Between electric-magnetic duality and the Langlands program

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with any corrections or concerns.

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CHAPTER 1

Overview

The geometric Langlands program is some kind of middle-ground between number theory and physics. Another point of view is that we will be navigating the narrow passage between the whirlpool Charybdis (physics) and the six-headed monster Scylla (number theory), as in Odysseus' travels.[1] Lecture 1; January 19, 2021

The inspiration for much of this course comes from [**Mac78**], which provides a historical account of harmonic analysis, focusing on the idea that function spaces can be decomposed using symmetry. This theme has long-standing connections to physics and number theory.

The spirit of what we will try to do is some kind of harmonic analysis (fancy version of Fourier theory) which will appear in different guises in both physics and number theory.

1. Modular/automorphic forms

1.1. Rough idea. The theory of modular forms is a kind of harmonic analysis/quantum mechanics on arithmetic locally symmetric spaces. The canonical example of a locally symmetric space is given by the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane $\mathbb{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2$. I.e. we are considering the quotient

$$\mathcal{M}_{\mathrm{SL}_2 \mathbb{R}} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \tag{1}$$

as in fig. 1.

[1]One can expand this analogy. Calypso's island is probably derived algebraic geometry (DAG), etc.

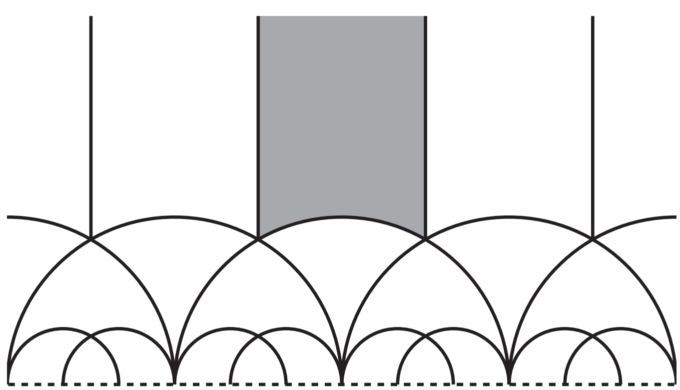

FIGURE 1. Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray.

For a general reductive algebraic group G we can consider the space

$$\mathcal{M}_G = \Gamma \backslash G / K \tag{2}$$

where Γ is an arithmetic lattice, and K is a maximal compact subgroup. For now we restrict to

$$G = \mathrm{SL}_2(\mathbb{R}) \qquad \Gamma = \mathrm{SL}_2(\mathbb{Z}) \qquad K = \mathrm{SO}_2 \ .$$

We want to do harmonic analysis on this space, i.e. we want to decompose spaces of functions on this in a meaningful way. In the case of quantum mechanics we're primarily interested in L^2 functions:

$$L^2(\Gamma \backslash G / K) \ , \tag{3}$$

and on this we have an action of the hyperbolic Laplace operator. I.e. we want to study the spectral theory of this operator.

The same information, possibly in a more accessible form, is given by getting rid of the K. That is, we can just study L^2 functions on

$$\Gamma \backslash G = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \ , \tag{4}$$

which is the unit tangent bundle, a circle bundle over the space we had before. Instead of studying the Laplacian, this is a homogeneous space so we can study the action of all of $\mathrm{SL}_2(\mathbb{R})$.

One can expand this to include differentials and pluri-differentials, i.e. sections of (powers of) the canonical bundle:

$$\Gamma\left(\Gamma \backslash \mathbb{H}, \omega^{k/2}\right) \ . \tag{5}$$

Definition 1. The Δ-eigenfunctions in $L^2(\Gamma \backslash G / K)$ are called *Maass forms*. *Modular forms* of weight k are holomorphic sections of $\omega^{k/2}$.

Remark 1. For a topologist, one might instead want to study (topological) cohomology (instead of forms) with coefficients in some local system (twisted coefficients). Indeed, modular forms can also arise by looking at the (twisted) cohomology of $\Gamma \backslash \mathbb{H}$. This is known as Eichler-Shimura theory.

One might worry that this leaves the world of quantum mechanics, but after passing to cohomology we're doing what is called *topological* quantum mechanics. We will be more concerned with this than honest quantum mechanics.

The idea is that there are no dynamics in this setting. We're just looking at the ground states, so the Laplacian is 0, and we're just looking at harmonic things. And this really has to do with topology and cohomology. But modular forms are some kind of ground states.

Remark 2. If we take general G, K, and Γ then we get the more general theory of *automorphic forms*.

Example 1. If we start with $G = \mathrm{Sp}_{2n}(\mathbb{R})$ and take $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, $K = \mathrm{SO}_n$ then we get *Siegel modular forms*.

1.2. Structure. There is a long history of thinking of this problem[2] as quantum mechanics on this locally symmetric space. But there is a lot more structure going on in the number theory than seems to be present in the quantum mechanics of a particle moving around on this locally symmetric space.

Restrict to the case $G = \mathrm{SL}_2(\mathbb{R})$.

[2] The problem of understanding L^2 functions on a locally symmetric space.

1.2.1. *Number field.* The question of understanding

$$L^2\left(\mathrm{SL}_2\,\mathbb{Z}\backslash\,\mathrm{SL}_2\,\mathbb{R}/\,\mathrm{SO}_2\right) \tag{6}$$

has an analogue for any number field. We can think of $\mathbb{Z}$ as being the ring of integers in the rational numbers:

$$\mathbb{Z} = \mathcal{O}_{\mathbb{Q}} \tag{7}$$

and from this we get a lattice $\mathrm{SL}_2\left(\mathcal{O}_{\mathbb{Q}}\right)$. Writing it this way, we see that we can replace $\mathbb{Q}$ by any finite extension F, and $\mathbb{Z}$ becomes the ring of integers $\mathcal{O}_F$:

$$\begin{aligned} \mathbb{Q} &\rightsquigarrow F \\ \mathbb{Z} &\rightsquigarrow \mathcal{O}_F \ . \end{aligned} \tag{8}$$

The upshot is that when we replace Q with some other number field $F/\mathbb{Q}$, then the space $\mathcal{M}_{G,\mathbb{Q}}$ becomes some space $\mathcal{M}_{G,F}$. Then we linearize by taking either L^2 or H^* of $\mathcal{M}_{G,F}$.

Example 2. This holds for all reductive algebraic groups G, but let $G = \mathrm{PSL}_2\,\mathbb{R}$. Then

$$\mathcal{M}_{G,\mathbb{Q}} = \mathrm{PSL}_2\,\mathbb{Z}\backslash\,\mathrm{PSL}_2\,\mathbb{R}/\,\mathrm{SO}_2 \tag{9}$$

is the locally symmetric space in fig. 1. If we replace $\mathbb{Q}$ with an arbitrary number field $F/\mathbb{Q}$, then we get

$$\mathcal{M}_{G,F} = \mathrm{PSL}_2\left(\mathcal{O}_F\right)\backslash\,\mathrm{PSL}_2\left(F\otimes_{\mathbb{Q}}\mathbb{R}\right)/K \ . \tag{10}$$

Note that

$$F\otimes_{\mathbb{Q}}\mathbb{R} \simeq \mathbb{R}^{\times r_1}\times\mathbb{C}^{\times r_2} \tag{11}$$

where r_1 is the number of real embeddings of F, and r_2 is the number of conjugate pairs of complex embeddings.

Example 3. Let $F = \mathbb{Q}\left(\sqrt{d}\right)$. If it is real ($d \geq 0$) then $r_1 = 2$ (corresponding to $\pm\sqrt{d}$) and $r_2 = 0$, so we get

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right)\otimes_{\mathbb{Q}}\mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{R}\times\mathrm{PSL}_2\,\mathbb{R} \ . \tag{12}$$

This leads to what are called *Hilbert modular forms.*

If it is imaginary ($d < 0$) then $r_1 = 0$, $r_2 = 1$, and

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right)\otimes_{\mathbb{Q}}\mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{C} \ . \tag{13}$$

In this case the maximal compact is $\mathrm{SO}_3\,\mathbb{R}$, and the quotient:

$$\mathbb{H}^3 = \mathrm{PSL}_2\,\mathbb{C}/\,\mathrm{SO}_3\,\mathbb{R} \tag{14}$$

is hyperbolic 3-space. Now we need to mod out (on the left) by a lattice, and the result is some hyperbolic manifold which is a 3-dimensional version of the picture in fig. 1.

Remark 3. The point is that the real group we get after varying the number field is not that interesting, just some copies of PSL_2. But the lattice we are modding out by depends more strongly on the number field, so this is the interesting part.

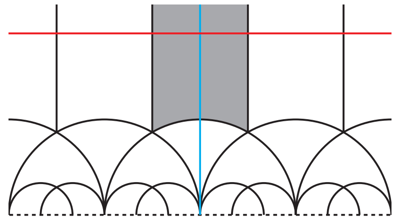

FIGURE 2. Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray. One can define a "period" as taking a modular form and integrating it, e.g. on the red or blue line.

1.2.2. *Conductor/ramification data.* Fixing the number field $F = \mathbb{Q}$, we can vary the "conductor" or "ramification data". The idea is as follows. The locally symmetric space $\Gamma\backslash\mathbb{H}$ has a bunch of covering spaces of the form $\Gamma'\backslash\mathbb{H}$, where Γ' is some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. So we can replace Γ by Γ'.

We won't define congruence subgroups in general, but there are basically two types. For $N \in \mathbb{Z}$, we fix subgroups:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N \right\} \tag{15}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \right\} . \tag{16}$$

The idea is that we start with the conductor N and the lattice Γ, and then we modify Γ at the divisors of N. Note that even in this setting we have the choice of $\Gamma(N)$ or $\Gamma_0(N)$. Really the collection of variants has a lot more structure. The local data at p has to do with the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

1.2.3. *Action of Hecke algebra.* We have seen that our Hilbert space depends on the group, the number field, and some ramification data. A very important aspect of this theory is that this vector space (of functions) carries a lot more structure. There is a huge "degeneracy" here in the sense that the eigenspaces of the Laplacian are much bigger than one might have guessed (not one-dimensional).

