Between electric-magnetic duality and the Langlands program

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with any corrections or concerns.

Contents

CHAPTER 1

Overview

The geometric Langlands program is some kind of middle-ground between number theory and physics. Another point of view is that we will be navigating the narrow passage between the whirlpool Charybdis (physics) and the six-headed monster Scylla (number theory), as in Odysseus' travels.[1]

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The inspiration for much of this course comes from [**Mac78**], which provides a historical account of harmonic analysis, focusing on the idea that function spaces can be decomposed using symmetry. This theme has long-standing connections to physics and number theory.

The spirit of what we will try to do is some kind of harmonic analysis (fancy version of Fourier theory) which will appear in different guises in both physics and number theory.

1. Modular/automorphic forms

1.1. Rough idea. The theory of modular forms is a kind of harmonic analysis/quantum mechanics on arithmetic locally symmetric spaces. The canonical example of a locally symmetric space is given by the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane $\mathbb{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2$. I.e. we are considering the quotient

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \tag{1}$$

as in fig. 1.

[1] One can expand this analogy. Calypso's island is probably derived algebraic geometry (DAG), etc.

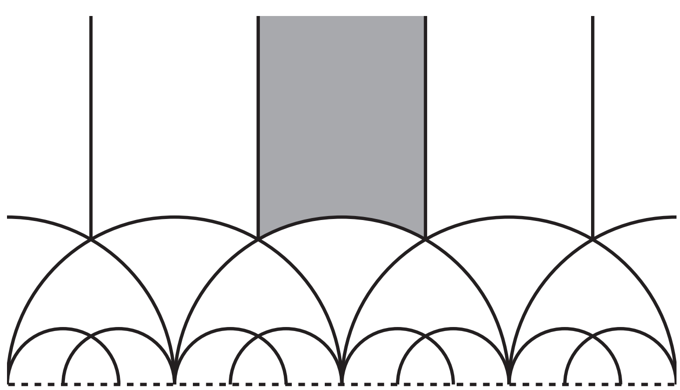

Figure 1. Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray.

For a general reductive algebraic group G we can consider the space

$$\Gamma \backslash G / K \tag{2}$$

where Γ is an arithmetic lattice, and K is a maximal compact subgroup. For now we restrict to

$$G = \mathrm{SL}_2(\mathbb{R}) \qquad \Gamma = \mathrm{SL}_2(\mathbb{Z}) \qquad K = \mathrm{SO}_2 \ .$$

We want to do harmonic analysis on this space, i.e. we want to decompose spaces of functions on this in a meaningful way. In the case of quantum mechanics we're primarily interested in L^2 functions:

$$L^2(\Gamma \backslash G / K) \ , \tag{3}$$

and on this we have an action of the hyperbolic Laplace operator. I.e. we want to study the spectral theory of this operator.

The same information, possibly in a more accessible form, is given by getting rid of the K. That is, we can just study L^2 functions on

$$\Gamma \backslash G = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \ , \tag{4}$$

which is the unit tangent bundle, a circle bundle over the space we had before. Instead of studying the Laplacian, this is a homogeneous space so we can study the action of all of $\mathrm{SL}_2(\mathbb{R})$.

One can expand this to include differentials and pluri-differentials, i.e. sections of (powers of) the canonical bundle:

$$\Gamma\left(\Gamma \backslash \mathbb{H}, \omega^{k/2}\right) \ . \tag{5}$$

Definition 1. The Δ-eigenfunctions in $L^2(\Gamma \backslash G / K)$ are called *Maass forms*. *Modular forms* of weight k are holomorphic sections of $\omega^{k/2}$.

Remark 1. For a topologist, one might instead want to study (topological) cohomology (instead of forms) with coefficients in some local system (twisted coefficients). Indeed, modular forms can also arise by looking at the (twisted) cohomology of $\Gamma \backslash \mathbb{H}$. This is known as Eichler-Shimura theory.

One might worry that this leaves the world of quantum mechanics, but after passing to cohomology we're doing what is called *topological* quantum mechanics. We will be more concerned with this than honest quantum mechanics.

The idea is that there are no dynamics in this setting. We're just looking at the ground states, so the Laplacian is 0, and we're just looking at harmonic things. And this really has to do with topology and cohomology. But modular forms are some kind of ground states.

Remark 2. If we take general G, K, and Γ then we get the more general theory of *automorphic forms*.

Example 1. If we start with $G = \mathrm{Sp}_{2n}(\mathbb{R})$ and take $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, $K = \mathrm{SO}_n$ then we get *Siegel modular forms*.

