LECTURE 3
ALGEBRAIC TOPOLOGY

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1. Deformation retractions and contractible spaces

Definition 1. X is *contractible* if id_X is null-homotopic, i.e. $\simeq$ to a constant map. Equivalently X is a deformation retraction to some point.

Example 1. If X is a convex subset of $\mathbb{R}^n$ then X strong deformation retracts to any point $x_0 \in X$. Therefore they are also contractible.

Example 2. S^1 is not contractible. We will see this later.

Lemma 1. *For a topological space X TFAE:*

(1) X is contractible,
(2) $\forall x_0 \in X$, X deformation retracts to $\{x_0\}$,
(3) $X \simeq \{pt\}$,
(4) $\forall Y$, any two maps $Y \to X$ are homotopic.
(5) $\forall Y$, any map $X \to Y$ is null-homotopic.

Proof. (1) $\implies$ (3): (1) is equivalent to saying that X deformation retracts to a point, so the inclusion map is certainly a homotopy equivalence.

(3) $\implies$ (4): Let $f : X \to \{z\}$ be a homotopy equivalence. By homework 1 exercise 3, we get an induced function:

$$f_* : [Y, X] \to [Y, \{z\}]$$

but there is only one map in the target set, so clearly there is only one homotopy class of maps $Y \to \{z\}$.

(4) $\implies$ (2): Take $Y = X$, and take any $x_0 \in X$. This means $\mathrm{id}_X \simeq c_{x_0}$, but this is exactly saying that X deformation retracts to x_0.

(2) $\implies$ (5): Let $f : X \to Y$ and $x_0 \in X$. Then (2) implies $\mathrm{id}_X \simeq_F c_{x_0}$. Then

$$f \circ \mathrm{id}_X \simeq_{f\circ F} f \circ c_{x_0}$$

i.e. f is nullhomotopic.

(5) $\implies$ (1): Take $Y = X$. $\square$

Corollary 1. *For X, Y contractible, then*

(1) $X \simeq Y$,
(2) any map $X \to Y$ is a homotopy equivalence.

Proof. (1) If $X, Y \simeq \{\mathrm{pt}\}$ then $X \simeq Y$.

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(2) Given $f : X \to Y$, let $g : Y \to X$ be any map. $gf : X \to X$, but X is contractible, so $gf \simeq \mathrm{id}_X$ by lemma 1.

□

Now we will give an example of a deformation retraction which is not a strong deformation retraction. Recall X strong deformation retracts to A implies X deformation retracts to A which implies $i : A \hookrightarrow X$ is a homotopy equivalence, but none of these implications are reversible.

Example 3 (Comb space)**.** Define the comb space $C \subset I \times I \subset \mathbb{R}^2$ to be:

$$C = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, y = 0, 0 \leq x \leq 1; 0 \leq y \leq 1, x = 0, 1/n\,(n = 1, 2, \ldots)\right\} .$$

This should be pictured as a bunch of vertical intervals.[1] The first thing to note is that C strong deformation retracts to $(0, 0)$. Therefore C is contractible. C also deformation retracts to $(0, 1)$. [More generally: if X deformation retracts to some $x_0 \in X$ and X is path connected, then X deformation retracts to any $x \in X$].

Claim 1. But it does not strong deformation retract to $(0, 1)$.

Proof. Let $F : C \times I \to C$ be such a strong deformation retraction. Let U be some open disc of radius $1/2$ centered at $(0, 1)$. $F^{-1}(U) \subset X \times I$ contains $(0, 1) \times I$. Therefore for all $t \in I$ there exists some neighborhood V_t of $(0, 1) \times \{t\}$ such that $V_t \subset F^{-1}(U)$. But $V_t = W_t \times Z_t$ for W_t some neighborhood of $(0, 1)$ in C and Z_t some neighborhood of t in I. I is compact which means $\exists t_1, \ldots, t_m$ such that

$$\bigcup_{i=1}^{m} Z_{t_i} = I .$$

Let

$$W = \bigcap_{i=1}^{m} W_{t_i} .$$

This is a neighborhood of $(0, 1)$ in C, and $W \times I \subset F^{-1}(U)$. (This is sometimes called the tube lemma). Pick n such that $(1/n, 1) \in W$. Then $F((1/n, 1), t)$, $0 \leq t \leq 1$, is a path in U from $(1/n, 1)$ to $(0, 1)$ but there clearly isn't such a path since these two points are in different path components. □

Corollary 2. *Let $X \subset I^2 \subset \mathbb{R}^2$ where C, I^2 are both contractible. Then the inclusion $i : C \to I^2$ is a homotopy equivalence. But there does not exist a deformation retraction $I^2 \to C$. In fact there is no retraction at all.*

Remark 1. There exists a space X such that X is contractible (therefore $\{x\} \hookrightarrow X$ is a homotopy equivalence for all $x \in X$) but there does not exist a deformation retraction from X to any $x \in X$. (e.g. Hatcher chapter 0, 6(b)).

