# HOMEWORK 3

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**Exercise 1** (Hatcher §1.1, 3)**.** For a path-connected space $X$, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms $\alpha_\#$ (where $\alpha$ is a path from a basepoint $x_0$ to $x_1$) depend only on the endpoints of the path $\alpha$.

**Exercise 2** (Hatcher §1.1, 17)**.** Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$.

**Exercise 3** (Hatcher §1.1, 20)**.** Suppose $f_t : X \to X$ is a homotopy such that $f_0$ and $f_1$ are both the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]

**Lemma 1** (Hatcher, 1.19)**.** *If $\varphi_t : X \to Y$ is a homotopy and $h$ is the path $\varphi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps in the diagram:*

$$\begin{tikzcd} & \pi_1(Y, \varphi_1(x_0)) \arrow[dd, "h_\#"] \\ \pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \\ & \pi_1(Y, \varphi_0(x_0)) \end{tikzcd}$$

*satisfy* $\varphi_{0*} = h_\# \varphi_{1*}$.

**Exercise 4.** Show that every $3 \times 3$ matrix with positive real entries has a positive real eigenvalue. [*Hint:* Consider $D = \left\{(x, y, z) \,\middle|\, x, y, z, \geq 0, x^2 + y^2 + z^2 = 1\right\}$ and use the Brouwer fixed point theorem.]

**Exercise 5.** Let $T$ be the 2-torus $T^2 = S^1 \times S^1$. For each of the following, is there a subspace $A \subset T$ such that $A \cong S^1$ and $A$ has the stated property:

(i) $A$ is a retract of $T$,
(ii) $A$ is not a retract of $T$,
(iii) $A$ is a deformation retract of $T$.

*Date*: September 18, 2019.