Between electric-magnetic duality and the Langlands program

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CHAPTER 1

Overview

Lecture 1; January 19, 2021

The geometric Langlands program is some kind of middle-ground between number theory and physics. Another point of view is that we will be navigating the narrow passage between the whirlpool Charybdis (physics) and the six-headed monster Scylla (number theory), as in Odysseus' travels.[1]

The inspiration for much of this course comes from [**Mac78**], which provides a historical account of harmonic analysis, focusing on the idea that function spaces can be decomposed using symmetry. This theme has long-standing connections to physics and number theory.

The spirit of what we will try to do is some kind of harmonic analysis (fancy version of Fourier theory) which will appear in different guises in both physics and number theory.

1. Modular/automorphic forms

1.1. Rough idea. The theory of modular forms is a kind of harmonic analysis/quantum mechanics on arithmetic locally symmetric spaces. The canonical example of a locally symmetric space is given by the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane $\mathbb{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2$. I.e. we are considering the quotient

$$\mathcal{M}_{\mathrm{SL}_2 \mathbb{R}} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \tag{1}$$

as in fig. 1.

[1]One can expand this analogy. Calypso's island is probably derived algebraic geometry (DAG), etc.

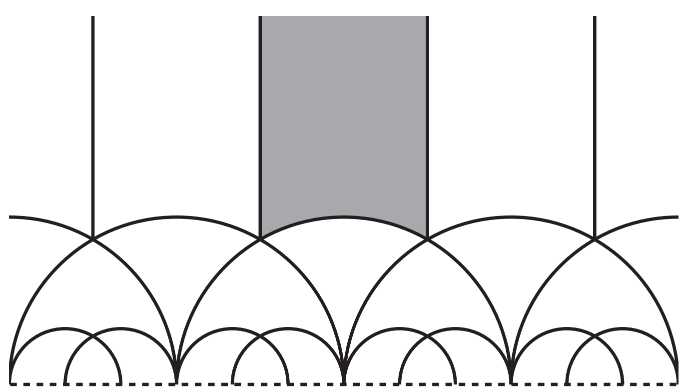

Figure 1. Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray.

For a general reductive algebraic group G we can consider the space

$$\mathcal{M}_G = \Gamma \backslash G / K \tag{2}$$

where Γ is an arithmetic lattice, and K is a maximal compact subgroup. For now we restrict to

$$G = \mathrm{SL}_2(\mathbb{R}) \qquad \Gamma = \mathrm{SL}_2(\mathbb{Z}) \qquad K = \mathrm{SO}_2 \ .$$

We want to do harmonic analysis on this space, i.e. we want to decompose spaces of functions on this in a meaningful way. In the case of quantum mechanics we're primarily interested in L^2 functions:

$$L^2(\Gamma \backslash G / K) \ , \tag{3}$$

and on this we have an action of the hyperbolic Laplace operator. I.e. we want to study the spectral theory of this operator.

The same information, possibly in a more accessible form, is given by getting rid of the K. That is, we can just study L^2 functions on

$$\Gamma \backslash G = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \ , \tag{4}$$

which is the unit tangent bundle, a circle bundle over the space we had before. Instead of studying the Laplacian, this is a homogeneous space so we can study the action of all of $\mathrm{SL}_2(\mathbb{R})$.

One can expand this to include differentials and pluri-differentials, i.e. sections of (powers of) the canonical bundle:

$$\Gamma\left(\Gamma \backslash \mathbb{H}, \omega^{k/2}\right) \ . \tag{5}$$

Definition 1. The Δ-eigenfunctions in $L^2(\Gamma \backslash G / K)$ are called *Maass forms*. *Modular forms* of weight k are holomorphic sections of $\omega^{k/2}$.

Remark 1. For a topologist, one might instead want to study (topological) cohomology (instead of forms) with coefficients in some local system (twisted coefficients). Indeed, modular forms can also arise by looking at the (twisted) cohomology of $\Gamma \backslash \mathbb{H}$. This is known as Eichler-Shimura theory.

One might worry that this leaves the world of quantum mechanics, but after passing to cohomology we're doing what is called *topological* quantum mechanics. We will be more concerned with this than honest quantum mechanics.

The idea is that there are no dynamics in this setting. We're just looking at the ground states, so the Laplacian is 0, and we're just looking at harmonic things. And this really has to do with topology and cohomology. But modular forms are some kind of ground states.

Remark 2. If we take general G, K, and Γ then we get the more general theory of *automorphic forms*.

Example 1. If we start with $G = \mathrm{Sp}_{2n}(\mathbb{R})$ and take $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, $K = \mathrm{SO}_n$ then we get *Siegel modular forms*.

1.2. Structure. There is a long history of thinking of this problem[2] as quantum mechanics on this locally symmetric space. But there is a lot more structure going on in the number theory than seems to be present in the quantum mechanics of a particle moving around on this locally symmetric space.

Restrict to the case $G = \mathrm{SL}_2(\mathbb{R})$.

[2] The problem of understanding L^2 functions on a locally symmetric space.

1.2.1. *Number field.* The question of understanding

$$L^2\left(\mathrm{SL}_2\,\mathbb{Z}\backslash\,\mathrm{SL}_2\,\mathbb{R}/\,\mathrm{SO}_2\right) \tag{6}$$

has an analogue for any number field. We can think of $\mathbb{Z}$ as being the ring of integers in the rational numbers:

$$\mathbb{Z} = \mathcal{O}_{\mathbb{Q}} \tag{7}$$

and from this we get a lattice $\mathrm{SL}_2\left(\mathcal{O}_{\mathbb{Q}}\right)$. Writing it this way, we see that we can replace $\mathbb{Q}$ by any finite extension F, and $\mathbb{Z}$ becomes the ring of integers $\mathcal{O}_F$:

$$\begin{aligned} \mathbb{Q} &\rightsquigarrow F \\ \mathbb{Z} &\rightsquigarrow \mathcal{O}_F\ . \end{aligned} \tag{8}$$

The upshot is that when we replace Q with some other number field $F/\mathbb{Q}$, then the space $\mathcal{M}_{G,\mathbb{Q}}$ becomes some space $\mathcal{M}_{G,F}$. Then we linearize by taking either L^2 or H^* of $\mathcal{M}_{G,F}$.

EXAMPLE 2. This holds for all reductive algebraic groups G, but let $G = \mathrm{PSL}_2\,\mathbb{R}$. Then

$$\mathcal{M}_{G,\mathbb{Q}} = \mathrm{PSL}_2\,\mathbb{Z}\backslash\,\mathrm{PSL}_2\,\mathbb{R}/\,\mathrm{SO}_2 \tag{9}$$

is the locally symmetric space in fig. 1. If we replace $\mathbb{Q}$ with an arbitrary number field $F/\mathbb{Q}$, then we get

$$\mathcal{M}_{G,F} = \mathrm{PSL}_2\left(\mathcal{O}_F\right)\backslash\,\mathrm{PSL}_2\left(F\otimes_{\mathbb{Q}}\mathbb{R}\right)/K\ . \tag{10}$$

Note that

$$F\otimes_{\mathbb{Q}}\mathbb{R} \simeq \mathbb{R}^{\times r_1}\times\mathbb{C}^{\times r_2} \tag{11}$$

where r_1 is the number of real embeddings of F, and r_2 is the number of conjugate pairs of complex embeddings.

EXAMPLE 3. Let $F = \mathbb{Q}\left(\sqrt{d}\right)$. If it is real $(d \geq 0)$ then $r_1 = 2$ (corresponding to $\pm\sqrt{d}$) and $r_2 = 0$, so we get

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right)\otimes_{\mathbb{Q}}\mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{R}\times\mathrm{PSL}_2\,\mathbb{R}\ . \tag{12}$$

This leads to what are called *Hilbert modular forms.*

If it is imaginary $(d < 0)$ then $r_1 = 0$, $r_2 = 1$, and

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right)\otimes_{\mathbb{Q}}\mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{C}\ . \tag{13}$$

In this case the maximal compact is $\mathrm{SO}_3\,\mathbb{R}$, and the quotient:

$$\mathbb{H}^3 = \mathrm{PSL}_2\,\mathbb{C}/\,\mathrm{SO}_3\,\mathbb{R} \tag{14}$$

is hyperbolic 3-space. Now we need to mod out (on the left) by a lattice, and the result is some hyperbolic manifold which is a 3-dimensional version of the picture in fig. 1.

REMARK 3. The point is that the real group we get after varying the number field is not that interesting, just some copies of PSL_2. But the lattice we are modding out by depends more strongly on the number field, so this is the interesting part.

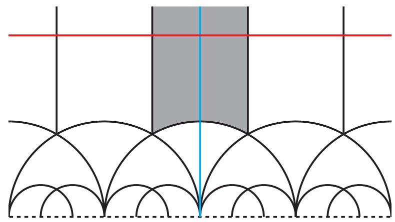

FIGURE 2. Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray. One can define a "period" as taking a modular form and integrating it, e.g. on the red or blue line.

1.2.2. *Conductor/ramification data.* Fixing the number field $F = \mathbb{Q}$, we can vary the "conductor" or "ramification data". The idea is as follows. The locally symmetric space $\Gamma\backslash\mathbb{H}$ has a bunch of covering spaces of the form $\Gamma'\backslash\mathbb{H}$, where Γ' is some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. So we can replace Γ by Γ'.

We won't define congruence subgroups in general, but there are basically two types. For $N \in \mathbb{Z}$, we fix subgroups:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N \right\} \tag{15}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \right\} . \tag{16}$$

The idea is that we start with the conductor N and the lattice Γ, and then we modify Γ at the divisors of N. Note that even in this setting we have the choice of $\Gamma(N)$ or $\Gamma_0(N)$. Really the collection of variants has a lot more structure. The local data at p has to do with the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

1.2.3. *Action of Hecke algebra.* We have seen that our Hilbert space depends on the group, the number field, and some ramification data. A very important aspect of this theory is that this vector space (of functions) carries a lot more structure. There is a huge "degeneracy" here in the sense that the eigenspaces of the Laplacian are much bigger than one might have guessed (not one-dimensional).

This degeneracy is given by the theory of *Hecke operators*. This says that the Laplacian Δ is actually a part of a huge commuting family of operators. In particular, these all act on the eigenspaces of the Laplacian. For p a prime (p unramified, i.e. $p \nmid N$) we have the Hecke operator at p, T_p. Then

$$\bigoplus_p \mathbb{C}[T_p] \circlearrowright L^2(\Gamma\backslash G/K) . \tag{17}$$

This is some kind of "quantum integrable system" because having so many operators commute with the Hamiltonian tells us that a lot of quantities are conserved.[3]

1.2.4. *Periods/states.* There is a special collection of measurements we can take of modular forms, called periods. A basic example is given by integrating a modular form on the line $i\mathbb{R}_+ \subset \mathbb{H}$ as in fig. 2. This is how Hecke defined the L-function.

[3]This example is often included in the literature as an example of quantum chaos (the opposite of integrability). The chaotic aspect has nothing to do with the discrete subgroup Γ. Specifically this fits into the study of "arithmetic quantum chaos" which more closely resembles the study of integral systems.

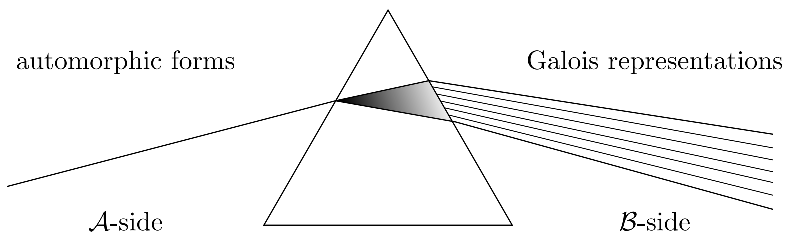

FIGURE 3. Just as light is decomposed by a prism, this spectral decomposition breaks automorphic forms ($\mathcal{A}$-side) up into Galois representations of number fields ($\mathcal{B}$-side).

The takeaway is that we have a collection of measurements/states with very good properties, and then we can study modular forms by measuring them with these periods.

1.2.5. *Langlands functoriality.* There is a collection of somewhat mysterious operators whose action corresponds to varying the group G.

2. The Langlands program and TFT

2.1. Overview. We have seen that for a choice of reductive algebraic group G and number field $F/\mathbb{Q}$, we get a locally symmetric space

$$\mathcal{M} = \mathcal{M}_{G,F} = \text{“arithmetic lattice”}\backslash\text{real group/maximal compact} . \tag{18}$$

This can be thought of as some space of G-bundles

$$\mathcal{M}_{G,F} = \text{“}\operatorname{Bun}_G(\operatorname{Spec}\mathcal{O}_F)'' . \tag{19}$$

Then we linearize this space by taking either L^2 or H^*.

Starting with this theory of automorphic forms, we spectrally decompose under the action of the Hecke algebra. Then the Langlands program says that the pieces of this decomposition correspond to Galois representations. We can think of the theory of automorphic forms as being fed into a prism, and the colors coming out on the other side are Galois representations as in fig. 3. More specifically, the “colors” are representations:

$$\operatorname{Gal}\left(\overline{F}/F\right) \to G^\vee_{\mathbb{C}} . \tag{20}$$

EXAMPLE 4. If $G = \mathrm{GL}_2\,\mathbb{R}$, then $G^\vee = \mathrm{GL}_2\,\mathbb{C}$. Let E be an elliptic curve. Then

$$H^1(E/\mathbb{Q}) \tag{21}$$

is a 2-dimensional representation of $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. This is the kind of representation you get in this setting.

EXAMPLE 5. The representations in example 4 are very specific to GL_2. If we started with $\mathrm{GL}_3(\mathbb{R})$ instead, the associated locally symmetric space $\mathrm{O}_3 \backslash \mathrm{GL}_3\,\mathbb{R}/\,\mathrm{GL}_3\,\mathbb{Z}$ is not a complex manifold.

The goal is to match all of this structure in section 1.2 with a problem in physics, but ordinary quantum mechanics will be too simple. On the physics side we will instead consider quantum field theory.

TABLE 1. Output of a four-dimensional topological field theory. Numbers are the easiest to understand, but are usually the trickiest to produce (often requires analysis). Vector spaces are also pretty simple, but three-manifolds are hard. So the sweet spot is kind of in 2-dimensions, since we understand surfaces and categories aren't that complicated.

Dimension	Output
4	$z \in \mathbb{C}$ (rarely well-defined algebraically, requires analysis)
3	(dg) vector space
2	(dg) category
1	$(\infty, 2)$-category
0	$(\infty, 3)$-category? (rarely understood)

Slogan: the Langlands program is part of the study of 4-dimensional (arithmetic, topological) quantum field theory.

The idea is that the Langlands program is an equivalence of 4-dimensional arithmetic topological field theories (TFTs):

$$\begin{array}{ccc} \mathcal{A}_G & \simeq & \mathcal{B}_{G^\vee} \\ \text{automorphic} & & \text{spectral} \\ \text{magnetic} & & \text{electric} \end{array} \tag{22}$$

called the $\mathcal{A}$ and $\mathcal{B}$-side theories.

REMARK 4. This is what one might call "four-dimensional mirror symmetry". The $\mathcal{A}$ and $\mathcal{B}$ are in the same sense as usual mirror symmetry.

An n-dimensional TFT is a beast which assign a quantum mechanics problem (or just a vector space, chain complex, etc.) to every $(n-1)$-manifold. So a 4-dimensional TFT sends a 3-manifold to some kind of vector space. It assigns more complicated data to lower-dimensional manifolds and less complicated data to higher-dimensional manifolds as in table 1.

The *topological* means we are throwing out the dynamics and only looking at the ground states. This is the analogue of only looking at the harmonic forms rather than the whole spectrum of the Laplacian. The *arithmetic* means that we're following the paradigm of *arithmetic topology.* The idea is that we will eventually make an analogy between number fields and three-manifolds. Then we can plug a number field into the TFT (instead of an honest manifold) to get a vector space which turns out to be $L^2(\mathcal{M}_{G,F})$ (or $H^*(\mathcal{M}_{G,F})$).

2.2. Arithmetic topology.

2.2.1. *Weil's Rosetta Stone.* In a letter to Simone Weil [**Kri05**], André Weil explained a beautiful analogy, now known as *Weil's Rosetta Stone.* This establishes a three-way analogy between number fields, function fields, and Riemann surfaces.

The general idea is as follows. $\operatorname{Spec}\mathbb{Z}$ is some version of a curve, with points $\operatorname{Spec}\mathbb{F}_p$ associated to different primes. $\operatorname{Spec}\mathbb{Z}_p$ is a version of a disk around the point, and $\operatorname{Spec}\mathbb{Q}_p$ is a version of a punctured disk around that point. This is analogous to the usual picture of an algebraic curve.

Curve	$\operatorname{Spec}\mathbb{F}_q[t]$	$\operatorname{Spec}\mathbb{Z}$
Point	$\operatorname{Spec}\mathbb{F}_p$	$\operatorname{Spec}\mathbb{F}_p$
Disk	$\operatorname{Spec}\mathbb{F}_t[[t]]$	$\operatorname{Spec}\mathbb{Z}_p$
Punctured disk	$\operatorname{Spec}\mathbb{F}_q((t))$	$\operatorname{Spec}\mathbb{Q}_p$

In general, let $F/\mathbb{Q}$ be a number field. Then we can consider $\mathcal{O}_F$, and $\operatorname{Spec}\mathcal{O}_F$ has points corresponding to primes in $\mathcal{O}_F$. The analogy between number fields and function fields is as follows. Start with a smooth projective curve $C/\mathbb{F}_q$ over a finite field. Then the analogue to F is the field of rational functions, $\mathbb{F}_q(C)$. The analogue to $\mathcal{O}_F$ is the ring of regular functions, $\mathbb{F}_q[C]$. Finally points of $\operatorname{Spec}\mathcal{O}_F$ correspond to points of C.

Now we might want to replace C with a Riemann surface. So let $\Sigma/\mathbb{C}$ be a compact Riemann surface. Then primes in $\mathcal{O}_F$ (and so points of C) correspond to points of Σ. The field of meromorphic rational functions on Σ, $\mathbb{C}(\Sigma)$, is the analogue of F. To get an analogue of $\mathcal{O}_F$ we have to remove some points of Σ (we wouldn't get any functions on the compact curve). The point is that number fields have some points at ∞, so the analogue isn't really a compact Riemann surface, but with some marked points. So the analogue of $\mathcal{O}_F$ consists of functions on Σ which are regular away from these points.

This is summarized in table 2.

Table 2. Weil's Rosetta stone, as it was initially developed, establishes an analogy between these three columns. We will eventually refine this dictionary. Let $F/\mathbb{Q}$ be a number field, $C/\mathbb{F}_q$ be a smooth projective curve over a finite field, and let $\Sigma/\mathbb{C}$ be a compact Riemann surface. $\mathbb{F}_q(C)$ denotes the field of rational functions, $\mathbb{F}_q[C]$ denotes the ring of regular functions, and $\mathbb{C}(\Sigma)$ denotes the meromorphic rational functions on Σ.

Number fields	Function fields	Riemann surfaces
$F/\mathbb{Q}$	$\mathbb{F}_q(C)$	$\mathbb{C}(\Sigma)$
$\mathcal{O}_F$	$\mathbb{F}_q[C]$	f'ns regular away from marked points of Σ
$\operatorname{Spec}\mathcal{O}_F$	points of C	$x \in \Sigma$

2.2.2. *Missing chip.* Now we want to take the point of view that there was a chip missing from this Rosetta stone, and we were supposed to consider 3-manifolds rather than Riemann surfaces. The idea is that $\Sigma/\mathbb{C}$ really corresponds to $C/\overline{\mathbb{F}_q}$. This is manifested in the following way. To study points, we study maps:

$$\operatorname{Spec}\mathbb{F}_q \hookrightarrow C\ . \tag{23}$$

But from the point of view of étale topology, $\operatorname{Spec}\mathbb{F}_q$ is not really a point. It is more like a circle in the sense that

$$\operatorname{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) = \widehat{\mathbb{Z}} = \pi_1^{\text{étale}}\left(\operatorname{Spec}\mathbb{F}_q\right) \tag{24}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion. So it's better to imagine this as a modified circle, where this $\widehat{\mathbb{Z}}$ is generated by the Frobenius. There is always a map

$$\operatorname{Spec}\overline{\mathbb{F}_q} \to \operatorname{Spec}\mathbb{F}_q \tag{25}$$

and we can lift our curve to $\overline{\mathbb{F}}_q$. This corresponds to unwrapping these circle, i.e. replacing them by their universal cover. So their is some factor of $\mathbb{R}$ which doesn't play into the topology/cohomology. So we have realized that curves over $\mathbb{F}_q$ have too much internal structure to match with a Riemann surface.

REMARK 5. The map $\operatorname{Spec}\mathbb{F}_{q^n} \to \operatorname{Spec}\mathbb{F}_q$ is analogous to the usual n-fold cover of the circle.

To fix the Rosetta Stone, we replace a Riemann surface Σ by certain a Σ-bundle over S^1. Explicitly, if we have Σ and a diffeomorphism φ, we can form the mapping torus:

$$\Sigma \times I / \left((x,0) \sim (\varphi(x), 1)\right) . \tag{26}$$

The idea is that if we start with a curve over a finite field, the diffeomorphism φ is like the Frobenius.

Similarly $\operatorname{Spec}\mathcal{O}_F$ looks like a curve where each "point" carries a circle. So this is again some kind of 3-manifold.

REMARK 6. These circles don't talk to one another because they all have to do with a Frobenius at a different prime. So they're less like a product or a fibration, and more like a 3-manifold with a foliation.

This fits with the existing theory of arithmetic topology, sometimes known as the "knots and primes" analogy. The theory was started in a letter from Mumford to Mazur, but can be attributed to many people such as Mazur [**Maz73**], Manin, Morishita [**Mor10**], Kapranov [**Kap95**], and Reznikov. The recent work [**Kim15, CKK$^+$19**] of Minhyong Kim plays a central role.

