Between electric-magnetic duality and the Langlands program

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Chapter 1

Overview

The geometric Langlands program is some kind of middle-ground between number theory and physics. Another point of view is that we will be navigating the narrow passage between the whirlpool Charybdis (physics) and the six-headed monster Scylla (number theory), as in Odysseus' travels.[1]

Lecture 1; January 19, 2021

The inspiration for much of this course comes from [Mac78], which provides a historical account of harmonic analysis, focusing on the idea that function spaces can be decomposed using symmetry. This theme has long-standing connections to physics and number theory.

The spirit of what we will try to do is some kind of harmonic analysis (fancy version of Fourier theory) which will appear in different guises in both physics and number theory.

1.1 Modular/automorphic forms

1.1.1 Rough idea

The theory of modular forms is a kind of harmonic analysis/quantum mechanics on arithmetic locally symmetric spaces. The canonical example of a locally symmetric space is given by the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane $\mathbb{H} = \mathrm{SL}_2(\mathbb{R}) \,/\, \mathrm{SO}_2$. I.e. we are considering the quotient

$$\mathcal{M}_{\mathrm{SL}_2 \mathbb{R}} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \,/\, \mathrm{SO}_2 \tag{1.1}$$

as in fig. 1.1.

For a general reductive algebraic group G we can consider the space

$$\mathcal{M}_G = \Gamma \backslash G / K \tag{1.2}$$

where Γ is an arithmetic lattice, and K is a maximal compact subgroup. For now we restrict to

$$G = \mathrm{SL}_2(\mathbb{R}) \qquad \Gamma = \mathrm{SL}_2(\mathbb{Z}) \qquad K = \mathrm{SO}_2 \ .$$

We want to do harmonic analysis on this space, i.e. we want to decompose spaces of functions on this in a meaningful way. In the case of quantum mechanics we're primarily interested

[1] One can expand this analogy. Calypso's island is probably derived algebraic geometry (DAG), etc.

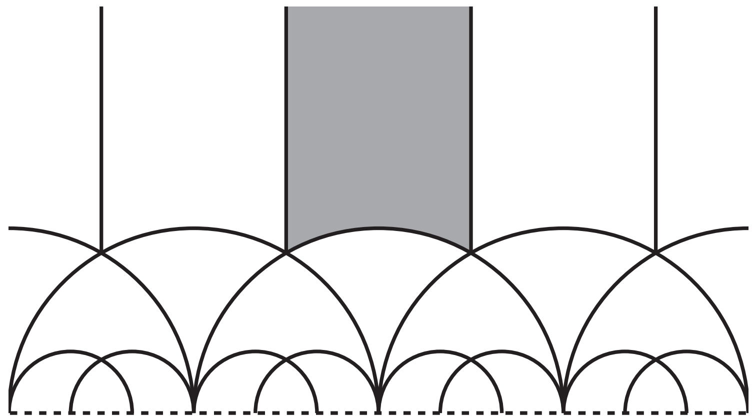

Figure 1.1: Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray.

in L^2 functions:

$$L^2(\Gamma\backslash G/K) \ , \tag{1.3}$$

and on this we have an action of the hyperbolic Laplace operator. I.e. we want to study the spectral theory of this operator.

The same information, possibly in a more accessible form, is given by getting rid of the K. That is, we can just study L^2 functions on

$$\Gamma\backslash G = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \ , \tag{1.4}$$

which is the unit tangent bundle, a circle bundle over the space we had before. Instead of studying the Laplacian, this is a homogeneous space so we can study the action of all of $\mathrm{SL}_2(\mathbb{R})$.

One can expand this to include differentials and pluri-differentials, i.e. sections of (powers of) the canonical bundle:

$$\Gamma\left(\Gamma\backslash\mathbb{H}, \omega^{k/2}\right) \ . \tag{1.5}$$

Definition 1. The Δ-eigenfunctions in $L^2(\Gamma\backslash G/K)$ are called *Maass forms. Modular forms* of weight k are holomorphic sections of $\omega^{k/2}$.

Remark 1. For a topologist, one might instead want to study (topological) cohomology (instead of forms) with coefficients in some local system (twisted coefficients). Indeed, modular forms can also arise by looking at the (twisted) cohomology of $\Gamma\backslash\mathbb{H}$. This is known as Eichler-Shimura theory.

One might worry that this leaves the world of quantum mechanics, but after passing to cohomology we're doing what is called *topological* quantum mechanics. We will be more concerned with this than honest quantum mechanics.

The idea is that there are no dynamics in this setting. We're just looking at the ground states, so the Laplacian is 0, and we're just looking at harmonic things. And this really has to do with topology and cohomology. But modular forms are some kind of ground states.

Remark 2. If we take general G, K, and Γ then we get the more general theory of *automorphic forms.*

Example 1. If we start with $G = \mathrm{Sp}_{2n}(\mathbb{R})$ and take $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, $K = \mathrm{SO}_n$ then we get *Siegel modular forms*.

1.1.2 Structure

There is a long history of thinking of this problem[2] as quantum mechanics on this locally symmetric space. But there is a lot more structure going on in the number theory than seems to be present in the quantum mechanics of a particle moving around on this locally symmetric space.

Restrict to the case $G = \mathrm{SL}_2(\mathbb{R})$.

Number field

The question of understanding

$$L^2\left(\mathrm{SL}_2 \mathbb{Z} \backslash \mathrm{SL}_2 \mathbb{R} / \mathrm{SO}_2\right) \tag{1.6}$$

has an analogue for any number field. We can think of $\mathbb{Z}$ as being the ring of integers in the rational numbers:

$$\mathbb{Z} = \mathcal{O}_{\mathbb{Q}} \tag{1.7}$$

and from this we get a lattice $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}})$. Writing it this way, we see that we can replace $\mathbb{Q}$ by any finite extension F, and $\mathbb{Z}$ becomes the ring of integers $\mathcal{O}_F$:

$$\begin{aligned} \mathbb{Q} &\rightsquigarrow F \\ \mathbb{Z} &\rightsquigarrow \mathcal{O}_F \ . \end{aligned} \tag{1.8}$$

The upshot is that when we replace Q with some other number field $F/\mathbb{Q}$, then the space $\mathcal{M}_{G,\mathbb{Q}}$ becomes some space $\mathcal{M}_{G,F}$. Then we linearize by taking either L^2 or H^* of $\mathcal{M}_{G,F}$.

Example 2. This holds for all reductive algebraic groups G, but let $G = \mathrm{PSL}_2 \mathbb{R}$. Then

$$\mathcal{M}_{G,\mathbb{Q}} = \mathrm{PSL}_2 \mathbb{Z} \backslash \mathrm{PSL}_2 \mathbb{R} / \mathrm{SO}_2 \tag{1.9}$$

is the locally symmetric space in fig. 1.1. If we replace $\mathbb{Q}$ with an arbitrary number field $F/\mathbb{Q}$, then we get

$$\mathcal{M}_{G,F} = \mathrm{PSL}_2\left(\mathcal{O}_F\right) \backslash \mathrm{PSL}_2\left(F \otimes_{\mathbb{Q}} \mathbb{R}\right) / K \ . \tag{1.10}$$

Note that

$$F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{\times r_1} \times \mathbb{C}^{\times r_2} \tag{1.11}$$

where r_1 is the number of real embeddings of F, and r_2 is the number of conjugate pairs of complex embeddings.

Example 3. Let $F = \mathbb{Q}\left(\sqrt{d}\right)$. If it is real ($d \geq 0$) then $r_1 = 2$ (corresponding to $\pm\sqrt{d}$) and $r_2 = 0$, so we get

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right) \otimes_{\mathbb{Q}} \mathbb{R}\right) = \mathrm{PSL}_2 \mathbb{R} \times \mathrm{PSL}_2 \mathbb{R} \ . \tag{1.12}$$

[2] The problem of understanding L^2 functions on a locally symmetric space.

This leads to what are called *Hilbert modular forms*.

If it is imaginary ($d < 0$) then $r_1 = 0$, $r_2 = 1$, and

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right) \otimes_{\mathbb{Q}} \mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{C}\ . \tag{1.13}$$

In this case the maximal compact is $\mathrm{SO}_3\,\mathbb{R}$, and the quotient:

$$\mathbb{H}^3 = \mathrm{PSL}_2\,\mathbb{C}/\,\mathrm{SO}_3\,\mathbb{R} \tag{1.14}$$

is hyperbolic 3-space. Now we need to mod out (on the left) by a lattice, and the result is some hyperbolic manifold which is a 3-dimensional version of the picture in fig. 1.1.

Remark 3. The point is that the real group we get after varying the number field is not that interesting, just some copies of PSL_2. But the lattice we are modding out by depends more strongly on the number field, so this is the interesting part.

Conductor/ramification data

Fixing the number field $F = \mathbb{Q}$, we can vary the "conductor" or "ramification data". The idea is as follows. The locally symmetric space $\Gamma\backslash\mathbb{H}$ has a bunch of covering spaces of the form $\Gamma'\backslash\mathbb{H}$, where Γ' is some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. So we can replace Γ by Γ'.

We won't define congruence subgroups in general, but there are basically two types. For $N \in \mathbb{Z}$, we fix subgroups:

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N\right\} \tag{1.15}$$

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N\right\}\ . \tag{1.16}$$

The idea is that we start with the conductor N and the lattice Γ, and then we modify Γ at the divisors of N. Note that even in this setting we have the choice of $\Gamma(N)$ or $\Gamma_0(N)$. Really the collection of variants has a lot more structure. The local data at p has to do with the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

Action of Hecke algebra

We have seen that our Hilbert space depends on the group, the number field, and some ramification data. A very important aspect of this theory is that this vector space (of functions) carries a lot more structure. There is a huge "degeneracy" here in the sense that the eigenspaces of the Laplacian are much bigger than one might have guessed (not one-dimensional).

This degeneracy is given by the theory of *Hecke operators*. This says that the Laplacian Δ is actually a part of a huge commuting family of operators. In particular, these all act on the eigenspaces of the Laplacian. For p a prime (p unramified, i.e. $p \not| N$) we have the Hecke operator at p, T_p. Then

$$\bigoplus_p \mathbb{C}[T_p] \circlearrowright L^2(\Gamma\backslash G/K)\ . \tag{1.17}$$

This is some kind of "quantum integrable system" because having so many operators commute with the Hamiltonian tells us that a lot of quantities are conserved.[3]

[3] This example is often included in the literature as an example of quantum chaos (the opposite of

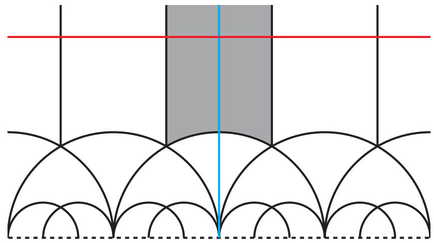

Figure 1.2: Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray. One can define a "period" as taking a modular form and integrating it, e.g. on the red or blue line.

Periods/states

There is a special collection of measurements we can take of modular forms, called periods. A basic example is given by integrating a modular form on the line $i\mathbb{R}_+ \subset \mathbb{H}$ as in fig. 1.2. This is how Hecke defined the L-function.

The takeaway is that we have a collection of measurements/states with very good properties, and then we can study modular forms by measuring them with these periods.

Langlands functoriality

There is a collection of somewhat mysterious operators whose action corresponds to varying the group G.

1.2 The Langlands program and TFT

1.2.1 Overview

We have seen that for a choice of reductive algebraic group G and number field $F/\mathbb{Q}$, we get a locally symmetric space

$$\mathcal{M} = \mathcal{M}_{G,F} = \text{“arithmetic lattice”}\backslash\text{real group/maximal compact} . \tag{1.18}$$

This can be thought of as some space of G-bundles

$$\mathcal{M}_{G,F} = \text{“}\mathrm{Bun}_G\left(\mathrm{Spec}\,\mathcal{O}_F\right)'' . \tag{1.19}$$

Then we linearize this space by taking either L^2 or H^*.

Starting with this theory of automorphic forms, we spectrally decompose under the action of the Hecke algebra. Then the Langlands program says that the pieces of this decomposition correspond to Galois representations. We can think of the theory of automorphic forms as being fed into a prism, and the colors coming out on the other side are Galois representations as in fig. 1.3. More specifically, the "colors" are representations:

$$\mathrm{Gal}\left(\overline{F}/F\right) \to G^\vee_{\mathbb{C}} . \tag{1.20}$$

integrability). The chaotic aspect has nothing to do with the discrete subgroup Γ. Specifically this fits into the study of "arithmetic quantum chaos" which more closely resembles the study of integral systems.

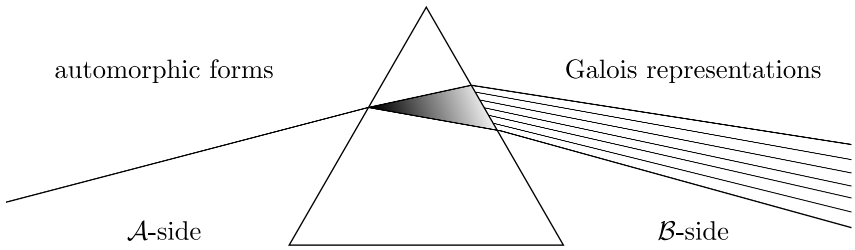

Figure 1.3: Just as light is decomposed by a prism, this spectral decomposition breaks automorphic forms ($\mathcal{A}$-side) up into Galois representations of number fields ($\mathcal{B}$-side).

Example 4. If $G = \mathrm{GL}_2 \mathbb{R}$, then $G^\vee = \mathrm{GL}_2 \mathbb{C}$. Let E be an elliptic curve. Then

$$H^1(E/\mathbb{Q}) \tag{1.21}$$

is a 2-dimensional representation of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. This is the kind of representation you get in this setting.

Example 5. The representations in example 4 are very specific to GL_2. If we started with $\mathrm{GL}_3(\mathbb{R})$ instead, the associated locally symmetric space $\mathrm{O}_3 \backslash \mathrm{GL}_3 \mathbb{R} / \mathrm{GL}_3 \mathbb{Z}$ is not a complex manifold.

The goal is to match all of this structure in section 1.1.2 with a problem in physics, but ordinary quantum mechanics will be too simple. On the physics side we will instead consider quantum field theory.

Slogan: the Langlands program is part of the study of 4-dimensional (arithmetic, topological) quantum field theory.

The idea is that the Langlands program is an equivalence of 4-dimensional arithmetic topological field theories (TFTs):

$$\begin{array}{ccc} \mathcal{A}_G & \simeq & \mathcal{B}_{G^\vee} \\ \text{automorphic} & & \text{spectral} \\ \text{magnetic} & & \text{electric} \end{array} \tag{1.22}$$

called the $\mathcal{A}$ and $\mathcal{B}$-side theories.

Remark 4. This is what one might call "four-dimensional mirror symmetry". The $\mathcal{A}$ and $\mathcal{B}$ are in the same sense as usual mirror symmetry.

An n-dimensional TFT is a beast which assign a quantum mechanics problem (or just a vector space, chain complex, etc.) to every $(n-1)$-manifold. So a 4-dimensional TFT sends a 3-manifold to some kind of vector space. It assigns more complicated data to lower-dimensional manifolds and less complicated data to higher-dimensional manifolds as in table 1.1.

Table 1.1: Output of a four-dimensional topological field theory. Numbers are the easiest to understand, but are usually the trickiest to produce (often requires analysis). Vector spaces are also pretty simple, but three-manifolds are hard. So the sweet spot is kind of in 2-dimensions, since we understand surfaces and categories aren't that complicated.

Dimension	Output
4	$z \in \mathbb{C}$ (rarely well-defined algebraically, requires analysis)
3	(dg) vector space
2	(dg) category
1	$(\infty, 2)$-category
0	$(\infty, 3)$-category? (rarely understood)

The *topological* means we are throwing out the dynamics and only looking at the ground states. This is the analogue of only looking at the harmonic forms rather than the whole spectrum of the Laplacian. The *arithmetic* means that we're following the paradigm of *arithmetic topology*. The idea is that we will eventually make an analogy between number fields and three-manifolds. Then we can plug a number field into the TFT (instead of an honest manifold) to get a vector space which turns out to be $L^2\left(\mathcal{M}_{G,F}\right)$ (or $H^*\left(\mathcal{M}_{G,F}\right)$).

1.2.2 Arithmetic topology

Weil's Rosetta Stone

In a letter to Simone Weil [Kri05], André Weil explained a beautiful analogy, now known as *Weil's Rosetta Stone*. This establishes a three-way analogy between number fields, function fields, and Riemann surfaces.

The general idea is as follows. $\operatorname{Spec}\mathbb{Z}$ is some version of a curve, with points $\operatorname{Spec}\mathbb{F}_p$ associated to different primes. $\operatorname{Spec}\mathbb{Z}_p$ is a version of a disk around the point, and $\operatorname{Spec}\mathbb{Q}_p$ is a version of a punctured disk around that point. This is analogous to the usual picture of an algebraic curve.

Curve	$\operatorname{Spec}\mathbb{F}_q[t]$	$\operatorname{Spec}\mathbb{Z}$
Point	$\operatorname{Spec}\mathbb{F}_p$	$\operatorname{Spec}\mathbb{F}_p$
Disk	$\operatorname{Spec}\mathbb{F}_t[[t]]$	$\operatorname{Spec}\mathbb{Z}_p$
Punctured disk	$\operatorname{Spec}\mathbb{F}_q((t))$	$\operatorname{Spec}\mathbb{Q}_p$

In general, let $F/\mathbb{Q}$ be a number field. Then we can consider $\mathcal{O}_F$, and $\operatorname{Spec}\mathcal{O}_F$ has points corresponding to primes in $\mathcal{O}_F$. The analogy between number fields and function fields is as follows. Start with a smooth projective curve $C/\mathbb{F}_q$ over a finite field. Then the analogue to F is the field of rational functions, $\mathbb{F}_q(C)$. The analogue to $\mathcal{O}_F$ is the ring of regular functions, $\mathbb{F}_q[C]$. Finally points of $\operatorname{Spec}\mathcal{O}_F$ correspond to points of C.

Now we might want to replace C with a Riemann surface. So let $\Sigma/\mathbb{C}$ be a compact Riemann surface. Then primes in $\mathcal{O}_F$ (and so points of C) correspond to points of Σ. The field of meromorphic rational functions on Σ, $\mathbb{C}(\Sigma)$, is the analogue of F. To get an analogue of $\mathcal{O}_F$ we have to remove some points of Σ (we wouldn't get any functions on the compact curve). The point is that number fields have some points at ∞, so the analogue isn't really a compact Riemann surface, but with some marked points. So the analogue of $\mathcal{O}_F$ consists of functions on Σ which are regular away from these points.

This is summarized in table 1.2.

Table 1.2: Weil's Rosetta stone, as it was initially developed, establishes an analogy between these three columns. We will eventually refine this dictionary. Let $F/\mathbb{Q}$ be a number field, $C/\mathbb{F}_q$ be a smooth projective curve over a finite field, and let $\Sigma/\mathbb{C}$ be a compact Riemann surface. $\mathbb{F}_q(C)$ denotes the field of rational functions, $\mathbb{F}_q[C]$ denotes the ring of regular functions, and $\mathbb{C}(\Sigma)$ denotes the meromorphic rational functions on Σ.

Number fields	Function fields	Riemann surfaces
$F/\mathbb{Q}$	$\mathbb{F}_q(C)$	$\mathbb{C}(\Sigma)$
$\mathcal{O}_F$	$\mathbb{F}_q[C]$	f'ns regular away from marked points of Σ
$\operatorname{Spec}\mathcal{O}_F$	points of C	$x \in \Sigma$

Missing chip

Now we want to take the point of view that there was a chip missing from this Rosetta stone, and we were supposed to consider 3-manifolds rather than Riemann surfaces. The idea is that $\Sigma/\mathbb{C}$ really corresponds to $C/\overline{\mathbb{F}_q}$. This is manifested in the following way. To study points, we study maps:

$$\operatorname{Spec}\mathbb{F}_q \hookrightarrow C\ . \tag{1.23}$$

But from the point of view of étale topology, $\operatorname{Spec}\mathbb{F}_q$ is not really a point. It is more like a circle in the sense that

$$\operatorname{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) = \widehat{\mathbb{Z}} = \pi_1^{\text{étale}}\left(\operatorname{Spec}\mathbb{F}_q\right) \tag{1.24}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion. So it's better to imagine this as a modified circle, where this $\widehat{\mathbb{Z}}$ is generated by the Frobenius. There is always a map

$$\operatorname{Spec}\overline{\mathbb{F}_q} \to \operatorname{Spec}\mathbb{F}_q \tag{1.25}$$

and we can lift our curve to $\overline{\mathbb{F}}_q$. This corresponds to unwrapping these circle, i.e. replacing them by their universal cover. So their is some factor of $\mathbb{R}$ which doesn't play into the topology/cohomology. So we have realized that curves over $\mathbb{F}_q$ have too much internal structure to match with a Riemann surface.

Remark 5. The map $\operatorname{Spec}\mathbb{F}_{q^n} \to \operatorname{Spec}\mathbb{F}_q$ is analogous to the usual n-fold cover of the circle.

To fix the Rosetta Stone, we replace a Riemann surface Σ by certain a Σ-bundle over S^1. Explicitly, if we have Σ and a diffeomorphism φ, we can form the mapping torus:

$$\Sigma \times I/\left((x,0) \sim (\varphi(x), 1)\right)\ . \tag{1.26}$$

The idea is that if we start with a curve over a finite field, the diffeomorphism φ is like the Frobenius.

Similarly $\operatorname{Spec}\mathcal{O}_F$ looks like a curve where each "point" carries a circle. So this is again some kind of 3-manifold.

Remark 6. These circles don't talk to one another because they all have to do with a Frobenius at a different prime. So they're less like a product or a fibration, and more like a 3-manifold with a foliation.

This fits with the existing theory of arithmetic topology, sometimes known as the "knots and primes" analogy. The theory was started in a letter from Mumford to Mazur, but can be attributed to many people such as Mazur [Maz73], Manin, Morishita [Mor10], Kapranov [Kap95], and Reznikov. The recent work [Kim15, CKK+19] of Minhyong Kim plays a central role.

Remark 7. Lots of aspects of this dictionary are spelled out, but one should be wary of using it too directly. Rather we should think of this as telling us that there are several classes of '3-manifolds': ordinary 3-manifolds, function fields over finite fields, and number fields.

Lecture 2; January 21, 2021

Updated Rosetta Stone

The upshot is that we are thinking of all three objects in the Rosetta stone as three-manifolds. In particular, we're thinking of $\operatorname{Spec}\mathcal{O}_F$ (e.g. $\operatorname{Spec}\mathbb{Z}$) as a 3-manifold, so for any prime p we have the loop $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathcal{O}_F$, which we can interpret as a knot in the 3-manifold. Let F_v be the completion of the local field F at the place v (e.g. $\mathbb{Q}_p$). Then $\operatorname{Spec}F_v$ turns out to be the boundary of a tubular neighborhood of the knot. The point is that if F_v is a non-Archimedean local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$) then the "fundamental group" is the Galois group, and it has a quotient:

$$\operatorname{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} \,. \tag{1.27}$$

This group is called a *Baumslag-Solitar* group. Explicitly it is:

$$\operatorname{BS}(1,p) = \left\{\sigma, u \,\middle|\, \sigma u \sigma^{-1} = u^p\right\} \tag{1.28}$$

where we think of σ as the Frobenius, so corresponding to $\widehat{\mathbb{Z}}$, and u as the generator of $\mathbb{Z}_\ell$. The kernel is

$$p\text{-group} \times \prod_{\ell^r \neq \ell, p} \mathbb{Z}_{\ell^r} \hookrightarrow \operatorname{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} \,. \tag{1.29}$$

This tells us that there is For $p = 1$, this group is $\mathbb{Z} \times \mathbb{Z} = \pi_1\left(T^2\right)$. For $p = -1$, this is the fundamental group of the Klein bottle. This is evidence that the étale fundamental group of $\operatorname{Spec}F_v$ looks like some kind of p-dependent version of the fundamental group of the torus. So we can think of $\operatorname{Spec}F_v$ as a 2-manifold (fibered over S^1).

After this discussion we can identify an updated (multi-dimensional) Rosetta Stone. In three-dimensions we have: $\operatorname{Spec}\mathcal{O}_F$, $C/\mathbb{F}_q$, and a mapping torus $T_\varphi(\Sigma)$. The first two comprise the *global arithmetic setting*. In two-dimensions we first have local fields, which come in two types. One is finite extensions $F_v/\mathbb{Q}_p$ and the other is $\mathbb{F}_q((t))$. Spec of either of these is "two-dimensional" and the latter is some kind of punctured disk D^*. These two comprise the *local arithmetic setting*. A curve $\overline{C/\mathbb{F}_q}$ over an algebraically closed field (of positive characteristic) and a Riemann surface (projective curve Σ over $\mathbb{C}$) are also both "two-dimensional". These comprise the *global geometric setting*. The only 4-manifolds we will consider are of the form $M^3 \times I$ or $M^3 \times S^1$ where M^3 is a three-dimensional object of arithmetic or geometric origin. This discussion is summarized in table 1.3.

Table 1.3: The columns correspond to the three aspects of Weil's Rosetta Stone, and the rows correspond to dimension. The four-dimensional objects we consider are just products of three-dimensional objects with S^1 or I. M^3 is a three-dimensional object of arithmetic or geometric origin. The three-dimensional objects are number fields, function fields and mapping tori of Riemann surfaces. F is a number field, φ is some diffeomorphism of Σ, and T_φ denotes the corresponding mapping torus construction. The two-dimensional objects are local fields and curves. F_v is a finite extension of $\mathbb{Q}_p$. The 1-dimensional objects are both versions of circles, and the 0-dimensional objects are points.

<table>
<tr><th>Dimension</th><th>Number fields</th><th>Function fields</th><th>Geometry</th></tr>
<tr><td>4</td><td colspan="3">$M^3 \times S^1$, $M^3 \times I$</td></tr>
<tr><td rowspan="2">3</td><td colspan="2">Global arithmetic</td><td>-</td></tr>
<tr><td>$\operatorname{Spec} \mathcal{O}_F$</td><td>$C/\mathbb{F}_q$</td><td>$T_\varphi(\Sigma)$</td></tr>
<tr><td rowspan="2">2</td><td colspan="2">Local arithmetic</td><td>Global geometric</td></tr>
<tr><td>$\operatorname{Spec} F_v$</td><td>$\operatorname{Spec} \mathbb{F}_q((t)) = D^*$</td><td>$\overline{C}/\overline{\mathbb{F}}_q, \quad \Sigma/\mathbb{C}$</td></tr>
<tr><td rowspan="2">1</td><td colspan="2">-</td><td>Local geometric</td></tr>
<tr><td colspan="2">$\operatorname{Spec} \mathbb{F}_q$</td><td>$D^*_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}((t))$,
$D^*_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q((t))$</td></tr>
<tr><td>0</td><td colspan="2">$\operatorname{Spec} \overline{\mathbb{F}_q}$</td><td>$\operatorname{Spec} \mathbb{C}$</td></tr>
</table>

1.2.3 $\mathcal{A}$-side

The $\mathcal{A}$-side (or automorphic/magnetic side) TFT $\mathcal{A}_G$ is a huge machine which does many things, as in table 1.1. So far, the only recognizable thing is that it sends a 3-manifold M to some vector space $\mathcal{A}_G(M)$. We're thinking of $\operatorname{Spec}\mathcal{O}_F$ as a 3-manifold, and the assignment is the vector space we've been discussing:

$$\mathcal{A}_G(\operatorname{Spec}\mathcal{O}_F) = L^2(\mathcal{M}_{G,F}) \text{ or } H^*(\mathcal{M}_{G,F}) \ . \tag{1.30}$$

Remark 8. As suggested in eq. (1.19), note that $\Gamma\backslash G/K$ is a moduli space of something over the 3-manifold in question, not the 3-manifold itself.

1.2.4 Structure (reprise)

As it turns out, all the bells and whistles from the theory of automorphic forms in section 1.1.2 line up perfectly with the bells and whistles of TFT.

Number field

The assignment in eq. (1.30) formalizes the idea that we got a vector space $L^2(\mathcal{M}_{G,F})$ labelled by a group and a number field.

Conductor/ramification data

Recall the ramification data was a series of primes. This is manifested as a link (collection of knots) in the 3-manifold, where we allow singularities. These appear as *defects* (of codimension 2) in the physics. So the structure we saw before is manifested as defects of the theory.

Hecke algebra

The Hecke operators correspond to line defects (codimension 3) in the field theory. Physically this is "creating magnetic monopoles" alone some loop in spacetime.

Periods/states

These correspond to boundary conditions, i.e. codimension 1 defects.

Langlands functoriality

Passing from G to H can be interpreted as crossing a domain walls (also a codimension 1 defect).

1.2.5 $\mathcal{B}$-side

The $\mathcal{B}$-side (or spectral side) is the hard part from the point of view of number theory because Galois groups of number fields (and their representations) are very hard. I.e. the $\mathcal{B}$-side is the question, and the $\mathcal{A}$-side is the answer. But from the point of view of geometry, it is the other way around because fundamental groups of Riemann surfaces are really easy.

The $\mathcal{B}$-wide is about studying the algebraic geometry of spaces of Galois representations.

Recall that given a three-manifold (or maybe a number field F) the $\mathcal{A}$-side is concerned with the topology of the arithmetic locally symmetric space $\mathcal{M}_{G,F}$. $\mathcal{M}_{G,F}$ has to do with the geometry of F, so the $\mathcal{A}$-side is concerned with the topology of the geometry of F.

The $\mathcal{B}$-side concerns itself with the algebra of the topology of F. This means the following. For a manifold M (of any dimension), we can construct $\pi_1(M)$. Then the collection of rank n local systems on M is:

$$\mathrm{Loc}_n M = \{\pi_1(M) \to \mathrm{GL}_n \mathbb{C}\} \ . \tag{1.31}$$

A local system looks like a locally constant sheaf of rank n (or vector bundles with flat connection). These are sometimes called *character varieties*. Then we can study $\mathbb{C}[\mathrm{Loc}_n M]$. We can also replace GL_n with our favorite complex Lie group G to get:

$$\mathrm{Loc}_{G^\vee} M = \{\pi_1(M) \to G^\vee\} \ . \tag{1.32}$$

This depends only on the topology of M.

If we're thinking of a number field as a three-manifold, then π_1 is a stand-in for the Galois group so this is a space of representations of Galois groups. The TFT sends any three-dimensional M^3 to functions on $\mathrm{Loc}_{G^\vee}$:

$$\mathcal{B}_{G^\vee}\left(M^3\right) = \mathbb{C}\left[\mathrm{Loc}_{G^\vee} M\right] \ . \tag{1.33}$$

Remark 9. This side was a lot easier to write down than the $\mathcal{A}$-side, but if M is a number field, the Galois group is potentially very hard to understand. All the other bells and whistles are also easy to define here.

1.2.6 All together

In all of the setting in table 1.3, we can either make and automorphic measurement (attach $\mathcal{M}_{G,F}$ and study its topology) or we could take the Galois group (or π_1), construct a variety out of it, and study algebraic functions on it. The idea we will explain is that the Langlands program is an equivalence of these giant packages, but for "Langlands dual groups" G and $G^\vee$:

$$\mathcal{A}_G \simeq \mathcal{B}_{G^\vee} \ . \tag{1.34}$$

Remark 10. More is proven in the geometric setting than the arithmetic, but even geometric Langlands for a Riemann surface is still an open question.

This is really a conjectural way of organizing a collection of conjectures.

Part I

Abelian

Chapter 2

Spectral decomposition

Lecture 3; January 26, 2021

2.1 What is a spectrum?

The basic idea is that we start in the world of geometry, meaning we have a notion of a "space" (e.g. algebraic geometry, topology, ...), and given one of these spaces X we attach some kind of collection of functions $\mathcal{O}(X)$. These functions can have many different flavors, but they always form some kind of commutative algebra, possibly with even more structure. We will access the geometry of spaces using these functions. The operation $\mathcal{O}$ turns out to be a functor, i.e. if we have a morphism $\pi\colon X \to Y$ of spaces, we get a pullback morphism $\pi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$.

A fundamental question about this setup is to what degree we can reverse this operation. So starting with any commutative algebra, we would like to understand the extent to which we can we get geometry out of it. Category theory tells us that the right sort of thing to consider is a right adjoint to $\mathcal{O}$, which we call Spec. The fact that they form an adjunction means:

$$\mathrm{Map}_{\mathbf{Spaces}}(X, \operatorname{Spec} R) = \mathrm{Hom}_{\mathbf{Ring}^{\mathrm{op}}}(\mathcal{O}(X), R) = \mathrm{Hom}_{\mathbf{Ring}}(R, \mathcal{O}(X)) \ . \tag{2.1}$$

In the language of analysis, we might regard $\operatorname{Spec} R$ as a "weak solution" in the sense that it's a formal solution to the problem of finding a space associated to R. It is a functor assigning a set to any test space, but there is no guarantee that there is an honest space out there which would agree with it.

Remark 11. We might have to adjust the categories we're considering so that

$$\mathcal{O}(\operatorname{Spec} R) = R \ , \tag{2.2}$$

since this doesn't just fall out of the adjunction.

The point is that for some nice class of spaces, one might hope that we can recover a space from functions on that space: $X = \operatorname{Spec} \mathcal{O}(X)$.

Remark 12. The word spectrum is used in many places in mathematics. They are basically all the same, except spectra from homotopy theory.

2.1.1 Finite set

Let X be a finite set and k a field. Then the k-valued functions $\mathcal{O}(X)$ can be expressed as

$$\mathcal{O}(X) = \prod_{x \in X} k \tag{2.3}$$

which can be thought of as diagonal $X \times X$ matrices.

2.1.2 Compactly supported continuous functions

Gelfand developed the following version of this philosophy. For our category of spaces, consider the category of locally compact Hausdorff spaces X with continuous maps as the morphisms. For our space of functions we take $C_v(X)$, which is the space of continuous $\mathbb{C}$-valued functions which vanish at ∞. This is in the category of commutative C^*-algebras. These are Banach $\mathbb{C}$-algebras with a $*$ operation (to be thought of as conjugation) which is $\mathbb{C}$-antilinear and compatible with the norm. Given any commutative C^*-algebra A, the associated spectrum $\text{m-Spec}\, A$ is called the *Gelfand spectral*, and as a set it consists of the maximal ideals in A. We can write this as:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) \tag{2.4}$$

i.e. the unitary 1-dimensional representations of A. The adjunction is then saying that:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) = \mathrm{Map}(\mathrm{pt}, \text{m-Spec}\, A)\ . \tag{2.5}$$

Theorem 1 (Gelfand-Naimark)**.** *C_v and* m-Spec *give an equivalence of categories.*

See e.g. [aHRW10] for more details on this theorem.

2.1.3 Measure space

There is a coarser version where we start with a measure space X, and take attach the bounded functions $L^\infty(X)$. This forms a commutative von Neumann algebra. Again this is an equivalence of categories.

