

**Between electric-magnetic duality and the  
Langlands program**

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## CHAPTER 1

### Overview

The geometric Langlands program is some kind of middle-ground between number theory and physics. Another point of view is that we will be navigating the narrow passage between the whirlpool Charybdis (physics) and the six-headed monster Scylla (number theory), as in Odysseus' travels.<sup>1</sup>

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The inspiration for much of this course comes from [Mac78], which provides a historical account of harmonic analysis, focusing on the idea that function spaces can be decomposed using symmetry. This theme has long-standing connections to physics and number theory.

The spirit of what we will try to do is some kind of harmonic analysis (fancy version of Fourier theory) which will appear in different guises in both physics and number theory.

#### 1. Modular/automorphic forms

**1.1. Rough idea.** The theory of modular forms is a kind of harmonic analysis/quantum mechanics on arithmetic locally symmetric spaces. The canonical example of a locally symmetric space is given by the fundamental domain for the action of  $\mathrm{SL}_2(\mathbb{Z})$  on the upper half-plane  $\mathbb{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2$ . I.e. we are considering the quotient

$$(1) \quad \mathcal{M}_{\mathrm{SL}_2 \mathbb{R}} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2$$

as in fig. 1.

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<sup>1</sup>One can expand this analogy. Calypso's island is probably derived algebraic geometry (DAG), etc.

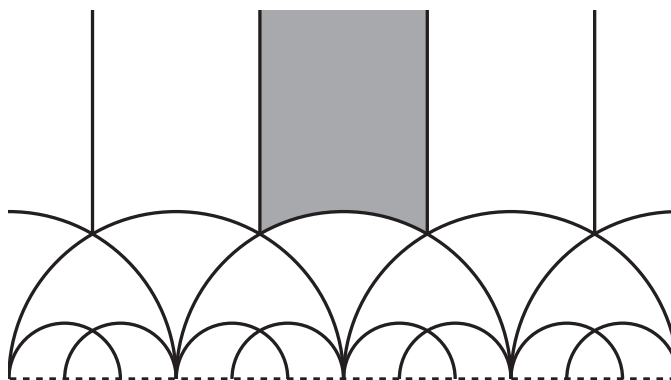


FIGURE 1. Fundamental domain for the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$  in gray.

For a general reductive algebraic group  $G$  we can consider the space

$$(2) \quad \mathcal{M}_G = \Gamma \backslash G / K$$

where  $\Gamma$  is an arithmetic lattice, and  $K$  is a maximal compact subgroup. For now we restrict to

$$G = \mathrm{SL}_2(\mathbb{R}) \quad \Gamma = \mathrm{SL}_2(\mathbb{Z}) \quad K = \mathrm{SO}_2 .$$

We want to do harmonic analysis on this space, i.e. we want to decompose spaces of functions on this in a meaningful way. In the case of quantum mechanics we're primarily interested in  $L^2$  functions:

$$(3) \quad L^2(\Gamma \backslash G / K) ,$$

and on this we have an action of the hyperbolic Laplace operator. I.e. we want to study the spectral theory of this operator.

The same information, possibly in a more accessible form, is given by getting rid of the  $K$ . That is, we can just study  $L^2$  functions on

$$(4) \quad \Gamma \backslash G = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) ,$$

which is the unit tangent bundle, a circle bundle over the space we had before. Instead of studying the Laplacian, this is a homogeneous space so we can study the action of all of  $\mathrm{SL}_2(\mathbb{R})$ .

One can expand this to include differentials and pluri-differentials, i.e. sections of (powers of) the canonical bundle:

$$(5) \quad \Gamma \left( \Gamma \backslash \mathbb{H}, \omega^{k/2} \right) .$$

DEFINITION 1. The  $\Delta$ -eigenfunctions in  $L^2(\Gamma \backslash G / K)$  are called *Maass forms*. *Modular forms* of weight  $k$  are holomorphic sections of  $\omega^{k/2}$ .

REMARK 1. For a topologist, one might instead want to study (topological) cohomology (instead of forms) with coefficients in some local system (twisted coefficients). Indeed, modular forms can also arise by looking at the (twisted) cohomology of  $\Gamma \backslash \mathbb{H}$ . This is known as Eichler-Shimura theory.

One might worry that this leaves the world of quantum mechanics, but after passing to cohomology we're doing what is called *topological* quantum mechanics. We will be more concerned with this than honest quantum mechanics.

