Mirror Symmetry

Lectures: Professor Bernd Siebert

Fall 2019; Notes by: Jackson Van Dyke; All errors introduced are my own.

Contents

Overview of mirror symmetry

The course is topics in algebraic geometry. We will be doing some sort of mirror symmetry. We will start with some historical overview. Lecture 1, August 29, 2019

0.1. Enumerative mirror symmetry. Let X be a CY manifold. In particular we will focus on CY 3-folds. This means $K_X = \det T_X^* \simeq \mathcal{O}_X$ is trivial as a holomorphic line bundle. Typically this means we want $b_1 = 0$ and irreducible.

EXAMPLE 0.1 (Quartic in $\mathbb{P}^4$). Take $f \in \mathbb{C}[x_0, \dots, x_4]$ homogeneous of degree 5. If it is sufficiently general, the zero locus is smooth inside $\mathbb{P}^4$ and is an example of a CY three-fold.

We have

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{\mathbb{P}^4}|_X \to \mathcal{O}_X \to 0 \tag{0.1}$$

where $\mathcal{I} = (f) \subset \mathcal{O}_{\mathbb{P}^4}$. We also have that $\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_X$ is an invertible sheaf so this first map sends $f \mapsto df$. This implies

$$K_{\mathbb{P}^4}|_X = \det \mathcal{O}_{\mathbb{P}^4}|_X = \mathcal{I}/\mathcal{I}^2 \otimes K_X \ , \tag{0.2}$$

and

$$K_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-5) \ . \tag{0.3}$$

Then $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^4}$ which has a section with poles of order 5. The point is we can make f into a five by dividing by x_0^5, so we have that, as an abstract sheaf,

$$\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^4}(-5) \ . \tag{0.4}$$

Then we have that

$$\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X(-5) \tag{0.5}$$

and we can just take the tensor product to get

$$K_{\mathbb{P}^4}|_X \simeq \mathcal{O}_X(-5) \tag{0.6}$$

so $K_X \simeq \mathcal{O}_X$ must be trivial.

Now we want to produce a string theory out of this. This is a very delicate process. There are things called $IIA(X)$ and $IIB(X)$ theories. These are the ones relevant in mirror symmetry. These come from the super-symmetric σ-models with target some 10-dimensional space[0.1] $\mathbb{R}^{1,3} \times X$. These are the so-called super conformal field theories $SCFT_A(X)$ and $SCFT_B(X)$. These are different theories which produce observables, e.g. the Hodge number of X can be computed from these theories. In particular we can compute $h_{1,1}(X)$ and $h_{2,1}(X)$ which correspond to some physical variables. On the B-side we make the same computation but get $h_{2,1}(X)$ and $h_{1,1}(X)$. Then we postulate that there is some other X' where these

[0.1] To have an anomaly free theory.

are not flipped. In particular the observation is, for very specific X, we can find a CY Y with

$$SCFT_A(X) = SCFT_B(Y) \qquad SCFT_B(X) = SCFT_A(Y) \ . \tag{0.7}$$

Somehow then the idea is that if this is really a model for string theory, we should really be swapping

$$IIA(X) \text{ ``}='' \ IIB(X) \qquad IIB(X) \text{ ``}='' \ IIA(Y) \ . \tag{0.8}$$

But this might be a bit much to ask.

0.2. Topological twists. There is something called a topologically twisted σ-model introduced by Witten in 1988. This produces a completely different theory. We get two theories, one called $A(X)$, and one $B(X)$.

Warning 0.1. As it turns out, $A(X)$ ends up computing things in certain limits of the $IIB(X)$ theory.

Remark 0.1. A priori these are unrelated to $SCFT_A(X)$ and $SCFT_B(X)$.

As it turns out, if we have $SCFT_A(X) = SCFT_B(Y)$, then we have

$$A(X) = B(Y) \qquad B(X) = A(Y) \ . \tag{0.9}$$

$A(X)$ and $B(X)$ compute certain limits, called Yukawa-couplings, for $SCFT_B(Y)$ and $SCFT_A(X)$.

Note that by this twisting procedure $A(X)$ sees (X, ω) (where ω is the Kähler form) only as a symplectic manifold, and $B(X)$ depends only on the complex manifold (X, I).

0.3. Useful calculations. The reason people really got excited about mirror symmetry is that it helps us make calculations we couldn't make before.

In [**2**] the Yukawa couplings for the quintic and the mirror quintic were computed. In particular they computed F_B of the mirror quintic Y_t, and claimed this is in fact equal to F_A of the quintic. Geometrically F_A has to do with counts of (genus 0) holomorphic curves. F_B has to do with period integrals

$$\int_\alpha \Omega_{Y_t} = F_B(t) \tag{0.10}$$

for $\alpha \in H_3(Y_t)$. So they predicted some counts, then someone computed it directly and they agreed.

Warning 0.2. This Ω_{Y_t} is only defined up to scale so really the case is that

$$F_B(t) = \exp\left(\int_{\alpha_1} \Omega_{Y_t} \Big/ \int_{\alpha_0} \Omega_{Y_t}\right) \tag{0.11}$$

for $\alpha_i \in H_3(Y_t)$.

Then Morrison/Deligne in 1992 described $F_B(Y)$ in terms of Hodge theory/more parameters for CY moduli. This is when Gromov-Witten theory entered the scene in 1993 to make $F_A(X)$ precise. So at least we had a mathematical statement.

0.4. Homological mirror symmetry. In 1994 Kontsevich gave his legendary ICM talk. This is where homological mirror symmetry took off. He said that as mathematicians we don't really know SCFTs. But what should be true is really:

$$D\,\mathrm{Fuk}\,(X) = D^b\,(\mathcal{O}_Y)\ . \tag{0.12}$$

This is a formulation, not an explanation.

Professor Siebert would like to convince us of a procedure to construct mirror pairs.

0.5. Proving numerical mirror symmetry. In 1996 Givental gave a proof that in the case of hypersurfaces F_A really is F_B of the mirror. This was somehow a computation showing that the sides do in fact agree. This is not very satisfying to Professor Siebert. In 1997 Lian, Liu, and Yan proved it more generally.

0.6. Proving HMS. In 2003 Paul Seidel proved HMS for the quartic in $\mathbb{P}^3$. Essentially he shows that both sides have enough rigidity to do a very minimal computation. This is also not very satisfying to Professor Siebert. It was then proved in 2011 by N. Sheridan for all CY hypersurfaces.

0.7. Modern state. There are many other manifestations of mirror symmetry. As it turns out even geometric Langlands can be viewed as some form of mirror symmetry.

As for HMS, some symplectic people are trying to prove this for so-called SYZ fibered symplectic manifolds with a rigid space as the mirror.

Then there are intrinsic constructions, things which Professor Siebert has worked on (with Mark Gross) with many applications. The idea is to use mirror symmetry as a tool in mathematics rather than just a phenomenon in physics. The point is one has to find a way of producing mirrors.