REMARK 7. Lots of aspects of this dictionary are spelled out, but one should be wary of using it too directly. Rather we should think of this as telling us that there are several classes of '3-manifolds': ordinary 3-manifolds, function fields over finite fields, and number fields.

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2.2.3. *Updated Rosetta Stone.* The upshot is that we are thinking of all three objects in the Rosetta stone as three-manifolds. In particular, we're thinking of $\operatorname{Spec}\mathcal{O}_F$ (e.g. $\operatorname{Spec}\mathbb{Z}$) as a 3-manifold, so for any prime p we have the loop $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathcal{O}_F$, which we can interpret as a knot in the 3-manifold. Let F_v be the completion of the local field F at the place v (e.g. $\mathbb{Q}_p$). Then $\operatorname{Spec} F_v$ turns out to be the boundary of a tubular neighborhood of the knot. The point is that if F_v is a non-Archimedean local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$) then the "fundamental group" is the Galois group, and it has a quotient:

$$\operatorname{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} . \tag{27}$$

This group is called a *Baumslag-Solitar* group. Explicitly it is:

$$\operatorname{BS}(1,p) = \left\{\sigma, u \,\middle|\, \sigma u \sigma^{-1} = u^p\right\} \tag{28}$$

where we think of σ as the Frobenius, so corresponding to $\widehat{\mathbb{Z}}$, and u as the generator of $\mathbb{Z}_\ell$. The kernel is

$$p\text{-group} \times \prod_{\ell^r \neq \ell, p} \mathbb{Z}_{\ell^r} \hookrightarrow \operatorname{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} \ . \tag{29}$$

This tells us that there is For $p = 1$, this group is $\mathbb{Z} \times \mathbb{Z} = \pi_1\left(T^2\right)$. For $p = -1$, this is the fundamental group of the Klein bottle. This is evidence that the étale fundamental group of $\operatorname{Spec} F_v$ looks like some kind of p-dependent version of the fundamental group of the torus. So we can think of $\operatorname{Spec} F_v$ as a 2-manifold (fibered over S^1).

After this discussion we can identify an updated (multi-dimensional) Rosetta Stone. In three-dimensions we have: $\operatorname{Spec} \mathcal{O}_F$, $C/\mathbb{F}_q$, and a mapping torus $T_\varphi(\Sigma)$. The first two comprise the *global arithmetic setting.* In two-dimensions we first have local fields, which come in two types. One is finite extensions $F_v/\mathbb{Q}_p$ and the other is $\mathbb{F}_q((t))$. Spec of either of these is "two-dimensional" and the latter is some kind of punctured disk D^*. These two comprise the *local arithmetic setting.* A curve $\overline{C}/\overline{\mathbb{F}_q}$ over an algebraically closed field (of positive characteristic) and a Riemann surface (projective curve Σ over $\mathbb{C}$) are also both "two-dimensional". These comprise the *global geometric setting.* The only 4-manifolds we will consider are of the form $M^3 \times I$ or $M^3 \times S^1$ where M^3 is a three-dimensional object of arithmetic or geometric origin. This discussion is summarized in table 3.

2.3. $\mathcal{A}$-side. The $\mathcal{A}$-side (or automorphic/magnetic side) TFT $\mathcal{A}_G$ is a huge machine which does many things, as in table 1. So far, the only recognizable thing is that it sends a 3-manifold M to some vector space $\mathcal{A}_G(M)$. We're thinking of $\operatorname{Spec} \mathcal{O}_F$ as a 3-manifold, and the assignment is the vector space we've been discussing:

$$\mathcal{A}_G(\operatorname{Spec} \mathcal{O}_F) = L^2(\mathcal{M}_{G,F}) \text{ or } H^*(\mathcal{M}_{G,F}) \ . \tag{30}$$

Remark 8. As suggested in eq. (19), note that $\Gamma \backslash G/K$ is a moduli space of something over the 3-manifold in question, not the 3-manifold itself.

2.4. Structure (reprise). As it turns out, all the bells and whistles from the theory of automorphic forms in section 1.2 line up perfectly with the bells and whistles of TFT.

2.4.1. *Number field.* The assignment in eq. (30) formalizes the idea that we got a vector space $L^2(\mathcal{M}_{G,F})$ labelled by a group and a number field.

2.4.2. *Conductor/ramification data.* Recall the ramification data was a series of primes. This is manifested as a link (collection of knots) in the 3-manifold, where we allow singularities. These appear as *defects* (of codimension 2) in the physics. So the structure we saw before is manifested as defects of the theory.

2.4.3. *Hecke algebra.* The Hecke operators correspond to line defects (codimension 3) in the field theory. Physically this is "creating magnetic monopoles" alone some loop in spacetime.

2.4.4. *Periods/states.* These correspond to boundary conditions, i.e. codimension 1 defects.

2.4.5. *Langlands functoriality.* Passing from G to H can be interpreted as crossing a domain walls (also a codimension 1 defect).

Table 3. The columns correspond to the three aspects of Weil's Rosetta Stone, and the rows correspond to dimension. The four-dimensional objects we consider are just products of three-dimensional objects with S^1 or I. M^3 is a three-dimensional object of arithmetic or geometric origin. The three-dimensional objects are number fields, function fields and mapping tori of Riemann surfaces. F is a number field, φ is some diffeomorphism of Σ, and T_φ denotes the corresponding mapping torus construction. The two-dimensional objects are local fields and curves. F_v is a finite extension of $\mathbb{Q}_p$. The 1-dimensional objects are both versions of circles, and the 0-dimensional objects are points.

Dimension	Number fields	Function fields	Geometry
4	$M^3 \times S^1$, $M^3 \times I$		
3	Global arithmetic		-
	$\operatorname{Spec} \mathcal{O}_F$	$C/\mathbb{F}_q$	$T_\varphi(\Sigma)$
2	Local arithmetic		Global geometric
	$\operatorname{Spec} F_v$	$\operatorname{Spec} \mathbb{F}_q((t)) = D^*$	$\overline{C}/\overline{\mathbb{F}}_q, \quad \Sigma/\mathbb{C}$
1	-		Local geometric
	$\operatorname{Spec} \mathbb{F}_q$		$D^*_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}((t))$, $D^*_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q((t))$
0	$\operatorname{Spec} \overline{\mathbb{F}_q}$		$\operatorname{Spec} \mathbb{C}$

2.5. $\mathcal{B}$-side. The $\mathcal{B}$-side (or spectral side) is the hard part from the point of view of number theory because Galois groups of number fields (and their representations) are very hard. I.e. the $\mathcal{B}$-side is the question, and the $\mathcal{A}$-side is the answer. But from the point of view of geometry, it is the other way around because fundamental groups of Riemann surfaces are really easy.

The $\mathcal{B}$-wide is about studying the algebraic geometry of spaces of Galois representations.

Recall that given a three-manifold (or maybe a number field F) the $\mathcal{A}$-side is concerned with the topology of the arithmetic locally symmetric space $\mathcal{M}_{G,F}$. $\mathcal{M}_{G,F}$ has to do with the geometry of F, so the $\mathcal{A}$-side is concerned with the topology of the geometry of F.

The $\mathcal{B}$-side concerns itself with the algebra of the topology of F. This means the following. For a manifold M (of any dimension), we can construct $\pi_1(M)$. Then the collection of rank n local systems on M is:

$$\operatorname{Loc}_n M = \{\pi_1(M) \to \operatorname{GL}_n \mathbb{C}\} \ . \tag{31}$$

A local system looks like a locally constant sheaf of rank n (or vector bundles with flat connection). These are sometimes called *character varieties.* Then we can study $\mathbb{C}[\operatorname{Loc}_n M]$. We can also replace GL_n with our favorite complex Lie group

G to get:

$$\mathrm{Loc}_{G^\vee} M = \{\pi_1(M) \to G^\vee\} \ . \tag{32}$$

This depends only on the topology of M.

If we're thinking of a number field as a three-manifold, then π_1 is a stand-in for the Galois group so this is a space of representations of Galois groups. The TFT sends any three-dimensional M^3 to functions on $\mathrm{Loc}_{G^\vee}$:

$$\mathcal{B}_{G^\vee}\left(M^3\right) = \mathbb{C}\left[\mathrm{Loc}_{G^\vee} M\right] \ . \tag{33}$$

Remark 9. This side was a lot easier to write down than the $\mathcal{A}$-side, but if M is a number field, the Galois group is potentially very hard to understand. All the other bells and whistles are also easy to define here.

2.6. All together. In all of the setting in table 3, we can either make and automorphic measurement (attach $\mathcal{M}_{G,F}$ and study its topology) or we could take the Galois group (or π_1), construct a variety out of it, and study algebraic functions on it. The idea we will explain is that the Langlands program is an equivalence of these giant packages, but for "Langlands dual groups" G and $G^\vee$:

$$\mathcal{A}_G \simeq \mathcal{B}_{G^\vee} \ . \tag{34}$$

Remark 10. More is proven in the geometric setting than the arithmetic, but even geometric Langlands for a Riemann surface is still an open question.

This is really a conjectural way of organizing a collection of conjectures.

CHAPTER 2

Spectral decomposition

Lecture 3; January 26, 2021

1. What is a spectrum?

The basic idea is that we start in the world of geometry, meaning we have a notion of a "space" (e.g. algebraic geometry, topology, ...), and given one of these spaces X we attach some kind of collection of functions $\mathcal{O}(X)$. These functions can have many different flavors, but they always form some kind of commutative algebra, possibly with even more structure. We will access the geometry of spaces using these functions. The operation $\mathcal{O}$ turns out to be a functor, i.e. if we have a morphism $\pi\colon X \to Y$ of spaces, we get a pullback morphism $\pi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$.

A fundamental question about this setup is to what degree we can reverse this operation. So starting with any commutative algebra, we would like to understand the extent to which we can we get geometry out of it. Category theory tells us that the right sort of thing to consider is a right adjoint to $\mathcal{O}$, which we call Spec. The fact that they form an adjunction means:

$$\mathrm{Map}_{\mathbf{Spaces}}(X, \mathrm{Spec}\, R) = \mathrm{Hom}_{\mathbf{Ring}^{\mathrm{op}}}(\mathcal{O}(X), R) = \mathrm{Hom}_{\mathbf{Ring}}(R, \mathcal{O}(X)) \ . \tag{35}$$

In the language of analysis, we might regard $\mathrm{Spec}\, R$ as a "weak solution" in the sense that it's a formal solution to the problem of finding a space associated to R. It is a functor assigning a set to any test space, but there is no guarantee that there is an honest space out there which would agree with it.

Remark 11. We might have to adjust the categories we're considering so that

$$\mathcal{O}(\mathrm{Spec}\, R) = R \ , \tag{36}$$

since this doesn't just fall out of the adjunction.

The point is that for some nice class of spaces, one might hope that we can recover a space from functions on that space: $X = \mathrm{Spec}\, \mathcal{O}(X)$.

Remark 12. The word spectrum is used in many places in mathematics. They are basically all the same, except spectra from homotopy theory.

1.1. Finite set. Let X be a finite set and k a field. Then the k-valued functions $\mathcal{O}(X)$ can be expressed as

$$\mathcal{O}(X) = \prod_{x \in X} k \tag{37}$$

which can be thought of as diagonal $X \times X$ matrices.

1.2. Compactly supported continuous functions. Gelfand developed the following version of this philosophy. For our category of spaces, consider the category of locally compact Hausdorff spaces X with continuous maps as the morphisms. For our space of functions we take $C_v(X)$, which is the space of continuous $\mathbb{C}$-valued functions which vanish at ∞. This is in the category of commutative C^*-algebras. These are Banach $\mathbb{C}$-algebras with a $*$ operation (to be thought of as conjugation) which is $\mathbb{C}$-antilinear and compatible with the norm. Given any commutative C^*-algebra A, the associated spectrum m-Spec A is called the *Gelfand spectral*, and as a set it consists of the maximal ideals in A. We can write this as:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) \tag{38}$$

i.e. the unitary 1-dimensional representations of A. The adjunction is then saying that:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) = \mathrm{Map}(\mathrm{pt}, \text{m-Spec}\, A) \ . \tag{39}$$

Theorem 1 (Gelfand-Naimark). *C_v and* m-Spec *give an equivalence of categories.*

See e.g. [**aHRW10**] for more details on this theorem.

1.3. Measure space. There is a coarser version where we start with a measure space X, and take attach the bounded functions $L^\infty(X)$. This forms a commutative von Neumann algebra. Again this is an equivalence of categories.

1.4. Algebraic geometry. We will focus on the setting of algebraic geometry. The category on the commutative algebra side will just be the category **cRing** of commutative rings. The geometric side will be the category of locally ringed spaces. This just means that there is a notion of evaluation at each point.

The functor:

$$(X, \mathcal{O}_X) \mapsto \mathcal{O}(X) \tag{40}$$

has an adjoint:

$$\mathrm{Spec}\, R \leftarrow\!\shortmid R \ . \tag{41}$$

Affine schemes comprise the image of Spec:

$$\begin{tikzcd} \text{locally-ringed spaces} \arrow[r] & \mathbf{cRing} \arrow[dl, "\mathrm{Spec}"] \\ \text{affine schemes} \arrow[u, hook] & \end{tikzcd} \ . \tag{42}$$

Then it is essentially built into the construction that affine schemes are equivalent to commutative rings.

This doesn't really capture a lot of what we want to study in algebraic geometry, because one is usually interested in general schemes, which are locally ringed spaces that locally look affine. One way to deal with this is to really think of our geometric objects as:

$$\mathrm{Fun}(\mathbf{Ring}^{\mathrm{op}}, \mathbf{Set}) = \mathrm{Fun}(\mathbf{Aff}, \mathbf{Set}) \subset \mathrm{Geometry} \ . \tag{43}$$

1.5. Topology. In homotopy theory we might start with the homotopy category of spaces and pass to some notion of functions, e.g. cohomology $H^*(X;\mathbb{Z})$. This sits in the category of graded commutative rings. This doesn't directly lead to a nice spectral theory, but it does if we remember a bit more structure. Instead, start with the category of spaces with continuous maps. Then we can take rational chains, $C^*_{\mathbb{Q}}$, and get a commutative differential graded $\mathbb{Q}$-algebra. This is the Quillen-Sullivan rational homotopy theory.[1] As it turns out, the category of simply connected spaces up to rational homotopy equivalence is equivalent to commutative differential graded algebras which are $\mathbb{Q}$ in degree 0 and 0 in degree 1.

2. Spectral decomposition

Let R be a commutative ring over some field k. Commutative rings usually arise via the study of modules over it. So let $V \in R$-**Mod**, i.e. a map

$$R \to \mathrm{End}(V) \ . \tag{44}$$

Now we want to decompose V into some sheaf $\underline{V}$ over $\mathrm{Spec}\, R$. We use that R-**Mod** is symmetric monoidal, so we have a tensor product $\otimes_R$ and we can define

$$\underline{V}(U) = V \otimes_R \mathcal{O}(U) \tag{45}$$

for open $U \subset \mathrm{Spec}\, R$. Then we can talk about the *support* of $v \in V$:

$$\mathrm{Supp}(v) \subset X \ . \tag{46}$$

Example 6. If X is finite, so $R = \prod_{x\in X} k$, then by asking for vectors supported at a single point, we get a decomposition $V = \oplus_{x\in X} V_x$. But to every point x, we get an evaluation morphism:

$$\lambda\colon R \xrightarrow{\mathrm{ev}_x} k \tag{47}$$

associated to the point x. Just like before we can think of this as a one-dimensional R-module, since $k = \mathrm{End}(k)$. Therefore, changing notation, we can write the decomposition as

$$V = \oplus_{\lambda\in X} V_\lambda w \ . \tag{48}$$

In this language the spaces in the decomposition are just the λ-eigenspaces:

$$V_\lambda = \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(k_\lambda, V) = \{v \in V \mid r\cdot v = \lambda(r)\, v\} \tag{49}$$

where k_λ is the one-dimensional module over R, where R acts via the map λ.

This description is via evaluation at a point, but points are both open and closed so we can also describe this as restriction:

$$V_\lambda = V \otimes_{\mathcal{O}(X)=R} k_\lambda \ , \tag{50}$$

so this vector spaces is realized as both a Hom and a tensor because these points happened to be both open and closed.

Define the category of quasi-coherent sheaves to be:

$$\begin{array}{ccc} \mathbf{QCoh}(X = \mathrm{Spec}\, R) & := & R\text{-}\mathbf{Mod} \\ \big\uparrow & & \\ \mathbf{Coh}(X) & := & R\text{-}\mathbf{Mod}_{\text{f.g.}} \end{array} \ . \tag{51}$$

[1]This should also be attributed to Mandell. See also Yuan's paper [**Yua19**].

Note that in general X is only locally of the form $\operatorname{Spec} R$, so $\mathbf{QCoh}(X)$ is only locally of the form R-**Mod**.

3. The spectral theorem

3.1. Algebraic geometry version. Let V be a vector space and $M \in \operatorname{End} V$. We can think of $M \in \operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{End} V)$, but $\operatorname{End} V$ is not just a set. It is an associative algebra over k, so really

$$M \in \operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{Forget} \operatorname{End} V) \ . \tag{52}$$

This sets us up for an adjunction with the free k-algebra construction. The free algebra in one generator is just $k[x]$ so the adjunction says that:

$$\operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{Forget} \operatorname{End} V) = \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(k[x], \operatorname{End} V) \ . \tag{53}$$

What we have seen here is that equipping V with some $M \in \operatorname{End} V$ is equivalent to making V a module over $k[x]$. I.e. equipping V with $M \in \operatorname{End} V$ is equivalent to V being global sections of some quasi-coherent sheaf on $\operatorname{Spec} k[x] = \mathbb{A}^1$. I.e. V spreads out over $\mathbb{A}^1$.

To make this precise, assume V is finitely generated, i.e. the sheaf $\underline{V}$ is coherent. Then V has a sort of decomposition as a quotient and a subspace:

$$V_{\text{torsion}} \hookrightarrow V \twoheadrightarrow V_{\text{tor. free}} \tag{54}$$

where

$$V_{\text{torsion}} = \bigcup_{\lambda \in \mathbb{A}^1} \{v \in V \mid \operatorname{Supp} v = \lambda\} \ . \tag{55}$$

For general modules over a PID ($k[x]$ is a PID) we have:

$$V \simeq \underbrace{V_{\text{tor}}}_{\text{discrete spectrum}} \oplus \underbrace{V_{\text{free}}}_{\text{continuous spectrum}} \tag{56}$$

where

$$V_{\text{free}} = k[x]^{\oplus r} \tag{57}$$

and

$$V_{\text{tor}} = \bigoplus_{\lambda \in \operatorname{Spec}} V_{\hat{\lambda}} \tag{58}$$

where $V_{\hat{\lambda}}$ is the subspace supported at λ. As it turns out

$$V_{\hat{\lambda}} = \bigoplus_i k[x] / (x-\lambda)^{\ell_i} \ , \tag{59}$$

i.e. for any element of this some power of $(x-\lambda)$ annihilates it, so these are generalized eigenspaces. This decomposition is precisely the Jordan normal form.

An eigenvector is some element $v \in V$ such that $\lambda v = xv$ (or by definition Mv). But this is the same as an element of:

$$\operatorname{Hom}_{k[x]}(k_\lambda, V) \ . \tag{60}$$

So this is what one might call a section supported "scheme-theoretically" at $\lambda \in \mathbb{A}^1$.

On the other hand, the fibers:

$$V \otimes_{k[x]} k_\lambda \tag{61}$$

are naturally quotients of V (rather than a sub), and so they're some kind of coeigenvectors.

In the continuous spectrum there are no eigenvectors: as a free module, $k[x]$ doesn't contain any eigenvectors.

EXAMPLE 7. Consider the free case. It is sufficient to consider $V = k[x]$, since otherwise it is just a direct sum of copies of this. Then $\underline{X} = \mathcal{O}_{\mathbb{A}^1}$. There are no generalized eigenvectors (because $(x-\lambda)^N = 0$ for some $N \gg 0$ and some $\lambda \in \mathbb{A}^1$). There are lots of coeigenvectors, though. For any $\lambda \in \mathbb{A}^1$ we have a map

$$V \twoheadrightarrow k_\lambda \ . \tag{62}$$

This is a distribution, i.e. an element of

$$\mathrm{Hom}(V, k) \ . \tag{63}$$

So for every $\lambda \in \mathbb{A}^1$, we get

$$\mathrm{Hom}(V, k) \ni \delta_\lambda \colon V \twoheadrightarrow k_\lambda \ . \tag{64}$$

The basic example is $V = L^2(\mathbb{R})$ and $M = x$. Then V_λ consists of functions supported at x, but there are none. We would like to say δ_λ, but this is not L^2. The dual operator $M^\vee = d/dx$ has eigenvectors which are roughly $e^{i\lambda x}$, but these are not L^2 either.

This is what continuous spectra look like. When you decompose functions on $\mathbb{A}^1$ under the action of x or d/dx, there is a sense in which it is a direct integral, which is different from a direct sum. The things you're integrating aren't actually subsets. So we can think of functions on $\mathbb{A}^1$ as being some kind of continuous direct sum of functions on a point, but those functions don't live as subspaces. In this case they lived as quotients, but not subspaces. This is simpler in the torsion-free case, but is a general feature of continuous spectra. This is not a weird/special fact about analysis, because we see it even at the level of algebra (polynomials).

3.2. Measurable version. Instead of the matrix M, we consider a self-adjoint operator A on a Hilbert space $V = \mathcal{H}$. Then von Neumann's spectral theorem tells us that there is a "sheaf" (projection valued measure) π_A on $\mathbb{R}$ and

$$A = \int_{\mathbb{R}} x d\pi_A \ . \tag{65}$$

A projective valued measure can be thought of as a sheaf $\underline{\pi_A}$ as follows. For $U \subset \mathbb{R}$ measurable, we attach the image under the projection:

$$\underline{\pi_A}(U) = \pi_A(U) \ . \tag{66}$$

So this is some kind of sheaf of Hilbert spaces. There is no topology to be compatible with, but it does satisfy the additivity property that the rule:

$$U \mapsto \langle w, \pi_A(U) v \rangle \tag{67}$$

defines a $\mathbb{C}$-valued measure on $\mathbb{R}$. So this is the version of a sheaf in the measurable world.