1.2. Structure. There is a long history of thinking of this problem[2] as quantum mechanics on this locally symmetric space. But there is a lot more structure going on in the number theory than seems to be present in the quantum mechanics of a particle moving around on this locally symmetric space.

Restrict to the case $G = \mathrm{SL}_2(\mathbb{R})$.

[2] The problem of understanding L^2 functions on a locally symmetric space.

1.2.1. *Number field.* The question of understanding

$$L^2\left(\mathrm{SL}_2\,\mathbb{Z}\backslash\,\mathrm{SL}_2\,\mathbb{R}/\,\mathrm{SO}_2\right) \tag{6}$$

has an analogue for any number field. We can think of $\mathbb{Z}$ as being the ring of integers in the rational numbers:

$$\mathbb{Z} = \mathcal{O}_{\mathbb{Q}} \tag{7}$$

and from this we get a lattice $\mathrm{SL}_2\left(\mathcal{O}_{\mathbb{Q}}\right)$. Writing it this way, we see that we can replace $\mathbb{Q}$ by any finite extension F, and $\mathbb{Z}$ becomes the ring of integers $\mathcal{O}_F$:

$$\begin{aligned} \mathbb{Q} &\rightsquigarrow F \\ \mathbb{Z} &\rightsquigarrow \mathcal{O}_F\ . \end{aligned} \tag{8}$$

The upshot is that when we replace Q with some other number field $F/\mathbb{Q}$, then the space $\mathcal{M}_{G,\mathbb{Q}}$ becomes some space $\mathcal{M}_{G,F}$. Then we linearize by taking either L^2 or H^* of $\mathcal{M}_{G,F}$.

Example 2. This holds for all reductive algebraic groups G, but let $G = \mathrm{PSL}_2\,\mathbb{R}$. Then

$$\mathcal{M}_{G,\mathbb{Q}} = \mathrm{PSL}_2\,\mathbb{Z}\backslash\,\mathrm{PSL}_2\,\mathbb{R}/\,\mathrm{SO}_2 \tag{9}$$

is the locally symmetric space in fig. 1. If we replace $\mathbb{Q}$ with an arbitrary number field $F/\mathbb{Q}$, then we get

$$\mathcal{M}_{G,F} = \mathrm{PSL}_2\left(\mathcal{O}_F\right)\backslash\,\mathrm{PSL}_2\left(F\otimes_{\mathbb{Q}}\mathbb{R}\right)/K\ . \tag{10}$$

Note that

$$F\otimes_{\mathbb{Q}}\mathbb{R} \simeq \mathbb{R}^{\times r_1}\times\mathbb{C}^{\times r_2} \tag{11}$$

where r_1 is the number of real embeddings of F, and r_2 is the number of conjugate pairs of complex embeddings.

Example 3. Let $F = \mathbb{Q}\left(\sqrt{d}\right)$. If it is real ($d \geq 0$) then $r_1 = 2$ (corresponding to $\pm\sqrt{d}$) and $r_2 = 0$, so we get

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right)\otimes_{\mathbb{Q}}\mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{R}\times\mathrm{PSL}_2\,\mathbb{R}\ . \tag{12}$$

This leads to what are called *Hilbert modular forms.*

If it is imaginary ($d < 0$) then $r_1 = 0$, $r_2 = 1$, and

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right)\otimes_{\mathbb{Q}}\mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{C}\ . \tag{13}$$

In this case the maximal compact is $\mathrm{SO}_3\,\mathbb{R}$, and the quotient:

$$\mathbb{H}^3 = \mathrm{PSL}_2\,\mathbb{C}/\,\mathrm{SO}_3\,\mathbb{R} \tag{14}$$

is hyperbolic 3-space. Now we need to mod out (on the left) by a lattice, and the result is some hyperbolic manifold which is a 3-dimensional version of the picture in fig. 1.

Remark 3. The point is that the real group we get after varying the number field is not that interesting, just some copies of PSL_2. But the lattice we are modding out by depends more strongly on the number field, so this is the interesting part.

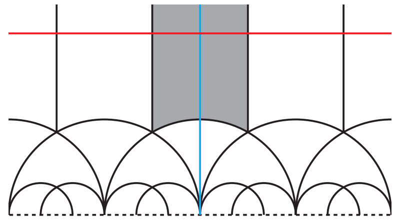

FIGURE 2. Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray. One can define a "period" as taking a modular form and integrating it, e.g. on the red or blue line.

