HOMEWORK 3

JACKSON VAN DYKE

Exercise 1 (Hatcher §1.1, 3)**.** For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms $\alpha_\#$ (where α is a path from a basepoint x_0 to x_1) depend only on the endpoints of the path α.

Exercise 2 (Hatcher §1.1, 17)**.** Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$.

Exercise 3 (Hatcher §1.1, 20)**.** Suppose $f_t : X \to X$ is a homotopy such that f_0 and f_1 are both the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$.]

Lemma 1 (Hatcher, 1.19)**.** *If $\varphi_t : X \to Y$ is a homotopy and h is the path $\varphi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps in the diagram:*

$$
\begin{array}{ccc}
 & & \pi_1(Y, \varphi_1(x_0)) \\
 & \nearrow^{\varphi_{1*}} & \\
\pi_1(X, x_0) & & \Big\downarrow \beta_h \\
 & \searrow_{\varphi_{0*}} & \\
 & & \pi_1(Y, \varphi_0(x_0))
\end{array}
$$

satisfy $\varphi_{0*} = \beta_H \varphi_{1*}$.

Exercise 4. Show that every 3×3 matrix with positive real entries has a positive real eigenvalue. [*Hint:* Consider $D = \left\{ (x, y, z) \,\middle|\, x, y, z, \geq 0, x^2 + y^2 + z^2 = 1 \right\}$ and use the Brouwer fixed point theorem.]

Exercise 5. Let T be the 2-torus $T^2 = S^1 \times S^1$. For each of the following, is there a subspace $A \subset T$ such that $A \cong S^1$ and A has the stated property:

   (i) A is a retract of T,
  (ii) A is not a retract of T,
 (iii) A is a deformation retract of T.