2.1.4 Algebraic geometry

We will focus on the setting of algebraic geometry. The category on the commutative algebra side will just be the category **cRing** of commutative rings. The geometric side will be the category of locally ringed spaces. This just means that there is a notion of evaluation at each point.

The functor:

$$(X, \mathcal{O}_X) \mapsto \mathcal{O}(X) \tag{2.6}$$

has an adjoint:

$$\mathrm{Spec}\, R \hookleftarrow R\ . \tag{2.7}$$

Affine schemes comprise the image of Spec:

$$\begin{tikzcd} \text{locally-ringed spaces} \arrow[r] & \mathbf{cRing} \arrow[dl, "\mathrm{Spec}"] \\ \text{affine schemes} \arrow[u, hook] & \end{tikzcd}\ . \tag{2.8}$$

Then it is essentially built into the construction that affine schemes are equivalent to commutative rings.

This doesn't really capture a lot of what we want to study in algebraic geometry, because one is usually interested in general schemes, which are locally ringed spaces that locally look affine. One way to deal with this is to really think of our geometric objects as:

$$\mathrm{Fun}\left(\mathbf{Ring}^{\mathrm{op}}, \mathbf{Set}\right) = \mathrm{Fun}\left(\mathbf{Aff}, \mathbf{Set}\right) \subset \text{Geometry} \ . \tag{2.9}$$

2.1.5 Topology

In homotopy theory we might start with the homotopy category of spaces and pass to some notion of functions, e.g. cohomology $H^*(X;\mathbb{Z})$. This sits in the category of graded commutative rings. This doesn't directly lead to a nice spectral theory, but it does if we remember a bit more structure. Instead, start with the category of spaces with continuous maps. Then we can take rational chains, $C^*_{\mathbb{Q}}$, and get a commutative differential graded $\mathbb{Q}$-algebra. This is the Quillen-Sullivan rational homotopy theory.[1] As it turns out, the category of simply connected spaces up to rational homotopy equivalence is equivalent to commutative differential graded algebras which are $\mathbb{Q}$ in degree 0 and 0 in degree 1.

2.2 Spectral decomposition

Let R be a commutative ring over some field k. Commutative rings usually arise via the study of modules over it. So let $V \in R$-**Mod**, i.e. a map

$$R \to \mathrm{End}\left(V\right) \ . \tag{2.10}$$

Now we want to decompose V into some sheaf $\underline{V}$ over $\mathrm{Spec}\, R$. We use that R-**Mod** is symmetric monoidal, so we have a tensor product $\otimes_R$ and we can define

$$\underline{V}\left(U\right) = V \otimes_R \mathcal{O}\left(U\right) \tag{2.11}$$

for open $U \subset \mathrm{Spec}\, R$. Then we can talk about the *support* of $v \in V$:

$$\mathrm{Supp}\left(v\right) \subset X \ . \tag{2.12}$$

Example 6. If X is finite, so $R = \prod_{x \in X} k$, then by asking for vectors supported at a single point, we get a decomposition $V = \oplus_{x \in X} V_x$. But to every point x, we get an evaluation morphism:

$$\lambda \colon R \xrightarrow{\mathrm{ev}_x} k \tag{2.13}$$

associated to the point x. Just like before we can think of this as a one-dimensional R-module, since $k = \mathrm{End}\left(k\right)$. Therefore, changing notation, we can write the decomposition as

$$V = \oplus_{\lambda \in X} V_\lambda w \ . \tag{2.14}$$

In this language the spaces in the decomposition are just the λ-eigenspaces:

$$V_\lambda = \mathrm{Hom}_{R\text{-}\mathbf{Mod}}\left(k_\lambda, V\right) = \left\{v \in V \mid r \cdot v = \lambda\left(r\right) v\right\} \tag{2.15}$$

[1] This should also be attributed to Mandell. See also Yuan's paper [Yua19].

where k_λ is the one-dimensional module over R, where R acts via the map λ.

This description is via evaluation at a point, but points are both open and closed so we can also describe this as restriction:

$$V_\lambda = V \otimes_{\mathcal{O}(X)=R} k_\lambda \ , \tag{2.16}$$

so this vector spaces is realized as both a Hom and a tensor because these points happened to be both open and closed.

Define the category of quasi-coherent sheaves to be:

$$\begin{array}{ccc} \mathbf{QCoh}\left(X = \operatorname{Spec} R\right) & := & R\text{-}\mathbf{Mod} \\ \big\uparrow & & \\ \mathbf{Coh}\left(X\right) & := & R\text{-}\mathbf{Mod}_{\text{f.g.}} \end{array} \ . \tag{2.17}$$

Note that in general X is only locally of the form $\operatorname{Spec} R$, so $\mathbf{QCoh}\left(X\right)$ is only locally of the form R-$\mathbf{Mod}$.

2.3 The spectral theorem

2.3.1 Algebraic geometry version

Let V be a vector space and $M \in \operatorname{End} V$. We can think of $M \in \operatorname{Hom}_{\mathbf{Set}}\left(\text{pt}, \operatorname{End} V\right)$, but $\operatorname{End} V$ is not just a set. It is an associative algebra over k, so really

$$M \in \operatorname{Hom}_{\mathbf{Set}}\left(\text{pt}, \operatorname{Forget} \operatorname{End} V\right) \ . \tag{2.18}$$

This sets us up for an adjunction with the free k-algebra construction. The free algebra in one generator is just $k\left[x\right]$ so the adjunction says that:

$$\operatorname{Hom}_{\mathbf{Set}}\left(\text{pt}, \operatorname{Forget} \operatorname{End} V\right) = \operatorname{Hom}_{k\text{-}\mathbf{Alg}}\left(k\left[x\right], \operatorname{End} V\right) \ . \tag{2.19}$$

What we have seen here is that equipping V with some $M \in \operatorname{End} V$ is equivalent to making V a module over $k\left[x\right]$. I.e. equipping V with $M \in \operatorname{End} V$ is equivalent to V being global sections of some quasi-coherent sheaf on $\operatorname{Spec} k\left[x\right] = \mathbb{A}^1$. I.e. V spreads out over $\mathbb{A}^1$.

To make this precise, assume V is finitely generated, i.e. the sheaf $\underline{V}$ is coherent. Then V has a sort of decomposition as a quotient and a subspace:

$$V_{\text{torsion}} \hookrightarrow V \twoheadrightarrow V_{\text{tor. free}} \tag{2.20}$$

where

$$V_{\text{torsion}} = \bigcup_{\lambda \in \mathbb{A}^1} \left\{v \in V \mid \operatorname{Supp} v = \lambda\right\} \ . \tag{2.21}$$

For general modules over a PID ($k\left[x\right]$ is a PID) we have:

$$V \simeq \underbrace{V_{\text{tor}}}_{\text{discrete spectrum}} \oplus \underbrace{V_{\text{free}}}_{\text{continuous spectrum}} \tag{2.22}$$

where

$$V_{\text{free}} = k\left[x\right]^{\oplus r} \tag{2.23}$$

and

$$V_{\mathrm{tor}} = \bigoplus_{\lambda \in \mathrm{Spec}} V_{\hat{\lambda}} \tag{2.24}$$

where $V_{\hat{\lambda}}$ is the subspace supported at λ. As it turns out

$$V_{\hat{\lambda}} = \bigoplus_{i} k[x] / (x-\lambda)^{\ell_i} \ , \tag{2.25}$$

i.e. for any element of this some power of $(x-\lambda)$ annihilates it, so these are generalized eigenspaces. This decomposition is precisely the Jordan normal form.

An eigenvector is some element $v \in V$ such that $\lambda v = xv$ (or by definition Mv). But this is the same as an element of:

$$\mathrm{Hom}_{k[x]}(k_\lambda, V) \ . \tag{2.26}$$

So this is what one might call a section supported "scheme-theoretically" at $\lambda \in \mathbb{A}^1$.

On the other hand, the fibers:

$$V \otimes_{k[x]} k_\lambda \tag{2.27}$$

are naturally quotients of V (rather than a sub), and so they're some kind of coeigenvectors.

In the continuous spectrum there are no eigenvectors: as a free module, $k[x]$ doesn't contain any eigenvectors.

Example 7. Consider the free case. It is sufficient to consider $V = k[x]$, since otherwise it is just a direct sum of copies of this. Then $\underline{X} = \mathcal{O}_{\mathbb{A}^1}$. There are no generalized eigenvectors (because $(x-\lambda)^N = 0$ for some $N \gg 0$ and some $\lambda \in \mathbb{A}^1$). There are lots of coeigenvectors, though. For any $\lambda \in \mathbb{A}^1$ we have a map

$$V \twoheadrightarrow k_\lambda \ . \tag{2.28}$$

This is a distribution, i.e. an element of

$$\mathrm{Hom}(V, k) \ . \tag{2.29}$$

So for every $\lambda \in \mathbb{A}^1$, we get

$$\mathrm{Hom}(V, k) \ni \delta_\lambda \colon V \twoheadrightarrow k_\lambda \ . \tag{2.30}$$

The basic example is $V = L^2(\mathbb{R})$ and $M = x$. Then V_λ consists of functions supported at x, but there are none. We would like to say δ_λ, but this is not L^2. The dual operator $M^\vee = d/dx$ has eigenvectors which are roughly $e^{i\lambda x}$, but these are not L^2 either.

This is what continuous spectra look like. When you decompose functions on $\mathbb{A}^1$ under the action of x or d/dx, there is a sense in which it is a direct integral, which is different from a direct sum. The things you're integrating aren't actually subsets. So we can think of functions on $\mathbb{A}^1$ as being some kind of continuous direct sum of functions on a point, but those functions don't live as subspaces. In this case they lived as quotients, but not subspaces. This is simpler in the torsion-free case, but is a general feature of continuous spectra. This is not a weird/special fact about analysis, because we see it even at the level of algebra (polynomials).

2.3.2 Measurable version

Instead of the matrix M, we consider a self-adjoint operator A on a Hilbert space $V = \mathcal{H}$. Then von Neumann's spectral theorem tells us that there is a "sheaf" (projection valued measure) π_A on $\mathbb{R}$ and

$$A = \int_{\mathbb{R}} x d\pi_A \ . \tag{2.31}$$

A projective valued measure can be thought of as a sheaf $\underline{\pi_A}$ as follows. For $U \subset \mathbb{R}$ measurable, we attach the image under the projection:

$$\underline{\pi_A}(U) = \pi_A(U) \ . \tag{2.32}$$

So this is some kind of sheaf of Hilbert spaces. There is no topology to be compatible with, but it does satisfy the additivity property that the rule:

$$U \mapsto \langle w, \pi_A(U) v \rangle \tag{2.33}$$

defines a $\mathbb{C}$-valued measure on $\mathbb{R}$. So this is the version of a sheaf in the measurable world.

So now eq. (2.31) is saying that the Hilbert space $\mathcal{H}$ sheafifies over $\mathbb{R}$ in such a way that A acts by the coordinate function x, just like in the algebro-geometric setting above. Then the spectrum is a measurable subset

$$\operatorname{Spec}(A) \coloneqq \operatorname{Supp} \pi_A \subset \mathbb{R} \ . \tag{2.34}$$

Example 8. If $\mathcal{H}$ is finite-dimensional then the spectrum is a discrete set of points, and the decomposition is just into eigenspaces.

2.3.3 Homotopical version

We saw that we had a quasi-coherent sheaf in algebraic geometry, a projection-valued measure in measure theory, and now in algebraic topology we have the following. If $R = C^*(X)$, then

$$R\text{-}\mathbf{Mod} \hookrightarrow \operatorname{Loc}(X) \tag{2.35}$$

where $\operatorname{Loc}(X)$ consists of locally-constant complexes on X.

The basic idea is that if we have a model for cochains on X:

$$\to R^{\oplus j} \to R^{\oplus i} \to M \tag{2.36}$$

then we get a presentation of $\underline{M}$ by constant sheaves:

$$\to \underline{k_X}^{\oplus j} \to \underline{k_X}^{\oplus i} \to \underline{M} \tag{2.37}$$

where the key point is that

$$C^*(X) = \operatorname{End}^*\left(\underline{k_X}, \underline{k_X}\right) \ . \tag{2.38}$$

Or more directly we can define the sheaf to be:

$$\underline{M}(U) = M \otimes_{C^*(X)} C^*(X) \ . \tag{2.39}$$

Lecture 4; January 28, 2021

2.3.4 Physics interpretation: observables and states

We have seen that, starting with some flavor of commutative algebra A, we can construct a geometric object $\operatorname{Spec} A$. Then a module M over A gets spread out into a sheaf over $\operatorname{Spec} A$. The algebra A can be thought of as the *algebra of observables* of some physical system. Then the *space of states* forms a module over A, and so fits into this framework.

Let's back up. The general idea is that we're trying to get a grip on the geometry of this space via the functions on it, i.e. making observations. Recall that the defining property of $\operatorname{Spec} A$ is that whenever we have a space X and a map $A \to \mathcal{O}(X)$, where A is commutative, then we get a map

$$X \to \operatorname{Spec} A \ . \tag{2.40}$$

We can think of the map $A \to \mathcal{O}(X)$ as picking out some functions (observables) on the space which satisfy the relations of A. E.g. if we just have one function, this is a map from the space down to the line, and then the space will decompose over this. In general the space will decompose over a higher-dimensional base. So this is a way of measuring the space with functions.

Spectral decomposition of modules is a linearized version of this. We replace X by a linearized version of it, e.g. $\mathcal{O}(X)$, and this becomes a module over A. I.e. a module M over A is a linearized version of a map $X \to \operatorname{Spec} A$. If we just have a single function, then this corresponds to a map $X \to \operatorname{Spec} k[x] = \mathbb{A}^1$. Likewise, a single matrix (endomorphism of a vector space) gives rise to a sheaf over $\mathbb{A}^1$.

In quantum mechanics, we don't have a phase space. We only have a vector space $\mathcal{H}$ (some linearized version of the phase space), called the *Hilbert space of states*. Observables are operators on $\mathcal{H}$. In physics we're interested in reality, so we might insist on the condition that observables are self-adjoint:

$$\mathcal{O} = \mathcal{O}^* \ . \tag{2.41}$$

Typically we won't impose this condition. For an observable $\mathcal{O} \circlearrowright \mathcal{H}$, spectral decomposition tells us that $\mathcal{H}$ sheafifies (as a projection-valued measure) over $\mathbb{R}$. The base is $\mathbb{R}$ because this is Spec of the algebra generated by a single operator. This is the analogue of starting with a classical phase space M and a single observable $M \to \mathbb{R}$, and then decomposing M over $\mathbb{R}$.

A *state* is an element $|\varphi\rangle \in \mathcal{H}$. Given a state and an operator $\mathcal{O}$, this state becomes a section of the sheaf $\mathcal{H}$, i.e. we get an eigenspace decomposition of this vector. Given a section, the first thing we can ask for is the support. This is just where the measurement we made "lives".

We can do something more precise by using the norm. As it turns out, $\|\varphi\|^2$ is a probability measure on $\mathbb{R}$ which tells us where to expect the state to be located. For example, we can take the expectation value of the observable $\mathcal{O}$ in the state φ:

$$\langle \mathcal{O} \rangle_\varphi = \frac{\langle \varphi | \mathcal{O} | \varphi \rangle}{\langle \varphi | \varphi \rangle} \ . \tag{2.42}$$

This is a continuous version of

$$\frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda \in \operatorname{Spec} \mathcal{O}} \lambda \left\| \operatorname{Proj}_{\mathcal{H}_\lambda} |\varphi\rangle \right\|^2 = \frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda, \psi_i} \lambda \langle \psi_i | \varphi \rangle |\psi_i\rangle \tag{2.43}$$

where ψ_i is a basis of eigenvectors.

Remark 13. To give a quantum-mechanical system, we also need to specify the *Hamiltonian* H. This is a specific observable (self-adjoint operator on $\mathcal{H}$) which plays the role of the energy functional. The eigenstates for H are the steady states of the system. This lets us spread $\mathcal{H}$ out over $\mathbb{R}$ to get the energy eigenstates. We will be working in the *topological* setting where $H = 0$, i.e. we're just looking at the 0 eigenspace. So this decomposition is kind of orthogonal to our interests.

Chapter 3

Fourier theory/abelian duality

We have seen that whenever we have a "spectral dictionary", we get a notion of spectral decomposition: modules become sheaves, where the notion of a sheaf depends on the context. This gives us a way of spreading out the algebra of modules over the geometry or topology of our space.

For this to be useful, we need interesting sources of commutative algebras. A natural source for commuting operators is when we have an abelian group G acting on a vector space V: given a morphism

$$\rho\colon G \to \mathrm{Aut}\,(V) \tag{3.1}$$

we get a family of operators $\{\rho\,(g)\}_{g\in G}$ and we can spectrally decompose V using these operators. This is what Fourier theory is about. So we're thinking of Fourier theory as some kind of special case of spectral decomposition.

3.1 Characters

Let V be a representation of an abelian group G, i.e. we have a map

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}\,(X) \ \subset \ \mathrm{End}\,(V) \\ g &\longmapsto T_g \end{aligned} \tag{3.2}$$

such that $T_g T_h = T_{gh}$.

Example 9. If G acts on a space X, and V is functions on X, then we get an action of G on V.

Example 10. G always acts on itself, so therefore it acts on functions on G itself. This is the *regular representation*.

Now we want to spectrally decompose. First we need to know what the spectrum is, so we ask the following question.

Question 1. What are the possible eigenvalues?

Let $v \in V$ be an eigenvector:

$$g \cdot v = \chi(g)\, v \tag{3.3}$$

where

$$\chi\colon G \to \operatorname{Aut} \mathbb{C}V = \mathbb{C}^{\times} \subset \mathbb{C} \tag{3.4}$$

is a group homomorphism, i.e. a *character* of G. So the possible eigenvalues are the characters:

$$G^{\vee} = \{\text{characters}\} = \operatorname{Hom}_{\mathbf{Grp}}\left(G, \mathbb{C}^{\times}\right) \ . \tag{3.5}$$

This is the spectrum, i.e. we will be performing spectral decomposition over $G^{\vee}$.

3.2 Finite Fourier transform

Now let G be a finite group.[1] We will eventually assume G is abelian, but we don't need this yet. We want to see $G^{\vee}$ appear at the spectrum. For a complex representation V we have a group map $G \to \operatorname{Aut} V$, but the composition with the inclusion

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right, "\text{monoid map}"'] & \operatorname{Aut} V \arrow[r, phantom, "\subset"] & \operatorname{End}(V) \end{tikzcd} \tag{3.6}$$

is a monoid map. In other words V gives rise to an element of

$$\operatorname{Hom}_{\mathbf{Monoid}}\left(G, \operatorname{Forget}(\operatorname{End} V)\right) \ . \tag{3.7}$$

Just like before, we have an adjunction:

$$\operatorname{Hom}_{\mathbf{Monoid}}\left(G, \operatorname{Forget}(\operatorname{End} V)\right) = \operatorname{Hom}_{\mathbb{C}\text{-}\mathbf{Alg}}\left(?, \operatorname{End} V\right) \ , \tag{3.8}$$

where the missing entry should be some kind of free construction. As it turns out, the answer is the *group algebra*:

$$? = \mathbb{C}G \ . \tag{3.9}$$

This is the algebra freely generated by scalar multiplication and sums of elements of G. Since the group is finite this is just:

$$\mathbb{C}G = \left\{ \sum_{g \in G} f(g) \cdot g \right\} \tag{3.10}$$

where $f\colon G \to \mathbb{C}$ is any function. Really we should think of f as a measure rather than a function. There is no difference when G is finite, but for any $g \in G$ would would like a canonical element

$$\delta_g = 1 \cdot g \in \mathbb{C}G \ , \tag{3.11}$$

which means the coefficients come from some f which is 1 at g and 0 elsewhere, which is not a function in general.

[1] This is a simplifying assumption so we don't need to worry about what "kind" of functions we're considering.

We can think of $\mathbb{C}G$ as being generated by the elements δ_g for $g \in G$. The algebra structure comes from convolution:

$$\delta_f * \delta_g = \delta_{fg} \ . \tag{3.12}$$

In general

$$\begin{aligned} f_1 * f_2 &= \sum_g f_1(g)\, g * \sum_h f_2(h)\, h && (3.13)\\ &= \sum_k \left(\sum_{gh=k} f_1(g)\, f_2(h) \right) k && (3.14)\\ &= \sum_k \sum_g f_1(g)\, f_2\left(kg^{-1}\right) \cdot k \ . && (3.15) \end{aligned}$$

We can express the convolution in terms of the multiplication map $\mu \colon G \times G \to G$ as follows. We have he two projections π_1 and π_2:

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\mu"] & & G \\ G & & G & \end{tikzcd} \ . \tag{3.16}$$

We can pull f_1 back along π_1 and f_2 back along π_2 to get a function on $G \times G$:

$$f_1 \boxtimes f_2 \coloneqq \pi_1^* f_1 \pi_2^* f_2 \ . \tag{3.17}$$

Then we can push this along μ, and the result is the convolution:

$$f_1 * f_2 = \mu_* \left(f_1 \boxtimes f_2\right) = \int_\mu f_1 \boxtimes f_2 \ . \tag{3.18}$$

The upshot is that we can define the group algebra in this way whenever we have things which can be pulled and pushed like this.

For $G \circlearrowright V$, we have extended this to an action of $\mathbb{C}G \circlearrowright V$. Then G is **abelian** iff $(\mathbb{C}G, *)$ is a **commutative** algebra. So now the fundamental object over which representation theory of G will sheafify is:

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) \ , \tag{3.19}$$

and as it turns out

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) = G^\vee \ , \tag{3.20}$$

i.e.

$$\left(\mathbb{C}G, *\right) \simeq \left(\mathcal{O}\left(G^\vee\right), \cdot\right) \ . \tag{3.21}$$

This is a first version of the Fourier transform. The idea is that a map $\operatorname{Spec} k \to \operatorname{Spec} A$ is the same as a morphism $A \to k$, which is exactly a 1-dimensional representation of G, i.e. a character. Under this correspondence, the characters $\chi_t \in \mathbb{C}G$ for $t \in G^\vee$ correspond to points $\delta_t \in \mathcal{O}\left(G^\vee\right)$ for $t \in G^\vee$. Moreover, translation by g corresponds to multiplication by $g^\vee$, i.e. the character

$$t \mapsto \chi_t(g) \ . \tag{3.22}$$

We can rephrase this equivalence slightly to make it more evident that this is some version of the Fourier transform. If f is a function on G then we can write

$$f = \sum_{t \in G^\vee} \widehat{f}(t) \cdot \chi_t \ . \tag{3.23}$$

This is just expressing f in terms of the basis of characters. Then we can recover $\widehat{f}(t)$ as the coefficient of f in this orthonormal basis. We also have that

$$f * (-) = \widehat{f} \cdot (-) \ . \tag{3.24}$$

3.2.1 Secret symmetry

There is a secret symmetry here. $G^\vee$ is an abelian group itself under the operation of pointwise multiplication. I.e.

$$\chi_{t \cdot s} \coloneqq \chi_t \cdot \chi_s \ . \tag{3.25}$$

Call the corresponding abelian group the *dual group to G*.

To see that this is a good duality, note that there is a tautological map

$$\begin{aligned} G &\longrightarrow G^{\vee\vee} \\ g &\overset{\sim}{\longmapsto} \{\chi \mapsto \chi(g)\} \ , \end{aligned} \tag{3.26}$$

which turns out to be an isomorphism.

We could have set this up in a more symmetric way. We have two projections:

$$\begin{tikzcd} & G \times G^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & G^\vee \end{tikzcd} \tag{3.27}$$

and there is a tautological object, called the universal character, living over $G \times G^\vee$:

$$\begin{array}{c} \chi(-,-) \\ \downarrow \\ G \times G^\vee \end{array} \ , \tag{3.28}$$

i.e. a function on $G \times G^\vee$ given by:

$$\chi(g,t) = \chi_t(g) = \chi_g(t) \ . \tag{3.29}$$

Then the Fourier transform of $f \in \mathrm{Fun}(G)$ is given by pulling up to $G \times G^\vee$, multiplying by χ, and then summing up by pushing forward by π_2:

$$f \mapsto \pi_{2*}(\pi_1^* f \cdot \chi) \ . \tag{3.30}$$

Explicitly the Fourier transform is:

$$\widehat{f}(t) = \sum_g f(g)\, \overline{\chi}(g,t) \tag{3.31}$$

$$f(g) = \sum_t \widehat{f}(t)\, \chi(g,t) \ . \tag{3.32}$$

We have simultaneously diagonalized the action of all $g \in G$ on $\mathrm{Fun}(G)$.

For any V with a G action, we get a $(\mathbb{C}G, *)$ action on V so V spectrally decomposes over $G^\vee$, i.e.

$$V = \bigoplus_{t \in G^\vee} V_{\chi_t} \ , \tag{3.33}$$

where G acts by the eigenvalue specified by χ_t on the subspace V_{χ_t}.

This gives a complete picture of the complex representation theory of finite abelian group. The exact same formalism works in any setting with abelian groups. We will focus on the setting of topological groups and algebraic groups. Everything will mostly look the same, with the difference being what kind of functions we consider.

3.3 Pontrjagin duality

Let G be a locally compact abelian (LCA) group.

Example 11. $\mathbb{Z}$, $\mathrm{U}(1)$, $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathbb{Q}_p^*$ are all (non-finite) examples.

Define the dual to be the collection of *unitary* characters

$$G^\vee = \mathrm{Hom}_{\mathbf{TopGrp}}(G, \mathrm{U}(1)) \ . \tag{3.34}$$

Remark 14. We shouldn't be too shocked by replacing $\mathbb{C}^\times$ by $\mathrm{U}(1) \subset \mathbb{C}^\times$. If G is finite, all of the character theory was captured by $\mathrm{U}(1)$ anyway.

3.3.1 Group algebra

The spectrum will again be Spec of the group algebra, but we need to determine the appropriate definition of the group algebra in this context.

Endow G with a Haar measure. Before we had the counting measure, and could translate freely between functions and measures (so, in particular, they could both push and pull). Now $L^1(G)$ has a convolution structure in exactly the same way as in eq. (3.18):

$$f_1 * f_2 = \int_{g \in G} f_1(h) f_2\left(gh^{-1}\right) dg \ . \tag{3.35}$$

Just as before, this convolution comes from an adjunction. I.e. it satisfies a universal property in the world of C^*-algebras. If we have a representation:

$$G \to \mathrm{End}(V) \tag{3.36}$$

then this will correspond to a morphism

$$\left(L^1(G), *\right) \to \mathrm{End}(V) \tag{3.37}$$

of C^*-algebra.

The spectrum is the Gelfand spectrum:

$$\text{m-Spec}\left(L^1(G), *\right) = G^\vee \ . \tag{3.38}$$

This is a version of the Fourier transform which says that:

$$\left(L^1(G), *\right) \simeq \left(C_v\left(G^\vee\right), \cdot\right) \ , \tag{3.39}$$

where C_v denotes functions vanishing at ∞. Another version says that there is a tautological map:

$$G \to G^{\vee\vee} \tag{3.40}$$

which is an isomorphism.

Again, this can be written in a symmetric way:

$$f \mapsto \pi_{2*}\left(\pi_1^* f \cdot \chi\right) \tag{3.41}$$

where

$$\begin{tikzcd} & \chi \arrow[d] & \\ & G \times G^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & G^\vee \end{tikzcd} . \tag{3.42}$$

For any notion of functions or distributions on G, we can perform this Fourier transform operation. The question is, given the type of functions we feed in, what type of functions do we get in the other side? For L^2 functions we simply get:

$$L^2(G) \xrightarrow{\sim} L^2(G) \ . \tag{3.43}$$

Any of these notions of a Fourier transform have the same general features. Some of which are as follows.

(i) Translation by a group element becomes pointwise multiplication.

(ii) Convolution also becomes pointwise multiplication.

(iii) Characters correspond to points.

3.3.2 Fourier series

Take $G = \mathrm{U}(1)$. Then

$$G^\vee = \mathrm{Hom}_{\mathbf{TopGrp}}\left(\mathrm{U}(1), \mathrm{U}(1)\right) = \mathbb{Z} \tag{3.44}$$

where $n \in \mathbb{Z}$ corresponds to

$$\left\{x \mapsto e^{2\pi i n x}\right\} \ . \tag{3.45}$$

Then the Fourier transform established an equivalence

$$L^2(\mathrm{U}(1)) \xrightarrow{\sim} L^2(\mathbb{Z}) = \ell^2 \ . \tag{3.46}$$

As before, this is symmetric. There is a universal character:

$$\chi\colon\ (x, n) \mapsto e^{2\pi i n x} \tag{3.47}$$

living over $\mathrm{U}(1) \times \mathbb{Z}$, and we have the usual projections:

$$\begin{tikzcd} & \mathrm{U}(1) \times \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathrm{U}(1) & & \mathbb{Z} \ . \end{tikzcd} \tag{3.48}$$

Then we can read it backwards. A character

$$\mathbb{Z} \to \mathrm{U}(1) \tag{3.49}$$

is determined by the image of $1 \in \mathbb{Z}$, so characters of $\mathbb{Z}$ are labelled by points of $\mathrm{U}(1)$.

In general Pontrjagin duality, G is compact iff $G^\vee$ is discrete. Concretely, for $n \in \mathbb{Z}$ we have

$$e^{2\pi i n x} \in L^2 \tag{3.50}$$

because G is compact. Similarly, because $\mathbb{Z}$ is discrete,

$$\delta_n \in \ell^2 \ . \tag{3.51}$$

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3.4 Fourier transform

Recall that the Pontrjagin dual of $\mathrm{U}(1)$ is $\mathbb{Z}$. The characters $\mathrm{U}(1) \to \mathrm{U}(1)$ are given by $z \mapsto z^n$ for any $n \in \mathbb{Z}$. Similarly, the dual of $\mathbb{Z}$ is $\mathrm{U}(1)$.

We can replace $\mathrm{U}(1)$ by a torus T, which is defined as:

$$T = \mathbb{R}^d/\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathrm{U}(1) \tag{3.52}$$

where $\Lambda \subset \mathbb{Z}^d$ is a full-rank lattice. The dual of T is the dual lattice:

$$T^\vee = \Lambda^\vee \ . \tag{3.53}$$

Similarly, the dual of the lattice Λ is the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathrm{U}(1) \ . \tag{3.54}$$

The classical Fourier transform takes G to be a real vector space. For $G = \mathbb{R}_x$ the dual is another copy of $\mathbb{R}$: $G^\vee = \mathbb{R}_t$. Performing the same operations as before, with universal character

$$\chi(x,t) = e^{2\pi i x t} \ , \tag{3.55}$$

we get the usual Fourier transform. For a general real vector space the dual is the dual vector space, and the character is

$$\chi(x,t) = e^{2\pi i \langle x,t\rangle} \ , \tag{3.56}$$

where the pairing is the usual one between the vector space and its dual.

Recall that in the context of the duality between $\mathrm{U}(1)$ and $\mathbb{Z}$, characters corresponded to points. This situation differs in the sense that characters are not L^2 anymore (since $\mathbb{R}$ is not compact, like $\mathrm{U}(1)$ is) and the points are not isolated (since $\mathbb{R}$ is not discrete like $\mathbb{Z}$). But we still have:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t)\, e^{2\pi i x t} dt \ . \tag{3.57}$$

The operation of differentiation d/dx corresponds with multiplication by t on the other side. We should think of d/dx as an infinitesimal version of group translation, which went to multiplication before. So this is part of the same framework where group theory on one side goes to geometry on the other.

In general, let G be a Lie group with abelian Lie algebra $\mathfrak{g}$. We have a map from $\mathfrak{g}$ to vector fields on G:

$$\mathfrak{g} \to \operatorname{Vect} G \subset \operatorname{Diff}(G) \tag{3.58}$$

and there is an adjunction between

$$\text{Forget}\colon \mathbf{Alg}_{\text{Assoc.}} \to \textbf{Lie-Alg} \tag{3.59}$$

and the functor

$$U\colon \textbf{Lie-Alg} \to \mathbf{Alg}_{\text{Assoc.}} \tag{3.60}$$

which sends a Lie algebra to the *universal enveloping algebra.* $\mathfrak{g}$ is abelian so the universal enveloping algebra is just the symmetric algebra

$$U(\mathfrak{g}) = \operatorname{Sym}^* \mathfrak{g} \ . \tag{3.61}$$

$U\mathfrak{g}$ is now a commutative algebra acting on $C^\infty(G)$. Therefore we can spectrally decompose/sheafify over

$$\operatorname{Spec} U\mathfrak{g} = \mathfrak{g}^* \ . \tag{3.62}$$

E.g. for d/dx acting on $\mathbb{R}_x$,

$$\mathfrak{g}^* = \mathbb{R}_t = \operatorname{Spec} RR\left[d/dx = t\right] \ . \tag{3.63}$$

Example 12. The dual of $\mathrm{U}(1)$ is $\mathbb{Z} \subset \mathbb{R}_t$.

3.5 In quantum mechanics

Before we see where the Fourier transform comes up in quantum mechanics, we consider classical mechanics. If we want to a model a particle moving around in a manifold M, then the phase space is the cotangent bundle T^*M with positive coordinates q on M, and momenta coordinates p in the fiber direction. The observables form:

$$\text{functions on } M \otimes \operatorname{Sym} TM \ . \tag{3.64}$$

In quantum mechanics the space of states is replaced by $\mathcal{H} = L^2(M)$. The observables contain $\operatorname{Diff}(M)$, the differential operators on M. Inside of this we have two pieces:

$$\begin{tikzcd}[column sep=small] \text{f'ns on } M \arrow[dr, hook] & \\ & \operatorname{Diff}(M) \\ TM \arrow[ur, hook] & \end{tikzcd} \ , \tag{3.65}$$

with

$$p_j = i\frac{d}{dq_j} \ . \tag{3.66}$$

In classical mechanics the analogous pieces commute. But here they commute up to a term: a tangent vector $\xi \in TM$ and a function $f \in \operatorname{Fun}(M)$ must satisfy

$$\xi f = f\xi + \hbar f' \ . \tag{3.67}$$

To summarize, states look like "$\sqrt{\text{observables}}$". The position operators:

$$q_j \cdot (-) \tag{3.68}$$

are diagonalized, on the other hand the momentum operators act as derivates.

We can also pass to the momentum picture where we diagonalize the p_i's (derivatives) instead. For $M = \mathbb{R}^n$ we have a natural basis of invariant vector fields (this is the advantage of having a group). Now we can simultaneously diagonalize

$$p_j = i \frac{d}{dx_j} \tag{3.69}$$

to identify

$$L^2\left(\mathbb{R}^n_q\right) \simeq L^2\left(\mathbb{R}^n_p\right) \ , \tag{3.70}$$

which is the Fourier transform. One might say that this is identifying quantum mechanics for G with quantum mechanics for the Pontrjagin dual $G^\vee$. This is the one-dimensional case of abelian duality, or "one-dimensional mirror symmetry".

3.6 Cartier duality

This is the same Fourier theory we've been doing, but in the context of algebraic geometry, i.e. instead of continuous, etc. functions we're considering algebraic functions. We will eventually see that this duality shows up in physics (electric-magnetic duality), as well as number theory (class field theory).