This degeneracy is given by the theory of *Hecke operators*. This says that the Laplacian Δ is actually a part of a huge commuting family of operators. In particular, these all act on the eigenspaces of the Laplacian. For p a prime (p unramified, i.e. $p \nmid N$) we have the Hecke operator at p, T_p. Then

$$\bigoplus_p \mathbb{C}[T_p] \circlearrowright L^2(\Gamma\backslash G/K) . \tag{17}$$

This is some kind of "quantum integrable system" because having so many operators commute with the Hamiltonian tells us that a lot of quantities are conserved.[3]

1.2.4. *Periods/states.* There is a special collection of measurements we can take of modular forms, called periods. A basic example is given by integrating a modular form on the line $i\mathbb{R}_+ \subset \mathbb{H}$ as in fig. 2. This is how Hecke defined the L-function.

[3]This example is often included in the literature as an example of quantum chaos (the opposite of integrability). The chaotic aspect has nothing to do with the discrete subgroup Γ. Specifically this fits into the study of "arithmetic quantum chaos" which more closely resembles the study of integral systems.

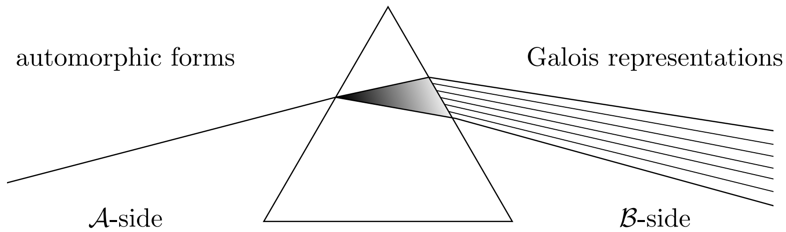

FIGURE 3. Just as light is decomposed by a prism, this spectral decomposition breaks automorphic forms ($\mathcal{A}$-side) up into Galois representations of number fields ($\mathcal{B}$-side).

The takeaway is that we have a collection of measurements/states with very good properties, and then we can study modular forms by measuring them with these periods.

1.2.5. *Langlands functoriality.* There is a collection of somewhat mysterious operators whose action corresponds to varying the group G.

2. The Langlands program and TFT

2.1. Overview. We have seen that for a choice of reductive algebraic group G and number field $F/\mathbb{Q}$, we get a locally symmetric space

$$\mathcal{M} = \mathcal{M}_{G,F} = \text{“arithmetic lattice”}\backslash\text{real group/maximal compact} . \tag{18}$$

This can be thought of as some space of G-bundles

$$\mathcal{M}_{G,F} = \text{“}\operatorname{Bun}_G(\operatorname{Spec}\mathcal{O}_F)'' . \tag{19}$$

Then we linearize this space by taking either L^2 or H^*.

Starting with this theory of automorphic forms, we spectrally decompose under the action of the Hecke algebra. Then the Langlands program says that the pieces of this decomposition correspond to Galois representations. We can think of the theory of automorphic forms as being fed into a prism, and the colors coming out on the other side are Galois representations as in fig. 3. More specifically, the “colors” are representations:

$$\operatorname{Gal}\left(\overline{F}/F\right) \to G^\vee_{\mathbb{C}} . \tag{20}$$

EXAMPLE 4. If $G = \mathrm{GL}_2\,\mathbb{R}$, then $G^\vee = \mathrm{GL}_2\,\mathbb{C}$. Let E be an elliptic curve. Then

$$H^1\left(E/\mathbb{Q}\right) \tag{21}$$

is a 2-dimensional representation of $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. This is the kind of representation you get in this setting.

EXAMPLE 5. The representations in example 4 are very specific to GL_2. If we started with $\mathrm{GL}_3\left(\mathbb{R}\right)$ instead, the associated locally symmetric space $\mathrm{O}_3 \setminus \mathrm{GL}_3\,\mathbb{R}/\,\mathrm{GL}_3\,\mathbb{Z}$ is not a complex manifold.

The goal is to match all of this structure in section 1.2 with a problem in physics, but ordinary quantum mechanics will be too simple. On the physics side we will instead consider quantum field theory.

TABLE 1. Output of a four-dimensional topological field theory. Numbers are the easiest to understand, but are usually the trickiest to produce (often requires analysis). Vector spaces are also pretty simple, but three-manifolds are hard. So the sweet spot is kind of in 2-dimensions, since we understand surfaces and categories aren't that complicated.

Dimension	Output
4	$z \in \mathbb{C}$ (rarely well-defined algebraically, requires analysis)
3	(dg) vector space
2	(dg) category
1	$(\infty, 2)$-category
0	$(\infty, 3)$-category? (rarely understood)

Slogan: the Langlands program is part of the study of 4-dimensional (arithmetic, topological) quantum field theory.

The idea is that the Langlands program is an equivalence of 4-dimensional arithmetic topological field theories (TFTs):

$$\begin{array}{ccc} \mathcal{A}_G & \simeq & \mathcal{B}_{G^\vee} \\ \text{automorphic} & & \text{spectral} \\ \text{magnetic} & & \text{electric} \end{array} \tag{22}$$

called the $\mathcal{A}$ and $\mathcal{B}$-side theories.

REMARK 4. This is what one might call "four-dimensional mirror symmetry". The $\mathcal{A}$ and $\mathcal{B}$ are in the same sense as usual mirror symmetry.

An n-dimensional TFT is a beast which assign a quantum mechanics problem (or just a vector space, chain complex, etc.) to every $(n-1)$-manifold. So a 4-dimensional TFT sends a 3-manifold to some kind of vector space. It assigns more complicated data to lower-dimensional manifolds and less complicated data to higher-dimensional manifolds as in table 1.

The *topological* means we are throwing out the dynamics and only looking at the ground states. This is the analogue of only looking at the harmonic forms rather than the whole spectrum of the Laplacian. The *arithmetic* means that we're following the paradigm of *arithmetic topology*. The idea is that we will eventually make an analogy between number fields and three-manifolds. Then we can plug a number field into the TFT (instead of an honest manifold) to get a vector space which turns out to be $L^2(\mathcal{M}_{G,F})$ (or $H^*(\mathcal{M}_{G,F})$).

2.2. Arithmetic topology.

2.2.1. *Weil's Rosetta Stone.* In a letter to Simone Weil [**Kri05**], André Weil explained a beautiful analogy, now known as *Weil's Rosetta Stone*. This establishes a three-way analogy between number fields, function fields, and Riemann surfaces.

The general idea is as follows. $\operatorname{Spec}\mathbb{Z}$ is some version of a curve, with points $\operatorname{Spec}\mathbb{F}_p$ associated to different primes. $\operatorname{Spec}\mathbb{Z}_p$ is a version of a disk around the point, and $\operatorname{Spec}\mathbb{Q}_p$ is a version of a punctured disk around that point. This is analogous to the usual picture of an algebraic curve.

Curve	$\operatorname{Spec}\mathbb{F}_q[t]$	$\operatorname{Spec}\mathbb{Z}$
Point	$\operatorname{Spec}\mathbb{F}_p$	$\operatorname{Spec}\mathbb{F}_p$
Disk	$\operatorname{Spec}\mathbb{F}_t[[t]]$	$\operatorname{Spec}\mathbb{Z}_p$
Punctured disk	$\operatorname{Spec}\mathbb{F}_q((t))$	$\operatorname{Spec}\mathbb{Q}_p$

In general, let $F/\mathbb{Q}$ be a number field. Then we can consider $\mathcal{O}_F$, and $\operatorname{Spec}\mathcal{O}_F$ has points corresponding to primes in $\mathcal{O}_F$. The analogy between number fields and function fields is as follows. Start with a smooth projective curve $C/\mathbb{F}_q$ over a finite field. Then the analogue to F is the field of rational functions, $\mathbb{F}_q(C)$. The analogue to $\mathcal{O}_F$ is the ring of regular functions, $\mathbb{F}_q[C]$. Finally points of $\operatorname{Spec}\mathcal{O}_F$ correspond to points of C.

Now we might want to replace C with a Riemann surface. So let $\Sigma/\mathbb{C}$ be a compact Riemann surface. Then primes in $\mathcal{O}_F$ (and so points of C) correspond to points of Σ. The field of meromorphic rational functions on Σ, $\mathbb{C}(\Sigma)$, is the analogue of F. To get an analogue of $\mathcal{O}_F$ we have to remove some points of Σ (we wouldn't get any functions on the compact curve). The point is that number fields have some points at ∞, so the analogue isn't really a compact Riemann surface, but with some marked points. So the analogue of $\mathcal{O}_F$ consists of functions on Σ which are regular away from these points.

This is summarized in table 2.

Table 2. Weil's Rosetta stone, as it was initially developed, establishes an analogy between these three columns. We will eventually refine this dictionary. Let $F/\mathbb{Q}$ be a number field, $C/\mathbb{F}_q$ be a smooth projective curve over a finite field, and let $\Sigma/\mathbb{C}$ be a compact Riemann surface. $\mathbb{F}_q(C)$ denotes the field of rational functions, $\mathbb{F}_q[C]$ denotes the ring of regular functions, and $\mathbb{C}(\Sigma)$ denotes the meromorphic rational functions on Σ.