So now eq. (65) is saying that the Hilbert space $\mathcal{H}$ sheafifies over $\mathbb{R}$ in such a way that A acts by the coordinate function x, just like in the algebro-geometric setting above. Then the spectrum is a measurable subset

$$\mathrm{Spec}(A) \coloneqq \mathrm{Supp}\, \pi_A \subset \mathbb{R} \ . \tag{68}$$

EXAMPLE 8. If $\mathcal{H}$ is finite-dimensional then the spectrum is a discrete set of points, and the decomposition is just into eigenspaces.

3.3. Homotopical version. We saw that we had a quasi-coherent sheaf in algebraic geometry, a projection-valued measure in measure theory, and now in algebraic topology we have the following. If $R = C^*(X)$, then

$$R\text{-}\mathbf{Mod} \hookrightarrow \operatorname{Loc}(X) \tag{69}$$

where $\operatorname{Loc}(X)$ consists of locally-constant complexes on X.

The basic idea is that if we have a model for cochains on X:

$$\to R^{\oplus j} \to R^{\oplus i} \to M \tag{70}$$

then we get a presentation of $\underline{M}$ by constant sheaves:

$$\to \underline{k_X}^{\oplus j} \to \underline{k_X}^{\oplus i} \to \underline{M} \tag{71}$$

where the key point is that

$$C^*(X) = \operatorname{End}^*\left(\underline{k_X}, \underline{k_X}\right) \ . \tag{72}$$

Or more directly we can define the sheaf to be:

$$\underline{M}(U) = M \otimes_{C^*(X)} C^*(X) \ . \tag{73}$$

Lecture 4; January 28, 2021

3.4. Physics interpretation: observables and states. We have seen that, starting with some flavor of commutative algebra A, we can construct a geometric object $\operatorname{Spec} A$. Then a module M over A gets spread out into a sheaf over $\operatorname{Spec} A$. The algebra A can be thought of as the *algebra of observables* of some physical system. Then the *space of states* forms a module over A, and so fits into this framework.

Let's back up. The general idea is that we're trying to get a grip on the geometry of this space via the functions on it, i.e. making observations. Recall that the defining property of $\operatorname{Spec} A$ is that whenever we have a space X and a map $A \to \mathcal{O}(X)$, where A is commutative, then we get a map

$$X \to \operatorname{Spec} A \ . \tag{74}$$

We can think of the map $A \to \mathcal{O}(X)$ as picking out some functions (observables) on the space which satisfy the relations of A. E.g. if we just have one function, this is a map from the space down to the line, and then the space will decompose over this. In general the space will decompose over a higher-dimensional base. So this is a way of measuring the space with functions.

Spectral decomposition of modules is a linearized version of this. We replace X by a linearized version of it, e.g. $\mathcal{O}(X)$, and this becomes a module over A. I.e. a module M over A is a linearized version of a map $X \to \operatorname{Spec} A$. If we just have a single function, then this corresponds to a map $X \to \operatorname{Spec} k[x] = \mathbb{A}^1$. Likewise, a single matrix (endomorphism of a vector space) gives rise to a sheaf over $\mathbb{A}^1$.

In quantum mechanics, we don't have a phase space. We only have a vector space $\mathcal{H}$ (some linearized version of the phase space), called the *Hilbert space of states*. Observables are operators on $\mathcal{H}$. In physics we're interested in reality, so we might insist on the condition that observables are self-adjoint:

$$\mathcal{O} = \mathcal{O}^* \ . \tag{75}$$

Typically we won't impose this condition. For an observable $\mathcal{O} \circlearrowright \mathcal{H}$, spectral decomposition tells us that $\mathcal{H}$ sheafifies (as a projection-valued measure) over $\mathbb{R}$. The base is $\mathbb{R}$ because this is Spec of the algebra generated by a single operator. This is the analogue of starting with a classical phase space M and a single observable $M \to \mathbb{R}$, and then decomposing M over $\mathbb{R}$.

A *state* is an element $|\varphi\rangle \in \mathcal{H}$. Given a state and an operator $\mathcal{O}$, this state becomes a section of the sheaf $\mathcal{H}$, i.e. we get an eigenspace decomposition of this vector. Given a section, the first thing we can ask for is the support. This is just where the measurement we made "lives".

We can do something more precise by using the norm. As it turns out, $\|\varphi\|^2$ is a probability measure on $\mathbb{R}$ which tells us where to expect the state to be located. For example, we can take the expectation value of the observable $\mathcal{O}$ in the state φ:

$$\langle \mathcal{O} \rangle_\varphi = \frac{\langle \varphi | \mathcal{O} | \varphi \rangle}{\langle \varphi | \varphi \rangle} \ . \tag{76}$$

This is a continuous version of

$$\frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda \in \operatorname{Spec} \mathcal{O}} \lambda \left\| \operatorname{Proj}_{\mathcal{H}_\lambda} |\varphi\rangle \right\|^2 = \frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda, \psi_i} \lambda \langle \psi_i | \varphi \rangle |\psi_i\rangle \tag{77}$$

where ψ_i is a basis of eigenvectors.

Remark 13. To give a quantum-mechanical system, we also need to specify the *Hamiltonian* H. This is a specific observable (self-adjoint operator on $\mathcal{H}$) which plays the role of the energy functional. The eigenstates for H are the steady states of the system. This lets us spread $\mathcal{H}$ out over $\mathbb{R}$ to get the energy eigenstates. We will be working in the *topological* setting where $H = 0$, i.e. we're just looking at the 0 eigenspace. So this decomposition is kind of orthogonal to our interests.

4. Fourier theory/abelian duality

We have seen that whenever we have a "spectral dictionary", we get a notion of spectral decomposition: modules become sheaves, where the notion of a sheaf depends on the context. This gives us a way of spreading out the algebra of modules over the geometry or topology of our space.

For this to be useful, we need interesting sources of commutative algebras. A natural source for commuting operators is when we have an abelian group G acting on a vector space V: given a morphism

$$\rho\colon G \to \operatorname{Aut}(V) \tag{78}$$

we get a family of operators $\{\rho(g)\}_{g \in G}$ and we can spectrally decompose V using these operators. This is what Fourier theory is about. So we're thinking of Fourier theory as some kind of special case of spectral decomposition.

4.1. Characters.

Let V be a representation of an abelian group G, i.e. we have a map

$$\begin{aligned} G &\longrightarrow \operatorname{Aut}(X) \ \subset \ \operatorname{End}(V) \\ g &\longmapsto T_g \end{aligned} \tag{79}$$

such that $T_g T_h = T_{gh}$.

EXAMPLE 9. If G acts on a space X, and V is functions on X, then we get an action of G on V.

EXAMPLE 10. G always acts on itself, so therefore it acts on functions on G itself. This is the *regular representation*.

Now we want to spectrally decompose. First we need to know what the spectrum is, so we ask the following question.

QUESTION 1. What are the possible eigenvalues?

Let $v \in V$ be an eigenvector:

$$g \cdot v = \chi(g) v \tag{80}$$

where

$$\chi \colon G \to \operatorname{Aut} \mathbb{C}V = \mathbb{C}^\times \subset \mathbb{C} \tag{81}$$

is a group homomorphism, i.e. a *character* of G. So the possible eigenvalues are the characters:

$$\widehat{G} = \{\text{characters}\} = \operatorname{Hom}_{\mathbf{Grp}}\left(G, \mathbb{C}^\times\right) \ . \tag{82}$$

This is the spectrum, i.e. we will be performing spectral decomposition over $\widehat{G}$.

4.2. Finite Fourier transform. Now let G be a finite group.[2] We will eventually assume G is abelian, but we don't need this yet. We want to see $\widehat{G}$ appear at the spectrum. For a complex representation V we have a group map $G \to \operatorname{Aut} V$, but the composition with the inclusion

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right, "\text{monoid map}"'] & \operatorname{Aut} V \arrow[r, phantom, "\subset"] & \operatorname{End}(V) \end{tikzcd} \tag{83}$$

is a monoid map. In other words V gives rise to an element of

$$\operatorname{Hom}_{\mathbf{Monoid}}\left(G, \operatorname{Forget}\left(\operatorname{End} V\right)\right) \ . \tag{84}$$

Just like before, we have an adjunction:

$$\operatorname{Hom}_{\mathbf{Monoid}}\left(G, \operatorname{Forget}\left(\operatorname{End} V\right)\right) = \operatorname{Hom}_{\mathbb{C}\text{-}\mathbf{Alg}}\left(?, \operatorname{End} V\right) \ , \tag{85}$$

where the missing entry should be some kind of free construction. As it turns out, the answer is the *group algebra*:

$$? = \mathbb{C}G \ . \tag{86}$$

This is the algebra freely generated by scalar multiplication and sums of elements of G. Since the group is finite this is just:

$$\mathbb{C}G = \left\{ \sum_{g \in G} f(g) \cdot g \right\} \tag{87}$$

where $f \colon G \to \mathbb{C}$ is any function. Really we should think of f as a measure rather than a function. There is no difference when G is finite, but for any $g \in G$ would would like a canonical element

$$\delta_g = 1 \cdot g \in \mathbb{C}G \ , \tag{88}$$

[2]This is a simplifying assumption so we don't need to worry about what "kind" of functions we're considering.

which means the coefficients come from some f which is 1 at g and 0 elsewhere, which is not a function in general.

We can think of $\mathbb{C}G$ as being generated by the elements δ_g for $g \in G$. The algebra structure comes from convolution:

$$\delta_f * \delta_g = \delta_{fg} \ . \tag{89}$$

In general

$$f_1 * f_2 = \sum_g f_1(g)\, g * \sum_h f_2(h)\, h \tag{90}$$

$$= \sum_k \left(\sum_{gh=k} f_1(g) f_2(h) \right) k \tag{91}$$

$$= \sum_k \sum_g f_1(g) f_2\left(kg^{-1}\right) \cdot k \ . \tag{92}$$

We can express the convolution in terms of the multiplication map $\mu\colon G \times G \to G$ as follows. We have he two projections π_1 and π_2:

$$\begin{tikzcd} & G \times G \arrow[r, "\mu"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & G \\ G & & G \end{tikzcd} \ . \tag{93}$$

We can pull f_1 back along π_1 and f_2 back along π_2 to get a function on $G \times G$:

$$f_1 \boxtimes f_2 \coloneqq \pi_1^* f_1 \pi_2^* f_2 \ . \tag{94}$$

Then we can push this along μ, and the result is the convolution:

$$f_1 * f_2 = \mu_* \left(f_1 \boxtimes f_2\right) = \int_\mu f_1 \boxtimes f_2 \ . \tag{95}$$

The upshot is that we can define the group algebra in this way whenever we have things which can be pulled and pushed like this.

For $G \circlearrowright V$, we have extended this to an action of $\mathbb{C}G \circlearrowright V$. Then G is **abelian** iff $(\mathbb{C}G, *)$ is a **commutative** algebra. So now the fundamental object over which representation theory of G will sheafify is:

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) \ , \tag{96}$$

and as it turns out

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) = \widehat{G} \ , \tag{97}$$

i.e.

$$\left(\mathbb{C}G, *\right) \simeq \left(\mathcal{O}\left(\widehat{G}\right), \cdot\right) \ . \tag{98}$$

This is a first version of the Fourier transform. The idea is that a map $\operatorname{Spec} k \to \operatorname{Spec} A$ is the same as a morphism $A \to k$, which is exactly a 1-dimensional representation of G, i.e. a character. Under this correspondence, the characters $\chi_t \in \mathbb{C}G$ for $t \in \widehat{G}$ correspond to points $\delta_t \in \mathcal{O}\left(\widehat{G}\right)$ for $t \in \widehat{G}$. Moreover, translation by g corresponds to multiplication by $\widehat{g}$, i.e. the character

$$t \mapsto \chi_t(g) \ . \tag{99}$$

We can rephrase this equivalence slightly to make it more evident that this is some version of the Fourier transform. If f is a function on G then we can write

$$f = \sum_{t \in \widehat{G}} \widehat{f}(t) \cdot \chi_t \ . \tag{100}$$

This is just expressing f in terms of the basis of characters. Then we can recover $\widehat{f}(t)$ as the coefficient of f in this orthonormal basis. We also have that

$$f * (-) = \widehat{f} \cdot (-) \ . \tag{101}$$

4.2.1. *Secret symmetry.* There is a secret symmetry here. $\widehat{G}$ is an abelian group itself under the operation of pointwise multiplication. I.e.

$$\chi_{t \cdot s} \coloneqq \chi_t \cdot \chi_s \ . \tag{102}$$

Call the corresponding abelian group the *dual group to G*.

To see that this is a good duality, note that there is a tautological map

$$\begin{aligned} G &\longrightarrow \widehat{\widehat{G}} \\ g &\xmapsto{\sim} \{\chi \mapsto \chi(g)\} \ , \end{aligned} \tag{103}$$

which turns out to be an isomorphism.

We could have set this up in a more symmetric way. We have two projections:

$$\begin{tikzcd} & G \times \widehat{G} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & \widehat{G} \end{tikzcd} \tag{104}$$

and there is a tautological object, called the universal character, living over $G \times \widehat{G}$:

$$\begin{matrix} \chi(-,-) \\ \downarrow \\ G \times \widehat{G} \end{matrix} \ , \tag{105}$$

i.e. a function on $G \times \widehat{G}$ given by:

$$\chi(g,t) = \chi_t(g) = \chi_g(t) \ . \tag{106}$$

Then the Fourier transform of $f \in \mathrm{Fun}(G)$ is given by pulling up to $G \times \widehat{G}$, multiplying by χ, and then summing up by pushing forward by π_2:

$$f \mapsto \pi_{2*}(\pi_1^* f \cdot \chi) \ . \tag{107}$$

Explicitly the Fourier transform is:

$$\widehat{f}(t) = \sum_g f(g)\, \overline{\chi}(g,t) \tag{108}$$

$$f(g) = \sum_t \widehat{f}(t)\, \chi(g,t) \ . \tag{109}$$

We have simultaneously diagonalized the action of all $g \in G$ on $\mathrm{Fun}(G)$.

For any V with a G action, we get a $(\mathbb{C}G, *)$ action on V so V spectrally decomposes over $\widehat{G}$, i.e.

$$V = \bigoplus_{t \in \widehat{G}} V_{\chi_t} \ , \tag{110}$$

where G acts by the eigenvalue specified by χ_t on the subspace V_{χ_t}.

This gives a complete picture of the complex representation theory of finite abelian group. The exact same formalism works in any setting with abelian groups. We will focus on the setting of topological groups and algebraic groups. Everything will mostly look the same, with the difference being what kind of functions we consider.

4.3. Pontrjagin duality.

Let G be a locally compact abelian (LCA) group.

EXAMPLE 11. $\mathbb{Z}$, $\mathrm{U}(1)$, $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathbb{Q}_p^*$ are all (non-finite) examples.

Define the dual to be the collection of *unitary* characters

$$\widehat{G} = \mathrm{Hom}_{\mathbf{TopGrp}}(G, \mathrm{U}(1)) \ . \tag{111}$$

REMARK 14. We shouldn't be too shocked by replacing $\mathbb{C}^\times$ by $\mathrm{U}(1) \subset \mathbb{C}^\times$. If G is finite, all of the character theory was captured by $\mathrm{U}(1)$ anyway.

4.3.1. *Group algebra.* The spectrum will again be Spec of the group algebra, but we need to determine the appropriate definition of the group algebra in this context.

Endow G with a Haar measure. Before we had the counting measure, and could translate freely between functions and measures (so, in particular, they could both push and pull). Now $L^1(G)$ has a convolution structure in exactly the same way as in eq. (95):

$$f_1 * f_2 = \int_{g \in G} f_1(h) f_2\left(gh^{-1}\right) dg \ . \tag{112}$$

Just as before, this convolution comes from an adjunction. I.e. it satisfies a universal property in the world of C^*-algebras. If we have a representation:

$$G \to \mathrm{End}(V) \tag{113}$$

then this will correspond to a morphism

$$\left(L^1(G), *\right) \to \mathrm{End}(V) \tag{114}$$

of C^*-algebra.

The spectrum is the Gelfand spectrum:

$$\text{m-Spec}\left(L^1(G), *\right) = \widehat{G} \ . \tag{115}$$

This is a version of the Fourier transform which says that:

$$\left(L^1(G), *\right) \simeq \left(C_v\left(\widehat{G}\right), \cdot\right) \ , \tag{116}$$

where C_v denotes functions vanishing at ∞. Another version says that there is a tautological map:

$$G \to \widehat{\widehat{G}} \tag{117}$$

which is an isomorphism.

Again, this can be written in a symmetric way:

$$f \mapsto \pi_{2*}\left(\pi_1^* f \cdot \chi\right) \tag{118}$$

where

$$\begin{tikzcd}[column sep=small]
 & \chi \arrow[d] & \\
 & G \times \widehat{G} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G & & \widehat{G}
\end{tikzcd} \quad . \tag{119}$$

For any notion of functions or distributions on G, we can perform this Fourier transform operation. The question is, given the type of functions we feed in, what type of functions do we get in the other side? For L^2 functions we simply get:

$$L^2(G) \xrightarrow{\sim} L^2(G) \ . \tag{120}$$

Any of these notions of a Fourier transform have the same general features. Some of which are as follows.

(i) Translation by a group element becomes pointwise multiplication.
(ii) Convolution also becomes pointwise multiplication.
(iii) Characters correspond to points.

4.3.2. *Fourier series.* Take $G = \mathrm{U}(1)$. Then

$$\widehat{G} = \mathrm{Hom}_{\mathbf{TopGrp}}\left(\mathrm{U}(1), \mathrm{U}(1)\right) = \mathbb{Z} \tag{121}$$

where $n \in \mathbb{Z}$ corresponds to

$$\left\{x \mapsto e^{2\pi i n x}\right\} \ . \tag{122}$$

Then the Fourier transform established an equivalence

$$L^2(\mathrm{U}(1)) \xrightarrow{\sim} L^2(\mathbb{Z}) = \ell^2 \ . \tag{123}$$

As before, this is symmetric. There is a universal character:

$$\chi\colon\ (x, n) \mapsto e^{2\pi i n x} \tag{124}$$

living over $\mathrm{U}(1) \times \mathbb{Z}$, and we have the usual projections:

$$\begin{tikzcd}[column sep=small]
 & \mathrm{U}(1) \times \mathbb{Z} \arrow[dl] \arrow[dr] & \\
\mathrm{U}(1) & & \mathbb{Z}
\end{tikzcd} \ . \tag{125}$$

Then we can read it backwards. A character

$$\mathbb{Z} \to \mathrm{U}(1) \tag{126}$$

is determined by the image of $1 \in \mathbb{Z}$, so characters of $\mathbb{Z}$ are labelled by points of $\mathrm{U}(1)$.

In general Pontrjagin duality, G is compact iff $\widehat{G}$ is discrete. Concretely, for $n \in \mathbb{Z}$ we have

$$e^{2\pi i n x} \in L^2 \tag{127}$$

because G is compact. Similarly, because $\mathbb{Z}$ is discrete,

$$\delta_n \in \ell^2 \ . \tag{128}$$

Lecture 5; February 2, 2021

4.4. Fourier transform. Recall that the Pontrjagin dual of $\mathrm{U}(1)$ is $\mathbb{Z}$. The characters $\mathrm{U}(1) \to \mathrm{U}(1)$ are given by $z \mapsto z^n$ for any $n \in \mathbb{Z}$. Similarly, the dual of $\mathbb{Z}$ is $\mathrm{U}(1)$.

We can replace $\mathrm{U}(1)$ by a torus T, which is defined as:

$$T = \mathbb{R}^d/\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathrm{U}(1) \tag{129}$$

where $\Lambda \subset \mathbb{Z}^d$ is a full-rank lattice. The dual of T is the dual lattice:

$$\widehat{T} = \Lambda^\vee \ . \tag{130}$$

Similarly, the dual of the lattice Λ is the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathrm{U}(1) \ . \tag{131}$$

The classical Fourier transform takes G to be a real vector space. For $G = \mathbb{R}_x$ the dual is another copy of $\mathbb{R}$: $\widehat{G} = \mathbb{R}_t$. Performing the same operations as before, with universal character

$$\chi(x,t) = e^{2\pi i x t} \ , \tag{132}$$

we get the usual Fourier transform. For a general real vector space the dual is the dual vector space, and the character is

$$\chi(x,t) = e^{2\pi i \langle x,t\rangle} \ , \tag{133}$$

where the pairing is the usual one between the vector space and its dual.

Recall that in the context of the duality between $\mathrm{U}(1)$ and $\mathbb{Z}$, characters corresponded to points. This situation differs in the sense that characters are not L^2 anymore (since $\mathbb{R}$ is not compact, like $\mathrm{U}(1)$ is) and the points are not isolated (since $\mathbb{R}$ is not discrete like $\mathbb{Z}$). But we still have:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t)\, e^{2\pi i x t} dt \ . \tag{134}$$

The operation of differentiation d/dx corresponds with multiplication by t on the other side. We should think of d/dx as an infinitesimal version of group translation, which went to multiplication before. So this is part of the same framework where group theory on one side goes to geometry on the other.