To say what a group is in the world of algebraic geometry, we need to review the notion of the functor of points. To a variety X we can associate a functor **cRing** $\to$ **Set** by sending a ring R to

$$X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X) \ . \tag{3.71}$$

As it turns out, specifying this functor is equivalent to specifying X itself.

A variety G is an *algebraic group* if the associated functor of points **cRing** $\to$ **Set** lifts to a functor landing in groups:

$$\begin{tikzcd} \mathbf{cRing} \arrow[r] \arrow[dr, dashed] & \mathbf{Set} \\ & \mathbf{Grp} \arrow[u] \end{tikzcd} \tag{3.72}$$

i.e. that

$$G(R) = \mathrm{Hom}(\mathrm{Spec}\, R, G) \tag{3.73}$$

is a group.

Remark 15. Sometimes other things are assumed in the definition of an algebraic group, which we do not assume here.

Example 13. Consider $\mathbb{A}^1 = \mathbb{G}_a$. As a functor, this sends

$$R \mapsto (R, +) \ . \tag{3.74}$$

This is saying that $\mathrm{Map}\left(X, \mathbb{A}^1\right) = \mathcal{O}(X)$.

Example 14. Consider $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m = \operatorname{Spec} k\left[t, t^{-1}\right]$. As a functor this sends

$$R \mapsto \left(R^\times, \cdot\right) \ . \tag{3.75}$$

Example 15. The integers form an algebraic group with functor of points given by:

$$\mathbb{Z}\colon R \mapsto (\mathbb{Z}, +) \ . \tag{3.76}$$

Let G be an abelian (algebraic) group. The *Cartier dual* of G is

$$G^\vee = \operatorname{Hom}_{\mathbf{Grp}_{\mathrm{Alg}}}\left(G, \mathbb{G}_m\right) \ . \tag{3.77}$$

where the dualizing object $\mathbb{G}_m$ comes from $\operatorname{Aut}_k = \mathbb{G}_m$.

Example 16. Let $G = \mathbb{Z}/n$. The Cartier dual is

$$\mathbb{Z}/n^\vee = \operatorname{Hom}\left(\mathbb{Z}/n, \mathbb{G}_m\right) \simeq \mu_n \tag{3.78}$$

where μ_n denotes the nth roots of unity. As a functor, this sends:

$$R \mapsto n\text{th roots of unity in } R \ . \tag{3.79}$$

Example 17. The dual of the integers is $\mathbb{Z}^\vee \simeq \mathbb{G}_m$, since $\mathbb{Z} \to \mathbb{G}_m$ is determined by the image of 1. Similarly:

$$\mathbb{G}_m^\vee = \operatorname{Hom}_{\mathbf{TopGrp}}\left(\mathbb{G}_m, \mathbb{G}_m\right) \simeq \mathbb{Z} = \{z \mapsto z^n\} \ . \tag{3.80}$$

Example 18. More generally, the dual to a lattice Λ will be the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m \ , \tag{3.81}$$

so (over $\mathbb{C}$) this looks roughly like $\left(\mathbb{C}^\times\right)^{\operatorname{rank}\Lambda}$. Similarly a torus T gets exchanged with the dual lattice $\Lambda^\vee$.

To avoid technicalities, assume G is finite.[2] Consider the collection of functions on G, $\mathcal{O}(G)$. Dualizing and taking Spec gives the dual group:

$$G^\vee = \operatorname{Spec}\mathcal{O}(G)^* \ . \tag{3.82}$$

Rather than talking about a group algebra, functions on G has the structure of a group coalgebra as follows. The multiplication map on G induces a coproduct on $\mathcal{O}(G)$:

$$\begin{gathered} G \times G \xrightarrow{\quad \mu \quad} G \\ \mathcal{O}(G) \xrightarrow{\Delta := \mu^*} \mathcal{O}(G) \otimes \mathcal{O}(G) \ . \end{gathered} \tag{3.83}$$

This makes $\mathcal{O}(G)$ into a *Hopf algebra*, i.e. $\mathcal{O}(G)$ has a multiplication, and a comultiplication Δ. In fact, this is a finite-dimensional commutative and cocommutative Hopf algebras. The study of these Hopf algebras turns out to be equivalent to the study of finite abelian group schemes.

[2] Formally we're assuming that G is a finite abelian group scheme.

Example 19 ("Fourier transform"). Let $\operatorname{char} k = 0$. The Cartier dual of $\mathbb{G}_a$ is the formal completion of itself:

$$\mathbb{G}_a^\vee = \widehat{\mathbb{G}_a} \ , \tag{3.84}$$

where the formal completion of $\mathbb{G}_a$ is given by:

$$\widehat{\mathbb{G}_a} = \bigcup \operatorname{Spec} k\left[t\right] / \left(t^n\right) \ . \tag{3.85}$$

The Cartier dual of an n-dimensional vector space V is the completion of the dual:

$$V^\vee = \widehat{V^*} \ . \tag{3.86}$$

The character of

$$\mathbb{G}_a = \operatorname{Spec} k\left[x\right] \tag{3.87}$$

should be

$$e^{xt} = \sum \frac{\left(xt\right)^n}{n!} \ , \tag{3.88}$$

but we need this to be a finite sum. This makes sense if t is nilpotent, i.e. there is $N \gg 0$ such that $t^N = 0$.

So when this is true, i.e. t is "close to 0", the function $e^{\langle x,t\rangle}$ is well-defined for $x \in V$ and $t \in V^*$, and is the character for V.

Similarly the dual of the completion is $\mathbb{G}_a$:

$$\mathbb{G}_a = \widehat{\mathbb{G}_a}^\vee \ . \tag{3.89}$$

Recall that Fourier duality exchanges

$$\left(\operatorname{Fun}\left(G\right), *\right) \simeq \left(\operatorname{Fun}\left(G^\vee\right), \cdot\right) \ . \tag{3.90}$$

Then $\mathbf{Rep}\left(G\right)$ became spectrally decomposed over $G^\vee$. In algebraic geometry, a representation V is a *comodule* for $\mathcal{O}\left(G\right)$. I.e. we have a map $G \times V \to V$, and passing to functions gives us a map

$$V \to \mathcal{O}\left(G\right) \otimes V \ . \tag{3.91}$$

Furthermore,

$$\mathbf{Rep}\left(G\right) \simeq \mathcal{O}\left(G\right)\text{-}\mathbf{coMod} \simeq \mathcal{O}\left(G^\vee\right)\text{-}\mathbf{Mod} \simeq \mathbf{QC}\left(G^\vee\right) \ . \tag{3.92}$$

3.6.1 Fourier series examples

Example 20. For $G = \mathbb{Z}$, the category of representations is given by

$$\mathbf{Rep}\left(\mathbb{Z}\right) = k\left[z, z^{-1}\right]\text{-}\mathbf{Mod} \tag{3.93}$$

where z is the action of $1 \in \mathbb{Z}$. This action must be invertible, which is why z^{-1} is included. Then the duality tells us that:

$$\mathbf{Rep}\left(\mathbb{Z}\right) = k\left[z, z^{-1}\right]\text{-}\mathbf{Mod} = \mathcal{O}\left(\mathbb{G}_m\right)\text{-}\mathbf{Mod} = \mathbf{QC}\left(\mathbb{G}_m\right) \ . \tag{3.94}$$

Example 21. A vector space and an endomorphism (matrix) gives us

$$\mathbf{QC}\left(\mathbb{A}^1 = \operatorname{Spec} k\,[z]\right) \tag{3.95}$$

but if we have an automorphism (invertible matrix) then we get

$$\mathbf{Rep}\,(\mathbb{Z}) \leftrightarrow \mathbf{QC}\left(\mathbb{A}^1 \setminus \{0\}\right)\ . \tag{3.96}$$

Example 22. In algebraic geometry

$$\mathbf{Rep}\,(\mathbb{G}_m) = \mathbb{Z}\text{-graded vector space} = \mathbf{QC}\,(\mathbb{Z}) \tag{3.97}$$

where

$$V \simeq \bigoplus_{n\in\mathbb{Z}} V_n \tag{3.98}$$

and $z \in \mathbb{G}_m$ acts on V_n by z^n.

Example 23 (Topological example)**.** The following is an example of Fourier series from topology. Let M^3 be a compact oriented three-manifold.[3] Let

$$G = \operatorname{Pic} M^3 \tag{3.99}$$

consist of complex line bundles (or $\mathrm{U}\,(1)$-bundles) on M^3 up to isomorphism. This forms an abelian group under tensor product. We can think of this as:

$$G = \operatorname{Map}\left(M^3, B\,\mathrm{U}\,(1)\right) \tag{3.100}$$

where $B\,\mathrm{U}\,(1)$ denotes the classifying space of $\mathrm{U}\,(1)$. Up to homotopy we can think of $B\,\mathrm{U}\,(1)$ as:

$$B\,\mathrm{U}\,(1) \simeq \mathbb{CP}^\infty \simeq K\,(\mathbb{Z}, 2)\ . \tag{3.101}$$

For $\mathcal{L} \in G$, we can attach the first Chern class:

$$c_1\,(\mathcal{L}) \in H^2\,(M, \mathbb{Z})\ , \tag{3.102}$$

which is a complete invariant of the line bundle. So we can take

$$G = \Lambda = H^2\,(M, \mathbb{Z})\ . \tag{3.103}$$

The Cartier dual is

$$G^\vee \simeq \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m = \operatorname{Hom}\left(H_1\,(M, \mathbb{Z})\,, \mathbb{G}_m\right) = \operatorname{Hom}\left(\pi_1\,(M)\,, \mathbb{G}_m\right) \tag{3.104}$$

where the last line follows from the fact that $H_1 = \pi_1^{\mathrm{ab}}$, and Hom from the abelianization is the same as Hom from the whole thing. But this is just flat $\mathbb{C}^\times$-bundles on M, i.e.

$$G^\vee = \operatorname{Loc}_{\mathbb{C}^\times}(M)\ . \tag{3.105}$$

This shouldn't be that surprising since G looks something like $\mathbb{Z}^n$, and the Cartier dual $G^\vee$ looks like $(\mathbb{C}^\times)^n$.

We can also replace line bundles by torus bundles, i.e. we can pass from $\mathrm{U}\,(1)$ to $\mathrm{U}\,(1)^n \simeq T$. Then

$$\operatorname{Bun}_T(M) \leftrightarrow \operatorname{Loc}_{T^\vee} M\ . \tag{3.106}$$

where $\operatorname{Loc}_{T^\vee} M$ consists of isomorphism classes of flat $T^\vee$-bundles over M. The LHS still looks like a lattice Λ, and the RHS still looks like a torus $(\mathbb{C}^\times)^n$.

transition between lectures

Example 24 (Fourier series)**.** Take $\mathbb{Z}$ to be my abelian group G. Then we have some notions of a dual. One is

$$\mathrm{Hom}_{\mathbf{TopGrp}}(\mathbb{Z}, \mathrm{U}(1)) = \mathrm{U}(1) \tag{3.107}$$

another is

$$\mathrm{Hom}_{\mathbf{TopGrp}}(\mathbb{Z}, \mathbb{G}_m) = \mathbb{G}_m \ , \tag{3.108}$$

i.e. $\mathbb{C}^\times$ if we're over $\mathbb{C}$.

The difference is the kind of function theory we're consider. In the first case we have an equivalence

$$\ell^2 = L^2(\mathbb{Z}) \simeq L^2(\mathrm{U}(1)) \tag{3.109}$$

and in the second case we have

$$\mathbb{C}\mathbb{Z} \simeq \mathcal{O}\left(\mathbb{C}^\times\right) \ . \tag{3.110}$$

Notice that

$$\mathcal{O}\left(\mathbb{C}^\times\right) = \mathbb{C}\left[z, z^{-1}\right] \subset L^2(\mathrm{U}(1)) \ , \tag{3.111}$$

and

$$\mathbb{C}\mathbb{Z} \subset \ell^2 \ . \tag{3.112}$$

So the algebraic version kind of its inside of the analytic story.

Digression 1. For any group G, there are many different version of representation theory, which we can think of as coming from different versions of the group algebra. On the dual side, this corresponds to a different structure on $G^\vee$.

For example, if we take the group algebra in the sense of a von Neumann algebra, then we get $G^\vee$ as a measure space. If we start with the group algebra as a C^*-algebra, then we get $G^\vee$ as a (locally compqct) topological space. If we start with a discrete/algebraic group algebra, then we get $G^\vee$ as an algebraic variety.

3.7 Pontrjagin-Poincaré duality

Now we want to give a series of examples of Pontrjagin/Fourier duality which are basically the same example, but they will be more interesting because we will introduce some topology. A summary of all of these duality statements is in table 3.1.

Let M^n be a compact oriented manifold. The cohomology group of M will be the abelian groups we study the duality theory of. This duality says that for G an abelian group

$$H^i(M, G)^\vee \simeq H^{n-i}(M, G^\vee) \ . \tag{3.113}$$

To convince ourselves that this is on the correct footing, we can check that the following is a nondegenerate pairing:

$$H^i(M, G) \otimes H^{n-i}(M, G^\vee) \to H^n(M, G \otimes G^\vee) \to H^n(M, \mathrm{U}(1)) \to \mathrm{U}(1) \tag{3.114}$$

where the last map is integration.

The "dimension" of the following examples will eventually correspond to the dimension of the appropriate field theory, i.e. the geometric objects we consider are actually a dimension less than the listed dimension so as to get a vector space. The side with $\mathbb{Z}$ coefficients will be called the A-side, and the side with $\mathbb{C}^\times$ coefficients will be called the B-side.

[3]This will be the three-manifold on which we do electromagnetism. This Cartier duality will give us electric-magnetic duality. If we think of a number field as a three-manifold, then this duality fits into the framework of class field theory.

3.7.1 One dimension

In this case we just have that the A-side $\mathrm{Map}(\mathrm{pt}, \mathbb{Z}) = \mathbb{Z}$ gets exchanged with the B-side $\mathrm{Map}(\mathrm{pt}, \mathbb{C}^\times) = \mathbb{C}^\times$. We can think of this as saying that quantum mechanics on $\mathbb{Z}$ is QM of $\mathbb{C}^\times$ in the sense that some kind of spaces of functions on these are identified by Fourier theory.

3.7.2 Two dimensions

One duality in two dimensions is between:

$$H^0\left(S^1, \mathbb{Z}\right) \quad \text{and} \quad H^1\left(S^1, \mathbb{C}^\times\right) , \tag{3.115}$$

in the sense that functions on these are identified. Note that we can write the A-side as:

$$H^0\left(S^1, \mathbb{Z}\right) = \pi_0\left(\mathrm{Map}\left(S^1, \mathbb{Z}\right)\right) = \left[S^1, \mathbb{Z}\right] . \tag{3.116}$$

The B-side is

$$H^1\left(S^1, \mathbb{C}^\times\right) = \left|\mathrm{Loc}_{\mathbb{C}^\times} S^1\right| , \tag{3.117}$$

where $\left|\mathrm{Loc}_{\mathbb{C}^\times} S^1\right|$ denotes the underlying space of $\mathrm{Loc}_{\mathbb{C}^\times} S^1$, which is a stack.

The A-side just looks like $\mathbb{Z}$, and the B-side just looks like $\mathbb{C}^\times$. This is the sense in which we're doing the "same" example as before.

The A-side is the theory of maps to $\mathbb{Z}$, and the B-side is some kind of $\mathrm{U}(1)$ Gauge theory. By Gauge theory we just mean that we're dealing with local systems or principal bundles.

Digression 2 (Local systems). Let X be a topological space. Let G be a group (with the discrete topology). Then $\mathrm{Loc}_G X$ consists of principal G-bundles on X. These are sheaves which are locally isomorphic to the constant sheaf $\underline{G}$. We can also think of this as:

$$\mathrm{Loc}_G X = \{\pi_1(X) \to G\} \tag{3.118}$$

where, given a principal G-bundle, the associated representation of π_1 is the monodromy representation.

If we only care about isomorphism classes, then this is

$$H^1(X, G) = |\mathrm{Loc}_G X| . \tag{3.119}$$

3.7.3 Two dimensions

Another version of this duality in two-dimensions has

$$H^1\left(S^1, \mathbb{Z}\right) \tag{3.120}$$

on the A-side. On the B-side we have

$$H^0\left(S^1, \mathbb{C}^\times\right) , \tag{3.121}$$

which is just the maps to $\mathbb{C}^\times$ with the discrete topology.

On the A-side:

$$H^1\left(S^1, \mathbb{Z}\right) = \left[S^1, B\mathbb{Z}\right] , \tag{3.122}$$

and

$$B\mathbb{Z} = K(\mathbb{Z}, 1) = S^1 \ . \tag{3.123}$$

So the A-side is maps to S^1, and the B-side is also maps to $\mathbb{C}^\times$ (some version of S^1). These two spaces consist of what are sometimes called *scalar fields*.

This duality theory is known as T-duality or mirror symmetry. It is also called $R \leftrightarrow 1/R$ duality.

Again, as a group, the A-side is just $\mathbb{Z}$ and the B-side is just some version of S^1. One thing that is useful, in all of these examples, is to replace $\mathbb{Z}$ by a lattice Λ. The duality turns out to be between

$$H^1\left(S^1, \Lambda\right) \qquad \text{and} \qquad H^0\left(S^1, T^\vee_{\mathbb{C}}\right) \ , \tag{3.124}$$

where we can think of

$$H^1\left(S^1, \Lambda\right) = \left[S^1, T\right] \tag{3.125}$$

where T is the compact lattice

$$\Lambda \otimes_{\mathbb{Z}} S^1 \ . \tag{3.126}$$

The upshot is that we are exchanging maps into the torus with maps into the dual torus.

3.7.4 Three dimensions

Let Σ be a compact oriented two-manifold. Then we have a duality between

$$H^1(\Sigma, \mathbb{Z}) \qquad \text{and} \qquad H^1\left(\Sigma, \mathbb{C}^\times\right) \ . \tag{3.127}$$

The B side is $|\mathrm{Loc}_{\mathbb{C}^\times} \Sigma|$, so the B-side is three-dimensional gauge theory (i.e. principal bundles are involved). The A-side is

$$H^1(\Sigma, \mathbb{Z}) = \left[\Sigma, B\mathbb{Z} = S^1\right] \ , \tag{3.128}$$

so this is a scalar field. So maps into a circle get exchanged with principal bundles.

3.7.5 Three dimensions

Alternatively, in three dimensions, we get a duality between

$$H^2(\Sigma, \mathbb{Z}) \qquad \text{and} \qquad H^0\left(\Sigma, \mathbb{C}^\times\right) \ . \tag{3.129}$$

We can think of

$$H^2(\Sigma, \mathbb{Z}) = H^1(\Sigma, \mathrm{U}(1)) = [\Sigma, B\,\mathrm{U}(1) = K(\mathbb{Z}, 2)] \tag{3.130}$$

as consisting of $\mathrm{U}(1)$-principal bundles (i.e. line bundles) so this is some kind of gauge theory. On the other hand,

$$H^0\left(\Sigma, \mathbb{C}^\times\right) = \left[\Sigma, \mathbb{C}^\times\right] \tag{3.131}$$

consists of scalar fields. So principal bundles got exchanged with maps into a circle. Again, this can be upgraded to tori and lattices.

Digression 3 (Dictionary between vector bundles and principal bundles). Let $\mathcal{P} \to X$ be a $\mathrm{U}(1)$-bundle. Then I can form a line bundle by taking the product $\mathcal{P} \times_{\mathrm{U}(1)} \mathbb{C}$. In general if I have a principal G-bundle I have a representation of the group, so I get a vector bundle.

Conversely, if $\mathcal{L} \to X$ is a line bundle we can take Hom with the trivial bundle: $\mathrm{Hom}(\mathcal{L}, X \times \mathbb{C})$ and this is a principal $G\,\mathrm{U}(1)$-bundle.

Table 3.1: Summary of examples of Pontrjagin-Poincaré duality in various dimensions.

dimension	A-side	B-side
1	$\mathrm{Map}\,(\mathrm{pt}, \mathbb{Z}) = \mathbb{Z}$	$\mathrm{Map}\,(\mathrm{pt}, \mathbb{C}^\times) = \mathbb{C}^\times$
2	$H^0\left(S^1, \mathbb{Z}\right)$	$H^1\left(S^1, \mathbb{C}^\times\right) = \left\lvert \mathrm{Loc}_{\mathbb{C}^\times} S^1 \right\rvert$
2	$H^1\left(S^1, \mathbb{Z}\right) = \left[S^1, S^1\right]$	$H^0\left(S^1, \mathbb{C}^\times\right)$
3	$H^1\left(\Sigma, \mathbb{Z}\right) = \left[\Sigma, S^1\right]$	$H^1\left(\Sigma, \mathbb{C}^\times\right) = \lvert \mathrm{Loc}_{\mathbb{C}^\times} \Sigma \rvert$
3	$H^2\left(\Sigma, \mathbb{Z}\right) = \left[\Sigma, B\,\mathrm{U}\,(1)\right]$	$H^0\left(\Sigma, \mathbb{C}^\times\right)$

3.7.6 Four dimensions

The case of four dimensions is important because both sides will be gauge theories. Assume there is no torsion in our cohomology. The duality is between:

$$H^2\left(M^3, \mathbb{Z}\right) \qquad \text{and} \qquad H^1\left(M^3, \mathbb{C}^\times\right)\ . \tag{3.132}$$

The B-side can be written as

$$H^1\left(M^3, \mathbb{C}^\times\right) = \lvert \mathrm{Loc}_{\mathbb{C}^\times} M \rvert\ , \tag{3.133}$$

i.e. some kind of gauge theory since we're dealing with $\mathbb{C}^\times$-bundles. The A-side is

$$[M, K\,(\mathbb{Z}, 2)] \tag{3.134}$$

and as a homotopy type:

$$K\,(\mathbb{Z}, 2) = B\,\mathrm{U}\,(1) = \mathbb{CP}^\infty\ . \tag{3.135}$$

So the A-side is

$$\begin{aligned} [M, K\,(\mathbb{Z}, 2)] &= \pi_0\,\mathrm{Map}\,(M, \mathbb{CP}^\infty) && (3.136)\\ &= \pi_0\,(\text{line bundles on } M) && (3.137)\\ &= \text{line bundles}/\sim\ , && (3.138) \end{aligned}$$

i.e. line bundles up to isomorphism, so also some kind of gauge theory.

Then the equivalent vector spaces are functions on the two sides. The formal statement is that finitely supported functions (or locally constant if we're not thinking of π_0 of the space) on the A-side are equivalent to algebraic functions on the B-side.

A summary of all of these duality statements is in table 3.1.

Chapter 4

Electric-magnetic duality

The idea is that these duality statements we have seen are a shadow of electric-magnetic duality. We will follow Witten [DEF+99], Freed [Fre00], and Freed-Moore-Segal [FMS07].

4.1 Classical field theory

4.1.1 dimension 2

We will first study classical field theory in dimension 2. Let Σ be a Riemannian 2-manifold. Consider a periodic scalar field, i.e. a smooth map:

$$\varphi \in \mathrm{Map}_{C^\infty}\left(\Sigma, S^1\right) \ . \tag{4.1}$$

Now there is a natural classical field equation for this to satisfy: the harmonic map equation. It is convenient to introduce the 1-form $u = d\varphi$ on Σ, so we can write down the set of equations:

$$\begin{cases} du = 0 \\ d \star u = 0 \end{cases} . \tag{4.2}$$

The first one is automatically satisfied, since u is already exact. The second equation is equivalent to $\star d \star u = 0$, which is equivalent to φ being harmonic.

We write eq. (4.2) like this because they are symmetric under sending $u \rightsquigarrow \star u$. In other words, the theory of a scalar φ (where $d\varphi = u$) is the same as the theory associated to some $\varphi^\vee$ (where $d\varphi^\vee = \star u$). If we were keeping track of metrics, then we would be studying harmonic maps into a circle of some radius R, and the dual theory is studying harmonic maps into a circle of radius $1/R$. This is $R \leftrightarrow 1/R$ duality.

Remark 16. For differential forms, Poincaré duality is realized by the Hodge star $\star$, so we are not so far from where we were before.

4.1.2 Three-dimensions

In dimension three, we can do the same trick. Let $\varphi \in \mathrm{Map}_{C^\infty}\left(\Sigma, S^1\right)$ be a scalar field, and let $u = d\varphi$. Again we write the harmonic map equation in this nice form:

$$\begin{cases} du = 0 \\ d \star u = 0 \end{cases} . \tag{4.3}$$

Now when we pass from u to $\star u$, we're passing from a 1-form to a 2-form so it's not even the same kind of beast anymore. Write $F = \star u$. Locally, we can write

$$F = dA \tag{4.4}$$

for A some 1-form, and we get the equations:

$$\begin{cases} dF = 0 \\ d * F = 0 \end{cases} , \tag{4.5}$$

which are the three-dimensional Maxwell equations. So the theory with a scalar field φ is dual to a theory with a field F satisfying Maxwell's equations.

4.2 Four-dimensions/Maxwell 101

The *electric field* is a 1-form $E \in \Omega^1\left(\mathbb{R}^3\right)$, and the magnetic field is a 2-form $B \in \Omega^2\left(\mathbb{R}^3\right)$. Relativistically, it is better to think of a composite beast, called the field strength:

$$F = B - \, dt \wedge E \in \Omega^2\left(\mathbb{M}^4\right) \ . \tag{4.6}$$

where $\mathbb{M}^4$ is Minkowski space. I.e. we need to turn E into a 2-form, so we wedge it with time so we can subtract it from B.

Maxwell's equations in a vacuum are:

$$\begin{cases} dF = 0 \\ d \star F = 0 \end{cases} . \tag{4.7}$$

Observe that this is symmetric under exchanging F and $\star F$. Some algebra reveals that:

$$\star F = B^\vee - \, dt \wedge E^\vee \ , \tag{4.8}$$

where $B^\vee$ is the three-dimensional hodge star of E:

$$B^\vee = - \star_3 E \tag{4.9}$$

and similarly

$$E^\vee = \star_3 B \ . \tag{4.10}$$

The upshot is that Maxwell's equations are symmetric under exchanging E with B.

If we're not in a vacuum, i.e. we have some currents, then Maxwell's equations become:

$$\begin{cases} dF = j_B \\ d \star F = j_E \end{cases} \tag{4.11}$$

where j_B is the magnetic current and j_E is the electric current. These are three-forms $j_B, j_E \in \Omega^3\left(\mathbb{M}^4\right)$.

The meaning of these currents is that they're related to the total charge:

$$\int_{M^3} j_B = Q_B \qquad\qquad \int_{M^3} j_E = Q_E \tag{4.12}$$

where we're taking $\mathbb{M}^4 = M^3 \times \mathbb{R}$. If we use Stokes' theorem, we get Gauss' law. This says that for a closed surface $\Sigma \subset M^3$, the magnetic flux can be expressed as:

$$b_\sigma = \int_\Sigma F = \# Q_{B,\Sigma} \tag{4.13}$$

for some scalar #, where $Q_{B,\Sigma}$ is the total magnetic charge in Σ. Similarly, the electric flux is:

$$e_\Sigma = \int_\Sigma \star F = \# Q_{E,\Sigma} \ . \tag{4.14}$$

If you've ever learned about electromagnetism, you probably remember that there is a 0 on the RHS of (4.13). This is because, in reality, there don't seem to be any magnetic monopoles, i.e. $Q_B = 0$ and Gauss' law says that $\int_\Sigma F = 0$. Because of this property, we can introduce A, the electromagnetic potential by writing:

$$F = dA \ . \tag{4.15}$$

Note that this is breaking the symmetry between electricity and magnetism since $dF = 0$ is automatically satisfied. In particular this means that $j_B = 0$. Recall we should consider

$$\nabla = d + A \ , \tag{4.16}$$

regarded as a connection on a $\mathrm{U}(1)$-bundle on M.

The field strength F was something meaningful, but the potential A has some kind of ambiguity, since it is a sort of anti-derivative of F. This ambiguity is captured by the gauge transformations

$$\nabla \rightsquigarrow g^{-1} \nabla g \tag{4.17}$$

where $g \colon M \to \mathrm{U}(1)$. Similarly

$$A \mapsto A + g^{-1} dg \ , \tag{4.18}$$

i.e. it shifts A by some derivative, which **does not affect** F.

In fact, this is something which can be experimentally measured. This is called the *Bohm-Aharonov effect*. Even when $F = 0$ (flat connection, i.e. $E = 0$, $B = 0$), the monodromy of this connection is observable, i.e. a charged particle acquires a phase when it travels along loops.

Now we can think of this as a connection on any oriented Riemannian four-manifold. When we pass from F to $\nabla = d + A$, this implements Dirac charge quantization, which roughly says that the charges of elementary particles have to be integer valued (up to some renormalization). Let $M^4 = M^3 \times \mathbb{R}$. So now ∇ is a connection on some arbitrary line bundle $\mathcal{L}$ and F is a 2-form just like before. Even without charged particles, we still have fluxes:

$$b_\Sigma = \frac{1}{2\pi i} \int_\sigma F = \langle c_1(\mathcal{L}), [Sigma] \rangle \tag{4.19}$$

where this is the pairing between H^2 and H_2 since

$$c_1(\mathcal{L}) \in H^2(M, \mathbb{Z}) \qquad\qquad [\Sigma](\mathcal{L}) \in H_2(M, \mathbb{Z}) \tag{4.20}$$

for $\Sigma \subset M^3$ a closed surface. Similarly

$$e_\Sigma = \int_\Sigma \star F \ . \tag{4.21}$$

4.3 Quantum field theory

We will work in the Hamiltonian formalism. Roughly speaking, quantum field theory (QFT) on $M^{d-1} \times \mathbb{R}$ (thought of as space crossed with time) will be quantum mechanics on some space of fields on M^3. So we will study a Hilbert space attached to a fixed time slide, which is roughly

$$L^2\left(\text{fields on } M^{d-1}\right) \ . \tag{4.22}$$

In electromagnetism we started with the field strength F, and replaced it with this connection A. Following the above heuristic, this means that in Maxwell theory, the Hilbert space is roughly:

$$\mathcal{H} = \text{“}L^{2\prime\prime}\left(\mathcal{C}\left(M^3\right)\right) \tag{4.23}$$

where $\mathcal{C}$ takes the isomorphism classes of line bundles $\mathcal{L}$ and $\mathrm{U}(1)$ connection ∇. I.e. this consists of isomorphism classes of connections $|\mathrm{conn}\,(M)|$. Connections always have automorphisms (circle rotation at least), but we'll just look at the set of isomorphism classes.

It is useful to note hat $\mathcal{C}(M)$ is an (∞-dimensional) abelian Lie group with group operation given by tensor product. Abelian Lie groups all look like:

$$\Lambda \times T \times V \tag{4.24}$$

for Λ some finite lattice, T some finite-dimensional torus, and V some infinite-dimensional vector space. We will kind of ignore V. Basically the idea is that we have both the lattice and torus present on both sides of the duality.

Now we follow [FMS07]. We can take the first Chern class (isomorphism class of underlying line bundle)

$$\mathcal{C}(M) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \ . \tag{4.25}$$

This is where the lattice comes from. This is the *total* magnetic flux in the sense that when we pair with a surface, we get the flux through the surface. We can also take the curvature

$$\mathcal{C}(M) \xrightarrow{F} \Omega^2_{\mathbb{Z}}(M) \ . \tag{4.26}$$

Integral differential forms “talk” to integral H^2, and we have a short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \arrow[r] & \mathcal{C}(M) \arrow[r, "c_1"] & H^2(M,\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{C}_{\text{flat}}(M) \arrow[r] \arrow[u] & \mathcal{C}(M) \arrow[r, "F"] & \Omega^2_{\mathbb{Z}}(M) \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{4.27}$$

where

$$\mathcal{C}_{\text{flat}}(M) = \left|\mathrm{Loc}_{\mathrm{U}(1)}\, M\right| = H^1(M, \mathbb{R}/\mathbb{Z}) = \mathrm{U}(1) \otimes_{\mathbb{Z}} H^1(M, \mathbb{Z}) \tag{4.28}$$

is a torus of dimension $b_1(M)$. There is a map

$$\mathcal{C}(M) \to \Lambda = H^2(M, \mathbb{Z}) \tag{4.29}$$

to the magnetic flux, and inside of here there is a torus:

$$T = \left|\mathrm{Loc}_{\mathrm{U}(1)}\, M\right| \to \mathcal{C}(M) \ . \tag{4.30}$$

Now we will see that electric-magnetic duality corresponds exactly to Pontrjagin duality on $\mathcal{C}(M)$. We will be able to write the same Hilbert space in two ways, and there is a sense in which magnetic measurements on one side are dual to electric measurements on the other side.

4.4 Quantum field theory

4.4.1 Lagrangian formalism

transition between lectures

Lecture 7; February 9, 2021

We will be doing euclidean quantum field theory in dimension d. First we will give a schematic overview of the Lagrangian formalism. The idea is that, to a d-manifold M, we will attach a space of *fields* $\mathcal{F}(M)$. These are some local quantities on our space, for example functions or sections of a bundle.

To any field $\varphi \in \mathcal{F}(M)$, we can attach the *action* $S(\varphi) \in \mathbb{C}$. This is a way of prescribing the classical equations of motion. Instead of finding solutions to some equations of motion one studies critical points of this function S.

In quantum field theory, we do some kind of "probability theory" on $\mathcal{F}(M)$ with "measure" given by

$$e^{-S(\varphi)/\hbar} D\varphi \ . \tag{4.31}$$

We can think of this thing as being a "vanilla" measure on the space of states that is then weighted by the action S. The idea is that, as $\hbar \to 0$, this concentrates on solutions to the equations of motion. We won't try to make mathematical sense of this, but this is the schematic.

For M^d closed, we can attach the partition function, which is the volume (total measure) of this space of fields:

$$Z(M) = \int_{\mathcal{F}(M)} e^{-S(\varphi)/\hbar} D\varphi \ . \tag{4.32}$$

This isn't a very interesting quantity, and we often normalize so that this is 1. The more interesting thing to calculate are the expectation values of operators.

Example 25. One example of an operator, specifically a local operator $\mathcal{O}_x$ at $x \in M$, is the functional on $\mathcal{F}(M)$ given by evaluation (making a measurement) at $x \in M$.

Now we can take the expectation value of this measurement:

$$\langle \mathcal{O}_x \rangle = \int_{\mathcal{F}(M)} \frac{\mathcal{O}_x(\varphi)\, e^{-S(\varphi)/\hbar} D\varphi}{Z(M)} \ , \tag{4.33}$$

where we're dividing by $Z(M)$ to normalize the measure. Then we might calculate correlation functions, where we're taking several different measurements at different points.

There are also "disorder" operators, where inserting the operator means we look at fields with a prescribed singularity at x. I.e. we're looking at all fields $\mathcal{F}(M \setminus \{x\})$, and the operator is something like the delta function on some prescribed singularity space, i.e. it's picking out fields with some prescribed singularity at x.

4.4.2 Time evolution

Let M be a Riemannian manifold with boundary:

$$\partial M = \partial M_{\text{in}} \sqcup \partial M_{\text{out}} \ . \tag{4.34}$$

This is a Riemannian *bordism* from ∂M_{in} to ∂M_{out}. See fig. 4.1 for a picture.