Number fields	Function fields	Riemann surfaces
$F/\mathbb{Q}$	$\mathbb{F}_q(C)$	$\mathbb{C}(\Sigma)$
$\mathcal{O}_F$	$\mathbb{F}_q[C]$	f'ns regular away from marked points of Σ
$\operatorname{Spec}\mathcal{O}_F$	points of C	$x \in \Sigma$

2.2.2. *Missing chip.* Now we want to take the point of view that there was a chip missing from this Rosetta stone, and we were supposed to consider 3-manifolds rather than Riemann surfaces. The idea is that $\Sigma/\mathbb{C}$ really corresponds to $C/\overline{\mathbb{F}_q}$. This is manifested in the following way. To study points, we study maps:

$$\operatorname{Spec}\mathbb{F}_q \hookrightarrow C\ . \tag{23}$$

But from the point of view of étale topology, $\operatorname{Spec}\mathbb{F}_q$ is not really a point. It is more like a circle in the sense that

$$\operatorname{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) = \widehat{\mathbb{Z}} = \pi_1^{\text{étale}}\left(\operatorname{Spec}\mathbb{F}_q\right) \tag{24}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion. So it's better to imagine this as a modified circle, where this $\widehat{\mathbb{Z}}$ is generated by the Frobenius. There is always a map

$$\operatorname{Spec}\overline{\mathbb{F}_q} \to \operatorname{Spec}\mathbb{F}_q \tag{25}$$

and we can lift our curve to $\overline{\mathbb{F}}_q$. This corresponds to unwrapping these circle, i.e. replacing them by their universal cover. So their is some factor of $\mathbb{R}$ which doesn't play into the topology/cohomology. So we have realized that curves over $\mathbb{F}_q$ have too much internal structure to match with a Riemann surface.

REMARK 5. The map $\operatorname{Spec}\mathbb{F}_{q^n} \to \operatorname{Spec}\mathbb{F}_q$ is analogous to the usual n-fold cover of the circle.

To fix the Rosetta Stone, we replace a Riemann surface Σ by certain a Σ-bundle over S^1. Explicitly, if we have Σ and a diffeomorphism φ, we can form the mapping torus:

$$\Sigma \times I / \left((x,0) \sim (\varphi(x), 1)\right) . \tag{26}$$

The idea is that if we start with a curve over a finite field, the diffeomorphism φ is like the Frobenius.

Similarly $\operatorname{Spec}\mathcal{O}_F$ looks like a curve where each "point" carries a circle. So this is again some kind of 3-manifold.

REMARK 6. These circles don't talk to one another because they all have to do with a Frobenius at a different prime. So they're less like a product or a fibration, and more like a 3-manifold with a foliation.

This fits with the existing theory of arithmetic topology, sometimes known as the "knots and primes" analogy. The theory was started in a letter from Mumford to Mazur, but can be attributed to many people such as Mazur [**Maz73**], Manin, Morishita [**Mor10**], Kapranov [**Kap95**], and Reznikov. The recent work [**Kim15, CKK$^+$19**] of Minhyong Kim plays a central role.

REMARK 7. Lots of aspects of this dictionary are spelled out, but one should be wary of using it too directly. Rather we should think of this as telling us that there are several classes of '3-manifolds': ordinary 3-manifolds, function fields over finite fields, and number fields.

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2.2.3. *Updated Rosetta Stone.* The upshot is that we are thinking of all three objects in the Rosetta stone as three-manifolds. In particular, we're thinking of $\operatorname{Spec}\mathcal{O}_F$ (e.g. $\operatorname{Spec}\mathbb{Z}$) as a 3-manifold, so for any prime p we have the loop $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathcal{O}_F$, which we can interpret as a knot in the 3-manifold. Let F_v be the completion of the local field F at the place v (e.g. $\mathbb{Q}_p$). Then $\operatorname{Spec} F_v$ turns out to be the boundary of a tubular neighborhood of the knot. The point is that if F_v is a non-Archimedean local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$) then the "fundamental group" is the Galois group, and it has a quotient:

$$\operatorname{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} . \tag{27}$$

This group is called a *Baumslag-Solitar* group. Explicitly it is:

$$\operatorname{BS}(1,p) = \left\{\sigma, u \,\middle|\, \sigma u \sigma^{-1} = u^p\right\} \tag{28}$$

where we think of σ as the Frobenius, so corresponding to $\widehat{\mathbb{Z}}$, and u as the generator of $\mathbb{Z}_\ell$. The kernel is

$$p\text{-group} \times \prod_{\ell^r \neq \ell, p} \mathbb{Z}_{\ell^r} \hookrightarrow \mathrm{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} \; . \tag{29}$$

This tells us that there is For $p = 1$, this group is $\mathbb{Z} \times \mathbb{Z} = \pi_1\left(T^2\right)$. For $p = -1$, this is the fundamental group of the Klein bottle. This is evidence that the étale fundamental group of $\operatorname{Spec} F_v$ looks like some kind of p-dependent version of the fundamental group of the torus. So we can think of $\operatorname{Spec} F_v$ as a 2-manifold (fibered over S^1).

After this discussion we can identify an updated (multi-dimensional) Rosetta Stone. In three-dimensions we have: $\operatorname{Spec} \mathcal{O}_F$, $C/\mathbb{F}_q$, and a mapping torus $T_\varphi(\Sigma)$. The first two comprise the *global arithmetic setting*. In two-dimensions we first have local fields, which come in two types. One is finite extensions $F_v/\mathbb{Q}_p$ and the other is $\mathbb{F}_q((t))$. Spec of either of these is "two-dimensional" and the latter is some kind of punctured disk D^*. These two comprise the *local arithmetic setting*. A curve $\overline{C}/\overline{\mathbb{F}_q}$ over an algebraically closed field (of positive characteristic) and a Riemann surface (projective curve Σ over $\mathbb{C}$) are also both "two-dimensional". These comprise the *global geometric setting*. The only 4-manifolds we will consider are of the form $M^3 \times I$ or $M^3 \times S^1$ where M^3 is a three-dimensional object of arithmetic or geometric origin. This discussion is summarized in table 3.

2.3. $\mathcal{A}$-side. The $\mathcal{A}$-side (or automorphic/magnetic side) TFT $\mathcal{A}_G$ is a huge machine which does many things, as in table 1. So far, the only recognizable thing is that it sends a 3-manifold M to some vector space $\mathcal{A}_G(M)$. We're thinking of $\operatorname{Spec} \mathcal{O}_F$ as a 3-manifold, and the assignment is the vector space we've been discussing:

$$\mathcal{A}_G(\operatorname{Spec} \mathcal{O}_F) = L^2(\mathcal{M}_{G,F}) \text{ or } H^*(\mathcal{M}_{G,F}) \; . \tag{30}$$

Remark 8. As suggested in eq. (19), note that $\Gamma \backslash G/K$ is a moduli space of something over the 3-manifold in question, not the 3-manifold itself.

2.4. Structure (reprise). As it turns out, all the bells and whistles from the theory of automorphic forms in section 1.2 line up perfectly with the bells and whistles of TFT.

2.4.1. *Number field.* The assignment in eq. (30) formalizes the idea that we got a vector space $L^2(\mathcal{M}_{G,F})$ labelled by a group and a number field.

2.4.2. *Conductor/ramification data.* Recall the ramification data was a series of primes. This is manifested as a link (collection of knots) in the 3-manifold, where we allow singularities. These appear as *defects* (of codimension 2) in the physics. So the structure we saw before is manifested as defects of the theory.

2.4.3. *Hecke algebra.* The Hecke operators correspond to line defects (codimension 3) in the field theory. Physically this is "creating magnetic monopoles" alone some loop in spacetime.

2.4.4. *Periods/states.* These correspond to boundary conditions, i.e. codimension 1 defects.

2.4.5. *Langlands functoriality.* Passing from G to H can be interpreted as crossing a domain walls (also a codimension 1 defect).

TABLE 3. The columns correspond to the three aspects of Weil's Rosetta Stone, and the rows correspond to dimension. The four-dimensional objects we consider are just products of three-dimensional objects with S^1 or I. M^3 is a three-dimensional object of arithmetic or geometric origin. The three-dimensional objects are number fields, function fields and mapping tori of Riemann surfaces. F is a number field, φ is some diffeomorphism of Σ, and T_φ denotes the corresponding mapping torus construction. The two-dimensional objects are local fields and curves. F_v is a finite extension of $\mathbb{Q}_p$. The 1-dimensional objects are both versions of circles, and the 0-dimensional objects are points.

Dimension	Number fields	Function fields	Geometry
4	$M^3 \times S^1$, $M^3 \times I$		
3	Global arithmetic		-
	$\operatorname{Spec} \mathcal{O}_F$	$C/\mathbb{F}_q$	$T_\varphi(\Sigma)$
2	Local arithmetic		Global geometric
	$\operatorname{Spec} F_v$	$\operatorname{Spec} \mathbb{F}_q((t)) = D^*$	$\overline{C}/\overline{\mathbb{F}}_q, \quad \Sigma/\mathbb{C}$
1	-		Local geometric
	$\operatorname{Spec} \mathbb{F}_q$		$D^*_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}((t))$, $D^*_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q((t))$
0	$\operatorname{Spec} \overline{\mathbb{F}_q}$		$\operatorname{Spec} \mathbb{C}$

2.5. $\mathcal{B}$-side. The $\mathcal{B}$-side (or spectral side) is the hard part from the point of view of number theory because Galois groups of number fields (and their representations) are very hard. I.e. the $\mathcal{B}$-side is the question, and the $\mathcal{A}$-side is the answer. But from the point of view of geometry, it is the other way around because fundamental groups of Riemann surfaces are really easy.

The $\mathcal{B}$-wide is about studying the algebraic geometry of spaces of Galois representations.

Recall that given a three-manifold (or maybe a number field F) the $\mathcal{A}$-side is concerned with the topology of the arithmetic locally symmetric space $\mathcal{M}_{G,F}$. $\mathcal{M}_{G,F}$ has to do with the geometry of F, so the $\mathcal{A}$-side is concerned with the topology of the geometry of F.

The $\mathcal{B}$-side concerns itself with the algebra of the topology of F. This means the following. For a manifold M (of any dimension), we can construct $\pi_1(M)$. Then the collection of rank n local systems on M is:

$$\mathbf{Loc}_n M = \{\pi_1(M) \to \mathrm{GL}_n \mathbb{C}\} \ . \tag{31}$$

A local system looks like a locally constant sheaf of rank n (or vector bundles with flat connection). These are sometimes called *character varieties*. Then we can study $\mathbb{C}[\mathbf{Loc}_n M]$. We can also replace GL_n with our favorite complex Lie group

G to get:

$$\mathbf{Loc}_{G^\vee} M = \{\pi_1(M) \to G^\vee\} \ . \tag{32}$$

This depends only on the topology of M.

