FLOER THEORY AND FUKAYA CATEGORIES

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1. Introduction to Floer homology

We start with a symplectic manifold $\left(M^{2m}, \omega\right)$ where $\omega \in \Omega^2(M)$ is a two form which is closed (that is, $d\omega = 0$). Note that ω^m is a volume form. Recall that a 2-form is a pairing on $T_P M$. Then this being a volume form just means that at any point this will be equal to the standard alternating form given by the block matrix:

$$\begin{pmatrix} 0 & 1 & \cdots & \cdots \\ -1 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & 1 \\ \cdots & \cdots & -1 & 0 \end{pmatrix}$$

Example 1. Any cotangent bundle T^*N is an example of this. Any Kähler manifold is also an example. Kähler forms are always symplectic forms.

There is a distinguished class of submanifolds, called Lagrangian submanifolds. We will be interested in compact ones. These are half dimensional embedded submanifolds such that $\left.\omega\right|_L \equiv 0$. This is a natural condition, because if we have a non-degenerate alternating 2-form, the maximum subspace on which it vanishes, is half dimensional. So Lagrangians integrate the local condition one might write down on the level of the tangent bundles.

Example 2. Inside a cotangent bundle, a zero section is always Lagrangian.

If you had a complex projective variety with real equations, then the real locus is also an Lagrangian submanifold.

These are the central things that symplectic topologists want to study. So a natural question is up to what equivalence? The simplest one is something called Hamiltonian isotopy. A Hamiltonian is just a time dependent function

$$[0,1] \times M \xrightarrow{C^\infty} \mathbb{R} \quad .$$

Now we can associate a vector field to this by using the symplectic form. We require that the vector field contracted with ω is the differential of H_t:

$$\omega\left(X_t, \cdot\right) = dH_t$$

so we end up with some time dependent vector field X_t. Now if we look at the time 1 flow φ of X_t, this is what we call a Hamiltonian isotopy. These naturally form a

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Any and all errors introduced are my own.

group under composition, which we will write $\mathrm{Ham}(M, \omega)$. It is easy to see these are all examples of symplectomorphisms:

$$\mathrm{Symp}(M, \omega) = \{\varphi \mid C^\infty, \varphi^*\omega = \omega\}$$

which are the diffeomorphisms which pull-back ω to itself.

The following is sort of the theorem that started it all.

Theorem 1 (Floer)**.** *Suppose that we have a compact Lagrangian manifold L', which is Hamiltonian isotopic to L. For simplicity, assume $L \pitchfork L'$ (they intersect transversely). Furthermore, assume $\omega_{\pi_2(M,L)} = 0$. Then*

$$|L \pitchfork L'| \leq \sum_i \dim H^2(L, \mathbb{Z}/2\mathbb{Z}) \ .$$

Example 3. Let's consider what this looks like for a basic example. In particular, we consider a torus, which is a symplectic manifold for ω any area form. Note that if we restrict ω to a one dimensional submanifold, it vanishes. So any embedded submanifold is Lagrangian.

Now the concept here, is if we have some L, say S^1, then the sort of displacements allowed by Hamiltonian isotopy, say $\varphi(L)$ where $\varphi \in \mathrm{Ham}$, then the condition on this displacement is that if we displace in one direction by area A_1, and in the other direction by A_2, then we have $A_1 = A_2$. More generally, Hamiltonian isotopies preserve flux.

There is also a type of symplectomorphism where if you take a sort of symplectic area form, we could rotate, and then this would preserve the area form. Then we could, for instance, get a displacement $\psi \in \mathrm{Symp} \setminus \mathrm{Ham}$. So what is happening here? Well if we restrict ourselves to Ham, then we can displace, but then the equal area displacement in each direction forces us to always have two intersections with L. Whereas with a displacement given by something like ψ, then we can sort of push it off. On the other hand, we also could have chosen a Lagrangian submanifold K which bounds a disk of positive area in the sense that it doesn't go through the hole. Then we have

$$\omega|_{\pi_2(T^2,K)}$$

since, as we said, K bounds a disk of positive area. Indeed, we can cook up a Hamiltonian isotopy, which completely pushes $\varphi(K)$ off of K. So already this shows us that this condition in Floer's theorem will be important.

So how did Floer do this? So he started with a pair L_0, L_1 of compact Lagrangians, with the homotopy condition:

$$\omega|_{\pi_2(M, L_0 \amalg L_1)} = 0$$

then he associated to this an invariant, now called *Floer cohomology* which we will write

$$\mathrm{HF}(L_0, L_1) = H^*(\mathrm{CF}(L_0, L_1), \partial) \ .$$

Before going into the definition, we discuss some features of this.

(1) Invariant under Hamiltonian isotopy of either of the L_0, L_1.
(2) If $L_0 \pitchfork L_1$, then $\mathrm{CF}(L_0, L_1) = \Lambda \langle L_0 \pitchfork L_1 \rangle$ where Λ is called the Novikov field (over $\mathbb{Z}/2\mathbb{Z}$). This is a sort of base field for this theory.
(3) $\mathrm{HF}(L, L) \cong H^*(L, \Lambda)$

Now we can check quite easily, that these three features give us Floer's theorem.