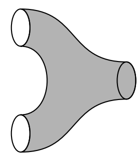


Figure 4.1: A bordism from the disjoint union of two copies of S^1 to a single copy of S^1.

Example 26. Let N be an $(n-1)$-manifold. Then $M = N \times I$ is a d-dimensional bordism from N to itself.

This gives us a correspondence of fields:

$$\begin{tikzcd} & \mathcal{F}(M) \arrow[dl] \arrow[dr] & \\ \mathcal{F}(\partial M_{\text{in}}) & & \mathcal{F}(\partial M_{\text{out}}) \end{tikzcd} . \tag{4.35}$$

The functional $e^{-S(\varphi)/\hbar} D\varphi$ lives over $\mathcal{F}(M)$, and so we get an integral transform given by pulling, multiplying by this functional, and integrate (push forward).

This integral transform gives us an operator $Z(M)$ between

$$\mathcal{H}_{\text{in}} = \text{functionals on } \mathcal{F}(\partial_{\text{in}}) \tag{4.36}$$

and

$$\mathcal{H}_{\text{out}} = \text{functionals on } \mathcal{F}(\partial_{\text{out}}) \ . \tag{4.37}$$

Then

$$\begin{aligned} (Z(M)(f))(\varphi_{\text{out}}) &= \int_{\varphi|_{\partial_{\text{out}}} = \varphi_{\text{out}}} f(\varphi|_{\partial_{\text{in}}}) \, e^{-S(\varphi)/\hbar} D\varphi \\ &= \pi_{\text{out}*} \left(\pi_{\text{in}}^* (f) \, e^{-S(\varphi)/\hbar} D\varphi \right) \ . \end{aligned}$$

This operator is the time evolution operator.

Remark 17 (Hamiltonian versus Lagrangian)*.* In the Hamiltonian formulation we start with a Hilbert space $\mathcal{H}$, which is associated to a time slice. Then we're also supposed to give a Hamiltonian operator H, and then time evolution is given by the operator $e^{iTH/\hbar}$. Then we have an algebra of observables and make various measurements.

On the other hand in the Lagrangian formalism we start with the space of fields and the action. The Lagrangian formulation is very flexible in the sense that it allows us to define these push-pull operators, which is how we recover a Hilbert space and time evolution operator (Hamiltonian formalism) from the Lagrangian formalism.

Example 27. One-dimensional QFT is quantum mechanics. Let X be a Riemannian target. Then the space of fields might be $\mathcal{F}(\mathbb{R}) = \text{Maps}(\mathbb{R}, X)$. The critical points of S will be geodesics in X. The Hilbert space is

$$\mathcal{H} = L^2(\text{Maps}(\text{pt}, X)) = L^2(X) \ , \tag{4.38}$$

and the Hamiltonian is the Laplace operator Δ.

To formalize all of this would require a great deal of work. We also don't want to perform any perturbative techniques since we don't necessarily want $\hbar$ to be small. We will use this as a schematic guide, and formally pass to the topological setting.

4.5 Quantum Maxwell theory

We will consider a four-dimensional quantum field theory. Our space of fields on a 4-manifold M^4 consists of $\mathrm{U}(1)$ bundles with connection $d+A$ (A is the electromagnetic potential) up to gauge equivalence, i.e. we mod out by the action of the gauge transformations.

The classical equations of motion are Maxwell's equations:

$$\begin{cases} dF = 0 \\ d \star F = 0 \ , \end{cases} \tag{4.39}$$

where F is the curvature. Then we get an action

$$S = \frac{g}{2\pi i} \int_M F \wedge \star F + \theta \int_M F \wedge F \ . \tag{4.40}$$

The integral in the second term is just calculating $c_1^2 = p_1$ of the line bundle. This term is called the topological term. constants?

The idea is that the Hilbert space is attached to a specific time slice. So assume we can write $M^4 = M^3 \times \mathbb{R}$, and then the Hilbert space is

$$\mathcal{H} = L^2\left(\mathcal{C}\left(M^3\right)\right) \tag{4.41}$$

where $\mathcal{C}\left(M^3\right)$ denotes the collection of line bundles with connections modulo gauge transformations on M^3. $\mathcal{C}\left(M^3\right)$ is an abelian Lie group, and has a map to the lattice of possible line bundles up to isomorphism:

$$\mathcal{C}\left(M^3\right) \xrightarrow{c_1} \Lambda = H^2(M, \mathbb{Z}) \tag{4.42}$$

and also a sub given by the torus of flat connections:

$$T = \text{flat connections} \simeq H^1(M, \mathbb{R}/\mathbb{Z} = \mathrm{U}(1)) \simeq BH^1(M, \mathbb{Z}) \to \mathcal{C}\left(M^3\right) \tag{4.43}$$

where $\mathrm{U}(1)$ is taken to have the discrete topology. Then the final factor of $\mathcal{H}$ is given by an infinite-dimensional vector space V. I.e. there is a (noncanonical) splitting $\mathcal{C}(M) \simeq \Lambda \times T \times V$.

4.5.1 Operators

Now we want to identify some operators on $\mathcal{H}$ (i.e. observables) that appear naturally from the setup. The first thing to say is that $\mathcal{H}$ has an obvious grading by $\Lambda = H^2(M, \mathbb{Z})$. This already picks out some operators, e.g. by telling you which component you're on. This is the magnetic flux, so it is said that this is a grading by magnetic fluxes.

Dirac/'t Hooft operators

Λ also acts on $\mathcal{H}$ to yield 't Hooft operators.[1] If we've already chosen a splitting $\mathcal{C}(M) \simeq \Lambda \times T \times V$, then the action is just by translation. I.e. these operators shift the magnetic

[1] Technically 't Hooft operators are the nonabelian generalization of these. These operators really have to do with Dirac monopoles, but this would be confusing since the term *Dirac operator* already has another meaning.

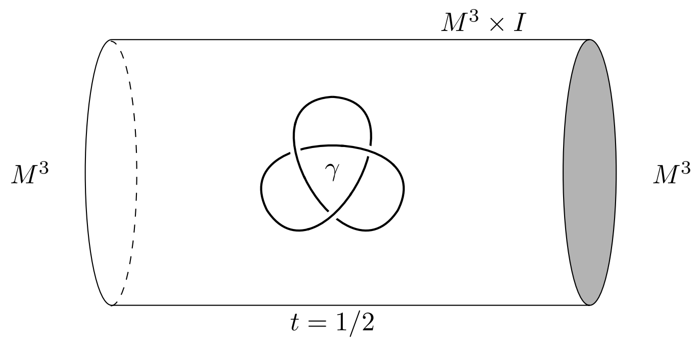


Figure 4.2: The loop γ lives in some time slice in $M^3 \times I$, say $t = 1/2$. We can excise this, and fields on the resulting 4-manifold will possibly have singularities along γ. Asking for the integral around the boundary of a neighborhood of γ to be 1 gives us a space of fields with controlled singularity inside the neighborhood of γ. Physically this introduces a magnetic monopole along γ.

flux. Recall the flux roughly records the number of enclosed monopoles. So physically these operators create magnetic monopoles.

More precisely, let γ be a simple closed curve in M. Note that this defines a class

$$[\gamma] \in H_1(M, \mathbb{Z}) \simeq H^2(M, \mathbb{Z}) \simeq \Lambda \ . \tag{4.44}$$

The associated operator introduces a monopole along γ as follows. The idea is that γ lives at a particular time slice in $M^3 \times I$, and we're studying electromagnetism on $M \times I$ with a monopole along γ as in fig. 4.2.

In other words, we're considering fields $\mathcal{C}(M \times I \setminus \gamma)$, i.e. the connections are possibly singular along γ. So we have excised the knot, and this has introduced a new boundary component of our 4-manifold: the link of the knot (boundary of tubular neighborhood of the knot) which looks like $S^2 \times S^1$. Then we can look at connections which have a specific integral over this. In particular, we can ask for

$$\frac{1}{2\pi i}\int_{S^2} F = 1 \ . \tag{4.45}$$

Write the resulting space of fields as:

$$\mathcal{C}(M \times I \setminus \gamma)_{c_1=1} \ . \tag{4.46}$$

Note that, physically, this is saying that this surface $S^2 \times S^1 \subset M^3 \times I$ contains magnetic charge 1. Mathematically we're seeing that we can't extend the connection over this curve. Physically we're seeing a monopole along this curve.

The actual operator is defined as follows. The 4-manifold $M \times I \setminus \gamma$ is a bordism from M^3 to itself, and therefore defines a correspondence and integral transform (Feynman path integral) as usual:

$$\begin{tikzcd}
& \mathcal{C}\left(M \times I \setminus \gamma\right)_{c_1=1} \arrow[dl, "\pi_{\mathrm{in}}"'] \arrow[dr, "\pi_{\mathrm{out}}"] & \\
\mathcal{C}\left(M^3_{\mathrm{in}}\right) & & \mathcal{C}\left(M^3_{\mathrm{out}}\right)
\end{tikzcd} \tag{4.47}$$

$$\mathcal{H} \xrightarrow{\pi_{\mathrm{out}*}\left(e^{-S(\varphi)/\hbar} D\varphi \pi^*_{\mathrm{in}}(-)\right)} \mathcal{H}\ .$$

Physically this operator takes the space of states and evolves it through time, during which a monopole is introduced along γ and then removed. Mathematically it is shifting the Chern class by $[\gamma]$, i.e. it is just tensoring with a flat line bundle with prescribed Chern class

$$c_1 = [\gamma] \in H^2\left(M, \mathbb{Z}\right)\ . \tag{4.48}$$

Wilson loop operators

On the other hand, there are much simpler observables called the Wilson loop operators. Again let $\gamma \subset M$ be a simple closed curve. This determines a function W_γ on $\mathcal{C}(M)$ given by:

$$W_\gamma\colon\ (\mathcal{L}, \nabla) \mapsto \text{holonomy along } \gamma \in \mathrm{U}\left(1\right) \subset \mathbb{C}\ . \tag{4.49}$$

We can draw the same picture as fig. 4.2, but we're doing something very different with it. We are still thinking of γ as living on some time slice inside $M \times I$. Now, as time evolves, we make the measurement of the holonomy along γ. So this is a measurement just like the example of a local operator in example 25 only now the measurement is along a loop, so this is called a line/loop operator. This still gives us a correspondence and an integral transform, only now the kernel of the transform is given by the function W_γ:

$$\begin{tikzcd}
& \mathcal{C}\left(M \times I\right) \arrow[dl, "\pi_{\mathrm{in}}"'] \arrow[dr, "\pi_{\mathrm{out}}"] & \\
\mathcal{C}\left(M^3_{\mathrm{in}}\right) & & \mathcal{C}\left(M^3_{\mathrm{out}}\right)
\end{tikzcd} \tag{4.50}$$

$$\mathcal{H} \xrightarrow{\pi_{\mathrm{out}*}\left(W_\gamma \pi^*_{\mathrm{in}}(-)\right)} \mathcal{H}\ .$$

The point is that we can multiply functions on this space of connections by this function W_γ. So these Wilson operators are already diagonalized given the way we've presented the Hilbert space.

The Wilson loops W_γ are eigenfunctions for the action of the space of flat connections:

$$T = \mathcal{C}_\flat\left(M\right) \simeq H^1\left(M, \mathrm{U}\left(1\right)\right)\ . \tag{4.51}$$

This acts on $\mathcal{C}(M)$ (by tensoring), so therefore it also acts on $L^2\left(\mathcal{C}\left(M\right)\right)$. The eigenvalues are given by multiplying by the monodromy along γ: given a flat connection (element of $H^1\left(M, \mathrm{U}\left(1\right)\right)$) we canonically get an element of $\mathrm{U}\left(1\right)$ by taking the monodromy along γ, and this is exactly doing Fourier series. These W_γ are the characters of this torus T.

4.5.2 Electric-magnetic duality

Notice that when we were studying 't Hooft operators we were paying attention to the action of the lattice, and we had a grading by magnetic fluxes. Now we're focusing on the torus action, and we can diagonalize. This gives a decomposition by characters of T, which comprise the lattice $H^2(M, \mathbb{Z})$:

$$\mathcal{H} = \bigoplus_{e \in H^2(M,\mathbb{Z})} \mathcal{H}_e \ , \tag{4.52}$$

and this can be interpreted as a grading by electric fluxes.

Passing from the magnetic grading to the electric grading is exactly performing a Fourier transform with respect to the torus part of our space of fields. In general, electric-magnetic duality can be thought of as doing Fourier series on our Lie group of fields. It specifically identifies:

$$L^2\left(\mathcal{C}_{\mathrm{U}(1)}(M)\right) \simeq L^2\left(\mathcal{C}_{\mathrm{U}(1)^\vee}(M)\right) \tag{4.53}$$

where the left side is the F side, and the right side is the $\star F$ side. The group always splits as a lattice, a torus, and a vector space (which we are ignoring). The F and $\star F$ sides are respectively associated with the splittings:

$$\Lambda_B \times T_E \times \mathbf{Vect} \qquad\qquad \Lambda_E \times T_B \times \mathbf{Vect} \ . \tag{4.54}$$

Λ_B is the lattice of magnetic fluxes, T_B is its dual torus, Λ_E is the lattice of electric fluxes, and T_E is its dual torus. On the F (left) side, Wilson loop operators are diagonalized, and on the $\star F$ (right) side, the 't Hooft operators are diagonalized. Similarly, the magnetic b grading goes to the electric e grading.

4.6 Abelian duality in two-dimensions

Recall this is called T-duality, or mirror symmetry. The space of fields is $\mathrm{Map}\left(M, S^1\right)$, or more generally into a torus T. Our Hilbert space $\mathcal{H}$ on S^1 was then

$$L^2\left(\mathrm{Map}\left(S^1, S^1_R\right)\right) \ . \tag{4.55}$$

Note that again $\mathrm{Map}\left(S^1, S^1\right)$ is an abelian Lie group under the operation of pointwise multiplication, i.e. we're using the group structure of the target not the source. There is also a map to a lattice: this is graded by $H^1\left(S^1, \mathbb{Z}\right)$ which is given by taking the winding number.

Dually we can study:

$$L^2\left(\mathrm{Maps}\left(S^1, \left(S^1_R\right)^\vee = S^1_{1/R}\right)\right) \ . \tag{4.56}$$

This is graded by $H^0\left(S^1, \mathrm{U}(1)\right)$, which is dual to the grading above. We have an operator given by shifting the winding number (analogues of the 't Hooft operators) and operators given by evaluation at a point (analogues of the Wilson operators) and the duality exchanges them.

4.7 Topological quantum mechanics/quantum field theory

The idea is that quantum mechanics is hard because it involves analysis. In ordinary quantum mechanics, we started with a point and assigned a Hilbert space $\mathcal{H}$. Then to an interval of length T, we assigned unitary time evolution operator

$$e^{iTH/\hbar} \ . \tag{4.57}$$

In *topological* quantum mechanics (TQM) we want to kill time, i.e. we want $H = 0$. The naive way to interpret this is to restrict to ground states. But a closed manifold does not admit any nonconstant bounded harmonic functions. So we killed the entire theory.

Witten [Wit82] introduced the following technique to kill time in a derived sense via super-symmetry (SUSY). Let X be a Riemannian manifold. Instead of $L^2(X)$, we expand our Hilbert space to be L^2 differential forms, i.e. we add new fields, and now we get a bigger symmetry group given by a *super Lie group*. So now we consider: what acts on differential forms? The first thing is the Laplace operator $H = \Delta$. We also have the de Rham differential $Q = d$ and its adjoint and $Q^* = d^*$. We also have a $\mathrm{U}(1)$ action which gives a $\mathbb{Z}$-grading. This package of operators is called the $\mathcal{N} = 1$ SUSY algebra. So we have operators where H has degree 0, Q has degree 1, and Q^* has degree -1. Most of these operators commute:

$$[Q^*, Q^*] = [Q, Q] = 0 \tag{4.58}$$
$$[Q, H] = [Q^*, H] = 0 \tag{4.59}$$

except

$$[Q, Q^*] = H \ . \tag{4.60}$$

$[Q, Q] = 0$ can be interpreted as $d^2 = 0$. The other relation is that

$$\Delta = dd^* + d^*d \ , \tag{4.61}$$

which says that Q^* gives a homotopy from H to 0. In particular, this implies that, on Q-cohomology, H acts by 0.

So we added new fields, and we got a larger symmetry algebra. One of the operators in this algebra is an odd operator squaring to 0, and we call it the differential. On the associated cohomology, H is zero. So we've killed time in a derived sense. So topological quantum mechanics assigns, to X, the complex

$$(\Omega^\bullet(X), d) \ , \tag{4.62}$$

or $H^*(X)$. In the first case H is homotopic to 0, and in the latter $H = 0$. This is the way de Rham cohomology gets recovered from quantum mechanics.

We will never be dealing with honest quantum field theories. Instead, we will be doing this topological twist. Note that in this setting, the theory only depends topologically on X. The upshot is that we're killing the dependence on Riemannian metrics, which will make life much easier.

Lecture 8; February 11, 2021

4.8 Topological quantum mechanics

The way to set this up formally, is that we have the $d = 1$, $\mathcal{N} = 1$ SUSY algebra acting on $\mathcal{H}$. The $d = 1$ means we're doing quantum mechanics and $\mathcal{N} = 1$ just means the smallest amount of supersymmetry.

transition between lectures

There are two natural realizations of TQM. The first is the A-type TQM. Take X to be a Riemannian manifold, and then the Hilbert space consists of differential forms:

$$\mathcal{H} = \Omega^{\bullet}(X) \ , \tag{4.63}$$

the Hamiltonian $H = \Delta$ is the Laplace operator, $Q = d$ is the de Rham differential, and $Q^* = d^*$ is its adjoint with respect to the metric. The grading is the usual one on forms, and the cohomology is

$$H^{\bullet}(\mathcal{H}) = H^{\bullet}_{\mathrm{dR}}(X) \ . \tag{4.64}$$

There is another realization, called B-type TQM. Now take X to be a complex manifold. The Hilbert space is

$$\mathcal{H} = \Omega^{0,\bullet}(X) \tag{4.65}$$

i.e. forms with $\overline{\partial}$ in them. The SUSY operators are given by $Q = \overline{\partial}$, $Q = \overline{\partial}^*$, and $H = \Delta_{\overline{\partial}}$. The cohomology is Dolbeault cohomology:

$$H^0(\mathcal{H}) = H^{0,*}(X) = R\Gamma(\mathcal{O}_X) \ . \tag{4.66}$$

These were both realizations of the 1-dimensional $\mathcal{N} = 1$ SUSY algebra. There is also $\mathcal{N} = 2$ TQM. One such example comes from studying a Kähler manifold X. The Hilbert space is still differential forms:

$$\Omega^{\bullet,\bullet}(X) \ . \tag{4.67}$$

This is bigraded, rather than having a single grading like before, and this has a bunch of operators, e.g. ∂, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$. The Kähler identities tell us that the associated Laplacians agree: $\Delta_{\overline{\partial}} = \Delta_d$.

For a mathematician, there is the following deep theorem. We have an action of $\mathrm{SU}(2)$ on $\Omega^{\bullet,\bullet}(X)$, which can be thought of as coming from its complexification $\mathrm{SU}(2) \subset \mathrm{SL}_2\mathbb{C}$. These are the Lefschetz operators. The diagonal part is giving a cohomological grading, and then there is a raising operator (intersecting with Kähler form) and a lowering operator. Mathematically this is a very deep statement about global cohomology of a Kähler manifold (the hard Lefschetz theorem). Physically this is a calculation of which SUSY algebra acts on our Hilbert space, and then this $\mathrm{SU}(2)$ actions is the R-symmetry. The point is that we found a big SUSY algebra.

The $\mathcal{N} = 1$ SUSY algebra encoded Hodge theory of Riemannian manifolds. This $\mathcal{N} = 2$ SUSY algebra encodes complex Hodge theory. There is also $\mathcal{N} = 4$ TQM. In this case we're studying differential forms on a compact hyperkähler manifold X: $\Omega^{\bullet}(X)$. The analogue of Lefschetz SL_2 action for hyperkähler is an action of $\mathrm{Spin}\,5$. The idea is that if we assemble all of the Lefschetz SL_2s together, you get a $\mathrm{Spin}\,5$. This is part of a super-Lie group that we won't write down. From the point of view of the physics, this is just a natural calculation about what operators are sitting around when we're studying the theory of particles in this manifold.

Remark 18. There is also $\mathcal{N} = 8$ SUSY. These different SUSY algebras are really distinguished by the number of odd operators (a.k.a. surpercharges Q) are present. E.g. $\mathcal{N} = 1$ has two, $\mathcal{N} = 2$ has 4, $\mathcal{N} = 4$ has 8, and $\mathcal{N} = 8$ had 16.

The general philosophy of TQM and eventually TQFT is to add fields so that we have an action of a big super-Lie group (SUSY algebra). We're looking for two things:

1. that the Hamiltonian is exact: $H = [Q, -]$ (i.e. we're killing time[2]), and
2. $T = [Q, -]$, i.e. the metric dependence is exact (i.e. we're killing geometry).

The stress energy tensor T is an object in QFT which measures the dependence on the metric. Mathematically T is a derived version of invariance under isometry. This is explained nicely in Costello-Gwilliam [CG17].

4.8.1 Topological Maxwell theory

We want to take the quantum theory of light (ordinary Maxwell theory), add some fields, and find the SUSY algebra. Then we will pick some Q and pass to cohomology. This will be an $\mathcal{N} = 4$ GL-twisted theory. In particular, this $\mathcal{N} = 4$ means that there is a lot of super-symmetry: there will be sixteen supercharges.

Before, our space of fields consisted of a line bundle and a choice of connection $\nabla = d+A$. Now we will add:

- a 1-form σ on the manifold (the Higgs field),
- a complex scalar u, and
- four fermions (odd fields) (we will not pay much attention to these).

The Hilbert space used to be $\mathcal{H} = L^2(\mathcal{C}(M))$. Now, in the A-twist, the 3-manifold M^3 gets attached to the cohomology of this space: $H^\bullet(\mathcal{C}(M))$. We have a much bigger space of fields now, but it doesn't actually make a difference at the level of the cohomology, since introducing these new fields didn't change the topology of the space. But as it turns out, we don't want ordinary cohomology, we want cohomology which is equivariant with respect to the automorphisms of the connections, i.e. we want:

$$\begin{aligned} \mathcal{H} &= H^\bullet(\text{connections/gauge equivalence}) && (4.68) \\ &= H^\bullet_{\mathrm{U}(1)}(\mathcal{C}(M)) \ . && (4.69) \end{aligned}$$

This is doing topological quantum mechanics in the *de Rham* sense on the space of connections.

There is another version called the B-twist. Now we modify the fields by thinking of:

$$\nabla + i\sigma = d + (A + i\sigma) \tag{4.70}$$

as a connection on a $\mathbb{C}^\times$-bundle rather than a $\mathrm{U}(1)$-bundle. The vector space attached to M^3 is Dolbeault TQM on $\mathrm{Loc}_{\mathbb{C}^\times}(M)$. This just means the vector space is (some derived version of) holomorphic functions on $\mathrm{Loc}_{\mathbb{C}^\times}(M)$.

Now we reformulate the A-side to connect with what we've seen. Write $\mathrm{Pic}(M)$ for the underlying topological space of the space of connections $\mathcal{C}(M)$, i.e. it doesn't detect the Riemannian geometry of M. $H^0(\mathrm{Pic}(M))$ consists of locally constant functions on the space of connections $\mathcal{C}(M)$, i.e.

$$\begin{aligned} H^0(\mathrm{Pic}(M)) &= \mathbb{C}[\pi_0(\mathcal{C}(M))] && (4.71) \\ &= \mathbb{C}\left[H^2(M, \mathbb{Z})\right] \ , && (4.72) \end{aligned}$$

[2] I.e. time evolution is made trivial.

which is the same vector space we were previously attaching to M in the discussion summarized in table 3.1. The full cohomology of this space is:

$$H^\bullet(\mathrm{Pic}(M)) = H^\bullet(\Lambda \times T \times V \times B\,\mathrm{U}(1)) \ . \tag{4.73}$$

Before we only saw $\Lambda = H^2(M, \mathbb{Z})$, but now we will see

- the cohomology of $T = H^1(M, \mathrm{U}(1))$ (an exterior algebra),
- the vector space doesn't contribute to cohomology, but we also see
- the cohomology of $B\,\mathrm{U}(1)$, which will look like $\mathbb{C}[u]$.

So this is the shape of our Hilbert space after making a topological twist, and on the B-side there will also be factors corresponding to these extra exterior and symmetric pieces.

On the B-side we're looking at the space $\mathrm{Loc}_{\mathbb{C}^\times} M$. Up to some derived factors, this looks like a complex torus $\mathrm{Loc}_{\mathbb{C}^\times} M \simeq T^\vee_{\mathbb{C}}$ of maps $\pi_1(M) \to \mathbb{C}^\times$ which factor as:

$$\begin{tikzcd} \pi_1(M) \arrow[rr] \arrow[dr] & & \mathbb{C}^\times \\ & H_1(M) \arrow[ur] & \end{tikzcd} \ . \tag{4.74}$$

$H_1(M)$ looks like Λ (by Poincaré duality), so this part is the dual torus to the lattice Λ. Recall the Fourier series identifies:

$$\mathbb{C}\left[H^2(M, \mathbb{Z})\right] \qquad \text{and} \qquad \mathbb{C}\left[T^\vee_{\mathbb{C}}\right] \ , \tag{4.75}$$

and now we have some extra (derived) factors, coming from $H^2(M, \mathbb{C})$ and $H^3(M, \mathbb{C})$. The $H^2(M, \mathbb{C})$ factor corresponds to the exterior algebra factor on the A-side, and the $H^3(M, \mathbb{C})$ factor corresponds to the symmetric algebra factor on the A-side.

So again, we had this notion of a Fourier transform, which exchanges:

$$\Lambda = H^2(M, \mathbb{Z}) \qquad \text{and} \qquad T^\vee_{\mathbb{C}} = |\mathrm{Loc}_{\mathbb{C}^\times}(M)| \ , \tag{4.76}$$

and this turns out to just be the degree 0 part of the duality between these A and B-twists, where we have this extra exterior algebra, and extra symmetric algebra. **Mottos:**

- The A-twisted super Maxwell theory with gauge group T studies the topology of $\mathrm{Pic}(M)$, and
- the B-twisted version of super Maxwell theory with gauge group $T^\vee$ studies the algebraic geometry of $\mathrm{Loc}_{\mathbb{C}^\times}(M)$.
- E-M duality switches these two.

See table 4.1 for a summary.

Remark 19. We looked at two different twists, i.e. two different realizations of the SUSY algebra, i.e. two different charges. We could also take any linear combination of these two and get a new topological theory. In other words, this is a $\mathbb{P}^1$ family of possible topological theories which all arise from quantum Maxwell theory. The A and B-twists are then two extreme points of $\mathbb{P}^1$. On one end only magnetic phenomena are left, and on the other only electric phenomena are left.

Table 4.1: Summary of the topological A and B twists of super-Maxwell theory.

A-side	B-side
topology of Pic	AG of Loc
$\Lambda = H^2(M,\mathbb{Z})$	$T^\vee_{\mathbb{C}} = \lvert \mathrm{Loc}_{\mathbb{C}^\times}(M)\rvert$
't Hooft operators (shifting lattice (magnetic flux) by $\gamma^\vee \in H^2(M,\mathbb{Z})$)	Wilson operators (multiply by monodromy along γ)
Create magnetic monopole	Create electric particle.
Magnetic side	Electric side

4.8.2 Defects

There are two other "physics operations" done in electromagnetism called defects, which we will describe at the topological level. These amount to changing the fields we're considering, e.g. introducing singularities.

Time-like line defects

The first type of defect we will consider is a "timelike" 't Hooft loop/line. Consider a point $x \in M^3$. Then we can consider electromagnetism in the presence of a monopole at x. Mathematically this means we look at the space of connections:

$$\mathrm{Pic}(M \setminus x) = |\mathcal{C}(M \setminus x)| = \coprod_n \mathrm{Pic}(M, nx) \tag{4.77}$$

where

$$\mathrm{Pic}(M, nx) = \{A \in \mathrm{Pic}(M \setminus x) \,|\, c_1 = n \text{ on sphere at } x\} \ . \tag{4.78}$$

E.g. $\mathcal{C}(M,x)$ is the charge 1 component. So now we get a new Hilbert space by linearizing $\mathcal{C}(M \setminus x)$ (via taking L^2 or H^*).

Dually (on the B-side) we have a "timelike" Wilson loop/line. For $x \in M$ we take our fields to be:

$$\mathrm{Loc}_{\mathbb{C}^\times}(M,x) \tag{4.79}$$

which are flat $\mathbb{C}^\times$ connections equipped with a trivialization of the fiber at x. Physically this is interpreted as adding a heavy[3] charged particle, but it breaks the gauge symmetry at this point. This new space of fields $\mathrm{Loc}_{\mathbb{C}^\times}(M,x)$ is a $\mathbb{C}^\times$-bundle over $\mathrm{Loc}_{\mathbb{C}^\times}(M)$:

$$\begin{tikzcd} \mathrm{Loc}_{\mathbb{C}^\times}(M,x) \arrow[d, "\mathbb{C}^\times"] \\ \mathrm{Loc}_{\mathbb{C}^\times}(M) \end{tikzcd} \ . \tag{4.80}$$

On the B-side we've introduced an extra factor of $\mathbb{C}^\times$, and on the A-side we've introduced an extra factor of $\mathbb{Z}$, by allowing different charges. Fourier series will identify these factors:

$$H^*(\mathrm{Pic}(M \setminus x)) \simeq \mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(M,x)] \ . \tag{4.81}$$

This is another instance of electric-magnetic duality: creating a magnetic monopole corresponds to creating an electrically charged particle.

[3]Heavy just means we're not adding a new field and doing field theory with it.

Surface defects

Another type of defect is a *surface defect*, also known as ramification, or introducing a solenoid. The idea is that we have a long tube, with a wire coiled around the outside, and then we send a current through it and this creates a magnetic field inside of the tube. So this is 1-dimensional subspace of a 3-manifold M^3, and inside of this crossed with time, $M \times I$, this defines a surface.

On the B-side, this introduces a singularity of the electric field, i.e. our fields are:

$$\mathrm{Loc}_{\mathbb{C}^\times}\left(M \setminus \beta\right) \ . \tag{4.82}$$

The holonomy around β introduces an extra factor of $\mathbb{Z}$ in H_1.

The magnetic (A-side) version of this has fields given by the same space of connections $\mathcal{C}\left(M\right)$, but equipped with a trivialization along β. How does this help you? A trivialization (or the difference between two trivializations) along a loop is a map from $\beta = S^1$ to $\mathrm{U}\left(1\right)$. This has a winding number, so we get $\mathbb{Z}$-many components to these fields. This is dual to the extra $\mathbb{C}^\times$ from the holonomy around β on the B-side. So this is another version of Fourier series. Explicitly this identifies:

$$\mathbb{C}\left[\mathrm{Loc}_{\mathbb{C}^\times}\left(M \setminus \beta\right)\right] \simeq H^*\left(\mathbb{C}\left(M \text{ trivialized along } \beta\right)\right) \ . \tag{4.83}$$

Chapter 5

Class field theory

Topology and physics gave us one source of interesting duality statements. Number theory, specifically class field theory (CFT), is another source. The topology (e.g. the locally constant functions) of Pic will still be exchanged with the algebraic geometry (e.g. the algebraic functions) of Loc. But we need to interpret Pic and Loc in this context.

Let $F/\mathbb{Q}$ be a number field. Write $\mathcal{C}\ell\,(F)$ for the ideal class group. We can think of this as:

$$\mathcal{C}\ell\,(F) = \mathrm{Pic}\,(\mathrm{Spec}\,\mathcal{O}_F) \ , \tag{5.1}$$

which consists of line bundles on $\mathrm{Spec}\,\mathcal{O}_F$, i.e. rank 1 projective $\mathcal{O}_F$-modules modulo isomorphism. Concretely, the *class group* is

$$\mathcal{C}\ell\,(F) = \text{fractional ideals of } \mathcal{O}_F/\text{principal ideals} \ . \tag{5.2}$$

Note that this is an abelian group. As it turns out $\mathcal{C}\ell\,(F)$ is also a finite group, and its order is an important invariant of F called the *class number*. The general philosophy of CFT is to relate the class group to the Galois group.

5.1 Unramified class field theory

Unramified CFT identifies:

$$\mathbb{C}\left[\mathcal{C}\ell\,(f)\right] \simeq \mathbb{C}\left[\mathrm{Loc}_{\mathbb{G}_m}(\mathcal{O}_F)\right] \ . \tag{5.3}$$

But we need to specify what Loc is in this context. We can always think of Loc as consisting of representations of π_1, and we can also write:

$$\pi_1\,(M) = \mathrm{Aut}\left(\widetilde{M}\right) \ , \tag{5.4}$$

where $\widetilde{M}$ is the universal cover of M. In this context, we have:

$$\mathrm{Gal}\,(E/F) = \mathrm{Aut}_F\,(E) \ , \tag{5.5}$$

and the analogue of the universal cover is the *maximal unramified extension* of F, written F^{ur}. So the analogue of π_1 is

$$\pi_1^{\mathrm{et}}\,(\mathrm{Spec}\,\mathcal{O}_F) = \mathrm{Gal}\,(F^{\mathrm{ur}}/F) \ . \tag{5.6}$$

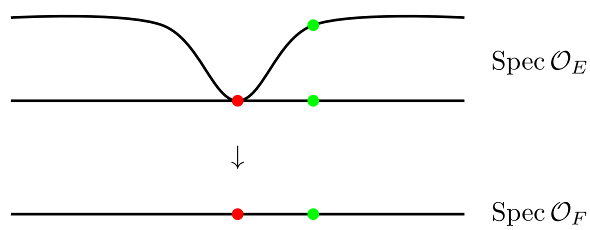


Figure 5.1: An extension which is ramified at the red point (on the left), and unramified at the green point (on the right).

We take F^{ur} because we want to consider coverings of $\operatorname{Spec}\mathcal{O}_F$, not of F itself. We say E/F is *unramified at a prime p* if

$$\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F/p \, (= \mathcal{O}_E/p) \tag{5.7}$$

is a product of fields, i.e. it has no nilpotents. The idea is that we have $\operatorname{Spec}\mathcal{O}_E$ living over $\operatorname{Spec}\mathcal{O}_F$, and we don't want there to be any branching as in fig. 5.1. I.e. if we look at the preimage of a points in the base, we want a product of fields. If we have some nilpotence, then this tells us there is some interesting geometry at that point, which we want to avoid for now. Arbitrary extensions will stay play the role of covering spaces away from the ramification point. E/F is unramified at ∞ if there are no $\mathbb{C}/\mathbb{R}$ extensions between them. E.g. if F is totally imaginary. The idea is that tensing with $\mathbb{R}$ corresponds to restricting to the neighborhood of infinity. But this is just a bunch of copies of $\mathbb{R}$ and $\mathbb{C}$:

$$\mathcal{O}_F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s \tag{5.8}$$

and then we insist that going from F to E never involves extending from $\mathbb{R}$ to $\mathbb{C}$.