If we're thinking of a number field as a three-manifold, then π_1 is a stand-in for the Galois group so this is a space of representations of Galois groups. The TFT sends any three-dimensional M^3 to functions on $\mathbf{Loc}_{G^\vee}$:

$$\mathcal{B}_{G^\vee}\left(M^3\right) = \mathbb{C}\left[\mathbf{Loc}_{G^\vee} M\right] \ . \tag{33}$$

REMARK 9. This side was a lot easier to write down than the $\mathcal{A}$-side, but if M is a number field, the Galois group is potentially very hard to understand. All the other bells and whistles are also easy to define here.

2.6. All together. In all of the setting in table 3, we can either make and automorphic measurement (attach $\mathcal{M}_{G,F}$ and study its topology) or we could take the Galois group (or π_1), construct a variety out of it, and study algebraic functions on it. The idea we will explain is that the Langlands program is an equivalence of these giant packages, but for "Langlands dual groups" G and $G^\vee$:

$$\mathcal{A}_G \simeq \mathcal{B}_{G^\vee} \ . \tag{34}$$

REMARK 10. More is proven in the geometric setting than the arithmetic, but even geometric Langlands for a Riemann surface is still an open question.

This is really a conjectural way of organizing a collection of conjectures.

CHAPTER 2

Spectral decomposition

Lecture 3; January 26, 2021

1. What is a spectrum?

The basic idea is that we start in the world of geometry, meaning we have a notion of a "space" (e.g. algebraic geometry, topology, ...), and given one of these spaces X we attach some kind of collection of functions $\mathcal{O}(X)$. These functions can have many different flavors, but they always form some kind of commutative algebra, possibly with even more structure. We will access the geometry of spaces using these functions. The operation $\mathcal{O}$ turns out to be a functor, i.e. if we have a morphism $\pi\colon X \to Y$ of spaces, we get a pullback morphism $\pi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$.

A fundamental question about this setup is to what degree we can reverse this operation. So starting with any commutative algebra, we would like to understand the extent to which we can we get geometry out of it. Category theory tells us that the right sort of thing to consider is a right adjoint to $\mathcal{O}$, which we call Spec. The fact that they form an adjunction means:

$$\operatorname{Map}_{\mathbf{Spaces}}(X, \operatorname{Spec} R) = \operatorname{Hom}_{\mathbf{Ring}^{\mathrm{op}}}(\mathcal{O}(X), R) = \operatorname{Hom}_{\mathbf{Ring}}(R, \mathcal{O}(X)) \ . \tag{35}$$

In the language of analysis, we might regard $\operatorname{Spec} R$ as a "weak solution" in the sense that it's a formal solution to the problem of finding a space associated to R. It is a functor assigning a set to any test space, but there is no guarantee that there is an honest space out there which would agree with it.

Remark 11. We might have to adjust the categories we're considering so that

$$\mathcal{O}(\operatorname{Spec} R) = R \ , \tag{36}$$

since this doesn't just fall out of the adjunction.

The point is that for some nice class of spaces, one might hope that we can recover a space from functions on that space: $X = \operatorname{Spec} \mathcal{O}(X)$.

Remark 12. The word spectrum is used in many places in mathematics. They are basically all the same, except spectra from homotopy theory.

1.1. Finite set. Let X be a finite set and k a field. Then the k-valued functions $\mathcal{O}(X)$ can be expressed as

$$\mathcal{O}(X) = \prod_{x \in X} k \tag{37}$$

which can be thought of as diagonal $X \times X$ matrices.

1.2. Compactly supported continuous functions. Gelfand developed the following version of this philosophy. For our category of spaces, consider the category of locally compact Hausdorff spaces X with continuous maps as the morphisms. For our space of functions we take $C_v(X)$, which is the space of continuous $\mathbb{C}$-valued functions which vanish at ∞. This is in the category of commutative C^*-algebras. These are Banach $\mathbb{C}$-algebras with a $*$ operation (to be thought of as conjugation) which is $\mathbb{C}$-antilinear and compatible with the norm. Given any commutative C^*-algebra A, the associated spectrum m-Spec A is called the *Gelfand spectral*, and as a set it consists of the maximal ideals in A. We can write this as:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) \tag{38}$$

i.e. the unitary 1-dimensional representations of A. The adjunction is then saying that:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) = \mathrm{Map}(\mathrm{pt}, \text{m-Spec}\, A)\ . \tag{39}$$

Theorem 1 (Gelfand-Naimark). *C_v and* m-Spec *give an equivalence of categories.*

See e.g. [**aHRW10**] for more details on this theorem.

1.3. Measure space. There is a coarser version where we start with a measure space X, and take attach the bounded functions $L^\infty(X)$. This forms a commutative von Neumann algebra. Again this is an equivalence of categories.

1.4. Algebraic geometry. We will focus on the setting of algebraic geometry. The category on the commutative algebra side will just be the category **cRing** of commutative rings. The geometric side will be the category of locally ringed spaces. This just means that there is a notion of evaluation at each point.

The functor:

$$(X, \mathcal{O}_X) \mapsto \mathcal{O}(X) \tag{40}$$

has an adjoint:

$$\mathrm{Spec}\, R \leftarrow\!\shortmid R\ . \tag{41}$$

Affine schemes comprise the image of Spec:

$$\begin{tikzcd} \text{locally-ringed spaces} \arrow[r] & \mathbf{cRing} \arrow[dl, "\mathrm{Spec}"] \\ \text{affine schemes} \arrow[u, hook] & \end{tikzcd} \ . \tag{42}$$

Then it is essentially built into the construction that affine schemes are equivalent to commutative rings.

This doesn't really capture a lot of what we want to study in algebraic geometry, because one is usually interested in general schemes, which are locally ringed spaces that locally look affine. One way to deal with this is to really think of our geometric objects as:

$$\mathrm{Fun}(\mathbf{Ring}^{\mathrm{op}}, \mathbf{Set}) = \mathrm{Fun}(\mathbf{Aff}, \mathbf{Set}) \subset \text{Geometry}\ . \tag{43}$$

1.5. Topology. In homotopy theory we might start with the homotopy category of spaces and pass to some notion of functions, e.g. cohomology $H^*(X;\mathbb{Z})$. This sits in the category of graded commutative rings. This doesn't directly lead to a nice spectral theory, but it does if we remember a bit more structure. Instead, start with the category of spaces with continuous maps. Then we can take rational chains, $C^*_{\mathbb{Q}}$, and get a commutative differential graded $\mathbb{Q}$-algebra. This is the Quillen-Sullivan rational homotopy theory.[1] As it turns out, the category of simply connected spaces up to rational homotopy equivalence is equivalent to commutative differential graded algebras which are $\mathbb{Q}$ in degree 0 and 0 in degree 1.

2. Spectral decomposition

Let R be a commutative ring over some field k. Commutative rings usually arise via the study of modules over it. So let $V \in R$-**Mod**, i.e. a map

$$R \to \mathrm{End}(V)\ . \tag{44}$$

Now we want to decompose V into some sheaf $\underline{V}$ over $\mathrm{Spec}\, R$. We use that R-**Mod** is symmetric monoidal, so we have a tensor product $\otimes_R$ and we can define

$$\underline{V}(U) = V \otimes_R \mathcal{O}(U) \tag{45}$$

for open $U \subset \mathrm{Spec}\, R$. Then we can talk about the *support* of $v \in V$:

$$\mathrm{Supp}(v) \subset X\ . \tag{46}$$

Example 6. If X is finite, so $R = \prod_{x\in X} k$, then by asking for vectors supported at a single point, we get a decomposition $V = \oplus_{x\in X} V_x$. But to every point x, we get an evaluation morphism:

$$\lambda\colon R \xrightarrow{\mathrm{ev}_x} k \tag{47}$$

associated to the point x. Just like before we can think of this as a one-dimensional R-module, since $k = \mathrm{End}(k)$. Therefore, changing notation, we can write the decomposition as

$$V = \oplus_{\lambda\in X} V_\lambda w\ . \tag{48}$$

In this language the spaces in the decomposition are just the λ-eigenspaces:

$$V_\lambda = \mathrm{Hom}_{R\text{-}\mathbf{Mod}}(k_\lambda, V) = \{v \in V \mid r\cdot v = \lambda(r)\, v\} \tag{49}$$

where k_λ is the one-dimensional module over R, where R acts via the map λ.

This description is via evaluation at a point, but points are both open and closed so we can also describe this as restriction:

$$V_\lambda = V \otimes_{\mathcal{O}(X)=R} k_\lambda\ , \tag{50}$$

so this vector spaces is realized as both a Hom and a tensor because these points happened to be both open and closed.

Define the category of quasi-coherent sheaves to be:

$$\begin{array}{ccc} \mathbf{QCoh}(X = \mathrm{Spec}\, R) & := & R\text{-}\mathbf{Mod} \\ \uparrow & & \\ \mathbf{Coh}(X) & := & R\text{-}\mathbf{Mod}_{\mathrm{f.g.}} \end{array}\ . \tag{51}$$

[1]This should also be attributed to Mandell. See also Yuan's paper [**Yua19**].

Note that in general X is only locally of the form $\operatorname{Spec} R$, so $\mathbf{QCoh}(X)$ is only locally of the form R-**Mod**.

3. The spectral theorem

3.1. Algebraic geometry version. Let V be a vector space and $M \in \operatorname{End} V$. We can think of $M \in \operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{End} V)$, but $\operatorname{End} V$ is not just a set. It is an associative algebra over k, so really

$$M \in \operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{Forget} \operatorname{End} V) \ . \tag{52}$$

This sets us up for an adjunction with the free k-algebra construction. The free algebra in one generator is just $k[x]$ so the adjunction says that:

$$\operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{Forget} \operatorname{End} V) = \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(k[x], \operatorname{End} V) \ . \tag{53}$$

What we have seen here is that equipping V with some $M \in \operatorname{End} V$ is equivalent to making V a module over $k[x]$. I.e. equipping V with $M \in \operatorname{End} V$ is equivalent to V being global sections of some quasi-coherent sheaf on $\operatorname{Spec} k[x] = \mathbb{A}^1$. I.e. V spreads out over $\mathbb{A}^1$.