Now we present some concrete definitions.

Definition 1. The Novikov field over $\mathbb{Z}/2\mathbb{Z}$, is

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{Z}/2\mathbb{Z}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = +\infty \right\}$$

where T is some formal variable. Note we can replace $\mathbb{Z}/2\mathbb{Z}$ with any other field.[1]

Definition 2. For $L_0 \pitchfork L_1$,

$$\mathrm{CF}\left(L_0, L_1\right) = \Lambda \left\langle L_0 \pitchfork L_1 \right\rangle \ .$$

If they don't intersect transversely, we would start by taking a Hamiltonian perturbation anyway. So we just stick to this case.

Definition 3. Suppose we have some intersection point p. Then the differential is

$$\partial_p = \sum_{\substack{q \in L_0 \pitchfork L_1 \\ \beta \in \pi_2(M, L_0, L_1)}} \#\mathcal{M}\left(p, q, \beta, J\right) T^{\omega(\beta)} q \ .$$

This is essentially the count of the points in some moduli space, which depends on p, q, β, and an almost complex structure J, which we will return to shortly. Remember ω is a closed two-form, so it defines a cohomology class, so in general it could eat $\pi_2(M)$, but since L_0, L_1, are Lagrangian, ω vanishes so it can eat these classes as well.

Definition 4. An *almost complex structure* $J \in \mathrm{End}(TM)$ is such that $J^2 = -1$. In particular, this is an actual complex structure just without needing to be integrable. It also must be compatible with ω in the sense that $\omega(\cdot, J\cdot)$ is a Riemannian metric.

Before we define the moduli space $\mathcal{M}$ itself, we define a related space:

Definition 5. The moduli space $\hat{\mathcal{M}}(p, q, J)$ is the set of smooth maps

$$u : \mathbb{R} \times [0,1] \xrightarrow{C^\infty} M$$

such that:

(1) It satisfies the J-holomorphic curve equation:[2]
$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0 = \bar{\partial}_J(u)$$
where $s \in \mathbb{R}$ and $t \in [0,1]$.

(2) It satisfies the boundary condition: $u(s, i) \in L_i$ for $i = 0, 1$.

(3) It satisfies the asymptotic conditions:
$$\lim_{s\to-\infty} u(s,t) = q \qquad\qquad \lim_{s\to\infty} u(s,t) = p \ .$$

Schematically, the boundary/asymptotic conditions are saying that u maps the domain strip to a shape made by the two curves L_0, L_1 intersecting at q on the left, and p on the right. This can be seen in fig. 1.

[1] Using $\mathbb{Z}/2\mathbb{Z}$ allows us to avoid a discussion of orientation.

[2] We can think of the domain of u as a sort of strip in $\mathbb{C}$, so this is a condition forcing it to be holomorphic in this region with these sort of generalized Cauchy-Riemann equations.

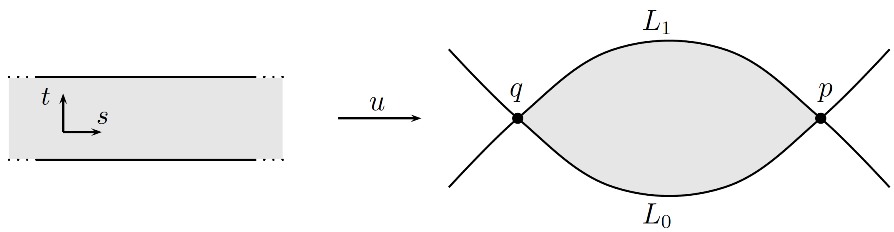

FIGURE 1. A pseudo-holomorphic strip contributing to the Floer differential on $\mathrm{CF}(L_0, L_1)$.

So far we have seen $\hat{\mathcal{M}}$ which doesn't depend on β. We can however get a version of $\hat{\mathcal{M}}$ which does depend on β by only considering the classes in $\hat{\mathcal{M}}$ which are in the relative class β:

$$\hat{\mathcal{M}}(p, q, J, \beta) = \left\{ u \in \hat{\mathcal{M}}(p, q, J) \mid [u] = \beta \in \pi_2(M, L_0 \cup L_1) \right\} .$$

Now to define $\mathcal{M}$ we can simply quotient out by the natural $\mathbb{R}$ action on $\hat{\mathcal{M}}$:

$$\mathcal{M}(p, q, J, \beta) = \hat{\mathcal{M}}(p, q, J, \beta) / \mathbb{R} .$$

This action is given by the fact that everything we have done is essentially translation invariant in the $\mathbb{R}$ direction. So if we have some solution $u(s, t)$, then we can apply $r \in \mathbb{R}$ to get:

$$r \cdot u(s, t) = u(r + s, t)$$

which is also a solution.