This entire story is genus 0, what physicists would call tree-level. There is also a higher genus case. From the representation theory side this has something to do with quantum groups. This is called second quantized mirror symmetry.[0.2] There is an entire field called topological recursion related to this.

0.8. Plan for the class.

(1) part of the COGP computation (periods)
(2) Gromov-Witten theory, virtual fundamental class/moduli stacks
(3) toric degenerations and mirror constructions[0.3]
(4) One strategy for proving HMS is to compute homogeneous coordinate rings of both sides. Polishchuk has shown that this ring determines $D^b\,(\mathcal{O}_Y)$. It would be nice to make the analogous symplectic calculation because this would be a very sneaky proof of HMS.
(5) Higher genus: Donaldson-Thomas invariants play some sort of unclear role in MS because they will have something to do with the higher genus story. One can make these computations using "crystal melting". This is some kind of statistical mechanics.

[0.2] This term comes from QFT.
[0.3] This will include some introduction to toric geometry.

CHAPTER 1

Mirror symmetry for the quintic

1. The quintic threefold, its mirror, and COGP

Take some quintic CY in $\mathbb{P}^4$, i.e. $V(f) \subset \mathbb{P}^4$ for some homogeneous degree 5 f. First let's do a dimension count for homogeneous polynomials in $x_0, \ldots, x_4$ of degree 5. This is just drawing with replacement, so we have

$$x_0^2 x_2 \qquad \leftrightarrow \qquad || \cdot \cdot | \tag{1.1}$$

$$x_0 x_1^2 x_2 \qquad \leftrightarrow \qquad | \cdot || \cdot | \tag{1.2}$$

and we get

$$\binom{9}{5} = 126 \tag{1.3}$$

which means

$$\dim_{\mathbb{C}} \left\{ Z(f) \subset \mathbb{P}^4 \right\} = 126 - 1 = 125 \ . \tag{1.4}$$

Now we mod out by $\mathrm{PGL}(5)$, which is of dimension $5 \cdot 5 - 1 = 24$. So we get

$$\dim \underbrace{\mathcal{M}_5}_{\text{moduli space of quintics}} = 101 \ . \tag{1.5}$$

Indeed: for a projective CY manifold X, the moduli space of CY manifolds deformation equivalent to X is a smooth orbifold of complex dimension $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle. We will compute this number as an exercise next time. For $V(5) \subset \mathbb{P}^4$ this is 101.

Lecture 2;
September 3, 2019

2. Quintic 3-folds

We will be looking at quintic 3-folds $V(5) \subset \mathbb{P}^4$ given by $Z(f)$ for some homogeneous degree 5 polynomial.

Theorem 1.1. *For X a projective CY manifold then the moduli space of CY deformation-equivalent to X is a smooth space of dimension $h^1(\Theta_X)$.*

By an elementary argument we saw that $\dim \mathcal{M}_{V(5)} = 101$.

2.1. Computation of $H^1(\Theta_X)$. We will compute in the case $X = V(5) \subset \mathbb{P}^4$. We start with the Euler sequence. We have $\mathbb{P}^4 = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_4]$ and

$$x_i \partial_{x_i} = x_i \frac{\partial}{\partial x_i} \tag{1.6}$$

are well-define logarithmic vector fields. Then the sequence is

$$
\begin{gathered}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow 0 \\
1 \longmapsto \sum e_i \\
e_i \longmapsto x_i \partial_{x_i}
\end{gathered} \tag{1.7}
$$

Then we have the conormal sequence

$$
0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{\mathbb{P}^4}\big|_X \xrightarrow{\mathrm{res}_x} \Omega^1_X \longrightarrow 0 \tag{1.8}
$$

where $\mathcal{I}$ is the ideal sheaf of X. This is dual to

$$
0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow N_{X/\mathbb{P}^4} \simeq \mathcal{O}_{\mathbb{P}^4}(5) \longrightarrow 0 \tag{1.9}
$$

$\mathcal{I}/\mathcal{I}^2$ can be computed because $\mathcal{I} \simeq \mathcal{O}(-5)$. The restriction sequences are:

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.10}
$$

$$
0 \longrightarrow \Theta_{\mathbb{P}^4}(-5) \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow 0 \tag{1.11}
$$

Now (1.9) gives us

$$
\begin{tikzcd}
H^0(\Theta_X) \arrow[r] \arrow[d, equal] & H^0(\Theta_{\mathbb{P}^4}|_X) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \arrow[r] \arrow[d, equal] & H^1(\Theta_X) \arrow[r] & H^1(\Theta_{\mathbb{P}^4}|_X) \arrow[d, equal] \\
0 & \mathbb{C}^{24} \arrow[u, "\simeq"] & \mathbb{C}^{125} & \mathbb{C}^{101} \arrow[u, "\simeq"] & 0
\end{tikzcd} . \tag{1.12}
$$

For $H^1(\Theta_{\mathbb{P}^4}|_X)$, (1.11) give us:

$$
H^1(\Theta_{\mathbb{P}^4}) \longrightarrow H^1(\Theta_{\mathbb{P}^4}|_X) \longrightarrow H^2(\Theta_{\mathbb{P}^4}(-5)) \quad . \tag{1.13}
$$

The Euler sequence gives us:

$$
\begin{tikzcd}
H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5} \arrow[r] \arrow[d, equal] & H^1(\Theta_{\mathbb{P}^4}) \arrow[r] & H^2(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\
0 & & 0
\end{tikzcd} , \tag{1.14}
$$

then we can tensor the Euler sequence with $\mathcal{O}(-5)$ to get

$$
\begin{tikzcd}
H^i(\mathcal{O}_{\mathbb{P}^4}(-4)) \arrow[r] \arrow[d, equal] & H^i(\Theta_{\mathbb{P}^4}(-5)) \arrow[r] \arrow[d, equal] & H^i(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\
0 \arrow[r, Rightarrow] & 0 & 0 \arrow[l, Rightarrow]
\end{tikzcd} . \tag{1.15}
$$

For $H^0\left(\mathcal{O}_{\mathbb{P}^4}(5)|_X\right)$ we can tensor (1.10) with $\mathcal{O}(5)$ to get
(1.16)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & H^0\left(\mathcal{O}_{\mathbb{P}^4}\right) \arrow[r] & H^0\left(\mathcal{O}_{\mathbb{P}^4}(5)\right) \arrow[r] \arrow[d, equal] & H^0\left(\mathcal{O}_X(5)\right) \arrow[r] \arrow[d, equal] & H^1\left(\mathcal{O}_{\mathbb{P}^4}\right) \\
\mathbb{C} & \mathbb{C}^{126} \arrow[u, "\simeq"] \arrow[r, Rightarrow] & \mathbb{C}^{125} & 0 &
\end{tikzcd} .$$