The idea is that there are no dynamics in this setting. We're just looking at the ground states, so the Laplacian is 0, and we're just looking at harmonic things. And this really has to do with topology and cohomology. But modular forms are some kind of ground states.

REMARK 2. If we take general  $G$ ,  $K$ , and  $\Gamma$  then we get the more general theory of *automorphic forms*.

EXAMPLE 1. If we start with  $G = \mathrm{Sp}_{2n}(\mathbb{R})$  and take  $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$ ,  $K = \mathrm{SO}_n$  then we get *Siegel modular forms*.

**1.2. Structure.** There is a long history of thinking of this problem<sup>2</sup> as quantum mechanics on this locally symmetric space. But there is a lot more structure going on in the number theory than seems to be present in the quantum mechanics of a particle moving around on this locally symmetric space.

Restrict to the case  $G = \mathrm{SL}_2(\mathbb{R})$ .

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<sup>2</sup> The problem of understanding  $L^2$  functions on a locally symmetric space.

1.2.1. *Number field.* The question of understanding

$$(6) \quad L^2(\mathrm{SL}_2 \mathbb{Z} \backslash \mathrm{SL}_2 \mathbb{R} / \mathrm{SO}_2)$$

has an analogue for any number field. We can think of  $\mathbb{Z}$  as being the ring of integers in the rational numbers:

$$(7) \quad \mathbb{Z} = \mathcal{O}_{\mathbb{Q}}$$

and from this we get a lattice  $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}})$ . Writing it this way, we see that we can replace  $\mathbb{Q}$  by any finite extension  $F$ , and  $\mathbb{Z}$  becomes the ring of integers  $\mathcal{O}_F$ :

$$(8) \quad \begin{aligned} \mathbb{Q} &\leadsto F \\ \mathbb{Z} &\leadsto \mathcal{O}_F . \end{aligned}$$

The upshot is that when we replace  $\mathbb{Q}$  with some other number field  $F/\mathbb{Q}$ , then the space  $\mathcal{M}_{G,\mathbb{Q}}$  becomes some space  $\mathcal{M}_{G,F}$ . Then we linearize by taking either  $L^2$  or  $H^*$  of  $\mathcal{M}_{G,F}$ .

EXAMPLE 2. This holds for all reductive algebraic groups  $G$ , but let  $G = \mathrm{PSL}_2 \mathbb{R}$ . Then

$$(9) \quad \mathcal{M}_{G,\mathbb{Q}} = \mathrm{PSL}_2 \mathbb{Z} \backslash \mathrm{PSL}_2 \mathbb{R} / \mathrm{SO}_2$$

is the locally symmetric space in [fig. 1](#). If we replace  $\mathbb{Q}$  with an arbitrary number field  $F/\mathbb{Q}$ , then we get

$$(10) \quad \mathcal{M}_{G,F} = \mathrm{PSL}_2(\mathcal{O}_F) \backslash \mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R}) / K .$$

Note that

$$(11) \quad F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{\times r_1} \times \mathbb{C}^{\times r_2}$$

where  $r_1$  is the number of real embeddings of  $F$ , and  $r_2$  is the number of conjugate pairs of complex embeddings.

EXAMPLE 3. Let  $F = \mathbb{Q}(\sqrt{d})$ . If it is real ( $d \geq 0$ ) then  $r_1 = 2$  (corresponding to  $\pm\sqrt{d}$ ) and  $r_2 = 0$ , so we get

$$(12) \quad \mathrm{PSL}_2(\mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{PSL}_2 \mathbb{R} \times \mathrm{PSL}_2 \mathbb{R} .$$

This leads to what are called *Hilbert modular forms*.

If it is imaginary ( $d < 0$ ) then  $r_1 = 0$ ,  $r_2 = 1$ , and

$$(13) \quad \mathrm{PSL}_2(\mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{PSL}_2 \mathbb{C} .$$

In this case the maximal compact is  $\mathrm{SO}_3 \mathbb{R}$ , and the quotient:

$$(14) \quad \mathbb{H}^3 = \mathrm{PSL}_2 \mathbb{C} / \mathrm{SO}_3 \mathbb{R}$$

is hyperbolic 3-space. Now we need to mod out (on the left) by a lattice, and the result is some hyperbolic manifold which is a 3-dimensional version of the picture in [fig. 1](#).

REMARK 3. The point is that the real group we get after varying the number field is not that interesting, just some copies of  $\mathrm{PSL}_2$ . But the lattice we are modding out by depends more strongly on the number field, so this is the interesting part.

