LECTURE 1
ALGEBRAIC TOPOLOGY

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We won't follow the book line by line, but it will be the reference for the course. Read the first three pages of chapter 0, talking about homotopy. We will really get started on chapter 1, section 1.1, about the fundamental group.

1. Introduction

Today will be an introductory account of what algebraic topology actually is. In topology the objects of interest are topological spaces where the natural equivalence relation is a homeomorphism, i.e. a bijection $f : X \to Y$ such that f and f^{-1} are continuous. Somehow the goal is classifying topological spaces up to homeomorphism, so the basic question is somehow:

Question 1. Given topological spaces X and Y, is $X \cong Y$.

In these terms, algebraic topology is somehow a way of translating this into an algebraic question. More specifically, *algebraic topology* is the construction and study of functors from **Top** to some categories of algebraic objects (e.g. groups [1], abelian groups, vector spaces, rings, modules, ...). Recall this means we have a map from topological spaces $X \to A(X)$ for some algebraic object $A(X)$. In addition, for every $f : X \to Y$ we get a morphism $f_* : A(X) \to A(Y)$. Then these have to satisfy the conditions that

$$(gf)_* = g_* f_* \ , \qquad\qquad (\mathrm{id})_* = \mathrm{id} \ .$$

Exercise 1. Show that $X \cong Y$ implies that $A(X) \cong A(Y)$.

Example 1 (Fundamental group)**.** Let X be a topological space. We will construct a group $\pi_1(X)$.

Example 2 (Higher homotopy groups)**.** There is also something, $\pi_n(X)$, called the nth homotopy group. As it turns out for $n \geq 2$ this is abelian.

Example 3 (Singular homology)**.** We will define abelian groups $H_n(X)$ (for $n \geq 0$) called the nth singular homology group.

We will also define real vector spaces $H_n(X; \mathbb{R})$ for $n \geq 0$ which are the nth singular homology with coefficients in $\mathbb{R}$.

Example 4 (Cohomology)**.** We will also have the nth (singular) cohomology rings $H^*(X)$. These is actually a graded ring.

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[1] Once Professor Gordon was giving a job talk about knot cobordisms. As it turns out these form a semigroup rather than a group. But if you add some sort of 4 dimensional equiv relation you get an honest group. So he was going on about how semigroups aren't so useful. After the talk he found out the chairman of the department worked on semigroups.

Warning 1. Above we actually should have said we're dealign with what are *covariant* functors, but in this case we are actually dealing with a *contravariant* functor. This just means we have:

$$f : X \to Y \rightsquigarrow f^* : H^*(Y) \to H^*(X) \ .$$

Remark 1. The point here is that problems about topological spaces and maps are "continuous" and "hard". But on the algebraic side these problems become somehow "discrete" and "easy".

2. A bit more specific

Recall in $\mathbb{R}^n$ we define:

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\} \qquad S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\} \ .$$

Example 5. Two examples of surfaces are S^2 and $T^2 = S^1 \times S^1$. They clearly aren't homeomorphic, but how are we supposed to prove such a fact? We will see that $\pi_1\left(S^2\right) = 1$ whereas $\pi^1\left(T^2\right) = \mathbb{Z} \times \mathbb{Z}$. Since these are not isomorphic, the spaces cannot be homeomorphic.

2.1. Retraction.

Let $A \subset X$ be a space and a subspace.

Definition 1. A *retraction* from X to A is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$, i.e. the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ A \arrow[ur, hook, "i"] \arrow[rr, "\mathrm{id}"] & & A \end{tikzcd} \ .$$

Note that r is certainly surjective since id is.

Example 6. If X is any nonempty space, $x_0 \in X$, define $r : X \to \{x_0\}$ as $r(x) = x_0$. So every nonempty space always retract onto a point.