Now for $H^0\left(\Theta_{\mathbb{P}^4}|_X\right)$ we have that (1.11) gives us
(1.17)

$$\begin{tikzcd}
H^0\left(\Theta_{\mathbb{P}^4}(-5)\right) \arrow[r] \arrow[d, equal] & H^0\left(\Theta_{\mathbb{P}^4}\right) \arrow[r] \arrow[d, equal] & H^0\left(\Theta_{\mathbb{P}^4}|_X\right) \arrow[r] & H^1\left(\Theta_{\mathbb{P}^4}(-5)\right) \arrow[d, equal] \\
0 & \mathbb{C}^{24} \arrow[r, Rightarrow] & \mathbb{C}^{24} \arrow[u, "\simeq"] & 0
\end{tikzcd} .$$

Then the result is that:

$$h^1\left(\Theta_X\right) = 101 \; . \tag{1.18}$$

3. Hodge diamond

3.1. Dolbeault cohomology. Let X be a compact Kähler manifold. Recall we have the Dolbeault cohomology $H^{i,j}_{\bar{\partial}}$ is the cohomology of the sequence

$$\mathcal{A}^{i,j} \xrightarrow{\bar{\partial}} \mathcal{A}^{i,j+1} \tag{1.19}$$

where this looks like

$$\sum h_{\mu\nu}\, dz_{\mu_1} \wedge \ldots \wedge dz_{\mu_i} \wedge d\bar{z}_{\nu_1} \wedge \ldots \wedge d\bar{z}_{\nu_j} \tag{1.20}$$

where the $h_{\mu\nu} \in \mathcal{C}^\infty$.

Then we have the following facts:

(1) We have a canonical isomorphism from the Dolbeault cohomology

$$H^{i,j}_{\bar{\partial}} = H^j\left(X, \Omega^i_X\right) = \overline{H^{j,i}_{\bar{\partial}}} \tag{1.21}$$

which implies

$$\dim H^{i,j}_{\bar{\partial}} = h_{ij} = h_{ji} \; . \tag{1.22}$$

(2)

$$H^{n-i,n-j}_{\bar{\partial}} = H^{n-j}\left(X, \Omega^{n-i}_X\right) = {}^{1.1} H^j\left(X, K_X \otimes \left(\mathcal{O}^{n-i}_X\right)^*\right)^* H^{i,j}_{\bar{\partial}} \tag{1.23}$$

which, in particular, means $h_{ij} = h_{n-i,n-j}$.

(3)

$$H^k(X,\mathbb{C}) = H^k_{dR}(X) = \bigoplus_{i+j=k} H^{i,j}_{\bar{\partial}} \; . \tag{1.24}$$

Recall $b_i = \dim_{\mathbb{C}} H^k(X,\mathbb{C})$. For CY n-folds, $b_1 = 0$ by the definition of CY. This implies $h_{10} = h_{01} = 0$. Moreover

$$H^{n,0} = H^0\left(X, \underbrace{K_X}_{\mathcal{O}_X}\right) \simeq \mathbb{C} \; . \tag{1.25}$$

[1.1] By Serre duality

This implies that

$$h_{n,0} = h_{0,n} = 1 \ . \tag{1.26}$$

We say a CY is irreducible if the universal cover $\tilde{X} \to X$ is not a nontrivial product of CY. This is equivalent to $H^{k,0} = 0$ for $k = 1, \ldots, n-1$.

Counterexample 1. The product $K3 \times K3$ is not irreducible.

3.2. Hodge diamond. The hodge diamond is the following:

$$\begin{array}{cccccccc}
 & & & h_{33} & & & & \leftarrow H^6 \\
 & & h_{23} & & h_{32} & & & \leftarrow H^5 \\
 & h_{13} & & h_{22} & & h_{31} & & \vdots \\
h_{01} & & h_{12} & & h_{21} & & h_{30} & \\
 & h_{02} & & h_{11} & & h_{20} & & \\
 & & h_{01} & & h_{10} & & & \\
 & & & h_{00} & & & &
\end{array} \tag{1.27}$$

The only interesting part is the center:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h_{22} & & 0 & \\
 & & & \Vert & & & \\
1 & & h_{12} & = & h_{21} & & 1 \\
 & & & \Vert & & & \\
 & 0 & & h_{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \tag{1.28}$$

since

$$H^1\left(\Theta_X\right) = H^1\left(\Theta_X \otimes \underbrace{K_X}_{\mathcal{O}_X}\right) = H^1\left(\mathcal{O}_X^{n-1}\right) = H^{n-1,1} \ . \tag{1.29}$$

So the question is reduced to making these calculations. In the $V(5)$ case $h_{21} = 101$ as we saw.

3.3. Lefschetz theorem on $(1,1)$-classes. Now we work at the generality of compact Kähler manifolds. The Néron-Severi group $NS(X)$, is the preimage of $H^{1,1}$ under $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$. So somehow morally $NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}$. Now we have the exponential sequence of abelian sheaves:

$$1 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1 \tag{1.30}$$

(1.31)

$$\begin{array}{ccccccccc} H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1\left(X,\mathcal{O}_X^\times\right) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_C) \\ & & \| & & \| & & & & \\ & & H^{0,1} & & \operatorname{Pic}(X) & & & & \end{array}.$$

So $c_1(\operatorname{Pic}(X)) = NS(X)$.

Now we know how to compute $\operatorname{Pic}(X)$ for CY n-folds. In particular, as long as $n \geq 3$, $h_{11} = \operatorname{rank}_{\mathbb{Z}} NS(X)$ and $\operatorname{Pic}(X) \simeq NS(X)$ since $\operatorname{Pic}^0(X) = 0$ in the CY case.

3.4. Hard Lefschetz theorem. Let $X \subset \mathbb{P}^n$ be a Kähler manifold. This tells us that

$$H^k(\mathbb{P}^n,\mathbb{Z}) \to H^k(X,\mathbb{Z}) \tag{1.32}$$

is an isomorphism for $k < n-1 = \dim X$ and surjective for $k = n-1$.

Now for a CY 3-fold we have $0 = H^1(\mathbb{P}^n,\mathbb{Z}) = H^1(X,\mathbb{Z})$ and $H^2(\mathbb{P}^n,\mathbb{Z}) \simeq \mathbb{Z} \twoheadrightarrow H^2(X,\mathbb{Z})$ which means $NS(X) \simeq \mathbb{Z}$. This is generated by the hyperplane class $c_1(\mathcal{O}(1))$ and then restricting to X gives the ample line bundle on X and this generates the group.

So today we learned that the interesting part of the diamond in the quintic case looks like

$$\begin{array}{ccc} & 1 & \\ 101 & & 101 \\ & 1 & \end{array} \tag{1.33}$$

and then as it turns out, the mirror quintic will look like

$$\begin{array}{ccc} & 101 & \\ 1 & & 1 \\ & 101 & \end{array} \tag{1.34}$$

so we have a huge Picard group.