Before getting into what it means to count $\mathcal{M}$, and why we can do this, we return to the example of the torus T^2 from before. Suppose we have L as before, just S^1, and L', some displacement of L allowed by Hamiltonian isotopy, with corresponding areas A_1, A_2. Call the intersection points p, q. Then these intersect transversely, so we can write the chain complex as:

$$\mathrm{CF}(L, L') = \Lambda \langle p, q \rangle \ .$$

Now we can see, first of all, that the Riemann mapping theorem gives us two sort of holomorphic strips between p, q. Then we can use some sort of complex analysis to convince ourselves that there is sort of nothing else. This means that

$$\partial p = \left(T^{A_1} + T^{A_2} \right) q \ .$$

First of all, if $A_1 = A_2$, then this is 0, recalling we are working in characteristic 2. This means that our Floer cohomology is:

$$\mathrm{HF}(L, L') = \lambda \langle p, q \rangle$$

and indeed this looks like cohomology of S^1. On the other hand, if $A_1 \neq A_2$, then it is a fairly easy exercise to show that $T^{A_1} + T^{A_2} \in \Lambda$ is invertible. As such, $\mathrm{HF} = 0$. Also, we can isotope through the Hamiltonian to displace them completely, and get no intersection points.

There is an invariant of a holomorphic strip which we will need later. This is the energy of u, written:

$$E(u) = \omega(\beta) = \int u^* \omega = \int \int \left| \frac{\partial \mu}{\partial s} \right|^2 ds \ dt \geq 0 \ .$$

The last equality shows that the energy is positive, and the only way in which we can have equality, is precisely when u is constant.

Now we still need to make better sense of this count of $\mathcal{M}$. We also need to see that the differential that we have seen is indeed a differential in the sense that it squares to zero. The things we will need for this are transversality, Gromov compactness, and gluing. We will basically just mention these to get a feel for what this is about.

1.1. **Transversality.** In general, to have well defined moduli spaces, what we really hope is for them to be manifolds. For us, in good cases this will be the case, and the dimension $i(\beta)$ will just depend on β. In order for this to be a manifold, in our world this is just an analysis problem. Schematically, this problem has the following flavor. We have some Banach manifold $\mathcal{B}$ of maps say from $\mathbb{R} \times [0,1] \to M$ with the same boundary and asymptotic (homotopy) conditions as u. The one crucial thing they don't have is the J-holomorphic curve equation. We are not quite looking at all smooth maps, because we want a space where sequences will converge. Now take $\mathcal{E}$ to be some Banach vector bundle of sections of $\Omega^{0,1} \otimes u^*TM$. The point is there is a natural section here, given by taking my operator $\bar{\partial}_J$. This means that my moduli space $\hat{\mathcal{M}}$ will be the preimage of 0 under this section:

$$\hat{\mathcal{M}}(p,q,\beta,J) = \bar{\partial}_J^{-1}(0) \subseteq \mathcal{B} \ .$$

This will be a manifold if this is cut out transversely:

$$\begin{tikzcd} \mathcal{E} \arrow[d] \\ \mathcal{B} \arrow[u, bend left, "\bar{\partial}_J"] \end{tikzcd} .$$

Essentially, this boils down to the fact that we need the linearization of this operator at the solution u, $D_{\bar{\partial}_J,u}$, to be a Fredholm operator. Recall that this means they are invertible up to compact operators. In particular, Fredholm operators have finite index, which is the dimension of the kernel of the operator minus the dimension of the cokernel, which is equal to $i(\beta)$.

It turns out, if you have a general J, which is the way we've set things up so far, you may just not be able to achieve transversality. There is however a fix: make J domain dependent. We will make this dependence fairly mild, because it will just depend on t: $J = \{J_t\}_{t\in[0,1]}$. Then our J-holomorphic curve condition becomes slightly different in this more general setting:

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = 0 \ .$$

Theorem 2 (Floer-Hofer-Salomon)**.** *For generic such domain dependent J, the corresponding linear operators are Fredholm.*[3]

1.2. **Gromov compactness.** Now we wish to show how to count elements of these moduli spaces, which are now manifolds, and show that $\partial^2 = 0$.

Definition 6. Gromov-compactness is when any sequence of J-holomorphic curves with bounded energy has a subsequence which "converges" to a tree of nodal holomorphic curves.

[3]In this setting, when coker $= 0$, J is called regular.

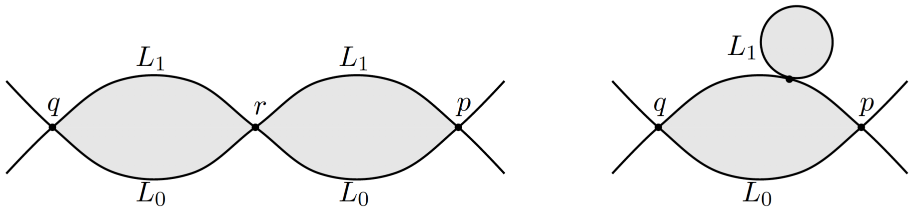

FIGURE 2. Possible limits of pseudo-holomorphic strips: a broken strip (left) and a disc bubble (right).