Now that we have established the basics, we can state that unramified CFT tells us that

$$\mathcal{C}\ell \simeq \operatorname{Gal}\left(F^{\mathrm{ur,ab}}/F\right) = \operatorname{Aut}\left(\text{Hilbert class field}\right) \ , \tag{5.9}$$

where the Hilbert class field is the maximal abelian extension of F which is unramified everywhere.

To make contact with what we have seen before, we should study characters of this Galois group. I.e. on one side we have an analogue to Pic: $\mathcal{C}\ell$, and on the other side our analogue to Loc is:

$$\operatorname{Hom}\left(\operatorname{Gal}\left(F^{\mathrm{ur}}/F\right), \mathbb{C}^\times\right) \ . \tag{5.10}$$

The point is that homomorphisms into an abelian group factor into the abelian quotient:

$$\operatorname{Hom}\left(\operatorname{Gal}\left(F^{\mathrm{ur}}/F\right), \mathbb{C}^\times\right) = \operatorname{Hom}\left(\operatorname{Gal}\left(F^{\mathrm{ur,ab}}/F\right), \mathbb{C}^\times\right) \ . \tag{5.11}$$

Remark 20. $\operatorname{Gal}\left(F^{\mathrm{ur}}/F\right)$ is the analogue of $\pi_1\left(\operatorname{Spec}\mathcal{O}_F\right)$, and $\operatorname{Gal}\left(F^{\mathrm{ur,ab}}/F\right)$ is the analogue of $H_1\left(\operatorname{Spec}\mathcal{O}_F\right)$. So the equality between these Hom spaces is analogous to when morphisms from $\pi_1\left(\operatorname{Spec}\mathcal{O}_F\right)$ factor through its abelianization $H_1\left(\operatorname{Spec}\mathcal{O}_F\right)$.

Therefore, from (5.9), this is the Pontrjagin dual of the class group:

$$\begin{aligned} \mathrm{Loc}_{\mathbb{G}_m}(\mathcal{O}_F) &= \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur}}/F\right), \mathbb{C}^\times\right) && (5.12)\\ &= \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right), \mathbb{C}^\times\right) && (5.13)\\ &= \mathrm{Hom}\left(\mathcal{C}\ell(F), \mathbb{C}^\times\right) && (5.14)\\ &= \mathcal{C}\ell(F)^\vee\ . && (5.15) \end{aligned}$$

Remark 21. Artin-Verdier duality is an analogue of Poincaré duality for $\operatorname{Spec}\mathcal{O}_F$. This was our basis for the arithmetic topology dictionary which says that a number field (or really $\operatorname{Spec}\mathcal{O}_F$) is analogous to a 3-manifold. Recall that the duality between Pic and Loc for a 3-manifold was Poincaré duality, and this reformulation of CFT is a statement of Artin-Verdier duality. But the development of Artin-Verdier duality depends on CFT, so this isn't the direction one takes to prove CFT.

Remark 22. The arithmetic topological dictionary told us that $\operatorname{Spec}\mathcal{O}_F$ corresponds to a 3-manifold. As it turns out, it is an *unoriented* one. That is, the dualizing object is not the constants. This lack of orientation is analogous to the difference between:

- $\mathbb{Z}/n$ and roots of unity μ_n,
- $\mathbb{Q}/\mathbb{Z}$ and μ_∞,
- $\mathbb{C}^\times$ and $\mathbb{G}_m$, etc.

As it turns out, rather than $\mathbb{Z}$ as the dualizing object, we have the Tate-twist $\mathbb{Z}(1)$ (μ_n with respect to the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$). In Gauss' law, or when calculating the Chern classes of line bundles, we encountered some factors of $2\pi i$. These are accounted for by this phenomenon.

The upshot of the identification of characters of $\mathrm{Gal}^{\mathrm{ur}}$ with $\mathcal{C}\ell_F^\vee$ is that we get a Fourier transform: Lecture 9; February 23, 2021

$$\mathbb{C}[\mathrm{Cl}_F] \simeq \mathbb{C}[\mathrm{Loc}_1(\operatorname{Spec}\mathcal{O}_F)] \tag{5.16}$$

where Loc_1 is defined as the characters of $\mathrm{Gal}^{\mathrm{ur}}$.

Example 28. If $F = \mathbb{Q}$ then both sides are trivial.

5.2 Function fields

Let C be a smooth projective curve over some field k. To this we can attach the function field $F = k(C)$, which is the field of rational functions on C. We can also consider the Picard group, $\mathrm{Pic}(C)$, which has k-points given by:

$$\begin{aligned} \mathrm{Pic}(C)(k) &= \{\text{line bundles on } C\}/\sim && (5.17)\\ &= \{\text{loc. free rank 1 } \mathcal{O}_C-\text{modules}\} && (5.18)\\ &= \{\text{divisors}\}/\{\text{pricipal divisors}\} && (5.19)\\ &= H^1\left(C, \mathcal{O}^\times\right) && (5.20) \end{aligned}$$

where in the last line we're describing line bundles by their transition functions, i.e. Čech cocycles. Each line bundle has a degree, which is an integer. If we think of a line bundle as being given by a sum $\sum_{x \in C} n_x x$, then this is explicitly given by the map:

$$\begin{array}{ccc} \mathrm{Pic} & \xrightarrow{\deg} & \mathbb{Z} \\ \sum_{x\in C} n_x x & \longmapsto & \sum n_x \end{array} , \tag{5.21}$$

which is surjective with kernel given by:

$$0 \to \mathrm{Jac} \to \mathrm{Pic} \to \mathbb{Z} \to 0 , \tag{5.22}$$

where Jac is the Jacobian, i.e. line bundles of degree 0. For any choice of $x \in C$, we get a copy $\mathbb{Z}x \hookrightarrow \mathrm{Pic}$, i.e. this extension is split, with a section for every point of the curve..

If $k = \mathbb{F}_q$ is finite, this is similar to something we can see from the (étale) fundamental group of C. Instead of having $\mathbb{Z}$ as a quotient, this has the profinite completion[1] $\widehat{\mathbb{Z}}$ as a quotient:

$$\pi_1^{\text{ét}}(C) \twoheadrightarrow \widehat{\mathbb{Z}} = \pi_1^{\text{ét}}(\mathrm{Spec}\, k) . \tag{5.23}$$

The idea is that the curve C lives over $\mathrm{Spec}\, k$, so for a covering of $\mathrm{Spec}\, k$, we can pull this back to a covering $\widetilde{C}$ of C. This corresponds to the map from the fundamental group from C to the fundamental group of $\mathrm{Spec}\, k$. This looks very similar to the above situation, where Pic has $\mathbb{Z}$ as a quotient, with two main differences:

1. $\pi_1^{\text{ét}}$ is not abelian, and
2. $\widehat{\mathbb{Z}}$ is not $\mathbb{Z}$.

We can address this in two different ways. The first thing we can do is replace $\pi_1^{\text{ét}} = \mathrm{Gal}^{\mathrm{ur}}(F)$ by the *unramified Weil group of* C, written W_C^{ur}. The idea is that we want to make $\mathrm{Spec}\, k$ look more like S^1, so we replace $\widehat{\mathbb{Z}}$ by $\mathbb{Z}$, and W_C^{ur} is the preimage of $\mathbb{Z}$ in $\pi_1^{\text{ét}}$:

$$\begin{tikzcd} \pi_1^{\text{ét}} \arrow[r] & \widehat{\mathbb{Z}} \\ W_C^{\mathrm{ur}} \arrow[u] \arrow[r] & \mathbb{Z} \arrow[u, hook] \end{tikzcd} . \tag{5.24}$$

The other thing we can do is pass to the abelianization: $W_C^{\mathrm{ur,ab}}$, which has $\mathbb{Z}$ as a quotient:

$$W_C^{\mathrm{ur,ab}} \twoheadrightarrow \mathbb{Z} . \tag{5.25}$$

There is also a section of this for every point $x \in C(k)$, just like in the Pic case above. A point is a map $x\colon \mathrm{Spec}\, k \to C$, and geometrically this gives us a map

$$\widehat{\mathbb{Z}} = \pi_1^{\text{ét}}(k) \to \pi_1^{\text{ét}}(C) , \tag{5.26}$$

which is the corresponding section. The picture is that $\mathrm{Spec}\, k$ is an analogue of the circle, and C is an analogue of a 3-manifold fibered over it. Then a k-point of C is a section, i.e. a loop in C. To summarize, the claim is that Pic and $W_C^{\mathrm{ur,ab}}$ both

[1] Recall the profinite completion is the inverse limit of all finite quotients.

- surject onto $\mathbb{Z}$,
- are abelian, and
- have a section for every $x \in C(k)$.

Instead of altering the finite field picture, we could have altered Pic. Recall that Pic was an extension of $\mathbb{Z}$ by the Jacobian, which is a finite group over a finite field. Therefore all of the "infinity" of Pic is coming from $\mathbb{Z}$. So we could replace Pic by its profinite completion $\widehat{\mathrm{Pic}}$. So now we get something with a surjection to $\widehat{\mathbb{Z}}$ and a section for every point $x \in C$. In any case, the theorem is that they match.

Theorem 2 (Unramified CFT)**.** *There is a map* $\mathrm{Pic} \to \pi_1^{\acute{e}t,ab}$ *which is an isomorphism on profinite completions. Equivalently we have an isomorphism*

$$\mathrm{Pic} \simeq W_C^{ur,ab} \tag{5.27}$$

respecting the operation of taking the degree of a line bundle and respecting the sections associated to $x \in C$*.*

Example 29. If $C = \mathbb{P}^1$ then both sides are just $\mathbb{Z}$. This is more interesting for higher-genus curves.

Now we want to reformulate this in a way which is more reminiscent of the Pontrjagin duality that we have seen. Theorem 2 implies that the characters of Pic are the same as $W_C^{\mathrm{ur}\ ,\ \mathrm{ab}}$, but once we take characters, we can't detect the abelianization, so

$$\begin{aligned}
\text{characters of } \mathrm{Pic} &\simeq \text{characters of } W_C^{\mathrm{ur,ab}} && (5.28)\\
&\simeq \text{characters of } W_C^{\mathrm{ur}} && (5.29)\\
&\simeq \text{characters of } \pi_1^{\text{ét}}(C) && (5.30)\\
&\simeq \text{rank 1 local systems on } C\ . && (5.31)
\end{aligned}$$

I.e. the Pontrjagin dual to $\mathrm{Pic}(C)$ is:

$$(\mathrm{Pic}(C))^\vee \simeq \mathrm{Loc}_1(C)\ . \tag{5.32}$$

One kind of statement we can make from this, is that there is a Fourier transform identifying functions on Pic with functions on Loc.

Inside of function on Pic, we have the characters. The character condition, which asks that $\chi(gh) = \chi(g)\chi(h)$, is equivalent to asking for

$$\mu^*\chi = \pi_1^*\chi \boxtimes \pi_2^*\chi \tag{5.33}$$

where

$$\begin{tikzcd}
& \mathrm{Pic}\times\mathrm{Pic} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\mu"] & & \mathrm{Pic} \\
\mathrm{Pic} & & \mathrm{Pic} &
\end{tikzcd}\ . \tag{5.34}$$

Equivalently these are eigenfunctions for the translation action of Pic on itself. But Pic is generated by $\mathbb{Z}_x$ for $x \in C$, so a character is the same as a function f on Pic which is an eigenfunction for the action of $\mathbb{Z}_x$ for all $x \in C$. Explicitly, for $x \in C$, this action sends

$\mathcal{L} \in \mathrm{Pic}$ to the line bundle $\mathcal{L}(x)$ which is $\mathcal{L}$ with an extra $1 \cdot x$ added to the divisor. Being an eigenfunction means that:

$$f(\mathcal{L}(x)) = \gamma_x \cdot f(\mathcal{L}) \ , \tag{5.35}$$

where γ_x is a number.

These eigenfunctions should go to something like delta functions on Loc under this Fourier transform. An element $\rho \in \mathrm{Loc}$ is a representation:

$$\rho\colon \pi_1^{\text{ét}} \to e^\times \tag{5.36}$$

for e some field of coefficients (e.g. $\mathbb{C}$). Then f is an eigenfunction if

$$f(\mathcal{L}(x)) = \rho(\mathrm{Fr}_x) \cdot f(\mathcal{L}) \ . \tag{5.37}$$

The idea is that a local system is giving you a collection of eigenvalues for each point of the curve $\gamma_x = \rho(\mathrm{Fr}_x)$, i.e. $\rho \in \mathrm{Loc}$ determines the eigenvalues for $\mathbb{Z}x$ (for $x \in C$).

The operator $\mathcal{L} \mapsto \mathcal{L}(x)$ is a *Hecke operator*. These are playing the role of the 't Hooft/Dirac monopole operators from section 4.5.1 which acted on $H^2(M) \simeq \Lambda$ by translation, and were labelled by a loop $\gamma \in H_1(M)$. Recall that we were thinking of this as $H^2(M) = \pi_0(\mathrm{Pic})$.

On the other hand, we had the Wilson operators from section 4.5.1. For M a 3-manifold we sent:

$$\mathrm{Loc}_1(M) \ni \rho \mapsto W_\gamma(\rho) = \text{monodromy of } \rho \text{ around } \gamma \ . \tag{5.38}$$

Now we're sending:

$$\mathrm{Loc}_1 \ni \rho \mapsto \rho(\mathrm{Fr}_x) \ . \tag{5.39}$$

Again, in all of these cases, we're studying the algebraic geometry of Pic as a realization of the topology of Loc and vice versa.

5.2.1 Loc

Now we explain a bit about how to think about Loc. In algebraic geometry, we can't fully "access" π_1 because we only have finite covers.

Example 30. Consider the punctured affine line $\mathbb{A}^1 \setminus \{0\}$. We cannot access the universal cover because it is the exponential map $\exp\colon \mathbb{A}^1 \to \mathbb{A}^1 \setminus \{0\}$, which is not an algebraic function. We can however access finite covers of this since $t^{1/n}$ is algebraic.

So the first think we might study is something like a map:

$$\pi_1 \to \mu_n \subset \mathbb{Q}/\mathbb{Z} \ . \tag{5.40}$$

Moreover, for $\mathrm{char}\, k = p$, only the theory of prime-to-p-order covers works "as expected". So we need to pick a prime $\ell \neq p = \mathrm{char}\, k$, and then we can study maps:

$$\pi_1 \to \mathbb{Z}/\ell^n \ . \tag{5.41}$$

But then taking the inverse limit over ℓ gives us $\mathbb{Z}_\ell$, so we can make sense of representations into $\mathbb{Z}_\ell$, but we can also tensor up, i.e. we can lift the representation along:

$$\mathbb{Z}/\ell^n \leftarrow \mathbb{Z}_\ell \subset \mathbb{Q}_\ell \subset \overline{\mathbb{Q}_\ell} \ . \tag{5.42}$$

The upshot is that this leads to a good theory of ℓ-adic representations:

$$\pi_1 \to \mathrm{GL}_n\left(\overline{\mathbb{Q}_\ell}\right) \ , \tag{5.43}$$

i.e. a theory of ℓ-adic local systems in characteristic p. One nice thing is that $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ as fields, but this does not respect the topology.

So when we say rank 1 local systems we really mean continuous morphisms:

$$\mathrm{Loc}_1 = \mathrm{Hom}_{\mathrm{cts}}\left(\mathrm{Gal}^{\mathrm{ur}}(C), \overline{\mathbb{Q}_\ell}^{\times}\right) \ . \tag{5.44}$$

More generally, whenever we're discussing $\mathbb{C}$-functions we should really be taking $\overline{\mathbb{Q}_\ell}$-valued functions. The resulting theory is independent of ℓ as long as $\ell \neq p$.

5.2.2 Pic

We want to describe $\mathrm{Pic}(C)$ for C/k a smooth projective curve as something along the lines of:

$$\mathrm{Pic}(C) = \text{Divisors/principal divisors} \tag{5.45}$$

$$= \bigoplus_{x \in C} \mathbb{Z} / \operatorname{unit} k(C) \ . \tag{5.46}$$

Let $\mathcal{L}$ be a line bundle. We want to describe it via its transition functions. We can trivialize $\mathcal{L}$ generically, i.e. there exists a meromorphic section s of $\mathcal{L}$ or equivalently there is an isomorphism of functions away from finitely many points $\{x_i\}_i$:

$$s\colon \ \mathcal{O}|_{C \setminus \{x_i\}_i} \xrightarrow{\sim} \mathcal{L}|_{C \setminus \{x_i\}_i} \ . \tag{5.47}$$

On the other hand, we can (more democratically) trivialize $\mathcal{L}$ very close to any $x \in C$. Formally, the disk around x is:

$$D_x = \mathrm{Spec}(\mathcal{O}_x) = \mathrm{Spec}(k[[t]]) \ , \tag{5.48}$$

where $\mathcal{O}_x$ is the completed local ring at x and we have chosen a coordinate t on C. And when we pull $\mathcal{L}$ back to D_x, it is automatically trivialized. Therefore the section s has a *nonzero* Laurent expansion around these finitely many points, i.e. we have

$$[s] \in K_x \simeq k((t)) \tag{5.49}$$

where K_x is the field of fractions of $\mathcal{O}_x$.

Now we can measure the degree of the section s at x:

$$\deg_x s \in K_x^\times / \mathcal{O}_x^\times \tag{5.50}$$

where we are quotienting out by changes of the trivialization of $\mathcal{L}$ on D_x. Expressed in the coordinate t, this is:

$$K_x^\times / \mathcal{O}_x^\times = k((t))^\times / k[[t]]^\times \tag{5.51}$$

$$= \left\{a_{-N} t^{-N} + \dots\right\} / \left\{b_0 + b_1 t + \dots \mid b_0 \neq 0\right\} \ . \tag{5.52}$$

Now it's an exercise in algebra to check that we can force $a_{-N} = 1$, and all other $a_i = 0$. Therefore this is identified with the integers:

$$K_x^\times / \mathcal{O}_x^\times \simeq \mathbb{Z} \; . \tag{5.53}$$

This description is not very "efficient". For varying line bundles, we have to vary the open set $U = C \setminus \{x_i\}$ where we are able to trivialize $\mathcal{L}$. The way we deal with this is by "removing all points". So consider the space of line bundles $\mathcal{L}$ equipped with a rational section[2] and a trivialization of $\mathcal{L}|_{D_x}$ for all $x \in C$. Now we want to describe the space of such data. We know we get a nonzero Laurent series for every point $x \in C$, so our first guess might be a product of $K_x^\times$, but this is actually a *restricted product* because for any particular line bundle, there are only finitely many points where there was a problem:

$$\prod_{x\in C}{}' K_x^\times = \left\{ (\gamma_x) \in K_x^\times \,\middle|\, \gamma_x \in \mathcal{O}_x^\times \text{ for all but finitely many } x \right\} \tag{5.54}$$

$$\subset \prod_{x\in C} K_x^\times \; . \tag{5.55}$$

So this was kind of "overkill", and we got a huge amount of data, and now we will kind of "strip it away". So consider Line bundles equipped only with just a rational section (not a trivialization everywhere). We can access these by quotienting out by changes of trivialization, i.e. line bundles equipped with a rational section comprise:

$$\prod{}' K_x^\times / \prod \mathcal{O}_x^\times = \prod{}' \left(K_x^\times / \operatorname{unit} \mathcal{O}_x \right) \tag{5.56}$$

$$= \prod{}' \mathbb{Z} \tag{5.57}$$

$$\left\{ \text{finite } \sum_{x\in C} a_x x \right\} \tag{5.58}$$

$$= \text{Divisors} \; . \tag{5.59}$$

Now we need to get rid of our choice of rational section by modding out on the left:

$$\operatorname{Pic} = \{\text{line bundles}\} \tag{5.60}$$

$$= F^\times \backslash \prod{}' K_x^\times / \prod \mathcal{O}_x^\times \tag{5.61}$$

$$= F^\times \backslash \prod{}' \mathbb{Z} \; , \tag{5.62}$$

which is exactly divisors moduli principal ones. The left quotient is by changes of the rational section, and the right quotient is by changes of the trivialization on D_x. Lecture 10; transition 2021

This is the adélic description of $\operatorname{Pic}(C)$. We can rewrite this as:

$$\operatorname{Pic}(C) = \mathrm{GL}_1(F) \backslash \mathrm{GL}_1(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}_F}) \tag{5.63}$$

where $\mathbb{A}_F$ is the adéles for F:

$$\mathbb{A}_F := \prod_{x\in C}{}' K_x \tag{5.64}$$

[2] Note we don't bother saying where the poles are. So we just trivialize $\mathcal{L}$ over the function field.

where K_x is the completed local field. Inside of this is the ring of integers:

$$\mathcal{O}_{\mathbb{A}} = \prod_{x \in C} \mathcal{O}_x \tag{5.65}$$

where $\mathcal{O}_x$ is the completed local ring.

Remark 23. This description might seem like overkill, in the sense that we're writing it has a quotient of something huge. This is the same sense in which defining Pic via divisors is overkill: the collection of all divisors is huge before we mod out by principal ones.

The *(unramified) idéle class group (idéles)* is:

$$\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times \; . \tag{5.66}$$

Arithmetic version

Now we will relate this discussion to the analogous ones in the arithmetic setting. For F a number field, we can construct the idéles of F as:

$$\mathbb{A}_F = \prod_{v \text{ places}}' F_v \; , \tag{5.67}$$

where F_v is a completed local field (e.g. $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}$, $\ldots$) and similarly

$$\mathcal{O}_{\mathbb{A}} = \prod_{\text{primes}} \mathcal{O}_v \; . \tag{5.68}$$

Then, in an attempt to get our hands on an analogue of Pic, we can follow our nose from the function field setting and guess that the unramified idéle class group is:

$$\mathrm{GL}_1(F) \backslash \mathrm{GL}_1(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) = F^\times \backslash {\prod}' F_v^\times / \prod_v \mathcal{O}_v^\times \; . \tag{5.69}$$

This is missing a factor coming from the place at infinity, which we can add as follows. At a finite place, the inclusion $F^\times \supset \mathcal{O}^\times$ looks something like $\mathbb{Q}_p^\times \supset \mathbb{Z}_p^\times$. One way of describing $\mathbb{Z}_p^\times$, is that is is a maximal compact subgroup with respect to the p-adic topology. So now we can use this to determine what to do at infinity. In particular, we let K_∞ be the maximal compact subgroup of $F_v^\times$, and we additionally quotient out by this on the right:

$$\mathrm{GL}_1(F) \backslash \mathrm{GL}_1(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) \cdot K_\infty = F^\times \backslash {\prod}' F_v^\times / \prod_v \mathcal{O}_v^\times \cdot K_\infty \; . \tag{5.70}$$

Note that, at infinity, $F_v^* \simeq \mathbb{R}^\times$ or $\mathbb{C}^\times$ so the maximal compacts are $\mathbb{Z}/2$ and $\mathrm{SO}(2)$ respectively. So we're only removing part of these extra copies of $\mathbb{R}^\times$ and $\mathbb{C}^\times$. Explicitly we still have some factors of $\mathbb{R}_+$.

The idea is that this is very close to $\mathcal{C}\ell$:

$$\mathcal{C}\ell_F = \pi_0 \left(F^\times \backslash \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty \right) \; . \tag{5.71}$$

In other words: there is a quotient from the unramified idéle class group to $\mathcal{C}\ell_F$ given by taking π_0, i.e. by quotienting out by the leftover copies of $\mathbb{R}_+$ from the places at infinity. Now recall $\mathcal{C}\ell$ is isomorphic to $\mathrm{Gal}^{\mathrm{ur,ab}}$:

$$F^\times \backslash \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty \twoheadrightarrow \mathcal{C}\ell_F \simeq \mathrm{Gal}^{\mathrm{ur,ab}} \; . \tag{5.72}$$

In particular if we look at characters that don't detect copies of $\mathbb{R}$ and $\mathbb{C}$ (e.g. finite order or continuous ones) then the characters on these groups will agree.

This is close to what we saw in topology. Recall we were consider the space of line bundles and connections, $\mathrm{Pic}\left(M^3\right) = \mathrm{Map}\left(M^3, B\,\mathrm{U}(1)\right)$, which had π_0 given by $H^2(M, \mathbb{Z})$. The identity component of the idéle class group can be described as follows. Our first guess is: Pic notation

$$\mathbb{R}_+^r \times \mathbb{R}_+^s \tag{5.73}$$

where r is the number of real embeddings of F, and s is the number of pairs of complex embeddings. This is because the infinite part of our field is:

$$F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s \ , \tag{5.74}$$

and then when we quotiented out by the maximal compacts, we just got copies of $\mathbb{R}_+$ for both $\mathbb{R}$ and $\mathbb{C}$. Then we quotient out by a boring diagonal copy on the right, and by the units in F on the left:

$$\mathcal{O}_F^\times \backslash \mathbb{R}_+^r \times \mathbb{R}_+^s / \mathbb{R}_+ \ . \tag{5.75}$$

Theorem 3 (Dirichlet unit theorem)**.** *$\mathcal{O}_F^\times \simeq \mu_\infty(\mathcal{O}_F) \times \mathbb{Z}^{r+s-1}$, i.e. the units are just the obvious ones (all roots of unity in F) and a lattice.*

Therefore the connected component of the idéle class group is:

$$\pi_0\left(F^\times \backslash \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty\right) = \simeq \mathrm{U}(1)^{r+s-1} \ . \tag{5.76}$$

So if we study the cohomology of the idéle class group, we get an exterior algebra of rank $r+s-1$.

This is the complete story for the unramified case. But this is a very restrictive condition to ask for, especially in number theory.

5.3 Ramification

5.3.1 Physics

Recall in the physics context (section 4.8.1) we had $\mathrm{Pic}\left(M^3\right)$, which consisted of line-bundles equipped with a connection. By line-bundles and connections we just mean $\mathrm{Map}(M, B\,\mathrm{U}(1))$. Pic notation
In particular this has

$$\pi_0 = H^2(M, \mathbb{Z}) \qquad\qquad \pi_1 = H^1(M, \mathbb{Z}) \ . \tag{5.77}$$

We matched this with

$$\mathrm{Loc}_1\left(M^3\right) = \mathrm{Hom}\left(\pi_1(M), \mathbb{C}^\times\right) \tag{5.78}$$

in the sense that functions on them were identified:

$$H^0\left(\mathrm{Pic}\left(M^3\right)\right) \leftrightarrow \mathbb{C}\left[\mathrm{Loc}_1 M^3\right] \ . \tag{5.79}$$

Then, in section 4.8.2, we had surface defects/solenoids where we looked at $\mathrm{Loc}_1\left(M^3 \setminus \gamma\right)$ for some closed 1-manifold $\gamma \hookrightarrow M^3$, e.g. a knot or link. I.e. we're allowing singularities along γ. Loc_1 now has an extra copy of $\mathbb{C}^\times$ for every component of the link, so we need to change Pic too.

To alter Pic, instead of introducing singularities along γ, we work relative to it, so we consider: $\mathrm{Pic}\left(M^3, \gamma\right)$, which consists of line bundles equipped with a trivialization on γ. This introduces an extra factor of $\mathbb{Z}$ in Pic per component of γ.

Remark 24. If we're thinking topologically, before we introduced ramification, we had the Poincaré duality (with coefficients) between: $\pi_0 \operatorname{Pic} = H^2(M, \mathbb{Z})$ and $H^1(M, \mathrm{U}(1))$. Introducing ramification corresponds to passing from Poincaré duality to Lefschetz-duality, which is a version for relative cohomology:

$$H^2((M,N), \mathbb{Z})^\vee \simeq H^1\left(M \setminus N, \mathbb{C}^\times\right) \ . \tag{5.80}$$

So we work relative to the knot on one side, and remove it on the other.

5.3.2 Arithmetic setting

Now we want to do the same thing in number theory, i.e. over curves over finite fields or number fields. We won't try to get the whole story, but just tame ramification.

Over $\mathbb{C}$,

$$D^\times = \operatorname{Spec} \mathbb{C}((t)) \tag{5.81}$$

is some version of a circle, and indeed

$$\pi_1\left(D^\times\right) = \mathbb{Z} \ . \tag{5.82}$$

In algebraic geometry, we should really think about:

$$\pi_1^{\text{ét}}\left(D^\times\right) = \widehat{\mathbb{Z}} \ . \tag{5.83}$$

This means that all extensions of $\mathbb{C}((t))$ correspond to taking roots of the coordinate: $\mathbb{C}((t)) \to \mathbb{C}\left(\left(t^{1/N}\right)\right)$.

In arithmetic, things are much more interesting. Analogues of the punctured disk are:

$$D^\times{}_{\mathbb{F}_q} = \operatorname{Spec} \mathbb{F}_q((t)) \qquad\qquad \operatorname{Spec} \mathbb{Q}_p \ , \tag{5.84}$$

only now $\pi_1^{\text{ét}}$ is richer. There are two sources of this richness. One is that we have a quotient to the Frobenius:

$$\pi_1^{\text{ét}}\left(D^\times{}_{\mathbb{F}_q}\right) \twoheadrightarrow \widehat{\mathbb{Z}} \ . \tag{5.85}$$

One might object that this is a consequence of working over a non-algebraically-closed field. But even when we pass to $\overline{\mathbb{F}_q}((t))$, this still has a complicated fundamental group. But there is part which mimics what we saw in geometry, called the Tame inertia group. The idea is that $\pi_1^{\text{ét}}\left(D^\times{}_{\mathbb{F}_q}\right)$ and $\operatorname{Gal}(\mathbb{Q}_p)$ both have a quotient given by

$$\Gamma = \left(\left\{F, m \,\middle|\, FmF^{-1} = m^q\right\}\right)^\wedge \ , \tag{5.86}$$

where q is the order of the residue field, and F stands for the Frobenius (and m stands for monodromy). This surjects onto the Frobenius part with the monodromy part sitting inside:

$$\widehat{\mathbb{Z}_m} \to \Gamma \twoheadrightarrow \widehat{\mathbb{Z}_F} \ . \tag{5.87}$$

This is something pretty geometric. We have something going around the point we removed (monodromy), and there is the Frobenius, and they relate via $FmF^{-1} = m^q$. The idea is that, in the arithmetic setting, this étale fundamental group tells us much more, but the tame inertia is the part that looks like the geometric setting.

In number theory our Loc is representations of the Galois group of F (here F is either rational functions on a curve $\mathbb{F}_q(C)$ or a number field):

$$\mathrm{Loc} = \left\{ \mathrm{Gal}_F \to \mathrm{GL}_1\left(\overline{\mathbb{Q}_\ell}\right) \right\} \ . \tag{5.88}$$

For a finite subset $S \subset C$ (or $S \subset$ primes of F) we can look at elements of Loc which are tamely ramified at the points of S, written $\mathrm{Loc}_1 (C \setminus S)^{\mathrm{tame}}$.

Remark 25. This is like when we allowed singularities along the knot in section 4.5.1. Only in the physics, we didn't need to control how singular these singularities are. Now we do need to get control, and we do so by asking for the ramification to be tame.

Then you might ask how to match this with some version of Pic. The claim is that the Pontrjagin duality exchanges:

$$\mathrm{Loc}_1 (C \setminus S)^{\mathrm{tame}} \qquad\qquad \text{and} \qquad\qquad \mathrm{Pic}(C, S) \ , \tag{5.89}$$

where $\mathrm{Pic}(C,S)$ consists of line bundles on C equipped with a trivialization on S.

Recall we were thinking of:

$$\mathrm{Pic}\, C = F^\times \backslash {\prod}' \mathbb{Z} \ . \tag{5.90}$$

Recall the basic comparison between Pic and $\pi_1^{\text{ét}}$ in Theorem 2 was based on matching these copies of $\mathbb{Z}$. Specifically these came from the fact that we had a section of $\mathrm{Pic} \to \mathbb{Z}$ for every point of the curve, and a copy of $\mathbb{Z}$ in the Galois group for each Frobenius. Now we have this richer description:

$$\mathrm{Pic}\, C = F^\times \backslash {\prod_{x\in C}}' K^\times / \prod_{x \in C} \mathcal{O}^\times \ . \tag{5.91}$$

Recall that this quotient description came from starting with ${\prod}' K^\times$, the collection of line bundles with a generic trivialization, and a trivialization near every point. Then we quotiented out by this extra data. Now we consider only quotienting on the left:

$$F^\times \backslash {\prod}' K^\times \twoheadrightarrow \mathrm{Pic}\, C \ , \tag{5.92}$$

which consists of line bundles with a trivialization near every $x \in C$. I.e. we have quotiented out by the choice of generic trivialization. The group in (5.92) is the *idéle class group*.[3]

Choosing a trivialization around every point is still a huge amount of data. The advantage of this is that it allows us to be flexible about what kind of data we pick where, e.g. we might ask for a finite order trivialization around some finite collection of points. So pick a finite subset $S \subset C$ with multiplicities n_x for $x \in S$. Then we can consider the collection of elements of Pic, equipped with an n_x-order trivialization at all $x \in S$. By trivialize to the order n_x, we mean the following. We have an nth order neighborhood of the point x sitting inside the disk around x:

$$\mathrm{Spec}\, k[t]/t^n \hookrightarrow D_x = \mathrm{Spec}\, k[[t]] \ , \tag{5.93}$$

[3] Before this, it was really the unramified idéle class group.

and instead of asking for an infinite Taylor series section of the bundle, we ask for a section of finite order n_x. This gives us an intermediate quotient Pic_S:

$$\begin{tikzcd} \mathrm{GL}_1(\mathrm{unit}\,F)\setminus\mathrm{GL}_1(\mathbb{A}_{\mathbb{C}}) \arrow[d] \arrow[dr] & \\ & \mathrm{Pic}(C,S) \arrow[dl] \\ \mathrm{Pic}=\mathrm{GL}_1(F^\times)\setminus\mathrm{GL}_1(\mathbb{A}_{\mathbb{C}})\,/\,\mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) & \end{tikzcd} \tag{5.94}$$

defined by

$$\mathrm{Pic}_S=\mathrm{GL}_1(F)\setminus\mathrm{GL}_1(\mathbb{A}_F)\,/\prod_{x\notin S}\mathrm{GL}_1(\mathcal{O}_X)\times\prod_{x\in S}\mathrm{GL}_1^{(n_x)}(\mathcal{O}_x)\ , \tag{5.95}$$

where $\mathrm{GL}_1^{(n)}(\mathcal{O}_x)$ consists of elements of $\mathrm{GL}_1(\mathcal{O}_x)$ that are congruent to $1 \bmod t^n$. I.e. they consist of changes of trivialization of a line bundle on the disk, constant to order n. So the top is elements of Pic equipped with infinite order trivialization everywhere, the bottom is just Pic, and the middle is elements of Pic equipped with an n_x order trivialization at every $x\in S$. Concretely:

$$\mathcal{O}_x^\times=\left\{a_0+a_1t+a_2t^2+\ldots\,\middle|\,a_0\neq 0\right\}\ , \tag{5.96}$$

and

$$\mathrm{GL}_1^{(n)}(\mathcal{O}_x)=\{1+a_nt^n+\ldots\}\ . \tag{5.97}$$

Example 31. Let S be a finite subset of points of C, and let $n_x=1$ for all $x\in S$. Then $\mathrm{Pic}(C,S)$ consists of line bundles whose fibers at $x\in S$ are trivialized. This maps down to $\mathrm{Pic}(C)$ with fiber $\mathcal{O}^\times/(\mathcal{O}^\times)^{(1)}=k^\times$.

