HOMEWORK 6
ALGEBRAIC TOPOLOGY

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Exercise 1 (Hatcher, §1.3, 7)**.** Let Y be the *quasi-circle* show in fig. 1, a closed subspace of $\mathbb{R}^2$ consisting of a portion of the graph of $y = \sin(1/x)$, the segment $[-1, 1]$ in the y-axis, and an arc connecting these two pieces. Collapsing the segment of Y in the y-axis to a point gives the quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.

Exercise 2 (Hatcher, §1.3, 9)**.** Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \to S^1$.]

Exercise 3 (Hatcher, §1.3, 10)**.** Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

Exercise 4 (Hatcher, §1.3, 20)**.** Construct nonnormal covering spaces of the Klein bottle by a Klein bottle and by a torus.

Exercise 5 (Hatcher, §1.A, 5)**.** Construct a connected graph X and maps $f, g : X \to X$ such that $fg = 1$ but f and g do not induce isomorphisms on π_1. [Note that $f_* g_* = 1$ implies that f_* is surjective and g_* is injective.]

Exercise 6. Show that a connected, locally path-connected space is path-connected.

Exercise 7. Prove the $\Longleftarrow$ direction of the following theorem.

Theorem 1. *Let $\left(\tilde{X}_i, p_i\right)$ be covering spaces of X with the X_i connected and lpc. Then $\left(\tilde{X}_1, p_1\right) \cong \left(\tilde{X}_2, p_2\right)$ iff for all $\tilde{x}_i \in \tilde{X}_i$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$ ($= x_0 \in X$, say)*

$$p_{1*}\left(\pi_1\left(\tilde{X}_1, \tilde{x}_1\right)\right) \qquad \text{and} \qquad p_{2*}\left(\pi_1\left(\tilde{X}_2, \tilde{x}_2\right)\right)$$

are conjugate.

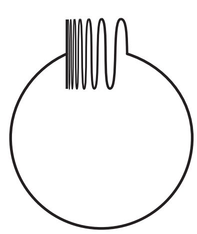

Figure 1. The quasi-circle.