The holomorphic curves we care about are these holomorphic strips which get mapped as in fig. 1. In practice, the way this compactness looks in this context is that this might start as a neat shape, and then things start to pinch and we get a sort of bubbling effect. Then eventually we get a sort of convergence where all of these degenerations become neater. One type of degeneration is called disk bubbling. This is when we get a disk which only connects with our “initial” disk at one point on the boundary. We can also get strip breaking, where the initial strip pinches into two separate disks. See fig. 2 for an example. We can also have sphere bubbles coming off of an interior point of the strip.

In the case we have, we've asked that $\omega|_{\pi_2(M,L_c)} = 0$. In particular, because of the energy condition we had before, we have no disk bubbling or sphere bubbling because such things would have to be constant. So what does this mean for strips? Well then the only allowed degeneration is strip breaking. The only sort of thing a strip could converge to is a sort of concatenation of k broken strips, each with an associated energy $\beta_0, \cdots, \beta_k$. We call the point where the ith strip pinches into the $i-1$st strip r_i. Then we have that the index of the Fredholm problem is additive over this:

$$i(\beta) = \sum_{l=0}^{k} i(\beta_l) \ .$$

Corollary 1. *Suppose we have some regular J_t, and the homotopy condition: $\omega|_{\pi_2(M,L_c)} = 0$, then:*

(1) If $i(\beta) = 1$, then $\mathcal{M}(p,q,\beta,J)$ is a compact 0 manifold.[4]

(2) If $i(\beta) = 2$, then we can define a compactification:[5]

$$\bar{\mathcal{M}}(p,q,\beta,J) = \mathcal{M}(p,q,\beta,J) \amalg \dagger \tag{1}$$

$$\dagger = \coprod_{\substack{r \in L_0 \pitchfork L_1 \\ \beta = \beta_0 + \beta_1 \\ i(\beta_0) = 1}} \left(\mathcal{M}(p,r,\beta_0,J) \times \mathcal{M}(r,q,\beta_1,J)\right) \tag{2}$$

where r is the point where the strip has broken into two strips with energies β_0, β_1.

1.3. **Gluing.** Now we have done nearly everything we need to show that it is indeed the case that $\partial^2 = 0$. The idea is that the terms in the expansion in corollary 1 are precisely the terms which come up when we need to count the number of points

[4] Before we took the quotient, the moduli space $\hat{\mathcal{M}}$ is a 1-manifold.

[5] We can do this since we know all of the things it might limit to.

in the boundary of p if we're doing the composition of two differentials. We've defined this as a compactification, but we don't actually know that every point in this expansion is actually a boundary point. There could just be some extra points floating around, which aren't actually boundary points. Luckily in good cases, this isn't the case, and they do indeed all arise as boundaries. This brings us to the subject of gluing. The basic idea is that we can take two holomorphic strips and associate a shared point at the pinched portion between them in a compatible and holomorphic way. We now make this more precise:

Theorem 3. *Suppose J_t is regular, and say we have two index one classes*

$$i\left(\beta_0\right) = i\left(\beta_1\right) = 1 \ .$$

Then we have a point in the compactification $\bar{\mathcal{M}}\left(p, q, \beta, J\right)$ where q is the pinched point on the far left, p on the far right, r in the middle, β_1 the energy of the left strip, and β_0 the energy of the right one. Then there is a neighborhood of this point which is homeomorphic to $[0, \delta)$ for some δ.

This theorem tells us that the compactification is a compact 1-manifold, and its boundary is precisely (†). Now we require two final propositions:

Proposition 1. *If we have a compact* 1*-manifold, its boundary points come in an even number.*

Proposition 2. *The decomposition* (†) *gives us precisely the coefficients of q in $\partial^2 p$.*

Finally putting these together, we see that $\partial^2 = 0$.

Thinking back to the statement of Floer's theorem from the start, the main theorem we haven't focused much on, is why if we have two Hamiltonian submanifolds which are Hamiltonian isotopic to each other the Floer cohomology is just the same as the standard cohomology ring. We also have not discussed the useful notion that if the Lagrangian satisfies some extra conditions, then we can equip HF with a grading. This is a very powerful tool.

2. Introduction to Fukaya categories

The primary focus of this section is to build up to the basic notion of a Fukaya category.