1.1. **Fixed point property.** A space X has the fixed point property (FPP) iff $\forall f : X \to x$, $\exists x \in X$ such that $f(x) = x$. X being contractible does not imply X has the FPP (e.g. $\mathbb{R}^1$).

Question 1 (Borsuk)**.** If X is compact and contractible does contractible imply FPP?

[1] Which is supposed to look like a comb.

2. The fundamental group

Definition 2. A *path* from x_0 to x is a map $\sigma : I \to X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x$.

Definition 3. Let σ be a path in X from x_0 to x_1, and τ a path in X from x_1 to x_2. Their *concatenation* $\sigma * \tau$ is a path from x_0 to x_2 given by:

$$(\sigma * \tau)(s) = \begin{cases} \sigma(2s) & 0 \le s \le 1/2 \\ \tau(2s-1) & 1/2 \le s \le 1 \end{cases} .$$

Definition 4. The *homotopy class* of σ is

$$[\sigma] = \{\sigma' \mid \sigma' \simeq \sigma \,(\mathrm{rel}\,\partial I)\} .$$

Lemma 2. *If $[\sigma] = [\sigma']$ and $[\tau] = [\tau']$ where $\sigma(1) = \tau(0)$ then $[\sigma * \tau] = [\sigma' * \tau']$.*

Proof. If $\sigma \simeq_{F_t} \sigma'$ and $\tau \simeq_{G_t} \tau'$ then $\sigma * \tau \simeq_{F_t * G_t} \sigma' * \tau' \,(\mathrm{rel}\,\partial I)$. □

This means we can define the product of two homotopy classes to the be the homotopy class of the concatenation. This is well defined by the lemma.

Lemma 3 (Reparameterization)**.** *Let $u : I \to I$ be a map such that $u|_{\partial I} = \mathrm{id}$. Then $u \simeq \mathrm{id}_I \,(\mathrm{rel}\,\partial I)$.*

Proof. Define $F : I \times I \to I$ by $F(s,t) = ts + (1-t)\,u(s)$. $F_0 = u$, $F_1 = \mathrm{id}_I$, $F_t|_{\partial I} = \mathrm{id}$ for all $t \in I$. □

Lemma 4 (Associativity)**.** *Let ρ, σ, τ be paths in X such that $\rho(1) = \sigma(0)$, $\sigma(1) = \tau(0)$. Then*

$$([\rho][\sigma])[\tau] = [\rho]([\sigma][\tau]) .$$

Proof. Define $u : I \to I$ by

$$u(s) = \begin{cases} 2s & 0 \le s \le 1/4 \\ s + 1/4 & 1/4 \le s \le 1/2 \\ (s+1)/2 & 1/2 \le s \le 1 \end{cases} .$$

Then

$$(\rho * (\sigma * \tau))\,u = (\rho * \sigma) * \tau .$$

but $u \simeq \mathrm{id}_I \,(\mathrm{rel}\,\partial I)$ so $(\rho * (\sigma * \tau)) = (\rho * \sigma) * \tau \,(\mathrm{rel}\,\partial I)$ □

Let $c_{x_0} : I \to X$ be the constant path given by $c_{x_0} = x_0$ for all $s \in I$.

Lemma 5. *For σ a path in X from x_0 to x_1 then*

$$[\sigma] = [\sigma][c_{x_1}] = [c_{x_0}][\sigma] .$$

Proof. Let $u : I \to I$ be

$$u(s) = \begin{cases} 2s & 0 \le s \le 1/2 \\ 1 & 1/2 \le s \le 1 \end{cases} .$$

Then $\sigma * c_{x_1} = \sigma * u$

$$[\sigma] = [\sigma][c_{x_1}]$$

by lemma 3. The proof is the same for the other part. □

If σ is a path from x_0 to x_1, the reverse of σ is the path $\bar{\sigma}$ from x_1 to x_0 given by

$$\bar{\sigma}(s) = \sigma(1-s) \ .$$

Note that immediately we have $\overline{(\bar{\sigma})} = \sigma$.

Lemma 6. $[\sigma][\bar{\sigma}] = [c_{x_0}]$.

Proof. Define $F : I \times I \to X$ by

$$F(s,t) = \begin{cases} \sigma(2st) & 0 \leq s \leq 1/2 \\ \sigma(2(1-s)t) & 1/2 \leq 2 \leq 1 \end{cases} \ .$$

Note that $F_0 = c_{x_0}$ and $F_1 = \sigma * \bar{\sigma}$ so we are done. □

Definition 5. Let X be a space and $x_0 \in X$. The fundamental group $\pi_1(X, x_0)$ of a space X is the collection of homotopy classes of paths starting and ending at x_0. The previous lemmas exactly tell us that this is a group under concatenation.