Example 7. Think of $A \subset A \times B$ by fixing some $b_0 \in B$ and sending

$$\begin{tikzcd} A \arrow[r, hook] & B \\ a \arrow[r, mapsto] & (a, b_0) \end{tikzcd} \ .$$

Then $r : A \to B \to$ defined by $r(a,b) = a$ is a retraction.

Recall that for $f : X \to Y$ for X path connected, then $f(X)$ is also path connected. Recall that $D^1 = [-1,1] \subset \mathbb{R}$ is path connected, whereas $S^0 = \{\pm 1\}$ is not. Therefore there cannot be a retraction $D^1 \to S^0$. This is a basic fact, but it motivates a more general statement which is not so clear.

Suppose there exists a retraction $r : D^1 \to S^0$. Then this means the following diagram commutes:

$$\begin{tikzcd} & D^1 \arrow[dr, "r"] & \\ S^0 \arrow[ur, hook] \arrow[rr, "\mathrm{id}"] & & S^0 \end{tikzcd} \ .$$

If we apply the functor H_0, we will see that

$$H_0(X) \cong \bigoplus_{\text{path components of } X} \mathbb{Z} \ .$$

So if we apply H_0 to the diagram we get:

$$\begin{tikzcd}[column sep=small] & H_0\left(D^1\right) \arrow[dr, "r_*"] & \\ H_0\left(S^0\right) \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & H_0\left(S^0\right)\end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=small] & \mathbb{Z} \arrow[dr, "r_*"] & \\ \mathbb{Z}\oplus\mathbb{Z} \arrow[ur, "i_*"] \arrow[rr, "\mathrm{id}"] & & \mathbb{Z}\oplus\mathbb{Z}\end{tikzcd}$$

but this is clearly impossible.

In the same way we will see the much harder fact:

Fact 1 (Brouwer). *There does not exist a retraction $D^n \to S^{n-1}$ (for $n \geq 2$).*

We will see this by applying H_{n-1}. The idea is that

$$H_{n-1}\left(D^n\right) = 0 \qquad\qquad H_{n-1}\left(S^{n-1}\right) = \mathbb{Z}$$

which means we would have the diagram:

$$\begin{tikzcd}[column sep=small] & 0 \arrow[dr] & \\ \mathbb{Z} \arrow[ur] \arrow[rr, "\mathrm{id}"] & & \mathbb{Z}\end{tikzcd}$$

which is a contradiction.

This turns out to imply the famous:

Theorem 1 (Brouwer fixed point theorem). *Every map $f : D^n \to D^n$ ($n \geq 1$) has a fixed point (i.e. a point $x \in D^n$ such that $f(x) = x$). In this case one says that D^n has the* fixed point property *(FPP).*

Proof. Suppose there exists an $f : D^n \to D^n$ such that $\forall x \in D^n$ $f(x) \neq x$. Now draw a straight line from x to $f(x)$ and continue it to the boundary S^{n-1}. Call this point $g(x)$. Then this defines a map $g : D^n \to S^{n-1}$. g is continuous and $g|_{S^{n-1}} = \mathrm{id}$. Therefore g is a retraction $D^n \to S^{n-1}$ which we saw cannot exist. □

2.2. **Dimension.** We know $\mathbb{R}^n$ somehow has dimension n. But what does this really mean? The intuition is that $\mathbb{R}^2$ somehow has more points than $\mathbb{R}$. But then in 1877 Cantor proved that there is in fact a bijection $\mathbb{R} \to \mathbb{R}^2$. But this is highly non-continuous, so this tells us continuity should have something to do with it. But then in 1890 Peano showed that there exists a continuous suggestion $\mathbb{R} \to \mathbb{R}^2$ as well. In 1910, using homology, Brouwer proved:[2]

Theorem 2. *For $m < n$ $\nexists$ continuous injection $\mathbb{R}^n \to \mathbb{R}^m$.*

We will prove this. A corollary of this is the famous invariance of dimension. I.e. $\mathbb{R}^m \cong \mathbb{R}^n$ iff $m = n$. The proof uses separation properties of $n-1$-sphere in $\mathbb{R}^n$, thus in turn uses H_*.