In general, let G be a Lie group with abelian Lie algebra $\mathfrak{g}$. We have a map from $\mathfrak{g}$ to vector fields on G:

$$\mathfrak{g} \to \operatorname{Vect} G \subset \operatorname{Diff}(G) \tag{135}$$

and there is an adjunction between

$$\text{Forget}\colon \mathbf{Alg}_{\text{Assoc.}} \to \textbf{Lie-Alg} \tag{136}$$

and the functor

$$U\colon \textbf{Lie-Alg} \to \mathbf{Alg}_{\text{Assoc.}} \tag{137}$$

which sends a Lie algebra to the *universal enveloping algebra*. $\mathfrak{g}$ is abelian so the universal enveloping algebra is just the symmetric algebra

$$U(\mathfrak{g}) = \operatorname{Sym}^* \mathfrak{g} \ . \tag{138}$$

$U\mathfrak{g}$ is now a commutative algebra acting on $C^\infty(G)$. Therefore we can spectrally decompose/sheafify over

$$\operatorname{Spec} U\mathfrak{g} = \mathfrak{g}^* \ . \tag{139}$$

E.g. for d/dx acting on $\mathbb{R}_x$,

$$\mathfrak{g}^* = \mathbb{R}_t = \operatorname{Spec} RR\left[d/dx = t\right] \ . \tag{140}$$

Example 12. The dual of $\mathrm{U}(1)$ is $\mathbb{Z} \subset \mathbb{R}_t$.

4.5. In quantum mechanics. Before we see where the Fourier transform comes up in quantum mechanics, we consider classical mechanics. If we want to a model a particle moving around in a manifold M, then the phase space is the cotangent bundle T^*M with positive coordinates q on M, and momenta coordinates p in the fiber direction. The observables form:

$$\text{functions on } M \otimes \operatorname{Sym} TM \ . \tag{141}$$

In quantum mechanics the space of states is replaced by $\mathcal{H} = L^2(M)$. The observables contain $\operatorname{Diff}(M)$, the differential operators on M. Inside of this we have two pieces:

$$\begin{tikzcd}[column sep=small] \text{f'ns on } M \arrow[dr, hook] & \\ & \operatorname{Diff}(M) \\ TM \arrow[ur, hook] & \end{tikzcd} \ , \tag{142}$$

with

$$p_j = i\frac{d}{dq_j} \ . \tag{143}$$

In classical mechanics the analogous pieces commute. But here they commute up to a term: a tangent vector $\xi \in TM$ and a function $f \in \operatorname{Fun}(M)$ must satisfy

$$\xi f = f\xi + \hbar f' \ . \tag{144}$$

To summarize, states look like "$\sqrt{\text{observables}}$". The position operators:

$$q_j \cdot (-) \tag{145}$$

are diagonalized, on the other hand the momentum operators act as derivates.

We can also pass to the momentum picture where we diagonalize the p_i's (derivatives) instead. For $M = \mathbb{R}^n$ we have a natural basis of invariant vector fields (this is the advantage of having a group). Now we can simultaneously diagonalize

$$p_j = i\frac{d}{dx_j} \tag{146}$$

to identify

$$L^2\left(\mathbb{R}^n_q\right) \simeq L^2\left(\mathbb{R}^n_p\right) \ , \tag{147}$$

which is the Fourier transform. One might say that this is identifying quantum mechanics for G with quantum mechanics for the Pontrjagin dual $\widehat{G}$. This is the one-dimensional case of abelian duality, or "one-dimensional mirror symmetry".

4.6. Cartier duality. This is the same Fourier theory we've been doing, but in the context of algebraic geometry, i.e. instead of continuous, etc. functions we're considering algebraic functions. We will eventually see that this duality shows up in physics (electric-magnetic duality), as well as number theory (class field theory).

To say what a group is in the world of algebraic geometry, we need to review the notion of the functor of points. To a variety X we can associate a functor **cRing** $\to$ **Set** by sending a ring R to

$$X(R) = \operatorname{Hom}(\operatorname{Spec} R, X) \ . \tag{148}$$

As it turns out, specifying this functor is equivalent to specifying X itself.

A variety G is an *algebraic group* if the associated functor of points **cRing** $\to$ **Set** lifts to a functor landing in groups:

$$\begin{tikzcd} \mathbf{cRing} \arrow[r] \arrow[dr, dashed] & \mathbf{Set} \\ & \mathbf{Grp} \arrow[u] \end{tikzcd} \tag{149}$$

i.e. that

$$G(R) = \operatorname{Hom}(\operatorname{Spec} R, G) \tag{150}$$

is a group.

Remark 15. Sometimes other things are assumed in the definition of an algebraic group, which we do not assume here.

Example 13. Consider $\mathbb{A}^1 = \mathbb{G}_a$. As a functor, this sends

$$R \mapsto (R, +) \ . \tag{151}$$

This is saying that $\operatorname{Map}\left(X, \mathbb{A}^1\right) = \mathcal{O}(X)$.

Example 14. Consider $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m = \operatorname{Spec} k\left[t, t^{-1}\right]$. As a functor this sends

$$R \mapsto \left(R^\times, \cdot\right) \ . \tag{152}$$

Example 15. The integers form an algebraic group with functor of points given by:

$$\mathbb{Z}\colon R \mapsto (\mathbb{Z}, +) \ . \tag{153}$$

Let G be an abelian (algebraic) group. The *Cartier dual* of G is

$$\widehat{G} = \operatorname{Hom}_{\mathbf{Grp}_{\mathrm{Alg}}}(G, \mathbb{G}_m) \ . \tag{154}$$

where the dualizing object $\mathbb{G}_m$ comes from $\operatorname{Aut}_k = \mathbb{G}_m$.

Example 16. Let $G = \mathbb{Z}/n$. The Cartier dual is

$$\widehat{\mathbb{Z}/n} = \operatorname{Hom}(\mathbb{Z}/n, \mathbb{G}_m) \simeq \mu_n \tag{155}$$

where μ_n denotes the nth roots of unity. As a functor, this sends:

$$R \mapsto n\text{th roots of unity in } R \ . \tag{156}$$

Example 17. The dual of the integers is $\widehat{\mathbb{Z}} \simeq \mathbb{G}_m$, since $\mathbb{Z} \to \mathbb{G}_m$ is determined by the image of 1. Similarly:

$$\widehat{\mathbb{G}_m} = \operatorname{Hom}_{\mathbf{TopGrp}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z} = \{z \mapsto z^n\} \ . \tag{157}$$

EXAMPLE 18. More generally, the dual to a lattice Λ will be the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m \ , \tag{158}$$

so (over $\mathbb{C}$) this looks roughly like $(\mathbb{C}^\times)^{\operatorname{rank}\Lambda}$. Similarly a torus T gets exchanged with the dual lattice $\Lambda^\vee$.

To avoid technicalities, assume G is finite.[3] Consider the collection of functions on G, $\mathcal{O}(G)$. Dualizing and taking Spec gives the dual group:

$$\widehat{G} = \operatorname{Spec} \mathcal{O}(G)^* \ . \tag{159}$$

Rather than talking about a group algebra, functions on G has the structure of a group coalgebra as follows. The multiplication map on G induces a coproduct on $\mathcal{O}(G)$:

$$\begin{gathered} G \times G \xrightarrow{\quad\mu\quad} G \\ \mathcal{O}(G) \xrightarrow{\Delta := \mu^*} \mathcal{O}(G) \otimes \mathcal{O}(G) \ . \end{gathered} \tag{160}$$

This makes $\mathcal{O}(G)$ into a *Hopf algebra*, i.e. $\mathcal{O}(G)$ has a multiplication, and a comultiplication Δ. In fact, this is a finite-dimensional commutative and cocommutative Hopf algebras. The study of these Hopf algebras turns out to be equivalent to the study of finite abelian group schemes.

EXAMPLE 19 ("Fourier transform"). Let $\operatorname{char} k = 0$. The Cartier dual of $\mathbb{G}_a$ is the formal completion of itself:

$$\widehat{\mathbb{G}_a} = \widehat{\mathbb{G}_a} \ , \tag{161}$$

where the formal completion of $\mathbb{G}_a$ is given by:

$$\widehat{\mathbb{G}_a} = \bigcup \operatorname{Spec} k[t] / (t^n) \ . \tag{162}$$

The Cartier dual of an n-dimensional vector space V is

$$\widehat{V} = \widehat{V^*} \ . \tag{163}$$

The character of

$$\mathbb{G}_a = \operatorname{Spec} k[x] \tag{164}$$

should be

$$e^{xt} = \sum \frac{(xt)^n}{n!} \ , \tag{165}$$

but we need this to be a finite sum. This makes sense if t is nilpotent, i.e. there is $N \gg 0$ such that $t^N = 0$.

So when this is true, i.e. t is "close to 0", the function $e^{\langle x,t\rangle}$ is well-defined for $x \in V$ and $t \in V^*$, and is the character for V.

Similarly the dual of the completion is $\mathbb{G}_a$:

$$\mathbb{G}_a = \widehat{\widehat{\mathbb{G}_a}} \ . \tag{166}$$

[3]Formally we're assuming that G is a finite abelian group scheme.

Recall that Fourier duality exchanges

$$(\mathrm{Fun}(G), *) \simeq \left(\mathrm{Fun}\left(\widehat{G}\right), \cdot\right) \ . \tag{167}$$

Then $\mathbf{Rep}(G)$ became spectrally decomposed over $\widehat{G}$. In algebraic geometry, a representation V is a *comodule* for $\mathcal{O}(G)$. I.e. we have a map $G \times V \to V$, and passing to functions gives us a map

$$V \to \mathcal{O}(G) \otimes V \ . \tag{168}$$

Furthermore,

$$\mathbf{Rep}(G) \simeq \mathcal{O}(G)\text{-}\mathbf{coMod} \simeq \mathcal{O}\left(\widehat{G}\right)\text{-}\mathbf{Mod} \simeq \mathbf{QC}\left(\widehat{G}\right) \ . \tag{169}$$

4.6.1. *Fourier series examples.*

Example 20. For $G = \mathbb{Z}$, the category of representations is given by

$$\mathbf{Rep}(\mathbb{Z}) = k\left[z, z^{-1}\right]\text{-}\mathbf{Mod} \tag{170}$$

where z is the action of $1 \in \mathbb{Z}$. This action must be invertible, which is why z^{-1} is included. Then the duality tells us that:

$$\mathbf{Rep}(\mathbb{Z}) = k\left[z, z^{-1}\right]\text{-}\mathbf{Mod} = \mathcal{O}(\mathbb{G}_m)\text{-}\mathbf{Mod} = \mathbf{QC}(\mathbb{G}_m) \ . \tag{171}$$

Example 21. A vector space and an endomorphism (matrix) gives us

$$\mathbf{QC}\left(\mathbb{A}^1 = \mathrm{Spec}\, k[z]\right) \tag{172}$$

but if we have an automorphism (invertible matrix) then we get

$$\mathbf{Rep}(\mathbb{Z}) \leftrightarrow \mathbf{QC}\left(\mathbb{A}^1 \setminus \{0\}\right) \ . \tag{173}$$

Example 22. In algebraic geometry

$$\mathbf{Rep}(\mathbb{G}_m) = \mathbb{Z}\text{-graded vector space} = \mathbf{QC}(\mathbb{Z}) \tag{174}$$

where

$$V \simeq \bigoplus_{n\in\mathbb{Z}} V_n \tag{175}$$

and $z \in \mathbb{G}_m$ acts on V_n by z^n.

Example 23 (Topological example). The following is an example of Fourier series from topology. Let M^3 be a compact oriented three-manifold.[4] Let

$$G = \mathrm{Pic}\, M^3 \tag{176}$$

consist of complex line bundles (or $\mathrm{U}(1)$-bundles) on M^3 up to isomorphism. This forms an abelian group under tensor product. We can think of this as:

$$G = \mathrm{Map}\left(M^3, B\,\mathrm{U}(1)\right) \tag{177}$$

where $B\,\mathrm{U}(1)$ denotes the classifying space of $\mathrm{U}(1)$. Up to homotopy we can think of $B\,\mathrm{U}(1)$ as:

$$B\,\mathrm{U}(1) \simeq \mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2) \ . \tag{178}$$

[4]This will be the three-manifold on which we do electromagnetism. This Cartier duality will give us electric-magnetic duality. If we think of a number field as a three-manifold, then this duality fits into the framework of class field theory.

For $\mathcal{L} \in G$, we can attach the first Chern class:

$$c_1(\mathcal{L}) \in H^2(M, \mathbb{Z}) \ , \tag{179}$$

which is a complete invariant of the line bundle. So we can take

$$G = \Lambda = H^2(M, \mathbb{Z}) \ . \tag{180}$$

The Cartier dual is

$$\widehat{G} \simeq \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m = \mathrm{Hom}\left(H_1(M, \mathbb{Z}), \mathbb{G}_m\right) = \mathrm{Hom}\left(\pi_1(M), \mathbb{G}_m\right) \tag{181}$$

where the last line follows from the fact that $H_1 = \pi_1^{\mathrm{ab}}$, and Hom from the abelianization is the same as Hom from the whole thing. But this is just flat $\mathbb{C}^\times$-bundles on M, i.e.

$$\widehat{G} = \mathrm{Loc}_{\mathbb{C}^\times}(M) \ . \tag{182}$$

This shouldn't be that surprising since G looks something like $\mathbb{Z}^n$, and this dual $\widehat{G}$ looks like $(\mathbb{C}^\times)^n$.

We can also replace line bundles by torus bundles, i.e. we can pass from $\mathrm{U}(1)$ to $\mathrm{U}(1)^n \simeq T$. Then

$$\mathrm{Bun}_T(M) \leftrightarrow \mathrm{Loc}_{T^\vee} M \ . \tag{183}$$

where $\mathrm{Loc}_{T^\vee} M$ consists of isomorphism classes of flat $T^\vee$-bundles over M. The LHS still looks like a lattice Λ, and the RHS still looks like a torus $(\mathbb{C}^\times)^n$.

transition between lectures

Lecture 6; February 4, 2021

Example 24 (Fourier series). Take $\mathbb{Z}$ to be my abelian group G. Then we have some notions of a dual. One is

$$\mathrm{Hom}_{\mathbf{TopGrp}}(\mathbb{Z}, \mathrm{U}(1)) = \mathrm{U}(1) \tag{184}$$

another is

$$\mathrm{Hom}_{\mathbf{TopGrp}}(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m = \ , \tag{185}$$

i.e. $\mathbb{C}^\times$ if we're over $\mathbb{C}$.

The difference is the kind of function theory we're consider. In the first case we have an equivalence

$$\ell^2 = L^2(\mathbb{Z}) \simeq L^2(\mathrm{U}(1)) \tag{186}$$

and in the second case we have

$$\mathbb{C}\mathbb{Z} \simeq \mathcal{O}\left(\mathbb{C}^\times\right) \ . \tag{187}$$

Notice that

$$\mathcal{O}\left(\mathbb{C}^\times\right) = \mathbb{C}\left[z, z^{-1}\right] \subset L^2(\mathrm{U}(1)) \ , \tag{188}$$

and

$$\mathbb{C}\mathbb{Z} \subset \ell^2 \ . \tag{189}$$

So the algebraic version kind of its inside of the analytic story.

DIGRESSION 1. For any group G, there are many different version of representation theory, which we can think of as coming from different versions of the group algebra. On the dual side, this corresponds to a different structure on $\widehat{G}$.

For example, if we take the group algebra in the sense of a von Neumann algebra, then we get $\widehat{G}$ as a measure space. If we start with the group algebra as a C^*-algebra, then we get $\widehat{G}$ as a (locally compqct) topological space. If we start with a discrete/algebraic group algebra, then we get $\widehat{G}$ as an algebraic variety.

4.7. Pontrjagin-Poincaré duality. Now we want to give a series of examples of Pontrjagin/Fourier duality which are basically the same example, but they will be more interesting because we will introduce some topology. A summary of all of these duality statements is in table 1.

$\widehat{G}$ versus $\check{G}$

Let M^n be a compact oriented manifold. The cohomology group of M will be the abelian groups we study the duality theory of. This duality says that for G an abelian group

$$H^i(M,G)^\vee \simeq H^{n-i}\left(M,\widehat{G}\right) \ . \tag{190}$$

To convince ourselves that this is on the correct footing, we can check that the following is a nondegenerate pairing:

$$H^i(M,G) \otimes H^{n-i}\left(M,\widehat{G}\right) \to H^n\left(M, G\otimes \widehat{G}\right) \to H^n(M,\mathrm{U}(1)) \to \mathrm{U}(1) \tag{191}$$

where the last map is integration.

The "dimension" of the following examples will eventually correspond to the dimension of the appropriate field theory, i.e. the geometric objects we consider are actually a dimension less than the listed dimension so as to get a vector space. The side with $\mathbb{Z}$ coefficients will be called the A-side, and the side with $\mathbb{C}^\times$ coefficients will be called the B-side.

4.7.1. *One dimension.* In this case we just have that the A-side $\mathrm{Map}(\mathrm{pt},\mathbb{Z}) = \mathbb{Z}$ gets exchanged with the B-side $\mathrm{Map}(\mathrm{pt},\mathbb{C}^\times) = \mathbb{C}^\times$. We can think of this as saying that quantum mechanics on $\mathbb{Z}$ is QM of $\mathbb{C}^\times$ in the sense that some kind of spaces of functions on these are identified by Fourier theory.

4.7.2. *Two dimensions.* One duality in two dimensions is between:

$$H^0\left(S^1,\mathbb{Z}\right) \qquad \text{and} \qquad H^1\left(S^1,\mathbb{C}^\times\right) \ , \tag{192}$$

in the sense that functions on these are identified. Note that we can write the A-side as:

$$H^0\left(S^1,\mathbb{Z}\right) = \pi_0\left(\mathrm{Map}\left(S^1,\mathbb{Z}\right)\right) = \left[S^1,\mathbb{Z}\right] \ . \tag{193}$$

The B-side is

$$H^1\left(S^1,\mathbb{C}^\times\right) = \left|\mathrm{Loc}_{\mathbb{C}^\times} S^1\right| \ , \tag{194}$$

where $\left|\mathrm{Loc}_{\mathbb{C}^\times} S^1\right|$ denotes the underlying space of $\mathrm{Loc}_{\mathbb{C}^\times} S^1$, which is a stack.

The A-side just looks like $\mathbb{Z}$, and the B-side just looks like $\mathbb{C}^\times$. This is the sense in which we're doing the "same" example as before.

The A-side is the theory of maps to $\mathbb{Z}$, and the B-side is some kind of $\mathrm{U}(1)$ Gauge theory. By Gauge theory we just mean that we're dealing with local systems or principal bundles.

DIGRESSION 2 (Local systems). Let X be a topological space. Let G be a group (with the discrete topology). Then $\mathrm{Loc}_G X$ consists of principal G-bundles on X. These are sheaves which are locally isomorphic to the constant sheaf $\underline{G}$. We can also think of this as:

$$\mathrm{Loc}_G X = \{\pi_1(X) \to G\} \tag{195}$$

where, given a principal G-bundle, the associated representation of π_1 is the monodromy representation.

If we only care about isomorphism classes, then this is

$$H^1(X, G) = |\mathrm{Loc}_G X| \ . \tag{196}$$

4.7.3. *Two dimensions.* Another version of this duality in two-dimensions has

$$H^1\left(S^1, \mathbb{Z}\right) \tag{197}$$

on the A-side. On the B-side we have

$$H^0\left(S^1, \mathbb{C}^\times\right) \ , \tag{198}$$

which is just the maps to $\mathbb{C}^\times$ with the discrete topology.

On the A-side:

$$H^1\left(S^1, \mathbb{Z}\right) = \left[S^1, B\mathbb{Z}\right] \ , \tag{199}$$

and

$$B\mathbb{Z} = K(\mathbb{Z}, 1) = S^1 \ . \tag{200}$$

So the A-side is maps to S^1, and the B-side is also maps to $\mathbb{C}^\times$ (some version of S^1). These two spaces consist of what are sometimes called *scalar fields.*

This duality theory is known as T-duality or mirror symmetry. It is also called $R \leftrightarrow 1/R$ duality.

Again, as a group, the A-side is just $\mathbb{Z}$ and the B-side is just some version of S^1. One thing that is useful, in all of these examples, is to replace $\mathbb{Z}$ by a lattice Λ. The duality turns out to be between

$$H^1\left(S^1, \Lambda\right) \qquad \text{and} \qquad H^0\left(S^1, T^\vee_{\mathbb{C}}\right) \ , \tag{201}$$

where we can think of

$$H^1\left(S^1, \Lambda\right) = \left[S^1, T\right] \tag{202}$$

where T is the compact lattice

$$\Lambda \otimes_{\mathbb{Z}} S^1 \ . \tag{203}$$

The upshot is that we are exchanging maps into the torus with maps into the dual torus.

4.7.4. *Three dimensions.* Let Σ be a compact oriented two-manifold. Then we have a duality between

$$H^1(\Sigma, \mathbb{Z}) \qquad \text{and} \qquad H^1\left(\Sigma, \mathbb{C}^\times\right) \ . \tag{204}$$

The B side is $|\mathrm{Loc}_{\mathbb{C}^\times} \Sigma|$, so the B-side is three-dimensional gauge theory (i.e. principal bundles are involved). The A-side is

$$H^1(\Sigma, \mathbb{Z}) = \left[\Sigma, B\mathbb{Z} = S^1\right] \ , \tag{205}$$

so this is a scalar field. So maps into a circle get exchanged with principal bundles.

4.7.5. *Three dimensions.* Alternatively, in three dimensions, we get a duality between

$$H^2(\Sigma,\mathbb{Z}) \qquad \text{and} \qquad H^0\left(\Sigma,\mathbb{C}^\times\right) \ . \tag{206}$$

We can think of

$$H^2(\Sigma,\mathbb{Z}) = H^1(\Sigma,\mathrm{U}(1)) = [\Sigma, B\,\mathrm{U}(1) = K(\mathbb{Z},2)] \tag{207}$$

as consisting of $\mathrm{U}(1)$-principal bundles (i.e. line bundles) so this is some kind of gauge theory. On the other hand,

$$H^0\left(\Sigma,\mathbb{C}^\times\right) = \left[\Sigma,\mathbb{C}^\times\right] \tag{208}$$

consists of scalar fields. So principal bundles got exchanged with maps into a circle. Again, this can be upgraded to tori and lattices.

Digression 3 (Dictionary between vector bundles and principal bundles). Let $\mathcal{P}\to X$ be a $\mathrm{U}(1)$-bundle. Then I can form a line bundle by taking the product $\mathcal{P}\times_{\mathrm{U}(1)}\mathbb{C}$. In general if I have a principal G-bundle I have a representation of the group, so I get a vector bundle.

