HOMEWORK 11

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Exercise 1 (Hatcher §2.2.29)**.** The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X into two copies of R. Also compute the relative groups $H_i(R, M_g)$.

Exercise 2 (Hatcher §2.B.3)**.** Let $(D, S) \subset \left(D^n, S^{n-1}\right)$ be a pair of suspaces homeomorphic to $\left(D^k, S^{k-1}\right)$ with $D \cap S^{n-1} = S$. Show the inclusion $S^{n-1} \setminus S \hookrightarrow D^n \setminus D$ induces an isomorphism on homology. [Glue two copies of (D^n, D) to the two ends of $\left(S^{n-1} \times I, S \times I\right)$ to produce a k-sphere in S^n and look at a Mayer-vietoris sequence for the complement of this k-sphere.]

Exercise 3 (Hatcher §2.C.2)**.** Use the Lefschetz fixed point theorem to show that a map $S^n \to S^n$ has a fixed point unless its degree is equal to the degree of the antipodal map $x \mapsto -x$.

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.
Let $0 = (0, \ldots, 0) \in \mathbb{R}^n_+ \subset \mathbb{R}^n$.
For questions 1 and 2, define an n-manifold to be a Hausdorff space M such that every $x \in M$ has a neighborhood U such that either

$$(U, x) \cong (\mathbb{R}^n, 0) \qquad \text{or} \qquad (U, x) \cong \left(\mathbb{R}^n_+, 0\right) . \tag{1}$$

Define

$$\operatorname{int}(M) = \{x \in M \mid x \text{ has a neighborhood of the first type}\} ; \tag{2}$$

$$\partial M = \{x \in M \mid x \text{ has a neighborhood of the second type}\} . \tag{3}$$

Exercise 4. Show that

(i) $(\operatorname{int}(M)) \cap \partial M = \emptyset$.
(ii) $\operatorname{int}(M)$ is an n-manifold and $\partial(\operatorname{int}(M)) = \emptyset$.
(iii) ∂M is an $(n-1)$-manifold and $\partial(\partial M) = \emptyset$.

Solution.

Exercise 5. Let M and N be n-manifolds without boundary, and let $f : M \to N$ be a continuous bijection. Show that f is a homeomorphism.

Exercise 6. Use the Borsuk-Ulam theorem to prove that if

$$S^n = \bigcup_{i=1}^{n+1} A_i \tag{4}$$

where A_i is closed in S^n, $1 \leq i \leq n+1$, then some A_i contains a pair of antipodal points.