Projective (symmetries of) TQFTs

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Anomalies

- **TQFT** = fully-extended symmetric-monoidal functor $\mathbf{Bord}_d^{\mathrm{fr}} \to \mathcal{T}$.
- **Relative/twisted/boundary theory** is a (lax) natural transformation[1] $F\colon 1 \to \beta$ (or $\beta \to 1$).
- An **anomaly**[2] is an invertible once-categorified d-dimensional TQFT α, and an anomalous d-dimensional TQFT is a relative theory $F\colon \alpha \to 1$.

Example

Let V be a finite-dimensional vector space.

- V classifies a TQFT $\mathbf{Bord}_1^{\mathrm{fr}} \to \mathbf{Vect}$.
- $G \to \mathrm{GL}(V)$ classifies a TQFT $\mathbf{Bord}_1^{BG} \to \mathbf{Vect}$.
- $G \to \mathrm{PGL}(V)$ classifies an anomalous 1-d TQFT on $\mathbf{Bord}_1^{BG}$.

[1] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories, 2017

[2] Daniel S. Freed. What is an anomaly?, 2023

Anomal(ous symmetr)ies of three-dimensional TQFTs

Building on existing results[3,4] I introduce:[5] $B^2\mu_q \hookrightarrow 3\mathrm{Pin} \twoheadrightarrow \mathrm{O}\left(L \oplus L^\vee\right)$.

Theorem (VD[3])

The framed Dijkgraaf-Witten theory for a finite group L canonically defines the following, which are equivalent:

- *an anomalous theory on* $\mathbf{Bord}_3^{B\,\mathrm{O}(L\oplus L^*)}$
- *a symmetric-monoidal functor* $\mathbf{Bord}_3^{B\,3\mathrm{Pin}(L\oplus L^*,\mathrm{ev})} \to \mathbf{Fus}$

- O acts via "twice-categorified integral transforms".[6].
- We can replace O with the 2-group $\mathrm{Aut}_{\mathbf{EqBr}}$ of $\sigma^3_{BL}\left(S^1\right)$, and then a certain "level" controls the non-triviality of the anomaly[5].

[3] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Fusion categories and homotopy theory (appendix by E. Meir), 2010

[4] Jürgen Fuchs, Jan Priel, Christoph Schweigert, and Alessandro Valentino. On the Brauer groups of symmetries of abelian Dijkgraaf-Witten theories, 2015

[5] Jackson Van Dyke. Projective symmetries of three-dimensional TQFTs, 2023. arXiv: 2311.01637 [math.QA]

[6] Jackson Van Dyke. Symmetries of quantization of finite groupoids, 2023. arXiv: 2312.00117 [math.QA]

1-**dimensional**	3-**dimensional**
(V, q)	(A, q)
$\mathrm{SO}(V, q) \subset \mathrm{O}(V, q)$	$\mathrm{SO}(A, q) \subset \mathrm{O}(A, q)$
$\mathbf{k}^{\times}$	$B^2\mathbf{k}^{\times}$
$\mathrm{Cliff}(V)$	$\mathcal{A} = (\mathbf{Vect}[A], *, \beta_q)$
$\{x, y\} = b_q(x, y)$	$\beta_q \colon \mathbf{k}_a * \mathbf{k}_b \xrightarrow{b_q(a,b)\,\mathrm{id}} \mathbf{k}_b * \mathbf{k}_a$
$V \rtimes \mathrm{O}(V, q)$	$\mathrm{Aut}_{\mathbf{EqBr}}(\mathcal{A})$
$\mathrm{Pin}(V, q)$	$3\mathrm{Pin}(A, q)$
$\mathrm{Spin}(V, q)$	$3\mathrm{Spin}(A, q)$
$V \simeq L \oplus L^*$	$A \simeq L \oplus L^*$
$\wedge^{\bullet} L^*$	$\mathcal{C} = (\mathbf{Vect}[L^*], *)$
$\mathrm{End}(\wedge^{\bullet} L^*) \simeq \mathrm{Cliff}$	$\mathrm{Aut}_{\mathbf{Fus}}(\mathcal{C}) \simeq \mathrm{Pic}(\mathcal{A})$

Future directions

- Analogous results for fusion 2-categories[7]?
- $\mathbb{P}\mathbf{Fus}$ replaced with the projectivization of the $(\infty, n+m+2)$-category $\mathbf{Alg}_n\left(m\,\mathrm{Pr}^L\right)$ á la JFS[1]?
- Gapped systems, topological phases of matter
- Non-semisimple finite ribbon categories
  - Link and manifold invariants[9]
  - Rozansky-Witten theory and relative Langlands

Conjecture

The truncation of $\mathrm{Aut}\left(\mathbf{RW}_M(*)\right)$ *to a group is* $\mathrm{Sp}(M)$*.*

Rmk: The k-invariant of $B\,\mathrm{Aut}$ in $H^4\left(B\,\mathrm{Sp}(M), \mathbb{C}^\times\right)$ would then be the projectivity/anomaly of the action $\mathrm{Sp}(M) \circlearrowright \mathbf{RW}_M$.

[7] Christopher L. Douglas and David J. Reutter. Fusion 2-categories and a state-sum invariant for 4-manifolds, 2018

[1] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories, 2017

[9] Johannes Berger, Azat M. Gainutdinov, and Ingo Runkel. Non-semisimple link and manifold invariants for symplectic fermions, 2023

Thank You!

References

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Higher projective symmetries

In Theorem C.16[5], I relate twisted quantization[7] with anomalies:

Theorem (VD[3])

$$\alpha_c \to 1 \qquad \iff \qquad 1 \to \sigma^{d+1}_{X,c}$$

E.g. (projective) group rep. mod. / (twisted) group alg.

Given a trivialization, they will reduce to (the same) X-theories:

$$\begin{tikzcd} 1 \arrow[r, "\sim"] \arrow[rr, bend right, "F_X"'] & \alpha_c \arrow[r, "F_\alpha"] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r] \arrow[rr, bend right, "F_X"'] & \sigma^{d+1}_{X,c} \arrow[r] & \sigma^{d+1}_X \end{tikzcd}$$

[5] Jackson Van Dyke. Projective symmetries of three-dimensional TQFTs, 2023. arXiv: 2311.01637 [math.QA]

[7] Daniel S. Freed, Gregory W. Moore, and Constantin Teleman. Topological symmetry in quantum field theory, 2022