Exercise 2. Find an easy proof for $m = 1$.

Theorem 3 (Jordan curve theorem). *For a subset $C \subset \mathbb{R}^2$ such that $C \simeq S^1$ then $\mathbb{R}^2 \setminus C$ has exactly 2 components A and B. In addition $C = Fr(A) = Fr(B)$. (Recall the* frontier *is defined as $Fr(X) = \overline{X} \cap \overline{(Y \setminus X)}$ for any $X \subset Y$.)*

[2] When he proved this Lebesgue contacted him saying that he could prove it too. So he sent him his proof, and Brouwer saw some errors. So over many years he eventually corrected it. In the end Brouwer summarized it by saying that it was really just his own proof.

Theorem 4 (Schönflies)**.** *Let A, B, and C be as above and assume A is a bounded component of $\mathbb{R}^2 \setminus X$. Then $\overline{A} \cong D^2$.*

The higher dimensional analog of JCT is true Brouwer). The proof will use H_*. The generalization is that for $\Sigma \subset \mathbb{R}^n$ then $\Sigma \cong S^{n-1}$ As it turns out the higher dimensional analog of the Schönflies theorem is false. The counterexample is the famous Alexander horned space. I.e. there exists $\Sigma \subset \mathbb{R}^3$, $\Sigma \cong S^2$; A and bounded component of $\mathbb{R}^2 \setminus \Sigma$ and $\overline{A} \ncong D^3$.

Recall

Theorem 5 (Heine-Borel)**.** *$X \subset \mathbb{R}^n$ compact is equivalent to X being closed and bounded.*

So consider $\emptyset \neq X \subset \mathbb{R}$ compact and connected. This is equivalent to being an interval $X = [a, b]$ $(a \leq b)$. In $\mathbb{R}^2$ things get much worse. In particular there exists compact, connected subset such that $\mathbb{R}^2 \setminus X$ has exactly three components A, B, and C, and in particular every neighborhood of every point in X meets all three components. This is known as the 'lakes of Wada'. We start with an island with two lakes, and then start digging canals which somehow get closer and closer to the lakes. In the end every point of what is left is arbitrarily close to all three lakes.

Question 2 (Open question)**.** Let X be a compact connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus X$ is connected. Does X have the fixed point property?

Chapter 1: Homotopy

Let $I = [0, 1]$.

Definition 2. Let $f, g : X \to Y$. Say f is homotopic to g (and write $f \simeq g$) iff there exists a map $F : X \times I \to Y$ such that for all $x \in X$

$$F(x, 0) = f(x) \qquad\qquad F(x, 1) = g(x) \ .$$

Define $F_t : X \to Y$ by $F_t(x) = F(x, t)$. Then F_t is a continuous 1-parameter family of maps $X \to Y$ such that $F_0 = f$ and $F_1 = g$.

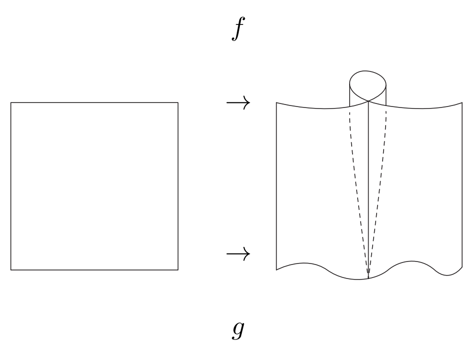

Figure 1.

Example 8. Let $F : S^{n-1} \times I \to \mathbb{R}^n$ be defined by $F(x, t) = (1 - t)x$. Then F_0 is the inclusion inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$ and F_1 is the constant map $S^{n-1} \to$ origin.

Lemma 1. *Homotopy is an equivalence relation on the set of all maps $X \to Y$.*