Remark 1. We start with a small aside. So far, we have been working only with characteristic 2. But what if we want to work over different characteristics? On monday, we were working over the Novikov field Λ which was really sort of $\Lambda_{\mathbb{Z}/2}$ where $\mathbb{Z}/2$ was the coefficient field for the formal power series. In general, given any other field, say $\mathbb{C}$ or $\mathbb{Z}/n$, we could still define a Novikov field over this in exactly the same way. Sometimes we can still do Floer theory over this, but we need extra conditions. For example, if L_0, L_1 are spin, then $\mathcal{M}\left(p, q, \beta, J\right)$ has a natural orientation. This is not obvious, we are simply asserting this to be true. But once we have moduli spaces which are somehow naturally compatibly oriented, then we can count the points in this zero dimensional moduli space with signs. Furthermore, the proof that the differential squares to zero is still going to apply using the signs, which means that we can define

$$\left(\mathrm{CF}\left(L_0, L_1, \Lambda_{\mathbb{K}}\right), \partial\right)$$

for $\mathbb{K}$ having any characteristic.

2.1. **Maslov indices and operator indices.** So in the last section we saw that these moduli spaces were manifolds, with dimensions given by some index which is defined as the index of some Fredholm operator. But let's imagine we are in a more concrete scenario where we have a disk and we want to find its index. It turns out there is a way of calculating this using topology, and something called Maslov indices.

So we are interested in something called the Lagrangian Grassmanian:

$$\mathrm{LGr}\,(n) \coloneqq \{\text{all Lagrangian vector subspaces of}\,(\mathbb{C}^n,\omega)\} = \mathrm{U}\,(n)\,/\,\mathrm{O}\,(n)$$

where ω is the standard symplectic pairing. So this is basically just linear algebra. It is an easy exercise to check the second equality.

In particular, we can calculate the fundamental group of this, which is explicitly isomorphic under:

$$\pi_1\,(\mathrm{LGr}\,(n)) \xrightarrow{\mu=\det^2} \mathbb{Z}$$

and this is called the Maslov index of a path in LGr.

So what does this have to do with Floer theory? Well let's suppose we have some J-holomorphic strip, u. Now this is contractible, so every vector bundle over it is trivializable. In particular, we can pullback:

$$\mu^*TM \cong \mathbb{R}\times[0,1]\times\mathbb{C}^n$$

where M is still the name of our symplectic manifold. So here we have fixed some trivialization as a complex vector space. Now we want to associate a path in LGr to this, but in order to do this we need to make some auxiliary choices. We can always insist that a local model for an intersection of two Lagrangian planes in $\mathbb{C}^n$ looks as follows. We can always arrange, after acting by a symplectic matrix, to have $T_pL_0 = \mathbb{R}^n$ and $T_pL_1 = (i\mathbb{R})^n$. Notice that both of these "axes" are points in $\mathrm{LGr}\,(n)$, and there is a canonical choice of path inside $\mathrm{LGr}\,(n)$ to go between them which is given by the powers to the n of little changes of angle between them.

The point is that we want to get a path in the whole of $\mathrm{LGr}\,(n)$ out of this disk. We do this as follows. Call this path $\gamma_u \subseteq \mathrm{LGr}\,(n)$. We get this by concatenating a series of paths. At every point on the boundary of this disk we have a point in LGr which is just the tangent space to either L_0 or L_1. First we go from $T_qL_0 \to T_pL_0$. This is moving from one intersection point to the other along L_0. Then we go from $T_pL_0 \to T_pL_1$. This is following the canonical path between these orthogonal points. Now we go from $T_pL_1 \to T_qL_1$, again along L_1. Now when we get back to q we want to subtract the canonical path from T_qL_0 to T_qL_1. In particular, if we are keeping track of orientation, by the end we have completed a positive half turn. So this disk has Maslov index $\mu = 1$.

So now we have the fact, that

$$\mu\,(\gamma_u) = i\,([u]) = i\,(\beta)\ .$$

In particular, things like the above example with $\mu = 1$ are precisely the sort of things we are counting. At this point, we might be wondering what happens when we try to treat higher index disks like this. As it turns out, once one knows what to look for, there are many such examples.

2.2. **Maslov indices and gradings.** As we saw in the Floer theory section, it is possible to equip our spaces with a grading. We now discuss the relationship between this and Maslov indices. First, instead of just looking at closed loops in $\mathrm{LGr}(n)$, we can look at the paths:

$$\mathcal{P}\,\mathrm{LGr}(n) = \{\gamma : [0,1] \to \mathrm{LGr}(n)\} \ .$$

Without giving a precise definition, we mention that we can also make sense of a Maslov index for such paths. In particular, this will recover the Maslov index from before when the path is closed.

Recall our manifold M is symplectic. So tautologically, there is a bundle $\mathrm{LGr}(M)$, which lives above it, which is the Lagrangian Grassmannian bundle of Lagrangian subspaces of the tangent bundle M. So the fiber above each point is $\mathrm{LGr}(n)$, where M has dimension $2n$.