Lecture 3; September 5, 2019

4. Lefschetz hyperplane theorem

This is an addendum to last lecture. Let $X \subset \mathbb{P}^n$ be a hypersurface. Then

$$H^k\left(\mathbb{P}^{n+1},\mathbb{Z}\right) \xrightarrow{\text{res}} H^k(X,\mathbb{Z}) \tag{1.35}$$

is an isomorphism for $k < n-1$ and surjective for $k = n-1$.

5. The mirror quintic

We should have mirrored Hodge numbers. Recall the interesting part of the Hodge diamond for the original quintic is

$$\begin{array}{ccc} & 1 & \\ 101 & & 101 \\ & 1 & \end{array} \tag{1.36}$$

and then the mirror diamond should be:

$$\begin{array}{ccc} & 101 & \\ 1 & & 1 \\ & 101 & \end{array} . \tag{1.37}$$

So $h_{21}(Y) = 1$, i.e. it has a one-dimensional moduli space, but $h_{11}(Y) = 101$ so it has a big $\mathrm{Pic}(Y)$.

5.1. Construction. This is physically motivated by the orbifold of the "minimal CFT" related to the Fermat quintic, i.e. the one given by $x_0^5 + \ldots + x_4^5 = 0$. $(\mathbb{Z}/5)^5$ acts diagonally on $\mathbb{P}^4$. This gives an effective action of $(\mathbb{Z}/5)^4 = (\mathbb{Z}/5)^5 / (\mathbb{Z}/5)$ since one copy of $\mathbb{Z}/5$ acts trivially. Now we take the finite quotient

$$\bar{Y} = X/(\mathbb{Z}/5)^4 . \tag{1.38}$$

The action is not free, so this is not a manifold, i.e. it has some orbifold singularities. (X is smooth by Jacobi criterion.) The stabilizer $G_Y \subset (\mathbb{Z}/5)^4$ is nontrivial. There are two cases:

- $x_i = x_j = 0$ $(i \neq j)$, in which case $G_Y \simeq \mathbb{Z}/5$. This gives quintic curves $\tilde{C}_{ij} = Z(x_i, x_j) \subset X$. The local action is given by $\zeta(z_1, z_2, z_3) = (\zeta z_1, \zeta^{-1} z_2, z_3)$. This gives rise to the singularity given by $uv = w^5$, the A_4 singularity in $\mathbb{C}^3$ (with coordinates u, v, w).
- $x_i = x_j = x_k = 0$ (i, j, k pairwise disjoint), where we get $G_Y \simeq (\mathbb{Z}/5)^2$. This gives us

$$\tilde{P}_{i,j,k} \to P_{ijk} \in \bar{Y} . \tag{1.39}$$

Now the local action looks like

$$(\zeta, \xi) \cdot (z_1, z_2, z_3) = (\zeta\xi z_1, \zeta^{-1} z_2, \xi^{-1} z_3) . \tag{1.40}$$

Remark 1.1. Inside $\bar{Y}$ we have

$$C_{01} = Z\left(x_0, x_1, x_2^5 + x_3^5 + x_4^5\right) / (\mathbb{Z}/5)^3 \simeq Z(u + v + w) \simeq \mathbb{P}^1 \subset \mathbb{P}^2 . \tag{1.41}$$

Any C_{ij} looks like a $\mathbb{P}^1$.

We want to blow these singularities up locally, but this is delicate if we want to stay in the world of projective algebraic varieties, i.e. we might just not have an ample line bundle after blowing up. So we have to prove something extra.

Proposition 1.2. *There exists a projective resolution $Y \to \bar{Y}$.*

This is done most efficiently by toric methods, but can be done by hand.

Let's count the independent exceptional divisors in T. We have 4 over each C_{ij} and 6 over each P_{ijk}. So we have 40 from the C_{ij} and 60 from the P_{ijk} and we have 100 in total. Together with the hyperplane class they span the H^2.

Proposition 1.3. *$h_{11}(Y) = 101$, $h_{21}(Y) = 1$.*

The proof was done directly by S.S. Roan and done by by toric methods by Batyrev.

5.2. Mirror families. This $\bmod G$ construction generalizes to what is called the "Dwork family". In particular we have $X_\psi = V(f_\psi)$ where

$$f_\psi = x_0^5 + \ldots + x_4^5 - 5\psi x_0 x_1 \ldots x_4 \tag{1.42}$$

and ψ is a complex parameter. So we have

$$\begin{array}{ccc} (\mathbb{Z}/5)^4 = \{(\zeta_0, \ldots, \zeta_4) \mid \prod \zeta_i = 1\} & & \subset (\mathbb{Z}/5)^5 \\ \downarrow & & \downarrow \\ G = (\mathbb{Z}/5)^3 & & (\mathbb{Z}/5)^4 \end{array} \tag{1.43}$$

and

$$\begin{tikzcd} X_\psi = Z(f_\psi) \arrow[d, "/(\mathbb{Z}/5)^3"] & \\ \bar{Y}_\psi & Y_\psi \arrow[l] \end{tikzcd} . \tag{1.44}$$

Note that $Y\psi \simeq Y_{\zeta\psi}$ for $\zeta^5 = 1$. Then we have

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "/\mathbb{Z}/5"] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathbb{P}^1_\psi \arrow[r, "/\mathbb{Z}/5"] & \mathbb{P}^1_z \end{tikzcd} \tag{1.45}$$

where $z := (5\psi)^{-5}$. This is a family of CY 3-folds which are smooth for $z \neq 0, \infty$.

The special fibers are as follows.

- $z = 0$: $x_0 \ldots x_4 = 0$ implies
$$Y_{z=0} \simeq \bigcup_5 \mathbb{P}^3 \subset \mathbb{P}^4 \tag{1.46}$$
is a union of coordinate hyperplanes.
- $z = 5^{-5}$, i.e. $\psi = 1$. This corresponds to one 3-fold A_1 singularity. X_1 has 125 three dimensional A_1-singularities. Which locally look like $x^2 + y^2 + z^2 + w^2 = 0$. They all lie in one $(\mathbb{Z}/5)^3$-orbit. $Y_{5^{-1}}$ has one three dimensional A_1 singularity sometimes called the "conifold".
- $z = \infty$, i.e. $\psi = 0$: this is the Fermat quintic. This has an additional $\mathbb{Z}/5$ symmetry because we drop the condition that the product of the x_i has to be 1. So this is really an orbifold point.

Now, at least from a physics point of view we are done.[1.2]

[1.2]Professor Siebert says that maybe mathematicians would be better off like this: not worrying so much and just seeing what life brings.

