$ to be the free R-module on $\Sigma_q(X)$.

Example 6.6. For $R = \mathbb{F}$ a field we would get the $\mathbb{F}$ vector space with basis $\Sigma_q(X)$. Then we get the singular homology of X with coefficients in R, $H(X; R)$.

Warning 6.1. If we take $R = \mathbb{Z}/2\mathbb{Z}$ we get $S_q(X; \mathbb{Z}/2) \simeq S_q(X; \mathbb{Z}) \otimes \mathbb{Z}/2$. So we might assume coefficients don't really matter. But when we pass to homology, $H_q(X; \mathbb{Z}/2) \not\simeq H_q(X; \mathbb{Z}) \otimes \mathbb{Z}/2$. We can however relate these two. This is called the universal coefficient theorem.

Everything we've done (homotopy axiom, M-V, etc.) goes through for $H(X; R)$.

2.2. Euler characteristic. Let $\mathbb{F}$ be a field. Let X be a space such that $H_q(X; \mathbb{F})$ is finite dimensional for all q and 0 for $q > m$ for some m. (E.g. X could be a cell-complex; by Theorem 6.13 with $\mathbb{F}$-coefficients.) Now define

$$\chi_{\mathbb{F}}(X) = \sum_{i=0}^{m} (-1)^q \dim H_q(X; \mathbb{F}) . \tag{6.58}$$

Theorem 6.16. *For X a finite cell complex then $\chi_{\mathbb{F}}(X) = \chi(X)$.*

Corollary 6.17. *If X and Y are finite cell complex and $X \simeq Y$, then $\chi(X) = \chi(Y)$.*

Proof of Theorem 6.16. This is clearly true for 0-complexes. Let Y be a finite cell complex. Take $X = Y \cup (n\text{-cell})$ for $n \geq 1$. Therefore

$$\chi(X) = \chi(Y) + (-1)^n . \tag{6.59}$$

Theorem 6.13 implies $\tilde{H}_q(X) \cong \tilde{H}_q(Y)$ for $q \neq n, n-1$, and that there exists an exact sequence

(6.60)

$$\begin{tikzcd} 0 \arrow[r] & \tilde{H}_n(Y; \mathbb{F}) \arrow[r] & \tilde{H}_n(X; \mathbb{F}) \arrow[r] & \tilde{H}_{n-1}(S^{n-1}; \mathbb{F}) \arrow[r, "f_*"] & \tilde{H}_{n-1}(Y; \mathbb{F}) \arrow[r] & \tilde{H}_{n-1}(X, \mathbb{F}) \arrow[r] & 0 \\ & & & & \mathbb{F} \arrow[u, "\cong"] & & \end{tikzcd} .$$

By a problem on homework 9, we have that the alternating sum of the dimensions is 0, i.e.

$$\dim \tilde{H}_n(X; \mathbb{F}) - \dim \tilde{H}_{n-1}(X; \mathbb{F}) = \dim \tilde{H}_n(Y, \mathbb{F}) - \dim \tilde{H}_{n-1}(Y, \mathbb{F}) + 1 . \tag{6.61}$$

Therefore

$$\chi_{\mathbb{F}}(X) = \chi_{\mathbb{F}}(Y) + (-1)^n \tag{6.62}$$

so by induction on the number of cells in X,

$$\chi(X) = \chi_{\mathbb{F}}(X) . \tag{6.63}$$

□