HOMEWORK 7

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Exercise 1 (Hatcher, §1.3, 12)**.** Let a and b be the generators of $\pi_1\left(S^1 \vee S^1\right)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.

Exercise 2 (Hatcher, §1.3, 15)**.** Let $p : \tilde{X} \to X$ be a simply-connected covering space of X and let $A \subset X$ be a path-connected, locally path-connected subspace, with $\tilde{A} \subset \tilde{X}$ a path-component of $p^{-1}(A)$. Show that $p : \tilde{A} \to A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \to \pi_1(X)$.

Exercise 3 (Hatcher, §1.3, 23)**.** Show that if a group G acts freely and properly discontinuously on a Hausdorff space X, then the action is a covering space action. (Here 'properly discontinuously' means that each $x \in X$ has a neighborhood U such that $\{g \in G \,|\, U \cap g(U) \neq \emptyset\}$ is finite.) In particular, a free action of a finite group on a Hausdorff space is a covering space action.

Exercise 4 (Hatcher, §1.A, 7)**.** If F is a finitely generated free group and N is a nontrivial normal subgroup of infinite index, show, using covering spaces, that N is not finitely generated.

Exercise 5 (Hatcher, §1.A, 8)**.** Show that a finitely generated group has only a finite number of subgroups of a given finite index. [First do the case of free groups, using covering spaces of graphs. The general case then follows since every group is a quotient group of a free group.]

Exercise 6. Show that the map $p_H : \tilde{X}/H \to X$ in the proof Theorem 4.18 is a covering projection.

Exercise 7. Prove (directly from the presentations) that the groups $\left\langle a, b \,\middle|\, a^2b^2 = 1\right\rangle$ and $\left\langle x, y \,\middle|\, x^{-1}yx = y^{-1}\right\rangle$ are isomorphic.

Exercise 8. Let $n \geq 2$, and let $b_0 = (1, 0, \ldots, 0) \in S^n$. Let $\pi_n(X, x_0)$ be the set of homotopy classes (rel b_0) of maps $\alpha : (S^n, b_0) \to (X, x_0)$. (In fact $\pi_n(X, x_0)$ can be given the structure of an abelian group.) A map $f : (X, x_0) \to (Y, y_0)$ induces a function $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined by $f_*([\alpha]) = [f\alpha]$. Show that if $p : \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ is a covering projection then $p_* : \pi_n\left(\tilde{X}, \tilde{x}_0\right) \to \pi_n(X, x_0)$ is a bijection.

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