Conversely, if $\mathcal{L}\to X$ is a line bundle we can take Hom with the trivial bundle: $\mathrm{Hom}(\mathcal{L}, X\times\mathbb{C})$ and this is a principal $G\,\mathrm{U}(1)$-bundle.

4.7.6. *Four dimensions.* The case of four dimensions is important because both sides will be gauge theories. Assume there is no torsion in our cohomology. The duality is between:

$$H^2\left(M^3,\mathbb{Z}\right) \qquad \text{and} \qquad H^1\left(M^3,\mathbb{C}^\times\right) \ . \tag{209}$$

The B-side can be written as

$$H^1\left(M^3,\mathbb{C}^\times\right) = \left|\mathrm{Loc}_{\mathbb{C}^\times} M\right| \ , \tag{210}$$

i.e. some kind of gauge theory since we're dealing with $\mathbb{C}^\times$-bundles. The A-side is

$$[M, K(\mathbb{Z},2)] \tag{211}$$

and as a homotopy type:

$$K(\mathbb{Z},2) = B\,\mathrm{U}(1) = \mathbb{CP}^\infty \ . \tag{212}$$

So the A-side is

$$[M, K(\mathbb{Z},2)] = \pi_0\,\mathrm{Map}(M,\mathbb{CP}^\infty) \tag{213}$$

$$= \pi_0\,(\text{line bundles on } M) \tag{214}$$

$$= \text{line bundles}/\sim \ , \tag{215}$$

i.e. line bundles up to isomorphism, so also some kind of gauge theory.

Then the equivalent vector spaces are functions on the two sides. The formal statement is that finitely supported functions (or locally constant if we're not thinking of π_0 of the space) on the A-side are equivalent to algebraic functions on the B-side.

A summary of all of these duality statements is in table 1.

5. Electric-magnetic duality

The idea is that these duality statements we have seen are a shadow of electric-magnetic duality. We will follow Witten [**DEF$^+$99**], Freed [**Fre00**], and Freed-Moore-Segal [**FMS07**].

TABLE 1. Summary of examples of Pontrjagin-Poincaré duality in various dimensions.

dimension	A-side	B-side
1	$\operatorname{Map}(\mathrm{pt}, \mathbb{Z}) = \mathbb{Z}$	$\operatorname{Map}(\mathrm{pt}, \mathbb{C}^\times) = \mathbb{C}^\times$
2	$H^0\left(S^1, \mathbb{Z}\right)$	$H^1\left(S^1, \mathbb{C}^\times\right) = \left\lvert \mathrm{Loc}_{\mathbb{C}^\times} S^1 \right\rvert$
2	$H^1\left(S^1, \mathbb{Z}\right) = \left[S^1, S^1\right]$	$H^0\left(S^1, \mathbb{C}^\times\right)$
3	$H^1(\Sigma, \mathbb{Z}) = \left[\Sigma, S^1\right]$	$H^1(\Sigma, \mathbb{C}^\times) = \lvert \mathrm{Loc}_{\mathbb{C}^\times} \Sigma \rvert$
3	$H^2(\Sigma, \mathbb{Z}) = [\Sigma, B\,\mathrm{U}(1)]$	$H^0(\Sigma, \mathbb{C}^\times)$

5.1. Classical field theory.

5.1.1. *dimension 2.* We will first study classical field theory in dimension 2. Let Σ be a Riemannian 2-manifold. Consider a periodic scalar field, i.e. a smooth map:

$$\varphi \in \operatorname{Map}_{C^\infty}\left(\Sigma, S^1\right) \ . \tag{216}$$

Now there is a natural classical field equation for this to satisfy: the harmonic map equation. It is convenient to introduce the 1-form $u = d\varphi$ on Σ, so we can write down the set of equations:

$$\begin{cases} du = 0 \\ d \star u = 0 \end{cases} . \tag{217}$$

The first one is automatically satisfied, since u is already exact. The second equation is equivalent to $\star d \star u = 0$, which is equivalent to φ being harmonic.

We write eq. (217) like this because they are symmetric under sending $u \rightsquigarrow \star u$. In other words, the theory of a scalar φ (where $d\varphi = u$) is the same as the theory associated to some $\varphi^\vee$ (where $d\varphi^\vee = \star u$). If we were keeping track of metrics, then we would be studying harmonic maps into a circle of some radius R, and the dual theory is studying harmonic maps into a circle of radius $1/R$. This is $R \leftrightarrow 1/R$ duality.

REMARK 16. For differential forms, Poincaré duality is realized by the Hodge star $\star$, so we are not so far from where we were before.

5.1.2. *Three-dimensions.* In dimension three, we can do the same trick. Let $\varphi \in \operatorname{Map}_{C^\infty}\left(\Sigma, S^1\right)$ be a scalar field, and let $u = d\varphi$. Again we write the harmonic map equation in this nice form:

$$\begin{cases} du = 0 \\ d \star u = 0 \end{cases} . \tag{218}$$

Now when we pass from u to $\star u$, we're passing from a 1-form to a 2-form so it's not even the same kind of beast anymore. Write $F = \star u$. Locally, we can write

$$F = dA \tag{219}$$

for A some 1-form, and we get the equations:

$$\begin{cases} dF = 0 \\ d * F = 0 \end{cases} , \tag{220}$$

which are the three-dimensional Maxwell equations. So the theory with a scalar field φ is dual to a theory with a field F satisfying Maxwell's equations.

5.2. Four-dimensions/Maxwell 101. The *electric field* is a 1-form $E \in \Omega^1\left(\mathbb{R}^3\right)$, and the magnetic field is a 2-form $B \in \Omega^2\left(\mathbb{R}^3\right)$. Relativistically, it is better to think of a composite beast, called the field strength:

$$F = B - dt \wedge E \in \Omega^2\left(\mathbb{M}^4\right) . \tag{221}$$

where $\mathbb{M}^4$ is Minkowski space. I.e. we need to turn E into a 2-form, so we wedge it with time so we can subtract it from B.

Maxwell's equations in a vacuum are:

$$\begin{cases} dF = 0 \\ d \star F = 0 \end{cases} . \tag{222}$$

Observe that this is symmetric under exchanging F and $\star F$. Some algebra reveals that:

$$\star F = B^\vee - dt \wedge E^\vee , \tag{223}$$

where $B^\vee$ is the three-dimensional hodge star of E:

$$B^\vee = - \star_3 E \tag{224}$$

and similarly

$$E^\vee = \star_3 B . \tag{225}$$

The upshot is that Maxwell's equations are symmetric under exchanging E with B.

If we're not in a vacuum, i.e. we have some currents, then Maxwell's equations become:

$$\begin{cases} dF = j_B \\ d \star F = j_E \end{cases} \tag{226}$$

where j_B is the magnetic current and j_E is the electric current. These are three-forms $j_B, j_E \in \Omega^3\left(\mathbb{M}^4\right)$.

The meaning of these currents is that they're related to the total charge:

$$\int_{M^3} j_B = Q_B \qquad\qquad \int_{M^3} j_E = Q_E \tag{227}$$

where we're taking $\mathbb{M}^4 = M^3 \times \mathbb{R}$. If we use Stokes' theorem, we get Gauss' law. This says that for a closed surface $\Sigma \subset M^3$, the magnetic flux can be expressed as:

$$b_\sigma = \int_\Sigma F = \# Q_{B,\Sigma} \tag{228}$$

for some scalar $\#$, where $Q_{B,\Sigma}$ is the total magnetic charge in Σ. Similarly, the electric flux is:

$$e_\Sigma = \int_\Sigma \star F = \# Q_{E,\Sigma} . \tag{229}$$

If you've ever learned about electromagnetism, you probably remember that there is a 0 on the RHS of (228). This is because, in reality, there don't seem to be any magnetic monopoles, i.e. $Q_B = 0$ and Gauss' law says that $\int_\Sigma F = 0$. Because of this property, we can introduce A, the electromagnetic potential by writing:

$$F = dA \ . \tag{230}$$

Note that this is breaking the symmetry between electricity and magnetism since $dF = 0$ is automatically satisfied. In particular this means that $j_B = 0$. Recall we should consider

$$\nabla = d + A \ , \tag{231}$$

regarded as a connection on a $\mathrm{U}(1)$-bundle on M.

The field strength F was something meaningful, but the potential A has some kind of ambiguity, since it is a sort of anti-derivative of F. This ambiguity is captured by the gauge transformations

$$\nabla \rightsquigarrow g^{-1}\nabla g \tag{232}$$

where $g\colon M \to \mathrm{U}(1)$. Similarly

$$A \mapsto A + g^{-1}dg \ , \tag{233}$$

i.e. it shifts A by some derivative, which **does not affect** F.

In fact, this is something which can be experimentally measured. This is called the *Bohm-Aharonov effect*. Even when $F = 0$ (flat connection, i.e. $E = 0$, $B = 0$), the monodromy of this connection is observable, i.e. a charged particle acquires a phase when it travels along loops.

Now we can think of this as a connection on any oriented Riemannian four-manifold. When we pass from F to $\nabla = d + A$, this implements Dirac charge quantization, which roughly says that the charges of elementary particles have to be integer valued (up to some renormalization). Let $M^4 = M^3 \times \mathbb{R}$. So now ∇ is a connection on some arbitrary line bundle $\mathcal{L}$ and F is a 2-form just like before. Even without charged particles, we still have fluxes:

$$b_\Sigma = \frac{1}{2\pi i}\int_\sigma F = \langle c_1(\mathcal{L}), [Sigma]\rangle \tag{234}$$

where this is the pairing between H^2 and H_2 since

$$c_1(\mathcal{L}) \in H^2(M,\mathbb{Z}) \qquad\qquad [\Sigma](\mathcal{L}) \in H_2(M,\mathbb{Z}) \tag{235}$$

for $\Sigma \subset M^3$ a closed surface. Similarly

$$e_\Sigma = \int_\Sigma \star F \ . \tag{236}$$

5.3. Quantum field theory. We will work in the Hamiltonian formalism. Roughly speaking, quantum field theory (QFT) on $M^{d-1}\times\mathbb{R}$ (thought of as space crossed with time) will be quantum mechanics on some space of fields on M^3. So we will study a Hilbert space attached to a fixed time slide, which is roughly

$$L^2\left(\text{fields on } M^{d-1}\right) \ . \tag{237}$$

In electromagnetism we started with the field strength F, and replaced it with this connection A. Following the above heuristic, this means that in Maxwell theory, the Hilbert space is roughly:

$$\mathcal{H} = \text{“}L^{2\prime\prime}\left(\mathcal{C}\left(M^3\right)\right) \tag{238}$$

where $\mathcal{C}$ takes the isomorphism classes of line bundles $\mathcal{L}$ and $\mathrm{U}(1)$ connection ∇. I.e. this consists of isomorphism classes of connections $|\mathrm{conn}(M)|$. Connections always have automorphisms (circle rotation at least), but we'll just look at the set of isomorphism classes.

It is useful to note hat $\mathcal{C}(M)$ is an (∞-dimensional) abelian Lie group with group operation given by tensor product. Abelian Lie groups all look like:

$$\Lambda \times T \times V \tag{239}$$

for Λ some finite lattice, T some finite-dimensional torus, and V some infinite-dimensional vector space. We will kind of ignore V. Basically the idea is that we have both the lattice and torus present on both sides of the duality.

Now we follow [**FMS07**]. We can take the first Chern class (isomorphism class of underlying line bundle)

$$\mathcal{C}(M) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \quad . \tag{240}$$

This is where the lattice comes from. This is the *total* magnetic flux in the sense that when we pair with a surface, we get the flux through the surface. We can also take the curvature

$$\mathcal{C}(M) \xrightarrow{F} \Omega^2_{\mathbb{Z}}(M) \quad . \tag{241}$$

Integral differential forms "talk" to integral H^2, and we have a short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \arrow[r] & \mathcal{C}(M) \arrow[r, "c_1"] & H^2(M,\mathbb{Z}) \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{C}_{\text{flat}}(M) \arrow[r] \arrow[u] & \mathcal{C}(M) \arrow[r, "F"] & \Omega^2_{\mathbb{Z}}(M) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{242}$$

where

$$\mathcal{C}_{\text{flat}}(M) = \left|\mathrm{Loc}_{\mathrm{U}(1)} M\right| = H^1(M,\mathbb{R}/\mathbb{Z}) = \mathrm{U}(1) \otimes_{\mathbb{Z}} H^1(M,\mathbb{Z}) \tag{243}$$

is a torus of dimension $b_1(M)$. There is a map

$$\mathcal{C}(M) \to \Lambda = H^2(M,\mathbb{Z}) \tag{244}$$

to the magnetic flux, and inside of here there is a torus:

$$T = \left|\mathrm{Loc}_{\mathrm{U}(1)} M\right| \to \mathcal{C}(M) \quad . \tag{245}$$

Now we will see that electric-magnetic duality corresponds exactly to Pontrjagin duality on $\mathcal{C}(M)$. We will be able to write the same Hilbert space in two ways, and there is a sense in which magnetic measurements on one side are dual to electric measurements on the other side.

6. Quantum field theory

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transition between lectures

6.1. Lagrangian formalism. Lecture 7; February 9, 2021

We will be doing euclidean quantum field theory in dimension d. First we will give a schematic overview of the Lagrangian formalism. The idea is that, to a d-manifold M, we will attach a space of *fields* $\mathcal{F}(M)$. These are some local quantities on our space, for example functions or sections of a bundle.

To any field $\varphi \in \mathcal{F}(M)$, we can attach the *action* $S(\varphi) \in \mathbb{C}$. This is a way of prescribing the classical equations of motion. Instead of finding solutions to some equations of motion one studies critical points of this function S.

In quantum field theory, we do some kind of "probability theory" on $\mathcal{F}(M)$ with "measure" given by

$$e^{-S(\varphi)/\hbar} D\varphi \ . \tag{246}$$

We can think of this thing as being a "vanilla" measure on the space of states that is then weighted by the action S. The idea is that, as $\hbar \to 0$, this concentrates on solutions to the equations of motion. We won't try to make mathematical sense of this, but this is the schematic.

For M^d closed, we can attach the partition function, which is the volume (total measure) of this space of fields:

$$Z(M) = \int_{\mathcal{F}(M)} e^{-S(\varphi)/\hbar} D\varphi \ . \tag{247}$$

This isn't a very interesting quantity, and we often normalize so that this is 1. The more interesting thing to calculate are the expectation values of operators.

Example 25. One example of an operator, specifically a local operator $\mathcal{O}_x$ at $x \in M$, is the functional on $\mathcal{F}(M)$ given by evaluation (making a measurement) at $x \in M$.

Now we can take the expectation value of this measurement:

$$\langle \mathcal{O}_x \rangle = \int_{\mathcal{F}(M)} \frac{\mathcal{O}_x(\varphi)\, e^{-S(\varphi)/\hbar} D\varphi}{Z(M)} \ , \tag{248}$$

where we're dividing by $Z(M)$ to normalize the measure. Then we might calculate correlation functions, where we're taking several different measurements at different points.

There are also "disorder" operators, where inserting the operator means we look at fields with a prescribed singularity at x. I.e. we're looking at all fields $\mathcal{F}(M \setminus \{x\})$, and the operator is something like the delta function on some prescribed singularity space, i.e. it's picking out fields with some prescribed singularity at x.

6.2. Time evolution. Let M be a Riemannian manifold with boundary:

$$\partial M = \partial M_{\text{in}} \sqcup \partial M_{\text{out}} \ . \tag{249}$$

This is a Riemannian *bordism* from ∂M_{in} to ∂M_{out}. See fig. 1 for a picture.

Example 26. Let N be an $(n-1)$-manifold. Then $M = N \times I$ is a d-dimensional bordism from N to itself.

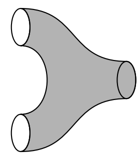

FIGURE 1. A bordism from the disjoint union of two copies of S^1 to a single copy of S^1.

This gives us a correspondence of fields:

$$\begin{tikzcd} & \mathcal{F}(M) \arrow[dl] \arrow[dr] & \\ \mathcal{F}(\partial M_{\text{in}}) & & \mathcal{F}(\partial M_{\text{out}}) \end{tikzcd} \quad . \tag{250}$$

The functional $e^{-S(\varphi)/\hbar}D\varphi$ lives over $\mathcal{F}(M)$, and so we get an integral transform given by pulling, multiplying by this functional, and integrate (push forward).

This integral transform gives us an operator $Z(M)$ between

$$\mathcal{H}_{\text{in}} = \text{functionals on } \mathcal{F}(\partial_{\text{in}}) \tag{251}$$

and

$$\mathcal{H}_{\text{out}} = \text{functionals on } \mathcal{F}(\partial_{\text{out}}) \ . \tag{252}$$

Then

$$\begin{aligned} (Z(M)(f))(\varphi_{\text{out}}) &= \int_{\varphi|_{\partial_{\text{out}}}=\varphi_{\text{out}}} f(\varphi|_{\partial_{\text{in}}})\, e^{-S(\varphi)/\hbar} D\varphi \\ &= \pi_{\text{out}*}\left(\pi_{\text{in}}^*(f)\, e^{-S(\varphi)/\hbar} D\varphi\right) \ . \end{aligned}$$

This operator is the time evolution operator.

REMARK 17 (Hamiltonian versus Lagrangian). In the Hamiltonian formulation we start with a Hilbert space $\mathcal{H}$, which is associated to a time slice. Then we're also supposed to give a Hamiltonian operator H, and then time evolution is given by the operator $e^{iTH/\hbar}$. Then we have an algebra of observables and make various measurements.

On the other hand in the Lagrangian formalism we start with the space of fields and the action. The Lagrangian formulation is very flexible in the sense that it allows us to define these push-pull operators, which is how we recover a Hilbert space and time evolution operator (Hamiltonian formalism) from the Lagrangian formalism.

EXAMPLE 27. One-dimensional QFT is quantum mechanics. Let X be a Riemannian target. Then the space of fields might be $\mathcal{F}(\mathbb{R}) = \text{Maps}(\mathbb{R}, X)$. The critical points of S will be geodesics in X. The Hilbert space is

$$\mathcal{H} = L^2(\text{Maps}(\text{pt}, X)) = L^2(X) \ , \tag{253}$$

and the Hamiltonian is the Laplace operator Δ.

To formalize all of this would require a great deal of work. We also don't want to perform any perturbative techniques since we don't necessarily want $\hbar$ to be small. We will use this as a schematic guide, and formally pass to the topological setting.

7. Quantum Maxwell theory

We will consider a four-dimensional quantum field theory. Our space of fields on a 4-manifold M^4 consists of $\mathrm{U}(1)$ bundles with connection $d + A$ (A is the electromagnetic potential) up to gauge equivalence, i.e. we mod out by the action of the gauge transformations.

The classical equations of motion are Maxwell's equations:

$$\begin{cases} dF = 0 \\ d \star F = 0 \ , \end{cases} \tag{254}$$

where F is the curvature. Then we get an action

$$S = \frac{g}{2\pi i} \int_M F \wedge \star F + \theta \int_M F \wedge F \ . \tag{255}$$

constants?

The integral in the second term is just calculating $c_1^2 = p_1$ of the line bundle. This term is called the topological term.

The idea is that the Hilbert space is attached to a specific time slice. So assume we can write $M^4 = M^3 \times \mathbb{R}$, and then the Hilbert space is

$$\mathcal{H} = L^2\left(\mathcal{C}\left(M^3\right)\right) \tag{256}$$

where $\mathcal{C}\left(M^3\right)$ denotes the collection of line bundles with connections modulo gauge transformations on M^3. $\mathcal{C}\left(M^3\right)$ is an abelian Lie group, and has a map to the lattice of possible line bundles up to isomorphism:

$$\mathcal{C}\left(M^3\right) \xrightarrow{c_1} \Lambda = H^2(M, \mathbb{Z}) \tag{257}$$

and also a sub given by the torus of flat connections:

$$T = \text{flat connections} \simeq H^1(M, \mathbb{R}/\mathbb{Z} = \mathrm{U}(1)) \simeq BH^1(M, \mathbb{Z}) \to \mathcal{C}\left(M^3\right) \tag{258}$$

where $\mathrm{U}(1)$ is taken to have the discrete topology. Then the final factor of $\mathcal{H}$ is given by an infinite-dimensional vector space V. I.e. there is a (noncanonical) splitting $\mathcal{C}(M) \simeq \Lambda \times T \times V$.

7.1. Operators. Now we want to identify some operators on $\mathcal{H}$ (i.e. observables) that appear naturally from the setup. The first thing to say is that $\mathcal{H}$ has an obvious grading by $\Lambda = H^2(M, \mathbb{Z})$. This already picks out some operators, e.g. by telling you which component you're on. This is the magnetic flux, so it is said that this is a grading by magnetic fluxes.

7.1.1. *Dirac/'t Hooft operators.* Λ also acts on $\mathcal{H}$ to yield 't Hooft operators.[5] If we've already chosen a splitting $\mathcal{C}(M) \simeq \Lambda \times T \times V$, then the action is just by translation. I.e. these operators shift the magnetic flux. Recall the flux roughly records the number of enclosed monopoles. So physically these operators create magnetic monopoles.

[5] Technically 't Hooft operators are the nonabelian generalization of these. These operators really have to do with Dirac monopoles, but this would be confusing since the term *Dirac operator* already has another meaning.

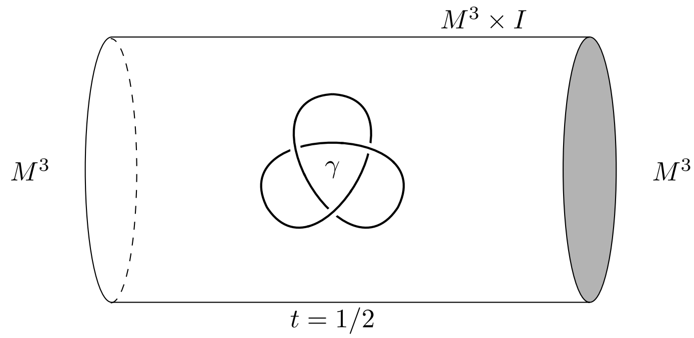

FIGURE 2. The loop γ lives in some time slice in $M^3 \times I$, say $t = 1/2$. We can excise this, and fields on the resulting 4-manifold will possibly have singularities along γ. Asking for the integral around the boundary of a neighborhood of γ to be 1 gives us a space of fields with controlled singularity inside the neighborhood of γ. Physically this introduces a magnetic monopole along γ.