6. Yukawa couplings

6.1. A-model. The A-model (symplectic) will deal with the quintic. So we have $H^2(X,\mathbb{Z}) = \mathbb{Z}\cdot h$ for $h = PD\,(\text{hyperplane})$. Then the Yukawa coupling is

$$\langle h,h,h\rangle_A = \sum_{d\in\mathbb{N}} N_d d^3 q^d \ . \tag{1.47}$$

At this point (historically) it was not clear what the N_d actually were because Gromov-Witten theory was sort of being developed in parallel. Nowadays we know that the N_d are Gromov-Witten (GW) counts of rational curves ($g = 0$) of degree d. If we write n_d for the primitive counts, then

$$5 + \sum_{d>0} d^3 n_d \qquad \underbrace{\frac{q^d}{1-q^d}}_{\text{multiple cover of }\deg d\text{ curves}} \qquad \in \mathbb{Q}\left[\!\left[q\right]\!\right] \ . \tag{1.48}$$

6.2. B-model. Now we consider the mirror quintic Y_z, $z = (5\psi)^{-5}$. Now the Yukawa coupling is given by:

$$\langle \partial_z,\partial_z,\partial_z\rangle_B := \int_{Y_z} \Omega^\nu(z)\wedge\partial_z^3\Omega^\nu(z) \tag{1.49}$$

where Ω^ν is a "normalized" holomorphic volume form:

$$\int_{\beta_0}\Omega^\nu = \text{constant} \tag{1.50}$$

where $\beta_0 \in H_2(Y,\mathbb{Z})$.

We need some kind of mirror to the vector field h on the moduli space of symplectic structures. What we really want is actually the exponentiated thing $e^{2\pi i h}$. So as it turns out on this side of mirror symmetry this looks like $\partial/\partial w$ which corresponds to a vector field on the complex moduli space of Y.

It turns out

$$w = \int_{\beta_1}\Omega^\nu(z) \tag{1.51}$$

where $\beta_1 \in H_3(Y,\mathbb{Z})$.

6.3. Mirror symmetry. Now the actual statement of mirror symmetry is that

$$\langle h,h,h\rangle_A = \langle \partial_z,\partial_z,\partial_z\rangle_B \tag{1.52}$$

where $q = e^{2\pi i w(z)}$ ($w = c\cdot z + \mathcal{O}\left(z^2\right)$).

So now we have to:

(1) write down the holomorphic 3 form (not too bad)
(2) do the normalization period integral (not too bad)
(3) computing this second integral (more bad)
(4) computing the Yukawa coupling.

6.4. Computation of the periods. There is an account of this in lecture notes by Mark Gross (Nordfjordeid). Recall we have:

$$\begin{array}{ccc} & & Y_\psi \\ & & \downarrow \\ X_\psi & \xrightarrow{/G} & \bar{Y}_\psi \end{array} \quad . \tag{1.53}$$

We know $H_3(Y_\psi, \mathbb{Z}) \simeq \mathbb{Z}^4$. Near $\psi = \infty$ (large complex structure limit) we have a vanishing cycle.

This looks like Professor Siebert's favorite picture of a degeneration. Consider $zw = t$. At $t = 0$ this looks like two disks meeting at a point. For $t \neq 0$ this looks like a cylinder. But now if we do a Dehn twist, we see that there is an S^1 which gets collapsed in this degeneration. A similar story holds in higher dimension.

In particular, our vanishing cycle looks like $\beta_0 = T^3$. Locally

$$u_1 \dots u_4 = z, \beta_0 = \left\{ |u_1| = \dots = |u_4| = |z|^{1/4}, \operatorname{Arg} u_1 \dots u_4 = 0 \right\} \tag{1.54}$$

where the u_i are holomorphic coordinates.

If we lift to X_ψ we get an explicit three-torus

$$T = \left\{ |x_0| = |x_1| = |x_2| = \delta \ll 1, x_3 = x_3(x_0, x_1, x_2) \text{ soln of } f_\psi(x_1, x_2, x_3, 1) = 0, \exists! \xrightarrow{z \to 0} 0 \right\} \tag{1.55}$$

Recall we have

Lecture 4; September 10, 2019

$$\begin{tikzcd} & Y_\psi \arrow[d] \\ X_\psi \arrow[r, "/G"] & \bar{Y}_\psi \end{tikzcd} \tag{1.56}$$

where $G = (\mathbb{Z}/5\mathbb{Z})^3$. Recall last time we discussed:

(1) Vanishing cycle T^3, $\beta_0 =\in H_3(Y_\psi, \mathbb{Z})$,
(2) Holomorphic 3-form,
(3) Normalization,
(4) Further periods, and
(5) Canonical coordinate/mirror map.

7. Holomorphic 3-form

We will construct the holomorphic 3-form as the residue of a meromorphic/rational 4-form on $\mathbb{P}^4$ with zeros along X_ψ:

$$\Omega(\psi) = 5\psi \operatorname{Res}_{X_\psi} \frac{\tilde{\Omega}}{f_\psi} \in \Gamma\left(X_\psi, \Omega^3_{X_\psi}\right) \tag{1.57}$$

where

$$\tilde{\Omega} = \sum x_i \, dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_4 \quad . \tag{1.58}$$

Locally $x_4 = 1$, $\partial_{x_3} f \neq 0$,

$$\Omega(\psi) = 5\psi \left. \frac{dx_0 \wedge dx_1 \wedge dx_2}{\partial_{x_3} f_\psi} \right|_{X_\psi} \quad . \tag{1.59}$$

8. Normalization

Now we deal with normalization. We have $\varphi_0 \coloneqq \int_{\beta_0} \Omega(\psi)$, then

$$\tilde{\Omega} = \varphi_0^{-1} \Omega(\psi) \tag{1.60}$$

is normalized with resides

$$2\pi i \int_{\beta_0} \Omega(\psi) = \int_{T^4} 5\psi \, dx_0 \; dx_1 \; dx_2 \; dx_3 \tag{1.61}$$

$$= \int_{T^4} \frac{dx_0 \; \ldots \; dx_3}{x_0 \ldots x_3} \frac{1}{\frac{\left(1+x_0^5+\ldots+x_3^5\right)}{5\psi x_0 \ldots x_3} - 1} \tag{1.62}$$

$$= -\sum_{n\geq 0} \int_{T^4} \frac{dx_0 \ldots dx_3}{x_0 \ldots x_3} \frac{\left(1+x_0^5+\ldots+x_3^5\right)^n}{(5\psi)^n (x_0 \ldots x_3)^n} \tag{1.63}$$

$$= -\sum_{n\geq 0} \int_{T^4} \frac{dx_0 \ldots dx_3}{x_0 \ldots x_3} \frac{\left(1+x_0^5+\ldots+x_3^5\right)^{5n}}{(5\psi)^{5n} (x_0 \ldots x_3)^{5n}} \tag{1.64}$$

$$\tag{1.65}$$

where all summands in both the numerator and denominator must be 5th powers to contribute. So from some combinatorics we have

$$2\pi i \int_{\beta_0} \Omega(\psi) = -(2\pi i)^4 \sum \frac{(5n)!}{(n!)^5 (5\psi)^{5n}} =: \varphi_0(z) \tag{1.66}$$

where $z = (1/5\psi)^5$. The number $(5n)!/(n!)^5$ is the number of terms

$$x_0^{5n} \ldots x_3^{5n} \left(1+x_0^5+\ldots+x_3^5\right)^{5n} \; . \tag{1.67}$$

9. Further periods

There is a procedure called Griffith's reduction of pole order. This involves the Picard-Fucks equation.