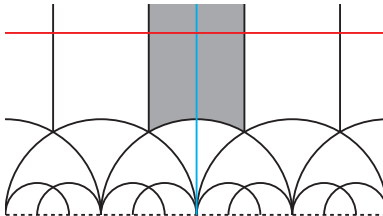


FIGURE 2. Fundamental domain for the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$  in gray. One can define a “period” as taking a modular form and integrating it, e.g. on the red or blue line.

1.2.2. *Conductor/ramification data.* Fixing the number field  $F = \mathbb{Q}$ , we can vary the “conductor” or “ramification data”. The idea is as follows. The locally symmetric space  $\Gamma \backslash \mathbb{H}$  has a bunch of covering spaces of the form  $\Gamma' \backslash \mathbb{H}$ , where  $\Gamma'$  is some congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ . So we can replace  $\Gamma$  by  $\Gamma'$ .

We won’t define congruence subgroups in general, but there are basically two types. For  $N \in \mathbb{Z}$ , we fix subgroups:

$$(15) \quad \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \pmod{N} \right\}$$

$$(16) \quad \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}.$$

The idea is that we start with the conductor  $N$  and the lattice  $\Gamma$ , and then we modify  $\Gamma$  at the divisors of  $N$ . Note that even in this setting we have the choice of  $\Gamma(N)$  or  $\Gamma_0(N)$ . Really the collection of variants has a lot more structure. The local data at  $p$  has to do with the representation theory of  $\mathrm{SL}_2(\mathbb{Q}_p)$ .

1.2.3. *Action of Hecke algebra.* We have seen that our Hilbert space depends on the group, the number field, and some ramification data. A very important aspect of this theory is that this vector space (of functions) carries a lot more structure. There is a huge “degeneracy” here in the sense that the eigenspaces of the Laplacian are much bigger than one might have guessed (not one-dimensional).

This degeneracy is given by the theory of *Hecke operators*. This says that the Laplacian  $\Delta$  is actually a part of a huge commuting family of operators. In particular, these all act on the eigenspaces of the Laplacian. For  $p$  a prime ( $p$  unramified, i.e.  $p \nmid N$ ) we have the Hecke operator at  $p$ ,  $T_p$ . Then

$$(17) \quad \bigoplus_p \mathbb{C}[T_p] \subset L^2(\Gamma \backslash G/K).$$

This is some kind of “quantum integrable system” because having so many operators commute with the Hamiltonian tells us that a lot of quantities are conserved.<sup>3</sup>

1.2.4. *Periods/states.* There is a special collection of measurements we can take of modular forms, called periods. A basic example is given by integrating a modular form on the line  $i\mathbb{R}_+ \subset \mathbb{H}$  as in fig. 2. This is how Hecke defined the  $L$ -function.

<sup>3</sup>This example is often included in the literature as an example of quantum chaos (the opposite of integrability). The chaotic aspect has nothing to do with the discrete subgroup  $\Gamma$ . Specifically this fits into the study of “arithmetic quantum chaos” which more closely resembles the study of integral systems.



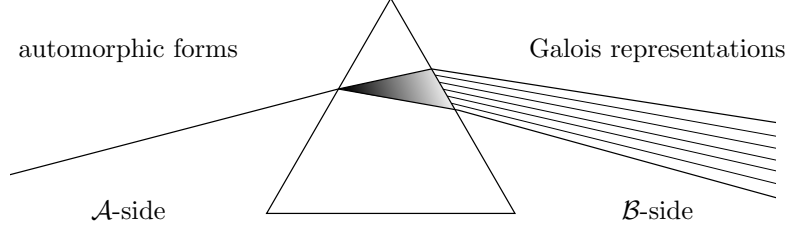


FIGURE 3. Just as light is decomposed by a prism, this spectral decomposition breaks automorphic forms ( $\mathcal{A}$ -side) up into Galois representations of number fields ( $\mathcal{B}$ -side).

The takeaway is that we have a collection of measurements/states with very good properties, and then we can study modular forms by measuring them with these periods.

1.2.5. *Langlands functoriality.* There is a collection of somewhat mysterious operators whose action corresponds to varying the group  $G$ .