More precisely, let γ be a simple closed curve in M. Note that this defines a class

$$[\gamma] \in H_1(M, \mathbb{Z}) \simeq H^2(M, \mathbb{Z}) \simeq \Lambda \; . \tag{259}$$

The associated operator introduces a monopole along γ as follows. The idea is that γ lives at a particular time slice in $M^3 \times I$, and we're studying electromagnetism on $M \times I$ with a monopole along γ as in fig. 2.

In other words, we're considering fields $\mathcal{C}(M \times I \setminus \gamma)$, i.e. the connections are possibly singular along γ. So we have excised the knot, and this has introduced a new boundary component of our 4-manifold: the link of the knot (boundary of tubular neighborhood of the knot) which looks like $S^2 \times S^1$. Then we can look at connections which have a specific integral over this. In particular, we can ask for

$$\frac{1}{2\pi i} \int_{S^2} F = 1 \; . \tag{260}$$

Write the resulting space of fields as:

$$\mathcal{C}(M \times I \setminus \gamma)_{c_1=1} \; . \tag{261}$$

Note that, physically, this is saying that this surface $S^2 \times S^1 \subset M^3 \times I$ contains magnetic charge 1. Mathematically we're seeing that we can't extend the connection over this curve. Physically we're seeing a monopole along this curve.

The actual operator is defined as follows. The 4-manifold $M \times I \setminus \gamma$ is a bordism from M^3 to itself, and therefore defines a correspondence and integral transform

(Feynman path integral) as usual:

$$\begin{tikzcd} & \mathcal{C}\left(M \times I \setminus \gamma\right)_{c_1=1} \arrow[dl, "\pi_{\mathrm{in}}"'] \arrow[dr, "\pi_{\mathrm{out}}"] & \\ \mathcal{C}\left(M^3_{\mathrm{in}}\right) & & \mathcal{C}\left(M^3_{\mathrm{out}}\right) \end{tikzcd} \tag{262}$$

$$\mathcal{H} \xrightarrow{\pi_{\mathrm{out}*}\left(e^{-S(\varphi)/\hbar} D\varphi \pi^*_{\mathrm{in}}(-)\right)} \mathcal{H}\ .$$

Physically this operator takes the space of states and evolves it through time, during which a monopole is introduced along γ and then removed. Mathematically it is shifting the Chern class by $[\gamma]$, i.e. it is just tensoring with a flat line bundle with prescribed Chern class

$$c_1 = [\gamma] \in H^2\left(M, \mathbb{Z}\right)\ . \tag{263}$$

7.1.2. *Wilson loop operators.* On the other hand, there are much simpler observables called the Wilson loop operators. Again let $\gamma \subset M$ be a simple closed curve. This determines a function W_γ on $\mathcal{C}\left(M\right)$ given by:

$$W_\gamma\colon\ (\mathcal{L}, \nabla) \mapsto \text{holonomy along } \gamma \in \mathrm{U}\left(1\right) \subset \mathbb{C}\ . \tag{264}$$

We can draw the same picture as fig. 2, but we're doing something very different with it. We are still thinking of γ as living on some time slice inside $M \times I$. Now, as time evolves, we make the measurement of the holonomy along γ. So this is a measurement just like the example of a local operator in example 25 only now the measurement is along a loop, so this is called a line/loop operator. This still gives us a correspondence and an integral transform, only now the kernel of the transform is given by the function W_γ:

$$\begin{tikzcd} & \mathcal{C}\left(M \times I\right) \arrow[dl, "\pi_{\mathrm{in}}"'] \arrow[dr, "\pi_{\mathrm{out}}"] & \\ \mathcal{C}\left(M^3_{\mathrm{in}}\right) & & \mathcal{C}\left(M^3_{\mathrm{out}}\right) \end{tikzcd} \tag{265}$$

$$\mathcal{H} \xrightarrow{\pi_{\mathrm{out}*}\left(W_\gamma \pi^*_{\mathrm{in}}(-)\right)} \mathcal{H}\ .$$

The point is that we can multiply functions on this space of connections by this function W_γ. So these Wilson operators are already diagonalized given the way we've presented the Hilbert space.

The Wilson loops W_γ are eigenfunctions for the action of the space of flat connections:

$$T = \mathcal{C}_\flat\left(M\right) \simeq H^1\left(M, \mathrm{U}\left(1\right)\right)\ . \tag{266}$$

This acts on $\mathcal{C}\left(M\right)$ (by tensoring), so therefore it also acts on $L^2\left(\mathcal{C}\left(M\right)\right)$. The eigenvalues are given by multiplying by the monodromy along γ: given a flat connection (element of $H^1\left(M, \mathrm{U}\left(1\right)\right)$) we canonically get an element of $\mathrm{U}\left(1\right)$ by taking the monodromy along γ, and this is exactly doing Fourier series. These W_γ are the characters of this torus T.

7.2. Electric-magnetic duality. Notice that when we were studying 't Hooft operators we were paying attention to the action of the lattice, and we had a grading by magnetic fluxes. Now we're focusing on the torus action, and we can diagonalize. This gives a decomposition by characters of T, which comprise the lattice $H^2(M,\mathbb{Z})$:

$$\mathcal{H} = \bigoplus_{e \in H^2(M,\mathbb{Z})} \mathcal{H}_e \ , \tag{267}$$

and this can be interpreted as a grading by electric fluxes.

Passing from the magnetic grading to the electric grading is exactly performing a Fourier transform with respect to the torus part of our space of fields. In general, electric-magnetic duality can be thought of as doing Fourier series on our Lie group of fields. It specifically identifies:

$$L^2\left(\mathcal{C}_{\mathrm{U}(1)}(M)\right) \simeq L^2\left(\mathcal{C}_{\mathrm{U}(1)^\vee}(M)\right) \tag{268}$$

where the left side is the F side, and the right side is the $\star F$ side. The group always splits as a lattice, a torus, and a vector space (which we are ignoring). The F and $\star F$ sides are respectively associated with the splittings:

$$\Lambda_B \times T_E \times \textbf{Vect} \qquad\qquad \Lambda_E \times T_B \times \textbf{Vect} \ . \tag{269}$$

Λ_B is the lattice of magnetic fluxes, T_B is its dual torus, Λ_E is the lattice of electric fluxes, and T_E is its dual torus. On the F (left) side, Wilson loop operators are diagonalized, and on the $\star F$ (right) side, the 't Hooft operators are diagonalized. Similarly, the magnetic b grading goes to the electric e grading.

7.3. Abelian duality in two-dimensions. Recall this is called T-duality, or mirror symmetry. The space of fields is $\mathrm{Map}\left(M, S^1\right)$, or more generally into a torus T. Our Hilbert space $\mathcal{H}$ on S^1 was then

$$L^2\left(\mathrm{Map}\left(S^1, S^1_R\right)\right) \ . \tag{270}$$

Note that again $\mathrm{Map}\left(S^1, S^1\right)$ is an abelian Lie group under the operation of pointwise multiplication, i.e. we're using the group structure of the target not the source. There is also a map to a lattice: this is graded by $H^1\left(S^1,\mathbb{Z}\right)$ which is given by taking the winding number.

Dually we can study:

$$L^2\left(\mathrm{Maps}\left(S^1, \left(S^1_R\right)^\vee = S^1_{1/R}\right)\right) \ . \tag{271}$$

This is graded by $H^0\left(S^1, \mathrm{U}(1)\right)$, which is dual to the grading above. We have an operator given by shifting the winding number (analogues of the 't Hooft operators) and operators given by evaluation at a point (analogues of the Wilson operators) and the duality exchanges them.

8. Topological quantum mechanics/quantum field theory

The idea is that quantum mechanics is hard because it involves analysis. In ordinary quantum mechanics, we started with a point and assigned a Hilbert space $\mathcal{H}$. Then to an interval of length T, we assigned unitary time evolution operator

$$e^{iTH/\hbar} \ . \tag{272}$$

In *topological* quantum mechanics (TQM) we want to kill time, i.e. we want $H = 0$. The naive way to interpret this is to restrict to ground states. But a closed manifold does not admit any nonconstant bounded harmonic functions. So we killed the entire theory.

Witten [**Wit82**] introduced the following technique to kill time in a derived sense via super-symmetry (SUSY). Let X be a Riemannian manifold. Instead of $L^2(X)$, we expand our Hilbert space to be L^2 differential forms, i.e. we add new fields, and now we get a bigger symmetry group given by a *super Lie group*. So now we consider: what acts on differential forms? The first thing is the Laplace operator $H = \Delta$. We also have the de Rham differential $Q = d$ and its adjoint and $Q^* = d^*$. We also have a $\mathrm{U}(1)$ action which gives a $\mathbb{Z}$-grading. This package of operators is called the $\mathcal{N} = 1$ SUSY algebra. So we have operators where H has degree 0, Q has degree 1, and Q^* has degree -1. Most of these operators commute:

$$[Q^*, Q^*] = [Q, Q] = 0 \tag{273}$$

$$[Q, H] = [Q^*, H] = 0 \tag{274}$$

except

$$[Q, Q^*] = H \ . \tag{275}$$

$[Q, Q] = 0$ can be interpreted as $d^2 = 0$. The other relation is that

$$\Delta = dd^* + d^*d \ , \tag{276}$$

which says that Q^* gives a homotopy from H to 0. In particular, this implies that, on Q-cohomology, H acts by 0.

So we added new fields, and we got a larger symmetry algebra. One of the operators in this algebra is an odd operator squaring to 0, and we call it the differential. On the associated cohomology, H is zero. So we've killed time in a derived sense. So topological quantum mechanics assigns, to X, the complex

$$(\Omega^\bullet(X), d) \ , \tag{277}$$

or $H^*(X)$. In the first case H is homotopic to 0, and in the latter $H = 0$. This is the way de Rham cohomology gets recovered from quantum mechanics.

We will never be dealing with honest quantum field theories. Instead, we will be doing this topological twist. Note that in this setting, the theory only depends topologically on X. The upshot is that we're killing the dependence on Riemannian metrics, which will make life much easier. Lecture 8; February 11, 2021

9. Topological quantum mechanics

The way to set this up formally, is that we have the $d = 1$, $\mathcal{N} = 1$ SUSY algebra acting on $\mathcal{H}$. The $d = 1$ means we're doing quantum mechanics and $\mathcal{N} = 1$ just means the smallest amount of supersymmetry.

transition between lectures

There are two natural realizations of TQM. The first is the A-type TQM. Take X to be a Riemannian manifold, and then the Hilbert space consists of differential forms:

$$\mathcal{H} = \Omega^\bullet(X) \ , \tag{278}$$

the Hamiltonian $H = \Delta$ is the Laplace operator, $Q = d$ is the de Rham differential, and $Q^* = d^*$ is its adjoint with respect to the metric. The grading is the usual one on forms, and the cohomology is

$$H^\bullet(\mathcal{H}) = H^\bullet_{\mathrm{dR}}(X) \ . \tag{279}$$

There is another realization, called B-type TQM. Now take X to be a complex manifold. The Hilbert space is

$$\mathcal{H} = \Omega^{0,\bullet}(X) \tag{280}$$

i.e. forms with $\overline{\partial}$ in them. The SUSY operators are given by $Q = \overline{\partial}$, $Q = \overline{\partial}^*$, and $H = \Delta_{\overline{\partial}}$. The cohomology is Dolbeault cohomology:

$$H^0(\mathcal{H}) = H^{0,*}(X) = R\Gamma(\mathcal{O}_X) \ . \tag{281}$$

These were both realizations of the 1-dimensional $\mathcal{N} = 1$ SUSY algebra. There is also $\mathcal{N} = 2$ TQM. One such example comes from studying a Kähler manifold X. The Hilbert space is still differential forms:

$$\Omega^{\bullet,\bullet}(X) \ . \tag{282}$$

This is bigraded, rather than having a single grading like before, and this has a bunch of operators, e.g. ∂, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$. The Kähler identities tell us that the associated Laplacians agree: $\Delta_{\overline{\partial}} = \Delta_d$.

For a mathematician, there is the following deep theorem. We have an action of $\mathrm{SU}(2)$ on $\Omega^{\bullet,\bullet}(X)$, which can be thought of as coming from its complexification $\mathrm{SU}(2) \subset \mathrm{SL}_2\,\mathbb{C}$. These are the Lefschetz operators. The diagonal part is giving a cohomological grading, and then there is a raising operator (intersecting with Kähler form) and a lowering operator. Mathematically this is a very deep statement about global cohomology of a Kähler manifold (the hard Lefschetz theorem). Physically this is a calculation of which SUSY algebra acts on our Hilbert space, and then this $\mathrm{SU}(2)$ actions is the R-symmetry. The point is that we found a big SUSY algebra.

The $\mathcal{N} = 1$ SUSY algebra encoded Hodge theory of Riemannian manifolds. This $\mathcal{N} = 2$ SUSY algebra encodes complex Hodge theory. There is also $\mathcal{N} = 4$ TQM. In this case we're studying differential forms on a compact hyperkähler manifold X: $\Omega^\bullet(X)$. The analogue of Lefschetz SL_2 action for hyperkähler is an action of $\mathrm{Spin}\,5$. The idea is that if we assemble all of the Lefschetz SL_2s together, you get a $\mathrm{Spin}\,5$. This is part of a super-Lie group that we won't write down. From the point of view of the physics, this is just a natural calculation about what operators are sitting around when we're studying the theory of particles in this manifold.

REMARK 18. There is also $\mathcal{N} = 8$ SUSY. These different SUSY algebras are really distinguished by the number of odd operators (a.k.a. surpercharges Q) are present. E.g. $\mathcal{N} = 1$ has two, $\mathcal{N} = 2$ has 4, $\mathcal{N} = 4$ has 8, and $\mathcal{N} = 8$ had 16.

The general philosophy of TQM and eventually TQFT is to add fields so that we have an action of a big super-Lie group (SUSY algebra). We're looking for two things:

(1) that the Hamiltonian is exact: $H = [Q, -]$ (i.e. we're killing time[6]), and
(2) $T = [Q, -]$, i.e. the metric dependence is exact (i.e. we're killing geometry).

The stress energy tensor T is an object in QFT which measures the dependence on the metric. Mathematically T is a derived version of invariance under isometry. This is explained nicely in Costello-Gwilliam [**CG17**].

[6] I.e. time evolution is made trivial.

9.1. Topological Maxwell theory. We want to take the quantum theory of light (ordinary Maxwell theory), add some fields, and find the SUSY algebra. Then we will pick some Q and pass to cohomology. This will be an $\mathcal{N}=4$ GL-twisted theory. In particular, this $\mathcal{N}=4$ means that there is a lot of super-symmetry: there will be sixteen supercharges.

Before, our space of fields consisted of a line bundle and a choice of connection $\nabla = d + A$. Now we will add:

- a 1-form σ on the manifold (the Higgs field),
- a complex scalar u, and
- four fermions (odd fields) (we will not pay much attention to these).

The Hilbert space used to be $\mathcal{H} = L^2(\mathcal{C}(M))$. Now, in the A-twist, the 3-manifold M^3 gets attached to the cohomology of this space: $H^\bullet(\mathcal{C}(M))$. We have a much bigger space of fields now, but it doesn't actually make a difference at the level of the cohomology, since introducing these new fields didn't change the topology of the space. But as it turns out, we don't want ordinary cohomology, we want cohomology which is equivariant with respect to the automorphisms of the connections, i.e. we want:

$$\mathcal{H} = H^\bullet(\text{connections/gauge equivalence}) \tag{283}$$
$$= H^\bullet_{\mathrm{U}(1)}(\mathcal{C}(M)) \ . \tag{284}$$

This is doing topological quantum mechanics in the *de Rham* sense on the space of connections.

There is another version called the B-twist. Now we modify the fields by thinking of:

$$\nabla + i\sigma = d + (A + i\sigma) \tag{285}$$

as a connection on a $\mathbb{C}^\times$-bundle rather than a $\mathrm{U}(1)$-bundle. The vector space attached to M^3 is Dolbeault TQM on $\mathrm{Loc}_{\mathbb{C}^\times}(M)$. This just means the vector space is (some derived version of) holomorphic functions on $\mathrm{Loc}_{\mathbb{C}^\times}(M)$.

Now we reformulate the A-side to connect with what we've seen. Write $\mathrm{Pic}(M)$ for the underlying topological space of the space of connections $\mathcal{C}(M)$, i.e. it doesn't detect the Riemannian geometry of M. $H^0(\mathrm{Pic}(M))$ consists of locally constant functions on the space of connections $\mathcal{C}(M)$, i.e.

$$H^0(\mathrm{Pic}(M)) = \mathbb{C}[\pi_0(\mathcal{C}(M))] \tag{286}$$
$$= \mathbb{C}\left[H^2(M,\mathbb{Z})\right] \ , \tag{287}$$

which is the same vector space we were previously attaching to M in the discussion summarized in table 1. The full cohomology of this space is:

$$H^\bullet(\mathrm{Pic}(M)) = H^\bullet(\Lambda \times T \times V \times B\,\mathrm{U}(1)) \ . \tag{288}$$

Before we only saw $\Lambda = H^2(M,\mathbb{Z})$, but now we will see

- the cohomology of $T = H^1(M, \mathrm{U}(1))$ (an exterior algebra),
- the vector space doesn't contribute to cohomology, but we also see
- the cohomology of $B\,\mathrm{U}(1)$, which will look like $\mathbb{C}[u]$.

So this is the shape of our Hilbert space after making a topological twist, and on the B-side there will also be factors corresponding to these extra exterior and symmetric pieces.

TABLE 2. Summary of the topological A and B twists of super-Maxwell theory.

A-side	B-side
topology of Pic	AG of Loc
$\Lambda = H^2(M, \mathbb{Z})$	$T_{\mathbb{C}}^{\vee} = \lvert \mathrm{Loc}_{\mathbb{C}^\times}(M) \rvert$
't Hooft operators (shifting lattice (magnetic flux) by $\gamma^\vee \in H^2(M, \mathbb{Z})$)	Wilson operators (multiply by monodromy along γ)
Create magnetic monopole	Create electric particle.
Magnetic side	Electric side

On the B-side we're looking at the space $\mathrm{Loc}_{\mathbb{C}^\times} M$. Up to some derived factors, this looks like a complex torus $\mathrm{Loc}_{\mathbb{C}^\times} M \simeq T_{\mathbb{C}}^{\vee}$ of maps $\pi_1(M) \to \mathbb{C}^\times$ which factor as:

$$\begin{tikzcd} \pi_1(M) \arrow[rr] \arrow[dr] & & \mathbb{C}^\times \\ & H_1(M) \arrow[ur] & \end{tikzcd} \quad . \tag{289}$$

$H_1(M)$ looks like Λ (by Poincaré duality), so this part is the dual torus to the lattice Λ. Recall the Fourier series identifies:

$$\mathbb{C}\left[H^2(M, \mathbb{Z})\right] \qquad \text{and} \qquad \mathbb{C}\left[T_{\mathbb{C}}^{\vee}\right] \ , \tag{290}$$

and now we have some extra (derived) factors, coming from $H^2(M, \mathbb{C})$ and $H^3(M, \mathbb{C})$. The $H^2(M, \mathbb{C})$ factor corresponds to the exterior algebra factor on the A-side, and the $H^3(M, \mathbb{C})$ factor corresponds to the symmetric algebra factor on the A-side.

So again, we had this notion of a Fourier transform, which exchanges:

$$\Lambda = H^2(M, \mathbb{Z}) \qquad \text{and} \qquad T_{\mathbb{C}}^{\vee} = \lvert \mathrm{Loc}_{\mathbb{C}^\times}(M) \rvert \ , \tag{291}$$

and this turns out to just be the degree 0 part of the duality between these A and B-twists, where we have this extra exterior algebra, and extra symmetric algebra.

Mottos:

- The A-twisted super Maxwell theory with gauge group T studies the topology of $\mathrm{Pic}(M)$, and
- the B-twisted version of super Maxwell theory with gauge group $T^\vee$ studies the algebraic geometry of $\mathrm{Loc}_{\mathbb{C}^\times}(M)$.
- E-M duality switches these two.

See table 2 for a summary.

REMARK 19. We looked at two different twists, i.e. two different realizations of the SUSY algebra, i.e. two different charges. We could also take any linear combination of these two and get a new topological theory. In other words, this is a $\mathbb{P}^1$ family of possible topological theories which all arise from quantum Maxwell theory. The A and B-twists are then two extreme points of $\mathbb{P}^1$. On one end only magnetic phenomena are left, and on the other only electric phenomena are left.

9.2. Defects. There are two other "physics operations" done in electromagnetism called defects, which we will describe at the topological level. These amount to changing the fields we're considering, e.g. introducing singularities.

9.2.1. *Time-like line defects.* The first type of defect we will consider is a "time-like" 't Hooft loop/line. Consider a point $x \in M^3$. Then we can consider electromagnetism in the presence of a monopole at x. Mathematically this means we look at the space of connections:

$$\mathrm{Pic}\,(M \setminus x) = |\mathcal{C}\,(M \setminus x)| = \coprod_n \mathrm{Pic}\,(M, nx) \tag{292}$$

where

$$\mathrm{Pic}\,(M, nx) = \{A \in \mathrm{Pic}\,(M \setminus x) \,|\, c_1 = n \text{ on sphere at } x\} \ . \tag{293}$$

E.g. $\mathcal{C}\,(M, x)$ is the charge 1 component. So now we get a new Hilbert space by linearizing $\mathcal{C}\,(M \setminus x)$ (via taking L^2 or H^*).

