HOMEWORK 4

JACKSON VAN DYKE

Exercise 1 (Hatcher §1.2, 1)**.** Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

Exercise 2 (Hatcher §1.2, 3)**.** Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

Exercise 3 (Hatcher §1.2, 4)**.** Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1\left(\mathbb{R}^3 \setminus X\right)$.

Exercise 4 (Hatcher §1.2, 7)**.** Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

Exercise 5 (Hatcher §1.2, 8)**.** Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Exercise 6 (Hatcher §1.2, 14)**.** Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order eight.

Exercise 7. Let X be obtained from Y by attaching an n-cell. Let $q : D^n \amalg Y \to X$ be the quotient map $i : D^n \to D^n \amalg Y$ inclusion, and $f = qi : D^n \to X$. Show that

1. $f|_{\mathrm{int}(D^n)}$ is a homeomorphism of $\mathrm{int}(D^n)$ onto $f(\mathrm{int}(D^n))$;
2. Y is Hausdorff $\implies$ X is Hausdorff.

Exercise 8.
- (i) Show that the group $\left\langle x, y \,\middle|\, x^2 = y^3 = (xy)^5 = 1 \right\rangle$ is non-trivial. (Hint: consider the symmetric group S_5.)
- (ii) Show that the group $\left\langle x, y \,\middle|\, x^{-1}yx = y^2, y^{-1}xy = x^2 \right\rangle$ is trivial.

Exercise 9. Express the solid torus $S^1 \times D^2$ as a cell-complex.

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