## 2. The Langlands program and TFT

**2.1. Overview.** We have seen that for a choice of reductive algebraic group  $G$  and number field  $F/\mathbb{Q}$ , we get a locally symmetric space

$$(18) \quad \mathcal{M} = \mathcal{M}_{G,F} = \text{“arithmetic lattice”} \backslash \text{real group/maximal compact} .$$

This can be thought of as some space of  $G$ -bundles

$$(19) \quad \mathcal{M}_{G,F} = \text{“Bun}_G(\text{Spec } \mathcal{O}_F)'' .$$

Then we linearize this space by taking either  $L^2$  or  $H^*$ .

Starting with this theory of automorphic forms, we spectrally decompose under the action of the Hecke algebra. Then the Langlands program says that the pieces of this decomposition correspond to Galois representations. We can think of the theory of automorphic forms as being fed into a prism, and the colors coming out on the other side are Galois representations as in [fig. 3](#). More specifically, the “colors” are representations:

$$(20) \quad \text{Gal}(\overline{F}/F) \rightarrow G_{\mathbb{C}}^{\vee} .$$

EXAMPLE 4. If  $G = \text{GL}_2 \mathbb{R}$ , then  $G^{\vee} = \text{GL}_2 \mathbb{C}$ . Let  $E$  be an elliptic curve. Then

$$(21) \quad H^1(E/\mathbb{Q})$$

is a 2-dimensional representation of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . This is the kind of representation you get in this setting.

EXAMPLE 5. The representations in [example 4](#) are very specific to  $\text{GL}_2$ . If we started with  $\text{GL}_3(\mathbb{R})$  instead, the associated locally symmetric space  $\text{O}_3 \backslash \text{GL}_3 \mathbb{R} / \text{GL}_3 \mathbb{Z}$  is not a complex manifold.

The goal is to match all of this structure in [section 1.2](#) with a problem in physics, but ordinary quantum mechanics will be too simple. On the physics side we will instead consider quantum field theory.

TABLE 1. Output of a four-dimensional topological field theory. Numbers are the easiest to understand, but are usually the trickiest to produce (often requires analysis). Vector spaces are also pretty simple, but three-manifolds are hard. So the sweet spot is kind of in 2-dimensions, since we understand surfaces and categories aren't that complicated.

Dimension	Output
4	$z \in \mathbb{C}$ (rarely well-defined algebraically, requires analysis)
3	(dg) vector space
2	(dg) category
1	$(\infty, 2)$ -category
0	$(\infty, 3)$ -category? (rarely understood)

**Slogan:** the Langlands program is part of the study of 4-dimensional (arithmetic, topological) quantum field theory.

The idea is that the Langlands program is an equivalence of 4-dimensional arithmetic topological field theories (TFTs):

$$\begin{array}{ccc}
 \mathcal{A}_G & \simeq & \mathcal{B}_{G^\vee} \\
 (22) \quad & \text{automorphic} & \text{spectral} \\
 & \text{magnetic} & \text{electric}
 \end{array}$$

called the  $\mathcal{A}$  and  $\mathcal{B}$ -side theories.

REMARK 4. This is what one might call “four-dimensional mirror symmetry”. The  $\mathcal{A}$  and  $\mathcal{B}$  are in the same sense as usual mirror symmetry.

An  $n$ -dimensional TFT is a beast which assign a quantum mechanics problem (or just a vector space, chain complex, etc.) to every  $(n-1)$ -manifold. So a 4-dimensional TFT sends a 3-manifold to some kind of vector space. It assigns more complicated data to lower-dimensional manifolds and less complicated data to higher-dimensional manifolds as in [table 1](#).

The *topological* means we are throwing out the dynamics and only looking at the ground states. This is the analogue of only looking at the harmonic forms rather than the whole spectrum of the Laplacian. The *arithmetic* means that we're following the paradigm of *arithmetic topology*. The idea is that we will eventually make an analogy between number fields and three-manifolds. Then we can plug a number field into the TFT (instead of an honest manifold) to get a vector space which turns out to be  $L^2(\mathcal{M}_{G,F})$  (or  $H^*(\mathcal{M}_{G,F})$ ).

## 2.2. Arithmetic topology.

2.2.1. *Weil's Rosetta Stone.* In a letter to Simone Weil [**Kri05**], André Weil explained a beautiful analogy, now known as *Weil's Rosetta Stone*. This establishes a three-way analogy between number fields, function fields, and Riemann surfaces.

The general idea is as follows.  $\text{Spec } \mathbb{Z}$  is some version of a curve, with points  $\text{Spec } \mathbb{F}_p$  associated to different primes.  $\text{Spec } \mathbb{Z}_p$  is a version of a disk around the point, and  $\text{Spec } \mathbb{Q}_p$  is a version of a punctured disk around that point. This is analogous to the usual picture of an algebraic curve.