1.2.2. *Conductor/ramification data.* Fixing the number field $F = \mathbb{Q}$, we can vary the "conductor" or "ramification data". The idea is as follows. The locally symmetric space $\Gamma\backslash\mathbb{H}$ has a bunch of covering spaces of the form $\Gamma'\backslash\mathbb{H}$, where Γ' is some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. So we can replace Γ by Γ'.

We won't define congruence subgroups in general, but there are basically two types. For $N \in \mathbb{Z}$, we fix subgroups:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N \right\} \tag{15}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \right\} . \tag{16}$$

The idea is that we start with the conductor N and the lattice Γ, and then we modify Γ at the divisors of N. Note that even in this setting we have the choice of $\Gamma(N)$ or $\Gamma_0(N)$. Really the collection of variants has a lot more structure. The local data at p has to do with the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

1.2.3. *Action of Hecke algebra.* We have seen that our Hilbert space depends on the group, the number field, and some ramification data. A very important aspect of this theory is that this vector space (of functions) carries a lot more structure. There is a huge "degeneracy" here in the sense that the eigenspaces of the Laplacian are much bigger than one might have guessed (not one-dimensional).

This degeneracy is given by the theory of *Hecke operators*. This says that the Laplacian Δ is actually a part of a huge commuting family of operators. In particular, these all act on the eigenspaces of the Laplacian. For p a prime (p unramified, i.e. $p \not| N$) we have the Hecke operator at p, T_p. Then

$$\bigoplus_p \mathbb{C}[T_p] \circlearrowright L^2(\Gamma\backslash G/K) . \tag{17}$$

This is some kind of "quantum integrable system" because having so many operators commute with the Hamiltonian tells us that a lot of quantities are conserved.[3]

1.2.4. *Periods/states.* There is a special collection of measurements we can take of modular forms, called periods. A basic example is given by integrating a modular form on the line $i\mathbb{R}_+ \subset \mathbb{H}$ as in fig. 2. This is how Hecke defined the L-function.

[3] This example is often included in the literature as an example of quantum chaos (the opposite of integrability). The chaotic aspect has nothing to do with the discrete subgroup Γ. Specifically this fits into the study of "arithmetic quantum chaos" which more closely resembles the study of integral systems.

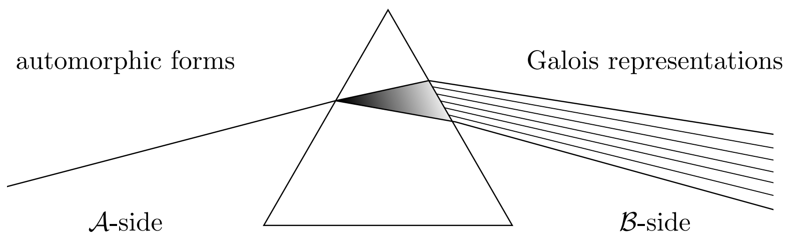

FIGURE 3. Just as light is decomposed by a prism, this spectral decomposition breaks automorphic forms ($\mathcal{A}$-side) up into Galois representations of number fields ($\mathcal{B}$-side).

The takeaway is that we have a collection of measurements/states with very good properties, and then we can study modular forms by measuring them with these periods.

1.2.5. *Langlands functoriality.* There is a collection of somewhat mysterious operators whose action corresponds to varying the group G.

2. The Langlands program and TFT

2.1. Overview. We have seen that for a choice of reductive algebraic group G and number field $F/\mathbb{Q}$, we get a locally symmetric space

$$\mathcal{M} = \mathcal{M}_{G,F} = \text{“arithmetic lattice”}\backslash\text{real group/maximal compact} . \tag{18}$$

This can be thought of as some space of G-bundles

$$\mathcal{M}_{G,F} = \text{“}\operatorname{Bun}_G(\operatorname{Spec}\mathcal{O}_F)\text{''} . \tag{19}$$

Then we linearize this space by taking either L^2 or H^*.

Starting with this theory of automorphic forms, we spectrally decompose under the action of the Hecke algebra. Then the Langlands program says that the pieces of this decomposition correspond to Galois representations. We can think of the theory of automorphic forms as being fed into a prism, and the colors coming out on the other side are Galois representations as in fig. 3. More specifically, the “colors” are representations:

$$\operatorname{Gal}\left(\overline{F}/F\right) \to G^\vee_{\mathbb{C}} . \tag{20}$$

EXAMPLE 4. If $G = \mathrm{GL}_2\,\mathbb{R}$, then $G^\vee = \mathrm{GL}_2\,\mathbb{C}$. Let E be an elliptic curve. Then

$$H^1(E/\mathbb{Q}) \tag{21}$$

is a 2-dimensional representation of $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. This is the kind of representation you get in this setting.