Locally $H^3(Y_\psi, \mathbb{C})$ is constant with dimension 4. This gives a trivial holomorphic vector bundle

$$\begin{array}{ccc} & & E = \mathcal{U} \times \mathbb{C}^4 \\ & & \downarrow \\ \mathcal{M}_Y & \hookleftarrow & \mathcal{U} \end{array} \quad . \tag{1.68}$$

This has a flat connection ∇^{GM} called the Gauß-Manin connection. Pointwise

$$E_\psi = \bigoplus_{p+q=3} H^{p,q}(X_\psi) \tag{1.69}$$

and we have seen $\Omega(\psi) \in H^{3,0}$. Now consider:

$$\Omega(z) \qquad \partial_z \Omega(z) \qquad \partial_z^2 \Omega(z) \qquad \partial_z^3 \Omega(z) \qquad \partial_z^4 \Omega(z) \tag{1.70}$$

which are related by a fourth order ODE with holomorphic coefficients called the Picard-Fuchs equation.

9.1. Derivation of the equation. Take $X = Z(f_\psi) \subset \mathbb{P}^4$. We can produce more 3-forms from forms with higher-order poles. Consider the long-exact sequence:
(1.71)

$$\begin{array}{ccccccc}
H^4\left(\mathbb{P}^4,\mathbb{C}\right) & \longrightarrow & H^4\left(\mathbb{P}^4\setminus X,\mathbb{C}\right) & \longrightarrow & H^5\left(\mathbb{P}^4,\mathbb{P}^4\setminus X;\mathbb{C},\right) & \longrightarrow & H^5\left(\mathbb{P}^4,\mathbb{C}\right)\\
 & & & & {\scriptstyle\text{excision}}\Big\| & & \\
 & & & & H^5(\mathcal{U},\mathcal{U}\setminus X;\mathbb{C}) & & \\
 & & & & \Big\| & & \\
 & & & & H^5(\mathcal{U},\partial\mathcal{U};\mathbb{C}) & & \\
 & & & & {\scriptstyle LD}\Big\| & & \\
 & & & & H_3(\mathcal{U}\setminus\partial\mathcal{U};\mathbb{C}) & & \\
 & & & & \Big\| & & \\
 & & & & H_3(X,\mathbb{C}) = H^3(X,\mathbb{C}) & &
\end{array}$$

where $X \subset \mathcal{U}$ is a tubular neighborhood and we are using the form of Lefschetz duality which states that $H^q(M,\partial M) = H_{n-q}(M\setminus\partial M)$ and in the last step we use Poincare duality. So we start with things of high pole order, this gives us some class in H^3, then in our case we take derivatives, and for certain classes we know they should be zero and this gives us some equations.

9.2. Griffiths' reduction of pole order. If we have

$$\frac{g\tilde{\Omega}}{f^l} \in H^0\left(\mathbb{P}^4, \Omega^4_{\mathbb{P}^4\setminus X}\right) \tag{1.72}$$

then we must have $\deg g = 5l-5$ ($l=0$ earlier so we had no g). The exact forms look like
(1.73)

$$d\left(\frac{1}{f^l}\left(\sum_{i<j}(-1)^{i+j}(x_ig_j - x_jg_i)\,dx_0\wedge\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots dx_4\right)\right)$$

$$= \left(l\sum g_j\partial_{x_j}f - f\sum\partial_{x_j}g_j\right)\frac{\tilde{\Omega}}{f^{l+1}}\ . \tag{1.74}$$

If $l\sum g_j\partial_{x_j}f \in \mathcal{J}(f) = (\partial_{x_i}f)$ then up to an exact form, it is of lower order since one copy of f cancels. I.e. the first term over f^{l+1} is of order $l+1$, and the second term over f^{l+1} is order l. The upshot is that the numerator $g \in \mathcal{J}(f)$ can reduce l.

So the algorithm is as follows: Compute $\Omega(z)$, $\partial_z\Omega(z)$, $\partial_z^2\Omega(z)$, …, $\partial_z^4\Omega(z) = g\tilde{\Omega}/f_\psi^5$ where $g \in \mathcal{J}(f_\psi)$. Then we express g modulo (1.74) as a linear combination of the $\partial_z^i\Omega(z)$.

Proposition 1.4. *Any period*

$$\varphi = \int_\alpha \Omega(\psi) \tag{1.75}$$

fulfills the ODE

$$\left[\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\right]\varphi(z) = 0 \tag{1.76}$$

where $\theta = z\partial_z$.

REMARK 1.2. This is easy to check for

$$\varphi = \varphi_0 = \sum_{n\geq 0} \frac{(5n)!}{(n!)^5} z^n \ . \tag{1.77}$$

(1.76) is an ODE with a regular singular pole:

$$\theta\varphi(z) = A(z)\varphi(z) \tag{1.78}$$

for $\psi(z) \in \mathbb{C}^s$.

THEOREM 1.5. (1.78) *has a fundamental system of equations of the form*

$$\Phi(z) = S(z)\, z^R \tag{1.79}$$

with $S(z) \in M(s, \mathcal{O}_0)$, $R \in M(S, \mathbb{C})$, *and*

$$z^R = I + (\log z) R + (\log z)^2 R^2 + \dots \ . \tag{1.80}$$

If the eigenvalues do not differ by integers, we may take $R = A(0)$.

For (1.76)

$$A(0) \simeq \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \tag{1.81}$$

and $S = (\psi_0, \dots, \psi_3)$ where $\psi_i \in \mathcal{O}^4_{\mathbb{C},0}$. This gives us a fundamental system of solutions:

$$\varphi_0(z) = \psi_0 \text{ (single-valued)} \tag{1.82}$$

$$\varphi_1(z) = \psi_0(z)\log z + \psi_1(z) \tag{1.83}$$

$$\varphi_2(z) = \psi_0(z)(\log z)^2 + \psi_1(z)\log z + \psi_2(z) \tag{1.84}$$

$$\varphi_3(z) = \psi_0(z)(\log z)^4 + \dots + \psi_3(z) \ . \tag{1.85}$$

This has something to do with monodromy. In particular, the monodromy of $z^{A(0)}$ reflects the monodromy T of $H^3(Y_z, \mathbb{C})$ about $z = 0$ (or $\psi = \infty$). In fact, one can show that there exists a symplectic basis β_0, β_1, α_1, $\alpha_0 \in H_3(Y_z, \mathbb{Q})$ with $N = T - I$. Then we have

$$\alpha_0 \mapsto \alpha_1 \mapsto \beta_1 \mapsto \beta_0 \mapsto 0 \tag{1.86}$$

which means

$$\varphi_0 = \int_{\beta_0} \Omega(z), \varphi_1 = \int_{\beta_1} \Omega(z), \varphi_2 = \int_{\alpha_1} \Omega(z), \varphi_3 = \int_{\alpha_0} \Omega(z) \ . \tag{1.87}$$

10. Canonical coordinate/mirror map

Looking at the solution set, we don't have much choice. The solution, when exponentiated should behave like z. Indeed, the canonical coordinate is

$$q = e^{2\pi i w} \tag{1.88}$$

where

$$w = \frac{\int_{\beta_1} \Omega(z)}{\int_{\beta_0} \Omega(z)} = \int_{\beta_1} \tilde{\Omega}(z) \ . \tag{1.89}$$

Then $\varphi_1(z) = \varphi_0(z)\log z + \psi_1(z)$ which is easy to obtain as series solution of (1.76).

$$\psi_1(z) = 5\sum_{n\geq 1}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)z^n \tag{1.90}$$

(up to constant c_2).