Curve	$\text{Spec } \mathbb{F}_q[t]$	$\text{Spec } \mathbb{Z}$
Point	$\text{Spec } \mathbb{F}_p$	$\text{Spec } \mathbb{F}_p$
Disk	$\text{Spec } \mathbb{F}_t[[t]]$	$\text{Spec } \mathbb{Z}_p$
Punctured disk	$\text{Spec } \mathbb{F}_q((t))$	$\text{Spec } \mathbb{Q}_p$

In general, let  $F/\mathbb{Q}$  be a number field. Then we can consider  $\mathcal{O}_F$ , and  $\text{Spec } \mathcal{O}_F$  has points corresponding to primes in  $\mathcal{O}_F$ . The analogy between number fields and function fields is as follows. Start with a smooth projective curve  $C/\mathbb{F}_q$  over a finite field. Then the analogue to  $F$  is the field of rational functions,  $\mathbb{F}_q(C)$ . The analogue to  $\mathcal{O}_F$  is the ring of regular functions,  $\mathbb{F}_q[C]$ . Finally points of  $\text{Spec } \mathcal{O}_F$  correspond to points of  $C$ .

Now we might want to replace  $C$  with a Riemann surface. So let  $\Sigma/\mathbb{C}$  be a compact Riemann surface. Then primes in  $\mathcal{O}_F$  (and so points of  $C$ ) correspond to points of  $\Sigma$ . The field of meromorphic rational functions on  $\Sigma$ ,  $\mathbb{C}(\Sigma)$ , is the analogue of  $F$ . To get an analogue of  $\mathcal{O}_F$  we have to remove some points of  $\Sigma$  (we wouldn't get any functions on the compact curve). The point is that number fields have some points at  $\infty$ , so the analogue isn't really a compact Riemann surface, but with some marked points. So the analogue of  $\mathcal{O}_F$  consists of functions on  $\Sigma$  which are regular away from these points.

This is summarized in [table 2](#).

TABLE 2. Weil's Rosetta stone, as it was initially developed, establishes an analogy between these three columns. We will eventually refine this dictionary. Let  $F/\mathbb{Q}$  be a number field,  $C/\mathbb{F}_q$  be a smooth projective curve over a finite field, and let  $\Sigma/\mathbb{C}$  be a compact Riemann surface.  $\mathbb{F}_q(C)$  denotes the field of rational functions,  $\mathbb{F}_q[C]$  denotes the ring of regular functions, and  $\mathbb{C}(\Sigma)$  denotes the meromorphic rational functions on  $\Sigma$ .

Number fields	Function fields	Riemann surfaces
$F/\mathbb{Q}$	$\mathbb{F}_q(C)$	$\mathbb{C}(\Sigma)$
$\mathcal{O}_F$	$\mathbb{F}_q[C]$	f'ns regular away from marked points of $\Sigma$
$\text{Spec } \mathcal{O}_F$	points of $C$	$x \in \Sigma$

**2.2.2. Missing chip.** Now we want to take the point of view that there was a chip missing from this Rosetta stone, and we were supposed to consider 3-manifolds rather than Riemann surfaces. The idea is that  $\Sigma/\mathbb{C}$  really corresponds to  $C/\mathbb{F}_q$ . This is manifested in the following way. To study points, we study maps:

$$(23) \quad \text{Spec } \mathbb{F}_q \hookrightarrow C.$$

But from the point of view of étale topology,  $\text{Spec } \mathbb{F}_q$  is not really a point. It is more like a circle in the sense that

$$(24) \quad \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \widehat{\mathbb{Z}} = \pi_1^{\text{étale}}(\text{Spec } \mathbb{F}_q)$$

where  $\widehat{\mathbb{Z}}$  denotes the profinite completion. So it's better to imagine this as a modified circle, where this  $\widehat{\mathbb{Z}}$  is generated by the Frobenius. There is always a map

$$(25) \quad \operatorname{Spec} \overline{\mathbb{F}_q} \rightarrow \operatorname{Spec} \mathbb{F}_q$$

and we can lift our curve to  $\overline{\mathbb{F}_q}$ . This corresponds to unwrapping these circle, i.e. replacing them by their universal cover. So there is some factor of  $\mathbb{R}$  which doesn't play into the topology/cohomology. So we have realized that curves over  $\mathbb{F}_q$  have too much internal structure to match with a Riemann surface.