On the other hand, as we just saw, the Lagrangian Grassmannian has fundamental group $\mathbb{Z}$, which means it has some $\mathbb{Z}$ universal cover, $\widetilde{\mathrm{LGr}}(n)$:

$$\widetilde{\mathrm{LGr}}(n) \to \mathrm{LGr}(n) \ .$$

So now we might wonder if we can find a lift from $\widetilde{\mathrm{LGr}}(n)$ to $\widetilde{\mathrm{LGr}}(M)$. As it turns out, if $2c_1(M) = 0$, then the whole thing extends, so

$$\begin{array}{ccc} \widetilde{\mathrm{LGr}}(n) & \longrightarrow & \widetilde{\mathrm{LGr}}(M) \\ & & \downarrow \\ & & M \end{array} \ .$$

So $\widetilde{\mathrm{LGr}}(M)$ is a fiber-wise universal cover. Of course we can always do something of this sort locally, and the obstruction to doing it globally is precisely the Chern class condition. One way to notice this is to see that to find such a cover, we precisely need a nowhere zero section of $(\Lambda^n_{\mathbb{C}}(T^*M))^{\otimes 2}$, which is just a holomorphic volume form.

Now suppose we have such an M with $2c_1(M) = 0$. In this situation we can always define this bundle. Now suppose we have a Lagrangian in M. This gives us a map to $\mathrm{LGr}(M)$, and we can then ask whether or not this can be lifted to the fiber-wise universal cover. The obstruction to this is something called the Maslov class of this Lagrangian L. We now see how this is defined. First we have the inclusion:

$$\pi_1(L) \to \pi_1(\mathrm{LGr}(M)) \to \mathbb{Z}$$

where the map from $L \to \mathrm{LGr}(M)$ is defined completely tautologically since L is a Lagrangian. Note this isn't quite an isomorphism of course, but this second map is precisely what is classifying the cover. Of course a class in $\mathrm{Hom}(\pi_1(L), \mathbb{Z})$ is the same as an element in $H^1(L, \mathbb{Z})$, and the class $\mu_L \in \mathrm{Hom}(\pi_1(L), \mathbb{Z})$ that we get by doing this composition, is called the Maslov class in L. Now just by sort of tracking the algebraic topology, this is precisely what is obstructing L to be lifted. In particular, $\mu_L = 0$ iff L can be lifted from $\mathrm{LGr}(M)$ to $\widetilde{\mathrm{LGr}}(M)$.

But what does this have to do with gradings? Formally, in our world, a grading of L is defined to be a choice of lift to $\widetilde{\mathrm{LGr}}(M)$. So what do we need to have this? Well first of all the whole manifold needs to have $2c_1(M) = 0$, and L has to have vanishing Maslov class $\mu_L = 0$. But how does this give me a grading on chain complexes?

Suppose we have some $p \in L_0 \pitchfork L_1$. Then if we look inside the fiber at p: $\widetilde{\mathrm{LGr}}(M)_p = \widetilde{\mathrm{LGr}}(n)$, then it is simply connected, so there is a unique path $\tilde{\gamma}_p$ between choices of lift from $\widetilde{T_p L_0}$, to $\widetilde{T_p L_1}$. Now we can look at the push-forward of the path:

$$\pi_* \tilde{\gamma}_p \in \mathcal{P}\,\mathrm{LGr}(n)$$

and we have asserted that whenever we have a path here, this carries some Maslov index, and in particular,

$$\deg p = \mu \left(\pi_* \tilde{\gamma}_p \in \mathcal{P}\,\mathrm{LGr}(n) \right) \ .$$

This is how we assign a grading to $\mathrm{CF}(L_0, L_1)$.

In any case, computationally, there are plenty of advantages to be had working with a theory that is graded. We know already, even without gradings, that $i([u])$ can be calculating using some Maslov index by taking a trivialization of the pullback. But in fact, we can check that as long as degrees are defined we have

$$i([u]) = \deg q - \deg p \ . \tag{3}$$

In particular, not only have we assigned a grading to this co-chain complex, but this grading is respected by the differential d. Since d has degree one, we not only have a graded chain complex, but we also have a graded cohomology theory.

Remark 2. We got a $\mathbb{Z}$ grading in the preceding treatment. If we would have restricted ourselves to orientable Lagrangians we would have gotten a $\mathbb{Z}/2\mathbb{Z}$ grading which can be very helpful.

2.3. **The Fukaya category.** Throughout the rest of this discussion we will assume that all of our Lagrangians satisfy the same property from the Floer theory section:

$$\omega|_{\pi_2(M, L_i)} = 0$$

for all i. So if we look at relative homotopy classes with boundary on them, ω vanishes identically.

The basic idea of the Fukaya category of M, is that the objects are Lagrangians, the morphisms between them are chain complexes, and then there are a bunch of extra operations. First we consider the product structure.

So far we have been counting holomorphic things where our domain is a strip. Now we want to consider some disk with these boundary punctures. We want these to map to M, with boundary conditions where the portions of the boundary between the punctures map to three Lagrangians L_0, L_1, and L_2, and we have asymptotic conditions p_1, p_2 and q at the punctures. Note q will become distinguished later.