To make this precise, assume V is finitely generated, i.e. the sheaf $\underline{V}$ is coherent. Then V has a sort of decomposition as a quotient and a subspace:

$$V_{\text{torsion}} \hookrightarrow V \twoheadrightarrow V_{\text{tor. free}} \tag{54}$$

where

$$V_{\text{torsion}} = \bigcup_{\lambda \in \mathbb{A}^1} \{v \in V \mid \operatorname{Supp} v = \lambda\} \ . \tag{55}$$

For general modules over a PID ($k[x]$ is a PID) we have:

$$V \simeq \underbrace{V_{\text{tor}}}_{\text{discrete spectrum}} \oplus \underbrace{V_{\text{free}}}_{\text{continuous spectrum}} \tag{56}$$

where

$$V_{\text{free}} = k[x]^{\oplus r} \tag{57}$$

and

$$V_{\text{tor}} = \bigoplus_{\lambda \in \operatorname{Spec}} V_{\hat{\lambda}} \tag{58}$$

where $V_{\hat{\lambda}}$ is the subspace supported at λ. As it turns out

$$V_{\hat{\lambda}} = \bigoplus_i k[x] / (x-\lambda)^{\ell_i} \ , \tag{59}$$

i.e. for any element of this some power of $(x-\lambda)$ annihilates it, so these are generalized eigenspaces. This decomposition is precisely the Jordan normal form.

An eigenvector is some element $v \in V$ such that $\lambda v = xv$ (or by definition Mv). But this is the same as an element of:

$$\operatorname{Hom}_{k[x]}(k_\lambda, V) \ . \tag{60}$$

So this is what one might call a section supported "scheme-theoretically" at $\lambda \in \mathbb{A}^1$.

On the other hand, the fibers:

$$V \otimes_{k[x]} k_\lambda \tag{61}$$

are naturally quotients of V (rather than a sub), and so they're some kind of coeigenvectors.

In the continuous spectrum there are no eigenvectors: as a free module, $k[x]$ doesn't contain any eigenvectors.

EXAMPLE 7. Consider the free case. It is sufficient to consider $V = k[x]$, since otherwise it is just a direct sum of copies of this. Then $\underline{X} = \mathcal{O}_{\mathbb{A}^1}$. There are no generalized eigenvectors (because $(x-\lambda)^N = 0$ for some $N \gg 0$ and some $\lambda \in \mathbb{A}^1$). There are lots of coeigenvectors, though. For any $\lambda \in \mathbb{A}^1$ we have a map

$$V \twoheadrightarrow k_\lambda \ . \tag{62}$$

This is a distribution, i.e. an element of

$$\operatorname{Hom}(V, k) \ . \tag{63}$$

So for every $\lambda \in \mathbb{A}^1$, we get

$$\operatorname{Hom}(V, k) \ni \delta_\lambda \colon V \twoheadrightarrow k_\lambda \ . \tag{64}$$

The basic example is $V = L^2(\mathbb{R})$ and $M = x$. Then V_λ consists of functions supported at x, but there are none. We would like to say δ_λ, but this is not L^2. The dual operator $M^\vee = d/dx$ has eigenvectors which are roughly $e^{i\lambda x}$, but these are not L^2 either.

This is what continuous spectra look like. When you decompose functions on $\mathbb{A}^1$ under the action of x or d/dx, there is a sense in which it is a direct integral, which is different from a direct sum. The things you're integrating aren't actually subsets. So we can think of functions on $\mathbb{A}^1$ as being some kind of continuous direct sum of functions on a point, but those functions don't live as subspaces. In this case they lived as quotients, but not subspaces. This is simpler in the torsion-free case, but is a general feature of continuous spectra. This is not a weird/special fact about analysis, because we see it even at the level of algebra (polynomials).

3.2. Measurable version. Instead of the matrix M, we consider a self-adjoint operator A on a Hilbert space $V = \mathcal{H}$. Then von Neumann's spectral theorem tells us that there is a "sheaf" (projection valued measure) π_A on $\mathbb{R}$ and

$$A = \int_{\mathbb{R}} x d\pi_A \ . \tag{65}$$

A projective valued measure can be thought of as a sheaf $\underline{\pi_A}$ as follows. For $U \subset \mathbb{R}$ measurable, we attach the image under the projection:

$$\underline{\pi_A}(U) = \pi_A(U) \ . \tag{66}$$

So this is some kind of sheaf of Hilbert spaces. There is no topology to be compatible with, but it does satisfy the additivity property that the rule:

$$U \mapsto \langle w, \pi_A(U) v \rangle \tag{67}$$

defines a $\mathbb{C}$-valued measure on $\mathbb{R}$. So this is the version of a sheaf in the measurable world.

So now eq. (65) is saying that the Hilbert space $\mathcal{H}$ sheafifies over $\mathbb{R}$ in such a way that A acts by the coordinate function x, just like in the algebro-geometric setting above. Then the spectrum is a measurable subset

$$\operatorname{Spec}(A) \coloneqq \operatorname{Supp} \pi_A \subset \mathbb{R} \ . \tag{68}$$

EXAMPLE 8. If $\mathcal{H}$ is finite-dimensional then the spectrum is a discrete set of points, and the decomposition is just into eigenspaces.

3.3. Homotopical version. We saw that we had a quasi-coherent sheaf in algebraic geometry, a projection-valued measure in measure theory, and now in algebraic topology we have the following. If $R = C^*(X)$, then

$$R\text{-}\mathbf{Mod} \hookrightarrow \mathbf{Loc}(X) \tag{69}$$

where $\mathbf{Loc}(X)$ consists of locally-constant complexes on X.

The basic idea is that if we have a model for cochains on X:

$$\to R^{\oplus j} \to R^{\oplus i} \to M \tag{70}$$

then we get a presentation of $\underline{M}$ by constant sheaves:

$$\to \underline{k_X}^{\oplus j} \to \underline{k_X}^{\oplus i} \to \underline{M} \tag{71}$$

where the key point is that

$$C^*(X) = \operatorname{End}^*\left(\underline{k_X}, \underline{k_X}\right) . \tag{72}$$

Or more directly we can define the sheaf to be:

$$\underline{M}(U) = M \otimes_{C^*(X)} C^*(X) . \tag{73}$$

Lecture 4; January 28, 2021

3.4. Physics interpretation: observables and states. We have seen that, starting with some flavor of commutative algebra A, we can construct a geometric object $\operatorname{Spec} A$. Then a module M over A gets spread out into a sheaf over $\operatorname{Spec} A$. The algebra A can be thought of as the *algebra of observables* of some physical system. Then the *space of states* forms a module over A, and so fits into this framework.

Let's back up. The general idea is that we're trying to get a grip on the geometry of this space via the functions on it, i.e. making observations. Recall that the defining property of $\operatorname{Spec} A$ is that whenever we have a space X and a map $A \to \mathcal{O}(X)$, where A is commutative, then we get a map

$$X \to \operatorname{Spec} A . \tag{74}$$

We can think of the map $A \to \mathcal{O}(X)$ as picking out some functions (observables) on the space which satisfy the relations of A. E.g. if we just have one function, this is a map from the space down to the line, and then the space will decompose over this. In general the space will decompose over a higher-dimensional base. So this is a way of measuring the space with functions.

Spectral decomposition of modules is a linearized version of this. We replace X by a linearized version of it, e.g. $\mathcal{O}(X)$, and this becomes a module over A. I.e. a module M over A is a linearized version of a map $X \to \operatorname{Spec} A$. If we just have a single function, then this corresponds to a map $X \to \operatorname{Spec} k[x] = \mathbb{A}^1$. Likewise, a single matrix (endomorphism of a vector space) gives rise to a sheaf over $\mathbb{A}^1$.

In quantum mechanics, we don't have a phase space. We only have a vector space $\mathcal{H}$ (some linearized version of the phase space), called the *Hilbert space of states*. Observables are operators on $\mathcal{H}$. In physics we're interested in reality, so we might insist on the condition that observables are self-adjoint:

$$\mathcal{O} = \mathcal{O}^* . \tag{75}$$

Typically we won't impose this condition. For an observable $\mathcal{O} \circlearrowright \mathcal{H}$, spectral decomposition tells us that $\mathcal{H}$ sheafifies (as a projection-valued measure) over $\mathbb{R}$. The base is $\mathbb{R}$ because this is Spec of the algebra generated by a single operator. This is the analogue of starting with a classical phase space M and a single observable $M \to \mathbb{R}$, and then decomposing M over $\mathbb{R}$.

A *state* is an element $|\varphi\rangle \in \mathcal{H}$. Given a state and an operator $\mathcal{O}$, this state becomes a section of the sheaf $\mathcal{H}$, i.e. we get an eigenspace decomposition of this vector. Given a section, the first thing we can ask for is the support. This is just where the measurement we made "lives".

We can do something more precise by using the norm. As it turns out, $\|\varphi\|^2$ is a probability measure on $\mathbb{R}$ which tells us where to expect the state to be located. For example, we can take the expectation value of the observable $\mathcal{O}$ in the state φ:

$$\langle \mathcal{O} \rangle_\varphi = \frac{\langle \varphi | \mathcal{O} | \varphi \rangle}{\langle \varphi | \varphi \rangle} \,. \tag{76}$$

This is a continuous version of

$$\frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda \in \operatorname{Spec} \mathcal{O}} \lambda \left\| \operatorname{Proj}_{\mathcal{H}_\lambda} |\varphi\rangle \right\|^2 = \frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda, \psi_i} \lambda \, \langle \psi_i | \varphi \rangle \, |\psi_i\rangle \tag{77}$$

where ψ_i is a basis of eigenvectors.

Remark 13. To give a quantum-mechanical system, we also need to specify the *Hamiltonian* H. This is a specific observable (self-adjoint operator on $\mathcal{H}$) which plays the role of the energy functional. The eigenstates for H are the steady states of the system. This lets us spread $\mathcal{H}$ out over $\mathbb{R}$ to get the energy eigenstates. We will be working in the *topological* setting where $H = 0$, i.e. we're just looking at the 0 eigenspace. So this decomposition is kind of orthogonal to our interests.