Recall $\pi_1^{\text{ét}}D^\times$ was complicated, but had this quotient Γ:

$$\pi_1^{\text{ét}}\left(D^\times\right)\twoheadrightarrow\Gamma=\left(\left\{F,m\,\middle|\,FmF^{-1}=m^q\right\}\right)^\wedge\ . \tag{5.98}$$

But we're only considering maps to some abelian group:

$$\begin{tikzcd} \pi_1^{\text{ét}}D^\times \arrow[r, two heads] \arrow[dr] & \Gamma \arrow[d] \\ & \text{abelian group} \end{tikzcd}\ , \tag{5.99}$$

so they factor through the abelianization:

$$\Gamma/[\Gamma,\Gamma]\simeq\mathbb{F}_q^\times\times\widehat{\mathbb{Z}}\ . \tag{5.100}$$

Now we want to compare this to $K^\times$. $K^\times/O^\times=\mathbb{Z}$ was our degree, and then:

$$\mathbb{F}_q^\times\times\mathbb{Z}\simeq K^\times/\mathcal{O}^{\times\,(1)}\to K^\times/\mathcal{O}^\times=\mathbb{Z}\ , \tag{5.101}$$

and this is the same $\mathbb{F}_q^\times\times\mathbb{Z}$ in the Galois group.

$$\overline{K^\times}\simeq\mathrm{Gal}^{\mathrm{ab}}\,\overline{K_x}/K_x\ . \tag{5.102}$$

This is the first instance of local class field which says that:

$$\widehat{K^{\times}} \simeq \mathrm{Gal}\,\overline{K_x}/K_x \ . \tag{5.103}$$

The RHS has a quotient to $\widehat{\mathbb{Z}}$ given by the Frobenius, and the LHS has a quotient to $\widehat{\mathbb{Z}}$ given by the degree. We also have quotients down to $\mathbb{F}_q^{\times}$. So these quotients reveal the part of these groups that we see in the topological context. Exactly as in the physics, this local class field theory gives us a duality between:

$$\mathrm{Pic}\,(C,S) \qquad \text{and} \qquad \mathrm{Loc}\,(C \setminus S)^{\mathrm{tame}} \ . \tag{5.104}$$

Chapter 6

Extended TFT

6.1 Ramification

We have seen the notion of ramification in extended TFT, and now we will consider higher codimension versions. This means we're "raising the category level" from vector spaces to categories. Lecture 11; March, 2021

This will be formalizing some notions that we've already seen in certain examples. We had two examples of ramification in physics. The first was the notion of a solenoid in Maxwell theory. In this situation we have some knot in a manifold M^3, and then we considered $H^2((M,N),\mathbb{Z})$ where N is the boundary of a tubular neighborhood of the knot. The dual of this was:

$$H^2((M,N),\mathbb{Z})^\vee \simeq H^1\left(M \setminus N, \mathbb{C}^\times\right) \ . \tag{6.1}$$

So the idea is that $H^2((M,N),\mathbb{Z})$ consists of line bundles equipped with a trivialization on the knot, whereas $H^1(M \setminus N, \mathbb{C}^\times)$ has flat connections with singularities along the knot.

We also had the Dirac monopole where we consider $H^2(M \setminus N, \mathbb{Z})$, where N is the boundary of a small ball. We can think of this as line bundles with a singularity at the point. The dual of this was:

$$H^2(M \setminus N, \mathbb{Z})^\vee \simeq H^1\left((M,N), \mathbb{C}^\times\right) \ , \tag{6.2}$$

which we can think of as flat connections with a trivialization at this point.

Something similar appeared in the arithmetic setting. We had this Cartier duality between line bundles and flat connections:

$$\operatorname{Pic}(C)^\vee \simeq \operatorname{Loc}_1 C \ . \tag{6.3}$$

On the left we can add level structure, i.e. we can add trivializations on a subset $S \subset C$. On the right we can add singularities at $S \subset C$. Then ramified class field theory tells us how adding a level structure on one side corresponds to adding singularities on the other.

Now we would like to explain, more systematically, how to add higher codimension data from the physics point of view.

Recall, in field theory, we are studying fields $\mathcal{F}(M)$ on a manifold M. If M has boundary $\partial M = N$, then we get a map:

$$\mathcal{F}(M) \xrightarrow{\pi_N} \mathcal{F}(N) \ . \tag{6.4}$$

In the examples (of a monopole and a solenoid) we then asked for conditions on N (either allowing a singularity or asking for a trivialization). We can say it "all at once" as follows. Let $\mathcal{E}_N$ be any *sheaf* on $\mathcal{F}(\partial M)$. Recall that we attached a vector space to M, given by the vector space of function (or cohomology etc.) on $\mathcal{F}(M)$. Now we want to modify this by the boundary data $\mathcal{E}_N$. Explicitly we can pull $\mathcal{E}_N$ back to M, to get $\pi_N^*\mathcal{E}_N$. This is a sheaf on M, so we can take global sections to get a new vector space:

$$\Gamma\left(M, \pi_N^*\mathcal{E}_N\right) \ . \tag{6.5}$$

This is a modified version of the Hilbert space we attach to M. So we get a vector space attached to the data of M and some $\mathcal{E}_N$, which plays the role of the conditions we were asking for on N (singularity or trivialization).

Remark 26. For any $\pi\colon X \to Y$ and $\mathcal{E} \in \mathbf{Shv}(Y)$ we have:

$$\begin{aligned}\Gamma\left(X, \pi^*\mathcal{E}\right) &= \Gamma\left(Y, \pi_*\pi^*\mathcal{E}\right) &(6.6)\\ &= \Gamma\left(Y, \mathcal{E} \otimes \pi_*\underline{k}_X\right) \ . &(6.7)\end{aligned}$$

So to any X we can attach $\pi_*\underline{k}_X \in \mathbf{Shv}(Y)$ and pairing with that object gives a vector space, or we can just think of this as the vector space attached to X by pulling back $\mathcal{E}$.

6.1.1 Topological Maxwell theory

In topological Maxwell theory section 4.8.1, the A-side dealt with

$$H^*\left(\mathcal{C}\left(M\right)\right) \ , \tag{6.8}$$

where $\mathcal{C}(M)$ is the space of line bundles with connections on M. Now we modify this to recover the relative version, i.e. either

$$H^2\left(\left(M, N\right), \mathbb{Z}\right) \tag{6.9}$$

or

$$H^2\left(M \setminus N, \mathbb{Z}\right) \ . \tag{6.10}$$

Recall the space of connections had components given by:

$$\pi_0\mathcal{C}\left(M\right) \simeq H^2\left(M, \mathbb{Z}\right) \ , \tag{6.11}$$

so we can think of $H^2(M, \mathbb{Z})$ as consisting of locally constant functions on the connected components. The point is that the space of connections is, topologically,

$$\mathcal{C}\left(M\right) = \mathrm{Map}\left(M, K\left(\mathbb{Z}, 2\right)\right) = \mathrm{Map}\left(M, B\,\mathrm{U}\left(1\right)\right) \ , \tag{6.12}$$

so the components are given by $H^2(M, \mathbb{Z})$.

So our Hilbert space in Maxwell theory was cohomology of the constant sheaf on $\mathcal{C}(M)$. Now we want to modify it by taking cohomology of different sheaves. We have a restriction from M to the boundary $\partial M = N$:

$$\mathrm{Map}\left(M, K\left(\mathbb{Z}, 2\right)\right) \xrightarrow{\pi_N} \mathrm{Map}\left(N, K\left(\mathbb{Z}, 2\right)\right) \ , \tag{6.13}$$

so we can start with a sheaf on the boundary and pull it back to M and take the cohomology.

For example, inside of

$$\mathrm{Map}\left(N, K\left(\mathbb{Z}, 2\right)\right) = \mathrm{Map}\left(N, B\,\mathrm{U}\left(1\right)\right) \ , \tag{6.14}$$

we have the constant maps, i.e. there is a point

$$i\colon \{\mathrm{pt}\} \hookrightarrow \mathrm{Map}\left(N, B\,\mathrm{U}\left(1\right)\right) \tag{6.15}$$

given by maps sending N to $\mathrm{pt} \in B\,\mathrm{U}\left(1\right)$. Consider the skyscraper at this point, $i_*\underline{\mathbb{C}}$, and take this as our sheaf:

$$\mathcal{E}_N = i_*\underline{\mathbb{C}} \ . \tag{6.16}$$

Then we pull this back to M to get $\pi^*\mathcal{E}_N$, which has cohomology:

$$H^*\left(\mathcal{C}\left(M\right), \pi^*\mathcal{E}\right) = H^*\left(\text{line bundles on } M + \text{trivialization on } N\right) \ . \tag{6.17}$$

This is the *Dirichlet boundary condition.* In particular, we can recover the relative cohomology $H^2\left(\left(M, N\right), \mathbb{Z}\right)$ as π_0 of the following fiber product:

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & \mathrm{Map}\left(M, K\left(\mathbb{Z}, 2\right)\right) \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{Maps}\left(M, K\left(\mathbb{Z}, 2\right)\right) \end{tikzcd} \ . \tag{6.18}$$

Let $K \subset M$ be a knot, and N the boundary of a tubular neighborhood of K. Write M_0 for the manifold with boundary $\partial M_0 = N$ obtained by removing the knot. So if we want to recover the cohomology of M_0, we just take the constant sheaf on $\mathcal{F}\left(N\right)$ and pull it back to $\mathcal{F}\left(M_0\right)$. This is the *von Neumann boundary condition.*

To reiterate, given fields on M we can put conditions on them along $K \subset M$ as follows. We have a map:

$$\mathcal{F}\left(M \setminus K =: M_0\right) \to \mathcal{F}\left(\partial M_0\right) \tag{6.19}$$

and we can take sheaves on the right, pull them back to $\mathcal{F}\left(M_0\right)$, and take the cohomology of $\mathcal{F}\left(M_0\right)$ with coefficients in this sheaf.

For example, the pullback of the constant sheaf $\underline{\mathbb{C}}$ is still the constant sheaf, and then we take

$$H^*\left(\mathcal{F}\left(M \setminus K\right), \pi^*\underline{\mathbb{C}} = \underline{\mathbb{C}}\right) = H^*\left(\mathcal{F}\left(M \setminus K\right)\right) \ . \tag{6.20}$$

I.e. we are allowing arbitrary singularities along K.

We can also consider the sky scraper $i_*\underline{\mathbb{C}}$, where $i\colon \pi \hookrightarrow \partial M_0$. This gives:

$$H^*\left(\mathcal{F}\left(M \setminus K\right), \pi^* i_*\underline{\mathbb{C}}\right) = H^*\left(\text{connections} + \text{trivialization on } N\right) \ . \tag{6.21}$$

In summary, we can think of sheaves on the boundary as being a source of ways to get a new vector space attached to M.

6.2 Extended TFT

The rough idea of an n-dimensional TFT, was to attach a number to a closed n-manifold P. We can think of this as some kind of volume of the space of all field on P. Then a closed $(n-1)$-manifold, we attached a vector space, which we think of as a space of functionals on the space of fields on M. Now we would like to attach some kind of category to a closed $(n-2)$-manifold. Again this is some kind of linearization of the space of fields, but now it's the category of sheaves on $\mathcal{F}\left(N\right)$.

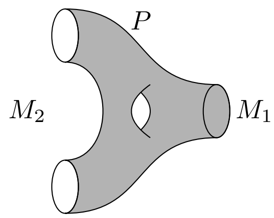


Figure 6.1: A morphism between two objects (closed $(n-1)$-manifolds) M_1 and M_2 in $\mathbf{Bord}_{n-1,m}$ is an n-manifold P with $M_1 \sqcup M_2$ as its boundary.

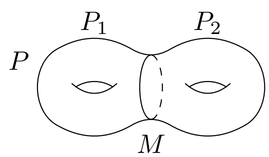


Figure 6.2: Closed n-manifold decomposed into P_1 and P_2 meeting along M.

6.2.1 Atiyah-Segal TFT

An *Atiyah-Segal (unexpected) TFT* is a symmetric monoidal functor

$$(\mathbf{Bord}_{n-1,n}, \sqcup) \xrightarrow{Z} (\mathbf{Vect}, \otimes) \quad . \tag{6.22}$$

The objects of $\mathbf{Bord}_{n-1,n}$ are (oriented[1]) closed $(n-1)$-manifolds. The objects of $\mathbf{Vect}$ are (complex) vector spaces. So a closed $(n-1)$-manifold M goes to some vector space $Z(M)$.

A morphism $P\colon M_1 \to M_2$ in $\mathbf{Bord}_{n-1,n}$ is an n-manifold P with boundary $\partial M = M_1 \sqcup M_2$ as in fig. 6.1.

To see why one would think this way we need to shift our interest away from the vector spaces of states, and briefly refocus on the partition function $Z(P)$ on a closed n-manifold P. The idea is that this Atiyah-Segal formalization of TFT encodes the locality of the partition function. I.e. if we cut P into two pieces P_1 and P_2, glued along M as in fig. 6.2, then $Z(P)$ factors as:

$$\begin{tikzcd} Z(\emptyset) = \mathbb{C} \arrow[r] \arrow[rr, bend left, "Z(P)"] & Z(M) \arrow[r] & \mathbb{C} = Z(\emptyset) \end{tikzcd} \quad . \tag{6.23}$$

We're more interested in the vector space of states on an $(n-1)$-manifold M. The idea is that we can similarly decompose $Z(M)$ into "local pieces" coming from a decomposition of M:

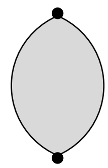
$$M = M_1 \sqcup_N M_2 \tag{6.24}$$

for N of codimension 2. We would like to write something like:

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle_{Z(N)} \quad , \tag{6.25}$$

but the Atiyah-Segal formalism doesn't tell us how to do this. This is the basic idea of extended field theory, due to Freed, Lawrence, Baez-Dolan, ... cite

[1] We can ask for different tangential structures on our bordisms. We're just taking oriented ones.



$$= \quad \begin{tikzcd} N_1 \arrow[d, bend right, "M_1"'] \arrow[d, bend left, "M_2"] \arrow[d, phantom, "\Longrightarrow"] \\ N_2 \end{tikzcd}$$

Figure 6.3: This manifold (with corners) is a 2-morphism between two 1-morphisms in $\mathbf{Bord}_{0,2}$.

Basically we need to associate some kind of object to spaces of codimension 2. To formalize this, we define some kind of 2-category $\mathbf{Bord}_{n-2,n}$. The objects are closed $(n-2)$-manifold N, morphisms are $(n-1)$-manifolds with boundary $N_1 \sqcup N_2$, and 2-morphisms are n-manifolds with corners. See fig. 6.3 for an example of a 2-morphism between two 1-morphisms in $\mathbf{Bord}_{0,2}$.

A 2-*tier TFT* is a symmetric monoidal functor:

$$Z\colon\ (\mathbf{Bord}_{n-2,n}, \sqcup) \to (\mathcal{C}, \otimes) \tag{6.26}$$

where $(\mathcal{C}, \otimes)$ is some symmetric monoidal 2-category. Beyond being symmetric monoidal, we want this functor to recover the "1-tiered" (i.e. Atiyah-Segal) formalism as follows. The empty $(n-2)$-manifold $\emptyset_{n-2}$ is the unit of $\mathbf{Bord}_{n-2,n}$. This functor being symmetric monoidal implies that this gets sent to the unit $1 \in \mathcal{C}$. The endomorphisms of the empty $(n-2)$-manifold are

$$\mathrm{End}_{\mathbf{Bord}_{n-2,n}}(\emptyset_{n-2}) = \mathbf{Bord}_{n,n-1}\ , \tag{6.27}$$

since bordisms between the empty $\{n-2\}$-manifold and itself comprise closed $(n-1)$-manifolds. So one thing we might require is that $\mathcal{C}$ is a "delooping" of $\mathbf{Vect}$. By this we just mean that:

$$\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}}) = \mathbf{Vect}\ . \tag{6.28}$$

In this case the functor Z determines a map:

$$\mathrm{End}_{\mathbf{Bord}_{n-2,n}}(\emptyset_{n-2}) = \mathbf{Bord}_{n,n-1} \to \mathbf{Vect} = \mathrm{End}(1_{\mathcal{C}})\ . \tag{6.29}$$

Whatever $\mathcal{C}$ is, we get the locality we were interested in: if $M = M_1 \sqcup_N M_2$, then functoriality tells us that

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "Z(M_1)"] \arrow[rr, bend right, "Z(M)"] & Z(N) \arrow[r, "Z(M_2)"] & 1_{\mathcal{C}} \end{tikzcd}\ . \tag{6.30}$$

A natural choice for $\mathcal{C}$ is $\mathbf{Cat}_{\mathbb{C}}$. The objects are categories enriched over $\mathbf{Vect}_{\mathbb{C}}$, the 1-morphisms are linear functors, and the 2-morphisms are linear natural transformations. There are other choices one can make, but this is somehow the most basic one. Now we want to extend our field theories to codimension 2 to encode ramification. I.e. we want a way to associate N of codimension 2 to some category, which we're thinking of (informally) as $\mathbf{Shv}(\mathcal{F}(N))$.

Let M be a closed 3-manifold, Z a 4-dimensional TFT, and $K \subset M$ a knot. Then the complement of a tubular knot, M_0, has boundary $\partial M_0 = N$. Then this gives us a functor

$$Z(N) \xrightarrow{Z(M_0)} Z(\emptyset) = \mathbf{Vect}\ . \tag{6.31}$$

The idea is that we start with some vector space $Z(M)$, modify it according to some condition along the boundary to get the vector space $Z(M_0)$. The interior of what we removed, i.e. the torus neighborhood of the knot, written M_1, defines a functor $\mathbf{Vect} \to Z(N)$, and functoriality gives us:

$$\begin{tikzcd} \mathbf{Vect} \arrow[r, "Z(M_1)"] \arrow[rr, bend right, "Z(M)"'] & Z(N) \arrow[r, "Z(M_0)"] & \mathbf{Vect} \end{tikzcd} . \tag{6.32}$$

This is the sense in which extended field theory encodes the boundary conditions we have been discussing.

6.3 Abelian duality examples

Recall abelian duality from chapter 3. Specifically the two-dimensional examples from sections 3.7.2, 3.7.3 and 4.6. Now we will discuss these examples with ramification of codimension 2.

6.3.1 Two-dimensions

Recall the first example exchanged the A-side $H^0\left(S^1, \mathbb{Z}\right)$ with the B-side $H^1\left(S^1, \mathbb{C}^\times\right)$.

On the A-side, fields are given by

$$\mathcal{F}(-) = \mathrm{Maps}(-, \mathbb{Z}) \ , \tag{6.33}$$

so on a connected manifold the fields just look like $\mathbb{Z}$. The only submanifold of codimension 2 is the point, and the assignment to the point is:

$$Z(\mathrm{pt}) = \mathbf{Shv}(\mathbb{Z}) = \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \ , \tag{6.34}$$

i.e. the category of graded vector spaces.

On the B-side, we interpreted $H^1\left(S^1, \mathbb{C}^\times\right)$ as isomorphism classes of $\mathbb{C}^\times$-local systems on S^1. This suggests that our space of fields should be

$$\mathcal{F}(-) = \mathrm{Loc}_{\mathbb{C}^\times}(-) \ . \tag{6.35}$$

If we really just think of the set of isomorphism classes, we won't get a category attached to a point: $\mathrm{Loc}_{\mathbb{C}^\times}(\mathrm{pt})$ is just a single point, so doesn't match the other side. Instead, we have to think of it as a stack. This is still a point, but equipped with an action of $\mathbb{C}^\times$ by automorphisms. I.e. it is the stack:

$$B\mathbb{C}^\times = \mathrm{pt}/\mathbb{C}^\times \ . \tag{6.36}$$

When we take sheaves we get:

$$\mathbf{Shv}(\mathcal{F}(\mathrm{pt})) = \mathbf{Shv}\left(\mathrm{pt}/\mathbb{C}^\times\right) = \mathbf{Rep}\left(\mathbb{C}^\times\right) \simeq \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \ . \tag{6.37}$$

The identification between representations of $\mathbb{C}^\times$ and graded vector spaces is given as follows. Given a representation V of $\mathbb{C}^\times$ we get a grading

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \tag{6.38}$$

where V_n is where $z \in \mathbb{C}^\times$ acts as z^n.

We can think of this as being some version of Pontrjagin duality as follows. We can think of $\mathbf{Shv}(\mathbb{Z})$ as being modules for $\mathbb{C}[\mathbb{Z}]$ equipped with pointwise multiplication:

$$\mathbf{Shv}(\mathbb{Z}) = (\mathbb{C}[\mathbb{Z}], \cdot)\text{-}\mathbf{Mod} . \tag{6.39}$$

Dually, representations of $\mathbb{C}^\times$ can be thought of as modules for $\mathbb{C}[\mathbb{C}^\times]$ with convolution:

$$\mathbf{Rep}\left(\mathbb{C}^\times\right) = \left(\mathbb{C}\mathbb{C}^\times, *\right)\text{-}\mathbf{Mod} . \tag{6.40}$$

And this duality is asserting that these categories are equivalent. On the B-side we're really looking at comodules over the algebraic functions on $\mathbb{C}^\times$, $\mathbb{C}[\mathbb{C}^\times]\text{-}\mathbf{coMod}$, i.e. modules for functions on $\mathbb{Z}$ are comodules for functions on $\mathbb{C}^\times$.

6.3.2 Two-dimensions

Recall sections 3.7.3 and 4.6, where we discuss another example of abelian duality in two dimensions. The A-side is given by functions on $H^1\left(S^1, \mathbb{Z}\right)$, and the B-side is given by functions on $H^0\left(S^1, \mathbb{C}^\times\right)$. This is a basic example of T-duality or mirror symmetry.

On the A-side, we can think of this as:

$$H^1\left(S^1, \mathbb{Z}\right) = H^0\left(S^1, B\mathbb{Z}\right) . \tag{6.41}$$

This suggests that the space of fields is given by:

$$\mathcal{F}(-) = \mathrm{Maps}\left(-, B\mathbb{Z} = S^1\right) , \tag{6.42}$$

so

$$\mathcal{F}(\mathrm{pt}) = S^1 = B\mathbb{Z} . \tag{6.43}$$

So the category we want is some kind of sheaves on this, which we take to be locally constant:

$$\mathbf{Shv}(B\mathbb{Z}) \coloneqq \mathrm{Loc}\left(S^1\right) = \mathbf{Rep}(\mathbb{Z}) = (\mathbb{C}\mathbb{Z}, *)\text{-}\mathbf{Mod} . \tag{6.44}$$

Another name for this category is the (wrapped) Fukaya category of T^*S^1. The punchline is that the symplectic topology of T^*S^1 is the ordinary topology of S^1.

On the B-side, fields on the circle were given by $H^0\left(S^1, \mathbb{C}^\times\right)$, which is telling us that the fields are maps to $\mathbb{C}^\times$ with the discrete topology:

$$\mathcal{F}(-) = \mathrm{Maps}\left(-, {\mathbb{C}^\times}_{\mathrm{disc}}\right) . \tag{6.45}$$

In particular:

$$\mathcal{F}(\mathrm{pt}) = \mathbb{C}^\times , \tag{6.46}$$

and the natural category of sheaves are going to be quasi-coherent:[2]

$$\mathbf{Shv}_B\left(\mathbb{C}^\times\right) \coloneqq \mathbf{QC}\left(\mathbb{C}^\times\right) = \left(\mathbb{C}\left[\mathbb{C}^\times\right], \cdot\right)\text{-}\mathbf{Mod} = \mathbb{C}\left[z, z^{-1}\right]\text{-}\mathbf{Mod} . \tag{6.47}$$

Again, we see that this is mimicking what we saw in the context of Pontrjagin duality. On the A-side we have modules for $\mathbb{C}[\mathbb{Z}]$ with convolution, and on the B-side we have modules

[2] This is in contrast to the locally-constant sheaves we had on the A-side.

for $\mathbb{C}[\mathbb{C}^\times]$ with pointwise multiplication. In other words, we can think of $\mathbb{C}\left[z, z^{-1}\right]$ as functions on $\mathbb{C}^\times$, or as the group algebra of $\mathbb{Z}$, $\mathbb{C}[\mathbb{Z}]$:

$$\text{f'ns on } \mathbb{C}^\times = \mathbb{C}\left[z, z^{-1}\right] = \mathbb{C}[\mathbb{Z}] \ , \tag{6.48}$$

so a module can be thought of either as a local system on the circle, or as a sheaf on $\mathbb{C}^\times$. This is the first example of homological mirror symmetry:

$$\mathrm{Fuk}^{\mathrm{wr}}\left(T^*S^1\right) = \mathrm{Loc}\left(S^1\right) \simeq \mathbf{QC}\left(\mathbb{C}^\times\right) \ . \tag{6.49}$$

6.4 Four-dimensional Maxwell theory

Recall the four-dimensional Maxwell theory from section 4.8.1. This duality exchanges the topology of line bundles on the A-side with the algebraic geometry of flat line bundles on the B-side. Recall that this duality exchanges the A-side $H^2\left(M^3, \mathbb{Z}\right)$ with the B-side $H^1\left(M^3, \mathbb{C}^\times\right)$. Now we want to extend our defects of codimension 1 from sections 4.5.1 and 4.5.1 down to codimension 2, i.e. down to surfaces.

6.4.1 A-side

Let Σ be a surface. On the A-side we had

$$H^2(-, \mathbb{Z}) = \pi_0\left(\mathrm{Map}\left(-, \underbrace{B\,\mathrm{U}(1)}_{K(\mathbb{Z},2)}\right)\right) \ , \tag{6.50}$$

i.e. we were detecting the connected components of some larger thing. This suggests our fields should be something like:

$$\mathcal{F}(-) = \mathrm{Map}(-, B\,\mathrm{U}(1)) \ . \tag{6.51}$$

We will write $\mathrm{Pic}\,\Sigma$ for these:

$$\mathcal{F}(\Sigma) = \mathrm{Pic}\,\Sigma := \mathrm{Map}(\Sigma, B\,\mathrm{U}(1)) \ . \tag{6.52}$$

To see the identification with line bundles more clearly, choose a complex structure on Σ (so it is now a Riemann surface). Then we can consider the space $\mathrm{Pic}\,\Sigma$ of holomorphic/algebraic line bundles on Σ. This is an object of algebraic geometry, but we're only concerning ourselves with the topology. So this space doesn't depend on the complex structure, but choosing one lets us realize this space explicitly. This fits into the short exact sequence:

$$0 \to \mathrm{Jac}\,\Sigma \to \mathrm{Pic}\,\Sigma \xrightarrow{\deg} \mathbb{Z} \to 0 \ , \tag{6.53}$$

where $\mathrm{Jac}\,\Sigma$ consists of degree 0 line bundles. The claim is that $\mathrm{Jac}\,\Sigma$ is a torus of real dimension $2g$, so topologically it is just:

$$\mathrm{Jac}\,\Sigma \simeq \mathbb{C}^g/\mathbb{Z}^{2g} \ . \tag{6.54}$$

Remark 27. Again we're ignoring the automorphisms of these line bundles, which would contribute an extra (stacky) factor of $B\,\mathrm{U}(1)$.

We get an explicit description as follows. The exponential short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^\times \longrightarrow 0 \tag{6.55}$$

gives us:

$$0 \longrightarrow H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}^\times) \longrightarrow H^2(\Sigma, \mathbb{Z}) = \mathbb{Z} \quad , \tag{6.56}$$

so

$$\mathrm{Jac} \simeq K\left(H^1(\Sigma, \mathbb{Z}), 1\right) \quad , \tag{6.57}$$

i.e. a torus with $\pi_1 = H^1(\Sigma, \mathbb{Z})$.

As a result, this abelian category can be written as:

$$\begin{aligned} \mathrm{Loc}(\mathrm{Pic}\,\Sigma) &= \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \otimes \mathrm{Loc}(\mathrm{Jac}\,\Sigma) & (6.58)\\ &= \mathbf{Rep}_{\mathbb{Z}\text{-gr}}\left(\Lambda = H^1(\Sigma, \mathbb{Z})\right) & (6.59)\\ &= \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \otimes \left(\mathbb{C}\left[H^1(\Sigma, \mathbb{Z})\right], *\right) \quad . & (6.60) \end{aligned}$$

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6.4.2 B-side

Now we will check what we do on the B-side, where we are studying the algebraic geometry of $\mathbb{C}^\times$-local systems. I.e. for Σ a surface (not necessarily with a complex structure) we consider:

$$\begin{aligned} \mathrm{Loc}_{\mathbb{C}^\times}(\Sigma) &= \mathrm{Hom}\left(\pi_1(\Sigma), \mathbb{C}^\times\right)/\mathbb{C}^\times & (6.61)\\ &= \mathrm{Hom}\left(H_1(\Sigma, \mathbb{Z}), \mathbb{C}^\times\right)/\mathbb{C}^\times & (6.62)\\ &\simeq \left(H^1(\Sigma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times\right)/\mathbb{C}^\times \quad . & (6.63) \end{aligned}$$

Now we're supposed to attach some kind of category (of sheaves). We're doing algebraic geometry, so the natural category of sheaves is:

$$\mathbf{QC}(\mathrm{Loc}_{\mathbb{C}^\times}\Sigma) = (\mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)], \cdot)\text{-}\mathbf{Mod} \otimes \mathbf{Rep}\left(\mathbb{C}^\times\right) \quad . \tag{6.64}$$

This ring of functions can be written as:

$$\mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)] = \mathbb{C}\left[H^1(\Sigma, \mathbb{Z})\right] \quad . \tag{6.65}$$

Now by Poincaré duality we have $H^1(\Sigma, \mathbb{Z}) \simeq H_1(\Sigma, \mathbb{Z})$. Also, as we have seen, $\mathbf{Rep}\mathbb{C}^\times = \mathbf{Vect}^{\mathbb{Z}\text{-gr}}$.

Now we can compare this to what we had on the A-side in eq. (6.60). The group algebra of a lattice with convolution $(\mathbb{C}\Lambda, *)$ gets identified, via Fourier series, with $(\mathbb{C}[T^\vee], \cdot)$. This means that modules over these two algebras are the same, i.e.

$$\mathrm{Loc}\,K(\Lambda, 1) \simeq \mathbf{QC}\left(T^\vee_{\mathbb{C}}\right) \quad . \tag{6.66}$$

The LHS here is $\mathrm{Loc}\,\mathrm{Jac}(\Sigma)$, and the RHS is $\mathbf{QC}(\mathrm{Loc})$ but ignoring the extra action of $\mathbb{C}^\times$. So the conclusion is that:

$$\mathrm{Loc}\,\mathrm{Pic}(\Sigma) \simeq \mathbf{QC}(\mathrm{Loc}) \quad . \tag{6.67}$$

This is our abelian duality for topological maxwell theory for codimension 2.

Remark 28. This is a first example of a *Fourier-Mukai* transform, i.e. a version of a Fourier transform for categories of sheaves.

Remark 29. This is the statement of the Betti geometric[3] Langlands correspondence for $G = \mathrm{GL}_1$.

6.4.3 Ramified E-M duality

We want to work out a general E-M duality with ramification. This will include Dirac monopoles (section 4.5.1) and solenoids (section 4.8.2). Recall the surface Σ comes to us as the boundary of the 3-manifold $M_0 = M \setminus K$ for $K \subset M$ a knot.

On the A-side, we have the restriction map:

$$\mathrm{Pic}\,(M_0) \xrightarrow{\pi} \mathrm{Pic}\,(\Sigma) \ , \tag{6.68}$$

where again we're just thinking of Pic topologically. Suppose $\mathcal{E} \in \mathrm{Loc}\,(\mathrm{Pic}\,\Sigma)$. Then we get:

$$\pi^*\mathcal{E} \in \mathrm{Loc}\,(\mathrm{Pic}\,(M_0)) \tag{6.69}$$

and we can attach the following vector space to this:

$$\Gamma\,(\mathrm{Pic}\,(M_0)\,, \pi^*\mathcal{E}) \ . \tag{6.70}$$

This was our way of saying that we're modifying $Z\,(M)$ by some ramification data.

On the B-side, we can also restrict:

$$\mathrm{Loc}_{\mathbb{C}^\times} M_0 \xrightarrow{\pi^\vee} \mathrm{Loc}_{\mathbb{C}^\times} \Sigma \tag{6.71}$$

and then we can pull $\mathcal{E}^\vee \in \mathbf{QC}\,(\mathrm{Loc}_{\mathbb{C}^\times} \Sigma)$ back to get:

$$\pi^{\vee *}\mathcal{E}^\vee \in \mathbf{QC}\,(\mathrm{Loc}_{\mathbb{C}^\times} M_0) \tag{6.72}$$

and we can attach the following vector space to this:

$$\Gamma\left(M_0, \pi^{\vee *}\mathcal{E}^\vee\right) \ . \tag{6.73}$$

Then ramified E-M duality exchanges $\mathcal{E}$ and $\mathcal{E}^\vee$ by a Fourier-Mukai transform. In particular it tells us that:

$$\Gamma\left(\mathrm{Pic}\, M_0^3, \pi^*\mathcal{E}\right) \simeq \Gamma\left(\mathrm{Loc}_{\mathbb{C}^\times} M, \pi^{\vee *}\mathcal{E}^\vee\right) \ . \tag{6.74}$$

The sheaves $\mathcal{E}$ and $\mathcal{E}^\vee$ prescribe boundary conditions, e.g. Dirichlet or Neumann. For example $\mathcal{E}^\vee$ might be the structure sheaf of some locus in $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$. And then we're looking at local systems with some particular singularity along that locus. I.e. elements of $\mathrm{Loc}\, M_0$ such that after restricting to the boundary Σ we land in a particular subset.

[3]The "geometric" just means we're looking at the level of categories.

6.5 Ramified class field theory

Let $C/\mathbb{F}_q$ be a curve over a finite field. For a point $x \in C$, we want to "allow ramification" at this point. Consider the punctured disk around x inside the punctured curve:

$$\operatorname{Spec} K_x = D_x^\times \hookrightarrow C \setminus x \ . \tag{6.75}$$

Note that $K_x \simeq \mathbb{F}_{q^n}((t))$. This picture is our analogue of the boundary of a tubular neighborhood of a knot sitting inside of our 3-manifold:

$$N = \partial\,(\text{neighborhood}\,(K)) \hookrightarrow M_0 = M \setminus K \ . \tag{6.76}$$

So now we're excising a point (instead of a knot), and we're trying to understand this punctured disk rather than the boundary of a tubular neighborhood of the knot.