Last time we learned how to do these period calculations, get canonical equations, and reduce pole order with the Picard Fucks equation. The remaining topic is the Yukawa coupling. Lecture 5; September 12, 2019

11. Yukawa coupling

We want to compute

$$\langle\partial_z,\partial_z,\partial_z\rangle_B = \int_{Y_z}\tilde{\Omega}(z)\wedge\partial_z^3\tilde{\Omega}(z) \tag{1.91}$$

where $\tilde{\Omega}(z) = \frac{1}{\varphi_0(z)}\Omega(z)$.

We introduce the auxiliary terms

$$W_k = \int_{Y_z}\Omega(z)\wedge\partial_2^k\Omega(z) \tag{1.92}$$

for $k = 0,\ldots,4$. So really we just want W_3. Rewrite the PF equation as

$$\left(\frac{d^4}{dz^4}+\sum_{k=0}^{3}c_k\frac{d^k}{dz^k}\right)\Omega(z) = 0\ . \tag{1.93}$$

This gives us

$$W_4 + \sum_{k=0}^{3}c_kW_k = 0\ . \tag{1.94}$$

11.1. Griffiths-transversality. Now we need to put some important information in called Griffiths-transvr. This has to do with how one defines the variation of Hodge structures. Let $\mathcal{U}$ be open inside the moduli space. Now define a decreasing filtration

$$\mathcal{F} = \underbrace{H^3(Y_z,\mathbb{C})}_{\mathbb{C}^4}\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{U}} = \mathcal{F}^0\supset\mathcal{F}^1\supset\mathcal{F}^2\supset\mathcal{F}^3 \tag{1.95}$$

where

$$\mathcal{F}^k = \bigoplus_{q\geq k}R^q\pi_*\Omega^{3-q}\ . \tag{1.96}$$

This is the Hodge filtration. Note that $\Omega\in\Gamma\left(\mathcal{U},\underbrace{R^3\pi_*\Omega^0}_{=\mathcal{F}^3\simeq\mathcal{O}}\right)$. So why do we write it this way instead of using the direct sum decomposition we seem to have? Abstractly we have that

$$\nabla^{GM}\mathcal{F}^k\subseteq\mathcal{F}^{k-1}\otimes\Omega^1_{\mathcal{M}}\ . \tag{1.97}$$

This inclusion comes from the definition/construction of ∇^{GM} and Hodge/Dolbeault theory. Moreover $H^{p,q}\perp H^{p',q'}$ unless $p+p'=3=q+q'$ (from $\int_Y\alpha\wedge\beta=0$).

Together this gives us that $W_0 = W_1 = W_2 = 0$. In particular

$$0 = \frac{d^2 W_2}{dz^2} = \ldots = 2W_3' - W_4 \tag{1.98}$$

and (1.94) tells us that

$$W_3' + \frac{1}{2} c_3 W_3 = 0 \; . \tag{1.99}$$

Now we compute

$$c_3\left(z\right) = \frac{6}{2} - \frac{25^5}{1 - 5^5 z} \tag{1.100}$$

which by separation of variables gives us

$$W_3 = \frac{c_1}{\left(2\pi i\right)^3 z^3 \left(5^5 z - 1\right)} \tag{1.101}$$

for c_1 some integration constant. Finally, reexpress in $q = e^{2\pi i w}$ where $w = \varphi_1\left(z\right) / \varphi_0\left(z\right)$. When we expand we get

$$\langle \partial_w, \partial_w, \partial_w \rangle_B = -c_1 - 575 \frac{c_1}{c_2} q - \frac{1950750}{2} \frac{c_1}{c_2^2} q^3 - \frac{10277490000}{6} \frac{c_1}{c_2^3} q^3 + \ldots \tag{1.102}$$

$$\stackrel{!}{=} 5 + \sum_{d \geq 1} d^3 n_d \frac{q^d}{1 - q^d} \tag{1.103}$$

$$= 5 + n_1 q + \left(8 n_2 + n_1\right) q^2 + \left(27 n_3 + n_1\right) q^3 + \ldots \; . \tag{1.104}$$

This predicts that $n_1 = 2875$ (which was classically known) and $n_2 = 6092500$ was also correct (as was shown just a few years earlier than this development by Katz in 1986). Then $n_3 = 317206375$ which originally disagreed with the result, but they found out there was an error in the computation so it also agreed. The first proof of this was in 1996 by Givental.

Lessons learned:[1.3]

(1) The prediction depends on the large complex structure limit because this has something to do with the Kähler cone:

$$\mathcal{K}_X = d\left\{ [\omega] \in H^2_{dR}\left(X\right) \,\middle|\, \omega \text{ Kähler} \right\} \; . \tag{1.105}$$

In particular the monodromy in $H^3\left(Y\right) = \bigoplus_{p=0}^{3} H^{p,3-p}$ corresponds to $\smile [\omega_X]$ on $\bigoplus_{p=0}^{3} H^{p,p}\left(X\right)$. Note however that $\langle \rangle_B$ is defined on all of $\mathcal{M}_B$!

(2) Orbifolding construction of the mirror is special to the quintic. Batyrev/Borisov consider a mirror for complete intersections in toric varieties.

[1.3] Besides that computations are hard...

CHAPTER 2

Stacks

We will learn some things about moduli spaces and stacks to motivate a discussion of Gromov-Witten theory. Hopefully also the logarithmic version and Donaldson-Thomas theory.

The general task is to make sense of curve counting, e.g. the number of genus 0 holomorphic curves in a quintic. Then we have the following problems:

- Classical enumerative algebraic geometry: "general position arguments" needed to make counts work. Transversaily can be very difficult.
- Translate into problem of topology, e.g. intersection theory in Grassmannian $\mathrm{Gr}(k,n)$ (Schubert calculus).
- Generally, spaces of curves, e.g. on a given quintic don't have the right dimension.

EXAMPLE 2.1. Consider the Dwork family $f_\psi = 0$. There are 375 isolated lines ($\simeq \mathbb{P}^1$), e.g. $\left(u, v, -\zeta^k u, -\zeta^k v, 0\right)$ for $u, v \in \mathbb{P}^1$, $\zeta^5 = 1$, $0 \le k,\ l \le 4$ and then two irreducible families. In degree > 1 we also always have multiple covers that come in families.