4. Fourier theory/abelian duality

We have seen that whenever we have a "spectral dictionary", we get a notion of spectral decomposition: modules become sheaves, where the notion of a sheaf depends on the context. This gives us a way of spreading out the algebra of modules over the geometry or topology of our space.

For this to be useful, we need interesting sources of commutative algebras. A natural source for commuting operators is when we have an abelian group G acting on a vector space V: given a morphism

$$\rho \colon G \to \operatorname{Aut}(V) \tag{78}$$

we get a family of operators $\{\rho(g)\}_{g \in G}$ and we can spectrally decompose V using these operators. This is what Fourier theory is about. So we're thinking of Fourier theory as some kind of special case of spectral decomposition.

4.1. Characters.

Let V be a representation of an abelian group G, i.e. we have a map

$$\begin{aligned} G &\longrightarrow \operatorname{Aut}(X) \subset \operatorname{End}(V) \\ g &\longmapsto T_g \end{aligned} \tag{79}$$

such that $T_g T_h = T_{gh}$.

Example 9. If G acts on a space X, and V is functions on X, then we get an action of G on V.

Example 10. G always acts on itself, so therefore it acts on functions on G itself. This is the *regular representation*.

Now we want to spectrally decompose. First we need to know what the spectrum is, so we ask the following question.

Question 1. What are the possible eigenvalues?

Let $v \in V$ be an eigenvector:

$$g \cdot v = \chi(g)\, v \tag{80}$$

where

$$\chi \colon G \to \operatorname{Aut} \mathbb{C}V = \mathbb{C}^{\times} \subset \mathbb{C} \tag{81}$$

is a group homomorphism, i.e. a *character* of G. So the possible eigenvalues are the characters:

$$\widehat{G} = \{\text{characters}\} = \operatorname{Hom}_{\mathbf{Grp}}\left(G, \mathbb{C}^{\times}\right) \ . \tag{82}$$

This is the spectrum, i.e. we will be performing spectral decomposition over $\widehat{G}$.

4.2. Finite Fourier transform. Now let G be a finite group.[2] We will eventually assume G is abelian, but we don't need this yet. We want to see $\widehat{G}$ appear at the spectrum. For a complex representation V we have a group map $G \to \operatorname{Aut} V$, but the composition with the inclusion

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right, "\text{monoid map}"'] & \operatorname{Aut} V \arrow[r, phantom, "\subset"] & \operatorname{End}(V) \end{tikzcd} \tag{83}$$

is a monoid map. In other words V gives rise to an element of

$$\operatorname{Hom}_{\mathbf{Monoid}}\left(G, \operatorname{Forget}\left(\operatorname{End} V\right)\right) \ . \tag{84}$$

Just like before, we have an adjunction:

$$\operatorname{Hom}_{\mathbf{Monoid}}\left(G, \operatorname{Forget}\left(\operatorname{End} V\right)\right) = \operatorname{Hom}_{\mathbb{C}\text{-}\mathbf{Alg}}\left(?, \operatorname{End} V\right) \ , \tag{85}$$

where the missing entry should be some kind of free construction. As it turns out, the answer is the *group algebra*:

$$? = \mathbb{C}G \ . \tag{86}$$

This is the algebra freely generated by scalar multiplication and sums of elements of G. Since the group is finite this is just:

$$\mathbb{C}G = \left\{ \sum_{g \in G} f(g) \cdot g \right\} \tag{87}$$

where $f \colon G \to \mathbb{C}$ is any function. Really we should think of f as a measure rather than a function. There is no difference when G is finite, but for any $g \in G$ would would like a canonical element

$$\delta_g = 1 \cdot g \in \mathbb{C}G \ , \tag{88}$$

[2] This is a simplifying assumption so we don't need to worry about what "kind" of functions we're considering.

which means the coefficients come from some f which is 1 at g and 0 elsewhere, which is not a function in general.

We can think of $\mathbb{C}G$ as being generated by the elements δ_g for $g \in G$. The algebra structure comes from convolution:

$$\delta_f * \delta_g = \delta_{fg} \ . \tag{89}$$

In general

$$f_1 * f_2 = \sum_g f_1(g)\, g * \sum_h f_2(h)\, h \tag{90}$$

$$= \sum_k \left(\sum_{gh=k} f_1(g)\, f_2(h) \right) k \tag{91}$$

$$= \sum_k \sum_g f_1(g)\, f_2\left(kg^{-1}\right) \cdot k \ . \tag{92}$$

We can express the convolution in terms of the multiplication map $\mu \colon G \times G \to G$ as follows. We have he two projections π_1 and π_2:

$$\begin{tikzcd} & G \times G \arrow[rr, "\mu"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & G \\ G & & G & \end{tikzcd} \ . \tag{93}$$

We can pull f_1 back along π_1 and f_2 back along π_2 to get a function on $G \times G$:

$$f_1 \boxtimes f_2 \coloneqq \pi_1^* f_1 \pi_2^* f_2 \ . \tag{94}$$

Then we can push this along μ, and the result is the convolution:

$$f_1 * f_2 = \mu_* \left(f_1 \boxtimes f_2\right) = \int_\mu f_1 \boxtimes f_2 \ . \tag{95}$$

The upshot is that we can define the group algebra in this way whenever we have things which can be pulled and pushed like this.

For $G \circlearrowright V$, we have extended this to an action of $\mathbb{C}G \circlearrowright V$. Then G is **abelian** iff $(\mathbb{C}G, *)$ is a **commutative** algebra. So now the fundamental object over which representation theory of G will sheafify is:

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) \ , \tag{96}$$

and as it turns out

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) = \widehat{G} \ , \tag{97}$$

i.e.

$$\left(\mathbb{C}G, *\right) \simeq \left(\mathcal{O}\left(\widehat{G}\right), \cdot\right) \ . \tag{98}$$

This is a first version of the Fourier transform. The idea is that a map $\operatorname{Spec} k \to \operatorname{Spec} A$ is the same as a morphism $A \to k$, which is exactly a 1-dimensional representation of G, i.e. a character. Under this correspondence, the characters $\chi_t \in \mathbb{C}G$ for $t \in \widehat{G}$ correspond to points $\delta_t \in \mathcal{O}\left(\widehat{G}\right)$ for $t \in \widehat{G}$. Moreover, translation by g corresponds to multiplication by $\widehat{g}$, i.e. the character

$$t \mapsto \chi_t(g) \ . \tag{99}$$

We can rephrase this equivalence slightly to make it more evident that this is some version of the Fourier transform. If f is a function on G then we can write

$$f = \sum_{t \in \widehat{G}} \widehat{f}(t) \cdot \chi_t \ . \tag{100}$$

This is just expressing f in terms of the basis of characters. Then we can recover $\widehat{f}(t)$ as the coefficient of f in this orthonormal basis. We also have that

$$f * (-) = \widehat{f} \cdot (-) \ . \tag{101}$$

4.2.1. *Secret symmetry.* There is a secret symmetry here. $\widehat{G}$ is an abelian group itself under the operation of pointwise multiplication. I.e.

$$\chi_{t \cdot s} \coloneqq \chi_t \cdot \chi_s \ . \tag{102}$$

Call the corresponding abelian group the *dual group to G*.

To see that this is a good duality, note that there is a tautological map

$$\begin{aligned} G &\longrightarrow \widehat{\widehat{G}} \\ g &\overset{\sim}{\longmapsto} \{\chi \mapsto \chi(g)\} \ , \end{aligned} \tag{103}$$

which turns out to be an isomorphism.

We could have set this up in a more symmetric way. We have two projections:

$$\begin{tikzcd} & G \times \widehat{G} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & \widehat{G} \end{tikzcd} \tag{104}$$

and there is a tautological object, called the universal character, living over $G \times \widehat{G}$:

$$\begin{array}{c} \chi(-,-) \\ \downarrow \\ G \times \widehat{G} \end{array} \ , \tag{105}$$

i.e. a function on $G \times \widehat{G}$ given by:

$$\chi(g, t) = \chi_t(g) = \chi_g(t) \ . \tag{106}$$

Then the Fourier transform of $f \in \mathrm{Fun}(G)$ is given by pulling up to $G \times \widehat{G}$, multiplying by χ, and then summing up by pushing forward by π_2:

$$f \mapsto \pi_{2*}(\pi_1^* f \cdot \chi) \ . \tag{107}$$

Explicitly the Fourier transform is:

$$\widehat{f}(t) = \sum_g f(g)\, \overline{\chi}(g, t) \tag{108}$$

$$f(g) = \sum_t \widehat{f}(t)\, \chi(g, t) \ . \tag{109}$$

We have simultaneously diagonalized the action of all $g \in G$ on $\mathrm{Fun}(G)$.

For any V with a G action, we get a $(\mathbb{C}G, *)$ action on V so V spectrally decomposes over $\widehat{G}$, i.e.

$$V = \bigoplus_{t \in \widehat{G}} V_{\chi_t} \ , \tag{110}$$

where G acts by the eigenvalue specified by χ_t on the subspace V_{χ_t}.

This gives a complete picture of the complex representation theory of finite abelian group. The exact same formalism works in any setting with abelian groups. We will focus on the setting of topological groups and algebraic groups. Everything will mostly look the same, with the difference being what kind of functions we consider.

4.3. Pontrjagin duality. Let G be a locally compact abelian (LCA) group.

EXAMPLE 11. $\mathbb{Z}$, $\mathrm{U}(1)$, $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathbb{Q}_p^*$ are all (non-finite) examples.

Define the dual to be the collection of *unitary* characters

$$\widehat{G} = \mathrm{Hom}_{\mathbf{TopGrp}}(G, \mathrm{U}(1)) \ . \tag{111}$$

REMARK 14. We shouldn't be too shocked by replacing $\mathbb{C}^\times$ by $\mathrm{U}(1) \subset \mathbb{C}^\times$. If G is finite, all of the character theory was captured by $\mathrm{U}(1)$ anyway.