REMARK 5. The map  $\operatorname{Spec} \mathbb{F}_{q^n} \rightarrow \operatorname{Spec} \mathbb{F}_q$  is analogous to the usual  $n$ -fold cover of the circle.

To fix the Rosetta Stone, we replace a Riemann surface  $\Sigma$  by certain a  $\Sigma$ -bundle over  $S^1$ . Explicitly, if we have  $\Sigma$  and a diffeomorphism  $\varphi$ , we can form the mapping torus:

$$(26) \quad \Sigma \times I / ((x, 0) \sim (\varphi(x), 1)) .$$

The idea is that if we start with a curve over a finite field, the diffeomorphism  $\varphi$  is like the Frobenius.

Similarly  $\operatorname{Spec} \mathcal{O}_F$  looks like a curve where each “point” carries a circle. So this is again some kind of 3-manifold.

REMARK 6. These circles don't talk to one another because they all have to do with a Frobenius at a different prime. So they're less like a product or a fibration, and more like a 3-manifold with a foliation.

This fits with the existing theory of arithmetic topology, sometimes known as the “knots and primes” analogy. The theory was started in a letter from Mumford to Mazur, but can be attributed to many people such as Mazur [Maz73], Manin, Morishita [Mor10], Kapranov [Kap95], and Reznikov. The recent work [Kim15, CKK<sup>+</sup>19] of Minhyong Kim plays a central role.

REMARK 7. Lots of aspects of this dictionary are spelled out, but one should be wary of using it too directly. Rather we should think of this as telling us that there are several classes of ‘3-manifolds’: ordinary 3-manifolds, function fields over finite fields, and number fields.

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2.2.3. *Updated Rosetta Stone.* The upshot is that we are thinking of all three objects in the Rosetta stone as three-manifolds. In particular, we're thinking of  $\operatorname{Spec} \mathcal{O}_F$  (e.g.  $\operatorname{Spec} \mathbb{Z}$ ) as a 3-manifold, so for any prime  $p$  we have the loop  $\operatorname{Spec} \mathbb{F}_p \rightarrow \operatorname{Spec} \mathcal{O}_F$ , which we can interpret as a knot in the 3-manifold. Let  $F_v$  be the completion of the local field  $F$  at the place  $v$  (e.g.  $\mathbb{Q}_p$ ). Then  $\operatorname{Spec} F_v$  turns out to be the boundary of a tubular neighborhood of the knot. The point is that if  $F_v$  is a non-Archimedean local field (e.g.  $\mathbb{Q}_p$  or  $\mathbb{F}_p((t))$ ) then the “fundamental group” is the Galois group, and it has a quotient:

$$(27) \quad \operatorname{Gal}(\overline{F_v}/F_v) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} .$$

This group is called a *Baumslag-Solitar* group. Explicitly it is:

$$(28) \quad \operatorname{BS}(1, p) = \{ \sigma, u \mid \sigma u \sigma^{-1} = u^p \}$$

where we think of  $\sigma$  as the Frobenius, so corresponding to  $\widehat{\mathbb{Z}}$ , and  $u$  as the generator of  $\mathbb{Z}_\ell$ . The kernel is

$$(29) \quad p\text{-group} \times \prod_{\ell' \neq \ell, p} \mathbb{Z}_{\ell'} \hookrightarrow \text{Gal}(\overline{F_v}/F_v) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}.$$

This tells us that there is For  $p = 1$ , this group is  $\mathbb{Z} \times \mathbb{Z} = \pi_1(T^2)$ . For  $p = -1$ , this is the fundamental group of the Klein bottle. This is evidence that the étale fundamental group of  $\text{Spec } F_v$  looks like some kind of  $p$ -dependent version of the fundamental group of the torus. So we can think of  $\text{Spec } F_v$  as a 2-manifold (fibered over  $S^1$ ).

After this discussion we can identify an updated (multi-dimensional) Rosetta Stone. In three-dimensions we have:  $\text{Spec } \mathcal{O}_F$ ,  $C/\mathbb{F}_q$ , and a mapping torus  $T_\varphi(\Sigma)$ . The first two comprise the *global arithmetic setting*. In two-dimensions we first have local fields, which come in two types. One is finite extensions  $F_v/\mathbb{Q}_p$  and the other is  $\mathbb{F}_q((t))$ . Spec of either of these is “two-dimensional” and the latter is some kind of punctured disk  $D^*$ . These two comprise the *local arithmetic setting*. A curve  $\overline{C}/\overline{\mathbb{F}_q}$  over an algebraically closed field (of positive characteristic) and a Riemann surface (projective curve  $\Sigma$  over  $\mathbb{C}$ ) are also both “two-dimensional”. These comprise the *global geometric setting*. The only 4-manifolds we will consider are of the form  $M^3 \times I$  or  $M^3 \times S^1$  where  $M^3$  is a three-dimensional object of arithmetic or geometric origin. This discussion is summarized in [table 3](#).

