HOMEWORK 1

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Exercise 1 (Chapter 0, 2; Hatcher)**.** Construct an explicit (strong) deformation retraction of $\mathbb{R}^{n+1} \setminus \{0\}$ to S^n.

Solution. Define $F : \mathbb{R}^{n+1} \times I \to \mathbb{R}^{n+1}$ as follows:

$$F(x,t) = x(1-t) + \frac{x}{\|x\|}t \ .$$

Then $F(x,0) = x$ and $F(x,1) = x/\|x\|$. This means the image of $F(x,1)$ is exactly S^n and for every $p \in S^n$

$$F(p,1) = \frac{p}{\|p\|} = p$$

since $\|p\| = 1$ for all $p \in S^n$.

Exercise 2 (Chapter 0, 11; Hatcher)**.** Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $hf \simeq \mathrm{id}_X$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Solution. Let $f : X \to Y$ be a map. Assume there are maps $g, h : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $hf \simeq \mathrm{id}_X$. It is sufficient to show that $fh \simeq \mathrm{id}_Y$. First we show that $g \simeq h$:

$$h(fg) \simeq (hf)g \simeq \mathrm{id}_X g \simeq g \ .$$

And now it follows that $\mathrm{id}_Y \simeq fg \simeq fh$ as desired.

Now assume $fg : Y \to Y$ is a homotopy equivalence with homotopy inverse $\psi : Y \to Y$ and $hf : Y \to Y$ is a homotopy equivalence with homotopy inverse $\varphi : Y \to Y$. We will show $g\psi$ is a homotopy inverse of f. By definition we already have $(fg)\psi \simeq \mathrm{id}_Y$ so it is sufficient to show that $(g\psi)f \simeq \mathrm{id}_Y$. To see this, note that $\psi(fg) \simeq \mathrm{id}_Y$ so $g\psi fg \simeq g$ which implies $(g\psi)f \simeq \mathrm{id}_Y$ as desired.

Exercise 3. Let $f : S^1 \to S^1$ be a map that is not homotopic to id_{S^1}. Show that there exists $x \in S^1$ such that $f(x) = -x$.

Solution. Assume we have some map $f : S^1 \to S^1$ such that for all $x \in S^1$ $f(x) \neq -x$. We will show that this must be homotopic to the identity. In fact we can explicitly define a homotopy $F : S^1 \times I \to S^1$ by:

$$F(x,t) = \frac{f(x)(1-t) + \mathrm{id}(x)t}{\|f(x)(1-t) + \mathrm{id}(x)t\|} \ .$$

Exercise 4. Let X, Y be closed subsets of $X \cup Y$. Let $f : X \to Z$ and $g : Y \to Z$ be maps such that $f|_{X\cap Y} = g|_{X\cap Y}$. Show that $f \cup g : X \cup Y \to Z$ is continuous.

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Solution. Let $U \subset Z$ be any closed subset. It is sufficient to show that the preimage $V = (f \cup g)^{-1}(U)$ is closed inside of $X \cup Y$. We know that

$$V \cap X = f^{-1}(V) \qquad\qquad V \cap Y = g^{-1}(V)$$

are both closed. But then

$$V = (V \cap X) \cup (V \cap Y)$$

is closed since it is a finite union of closed sets.

Exercise 5. Let $f : X \to Y$ be a map and W a space. Define

$$f_* : [W, X] \to [W, Y]$$

by $f_*([h]) = [fh]$. Show

(i) f_* is well-defined.
(ii) if $f : X \to Y$, $g : Y \to Z$ are maps and W a space then

$$(fg)_* = f_* g_* : [W, X] \to [W, Z]$$

(iii) $(\mathrm{id}_X)_* = \mathrm{id}_{[W,X]}$.
(iv) if $f : X \to Y$ is a homotopy equivalence then f_* is a bijection.

(Corresponding dual properties hold for

$$f^* [Y, W] \to [X, W]$$

defined by $f^*([h]) = [hf]$.)

Solution. (i)

Exercise 6. Recall that a space X has the fixed point property (FPP) if for every map $f : X \to X$ there exists $x \in X$ such that $f(x) = x$.

(i) Suppose $X \simeq Y$ and X has the FPP. Does Y have the FPP?
(ii) If A is a retract of X and X has the FPP does A have the FPP?
(iii) If A is a retract of X and A has the FPP does X have the FPP?

Solution. (i) No. Consider the real line and $\{0\} \subset \mathbb{R}$. We know $\{0\} \simeq \mathbb{R}$, and all singletons have the FPP. However, $\mathbb{R}$ does not. For example the map $f(x) = x + 1$ has no fixed points and is perfectly continuous.
(ii) Yes. Assume X has the FPP. Let $i : A \hookrightarrow X$ be the inclusion $\rho : X \to A$ be a retraction. Let $f : A \to A$ be any map. Then the map $i \circ f \circ \rho$ is still continuous and therefore there is some $x \in X$ such that $i \circ f \circ \rho(x) = x$. But in particular this x must be in A. So really there is some $x \in A$ such that $f \circ \rho(x) = x$. But ρ is the identity on A, so in fact $f(x) = x$ as desired.
(iii) Yes. Assume A has the FPP. Then consider any map $f : X \to X$. The restriction $f|_A$ is a function on A and therefore there is some $x \in A$ such that $f(x) = x$. But since A is a retract of X, $x \in X$ as well, so X inherits the FPP.

Exercise 7. Use path-connectedness to show that there is no continuous injection from $\mathbb{R}^n$ to $\mathbb{R}^1$ for $n > 1$.

Solution. Assume there is some such map $f : \mathbb{R}^n \to \mathbb{R}$. Recall that the image of a connected set under a continuous map is connected. Therefore the image of f is a connected subset of $\mathbb{R}$, i.e. an interval I. Take some point p such that $f(p)$ is in the interior of the interval.[1] Then we have

$$f\left(\mathbb{R}^3 \setminus \{p\}\right) = f\left(\mathbb{R}^3\right) \setminus f(p) = I \setminus f(p)$$

but since $f(p)$ is in the interior this means the image of a connected set $(\mathbb{R}^3 \setminus \{p\})$ is not connected.

[1]We know such a point exists since if not, it would have to be a singleton and therefore not injective.