4.3.1. *Group algebra.* The spectrum will again be Spec of the group algebra, but we need to determine the appropriate definition of the group algebra in this context.

Endow G with a Haar measure. Before we had the counting measure, and could translate freely between functions and measures (so, in particular, they could both push and pull). Now $L^1(G)$ has a convolution structure in exactly the same way as in eq. (95):

$$f_1 * f_2 = \int_{g \in G} f_1(h)\, f_2\left(gh^{-1}\right) dg \ . \tag{112}$$

Just as before, this convolution comes from an adjunction. I.e. it satisfies a universal property in the world of C^*-algebras. If we have a representation:

$$G \to \mathrm{End}(V) \tag{113}$$

then this will correspond to a morphism

$$\left(L^1(G), *\right) \to \mathrm{End}(V) \tag{114}$$

of C^*-algebra.

The spectrum is the Gelfand spectrum:

$$\text{m-Spec}\left(L^1(G), *\right) = \widehat{G} \ . \tag{115}$$

This is a version of the Fourier transform which says that:

$$\left(L^1(G), *\right) \simeq \left(C_v\left(\widehat{G}\right), \cdot\right) \ , \tag{116}$$

where C_v denotes functions vanishing at ∞. Another version says that there is a tautological map:

$$G \to \widehat{\widehat{G}} \tag{117}$$

which is an isomorphism.

Again, this can be written in a symmetric way:

$$f \mapsto \pi_{2*}\left(\pi_1^* f \cdot \chi\right) \tag{118}$$

where

$$\begin{tikzcd} & \chi \arrow[d] & \\ & G \times \widehat{G} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & \widehat{G} \end{tikzcd} \quad . \tag{119}$$

For any notion of functions or distributions on G, we can perform this Fourier transform operation. The question is, given the type of functions we feed in, what type of functions do we get in the other side? For L^2 functions we simply get:

$$L^2\left(G\right) \xrightarrow{\sim} L^2\left(G\right) \ . \tag{120}$$

Any of these notions of a Fourier transform have the same general features. Some of which are as follows.

(i) Translation by a group element becomes pointwise multiplication.
(ii) Convolution also becomes pointwise multiplication.
(iii) Characters correspond to points.

4.3.2. *Fourier series.* Take $G = \mathrm{U}\left(1\right)$. Then

$$\widehat{G} = \mathrm{Hom}_{\mathbf{TopGrp}}\left(\mathrm{U}\left(1\right), \mathrm{U}\left(1\right)\right) = \mathbb{Z} \tag{121}$$

where $n \in \mathbb{Z}$ corresponds to

$$\left\{x \mapsto e^{2\pi i n x}\right\} \ . \tag{122}$$

Then the Fourier transform established an equivalence

$$L^2\left(\mathrm{U}\left(1\right)\right) \xrightarrow{\sim} L^2\left(\mathbb{Z}\right) = \ell^2 \ . \tag{123}$$

As before, this is symmetric. There is a universal character:

$$\chi\colon \ (x, n) \mapsto e^{2\pi i n x} \tag{124}$$

living over $\mathrm{U}\left(1\right) \times \mathbb{Z}$, and we have the usual projections:

$$\begin{tikzcd} & \mathrm{U}\left(1\right) \times \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathrm{U}\left(1\right) & & \mathbb{Z} \end{tikzcd} \ . \tag{125}$$

Then we can read it backwards. A character

$$\mathbb{Z} \to \mathrm{U}\left(1\right) \tag{126}$$

is determined by the image of $1 \in \mathbb{Z}$, so characters of $\mathbb{Z}$ are labelled by points of $\mathrm{U}\left(1\right)$.

In general Pontrjagin duality, G is compact iff $\widehat{G}$ is discrete. Concretely, for $n \in \mathbb{Z}$ we have

$$e^{2\pi i n x} \in L^2 \tag{127}$$

because G is compact. Similarly, because $\mathbb{Z}$ is discrete,

$$\delta_n \in \ell^2 \ . \tag{128}$$

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4.4. Fourier transform. Recall that the Pontrjagin dual of $\mathrm{U}(1)$ is $\mathbb{Z}$. The characters $\mathrm{U}(1) \to \mathrm{U}(1)$ are given by $z \mapsto z^n$ for any $n \in \mathbb{Z}$. Similarly, the dual of $\mathbb{Z}$ is $\mathrm{U}(1)$.

We can replace $\mathrm{U}(1)$ by a torus T, which is defined as:

$$T = \mathbb{R}^d/\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathrm{U}(1) \tag{129}$$

where $\Lambda \subset \mathbb{Z}^d$ is a full-rank lattice. The dual of T is the dual lattice:

$$\widehat{T} = \Lambda^\vee \ . \tag{130}$$

Similarly, the dual of the lattice Λ is the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathrm{U}(1) \ . \tag{131}$$

The classical Fourier transform takes G to be a real vector space. For $G = \mathbb{R}_x$ the dual is another copy of $\mathbb{R}$: $\widehat{G} = \mathbb{R}_t$. Performing the same operations as before, with universal character

$$\chi(x,t) = e^{2\pi i x t} \ , \tag{132}$$

we get the usual Fourier transform. For a general real vector space the dual is the dual vector space, and the character is

$$\chi(x,t) = e^{2\pi i \langle x,t\rangle} \ , \tag{133}$$

where the pairing is the usual one between the vector space and its dual.

Recall that in the context of the duality between $\mathrm{U}(1)$ and $\mathbb{Z}$, characters corresponded to points. This situation differs in the sense that characters are not L^2 anymore (since $\mathbb{R}$ is not compact, like $\mathrm{U}(1)$ is) and the points are not isolated (since $\mathbb{R}$ is not discrete like $\mathbb{Z}$). But we still have:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t)\, e^{2\pi i x t} dt \ . \tag{134}$$

The operation of differentiation d/dx corresponds with multiplication by t on the other side. We should think of d/dx as an infinitesimal version of group translation, which went to multiplication before. So this is part of the same framework where group theory on one side goes to geometry on the other.

In general, let G be a Lie group with abelian Lie algebra $\mathfrak{g}$. We have a map from $\mathfrak{g}$ to vector fields on G:

$$\mathfrak{g} \to \operatorname{Vect} G \subset \operatorname{Diff}(G) \tag{135}$$

and there is an adjunction between

$$\text{Forget}\colon \mathbf{Alg}_{\text{Assoc.}} \to \textbf{Lie-Alg} \tag{136}$$

and the functor

$$U\colon \textbf{Lie-Alg} \to \mathbf{Alg}_{\text{Assoc.}} \tag{137}$$

which sends a Lie algebra to the *universal enveloping algebra*. $\mathfrak{g}$ is abelian so the universal enveloping algebra is just the symmetric algebra

$$U(\mathfrak{g}) = \operatorname{Sym}^* \mathfrak{g} \ . \tag{138}$$

$U\mathfrak{g}$ is now a commutative algebra acting on $C^{\infty}(G)$. Therefore we can spectrally decompose/sheafify over

$$\operatorname{Spec} U\mathfrak{g} = \mathfrak{g}^* \ . \tag{139}$$

E.g. for d/dx acting on $\mathbb{R}_x$,

$$\mathfrak{g}^* = \mathbb{R}_t = \operatorname{Spec} RR\left[d/dx = t\right] \ . \tag{140}$$

EXAMPLE 12. The dual of $\mathrm{U}(1)$ is $\mathbb{Z} \subset \mathbb{R}_t$.

4.5. In quantum mechanics. Before we see where the Fourier transform comes up in quantum mechanics, we consider classical mechanics. If we want to a model a particle moving around in a manifold M, then the phase space is the cotangent bundle T^*M with positive coordinates q on M, and momenta coordinates p in the fiber direction. The observables form:

$$\text{functions on } M \otimes \operatorname{Sym} TM \ . \tag{141}$$

In quantum mechanics the space of states is replaced by $\mathcal{H} = L^2(M)$. The observables contain $\operatorname{Diff}(M)$, the differential operators on M. Inside of this we have two pieces:

$$\begin{tikzcd} \text{f'ns on } M \arrow[dr, hook] & \\ & \operatorname{Diff}(M) \\ TM \arrow[ur, hook] & \end{tikzcd} \ , \tag{142}$$

with

$$p_j = i\frac{d}{dq_j} \ . \tag{143}$$

In classical mechanics the analogous pieces commute. But here they commute up to a term: a tangent vector $\xi \in TM$ and a function $f \in \operatorname{Fun}(M)$ must satisfy

$$\xi f = f\xi + \hbar f' \ . \tag{144}$$

To summarize, states look like "$\sqrt{\text{observables}}$". The position operators:

$$q_j \cdot (-) \tag{145}$$

are diagonalized, on the other hand the momentum operators act as derivates.

We can also pass to the momentum picture where we diagonalize the p_i's (derivatives) instead. For $M = \mathbb{R}^n$ we have a natural basis of invariant vector fields (this is the advantage of having a group). Now we can simultaneously diagonalize

$$p_j = i\frac{d}{dx_j} \tag{146}$$

to identify

$$L^2\left(\mathbb{R}^n_q\right) \simeq L^2\left(\mathbb{R}^n_p\right) \ , \tag{147}$$

which is the Fourier transform. One might say that this is identifying quantum mechanics for G with quantum mechanics for the Pontrjagin dual $\widehat{G}$. This is the one-dimensional case of abelian duality, or "one-dimensional mirror symmetry".

4.6. Cartier duality. This is the same Fourier theory we've been doing, but in the context of algebraic geometry, i.e. instead of continuous, etc. functions we're considering algebraic functions. We will eventually see that this duality shows up in physics (electric-magnetic duality), as well as number theory (class field theory).

To say what a group is in the world of algebraic geometry, we need to review the notion of the functor of points. To a variety X we can associate a functor **cRing** $\to$ **Set** by sending a ring R to

$$X(R) = \operatorname{Hom}(\operatorname{Spec} R, X) \ . \tag{148}$$

As it turns out, specifying this functor is equivalent to specifying X itself.