So we want to consider such maps which are J-holomorphic. In the previous case, these are called J-holomorphic triangles. This is formally still the same condition as before:

$$\bar{\partial}_J u = 0 \ .$$

So we don't have nice coordinates s, t on D, but we still get a generalization of the Cauchy-Riemann equation. In particular, if we look at the differential of u, we get

$$Du \circ j = J \circ Du$$

where j is the standard complex structure on $\mathbb{D}$, and J is some choice of almost complex structure on the target. So this moduli space

$$\mathcal{M}(p_1, p_2, q, \beta, J) = \{ u \mid [u] = \beta \}$$

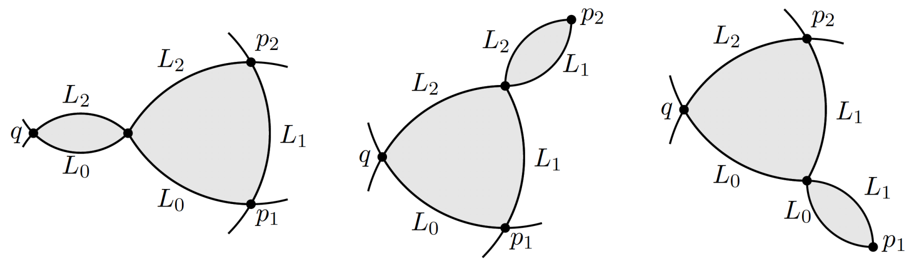

FIGURE 3. The ends of a 1-dimensional moduli space $\mathcal{M}(p_1, p_2, q; [u], J)$.

where β is a relative homotopy class with boundary on the union of the three L_i. Now in order for this to be a manifold, as in the Floer theory section, we need to consider a bigger class of J. So it will be a manifold for any generic domain dependent J.

The dimension of $\mathcal{M}$ is the index of the linearized $\bar{\partial}$ operator $D_{\bar{\partial}_{J,u}}$ which is just $i(\beta)$. Now this can still be calculated in terms of Maslov indices. In particular, this is the Maslov index of a path in $\mathrm{LGr}(n)$.

So now recalling our setup with the J-holomorphic triangles, we can get a path by first moving from q to p_1 along L_0, then from L_0 to L_1 canonically, then along L_1 to p_2, then from L_1 to L_2 canonically, then from p_2 to q along L_2. Now we do the same as in the case with a strip, and subtract the canonical path from L_0 to L_2. Now in the case where we have gradings we have

$$\dim \mathcal{M} = \deg q - \deg p_1 - \deg p_2 \ .$$

So when the index is zero and this manifold is zero dimensional we can count how many elements it has. Now we can use this to define an operation:

$$\begin{array}{ccc} \mathrm{CF}(L_0, L_1) \otimes \mathrm{CF}(L_1, L_2) & \xrightarrow{\quad \mu_2 \quad} & \mathrm{CF}(L_0, L_2) \\ (p_1, p_2) & \longmapsto & \sum_{q,\beta} \#\mathcal{M}(p_1, p_2, q, \mathcal{B}, J) \times T^{\omega(\beta)} q \end{array} .$$

The reason q is getting treated differently here is that p_1 and p_2 are sort of input points for the operation μ_2, whereas q is an output point.

Recall in the section on Floer theory, that once we defined the differential, we used Gromov compactness to get that $\partial^2 = 0$. So let's see what Gromov compactness gives us in this case. Suppose we start with a sequence of such triangles as we had before. Now what kind of degenerations are allowed by Gromov compactness? Recall we have assumed that the restriction of ω vanishes on $\pi_2(M, L_i)$, which means we immediately rule out disk bubbling and sphere bubbling. This leaves us with just strip breaking. So we could have strip breaking of p_1, strip breaking of p_2, and strip breaking of q. This can be seen in fig. 3.

In particular, if we look at the single dimensional part of these kind of moduli spaces, we get the relation that if the interior is a one dimensional path, then this breaking is zero dimensional, and counting it gives 0. The first kind of breaking is a term which looks like we're counting $\mu_2(\partial p_1, p_2)$, since ∂p_2 is the new point closing off the triangle with p_2 and q after this breaking. We get similar terms for the other types of breaking, so we can write:

$$\mu_2(\partial p_1, p_2) + \mu_2(p_1, \partial p_2) + \delta(\mu_2(p_1, p_2)) = 0$$

where the last term is different, since we first take $\mu_2(p_1, p_2)$, which is going to be this new point, and then the second strip between this point and q is giving us a differential. Again this is zero because it is the boundary of a single dimensional thing. Now once we pass to cohomology, we see that μ_2 induces an honest product on HF.

Remark 3. In the graded case we have already seen that this index is the difference expression in eq. (3). But now since we are counting zero dimensional things, we are counting cases where the index is 0. This means precisely that the sum of the degrees of the p_i is equal to the degree of q. In other words the product must respect gradings.