On the B/Galois side, we were looking at. $\mathrm{Loc}_1\,(C \setminus x)$. Whatever these are, they can be restricted to the punctured disk:

$$\begin{aligned}\mathrm{Loc}_1\,(C \setminus x) \to \mathrm{Loc}_1\, D^\times &= \mathrm{Hom}_{\mathrm{cts}}\left(\mathrm{Gal}\,\overline{K}_x/K_x, \overline{\mathbb{Q}_\ell^\times}\right) &(6.77)\\ &= \mathrm{Hom}_{\mathrm{cts}}\left(\mathrm{Gal}^{\mathrm{ab}}\,\overline{K}_x/K_x, \overline{\mathbb{Q}_\ell^\times}\right)\ . &(6.78)\end{aligned}$$

I.e. we're studying characters of the abelianized Galois group, but they're arising as local systems on $D^\times$. So now the idea is that we're going to study local systems on $C \setminus x$ which are equipped with particular ramification data around the puncture.

On the A-side we are looking at $\operatorname{Pic} C$. Recall we had the adélic description of this from section 5.2.2:

$$\begin{aligned}\operatorname{Pic} C &= F^\times \backslash \prod_{y\in C}{}' K_y^\times \Big/ \prod_{y \in C} \mathcal{O}_Y^\times &(6.79)\\ &= F^\times \backslash \prod{}' \mathbb{Z}\ . &(6.80)\end{aligned}$$

We have a similar description for the punctured curve:

$$\operatorname{Pic}\,(C \setminus x) = F^\times \backslash \prod_{y \in C\setminus x}{}' K_y^\times \Big/ \prod_{y\in C\setminus x} \mathcal{O}_y^\times\ . \tag{6.81}$$

We can write this as:

$$\begin{aligned}\operatorname{Pic}\,(C \setminus x) &= F^\times \backslash \prod_{y \in C\setminus x}{}' K_y^\times \Big/ \prod_{y\in C\setminus x} \mathcal{O}_y^\times &(6.82)\\ &= \operatorname{Pic} C / \left(\mathbb{Z}_x = K_x^\times/\mathcal{O}_x^\times\right)\ , &(6.83)\end{aligned}$$

where we're modding out by "adding or subtracting copies of this point x". Then we can also write this as:

$$\begin{aligned}\operatorname{Pic}\,(C \setminus x) &= F^\times \backslash \prod_{y \in C\setminus x}{}' K_y^\times \Big/ \prod_{y\in C\setminus x} \mathcal{O}_y^\times &(6.84)\\ &= \operatorname{Pic} C / \left(\mathbb{Z}_x = K_x^\times/\mathcal{O}_x^\times\right) &(6.85)\\ &= \operatorname{Pic}\,(C, \widehat{x})\,/K_x^\times &(6.86)\end{aligned}$$

where $\mathrm{Pic}\,(C, \widehat{x})$ is defined to be line bundles on C equipped with a trivialization on D_x. So we're modding out by this trivialization on the right.

$\mathrm{Pic}\,(C, \widehat{x})$ maps to $\mathrm{Pic}\, C$ with fiber given by changes of trivialization, and with intermediates $\mathrm{Pic}\,(C, nx)$ for $n \in \mathbb{Z}^+$:

$$\begin{tikzcd} \mathrm{Pic}\,(C, \widehat{c}) \arrow[rr, "\mathcal{O}_X^\times"] \arrow[dr] & & \mathrm{Pic}\,(C) \\ & \mathrm{Pic}\,(C, nx) \arrow[ur] & \end{tikzcd} , \tag{6.87}$$

where $\mathrm{Pic}\,(C, nx)$ consists of objects of $\mathrm{Pic}\,(C)$ with trivialization of order n around x. This story is telling us that to study the punctured curve, we don't need to puncture. We can study bundles over C equipped with a trivialization, as long as we understand how $K_x^\times$ acts.

In any case, $\mathrm{Pic}\,(C, \widehat{x})$ has an action of $K_x^\times$. So we're going to look at functions on $\mathrm{Pic}\,(C, \widehat{x})$ and decompose them under this action. From the TFT point of view we said this a little differently. Namely, we have a restriction:

$$\begin{tikzcd} \mathrm{Pic}\,(C \setminus x) \arrow[r, equal] \arrow[d] & \mathrm{Pic}\,(C, \widehat{x})\,/K^\times \arrow[d] \\ \mathrm{Pic}\,(D_x^\times) \arrow[r, equal] & \mathrm{pt}/K^\times \end{tikzcd} , \tag{6.88}$$

and we can take a sheaf and take global sections on $\mathrm{Pic}\,(C, \widehat{x})$, i.e. push forward to $\mathrm{pt}/K_x^\times$. Studying this picture is equivalent to studying the action of $K^\times$ on $\mathrm{Pic}\,(C, \widehat{x})$.

Number theorists will tell you that you don't want "infinite level structure". We have $K = \mathbb{F}_q\,((t))$. Then we have:

$$K^\times = \mathbb{F}_q\,((t))^\times \supset \mathcal{O}^\times = \mathbb{F}_q\,[[t]] \ . \tag{6.89}$$

This also contains these subgroups:

$$K^\times \supset \mathcal{O}^\times \supset \mathcal{O}^{\times\,(1)} \supset \cdots \supset \mathcal{O}^{\times\,(n)} \supset \cdots \tag{6.90}$$

where being in $\mathcal{O}^{\times\,(n)}$ just means you're of the form $(1 + t^n)\,(-)$. Before, the quotient $K^\times/\mathcal{O}^\times = \mathbb{Z}$ gave us divisors. So in the unramified situation we had $\mathbb{Z}$ acting on $\mathrm{Pic}\,(C)$. Now when we add level structure, i.e. we pass from $\mathrm{Pic}\,(C)$ to $\mathrm{Pic}\,(C, nx)$, we have an action of $K^\times/\mathcal{O}^{\times\,(n)}$.

So what we do is look at *smooth* representations of $K^\times$, which means every vector is fixed by $\mathcal{O}^{\times\,(N)}$ for $N \gg 0$. This is equivalent to topological continuity if we give the thing it's acting on the discrete topology.

Now we want to establish an analogue to the picture that for a 3-manifold M_0 with boundary $\partial M_0 = \Sigma$, we get

$$Z\,(M_0) \in Z\,(\Sigma) \ . \tag{6.91}$$

The analogue here is as follows. To $C/\mathbb{F}_q$ and $x \in C$ we can attach

$$\mathrm{Pic}\,(C, \widehat{x}) \ . \tag{6.92}$$

Now we want to consider

$$\mathbb{C}\,[\mathrm{Pic}\,(C, \widehat{x})] \ , \tag{6.93}$$

specifically thought of as a representation of $K_x^\times$. This can be thought of as taking the constant sheaf $\underline{k}_{\mathrm{Pic}(C\setminus x)}$ and pushing it forward to get

$$\pi_* \underline{k} \in \mathbf{Shv}\left(\mathrm{pt}/K^\times\right) = \mathrm{Pic}\left(D^\times\right) \ , \tag{6.94}$$

i.e. a representation of $K^\times$.

So on the A-side we have a representation of $K^\times$ coming from Pic with infinite level structure, $\mathrm{Pic}\,(C, \widehat{x})$ (or some finite level structure $\mathrm{Pic}\,(C, nx)$). On the B-side we have a sheaf $\mathcal{O}_{\mathrm{Loc}(C\setminus x)}$ which we pushforward to get a sheaf on $\mathrm{Loc}_1\,(D^\times)$, i.e. a module for functions on $D^\times$. Now we need the analogue of the Fourier-Mukai transform from remark 28. This is the content of local class field theory.

Global class field theory says that:

$$\mathrm{Gal}^{\text{ab, ur}} \simeq \widehat{\mathrm{Pic}} \ , \tag{6.95}$$

where $\widehat{\mathrm{Pic}}$ denotes the profinite completion. The statement of local class field theory says that there is a canonical map:

$$K_x^\times \to \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K_x}/K_x\right) \ , \tag{6.96}$$

which is an isomorphism after profinitely completing:

$$\widehat{K_x^\times} \xrightarrow{\sim} \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K_x}/K_x\right) \ . \tag{6.97}$$

This tells us that the group ring of $K^\times$ is Pontrjagin dual to:

$$\mathrm{Hom}\left(\mathrm{Gal}^{\mathrm{ab}}\,\overline{K_x}/K_x, \overline{\mathbb{Q}_\ell}^\times\right) = \mathrm{Hom}\left(\mathrm{Gal}\,\overline{K_x}/K_x, \overline{\mathbb{Q}_\ell}^\times\right) = \mathrm{Loc}_1\,(C) \ . \tag{6.98}$$

In other words:

$$\left(\overline{\mathbb{Q}_\ell}^\times\left[K^\times\right], *\right) \simeq \left(\overline{\mathbb{Q}_\ell}^\times\left[\mathrm{Loc}_1\right], \cdot\right) \ . \tag{6.99}$$

This is analogous to when we exchanged $\pi_1\,\mathrm{Jac} = \Lambda$ with $\mathbb{C}\left[T^\vee\right]$ in the case of a Riemann surface. Now passing to modules we get:

$$\mathbf{Rep}\left(K^\times\right) = \mathbf{QC}\left(\mathrm{Loc}_1\, C\right) \ . \tag{6.100}$$

To make a statement like this technically precise requires a lot of work. So the local statement is that local systems on the punctured disk have to do with representation of $K^\times$. And the global statement is that the vector space attached to a punctured surface, i.e. a representation of $K^\times$, is identified with local systems with a prescribed ramification.

6.6 Ramification for number fields

The statements are very similar for number fields. Local class field theory says that for a non-archimedean place v of F, i.e. a prime in $\mathcal{O}_F$, there is a natural map:

$$K_v^\times \to \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K}/K\right) \tag{6.101}$$

which is an isomorphism on the profinite completion:

$$\widehat{K^\times} \xrightarrow{\sim} \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K}/K\right) \ . \tag{6.102}$$

E.g. think $K_v = \mathbb{Q}_p$. We can interpret this as identifying representations of $K^\times$ with modules for functions on GL_1 Galois representations.

The global class field theory says the following. The analogue to $\mathrm{Pic}\,(C, \widehat{x})$ is:

$$F^\times \backslash \prod_v K^\times / \prod_{v \notin S} \mathcal{O}^\times \ , \tag{6.103}$$

and functions on this has an action of:

$$\prod_{v \in S} K_v^\times \ . \tag{6.104}$$

So we can decompose them under this action. This will match with functions on characters of Galois groups ramified at S. Lecture 13; March 9, 2021

Chapter 7

Geometric class field theory

Let Σ be a Riemann surface. Then $\operatorname{Pic}\Sigma$ fits into the following short exact sequence

$$0 \to \operatorname{Jac}\Sigma \to \operatorname{Pic}\Sigma \to \mathbb{Z} \to 0 \tag{7.1}$$

and the Jacobian is a lattice, i.e. it is

$$V/H^1(\Sigma,\mathbb{Z}) \tag{7.2}$$

for some vector space V. This means that, by Poincaré duality:

$$\pi_1(\mathrm{Jac}) \simeq H_1(\Sigma,\mathbb{Z}) = \pi_1^{\mathrm{ab}}(\Sigma)\ , \tag{7.3}$$

so $\pi_1(\mathrm{Jac})$ is just the abelianization of $\pi_1(\Sigma)$.

The upshot is that for an abelian group A we have:

$$\operatorname{Loc}_A(\Sigma) = \{\pi_1(\Sigma) \to A\}/\sim \tag{7.4}$$

which is the same as

$$\operatorname{Loc}_A(\operatorname{Jac}\Sigma) = \{\pi_1(\mathrm{Jac}) \to A\}/\sim \tag{7.5}$$

since maps to an abelian group do not detect if the source is abelianized or not. I.e. abelian A-covers of Σ and abelian A-covers of $\operatorname{Jac}(\Sigma)$ are in bijection.

We can understand a finite abelian group via its characters, so we might as well take $A = \mathrm{GL}_1\mathbb{C} = \mathbb{C}^\times$. E.g. we can write

$$\operatorname{Loc}_A\Sigma = \operatorname{Hom}(A^\vee, \operatorname{Loc}_1\Sigma)\ . \tag{7.6}$$

This is just to say that the main content here is that:

$$\operatorname{Loc}_1\Sigma \simeq \operatorname{Loc}_1\left(\mathrm{Jac} = \mathrm{Pic}^0\right)\ . \tag{7.7}$$

We can express this in a more canonical way. The most important ingredient is the Abel-Jacobi map:

$$\Sigma \to \operatorname{Pic}^1\Sigma\ . \tag{7.8}$$

A point $x \in \Sigma$ goes to the divisor $1 \cdot x \in \operatorname{Pic}^1\Sigma$. Of course $\mathrm{Pic}^1 \simeq \mathrm{Pic}^0$. The way Abel and Jacobi would have written this is: given a basepoint $e \in \Sigma$ we get a map (which is also commonly called the Abel-Jacobi map)

$$\Sigma \to \mathrm{Jac} \tag{7.9}$$

where x maps to $x - e$.

The pullback is:

$$\mathrm{AJ}^* \colon \mathrm{Loc}_1\left(\mathrm{Pic}^1\right) \xrightarrow{\sim} \mathrm{Loc}_1(\Sigma) \ , \tag{7.10}$$

so AJ^* identifies A-covers of Pic^1 with A-covers of Σ. So this is a more canonical way to write this bijection from before.

7.1 Albanese property

What we have seen so far is really expressing something called the *Albanese property* of the Jacobian. For higher dimensional varieties, there are two different abelian varieties we can attach to them: the Picard and the Albanese, and they are dual. The idea is that this is not fundamentally a fact about divisors/line bundles but rather about 0-dimensional cycles. They just turn out to coincide here in dimension 1.

Let A be an abelian group scheme.[1] Given a map of schemes $\Sigma \to A$ for A an abelian group, the *Albanese property* says that this factors as:

$$\begin{tikzcd} \Sigma \arrow[rr] \arrow[dr, "\mathrm{AJ}_e"'] & & A \\ & \mathrm{Pic} \arrow[ur, dashed] & \end{tikzcd} \ . \tag{7.11}$$

In other words

$$\mathrm{AJ}^* \colon \mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Pic}\,\Sigma, A) \xrightarrow{\sim} \mathrm{Map}(\Sigma, A) \ . \tag{7.12}$$

We can think of this as saying that Pic is the abelian group scheme "freely generated" by Σ.

Remark 30. For X smooth and projective there is an abelian variety $\mathrm{Alb}(X)$, called the *Albanese*, satisfying the above universal property. (Usually this is defines as 0-cycles modulo some equivalence relation.) Then the classical statement is that there is a canonical map $X \to \mathrm{Alb}(X)$. For a curve the Albanese happens to be the same as the Jacobian. The Albanese is a torus whose fundamental group is H_1. This is the dual abelian variety to the Picard. Accordingly, Pic^0 has fundamental group H^1.

7.2 Dold-Thom theorem

The Dold-Thom theorem relates homology of a pointed space (X, x) to $\mathrm{Sym}^\infty(X, x)$. We can think of $\mathrm{Sym}^\infty(X, x)$ as the "free commutative monoid" built out of X. Specifically the theorem says that the homotopy groups of this are the homology of X. Now we want to state the algebro-geometric analogue of this.

[1] Classically one might have taken A to be an abelian variety. The most modern version might be to take A to be an abelian group stack.

We have the following maps:

$$\begin{aligned}
\Sigma &\hookrightarrow \mathrm{Pic}^1 \\
\mathrm{Sym}^2\,\Sigma &\longrightarrow \mathrm{Pic}^2 \\
x, y &\longmapsto x + y \\
&\cdots \\
\mathrm{Sym}^d\,\Sigma &\longrightarrow \mathrm{Pic}^d
\end{aligned} \tag{7.13}$$

which assemble to give a map:

$$\mathrm{AJ}^\bullet\colon\ \mathrm{Sym}^\bullet(\Sigma) \to \mathrm{Pic}^\bullet(\Sigma)\ . \tag{7.14}$$

An element of $\mathrm{Sym}^\bullet\,\Sigma$ is what is known as an *effective divisor*. I.e. it's some linear combination of points with positive coefficients. This is the same as a line bundle $\mathcal{L}$ and a holomorphic section $\sigma \in \Gamma(\mathcal{L})$. I.e. it has no poles, but it does have some zeros determining a weighted combination of points. From this point of view, the $\mathrm{AJ}^\bullet$ map is just forgetting the section. In other words, $\mathrm{AJ}^\bullet$ has fibers given by:

$$\left(\mathrm{AJ}^\bullet\right)^{-1}(\mathcal{L}) = \mathbb{P}\left(H^0(\Sigma, \mathcal{L})\right)\ , \tag{7.15}$$

where we take the projectivization since a divisor only determines a section up to a scalar.

The Riemann-Roch theorem says that for $d > 2g - 2$ then $H^1(\Sigma, \mathcal{L}) = 0$, so we just get a projective space bundle $\mathbb{P}^{d-g}$. So if we look at high enough degrees, these spaces $\mathrm{Sym}^\bullet$ and Pic look more similar. The point is that these projective spaces are simple-connected, so $\mathrm{AJ}^\bullet$ is an isomorphism on π_1 for $d \gg 0$.

The idea is that we want to form a better understanding of the relationship between $\mathrm{Loc}(\mathrm{Pic})$ and $\mathrm{Loc}_1\,\Sigma$. Recall that:

$$\mathrm{Loc}(\mathrm{Pic}) \simeq \mathbf{QC}(\mathrm{Loc}_1\,\Sigma)\ . \tag{7.16}$$

This is what we want to wrap our heads around. The problem was that Σ only saw a single component of Pic. But to understand $\mathrm{Loc}_1\,\Sigma$, we have to relate the different components together. This is what the statement of geometric class field theory will say.

7.3 Recall: unramified CFT

Recall the unramified CFT for a curve $C/\mathbb{F}_q$. The statement was that rank 1 local systems on C correspond to characters of Pic. Explicitly we started with $E \in \mathrm{Loc}_1(C)$ and we attached χ_E, a character of Pic. Recall that being a character means:

$$\chi_E(\mathcal{L} \otimes \mathcal{M}) \simeq \chi_E(\mathcal{L}) \cdot \chi_E(\mathcal{M})\ . \tag{7.17}$$

But how does this relate to E? Recall Pic is generated by $\mathbb{Z}x$ for $x \in C$. Then we can characterize χ_E as satisfying the following. We know:

$$\chi_E(\mathcal{L}(x)) = \chi_E(\mathcal{L} \otimes \mathcal{O}(x)) \tag{7.18}$$

where $\mathcal{O}(x)$ is the line bundle with associated divisor $1 \cdot x$. By the character property:

$$\chi_E(\mathcal{L}(x)) = \chi_E(\mathcal{L} \otimes \mathcal{O}(x)) \tag{7.19}$$
$$= \chi_E(\mathcal{O}(x)) \cdot \chi_E(\mathcal{L}) \ , \tag{7.20}$$

and this number $\chi_E(\mathcal{O}(x))$ should be the value of the local system E on Fr_x, the Frobenius element at x. Note that $\mathcal{O}(x) = \mathrm{AJ}(x)$, so this value $\chi_E(\mathcal{O}(x))$ is really something about the curve.

7.4 Character sheaves

Now we want to state a geometric[2] version. Let C/k be a curve over any field (or Σ a Riemann surface if we just want $k = \mathbb{C}$). We don't have Frobenius elements anymore, so we can't automatically form this kind of statement, but we can formulate something else. Consider *character local systems* on $\mathrm{Pic}(C)$. These satisfy a multiplicative property, just as ordinary characters did. Specifically a *character local system* is a local system $\chi \in \mathrm{Loc}(\mathrm{Pic})$ equipped with an isomorphism:

$$\mu^* \chi \xrightarrow{\sim} \chi \boxtimes \chi \tag{7.21}$$

where $\mu\colon \mathrm{Pic} \times \mathrm{Pic} \to \mathrm{Pic}$. This also has to satisfy an associativity relation.

Digression 4. We can rewrite the definition (7.17) of a character χ_E as insisting that:

$$\mu^* \chi_E = \chi_E \boxtimes \chi_E \ . \tag{7.22}$$

So we can insist on the same condition on sheaves. I.e. that

$$\mu^* \chi \xrightarrow{\sim} \chi \boxtimes \chi \tag{7.23}$$

or equivalently

$$\chi|_{\mathcal{L} \otimes \mathcal{M}} \simeq \chi|_{\mathcal{L}} \otimes \chi|_{\mathcal{M}} \ . \tag{7.24}$$

We can make this definition of a character sheaf on anything with sheaves living on it. The claim is that they are always rank 1. The idea is that

$$\chi(0 + a) \cong \chi(a) \ , \tag{7.25}$$

so $\chi(0)$ is 1-dimensional, and there is a unital property that identifies it with k. Then $\chi(a - a) \cong \chi(0) \cong \chi(a) \to \chi(-a)$. This means the total space is a $k^\times$ torsor over A:

$$\begin{tikzcd} A_\chi = \mathrm{Tot}(\chi)^\times \arrow[d, "k^\times"] \\ A \end{tikzcd} \ . \tag{7.26}$$

So the statement is that χ is a character iff A_χ is an abelian group, i.e. it defines an extension:

$$0 \to k^\times \to A_\chi \to A \to 0 \ . \tag{7.27}$$

So really we're saying $\chi \in \mathrm{Ext}^1(\mathrm{Pic}, \mathbb{G}_m)$, i.e. χ is a character iff it is a homomorphism

$$\chi\colon \mathrm{Pic} \to B\mathbb{G}_m \ . \tag{7.28}$$

[2]This means we're looking at sheaves.

Let χ be a character local system on $\operatorname{Pic} C$. Then

$$\chi(\mathcal{L} \otimes \mathcal{M}) \simeq \chi(\mathcal{L}) \otimes \chi(\mathcal{M}) \ . \tag{7.29}$$

Just like with ordinary characters, we can look at $\mathcal{L}$ modified at a single point $x \in C$:

$$\chi(\mathcal{L}(x)) = \chi\left(\underbrace{\mathcal{O}(x)}_{=\mathrm{AJ}(x)}\right) \otimes \chi(\mathcal{L}) \ . \tag{7.30}$$

Let $L_\chi = \mathrm{AJ}^* \chi \in \mathrm{Loc}_1 C$. Then this is saying that

$$\chi(\mathcal{L}(x)) \simeq L_\chi|_x \to \chi(\mathcal{L}) \ . \tag{7.31}$$

We can say this in families. We have a diagram:

$$\begin{tikzcd} \operatorname{Pic} \times \operatorname{Pic} \arrow[r, "\otimes"] & \operatorname{Pic} & \mathcal{L}(x) \\ C \times \operatorname{Pic} \arrow[u, hook, "\mathrm{AJ} \times \mathrm{id}"] \arrow[ur, "\mu_C"'] & & \\ (x, \mathcal{L}) \arrow[uurr, mapsto] & & \end{tikzcd} \ , \tag{7.32}$$

and the statement is that we have an identification

$$\mu_C^* \chi \simeq L \boxtimes \chi \ . \tag{7.33}$$

This is called the *Hecke eigen-property*. This should be enough to completely determine the multiplicativity of χ.

7.5 Geometric CFT

Geometric CFT says that the map

$$\chi \mapsto L_\chi = \mathrm{AJ}^* \chi \tag{7.34}$$

gives an equivalence between:

$$\{\text{character local systems on } \operatorname{Pic}\} \simeq \{\text{rank 1 local systems on } C\} \ . \tag{7.35}$$

Before we used some π_1 calculations to learn that local systems on one component of Pic and local systems on C are the same. But now we're looking at local systems on all of Pic which have this very strong property of being characters. And the claim is that if we take such a character, then it is completely determined by the rank 1 local system on C we get by restricting.

We already have a map one way. Inside of Pic we have:

$$\mathrm{AJ}\colon C \hookrightarrow \operatorname{Pic}^1 \subset \operatorname{Pic} \ . \tag{7.36}$$

But we should imagine that we have all of $\mathrm{Sym}^\bullet C \to \mathrm{Pic}$:

$$\begin{tikzcd} \mathrm{Sym}^\bullet C \arrow[r] & \mathrm{Pic} \\ C \arrow[u, hook] \arrow[r, hook, "\mathrm{AJ}"] & \mathrm{Pic}^1 \arrow[u, hook] \end{tikzcd} \quad . \tag{7.37}$$

$\mathrm{Sym}^\bullet C$ is like the free monoid generated by C, whereas $\mathrm{Pic}\,(C)$ is like the free abelian group generated by C. So this map is a kind of group completion. The claim is going to be that any rank 1 local system L on C extends canonically to $\mathrm{Sym}^\bullet C$, and then this descends to Pic, the "group completion".

Proof (Deligne). Start with a local system L on C. First we extend this to $\mathrm{Sym}^\bullet C$. Write a local system $L^{(d)}$ on $\mathrm{Sym}^d C$ defined by:

$$L^{(d)}\left(\sum_1^d x_i\right) = \bigotimes_i L\,(x_i) \ . \tag{7.38}$$

This defines a rank 1 local system on $\mathrm{Sym}^d X$ for any d. More formally, we start L and construct $L^{\boxtimes d}$ on C^d. Then we can project down to $\pi_* L^{\boxtimes d}$ where $\pi\colon C^d \to \mathrm{Sym}^d$. Then we take invariants under the symmetric group:

$$\left(\pi_* L^{\boxtimes d}\right)^{S_d} \ . \tag{7.39}$$

Now we need to descend down to $\mathrm{Pic}\, C$.

The key point is that $\mathbb{P}^{d-g}$ (the fibers of the map $\mathrm{Sym}^d \to \mathrm{Pic}^d$ for $d \gg 0$) are simply connected, so $\mathrm{Sym}^\bullet C \to \mathrm{Pic}\, C$ is an isomorphism in high enough degree. Therefore $L^{(d)}$ descends to Pic^d, i.e.

$$L^{(d)} \simeq \mathrm{AJ}^{d^*}\left(\chi_L^d\right) \tag{7.40}$$

for χ_L^d a local system on Pic^d.

So starting with L we constructed $\chi_L^\bullet$ on Pic^d (for $d \gg 0$) which satisfies the multiplicative property. Then there exists a unique extension of $\chi_L^\bullet$ to all of Pic satisfying the multiplicative property. Then this gives the formula:

$$\chi\,(D) = \chi\,(D_1) \otimes \chi\,(D_2)^{-1} \ , \tag{7.41}$$

for $D = D_1 - D_2$ where D_1 and D_2 are effective divisors of degree $d \gg 0$. $\square$

Remark 31. We followed [Bha] for this proof. There he also mentions that the same proof works for *ramified* geometric CFT. So for a subset of points $S \subset \sigma$:

$$\mathrm{AJ}\colon C \setminus S \hookrightarrow \mathrm{Pic}\,(C, S) \tag{7.42}$$

where $\mathrm{Pic}\,(C, S)$ consists of line bundles trivialized near S.

We want to think of this geometric CFT as a spectral decomposition of $\mathrm{Loc}\,(\mathrm{Pic})$. Lecture 14; March 11, 2021

Chapter 8

Spectral decomposition (take two): Tannakian reconstruction

The basic theme is: if you see commutativity then this indicates geometry. Now we are taking it up a notch. If R is commutative, we can consider the category R-**Mod**. This category is commutative itself under the operation:

$$(M, N) \mapsto M \otimes_R N \ . \tag{8.1}$$

This makes R-**Mod** into a symmetric monoidal category (or $\otimes$-category). In brief this is a category $\mathcal{C}$ which is abelian and has some "niceness properties", e.g. we might insist on including arbitrary direct sums. The main example we will have in mind, and in particular satisfies any niceness properties, is R-**Mod**. So we have a tensor product:

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{8.2}$$

and a unit $1 \in \mathcal{C}$, and then they have to satisfy some axioms: the identity axiom,

$$1 \otimes (-) \simeq (-) \otimes 1 \ , \tag{8.3}$$

if you switch twice you come back to the identity:

$$\begin{tikzcd} M \otimes N \arrow[r, "\sim"] \arrow[rr, bend right, "\mathrm{id}"'] & N \otimes M \arrow[r, "\sim"] & M \otimes N \end{tikzcd} \ , \tag{8.4}$$

and then we have associativity (pentagon identity) and commutativity (hexagon identity) conditions on $\otimes$.

Before we just used R-modules for spectral decomposition. Now we're viewing R-**Mod** itself as some kind of (categorified) commutative ring. For an affine scheme $X = \operatorname{Spec} R$, we can basically define:

$$\mathbf{QC}\,(X) = R\text{-}\mathbf{Mod} \ , \tag{8.5}$$

and the tensor products match too. But not many schemes are affine. This is, in some sense, the main difference between algebraic geometry and differential topology: in differential topology C^∞ functions can capture any space, but in algebraic geometry (algebraic) functions only capture a small collection of varieties. However, in algebraic geometry, when we replace functions by sheaves we get a lot more flexibility. For any scheme X we have

$$X = \varinjlim \operatorname{Spec} R_i \ , \tag{8.6}$$

i.e. X is built out of affines. Similarly:

$$\mathbf{QC}\,(X) = \varprojlim R_i\text{-}\mathbf{Mod} \ . \tag{8.7}$$

The key point is that if we have a map:

$$\operatorname{Spec} S = Y \xrightarrow{f} X = \operatorname{Spec} R, \tag{8.8}$$

i.e. a map $R \to S$, then we get a functor:

$$(-) \otimes_R S \colon R\text{-}\mathbf{Mod} \xrightarrow{f^*} S\text{-}\mathbf{Mod} \ , \tag{8.9}$$

i.e. a functor:

$$f^* \colon \mathbf{QC}\,(X) \to \mathbf{QC}\,(Y) \ . \tag{8.10}$$

The key fact is that f^* is actually a *symmetric monoidal functor*. I.e. the tensor product goes to the tensor product, the unit R goes to the unit S, preserving all the extra structure. I.e. we can build $\mathbf{QC}\,(X)$ out of the local pieces R_i-$\mathbf{Mod}$ as a *symmtric monoidal* category, i.e. the tensor products for R_i-$\mathbf{Mod}$ induce the correct tensor product globally.

This is all to say that $\mathbf{QC}\,(X)$ has the universal property that whenever I have an affine $f \colon \operatorname{Spec} R \to X$, we get a functor $f^* \colon \mathbf{QC}\,(X) \to R$-$\mathbf{Mod}$, and this preserves tensor the product. In other words we have an operation:

$$\mathbf{QC} \colon \mathbf{Sch} \to (\otimes\text{-}\mathbf{Cat})^{\mathrm{op}} \ . \tag{8.11}$$

This is the categorified analogue of:

$$\mathcal{O} \colon \mathbf{Spaces} \to (\mathbf{cRing})^{\mathrm{op}} \ . \tag{8.12}$$

Remark 32. We can define $\mathbf{QC}$ on a bigger source category, e.g. a category of stacks, **Stacks**, or even prestacks. The point is just that the objects can be formally built as colimits of affines.

So we have a big source of $\otimes$-categories from algebraic geometry. We can think of this as the *geometric* source of $\otimes$-categories. This is our home base. For us, we will “understand” a $\otimes$-category when it is sheaves on some kind of space. We will test arbitrary “commutative rings” by comparing them to these “basic” geometric examples of the form R-$\mathbf{Mod}$. This means that the “questions” we ask will be geometrically biased. We have this functor:

$$\mathbf{QC} \colon \mathbf{Stacks} \to (\otimes\text{-}\mathbf{Cat})^{\mathrm{op}} \ , \tag{8.13}$$

and we want to construct a right adjoint:

$$\operatorname{Spec} \colon (\otimes\text{-}\mathbf{Cat})^{\mathrm{op}} \to \mathbf{Stacks} \ . \tag{8.14}$$

So we start with a $\otimes$-category $\mathcal{C}$, and we want to construct some algebro-geometric space (e.g. a stack) called $\operatorname{Spec} \mathcal{C}$. This is the best approximation of $\mathcal{C}$ within the world of algebraic geometry. Technically the adjunction says that $\operatorname{Spec}(\mathcal{C}, \otimes, 1)$ is characterized by the property that:

$$\operatorname{Hom}_{\mathbf{Stacks}}(X, \operatorname{Spec}\mathcal{C}) = \operatorname{Hom}_{\otimes\text{-}\mathbf{Cat}}(\mathcal{C}, \mathbf{QC}(X)) \ . \tag{8.15}$$

This is analogous to how we characterized Spec of a ring in eq. (2.1) in section 2.1. This boils down to:

$$\operatorname{Hom}_{\mathbf{Stacks}}(\operatorname{Spec} R, \operatorname{Spec}\mathcal{C}) = \operatorname{Hom}_{\otimes\text{-}\mathbf{Cat}}(\mathcal{C}, R\text{-}\mathbf{Mod}) \ , \tag{8.16}$$

so we define this as a functor on rings:

$$\mathbf{cRing} \to \mathbf{Grpd} \ , \tag{8.17}$$

by sending

$$R \mapsto \operatorname{Spec}\mathcal{C}(R) \tag{8.18}$$

where

$$\operatorname{Spec}\mathcal{C}(R) := \operatorname{Hom}_{\otimes\text{-}\mathbf{Cat}}(\mathcal{C}, R\text{-}\mathbf{Mod}) \ . \tag{8.19}$$

For references see [Lur04] and [BHL17]. cite

The adjunction implies that there is a canonical map for any X:

$$X \to \operatorname{Spec}\mathbf{QC}(X) \ . \tag{8.20}$$

We can think of this as a sort of a 1-affinization (or "Tannakization") of X. This is just like how we always have a map to the usual affinization:

$$X \to \operatorname{Spec}\mathcal{O}(X) \ , \tag{8.21}$$

which was the best affine approximation of X.

Now let's check that this recovers our old notion of Spec. Suppose $\mathcal{C} = S\text{-}\mathbf{Mod}$ for $S \in \mathbf{cRing}$, and consider a $\otimes$-functor:

$$F : \mathcal{C} \to R\text{-}\mathbf{Mod} \ . \tag{8.22}$$

This tells us that $1 \in \mathcal{C}$ goes to $R \in R\text{-}\mathbf{Mod}$. Because this is a functor we get a map

$$\operatorname{End}(1) \to \operatorname{End}_{R\text{-}\mathbf{Mod}}(R) = R \ . \tag{8.23}$$

We can think of $\operatorname{End}(1)$ as being a commutative ring attached to $\mathcal{C}$. I.e. we have a map:

$$\operatorname{Hom}_{\otimes}(\mathcal{C}, R\text{-}\mathbf{Mod}) \to \operatorname{Hom}_{\mathbf{cRing}}(\operatorname{End}(1), R) \ , \tag{8.24}$$

and elements of the target are easier to understand. If $\mathcal{C} = S\text{-}\mathbf{Mod}$, then $\mathcal{C}$ is *generated* by $1 = S$. This means we can build (resolve) any S-modules by resolutions starting with S. Another way to think about this is that: $\operatorname{Hom}_{S\text{-}\mathbf{Mod}}(S, -)$ is the underlying abelian group. The upshot is that any $\otimes$-functor $S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ comes from a ring map $S \to R$, so

$$\operatorname{Spec} S\text{-}\mathbf{Mod} = \operatorname{Spec} S \ . \tag{8.25}$$

I.e. on $\mathcal{C} = S\text{-}\mathbf{Mod}$ our new Spec recovers our old notion of Spec.