Dually (on the B-side) we have a "timelike" Wilson loop/line. For $x \in M$ we take our fields to be:

$$\mathrm{Loc}_{\mathbb{C}^\times}\,(M, x) \tag{294}$$

which are flat $\mathbb{C}^\times$ connections equipped with a trivialization of the fiber at x. Physically this is interpreted as adding a heavy[7] charged particle, but it breaks the gauge symmetry at this point. This new space of fields $\mathrm{Loc}_{\mathbb{C}^\times}\,(M, x)$ is a $\mathbb{C}^\times$-bundle over $\mathrm{Loc}_{\mathbb{C}^\times}\,(M)$:

$$\begin{tikzcd} \mathrm{Loc}_{\mathbb{C}^\times}\,(M, x) \arrow[d, "\mathbb{C}^\times"] \\ \mathrm{Loc}_{\mathbb{C}^\times}\,(M) \end{tikzcd} \quad . \tag{295}$$

On the B-side we've introduced an extra factor of $\mathbb{C}^\times$, and on the A-side we've introduced an extra factor of $\mathbb{Z}$, by allowing different charges. Fourier series will identify these factors:

$$H^*\,(\mathrm{Pic}\,(M \setminus x)) \simeq \mathbb{C}\,[\mathrm{Loc}_{\mathbb{C}^\times}\,(M, x)] \ . \tag{296}$$

This is another instance of electric-magnetic duality: creating a magnetic monopole corresponds to creating an electrically charged particle.

9.2.2. *Surface defects.* Another type of defect is a *surface defect*, also known as ramification, or introducing a solenoid. The idea is that we have a long tube, with a wire coiled around the outside, and then we send a current through it and this creates a magnetic field inside of the tube. So this is 1-dimensional subspace of a 3-manifold M^3, and inside of this crossed with time, $M \times I$, this defines a surface.

On the B-side, this introduces a singularity of the electric field, i.e. our fields are:

$$\mathrm{Loc}_{\mathbb{C}^\times}\,(M \setminus \beta) \ . \tag{297}$$

The holonomy around β introduces an extra factor of $\mathbb{Z}$ in H_1.

The magnetic (A-side) version of this has fields given by the same space of connections $\mathcal{C}\,(M)$, but equipped with a trivialization along β. How does this help you? A trivialization (or the difference between two trivializations) along a loop is a map from $\beta = S^1$ to $\mathrm{U}\,(1)$. This has a winding number, so we get $\mathbb{Z}$-many

[7] Heavy just means we're not adding a new field and doing field theory with it.

components to these fields. This is dual to the extra $\mathbb{C}^\times$ from the holonomy around β on the B-side. So this is another version of Fourier series. Explicitly this identifies:

$$\mathbb{C}\left[\mathrm{Loc}_{\mathbb{C}^\times}\left(M \setminus \beta\right)\right] \simeq H^*\left(\mathbb{C}\left(M \text{ trivialized along } \beta\right)\right) \ . \tag{298}$$

CHAPTER 3

Class field theory

Topology and physics gave us one source of interesting duality statements. Number theory, specifically class field theory (CFT), is another source. The topology (e.g. the locally constant functions) of Pic will still be exchanged with the algebraic geometry (e.g. the algebraic functions) of Loc. But we need to interpret Pic and Loc in this context.

Let $F/\mathbb{Q}$ be a number field. Write $\mathcal{C}\ell\,(F)$ for the ideal class group. We can think of this as:

$$\mathcal{C}\ell\,(F) = \operatorname{Pic}\left(\operatorname{Spec}\mathcal{O}_F\right) \ , \tag{299}$$

which consists of line bundles on $\operatorname{Spec}\mathcal{O}_F$, i.e. rank 1 projective $\mathcal{O}_F$-modules modulo isomorphism. Concretely, the *class group* is

$$\mathcal{C}\ell\,(F) = \text{fractional ideals of } \mathcal{O}_F/\text{principal ideals} \ . \tag{300}$$

Note that this is an abelian group. As it turns out $\mathcal{C}\ell\,(F)$ is also a finite group, and its order is an important invariant of F called the *class number*. The general philosophy of CFT is to relate the class group to the Galois group.

1. Unramified class field theory

Unramified CFT identifies:

$$\mathbb{C}\left[\mathcal{C}\ell\,(f)\right] \simeq \mathbb{C}\left[\operatorname{Loc}_{\mathbb{G}_m}(\mathcal{O}_F)\right] \ . \tag{301}$$

But we need to specify what Loc is in this context. We can always think of Loc as consisting of representations of π_1, and we can also write:

$$\pi_1(M) = \operatorname{Aut}\left(\widetilde{M}\right) \ , \tag{302}$$

where $\widetilde{M}$ is the universal cover of M. In this context, we have:

$$\operatorname{Gal}(E/F) = \operatorname{Aut}_F(E) \ , \tag{303}$$

and the analogue of the universal cover is the *maximal unramified extension* of F, written F^{ur}. So the analogue of π_1 is

$$\pi_1^{\mathrm{et}}\left(\operatorname{Spec}\mathcal{O}_F\right) = \operatorname{Gal}\left(F^{\mathrm{ur}}/F\right) \ . \tag{304}$$

We take F^{ur} because we want to consider coverings of $\operatorname{Spec}\mathcal{O}_F$, not of F itself. We say E/F is *unramified at a prime p* if

$$\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F/p \left(= \mathcal{O}_E/p\right) \tag{305}$$

is a product of fields, i.e. it has no nilpotents. The idea is that we have $\operatorname{Spec}\mathcal{O}_E$ living over $\operatorname{Spec}\mathcal{O}_F$, and we don't want there to be any branching as in fig. 1. I.e. if we look at the preimage of a points in the base, we want a product of fields. If we have some nilpotence, then this tells us there is some interesting geometry at

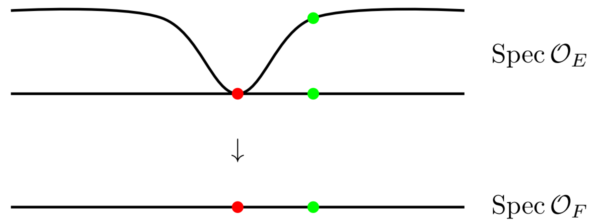

FIGURE 1. An extension which is ramified at the red point (on the left), and unramified at the green point (on the right).

that point, which we want to avoid for now. Arbitrary extensions will stay play the role of covering spaces away from the ramification point. E/F is unramified at ∞ if there are no $\mathbb{C}/\mathbb{R}$ extensions between them. E.g. if F is totally imaginary. The idea is that tensing with $\mathbb{R}$ corresponds to restricting to the neighborhood of infinity. But this is just a bunch of copies of $\mathbb{R}$ and $\mathbb{C}$:

$$\mathcal{O}_F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s \tag{306}$$

and then we insist that going from F to E never involves extending from $\mathbb{R}$ to $\mathbb{C}$.

Now that we have established the basics, we can state that unramified CFT tells us that

$$\mathcal{C}\ell \simeq \mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right) = \mathrm{Aut}\left(\text{Hilbert class field}\right) \ , \tag{307}$$

where the Hilbert class field is the maximal abelian extension of F which is unramified everywhere.

To make contact with what we have seen before, we should study characters of this Galois group. I.e. on one side we have an analogue to Pic: $\mathcal{C}\ell$, and on the other side our analogue to Loc is:

$$\mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur}}/F\right), \mathbb{C}^\times\right) \ . \tag{308}$$

The point is that homomorphisms into an abelian group factor into the abelian quotient:

$$\mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur}}/F\right), \mathbb{C}^\times\right) = \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right), \mathbb{C}^\times\right) \ . \tag{309}$$

REMARK 20. $\mathrm{Gal}\left(F^{\mathrm{ur}}/F\right)$ is the analogue of $\pi_1\left(\mathrm{Spec}\,\mathcal{O}_F\right)$, and $\mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right)$ is the analogue of $H_1\left(\mathrm{Spec}\,\mathcal{O}_F\right)$. So the equality between these Hom spaces is analogous to when morphisms from $\pi_1\left(\mathrm{Spec}\,\mathcal{O}_F\right)$ factor through its abelianization $H_1\left(\mathrm{Spec}\,\mathcal{O}_F\right)$.

Therefore, from (307), this is the Pontrjagin dual of the class group:

$$\mathrm{Loc}_{\mathbb{G}_m}\left(\mathcal{O}_F\right) = \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur}}/F\right), \mathbb{C}^\times\right) \tag{310}$$
$$= \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right), \mathbb{C}^\times\right) \tag{311}$$
$$= \mathrm{Hom}\left(\mathcal{C}\ell\left(F\right), \mathbb{C}^\times\right) \tag{312}$$
$$= \mathcal{C}\ell\left(F\right)^\wedge \ . \tag{313}$$

REMARK 21. Artin-Verdier duality is an analogue of Poincaré duality for Spec $\mathcal{O}_F$. This was our basis for the arithmetic topology dictionary which says that a number field (or really Spec $\mathcal{O}_F$) is analogous to a 3-manifold. Recall that the duality between Pic and Loc for a 3-manifold was Poincaré duality, and this reformulation of

CFT is a statement of Artin-Verdier duality. But the development of Artin-Verdier duality depends on CFT, so this isn't the direction one takes to prove CFT.

REMARK 22. The arithmetic topological dictionary told us that $\operatorname{Spec} \mathcal{O}_F$ corresponds to a 3-manifold. As it turns out, it is an *unoriented* one. That is, the dualizing object is not the constants. This lack of orientation is analogous to the difference between:

- $\mathbb{Z}/n$ and roots of unity μ_n,
- $\mathbb{Q}/\mathbb{Z}$ and μ_∞,
- $\mathbb{C}^\times$ and $\mathbb{G}_m$, etc.

As it turns out, rather than $\mathbb{Z}$ as the dualizing object, we have the Tate-twist $\mathbb{Z}(1)$ (μ_n with respect to the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$). In Gauss' law, or when calculating the Chern classes of line bundles, we encountered some factors of $2\pi i$. These are accounted for by this phenomenon.

Lecture 9; February 23, 2021

The upshot of the identification of characters of $\mathrm{Gal}^{\mathrm{ur}}$ with $\mathcal{C}\ell_F^\vee$ is that we get a Fourier transform:

$$\mathbb{C}[\mathrm{Cl}_F] \simeq \mathbb{C}[\mathrm{Loc}_1(\operatorname{Spec} \mathcal{O}_F)] \tag{314}$$

where Loc_1 is defined as the characters of $\mathrm{Gal}^{\mathrm{ur}}$.

EXAMPLE 28. If $F = \mathbb{Q}$ then both sides are trivial.

2. Function fields

Let C be a smooth projective curve over some field k. To this we can attach the function field $F = k(C)$, which is the field of rational functions on C. We can also consider the Picard group, $\mathrm{Pic}(C)$, which has k-points given by:

$$\begin{aligned}
\mathrm{Pic}(C)(k) &= \{\text{line bundles on } C\} / \sim && (315)\\
&= \{\text{loc. free rank 1 } \mathcal{O}_C - \text{modules}\} && (316)\\
&= \{\text{divisors}\} / \{\text{pricipal divisors}\} && (317)\\
&= H^1\left(C, \mathcal{O}^\times\right) && (318)
\end{aligned}$$

where in the last line we're describing line bundles by their transition functions, i.e. Čech cocycles. Each line bundle has a degree, which is an integer. If we think of a line bundle as being given by a sum $\sum_{x\in C} n_x x$, then this is explicitly given by the map:

$$\begin{array}{ccc} \mathrm{Pic} & \xrightarrow{\deg} & \mathbb{Z} \\ \sum_{x\in C} n_x x & \longmapsto & \sum n_x \end{array}, \tag{319}$$

which is surjective with kernel given by:

$$0 \to \mathrm{Jac} \to \mathrm{Pic} \to \mathbb{Z} \to 0\ , \tag{320}$$

where Jac is the Jacobian, i.e. line bundles of degree 0. For any choice of $x \in C$, we get a copy $\mathbb{Z}x \hookrightarrow \mathrm{Pic}$, i.e. this extension is split, with a section for every point of the curve..

If $k = \mathbb{F}_q$ is finite, this is similar to something we can see from the (étale) fundamental group of C. Instead of having $\mathbb{Z}$ as a quotient, this has the profinite completion[1] $\widehat{\mathbb{Z}}$ as a quotient:

$$\pi_1^{\text{ét}}(C) \twoheadrightarrow \widehat{\mathbb{Z}} = \pi_1^{\text{ét}}(\operatorname{Spec} k) \ . \tag{321}$$

The idea is that the curve C lives over $\operatorname{Spec} k$, so for a covering of $\operatorname{Spec} k$, we can pull this back to a covering $\widetilde{C}$ of C. This corresponds to the map from the fundamental group from C to the fundamental group of $\operatorname{Spec} k$. This looks very similar to the above situation, where Pic has $\mathbb{Z}$ as a quotient, with two main differences:

(1) $\pi_1^{\text{ét}}$ is not abelian, and
(2) $\widehat{\mathbb{Z}}$ is not $\mathbb{Z}$.

We can address this in two different ways. The first thing we can do is replace $\pi_1^{\text{ét}} = \mathrm{Gal}^{\mathrm{ur}}(F)$ by the *unramified Weil group of* C, written W_C^{ur}. The idea is that we want to make $\operatorname{Spec} k$ look more like S^1, so we replace $\widehat{\mathbb{Z}}$ by $\mathbb{Z}$, and W_C^{ur} is the preimage of $\mathbb{Z}$ in $\pi_1^{\text{ét}}$:

$$\begin{tikzcd} \pi_1^{\text{ét}} \arrow[r] & \widehat{\mathbb{Z}} \\ W_C^{\mathrm{ur}} \arrow[u] \arrow[r] & \mathbb{Z} \arrow[u, hook] \end{tikzcd} \ . \tag{322}$$

The other thing we can do is pass to the abelianization: $W_C^{\mathrm{ur,ab}}$, which has $\mathbb{Z}$ as a quotient:

$$W_C^{\mathrm{ur,ab}} \twoheadrightarrow \mathbb{Z} \ . \tag{323}$$

There is also a section of this for every point $x \in C(k)$, just like in the Pic case above. A point is a map $x\colon \operatorname{Spec} k \to C$, and geometrically this gives us a map

$$\widehat{\mathbb{Z}} = \pi_1^{\text{ét}}(k) \to \pi_1^{\text{ét}}(C) \ , \tag{324}$$

which is the corresponding section. The picture is that $\operatorname{Spec} k$ is an analogue of the circle, and C is an analogue of a 3-manifold fibered over it. Then a k-point of C is a section, i.e. a loop in C. To summarize, the claim is that Pic and $W_C^{\mathrm{ur,ab}}$ both

- surject onto $\mathbb{Z}$,
- are abelian, and
- have a section for every $x \in C(k)$.

Instead of altering the finite field picture, we could have altered Pic. Recall that Pic was an extension of $\mathbb{Z}$ by the Jacobian, which is a finite group over a finite field. Therefore all of the "infinity" of Pic is coming from $\mathbb{Z}$. So we could replace Pic by its profinite completion $\widehat{\mathrm{Pic}}$. So now we get something with a surjection to $\widehat{\mathbb{Z}}$ and a section for every point $x \in C$. In any case, the theorem is that they match.

THEOREM 2 (Unramified CFT). *There is a map* Pic $\to \pi_1^{\acute{e}t,ab}$ *which is an isomorphism on profinite completions. Equivalently we have an isomorphism*

$$\mathrm{Pic} \simeq W_C^{ur,ab} \tag{325}$$

respecting the operation of taking the degree of a line bundle and respecting the sections associated to $x \in C$.

[1] Recall the profinite completion is the inverse limit of all finite quotients.

EXAMPLE 29. If $C = \mathbb{P}^1$ then both sides are just $\mathbb{Z}$. This is more interesting for higher-genus curves.

Now we want to reformulate this in a way which is more reminiscent of the Pontrjagin duality that we have seen. Theorem 2 implies that the characters of Pic are the same as $W_C^{\mathrm{ur}\,,\,\mathrm{ab}}$, but once we take characters, we can't detect the abelianization, so

$$\text{characters of Pic} \simeq \text{characters of } W_C^{\mathrm{ur,ab}} \tag{326}$$
$$\simeq \text{characters of } W_C^{\mathrm{ur}} \tag{327}$$
$$\simeq \text{characters of } \pi_1^{\text{ét}}(C) \tag{328}$$
$$\simeq \text{rank 1 local systems on } C\ . \tag{329}$$

I.e. the dual to $\mathrm{Pic}(C)$ is:

$$(\mathrm{Pic}(C))^\vee \simeq \mathrm{Loc}_1(C)\ . \tag{330}$$

One kind of statement we can make from this, is that there is a Fourier transform identifying functions on Pic with functions on Loc.

Inside of function on Pic, we have the characters. The character condition, which asks that $\chi(gh) = \chi(g)\chi(h)$, is equivalent to asking for

$$\mu^*\chi = \pi_1^*\chi \boxtimes \pi_2^*\chi \tag{331}$$

where

$$\begin{tikzcd} & \mathrm{Pic}\times\mathrm{Pic} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\mu"] & & \mathrm{Pic} \\ \mathrm{Pic} & & \mathrm{Pic} & \end{tikzcd} \tag{332}$$

Equivalently these are eigenfunctions for the translation action of Pic on itself. But Pic is generated by $\mathbb{Z}_x$ for $x \in C$, so a character is the same as a function f on Pic which is an eigenfunction for the action of $\mathbb{Z}_x$ for all $x \in C$. Explicitly, for $x \in C$, this action sends $\mathcal{L} \in \mathrm{Pic}$ to the line bundle $\mathcal{L}(x)$ which is $\mathcal{L}$ with an extra $1 \cdot x$ added to the divisor. Being an eigenfunction means that:

$$f(\mathcal{L}(x)) = \gamma_x \cdot f(\mathcal{L})\ , \tag{333}$$

where γ_x is a number.

These eigenfunctions should go to something like delta functions on Loc under this Fourier transform. An element $\rho \in \mathrm{Loc}$ is a representation:

$$\rho\colon \pi_1^{\text{ét}} \to e^\times \tag{334}$$

for e some field of coefficients (e.g. $\mathbb{C}$). Then f is an eigenfunction if

$$f(\mathcal{L}(x)) = \rho(\mathrm{Fr}_x) \cdot f(\mathcal{L})\ . \tag{335}$$

The idea is that a local system is giving you a collection of eigenvalues for each point of the curve $\gamma_x = \rho(\mathrm{Fr}_x)$, i.e. $\rho \in \mathrm{Loc}$ determines the eigenvalues for $\mathbb{Z}x$ (for $x \in C$).

The operator $\mathcal{L} \mapsto \mathcal{L}(x)$ is a *Hecke operator*. These are playing the role of the 't Hooft/Dirac monopole operators from section 7.1.1 which acted on $H^2(M) \simeq \Lambda$ by translation, and were labelled by a loop $\gamma \in H_1(M)$. Recall that we were thinking of this as $H^2(M) = \pi_0(\mathrm{Pic})$.

On the other hand, we had the Wilson operators from section 7.1.2. For M a 3-manifold we sent:

$$\mathrm{Loc}_1(M) \ni \rho \mapsto W_\gamma(\rho) = \text{monodromy of } \rho \text{ around } \gamma\ . \tag{336}$$

Now we're sending:

$$\mathrm{Loc}_1 \ni \rho \mapsto \rho(\mathrm{Fr}_x)\ . \tag{337}$$

Again, in all of these cases, we're studying the algebraic geometry of Pic as a realization of the topology of Loc and vice versa.

2.1. Loc**.** Now we explain a bit about how to think about Loc. In algebraic geometry, we can't fully "access" π_1 because we only have finite covers.

Example 30. Consider the punctured affine line $\mathbb{A}^1 \setminus \{0\}$. We cannot access the universal cover because it is the exponential map $\exp\colon \mathbb{A}^1 \to \mathbb{A}^1 \setminus \{0\}$, which is not an algebraic function. We can however access finite covers of this since $t^{1/n}$ is algebraic.

So the first think we might study is something like a map:

$$\pi_1 \to \mu_n \subset \mathbb{Q}/\mathbb{Z}\ . \tag{338}$$

Moreover, for $\mathrm{char}\, k = p$, only the theory of prime-to-p-order covers works "as expected". So we need to pick a prime $\ell \neq p = \mathrm{char}\, k$, and then we can study maps:

$$\pi_1 \to \mathbb{Z}/\ell^n\ . \tag{339}$$

But then taking the inverse limit over ℓ gives us $\mathbb{Z}_\ell$, so we can make sense of representations into $\mathbb{Z}_\ell$, but we can also tensor up, i.e. we can lift the representation along:

$$\mathbb{Z}/\ell^n \leftarrow \mathbb{Z}_\ell \subset \mathbb{Q}_\ell \subset \overline{\mathbb{Q}_\ell}\ . \tag{340}$$

The upshot is that this leads to a good theory of ℓ-adic representations:

$$\pi_1 \to \mathrm{GL}_n\left(\overline{\mathbb{Q}_\ell}\right)\ , \tag{341}$$

i.e. a theory of ℓ-adic local systems in characteristic p. One nice thing is that $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ as fields, but this does not respect the topology.

So when we say rank 1 local systems we really mean continuous morphisms:

$$\mathrm{Loc}_1 = \mathrm{Hom}_{\mathrm{cts}}\left(\mathrm{Gal}^{\mathrm{ur}}(C), \overline{\mathbb{Q}_\ell}^\times\right)\ . \tag{342}$$

More generally, whenever we're discussing $\mathbb{C}$-functions we should really be taking $\overline{\mathbb{Q}_\ell}$-valued functions. The resulting theory is independent of ℓ as long as $\ell \neq p$.