EXAMPLE 5. The representations in example 4 are very specific to GL_2. If we started with $\mathrm{GL}_3(\mathbb{R})$ instead, the associated locally symmetric space $\mathrm{O}_3 \backslash \mathrm{GL}_3\,\mathbb{R}/\,\mathrm{GL}_3\,\mathbb{Z}$ is not a complex manifold, it's not the moduli space of anything.

The goal is to match all of this structure in section 1.2 with a problem in physics, but ordinary quantum mechanics will be too simple. On the physics side we will instead consider quantum field theory.

Slogan: the Langlands program is part of the study of 4-dimensional (arithmetic, topological) quantum field theory.

The idea is that the Langlands program is an equivalence of 4-dimensional arithmetic TFTs

$$\begin{array}{ccc} \mathcal{A}_G & \simeq & \mathcal{B}_{G^\vee} \\ \text{automorphic} & & \text{spectral} \end{array} \tag{22}$$

called the $\mathcal{A}$ and $\mathcal{B}$-side theories.

The *topological* means we are throwing out the dynamics and only looking at the ground states. This is the analogue of only looking at the harmonic forms rather than the whole spectrum of the Laplacian. The *arithmetic* means that we're following the paradigm of *arithmetic topology*. The idea is that we're trying to make an analogy between number fields and some geometric objects.

2.2. Arithmetic topology.

2.2.1. *Weil's Rosetta Stone.* In a letter to Simone Weil [**Kri05**], André Weil explained a beautiful analogy, now known as *Weil's Rosetta Stone*. This establishes a three-way analogy between number fields, function fields, and Riemann surfaces.

The general idea is as follows. $\operatorname{Spec}\mathbb{Z}$ is some version of a curve, with points $\operatorname{Spec}\mathbb{F}_p$ associated to different primes. $\operatorname{Spec}\mathbb{Z}_p$ is a version of a disk around the point, and $\operatorname{Spec}\mathbb{Q}_p$ is a version of a punctured disk around that point. This is analogous to the usual picture of an algebraic curve.

Curve	$\operatorname{Spec}\mathbb{F}_q[t]$	$\operatorname{Spec}\mathbb{Z}$
Point	$\operatorname{Spec}\mathbb{F}_p$	$\operatorname{Spec}\mathbb{F}_p$
Disk	$\operatorname{Spec}\mathbb{F}_t[[t]]$	$\operatorname{Spec}\mathbb{Z}_p$
Punctured disk	$\operatorname{Spec}\mathbb{F}_q((t))$	$\operatorname{Spec}\mathbb{Q}_p$

In general, let $F/\mathbb{Q}$ be a number field. Then we can consider $\mathcal{O}_F$, and $\operatorname{Spec}\mathcal{O}_F$ has points corresponding to primes in $\mathcal{O}_F$. The analogy between number fields and function fields is as follows. Start with a smooth projective curve $C/\mathbb{F}_q$ over a finite field. Then the analogue to F is the field of rational functions, $\mathbb{F}_q(C)$. The analogue to $\mathcal{O}_F$ is the ring of regular functions, $\mathbb{F}_q[C]$. Finally points of $\operatorname{Spec}\mathcal{O}_F$ correspond to points of C.

Now we might want to replace C with a Riemann surface. So let $\Sigma/\mathbb{C}$ be a compact Riemann surface. Then primes in $\mathcal{O}_F$ (and so points of C) correspond to points of Σ. The field of meromorphic rational functions on Σ, $\mathbb{C}(\Sigma)$, is the analogue of F. To get an analogue of $\mathcal{O}_F$ we have to remove some points of Σ (we wouldn't get any functions on the compact curve). The point is that number fields have some points at ∞, so the analogue isn't really a compact Riemann surface, but with some marked points. So the analogue of $\mathcal{O}_F$ consists of functions on Σ which are regular away from these points.

This is summarized in table 1.

2.2.2. *Missing chip.* Now we want to take the point of view that there was a chip missing from this Rosetta stone, and we were supposed to consider 3-manifolds rather than Riemann surfaces. The idea is that $\Sigma/\mathbb{C}$ really corresponds to $C/\overline{\mathbb{F}_q}$. This is manifested in the following way. To study points, we study maps:

$$\operatorname{Spec}\mathbb{F}_q \hookrightarrow C\ . \tag{23}$$

TABLE 1. Let $F/\mathbb{Q}$ be a number field, $C/\mathbb{F}_q$ be a smooth projective curve over a finite field, and let $\Sigma/\mathbb{C}$ be a compact Riemann surface. $\mathbb{F}_q(C)$ denotes the field of rational functions, $\mathbb{F}_q[C]$ denotes the ring of regular functions, and $\mathbb{C}(\Sigma)$ denotes the meromorphic rational functions on Σ.