The key point is that most schemes/stacks don't have enough functions, so

$$\operatorname{Spec} \mathcal{O}(X) \neq X \ . \tag{8.26}$$

So we want to see if we are any luckier a categorical level up. And it turns out we are: we *do* have enough quasi-coherent sheaves to see most stacks. The main issue is that functions don't push, only distributions/measures do. But once we have categorified we do have a pushforward functor.

Theorem 4. *If X is a* geometric stack *then*

$$X \xrightarrow{\sim} \operatorname{Spec} \mathbf{QC}(X) \tag{8.27}$$

is an isomorphism. I.e. X is Tannakian/categorically affine*.*

The notion of a geometric stack includes anything we encounter in practice. In particular it includes any quasicompact scheme. The definition we will use is that a *geometric stack* is a quasicompact stack with an affine diagonal. In practice this means that we can explicitly build X out of quotients of some affine U by the action of some affine groupoid $\mathcal{G}$. The idea is that we can build it out of affines as long as we insist on some relations coming to us from the groupoids $\mathcal{G}$.

Example 32. If X is a scheme then maps $f\colon \operatorname{Spec} R \to X$ are the same as maps $f^*\colon \mathbf{QC}(X) \to R\text{-}\mathbf{Mod}$.

Example 33. Let $f\colon \operatorname{Spec} k \to X$ be a point of X. Then this corresponds to a map

$$f^*\colon \mathbf{QC}(X) \to \mathbf{Vect}_k \ . \tag{8.28}$$

8.1 Category of representations

Besides $\mathcal{C} = \mathbf{QC}(X)$, the category of representations of a group is the quintessential example of Tannakian duality. Let G be a group. Then the category $\mathbf{Rep}_k(G)$ is a $\otimes$-category. I.e. the objects are k-vector spaces with a G-action, and morphisms are linear maps respecting the G-action. The unit is the trivial rep $1 = k$, and the tensor product is the usual one of vector spaces with the diagonal G-action.

To calculate $\operatorname{Spec}(\mathbf{Rep}(G))$ we will consider some extra structure on this category. We have a *faithful $\otimes$-functor*

$$\mathbf{Rep}_k G \to \mathbf{Vect}_k \ , \tag{8.29}$$

i.e. maps in the source inject into maps in the target. This is called a *fiber functor*, i.e. for any monoidal category $(\mathcal{C}, \otimes, 1)$ a faithful $\otimes$-functor from $\mathcal{C} \to \mathbf{Vect}_k$ is called a *fiber functor*. Having a fiber functor is actually saying that $\operatorname{Spec} \mathcal{C}$ is covered by a point, i.e. we have a faithfully flat map

$$\mathrm{pt} \to \operatorname{Spec} \mathcal{C} \ , \tag{8.30}$$

i.e. as a space this is a point modulo the action of some groupoid.

Remark 33. Having a fiber functor is kind of orthogonal to the example $\mathcal{C} = \mathbf{QC}(X)$. Here we get functors

$$i^*\colon \mathcal{C} = \mathbf{QC}(X) \to \mathbf{Vect}_k \tag{8.31}$$

where $i\colon \operatorname{Spec} k \to X$, but they kill most things. But this makes sense: $\operatorname{Spec}(\mathbf{QC}(X))$ should not be covered by a point in general.

Let G be an affine algebraic group, i.e. G is an affine variety. This implies:

$$\operatorname{Spec}(\mathbf{Rep}(G)) = BG = \mathrm{pt}/G \ . \tag{8.32}$$

This is indeed covered by a point

$$G \rightrightarrows \mathrm{pt} \longrightarrow \mathrm{pt}/G \quad . \tag{8.33}$$

For general $\mathcal{C}$ equipped with some fiber functor F, we get a group $G = \operatorname{Aut}(F)$, and

$$\operatorname{Spec}\mathcal{C} = BG^{\mathrm{proalg}} \ , \tag{8.34}$$

where G^{proalg} is the pro-algebraic completion of G. This is the thing we see out of G by studying algebraic representations.

Now we're going to look for $\otimes$-categories (in nature) which naturally have a fiber functor. From this we can produce a group. This sounds a bit abstract until we realize that this comes up all over the place.

Example 34. Again, we want to consider vector spaces with some extra structure. So consider $\mathbf{Vect}^{\mathbb{Z}\text{-gr}}$, the category of $\mathbb{Z}$-graded vector spaces. The tensor product is given by:

$$(M^\bullet \otimes N^\bullet)_k = \bigoplus M^i \otimes N^{k-i} \ . \tag{8.35}$$

This has a functor:

$$\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \xrightarrow{F} \mathbf{Vect} \ , \tag{8.36}$$

which is a fiber functor. This means we get a group G such that $\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \simeq \mathbf{Rep}G$. In this case $G = \mathbb{G}_m$:

$$\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \simeq \mathbf{Rep}\mathbb{G}_m \simeq \mathbf{QC}(B\mathbb{G}_m) \ , \tag{8.37}$$

which is an equivalence we have seen before.

This is a categorical version of the Fourier duality between $\mathbb{Z}$ and $\mathbb{G}_m$. Similarly we have an equivalence between lattice-graded representations and representations of $T^\vee$:

$$\mathbf{Vect}^{\Lambda\text{-gr}} \simeq \mathbf{QC}(BT^\vee) \ . \tag{8.38}$$

Pontrjagin duality told us that G being compact was dual to $G^\vee$ being discrete. Now the discreteness of $\mathbb{Z}$ is manifested in the semisimplicity of $\mathbf{Vect}^{\mathbb{Z}\text{-gr}}$. This is equivalent to G being reductive. In characteristic 0, being reductive can even be defined to mean that $\mathbf{QC}(\mathrm{pt}/G) = \mathbf{Rep}G$ is semisimple.

Example 35. This is where the name "fiber functor" comes from. Let X be a connected topological space. Out of this we will construct a $\otimes$-category with a fiber functor. The category is $\operatorname{Loc}(X)$, the category of local systems on X. This has a tensor structure given by pointwise tensoring. Now choose a basepoint $x \in X$. This gives a functor:

$$F_x \colon \operatorname{Loc}(X) \xrightarrow{\text{fiber at } x} \mathbf{Vect}_k \ . \tag{8.39}$$

The way we defined the tensor product makes this a tensor functor. X being (reasonable and) connected means F_x is faithful (i.e. if a local system is 0 at x then it is 0 everywhere). This tells us that we get a Tannakian group G so that $\operatorname{Loc}(X) = \mathbf{Rep}(G)$. G is essentially the fundamental group based at x, $\pi_1(X, x)$. The idea is that representations of π_1 on a

k-vector space are the same as k-local systems on X where we send a local system to its monodromy. The formal statement is that:

$$G = \pi_1^{\text{pro-alg}}(X, x) \ , \tag{8.40}$$

i.e. the part of π_1 you see by studying only its finite-dimensional representations.

Example 36. Let $\mathcal{C} = \mathbf{Vect}_k$ for $k \neq \overline{k}$. Then we can tensor with $\overline{k}$ to get a functor:

$$\mathbf{Vect}_k \xrightarrow{(-)\otimes_k \overline{k}} \mathbf{Vect}_{\overline{k}} \ . \tag{8.41}$$

This is a fiber functor and:

$$\mathbf{Vect}_k \simeq \mathbf{Rep}\left(\operatorname{Gal} \overline{k}/k\right) \ . \tag{8.42}$$

8.2 Fourier-Mukai theory

Our next source of examples will come from "Fourier-Mukai" theory. We will follow the same logic from Pontrjagin duality. The point was that a nice source of commutative rings is group algebras of abelian groups. Likewise we can play the Tannakian game for particular tensor-categories. Let G be commutative with product $\mu \colon G \times G \to G$. Then we can always push sheaves, so we get a functor:

$$\mathbf{Shv}(G) \times \mathbf{Shv}(G) \xrightarrow{\mu_*} \mathbf{Shv}(G) \ . \tag{8.43}$$

This sends a pair of sheaves $\mathcal{F}$ and $\mathcal{G}$ to the convolution product $\mathcal{F} * \mathcal{G}$. This operation makes sense for any group, but for G abelian this makes $\mathbf{Shv}(G)$ a $\otimes$-category (i.e. convolution is commutative).

Remark 34. We're being intentionally vague about what we mean by **Shv**. E.g. in topology we might take local systems, in algebraic geometry we might take **QC**, etc. but in any setting this will still be a $\otimes$-category.

We can define the 1-*shifted Cartier/Fourier-Mukai dual* of G to be:

$$G^\vee[1] \coloneqq \operatorname{Spec}(\mathbf{Shv}(G), *) \ . \tag{8.44}$$

Now we're in more of a "Fourier Transform-like setting". We start with G, we pass to $\mathbf{Shv}(G)$, which is a kind of categorified group algebra, and we get a new space/group out of this Tannakian machinery.

Example 37. Let $G = \mathbb{Z}$. Then $\mathbf{Shv}(\mathbb{Z}) = \mathbf{Vect}^{\mathbb{Z}\text{-gr}}$, and the convolution operation is the tensor product of graded vector spaces:

$$(M \otimes N)_n = \bigoplus M_i \otimes N_{n-i} \ . \tag{8.45}$$

Then

$$\operatorname{Spec}(\mathbf{Shv}(\mathbb{Z}), *) = B\mathbb{G}_m = \mathbb{Z}^\vee[1] \ . \tag{8.46}$$

Recall that the ordinary dual was $\mathbb{Z}^\vee \simeq \mathbb{G}_m$.

Lecture 15;
March 23, 2021

8.2.1 Cartier

Consider a setting where Cartier duality works, e.g. G a finite group scheme. Then recall from section 3.6 that the Cartier dual of G is

$$G^{\vee} = \mathrm{Spec}\left(\mathbb{C}G, *\right) \ . \tag{8.47}$$

The point being that functions on G with $*$ are identified with functions on $G^{\vee}$ with $\cdot$. Passing to modules we get:

$$\mathbf{Rep}\left(G\right) \simeq \mathbf{QC}\left(G^{\vee}\right) \ . \tag{8.48}$$

But we can understand $\mathbf{Rep}\left(G\right)$ as sheaves on the classifying space:

$$\mathbf{Rep}\left(G\right) \simeq \mathbf{QC}\left(BG\right) \ . \tag{8.49}$$

If we really want an equivalence of sheaves (like we have an equivalence of functions), then we get: The upshot is that we get an equivalence of sheaves:

$$\mathbf{QC}\left(G^{\vee}\right) \simeq \mathbf{QC}\left(BG\right) \ , \tag{8.50}$$

so this Fourier-Mukai duality exchanges G with B of its ordinary Cartier dual, so this is why we also called the Fourier-Mukai dual the 1-shifted Cartier dual. This is exactly what we saw in example 37 where $\mathbb{Z}$ got exchanged with $B\mathbb{G}_m$.

Let's try to say this a bit more methodically. We want to form a better understanding of $G^{\vee}[1]$. The k-points are $\otimes$-functors from sheaves on G (with convolution) to k-vector spaces:

$$\left(\mathbf{QC}\left(G\right), *\right) \to \left(\mathbf{Vect}_k, \otimes\right) \ . \tag{8.51}$$

Now we want to see what this tells us about the group itself. We're thinking about $\mathbf{Vect}_k$ as:

$$\mathbf{Vect}_k = \mathrm{End}\left(\mathbf{Vect}_k\right) \ . \tag{8.52}$$

The idea is that this map (8.51) is a "categorical 1-dimensional representation" of $\left(\mathbf{QC}\left(G\right), *\right)$, in analogy with a 1-dimensional representation $G \to \mathrm{GL}\left(k\right) \simeq k^{\times}$, or equivalently:

$$\mathbb{C}G \to \mathrm{End}\left(k\right) \simeq k \ . \tag{8.53}$$

So a k-point:

$$\mathbf{QC}\left(G\right) \to \mathrm{End}\left(\mathbf{Vect}_k\right) \ , \tag{8.54}$$

and we can restrict this to the invertible elements G to get a map:

$$G \to \mathrm{Aut}\left(\mathbf{Vect}_k\right) \ . \tag{8.55}$$

Now notice that these automorphisms are just given by tensoring with lines:

$$\mathrm{Aut}\left(\mathbf{Vect}_k\right) = \left(\text{lines}, \otimes\right) = B\mathbb{G}_m \ . \tag{8.56}$$

$B\mathbb{G}_m$ is a group object in the world of stacks, i.e. as a functor it is valued in Picard groupoids (symmetric monoidal groupoids with invertible objects). It has R-points given by:

$$B\mathbb{G}_m\left(R\right) = \left(R\text{-lines}, \otimes\right) \ . \tag{8.57}$$

The upshot of this discussion is that a k-point of $G^\vee[1]$ amounts to a map:

$$G \to B\mathbb{G}_m \ . \tag{8.58}$$

Recall $B\mathbb{G}_m$ showed up in example 37 as the 1-shifted Cartier dual of $\mathbb{Z}$. Now we're saying that $B\mathbb{G}_m$ plays in important role in general.

Recall the points of $\mathrm{Spec}\left(\mathbb{C}G, *\right)$ are *characters* of G:

$$G \to \mathbb{G}_m = \mathrm{Aut}\left(k\right) \ . \tag{8.59}$$

The claim is that the same analysis works a level up. I.e. the points of $\mathrm{Spec}\left(\mathbf{Shv}\left(G\right), *\right)$ are categorical characters/character sheaves:

$$G \to B\mathbb{G}_m = (\mathrm{lines}, \otimes) \ . \tag{8.60}$$

This is the same notion that we discussed in section 7.4. The idea is that it is a consistent attachment of lines $\mathcal{L}_g$ to the elements $g \in G$. I.e. this is a line bundle $\mathcal{L}$ over G. This map is actually a homomorphism so this line bundle is multiplicative, which we wrote before as the condition that:

$$\mu^* \mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L} \ . \tag{8.61}$$

This is just writing $\mathcal{L}_{gh} = \mathcal{L}_g \otimes \mathcal{L}_h$ in families. So the claim is that:

$$G^\vee[1] \cong \mathrm{Hom}_{\mathbf{Grp}}\left(G, B\mathbb{G}_m\right) \tag{8.62}$$

$$= \text{mult. line bundles.} \tag{8.63}$$

Then multiplicative line bundles are given by Ext^1 to $\mathbb{G}_m$:

$$G^\vee[1] = \mathrm{Ext}^1_{\mathbf{Grp}_{\mathrm{Ab}}}\left(G, \mathbb{G}_m\right) \tag{8.64}$$

$$= \text{extensions of } G \text{ by } \mathbb{G}_m \ . \tag{8.65}$$

Remark 35. Usual commutative group schemes and commutative group stacks live in the derived category of sheaves of abelian groups on the category $\mathbf{Sch}^{\mathrm{aff}} = \mathbf{Ring}^{\mathrm{op}}$. This shift in the notation for the Fourier-Mukai dual is literally a shift in this derived category.

8.2.2 Fourier

With this setup we can now write down a version of the Fourier transform. On the product $G \times G^\vee[1]$, there is a tautological line bundle $\mathcal{L}$, because a point in $G^\vee[1]$ specifies a line bundle on G:

$$\begin{tikzcd} & \mathcal{L} \arrow[d] & \\ & G \times G^\vee[1] \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & \\ G & & G^\vee[1] \end{tikzcd} \ . \tag{8.66}$$

Now, for decent[1] G, we can write the *Fourier-Mukai transform*, which is:

$$\begin{aligned} \mathbf{QC}\left(G\right) &\longrightarrow \mathbf{QC}\left(G^\vee[1]\right) \\ F &\longmapsto \pi_{2*}\left(\pi_1^* \mathcal{F} \otimes \mathcal{L}\right) \end{aligned} \ . \tag{8.67}$$

[1] See remark 36.

This is a sheaf-theoretic analogue of the Fourier transform from sections 3.2 and 3.4.

Remark 36 (What do we mean by decent G?). Recall, from section 3.6, that we needed really strict conditions for ordinary Cartier duality. E.g. we might ask for a finite group scheme (Spec of a finite-dimensional k-algebra) in order to really make it really work. Now this duality works in much greater generality. This is because sheaves are much more robust than functions: there are a lot more sheaves and they behave much better than functions.

8.2.3 Mukai

The original example introduced of this, by Mukai, took $G = A$ to be an abelian variety, i.e. an abelian, connected, projective algebraic group. So something like an elliptic curve (compact complex torus with an algebraic structure). For example, we might take the Jacobian of a Riemann surface, $\mathrm{Jac}\,(C)$. The *dual abelian variety*, written A^*, is the dual compact torus. It turns out that:

$$A^* \simeq A^\vee\,[1] \ , \tag{8.68}$$

i.e. A^* is the collection of multiplicative line bundles on A. This means we have a Fourier-Mukai transform which is an equivalence of derived categories

$$D\,(A) \xrightarrow{\sim} D\,(A^*) \ , \tag{8.69}$$

but from now on we will just write $\mathbf{QC}$ for the derived category of quasi-coherent sheaves. The basic problem is that the pushforward in the formula eq. (8.67) is not an exact functor because the fibers are tori, so not affine. So the formula eq. (8.67) is not good to write naively if G is not affine, but we can fix it by deriving everything.

So we have an equivalence:

$$\mathbf{QC}\,(A) \xrightarrow{\sim} \mathbf{QC}\,(A^\vee\,[1]) \ . \tag{8.70}$$

So this is much more robust than down a level. We don't need to be finite, or affine, or anything. This works for everything we will run into. There is a nice article by Laumon, [Lau91], where a lot of examples are worked out from this point of view in great generality.

Remark 37. Note that we're just saying Fourier-Mukai transforms are a certain special type of integral transforms. In some places in the literature, "Fourier-Mukai transform" is used to refer to arbitrary integral transforms. The problem is that *all* reasonable functors on derived categories can be expressed as integral transforms, so it doesn't make sense to call them all Fourier-Mukai. We will only use Fourier-Mukai transform to refer to a transform as above. In particular it should exchange multiplication with convolution, as the ordinary Fourier transform does.

Remark 38 (Spectral decomposition). For $(\mathcal{C}, *)$ acting on $\mathcal{M}$, we get a module category for

$$\mathbf{QC}\,(\mathrm{Spec}\,\mathcal{C}) \ . \tag{8.71}$$

This gives rise to a *sheaf of categories* over $\mathrm{Spec}\,\mathcal{C}$. By Gaitsgory's 1-affineness, this is actually an equivalence between modules over $\mathbf{QC}\,(\mathrm{Spec}\,\mathcal{C})$ and sheaves of categories over $\mathrm{Spec}\,\mathcal{C}$.

Example 38. Let C be a Riemann surface. Then Riemann, Abel, and Jacobi showed that Jac is self-dual:

$$\mathrm{Jac}\,(C) \simeq (\mathrm{Jac}\,(C))^* \ . \tag{8.72}$$

This is a version of Poincaré duality between H_1 and H^1.

We will use $\mathrm{Pic}\,(C)$ to denote the stack of line bundles on C, i.e.

$$\mathrm{Pic}\,(C) = |\mathrm{Pic}\,(C)| \times B\mathbb{G}_m \ , \tag{8.73}$$

where $|\mathrm{Pic}\,(C)|$ is the usual Picard scheme. We can describe $\mathrm{Pic}\,(C)$ as consisting of three parts: the degree, the degree 0 line bundles, and the automorphisms:

$$\mathrm{Pic}\, C \cong \mathbb{Z} \times \mathrm{Jac} \times B\mathbb{G}_m \ . \tag{8.74}$$

When we take the dual we get:

$$(\mathrm{Pic}\,(C))^* = \mathrm{Pic}\,(C)^\vee\,[1] \simeq B\mathbb{G}_m \times \mathrm{Jac} \times \mathbb{Z} \ , \tag{8.75}$$

i.e. Pic is self-dual just like Jac, but in this more interesting way where the automorphisms are exchanged with the degrees. So there is a Fourier-Mukai transform between sheaves on $\mathrm{Pic}\, C$ and itself, where different degrees correspond to a different action of this stabilizer $\mathbb{G}_m$. This example has to do with class field theory. The discussion continues in section 8.2.4.

8.2.4 Betti CFT

The previous example is related to class field theory. So far, e.g. in chapter 7, we have discussed what one might call *Betti class field theory (CFT)*. We studied Pic, as a commutative group, and we attached the category of local systems with convolution:

Starting here, **Loc** denotes the category and Loc denotes the stack. Need to fix earlier.

$$(\mathbf{Loc}\,(\mathrm{Pic})\,, *) \ , \tag{8.76}$$

which is a $\otimes$-category. Taking Spec we get:

$$\mathrm{Spec}\,(\mathbf{Loc}\,(\mathrm{Pic})\,, *) \simeq \mathrm{Loc}_1\, C \ , \tag{8.77}$$

the stack of rank 1 local systems on C.

Maybe more suggestively, this is saying that:

$$(\mathbf{Loc}\,(\mathrm{Pic})\,, *) \simeq (\mathbf{QC}\,(\mathrm{Loc}_1\, C)\,, \otimes) \ . \tag{8.78}$$

We didn't state this, but what we did state, in chapter 7, was that given L a local system, it corresponds to a character local system χ_L. This extends to a Fourier-Mukai transform, which is the equivalence we have above. So we can identify this as saying that the 1-shifted Cartier dual of Pic is Loc.

We can write this more suggestively as follows. Over $\mathrm{Pic} \times \mathrm{Loc}_1$, there is a universal character local system χ:

$$\begin{tikzcd} & \chi \arrow[d] & \\ & \mathrm{Pic} \times \mathrm{Loc}_1 \arrow[dl] \arrow[dr] & \\ \mathrm{Pic} & & \mathrm{Loc}_1 \end{tikzcd} \ . \tag{8.79}$$

This means χ is a local system in the Pic "direction" and a quasi-coherent sheaf in the Loc_1 direction.

Remark 39. This is the sense in which the Betti picture is less symmetric than the de Rham one. In the Betti world, we're treating Pic as something purely topological, and Loc_1 as an object of algebraic geometry. In the de Rham version, the two sides will be much more symmetric, but both sides will be more complicated.

In any case, we get an identification:

$$\mathbf{Loc}\,(\mathrm{Pic}) \simeq \mathbf{QC}\,(\mathrm{Loc}_1)\ , \tag{8.80}$$

by the same integral transform formalism. One of the features of this is as follows. For $x \in C$ we get a homomorphism $\mathbb{Z} \to \mathrm{Pic}$ given by multiples of that point. Correspondingly we have a homomorphism on the other side: $\mathrm{Loc}_1\, C \to B\mathbb{G}_m$ given by $L \mapsto L|_x$. So the above identification is realised in such a way that a map from $\mathbb{Z}$ is dual to a map to $B\mathbb{G}_m$. More generally we can act by $\mathbb{Z}x$ on local systems, i.e. the Hecke modifications $\mathcal{L} \mapsto \mathcal{L}\,(x)$ for $\mathcal{L}$ a line bundle. These are identified under the duality with tensoring with the line bundle W_x on Loc_1. This line bundle is exactly the map $W_x\colon \mathrm{Loc}_1 \to B\mathbb{G}_m$, since a map to $B\mathbb{G}_m$ is the same as a line bundle over the source.

8.2.5 de Rham CFT

Now we will do some cool tricks with this 1-shifted Cartier duality, based on Laumon [Lau91] and Rothstein [Rot96]. The idea is that we will start from this self-duality of Jac, and build up to something that looks like class field theory.

The de Rham space

To do this, we need to find new abelian varieties, which we will do by introducing something called the de Rham space/functor. For X a variety (or scheme or stack, etc.) there is an associated object X_{dR}, called the de Rham space. We can think of it as the quotient of X by the equivalence relation of being infinitesimally close. More formally:

$$X_{\mathrm{dR}} = X/\widehat{\Delta} \tag{8.81}$$

where $\widehat{\Delta}$ is a formal neighborhood of the diagonal. This is not representable as a scheme, but it still makes sense as a functor of points, i.e. we can at least do algebraic geometry on it. But for us, it's really just a placeholder for $\mathcal{D}$-modules in the following sense. Since this is an object of algebraic geometry, we can consider sheaves on it, i.e. sheaves on X which are equivariant for this equivalence relation:

$$\mathbf{QC}\,(X_{\mathrm{dR}}) = \mathbf{QC}\,(X)^{\widehat{\Delta}}\ . \tag{8.82}$$

In other words it has objects given by $\mathcal{F} \in \mathbf{QC}\,(X)$, and an identification

$$\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_y \tag{8.83}$$

for all x and y which are infinitesimally close. This is reminiscent of a flat connection: it gives you exactly this kind of data. So this is a version of sheaves on X equipped with a flat connection. This turns out to be identified with the category of *$\mathcal{D}$-modules on X*:

$$\mathbf{QC}\,(X_{\mathrm{dR}}) = \mathcal{D}_X\text{-}\mathbf{Mod}\ . \tag{8.84}$$

$\mathcal{D}_X$-**Mod** is the category of quasicoherent sheaves $\mathcal{F} \in \mathbf{QC}(X)$ with an action of rings:

$$\mathcal{D}(U) \circlearrowleft \mathcal{F}(U) \ , \tag{8.85}$$

for affine opens U. This is just saying that we have a map:

$$T_X \to \mathrm{End}(\mathcal{F}) \tag{8.86}$$

satisfying the Leibniz rule: for any two vector fields ξ_1 and ξ_2, $[\xi_1, \xi_2]$ acts by the commutator of ξ_1 and ξ_2 (flatness), and they satisfy:

$$\xi f = f\xi + f' \ . \tag{8.87}$$

In other words $\mathcal{F}$ is a quasi-coherent sheaf with a flat connection:

$$\nabla \colon \mathcal{F} \to \mathcal{F} \otimes \Omega^1 \tag{8.88}$$

where $\nabla^2 = 0$.

Back to Fourier-Mukai theory

Let G be an abelian group. Then G_{dR} can be written as a quotient:

$$G_{\mathrm{dR}} = G/\widehat{G} \tag{8.89}$$

where $\widehat{G}$ is the *formal group of* G, i.e. the formal neighborhood of the identity. This is saying that, in the case of a group, x is infinitesimally close to y iff $xy^{-1} \in \widehat{G}$, i.e. xy^{-1} is infinitesimally close to the identity. This allows us to add some "spice" (de Rham) to all of our favorite examples of Cartier duality from section 3.6. Now we can add some spice to all of our favorite examples of Cartier duality.

Recall from example 19 that the Cartier dual of $\mathbb{A}^1$ is given by the formal completion of $\mathbb{A}^1$:

$$\left(\mathbb{A}^1\right)^{\vee} = \widehat{\mathbb{A}^1} \tag{8.90}$$

and more generally:

$$V^{\vee} = \widehat{(V^*)} \ . \tag{8.91}$$

This means the Fourier-Mukai dual is:

$$V^{\vee}[1] = B\widehat{(V^*)} \ . \tag{8.92}$$

Likewise:

$$\widehat{V}^{\vee}[1] = BV^* \ . \tag{8.93}$$

This becomes interesting when we do the following trick. Let V be some vector space. V is an abelian group, so V_{dR} is the quotient:

$$V_{\mathrm{dR}} = V/\widehat{V} \tag{8.94}$$

or we can think of this as a chain complex:

$$V_{\mathrm{dR}} = \left[\widehat{V} \to V\right] \ . \tag{8.95}$$

Now we get something beautifully self-dual:

$$\begin{aligned} V_{\mathrm{dR}}^{\vee}[1] &= \left[\widehat{V} \to V\right]^{\vee}[1] && (8.96)\\ &= \left[\widehat{V^*} \to V^*\right] && (8.97)\\ &= V_{\mathrm{dR}}^* \ . && (8.98) \end{aligned}$$

Writing this another way, we have:

$$\left(V/\widehat{V}\right)^{\vee}[1] = V^*/\widehat{V^*} \ . \tag{8.99}$$

This tells us that $\mathcal{D}$-modules on V and V^* are identified:

$$\mathcal{D}(V) \simeq \mathcal{D}(V^*) \ . \tag{8.100}$$

Example 39. For example:

$$\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle x, \partial_x\rangle / (\partial x - x\partial = 1) \ , \tag{8.101}$$

and on the other hand

$$\mathcal{D}_{\mathbb{A}^{1*}} = \mathbb{C}\langle y, \partial_y\rangle / (\partial y - y\partial = 1) \ . \tag{8.102}$$

But $\mathcal{D}_{\mathbb{A}^1}$ has an interesting automorphism:

$$\begin{aligned} x &\longmapsto \partial_x \\ \partial_x &\longmapsto -x \end{aligned} \ , \tag{8.103}$$

which is exactly what happens under the Fourier transform. This automorphism gives an equivalence between the categories of modules:

$$\mathbb{F}\colon \mathcal{D}_{\mathbb{A}^1}\text{-}\mathbf{Mod} \to \mathcal{D}_{\mathbb{A}^1}\text{-}\mathbf{Mod} \ , \tag{8.104}$$

called the Fourier transform, which is the identity on objects as vector spaces, but changes the action of x and ∂_x by switching them (with a sign).

In general we get a Fourier transform:

$$\mathcal{D}_V\text{-}\mathbf{Mod} \simeq \mathcal{D}_{V^*}\text{-}\mathbf{Mod} \ . \tag{8.105}$$

Example 40 (Relation to usual Fourier transform on $\mathbb{R}$.)**.** For φ a generalized function, it defines a module for $\mathcal{A}_{\mathbb{A}^1}$, $\mathcal{M}_\varphi \in \mathcal{D}_{\mathbb{A}^1}$-**Mod**, which is

$$\mathcal{D}_{\mathbb{A}^1}\text{-}\mathbf{Mod} = \{L\varphi \,|\, L \in \mathcal{D}_{\mathbb{A}^1}\} \subset \text{generalized functions on } \mathbb{R} \ . \tag{8.106}$$

The claim is:

$$\mathbb{F}(\mathcal{M}_\varphi) = \mathcal{M}_{\mathbb{F}(\varphi)} \ . \tag{8.107}$$

So this really is the usual Fourier transform of vector spaces.

Laumon and Rothstein, [Lau91, Rot96], pointed out that you can do the same thing on any abelian variety, so in particular on $\mathrm{Jac}(C)$. So let $A = \mathrm{Jac}(C)$. Then, as usual, $A_{\mathrm{dR}} = A/\widehat{A}$, and the claim is that:

$$A_{\mathrm{dR}}^{\vee}[1] \simeq \mathrm{Conn}_1(C) \ , \tag{8.108}$$

the space of rank 1 flat connections on C. In fact, what we have already seen is enough to show that:

$$A_{\mathrm{dR}} = \left(\mathrm{Jac}/\widehat{\mathbb{A}^1}\right) \ , \tag{8.109}$$

so the dual is an $\mathbb{A}^1$-bundle over Jac, which is what $\mathrm{Conn}_1(C)$ is. So starting from the self-duality of the Jacobian, and formally applying this Cartier duality, we get a statement, $A_{\mathrm{dR}}^{\vee}[1] \simeq \mathrm{Conn}_1(C)$, which looks a lot like in the Betti setting. This implies that (after taking **QC**) we have

$$\mathcal{D}_{\mathrm{Jac}}\text{-}\mathbf{Mod} \simeq \mathbf{QC}(\mathrm{Conn}_1(C)) \ , \tag{8.110}$$

where again there is some χ_L on the left corresponding to any L on the right. This is de Rham geometric CFT.

Part II

Nonabelian

Chapter 9

Nonabelian

Lecture 16;
March 25, 2021

Bibliography

[aHRW10] Astrid an Huef, Iain Raeburn, and Dana P. Williams, *Functoriality of Rieffel's generalised fixed-point algebras for proper actions*, Superstrings, geometry, topology, and C^*-algebras, Proc. Sympos. Pure Math., vol. 81, Amer. Math. Soc., Providence, RI, 2010, pp. 9–25. MR 2681756 19

[Bha] Bhargav Bhatt, *Geometric class field theory.* 91

[BHL17] Bhargav Bhatt and Daniel Halpern-Leistner, *Tannaka duality revisited*, Adv. Math. **316** (2017), 576–612. MR 3672914 94

[CG17] Kevin Costello and Owen Gwilliam, *Factorization algebras in quantum field theory. Vol. 1*, New Mathematical Monographs, vol. 31, Cambridge University Press, Cambridge, 2017. MR 3586504 54

[CKK+19] Hee-Joong Chung, Dohyeong Kim, Minhyong Kim, George Pappas, Jeehoon Park, and Hwajong Yoo, *Erratum: "Abelian arithmetic Chern-Simons theory and arithmetic linking numbers"*, Int. Math. Res. Not. IMRN (2019), no. 18, 5854–5857. MR 4012129 13

[DEF+99] Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten (eds.), *Quantum fields and strings: a course for mathematicians. Vol. 1, 2*, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999, Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. MR 1701618 42

[FMS07] Daniel S. Freed, Gregory W. Moore, and Graeme Segal, *The uncertainty of fluxes*, Comm. Math. Phys. **271** (2007), no. 1, 247–274. MR 2283960 42, 45

[Fre00] Daniel S. Freed, *Dirac charge quantization and generalized differential cohomology*, Surveys in differential geometry, Surv. Differ. Geom., vol. 7, Int. Press, Somerville, MA, 2000, pp. 129–194. MR 1919425 42

[Kap95] M. M. Kapranov, *Analogies between the Langlands correspondence and topological quantum field theory*, Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), Progr. Math., vol. 131, Birkhäuser Boston, Boston, MA, 1995, pp. 119–151. MR 1373001 13

[Kim15] Minhyong Kim, *Arithmetic chern-simons theory i*, 2015. 13

[Kri05] Martin H. Krieger, *A 1940 letter of André Weil on analogy in mathematics*, Notices Amer. Math. Soc. **52** (2005), no. 3, 334–341, Excerpted from ıt Doing mathematics [World Scientific Publishing Co., Inc., River Edge, NJ, 2003; MR1961400]. MR 2125268 11

[Lau91] Gérard Laumon, *La transformation de Fourier géométrique et ses applications*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 437–445. MR 1159232 100, 102, 105

[Lur04] Jacob Lurie, *Tannaka Duality for Geometric Stacks*, arXiv Mathematics e-prints (2004), math/0412266. 94

[Mac78] George W. Mackey, *Harmonic analysis as the exploitation of symmetry—a historical survey*, Rice Univ. Stud. **64** (1978), no. 2-3, 73–228, History of analysis (Proc. Conf., Rice Univ., Houston, Tex., 1977). MR 526217 5

[Maz73] Barry Mazur, *Notes on étale cohomology of number fields*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 521–552 (1974). MR 344254 13

[Mor10] Masanori Morishita, *Analogies between knots and primes, 3-manifolds and number rings [translation of mr2208305]*, vol. 23, 2010, Sugaku expositions, pp. 1–30. MR 2605747 13

[Rot96] Mitchell Rothstein, *Sheaves with connection on abelian varieties*, Duke Math. J. **84** (1996), no. 3, 565–598. MR 1408538 102, 105

[Wit82] Edward Witten, *Supersymmetry and Morse theory*, J. Differential Geometry **17** (1982), no. 4, 661–692 (1983). MR 683171 52

[Yua19] Allen Yuan, *Integral models for spaces via the higher frobenius*, 2019. 20