HOMEWORK 5
ALGEBRAIC TOPOLOGY

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Exercise 1 (Hatcher §1.2, 9)**.** In the surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M_h' and M_k' obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M_h' does not retract onto its boundary circle C, and hence M_g does not retract onto C. [Hint: abelianize π_1.] But show that Mg does retract onto the nonseparating circle C' in fig. 1.

Exercise 2 (Hatcher §1.2, 10)**.** Consider two arcs α and β embedded in $D^2 \times I$ as shown in fig. 2. The loop γ is obviously nullhomotopic in $D^2 \times I$, but show that there is no nullhomotopy of γ in the complement of $\alpha \cup \beta$.

Exercise 3 (Hatcher §1.2, 16)**.** Show that the fundamental group of the surface of infinite genus shown in fig. 3 is free on an infinite number of generators.

Exercise 4 (Hatcher §1.2, 20)**.** Let X be the subspace of $\mathbb{R}^2$ that is the union of the circles C_n of radius n and center $(n, 0)$ for $n = 1, 2, \ldots$. Show that $\pi_1(X)$ is the

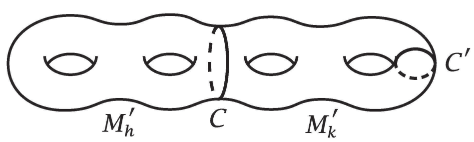

FIGURE 1. Figure for exercise 1.

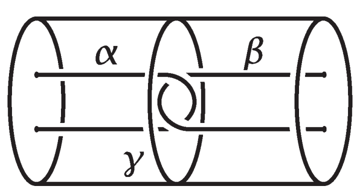

FIGURE 2. Figure for exercise 2.

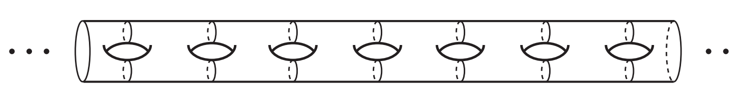

FIGURE 3. Figure for exercise 3.

free group $*_n \pi_1(X_n)$, the same as for the infinite wedge sum $\bigvee_\infty S_1$. Show that X and $\bigvee_\infty S_1$ are in fact homotopy equivalent, but not homeomorphic.

Exercise 5 (Hatcher §1.3, 3)**.** Let $p : \tilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\tilde{X}$ is compact Hausdorff iff X is compact Hausdorff.

Exercise 6 (Hatcher §1.3, 4)**.** Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Exercise 7. Let $T_g = \#_g T^2$ and $P_k = \#_k P^2$. Show that if $g \geq 2$ and $k \geq 2$ then $\pi_1(T_g)$ and $\pi_1(P_k)$ are not abelian. (Hint: show that the group maps onto a non-trivial free product.)

Exercise 8. Show that P_{2k} is a covering space of P_{k+1}, and that if $g \geq 1$ then T_{2g-1} is a covering space of T_g.