HOMEWORK 10

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Exercise 1 (Hatcher §2.1.12)**.** Show that chain homotopy of chain maps is an equivalence relation.

Solution. Recall that chain maps $\varphi, \psi : C \to D$ are chain homotopic, written $\varphi \overset{c}{\cong} \psi$ iff there exists a homomorphism $T : C \to D$ such that $\partial T + T\partial = \varphi - \psi$.

Reflexive: Let $\varphi : C \to D$ be a chain map. Then take $T : C \to D$ to be the zero map. Then $\partial T + T\partial = 0 = \varphi - \varphi$.

Symmetric: Let $\varphi \overset{c}{\cong} \psi$, i.e. there is some $T : C \to D$ such that $\partial T + T\partial = \varphi - \psi$. Then $-T$ is a chain homotopy in the other direction, since

$$\partial\left(-T\right) - T\partial = -\left(\varphi - \psi\right) = \psi - \varphi \ . \tag{1}$$

Transitive: Let $\varphi_1 \overset{c}{\cong} \varphi_2$ and $\varphi_2 \overset{c}{\cong} \varphi_3$, i.e. we have $T_1, T_2 : C \to D$ such that

$$T_1\partial + \partial T_1 = \varphi_1 - \varphi_2 \qquad\qquad T_2\partial + \partial T_2 = \varphi_2 - \varphi_3 \ . \tag{2}$$

Then $T \coloneqq T_1 + T_2$ is a chain homotopy from φ_1 to φ_3, since

$$T\partial + \partial T = T_1\partial + \partial T_1 + T_2\partial + \partial T_2 = \varphi_1 - \varphi_2 + \varphi_2 - \varphi_3 = \varphi_1 - \varphi_3 \ . \tag{3}$$

Exercise 2 (Hatcher §2.1.17)**.**
(a) Compute the homology groups $H_n\left(X, A\right)$ when X is S^2 or $S^1 \times S^1$, and A is a finite set of points in X.
(b) Compute the groups $H_n\left(X, A\right)$ and $H_n\left(X, B\right)$ for X a closed orientable surface of genus two with A and B the circles shown in Fig. 1. [What are X/A and X/B?]

Exercise 3 (Hatcher §2.1.27)**.** Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.
(a) Show that $f_* : H_n\left(X, A\right) \to H_n\left(Y, B\right)$ is an isomorphism for all n.
(b) For the case of the inclusion $f : \left(D^n, S^{n-1}\right) \hookrightarrow \left(D^n, D^n \setminus \{0\}\right)$ show that f is not a homotopy equivalence of pairs - there is no $g : \left(D^n, D^n \setminus \{0\}\right) \to \left(D^n, S^{n-1}\right)$ such that fg and gf are homotopic to the identity through maps of pairs. [Observe that a homotopy equivalence of pairs $(X, A) \to (Y, B)$ is also a homotopy equivalence for the pairs obtained by replacing A and B by their closures.]

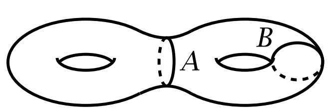

FIGURE 1. Figure from Hatcher.

Exercise 4 (Hatcher §2.1.29)**.** Show that $S^1 \times S^1$ and $S^2 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Exercise 5 (Hatcher §2.2.9(a,b))**.** Compute the homology groups of the following 2-complexes:

(a) The quotient of S^2 obtained by identifying north and south poles to a point.
(b) $S^1 \times \left(S^1 \vee S^1\right)$.

Exercise 6 (Hatcher §2.2.12)**.** Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^1 \to S^1 \times S^1$ is nullhomotopic.

Exercise 7 (Hatcher §2.2.28)**.** (a) Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained form a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.
(b) Do the same for the space obtained by attaching a Möbius band to $\mathbb{RP}^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

Exercise 8. Let A be the interval

$$\left\{ e^{i\theta} \,\middle|\, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\} \subset S^1 \; . \tag{4}$$

Note that $1 \in A$. Regarding $S^1 \subset S^2$ in the usual way, show that excision fails for $\{1\} \subset A \subset S^2$, i.e. inclusion does not induce an isomorphism

$$H\left(S^2 \setminus \{1\}, A \setminus \{1\}\right) \to H\left(S^2, A\right) \; . \tag{5}$$