We now consider a more general family of operations, called the A_∞ operations, which are a sort of extension of these. The idea is that we are going to have some moduli spaces, where we have "input points" $\{p_i\}_{i=1}^k$, we still have q, and these still depend on some class β. J will also be domain dependent. Now we are counting sort of "holomorphic polygons" where the domain is $\mathbb{D}$ with $k+1$ punctures, and we have a J-holomorphic map u with boundary conditions where the segments between the punctures map to Lagrangians $L_0, \cdots, L_k$. We call the intersection point between L_i and L_{i+1}, p_{i+1}. Then the intersection between L_0 and L_k is q. When we just had 3 marked points, we had a nice unique disk, but now this whole story is just true up to holomorphic reparameterization.

As before, the index of the linearized operator $D_{\bar{\partial}_{J,u}}$ is still a Maslov index of some path that we get from $\mu^* TM$, and the conventions at each of the points are precisely the same as before. In the graded case, this is

$$\deg q - \sum \deg p_i \ .$$

Then it turns out that

$$\dim \mathcal{M} = k - 2 + i([u]) = k - 2 + \deg q - \sum \deg p_i$$

which is a bit more complicated because of the fact that there are moduli for the space of such marked disks. Again, we will define an operation by counting zero dimensional paths. So if we count zero dimensional such $\mathcal{M}$s, we get operations of the following form:

$$\begin{array}{rcl} \mathrm{CF}(L_0, L_1) \otimes \mathrm{CF}(L_1, L_2) \otimes \cdots \otimes \mathrm{CF}(L_{k-1}, L_k) & \xrightarrow{\mu_k} & \mathrm{CF}(L_0, L_k) \\ (p_1, p_2, \cdots, p_k) & \longmapsto & \text{combination of } q\text{'s} \end{array} .$$

While the product had degree 0 in the graded case, for this moduli space to be zero dimensional the index has to be $2-k$.

Remark 4. In the case of Floer cochain complexes where we just have a differential, we call it ∂, but ∂ naturally fits into this pattern as a μ_1.

Plenty is being swept under the rug, but it is true that once we take completely domain dependent J as we have, we can achieve transversality, which means these moduli spaces are indeed manifolds so we can do this count.

Let's see what Gromov compactness gives us in this more general situation. Let's play the same game as before, where we look at the one dimensional part of the moduli space, and we want to see what all of its boundary components look like. So if we count the boundary points of each one dimensional such $\mathcal{M}$, we get a generalization of the picture from before. We get no bubbles, and only breaking. But now this breaking can happen at any of the points. Any such breaking is allowed in the sense that we can have one break, all the way through all of them apart from one. In any case, the sum over all of these will be 0.

So this is what the boundary of the compactification of the one dimensional moduli space looks like. What does this mean for our μ_k operations? In general, we get equations of the following form:

$$\sum \mu_* \left(p_1, \cdots, p_i, \mu_\star \left(p_{i+1}, \cdots, p_j\right), p_{j+1}, \cdots, p_k\right) = 0$$

where $\mu_\star$ is doing the infinity operation on the broken portion, and μ_* is doing the operation on the main disk. Now these are what are called A_∞ relations.

Finally we have our definition:

Definition 7 (Fukaya category)**.** A category with objects L, which are compact Lagrangians with $\omega|_{\pi_2(M,L)} = 0$. The morphisms between L_0, L_1 are given by $\mathrm{CF}\left(L_0, L_1\right)$. We also have that the morphism spaces carry these A_∞ operations.

Remark 5. We have discussed/hinted at two variations throughout:

(1) We can specialize to graded objects, if $2c_2\left(M\right) = 0$.[6]
(2) We can specialize to spin Lagrangians, if we want to work with $\operatorname{char} \mathbb{K} \neq 2$.

Note that in everything so far, we have been effectively assuming our L_i are transverse, even though we saw in the Floer theory section, that the Floer cochain complex should be defined for anything. In particular, we said that if L_0, L_1 are not transverse, we need to start by applying a small Hamiltonian perturbation to one of them to make them transverse. So if we are just doing Floer cohomology of two things, it is fine to do that in a fairly ad hoc manner, but now that we have this category with all of these operations, we have to sort of set things up so we can do this sort of systematically. This is sort of heavy bookkeeping just making sure everything is consistent.

Similarly, we have been picking sort of generic almost complex Js here, but of course, for instance, when we get this splitting, the almost complex structures get inherited on either side. So when we choose generic domain dependent J, we sort of need to do this in a coherent manner. This is somehow nontrivial when we are doing the setup for this theory. And if we want to calculate something, we need to make sure we've make our choices for our calculation in a consistent manner.

Perhaps unsurprisingly, there aren't canonical choices in general for how to do this. So what happens when we make two different choices? We would get not the same A_∞ category exactly, but something called a quasi-isomorphic A_∞ category. This is a sort of natural equivalence on the level of A_∞ categories. So once we have chosen a consistent J, up to quasi-isomorphism, the Fukaya category really is an invariant of our manifold M.

[6] So we can only consider Lagrangians with 0 Maslov class.