2.2. Pic**.** We want to describe $\mathrm{Pic}(C)$ for C/k a smooth projective curve as something along the lines of:

$$\mathrm{Pic}(C) = \text{Divisors/principal divisors} \tag{343}$$

$$= \bigoplus_{x \in C} \mathbb{Z}/\,\mathrm{unit}\, k(C)\ . \tag{344}$$

Let $\mathcal{L}$ be a line bundle. We want to describe it via its transition functions. We can trivialize $\mathcal{L}$ generically, i.e. there exists a meromorphic section s of $\mathcal{L}$ or equivalently there is an isomorphism of functions away from finitely many points $\{x_i\}_i$:

$$s\colon\ \mathcal{O}|_{C\setminus\{x_i\}_i} \xrightarrow{\sim} \mathcal{L}|_{C\setminus\{x_i\}_i}\ . \tag{345}$$

On the other hand, we can (more democratically) trivialize $\mathcal{L}$ very close to any $x \in C$. Formally, the disk around x is:

$$D_x = \operatorname{Spec}(\mathcal{O}_x) = \operatorname{Spec}(k\,[[t]])\ , \tag{346}$$

where $\mathcal{O}_x$ is the completed local ring at x and we have chosen a coordinate t on C. And when we pull $\mathcal{L}$ back to D_x, it is automatically trivialized. Therefore the section s has a *nonzero* Laurent expansion around these finitely many points, i.e. we have

$$[s] \in K_x \simeq k\,((t)) \tag{347}$$

where K_x is the field of fractions of $\mathcal{O}_x$.

Now we can measure the degree of the section s at x:

$$\deg_x s \in K_x^\times/\mathcal{O}_x^\times \tag{348}$$

where we are quotienting out by changes of the trivialization of $\mathcal{L}$ on D_x. Expressed in the coordinate t, this is:

$$K_x^\times/\mathcal{O}_x^\times = k\,((t))^\times/k\,[[t]]^\times \tag{349}$$

$$= \left\{a_{-N}t^{-N}+\ldots\right\}/\left\{b_0+b_1t+\ldots \,|\, b_0\neq 0\right\}\ . \tag{350}$$

Now it's an exercise in algebra to check that we can force $a_{-N}=1$, and all other $a_i = 0$. Therefore this is identified with the integers:

$$K_x^\times/\mathcal{O}_x^\times \simeq \mathbb{Z}\ . \tag{351}$$

This description is not very "efficient". For varying line bundles, we have to vary the open set $U = C\setminus\{x_i\}$ where we are able to trivialize $\mathcal{L}$. The way we deal with this is by "removing all points". So consider the space of line bundles $\mathcal{L}$ equipped with a rational section[2] and a trivialization of $\mathcal{L}|_{D_x}$ for all $x \in C$. Now we want to describe the space of such data. We know we get a nonzero Laurent series for every point $x \in C$, so our first guess might be a product of $K_x^\times$, but this is actually a *restricted product* because for any particular line bundle, there are only finitely many points where there was a problem:

$$\prod_{x\in C}{}' K_x^\times = \left\{(\gamma_x)\in K_x^\times \,\middle|\, \gamma_x\in\mathcal{O}_x^\times \text{ for all but finitely many } x\right\} \tag{352}$$

$$\subset \prod_{x\in C} K_x^\times\ . \tag{353}$$

So this was kind of "overkill", and we got a huge amount of data, and now we will kind of "strip it away". So consider Line bundles equipped only with just a rational section (not a trivialization everywhere). We can access these by

[2] Note we don't bother saying where the poles are. So we just trivialize $\mathcal{L}$ over the function field.

quotienting out by changes of trivialization, i.e. line bundles equipped with a rational section comprise:

$$\prod{}' K_x^\times / \prod \mathcal{O}_x^\times = \prod{}' \left(K_x^\times / \operatorname{unit} \mathcal{O}_x\right) \tag{354}$$

$$= \prod{}' \mathbb{Z} \tag{355}$$

$$\left\{\text{finite } \sum_{x \in C} a_x x\right\} \tag{356}$$

$$= \text{Divisors} \ . \tag{357}$$

Now we need to get rid of our choice of rational section by modding out on the left:

$$\operatorname{Pic} = \{\text{line bundles}\} \tag{358}$$

$$= F^\times \backslash \prod{}' K_x^\times / \prod \mathcal{O}_x^\times \tag{359}$$

$$= F^\times \backslash \prod{}' \mathbb{Z} \ , \tag{360}$$

which is exactly divisors moduli principal ones. The left quotient is by changes of the rational section, and the right quotient is by changes of the trivialization on D_x. Lecture 10; February 25, 2021

transition

This is the adélic description of $\operatorname{Pic}(C)$. We can rewrite this as:

$$\operatorname{Pic}(C) = \mathrm{GL}_1(F) \backslash \mathrm{GL}_2(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}_F}) \tag{361}$$

where $\mathbb{A}_F$ is the adéles for F:

$$\mathbb{A}_F \coloneqq \prod_{x \in C}{}' K_x \tag{362}$$

where K_x is the completed local field. Inside of this is the ring of integers:

$$\mathcal{O}_{\mathbb{A}} = \prod_{x \in C} \mathcal{O}_x \tag{363}$$

where $\mathcal{O}_x$ is the completed local ring.

Remark 23. This description might seem like overkill, in the sense that we're writing it has a quotient of something huge. This is the same sense in which defining Pic via divisors is overkill: the collection of all divisors is huge before we mod out by principal ones.

The *(unramified) idéle class group (idéles)* is:

$$\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times \ . \tag{364}$$

2.2.1. *Arithmetic version.* Now we will relate this discussion to the analogous ones in the arithmetic setting. For F a number field, we can construct the idéles of F as:

$$\mathbb{A}_F = \prod_{v \text{ places}}{}' F_v \ , \tag{365}$$

where F_v is a completed local field (e.g. $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}$, ...) and similarly

$$\mathcal{O}_{\mathbb{A}} = \prod_{\text{primes}} \mathcal{O}_v \ . \tag{366}$$

Then, in an attempt to get our hands on an analogue of Pic, we can follow our nose from the function field setting and guess that the unramified idéle class group is:

$$\mathrm{GL}_1(F) \setminus \mathrm{GL}_1(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) = F^\times \setminus {\prod}' F_v^\times / \prod_v \mathcal{O}_v^\times \ . \tag{367}$$

This is missing a factor coming from the place at infinity, which we can add as follows. At a finite place, the inclusion $F^\times \supset \mathcal{O}^\times$ looks something like $\mathbb{Q}_p^\times \supset \mathbb{Z}_p^\times$. One way of describing $\mathbb{Z}_p^\times$, is that is is a maximal compact subgroup with respect to the p-adic topology. So now we can use this to determine what to do at infinity. In particular, we let K_∞ be the maximal compact subgroup of $F_v^\times$, and we additionally quotient out by this on the right:

$$\mathrm{GL}_1(F) \setminus \mathrm{GL}_1(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) \cdot K_\infty = F^\times \setminus {\prod}' F_v^\times / \prod_v \mathcal{O}_v^\times \cdot K_\infty \ . \tag{368}$$

Note that, at infinity, $F_v^* \simeq \mathbb{R}^\times$ or $\mathbb{C}^\times$ so the maximal compacts are $\mathbb{Z}/2$ and $\mathrm{SO}(2)$ respectively. So we're only removing part of these extra copies of $\mathbb{R}^\times$ and $\mathbb{C}^\times$. Explicitly we still have some factors of $\mathbb{R}_+$.

The idea is that this is very close to $\mathcal{C}\ell$:

$$\mathcal{C}\ell_F = \pi_0\left(F^\times \setminus \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty\right) \ . \tag{369}$$

In other words: there is a quotient from the unramified idéle class group to $\mathcal{C}\ell_F$ given by taking π_0, i.e. by quotienting out by the leftover copies of $\mathbb{R}_+$ from the places at infinity. Now recall $\mathcal{C}\ell$ is isomorphic to $\mathrm{Gal}^{\mathrm{ur,ab}}$:

$$F^\times \setminus \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty \twoheadrightarrow \mathcal{C}\ell_F \simeq \mathrm{Gal}^{\mathrm{ur,ab}} \ . \tag{370}$$

In particular if we look at characters that don't detect copies of $\mathbb{R}$ and $\mathbb{C}$ (e.g. finite order or continuous ones) then the characters on these groups will agree.

This is close to what we saw in topology. Recall we were consider the space of line bundles and connections, $\mathrm{Pic}\left(M^3\right) = \mathrm{Map}\left(M^3, B\,\mathrm{U}(1)\right)$, which had π_0 given by $H^2(M, \mathbb{Z})$. The identity component of the idéle class group can be described as follows. Our first guess is:

Pic notation

$$\mathbb{R}_+^r \times \mathbb{R}_+^s \tag{371}$$

where r is the number of real embeddings of F, and s is the number of pairs of complex embeddings. This is because the infinite part of our field is:

$$F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s \ , \tag{372}$$

and then when we quotiented out by the maximal compacts, we just got copies of $\mathbb{R}_+$ for both $\mathbb{R}$ and $\mathbb{C}$. Then we quotient out by a boring diagonal copy on the right, and by the units in F on the left:

$$\mathcal{O}_F^\times \backslash \mathbb{R}_+^r \times \mathbb{R}_+^s / \mathbb{R}_+ \ . \tag{373}$$

THEOREM 3 (Dirichlet unit theorem). *$\mathcal{O}_F^\times \simeq \mu_\infty(\mathcal{O}_F) \times \mathbb{Z}^{r+s-1}$, i.e. the units are just the obvious ones (all roots of unity in F) and a lattice.*

Therefore the connected component of the idéle class group is:

$$\pi_0\left(F^\times \setminus \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty\right) = \simeq \mathrm{U}(1)^{r+s-1} \ . \tag{374}$$

So if we study the cohomology of the idéle class group, we get an exterior algebra of rank $r + s - 1$.

This is the complete story for the unramified case. But this is a very restrictive condition to ask for, especially in number theory.

3. Ramification

3.1. Physics. Recall in the physics context (section 9.1) we had $\mathrm{Pic}\left(M^3\right)$, which consisted of line-bundles equipped with a connection. By line-bundles and connections we just mean $\mathrm{Map}\left(M, B\,\mathrm{U}\left(1\right)\right)$. In particular this has

Pic notation

$$\pi_0 = H^2\left(M, \mathbb{Z}\right) \qquad\qquad \pi_1 = H^1\left(M, \mathbb{Z}\right) \ . \tag{375}$$

We matched this with

$$\mathrm{Loc}_1\left(M^3\right) = \mathrm{Hom}\left(\pi_1\left(M\right), \mathbb{C}^\times\right) \tag{376}$$

in the sense that functions on them were identified:

$$H^0\left(\mathrm{Pic}\left(M^3\right)\right) \leftrightarrow \mathbb{C}\left[\mathrm{Loc}_1 M^3\right] \ . \tag{377}$$

Then, in section 9.2.2, we had surface defects/solenoids where we looked at $\mathrm{Loc}_1\left(M^3 \setminus \gamma\right)$ for some closed 1-manifold $\gamma \hookrightarrow M^3$, e.g. a knot or link. I.e. we're allowing singularities along γ. Loc_1 now has an extra copy of $\mathbb{C}^\times$ for every component of the link, so we need to change Pic too.

To alter Pic, instead of introducing singularities along γ, we work relative to it, so we consider: $\mathrm{Pic}\left(M^3, \gamma\right)$, which consists of line bundles equipped with a trivialization on γ. This introduces an extra factor of $\mathbb{Z}$ in Pic per component of γ.

Remark 24. If we're thinking topologically, before we introduced ramification, we had the Poincaré duality (with coefficients) between: $\pi_0 \,\mathrm{Pic} = H^2\left(M, \mathbb{Z}\right)$ and $H^1\left(M, \mathrm{U}\left(1\right)\right)$. Introducing ramification corresponds to passing from Poincaré duality to Lefschetz-duality, which is a version for relative cohomology:

$$H^2\left(\left(M, N\right), \mathbb{Z}\right)^\vee \simeq H^1\left(M \setminus N, \mathbb{C}^\times\right) \ . \tag{378}$$

So we work relative to the knot on one side, and remove it on the other.

3.2. Arithmetic setting. Now we want to do the same thing in number theory, i.e. over curves over finite fields or number fields. We won't try to get the whole story, but just tame ramification.

Over $\mathbb{C}$,

$$D^\times = \mathrm{Spec}\,\mathbb{C}\left(\left(t\right)\right) \tag{379}$$

is some version of a circle, and indeed

$$\pi_1\left(D^\times\right) = \mathbb{Z} \ . \tag{380}$$

In algebraic geometry, we should really think about:

$$\pi_1^{\text{ét}}\left(D^\times\right) = \widehat{\mathbb{Z}} \ . \tag{381}$$

This means that all extensions of $\mathbb{C}\left(\left(t\right)\right)$ correspond to taking roots of the coordinate: $\mathbb{C}\left(\left(t\right)\right) \to \mathbb{C}\left(\left(t^{1/N}\right)\right)$.

In arithmetic, things are much more interesting. Analogues of the punctured disk are:

$$D^{\times}{}_{\mathbb{F}_q} = \mathrm{Spec}\,\mathbb{F}_q\left(\left(t\right)\right) \qquad\qquad \mathrm{Spec}\,\mathbb{Q}_p \ , \tag{382}$$

only now $\pi_1^{\text{ét}}$ is richer. There are two sources of this richness. One is that we have a quotient to the Frobenius:

$$\pi_1^{\text{ét}}\left(D^{\times}{}_{\mathbb{F}_q}\right) \twoheadrightarrow \widehat{\mathbb{Z}} \; . \tag{383}$$

One might object that this is a consequence of working over a non-algebraically-closed field. But even when we pass to $\overline{\mathbb{F}_q}\left((t)\right)$, this still has a complicated fundamental group. But there is part which mimics what we saw in geometry, called the Tame inertia group. The idea is that $\pi_1^{\text{ét}}\left(D^{\times}{}_{\mathbb{F}_q}\right)$ and $\mathrm{Gal}\left(\mathbb{Q}_p\right)$ both have a quotient given by

$$\Gamma = \left(\left\{F, m \,\middle|\, FmF^{-1} = m^q\right\}\right)^{\wedge} \;, \tag{384}$$

where q is the order of the residue field, and F stands for the Frobenius (and m stands for monodromy). This surjects onto the Frobenius part with the monodromy part sitting inside:

$$\widehat{\mathbb{Z}_m} \to \Gamma \twoheadrightarrow \widehat{\mathbb{Z}_F} \; . \tag{385}$$

This is something pretty geometric. We have something going around the point we removed (monodromy), and there is the Frobenius, and they relate via $FmF^{-1} = m^q$. The idea is that, in the arithmetic setting, this étale fundamental group tells us much more, but the tame inertia is the part that looks like the geometric setting.

In number theory our Loc is representations of the Galois group of F (here F is either rational functions on a curve $\mathbb{F}_q\left(C\right)$ or a number field):

$$\mathrm{Loc} = \left\{\mathrm{Gal}_F \to \mathrm{GL}_1\left(\overline{\mathbb{Q}_\ell}\right)\right\} \; . \tag{386}$$

For a finite subset $S \subset C$ (or $S \subset$ primes of F) we can look at elements of Loc which are tamely ramified at the points of S, written $\mathrm{Loc}_1\left(C \setminus S\right)^{\text{tame}}$.

Remark 25. This is like when we allowed singularities along the knot in section 7.1.1. Only in the physics, we didn't need to control how singular these singularities are. Now we do need to get control, and we do so by asking for the ramification to be tame.

Then you might ask how to match this with some version of Pic. The claim is that the Pontrjagin duality exchanges:

$$\mathrm{Loc}_1\left(C \setminus S\right)^{\text{tame}} \qquad \text{and} \qquad \mathrm{Pic}\left(C, S\right) \;, \tag{387}$$

where $\mathrm{Pic}\left(C,S\right)$ consists of line bundles on C equipped with a trivialization on S.

Recall we were thinking of:

$$\mathrm{Pic}\, C = F^{\times} \backslash \prod{}' \mathbb{Z} \; . \tag{388}$$

Recall the basic comparison between Pic and $\pi_1^{\text{ét}}$ in Theorem 2 was based on matching these copies of $\mathbb{Z}$. Specifically these came from the fact that we had a section of $\mathrm{Pic} \to \mathbb{Z}$ for every point of the curve, and a copy of $\mathbb{Z}$ in the Galois group for each Frobenius. Now we have this richer description:

$$\mathrm{Pic}\, C = F^{\times} \backslash \prod_{x \in C}{}' K^{\times} / \prod_{x \in C} \mathcal{O}^{\times} \; . \tag{389}$$

Recall that this quotient description came from starting with $\prod{}' K^{\times}$, the collection of line bundles with a generic trivialization, and a trivialization near every point.

Then we quotiented out by this extra data. Now we consider only quotienting on the left:

$$F^{\times} \backslash \prod{}' K^{\times} \twoheadrightarrow \operatorname{Pic} C \ , \tag{390}$$

which consists of line bundles with a trivialization near every $x \in C$. I.e. we have quotiented out by the choice of generic trivialization. The group in (390) is the *idéle class group*.[3]

Choosing a trivialization around every point is still a huge amount of data. The advantage of this is that it allows us to be flexible about what kind of data we pick where, e.g. we might ask for a finite order trivialization around some finite collection of points. So pick a finite subset $S \subset C$ with multiplicities n_x for $x \in S$. Then we can consider the collection of elements of Pic, equipped with an n_x-order trivialization at all $x \in S$. By trivialize to the order n_x, we mean the following. We have an nth order neighborhood of the point x sitting inside the disk around x:

$$\operatorname{Spec} k\,[t]\,/t^n \hookrightarrow D_x = \operatorname{Spec} k\,[[t]] \ , \tag{391}$$

and instead of asking for an infinite Taylor series section of the bundle, we ask for a section of finite order n_x. This gives us an intermediate quotient Pic_S:

$$\begin{tikzcd}
\mathrm{GL}_1\,(\mathrm{unit}\,F) \,\backslash\, \mathrm{GL}_1\,(\mathbb{A}_{\mathbb{C}}) \arrow[dd] \arrow[dr] & \\
& \operatorname{Pic}\,(C, S) \arrow[dl] \\
\operatorname{Pic} = \mathrm{GL}_1\,(F^{\times}) \,\backslash\, \mathrm{GL}_1\,(\mathbb{A}_{\mathbb{C}}) \,/\, \mathrm{GL}_1\,(\mathcal{O}_{\mathbb{A}}) &
\end{tikzcd} \tag{392}$$

defined by

$$\operatorname{Pic}_S = \mathrm{GL}_1\,(F) \,\backslash\, \mathrm{GL}_1\,(\mathbb{A}_F) \,/ \prod_{x \notin S} \mathrm{GL}_1\,(\mathcal{O}_X) \times \prod_{x \in S} \mathrm{GL}_1^{(n_x)}\,(\mathcal{O}_x) \ , \tag{393}$$

where $\mathrm{GL}_1^{(n)}\,(\mathcal{O}_x)$ consists of elements of $\mathrm{GL}_1\,(\mathcal{O}_x)$ that are congruent to $1 \bmod t^n$. I.e. they consist of changes of trivialization of a line bundle on the disk, constant to order n. So the top is elements of Pic equipped with infinite order trivialization everywhere, the bottom is just Pic, and the middle is elements of Pic equipped with an n_x order trivialization at every $x \in S$. Concretely:

$$\mathcal{O}_x^{\times} = \left\{ a_0 + a_1 t + a_2 t^2 + \ldots \,\middle|\, a_0 \neq 0 \right\} \ , \tag{394}$$

and

$$\mathrm{GL}_1^{(n)}\,(\mathcal{O}_x) = \{1 + a_n t^n + \ldots\} \ . \tag{395}$$

Example 31. Let S be a finite subset of points of C, and let $n_x = 1$ for all $x \in S$. Then $\operatorname{Pic}\,(C, S)$ consists of line bundles whose fibers at $x \in S$ are trivialized. This maps down to $\operatorname{Pic}\,(C)$ with fiber $\mathcal{O}^{\times} / \left(\mathcal{O}^{\times}\right)^{(1)} = k^{\times}$.

Recall $\pi_1^{\text{ét}} D^{\times}$ was complicated, but had this quotient Γ:

$$\pi_1^{\text{ét}}\left(D^{\times}\right) \twoheadrightarrow \Gamma = \left(\left\{ F, m \,\middle|\, F m F^{-1} = m^q \right\}\right)^{\wedge} \ . \tag{396}$$

[3] Before this, it was really the unramified idéle class group.

But we're only considering maps to some abelian group:

$$
\begin{tikzcd}
\pi_1^{\text{ét}} D^\times \arrow[r, two heads] \arrow[dr] & \Gamma \arrow[d] \\
& \text{abelian group}
\end{tikzcd}
\tag{397}
$$

so they factor through the abelianization:

$$\Gamma / [\Gamma, \Gamma] \simeq \mathbb{F}_q^\times \times \widehat{\mathbb{Z}} \; . \tag{398}$$

Now we want to compare this to $K^\times$. $K^\times/O^\times = \mathbb{Z}$ was our degree, and then:

$$\mathbb{F}_q^\times \times \mathbb{Z} \simeq K^\times / \mathcal{O}^{\times (1)} \to K^\times / \mathcal{O}^\times = \mathbb{Z} \; , \tag{399}$$

and this is the same $\mathbb{F}_q^\times \times \mathbb{Z}$ in the Galois group.

$$\overline{K^\times} \simeq \mathrm{Gal}^{\mathrm{ab}} \, \overline{K_x} / K_x \; . \tag{400}$$

This is the first instance of local class field which says that:

$$\widehat{K^\times} \simeq \mathrm{Gal} \, \overline{K_x} / K_x \; . \tag{401}$$

The RHS has a quotient to $\widehat{\mathbb{Z}}$ given by the Frobenius, and the LHS has a quotient to $\widehat{\mathbb{Z}}$ given by the degree. We also have quotients down to $\mathbb{F}_q^\times$. So these quotients reveal the part of these groups that we see in the topological context. Exactly as in the physics, this local class field theory gives us a duality between:

$$\mathrm{Pic}\,(C, S) \qquad \text{and} \qquad \mathrm{Loc}\,(C \setminus S)^{\mathrm{tame}} \; . \tag{402}$$

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