**2.3.  $\mathcal{A}$ -side.** The  $\mathcal{A}$ -side (or automorphic/magnetic side) TFT  $\mathcal{A}_G$  is a huge machine which does many things, as in [table 1](#). So far, the only recognizable thing is that it sends a 3-manifold  $M$  to some vector space  $\mathcal{A}_G(M)$ . We’re thinking of  $\text{Spec } \mathcal{O}_F$  as a 3-manifold, and the assignment is the vector space we’ve been discussing:

$$(30) \quad \mathcal{A}_G(\text{Spec } \mathcal{O}_F) = L^2(\mathcal{M}_{G,F}) \text{ or } H^*(\mathcal{M}_{G,F}).$$

REMARK 8. As suggested in [eq. \(19\)](#), note that  $\Gamma \backslash G/K$  is a moduli space of something over the 3-manifold in question, not the 3-manifold itself.

**2.4. Structure (reprise).** As it turns out, all the bells and whistles from the theory of automorphic forms in [section 1.2](#) line up perfectly with the bells and whistles of TFT.

2.4.1. *Number field.* The assignment in [eq. \(30\)](#) formalizes the idea that we got a vector space  $L^2(\mathcal{M}_{G,F})$  labelled by a group and a number field.

2.4.2. *Conductor/ramification data.* Recall the ramification data was a series of primes. This is manifested as a link (collection of knots) in the 3-manifold, where we allow singularities. These appear as *defects* (of codimension 2) in the physics. So the structure we saw before is manifested as defects of the theory.

2.4.3. *Hecke algebra.* The Hecke operators correspond to line defects (codimension 3) in the field theory. Physically this is “creating magnetic monopoles” alone some loop in spacetime.

2.4.4. *Periods/states.* These correspond to boundary conditions, i.e. codimension 1 defects.

2.4.5. *Langlands functoriality.* Passing from  $G$  to  $H$  can be interpreted as crossing a domain walls (also a codimension 1 defect).

TABLE 3. The columns correspond to the three aspects of Weil's Rosetta Stone, and the rows correspond to dimension. The four-dimensional objects we consider are just products of three-dimensional objects with  $S^1$  or  $I$ .  $M^3$  is a three-dimensional object of arithmetic or geometric origin. The three-dimensional objects are number fields, function fields and mapping tori of Riemann surfaces.  $F$  is a number field,  $\varphi$  is some diffeomorphism of  $\Sigma$ , and  $T_\varphi$  denotes the corresponding mapping torus construction. The two-dimensional objects are local fields and curves.  $F_v$  is a finite extension of  $\mathbb{Q}_p$ . The 1-dimensional objects are both versions of circles, and the 0-dimensional objects are points.

Dimension	Number fields	Function fields	Geometry
4	$M^3 \times S^1, M^3 \times I$		
3	Global arithmetic		-
	$\text{Spec } \mathcal{O}_F$	$C/\mathbb{F}_q$	$T_\varphi(\Sigma)$
2	Local arithmetic		Global geometric
	$\text{Spec } F_v$	$\text{Spec } \mathbb{F}_q((t)) = D^*$	$\overline{C}/\overline{\mathbb{F}}_q, \Sigma/\mathbb{C}$
1	-		Local geometric
	$\text{Spec } \mathbb{F}_q$		$D_{\mathbb{C}}^* = \text{Spec } \mathbb{C}((t))$ , $D_{\overline{\mathbb{F}}_q}^* = \text{Spec } \overline{\mathbb{F}}_q((t))$
0	$\text{Spec } \overline{\mathbb{F}}_q$		$\text{Spec } \mathbb{C}$

**2.5.  $\mathcal{B}$ -side.** The  $\mathcal{B}$ -side (or spectral side) is the hard part from the point of view of number theory because Galois groups of number fields (and their representations) are very hard. I.e. the  $\mathcal{B}$ -side is the question, and the  $\mathcal{A}$ -side is the answer. But from the point of view of geometry, it is the other way around because fundamental groups of Riemann surfaces are really easy.