A variety G is an *algebraic group* if the associated functor of points **cRing** $\to$ **Set** lifts to a functor landing in groups:

$$\begin{tikzcd} \mathbf{cRing} \arrow[r] \arrow[dr, dashed] & \mathbf{Set} \\ & \mathbf{Grp} \arrow[u] \end{tikzcd} \tag{149}$$

i.e. that

$$G(R) = \operatorname{Hom}(\operatorname{Spec} R, G) \tag{150}$$

is a group.

Remark 15. Sometimes other things are assumed in the definition of an algebraic group, which we do not assume here.

Example 13. Consider $\mathbb{A}^1 = \mathbb{G}_a$. As a functor, this sends

$$R \mapsto (R, +) \ . \tag{151}$$

This is saying that $\operatorname{Map}\left(X, \mathbb{A}^1\right) = \mathcal{O}(X)$.

Example 14. Consider $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m = \operatorname{Spec} k\left[t, t^{-1}\right]$. As a functor this sends

$$R \mapsto \left(R^\times, \cdot\right) \ . \tag{152}$$

Example 15. The integers form an algebraic group with functor of points given by:

$$\mathbb{Z}\colon R \mapsto (\mathbb{Z}, +) \ . \tag{153}$$

Let G be an abelian (algebraic) group. The *Cartier dual* of G is

$$\widehat{G} = \operatorname{Hom}_{\mathbf{Grp}_{\mathrm{Alg}}}(G, \mathbb{G}_m) \ . \tag{154}$$

where the dualizing object $\mathbb{G}_m$ comes from $\operatorname{Aut}_k = \mathbb{G}_m$.

Example 16. Let $G = \mathbb{Z}/n$. The Cartier dual is

$$\widehat{\mathbb{Z}/n} = \operatorname{Hom}(\mathbb{Z}/n, \mathbb{G}_m) \simeq \mu_n \tag{155}$$

where μ_n denotes the nth roots of unity. As a functor, this sends:

$$R \mapsto n\text{th roots of unity in } R \ . \tag{156}$$

Example 17. The dual of the integers is $\widehat{\mathbb{Z}} \simeq \mathbb{G}_m$, since $\mathbb{Z} \to \mathbb{G}_m$ is determined by the image of 1. Similarly:

$$\widehat{\mathbb{G}_m} = \operatorname{Hom}_{\mathbf{TopGrp}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z} = \{z \mapsto z^n\} \ . \tag{157}$$

EXAMPLE 18. More generally, the dual to a lattice Λ will be the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m \ , \tag{158}$$

so (over $\mathbb{C}$) this looks roughly like $(\mathbb{C}^\times)^{\operatorname{rank}\Lambda}$. Similarly a torus T gets exchanged with the dual lattice $\Lambda^\vee$.

To avoid technicalities, assume G is finite.[3] Consider the collection of functions on G, $\mathcal{O}(G)$. Dualizing and taking Spec gives the dual group:

$$\widehat{G} = \operatorname{Spec} \mathcal{O}(G)^* \ . \tag{159}$$

Rather than talking about a group algebra, functions on G has the structure of a group coalgebra as follows. The multiplication map on G induces a coproduct on $\mathcal{O}(G)$:

$$\begin{gathered} G \times G \xrightarrow{\quad \mu \quad} G \\ \mathcal{O}(G) \xrightarrow{\Delta := \mu^*} \mathcal{O}(G) \otimes \mathcal{O}(G) \ . \end{gathered} \tag{160}$$

This makes $\mathcal{O}(G)$ into a *Hopf algebra*, i.e. $\mathcal{O}(G)$ has a multiplication, and a comultiplication Δ. In fact, this is a finite-dimensional commutative and cocommutative Hopf algebras. The study of these Hopf algebras turns out to be equivalent to the study of finite abelian group schemes.

EXAMPLE 19 ("Fourier transform"). Let $\operatorname{char} k = 0$. The Cartier dual of $\mathbb{G}_a$ is the formal completion of itself:

$$\widehat{\mathbb{G}_a} = \widehat{\mathbb{G}_a} \ , \tag{161}$$

where the formal completion of $\mathbb{G}_a$ is given by:

$$\widehat{\mathbb{G}_a} = \bigcup \operatorname{Spec} k[t] / (t^n) \ . \tag{162}$$

The Cartier dual of an n-dimensional vector space V is

$$\widehat{V} = \widehat{V^*} \ . \tag{163}$$

The character of

$$\mathbb{G}_a = \operatorname{Spec} k[x] \tag{164}$$

should be

$$e^{xt} = \sum \frac{(xt)^n}{n!} \ , \tag{165}$$

but we need this to be a finite sum. This makes sense if t is nilpotent, i.e. there is $N \gg 0$ such that $t^N = 0$.

So when this is true, i.e. t is "close to 0", the function $e^{\langle x,t \rangle}$ is well-defined for $x \in V$ and $t \in V^*$, and is the character for V.

Similarly the dual of the completion is $\mathbb{G}_a$:

$$\mathbb{G}_a = \widehat{\widehat{\mathbb{G}_a}} \ . \tag{166}$$

[3]Formally we're assuming that G is a finite abelian group scheme.

Recall that Fourier duality exchanges

$$(\mathrm{Fun}\,(G)\,, *) \simeq \left(\mathrm{Fun}\left(\widehat{G}\right), \cdot\right) \ . \tag{167}$$

Then $\mathbf{Rep}\,(G)$ became spectrally decomposed over $\widehat{G}$. In algebraic geometry, a representation V is a *comodule* for $\mathcal{O}\,(G)$. I.e. we have a map $G \times V \to V$, and passing to functions gives us a map

$$V \to \mathcal{O}\,(G) \otimes V \ . \tag{168}$$

Furthermore,

$$\mathbf{Rep}\,(G) \simeq \mathcal{O}\,(G)\,\textbf{-coMod} \simeq \mathcal{O}\left(\widehat{G}\right)\textbf{-Mod} \simeq \mathbf{QC}\left(\widehat{G}\right) \ . \tag{169}$$

4.6.1. *Fourier series examples.*

Example 20. For $G = \mathbb{Z}$, the category of representations is given by

$$\mathbf{Rep}\,(\mathbb{Z}) = k\left[z, z^{-1}\right]\textbf{-Mod} \tag{170}$$

where z is the action of $1 \in \mathbb{Z}$. This action must be invertible, which is why z^{-1} is included. Then the duality tells us that:

$$\mathbf{Rep}\,(\mathbb{Z}) = k\left[z, z^{-1}\right]\textbf{-Mod} = \mathcal{O}\,(\mathbb{G}_m)\,\textbf{-Mod} = \mathbf{QC}\,(\mathbb{G}_m) \ . \tag{171}$$

Example 21. A vector space and an endomorphism (matrix) gives us

$$\mathbf{QC}\left(\mathbb{A}^1 = \mathrm{Spec}\, k\,[z]\right) \tag{172}$$

but if we have an automorphism (invertible matrix) then we get

$$\mathbf{Rep}\,(\mathbb{Z}) \leftrightarrow \mathbf{QC}\left(\mathbb{A}^1 \setminus \{0\}\right) \ . \tag{173}$$

Example 22. In algebraic geometry

$$\mathbf{Rep}\,(\mathbb{G}_m) = \mathbb{Z}\text{-graded vector space} = \mathbf{QC}\,(\mathbb{Z}) \tag{174}$$

where

$$V \simeq \bigoplus_{n \in \mathbb{Z}} V_n \tag{175}$$

and $z \in \mathbb{G}_m$ acts on V_n by z^n.

Example 23 (Topological example). The following is an example of Fourier series from topology. Let M^3 be a compact oriented three-manifold.[4] Let

$$G = \mathrm{Pic}\, M^3 \tag{176}$$

consist of complex line bundles (or $\mathrm{U}\,(1)$-bundles) on M^3 up to isomorphism. This forms an abelian group under tensor product. We can think of this as:

$$G = \mathrm{Map}\left(M^3, B\,\mathrm{U}\,(1)\right) \tag{177}$$

where $B\,\mathrm{U}\,(1)$ denotes the classifying space of $\mathrm{U}\,(1)$. Up to homotopy we can think of $B\,\mathrm{U}\,(1)$ as:

$$B\,\mathrm{U}\,(1) \simeq \mathbb{CP}^\infty \simeq K\,(\mathbb{Z}, 2) \ . \tag{178}$$

[4] This will be the three-manifold on which we do electromagnetism. This Cartier duality will give us electric-magnetic duality. If we think of a number field as a three-manifold, then this duality fits into the framework of class field theory.

For $\mathcal{L} \in G$, we can attach the first Chern class:

$$c_1(\mathcal{L}) \in H^2(M, \mathbb{Z}) \ , \tag{179}$$

which is a complete invariant of the line bundle. So we can take

$$G = \Lambda = H^2(M, \mathbb{Z}) \ . \tag{180}$$

The Cartier dual is

$$\widehat{G} \simeq \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m = \mathrm{Hom}\left(H_1(M, \mathbb{Z}), \mathbb{G}_m\right) = \mathrm{Hom}\left(\pi_1(M), \mathbb{G}_m\right) \tag{181}$$

where the last line follows from the fact that $H_1 = \pi_1^{\mathrm{ab}}$, and Hom from the abelianization is the same as Hom from the whole thing. But this is just flat $\mathbb{C}^\times$-bundles on M, i.e.

$$\widehat{G} = \mathbf{Loc}_{\mathbb{C}^\times}(M) \ . \tag{182}$$

This shouldn't be that surprising since G looks something like $\mathbb{Z}^n$, and this dual $\widehat{G}$ looks like $(\mathbb{C}^\times)^n$.

We can also replace line bundles by torus bundles, i.e. we can pass from $\mathrm{U}(1)$ to $\mathrm{U}(1)^n \simeq T$. Then

$$\mathrm{Bun}_T(M) \leftrightarrow \mathbf{Loc}_{T^\vee} M \ . \tag{183}$$

where $\mathbf{Loc}_{T^\vee} M$ consists of isomorphism classes of flat $T^\vee$-bundles over M. The LHS still looks like a lattice Λ, and the RHS still looks like a torus $(\mathbb{C}^\times)^n$.

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