Number fields	Function fields	Riemann surfaces
$F/\mathbb{Q}$	$\mathbb{F}_q(C)$	$\mathbb{C}(\Sigma)$
$\mathcal{O}_F$	$\mathbb{F}_q[C]$	f'ns regular away from marked points of Σ
$\operatorname{Spec}\mathcal{O}_F$	points of C	$x \in \Sigma$

But from the point of view of étale topology, $\operatorname{Spec}\mathbb{F}_q$ is not really a point. It is more like a circle in the sense that

$$\operatorname{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) = \widehat{\mathbb{Z}} = \pi_1^{\text{étale}}\left(\operatorname{Spec}\mathbb{F}_q\right) \tag{24}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion. So it's better to imagine this as a modified circle, where this $\widehat{\mathbb{Z}}$ is generated by the Frobenius.

So we have realized that curves over $\mathbb{F}_q$ have too much internal structure to match with a Riemann surface. There is always a map

$$\operatorname{Spec}\overline{\mathbb{F}_q} \to \operatorname{Spec}\mathbb{F}_q \tag{25}$$

and we can lift our curve to $\mathbb{F}_q$. This corresponds to unwrapping these circle, i.e. replacing them by their universal cover. So their is some factor of $\mathbb{R}$ which doesn't play into the topology/cohomology.

REMARK 4. The map $\operatorname{Spec}\mathbb{F}_{q^n} \to \operatorname{Spec}\mathbb{F}_q$ is analogous to the usual n-fold cover of the circle.

To fix the Rosetta Stone, we replace a Riemann surface Σ by certain a Σ-bundle over S^1. Explicitly, if we have Σ and a diffeomorphism φ, we can form the mapping torus:

$$\Sigma \times I/\left((x,0) \sim (\varphi(x), 1)\right) \ . \tag{26}$$

The idea is that if we start with a curve over a finite field, the diffeomorphism φ is like the Frobenius.

This fits with the existing theory of arithmetic topology, sometimes known as the "knots and primes" analogy. The theory was started in a letter from Mumford to Mazur, but can be attributed to many people such as Mazur [**Maz73**], Manin, Morishita [**Mor10**], Kapranov [**Kap95**], and Reznikov. The recent work [**Kim15, CKK+19**] of Minhyong Kim plays a central role.

The upshot is that we are thinking of all three objects in the Rosetta stone as three-manifolds. Then primes (on the number field side) correspond to circles (rather than points), they correspond to knots in the 3-manifold.

REMARK 5. Lots of aspects of this dictionary are spelled out, but one should be wary of using it too directly. Rather we should think of this as telling us that there are several classes of '3-manifolds': ordinary 3-manifolds, function fields over finite fields, and number fields.

Bibliography

[CKK+19] Hee-Joong Chung, Dohyeong Kim, Minhyong Kim, George Pappas, Jeehoon Park, and Hwajong Yoo, *Erratum: "Abelian arithmetic Chern-Simons theory and arithmetic linking numbers"*, Int. Math. Res. Not. IMRN (2019), no. 18, 5854–5857. MR 4012129 11

[Kap95] M. M. Kapranov, *Analogies between the Langlands correspondence and topological quantum field theory*, Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), Progr. Math., vol. 131, Birkhäuser Boston, Boston, MA, 1995, pp. 119–151. MR 1373001 11

[Kim15] Minhyong Kim, *Arithmetic chern-simons theory i*, 2015. 11

[Kri05] Martin H. Krieger, *A 1940 letter of André Weil on analogy in mathematics*, Notices Amer. Math. Soc. **52** (2005), no. 3, 334–341, Excerpted from ıt Doing mathematics [World Scientific Publishing Co., Inc., River Edge, NJ, 2003; MR1961400]. MR 2125268 10

[Mac78] George W. Mackey, *Harmonic analysis as the exploitation of symmetry—a historical survey*, Rice Univ. Stud. **64** (1978), no. 2-3, 73–228, History of analysis (Proc. Conf., Rice Univ., Houston, Tex., 1977). MR 526217 5

[Maz73] Barry Mazur, *Notes on étale cohomology of number fields*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 521–552 (1974). MR 344254 11

[Mor10] Masanori Morishita, *Analogies between knots and primes, 3-manifolds and number rings [translation of mr2208305]*, vol. 23, 2010, Sugaku expositions, pp. 1–30. MR 2605747 11