The  $\mathcal{B}$ -wide is about studying the algebraic geometry of spaces of Galois representations.

Recall that given a three-manifold (or maybe a number field  $F$ ) the  $\mathcal{A}$ -side is concerned with the topology of the arithmetic locally symmetric space  $\mathcal{M}_{G,F}$ .  $\mathcal{M}_{G,F}$  has to do with the geometry of  $F$ , so the  $\mathcal{A}$ -side is concerned with the topology of the geometry of  $F$ .

**The  $\mathcal{B}$ -side concerns itself with the algebra of the topology of  $F$ .**

This means the following. For a manifold  $M$  (of any dimension), we can construct  $\pi_1(M)$ . Then the collection of rank  $n$  local systems on  $M$  is:

$$(31) \quad \mathbf{Loc}_n M = \{ \pi_1(M) \rightarrow \text{GL}_n \mathbb{C} \} .$$

A local system looks like a locally constant sheaf of rank  $n$  (or vector bundles with flat connection). These are sometimes called *character varieties*. Then we can study  $\mathbb{C}[\mathbf{Loc}_n M]$ . We can also replace  $\text{GL}_n$  with our favorite complex Lie group

$G$  to get:

$$(32) \quad \mathbf{Loc}_{G^\vee} M = \{\pi_1(M) \rightarrow G^\vee\} .$$

This depends only on the topology of  $M$ .

If we're thinking of a number field as a three-manifold, then  $\pi_1$  is a stand-in for the Galois group so this is a space of representations of Galois groups. The TFT sends any three-dimensional  $M^3$  to functions on  $\mathbf{Loc}_{G^\vee}$ :

$$(33) \quad \mathcal{B}_{G^\vee}(M^3) = \mathbb{C}[\mathbf{Loc}_{G^\vee} M] .$$

REMARK 9. This side was a lot easier to write down than the  $\mathcal{A}$ -side, but if  $M$  is a number field, the Galois group is potentially very hard to understand. All the other bells and whistles are also easy to define here.

**2.6. All together.** In all of the setting in [table 3](#), we can either make and automorphic measurement (attach  $\mathcal{M}_{G,F}$  and study its topology) or we could take the Galois group (or  $\pi_1$ ), construct a variety out of it, and study algebraic functions on it. The idea we will explain is that the Langlands program is an equivalence of these giant packages, but for “Langlands dual groups”  $G$  and  $G^\vee$ :

$$(34) \quad \mathcal{A}_G \simeq \mathcal{B}_{G^\vee} .$$

REMARK 10. More is proven in the geometric setting than the arithmetic, but even geometric Langlands for a Riemann surface is still an open question.

This is really a conjectural way of organizing a collection of conjectures.





## Bibliography

- [CKK<sup>+</sup>19] Hee-Joong Chung, Dohyeong Kim, Minhyong Kim, George Pappas, Jeehoon Park, and Hwajong Yoo, *Erratum: “Abelian arithmetic Chern-Simons theory and arithmetic linking numbers”*, Int. Math. Res. Not. IMRN (2019), no. 18, 5854–5857. MR 4012129 [12](#)
- [Kap95] M. M. Kapranov, *Analogies between the Langlands correspondence and topological quantum field theory*, Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), Progr. Math., vol. 131, Birkhäuser Boston, Boston, MA, 1995, pp. 119–151. MR 1373001 [12](#)
- [Kim15] Minhyong Kim, *Arithmetic chern-simons theory i*, 2015. [12](#)
- [Kri05] Martin H. Krieger, *A 1940 letter of André Weil on analogy in mathematics*, Notices Amer. Math. Soc. **52** (2005), no. 3, 334–341, Excerpted from it Doing mathematics [World Scientific Publishing Co., Inc., River Edge, NJ, 2003; MR1961400]. MR 2125268 [10](#)
- [Mac78] George W. Mackey, *Harmonic analysis as the exploitation of symmetry—a historical survey*, Rice Univ. Stud. **64** (1978), no. 2-3, 73–228, History of analysis (Proc. Conf., Rice Univ., Houston, Tex., 1977). MR 526217 [5](#)
- [Maz73] Barry Mazur, *Notes on étale cohomology of number fields*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 521–552 (1974). MR 344254 [12](#)
- [Mor10] Masanori Morishita, *Analogies between knots and primes, 3-manifolds and number rings [translation of mr2208305]*, vol. 23, 2010, Sugaku expositions, pp. 1–30. MR 2605747 [12](#)