So how do we count in the absence of general deformations? The solution is that there is a virtual formalism. This is exactly what Gromov-Witten theory does. Invariants are constant in families of targets.

1. Moduli spaces

So we are interested in a set of closed points. In particular, this consists of isomorphism classes of certain algebraic geometric objects e.g. varieties, subvarieties. Then we want so somehow give it some extra structure. The best scenario would be to view it as a variety.

Let $T \to \mathcal{M}$ be the structure sheaf. Then the point is that holomorphic maps correspond to families of objects over T.

EXAMPLE 2.2. Fix some N. The Hilbert scheme $\mathrm{Hilb}\left(\mathbb{P}^N\right)$ will somehow classify closed subschemes. In particular it has the universal property that for any

$$\begin{array}{c} Z \ \subset \ T \times \mathbb{P}^N \\ \big\downarrow \text{flat, proper} \\ T \end{array} \tag{2.1}$$

we have a unique map φ such that

$$\begin{array}{ccc} Z = \mathcal{Z}_T & \longrightarrow & \mathcal{Z} \\ \downarrow & & \hookdownarrow \\ T \times \mathbb{P}^N & \longrightarrow & \operatorname{Hilb}\left(\mathbb{P}^N\right) \times \mathbb{P}^N \\ \big\downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & \operatorname{Hilb}\left(\mathbb{P}^N\right) \end{array} \quad . \tag{2.2}$$

In categorical terms this says that we have a functor

$$\begin{gathered} \mathbf{Sch} \xrightarrow{\quad F \quad} \mathbf{Set} \\ T \longmapsto \left\{Z \to T, Z \hookrightarrow T \times \mathbb{P}^N \text{ flat, proper}\right\} \end{gathered} \tag{2.3}$$

which is corepresented by $\operatorname{Hilb}\left(\mathbb{P}^N\right)$. This means we have a natural isomorphism $F \to \hom\left(\cdot, \operatorname{Hilb}\left(\mathbb{P}^N\right)\right)$. In particular we get that $\mathrm{id}_{\operatorname{Hilb}(\mathbb{P}^n)} \in \hom\left(\operatorname{Hilb}\left(\mathbb{P}^N\right), \operatorname{Hilb}\left(\mathbb{P}^N\right)\right)$ corresponds to $F\left(\operatorname{Hilb}\left(\mathbb{P}^N\right)\right)$, e.g. the universal family.

Discouraging observation: For families of curves[2.1] we cannot have (co-)representability.

The reason is that there are families of curves where all of the fibers are isomorphic but they are not globally a product. So if we had such a corepresentation then this can't pull back to the identity.

Lecture 6; September 17, 2019

1.1. The problem of moduli for curves. We will kind of follow [**4**]. [**3**] is where stacks were first really worked out.

We want to repeat the story of Hilb for complete curves of genus g.

Recall the notion of families. If S is a scheme (think of this as some sort of parameter space), then a *curve* of genus g over S is a morphism $\pi : C \to S$ such that:

(1) π is proper, flat;
(2) recall the geometric fibers

$$C_S = \operatorname{Spec} K \times_S C \tag{2.4}$$

for K algebraically closed, fit into the diagram

$$\begin{array}{ccc} C_S & \longrightarrow & C \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & S \end{array} \quad . \tag{2.5}$$

Then we ask that:
(a) C_S is reduced, connected, $\dim C_S = 1$,
(b) $h^1\left(C_S, \mathcal{O}_C\right) = g$ (arithmetic genus).

One can also restrict the allowed singularities of these fibers, but this is not necessary for us at the moment. E.g. one might ask for C_S to be non-singular.

[2.1] or any other moduli proble with objects with automorphisms

Now we have a fundamental problem. The functor

$$\begin{array}{ccc} \mathbf{Sch} & \xrightarrow{\quad F_g \quad} & \mathbf{Set} \\ S & \longmapsto & \{X \to S \mid \text{ non-singular curve of genus } g\} \end{array} \tag{2.6}$$

cannot be representable since there are nontrivial isotrivial families of curves, i.e. $\pi : X \to S$ which becomes trivial only after (finite) base change. To see that this is the case, proceed by contradiction. Suppose it is representable by some $\mathcal{M}_g$, then

$$\begin{tikzcd} T_X C_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & C_g \arrow[d] \\ T \arrow[r, "\text{finite}"] \arrow[rr, bend right, "\text{constant}"] & S \arrow[r, "\varphi"] & \mathcal{M}_g \end{tikzcd} \tag{2.7}$$

but this map being constant implies φ is constant which is a contradiction.

Example 2.3. Let C_0 be a curve with $\mathrm{Aut}\,(C_0) \neq \{1\}$, e.g. $C_0 \to \mathbb{P}^1$ a two-to-one hyperelliptic curve, e.g. the projective closure of

$$\left(y^2 - (x-g-1)(x-g)\dots(x-1)(x+1)\dots(x+g+1) = 0\right) \ . \tag{2.8}$$

We have one automorphism which swaps the two branches and one which sends $x \to -x$ so we have $(\mathbb{Z}/2)^2$ symmetry. Take φ to be any automorphism such that $\varphi^a = \mathrm{id}$. Recall $\mathbb{G}_M = \mathrm{Spec}\,\mathbb{C}\left[x, x^{-1}\right]$. Take $C_0 \times \mathbb{G}_m / (\mathbb{Z}/a)$. The action is as follows. For $\mathbb{C}^\times \ni \zeta \neq 1$, $\zeta^a = 1$ we have

$$(\varphi, \zeta) : (z, t) \mapsto (\varphi(z), \zeta \cdot t) \ . \tag{2.9}$$

So this is a nontrivial bundle over $\mathbb{G}_m$.

Remark 2.1. If automorphisms are the problem, then why not just stick to ones without them. As it turns out, restricting to C_S with $\mathrm{Aut}\,(C_S) = \{1\}$ would indeed make F_g representable, but it is not very useful.

Instead, we will construct the moduli space as an *algebraic stack*, which is a generalization of the notion of a scheme accomodating automorphisms from the beginning.

2. Stacks

Another good reference (which is unfortunately only in French[2.2]) is [**8**].

The idea here is to formalize the notion of a "family of objects parameterized by a scheme along fibrewise automorphisms".

Fix a base scheme S (think $\mathbb{C}$). Write $\mathcal{S} = \mathbf{Sch}/S$ for the category of schemes over S.

Definition 2.1. (1) A *category over* $\mathcal{S}$ is a category $\mathcal{F}$ together with a functor $p_\mathcal{F} : \mathcal{F} \to \mathcal{S}$. For $B \in \mathrm{Obj}\,(\mathcal{S})$, we have a fibre category $\mathcal{F}(B)$ which is a subcategory of $\mathcal{F}$ with objects

$$\{X \in \mathrm{Obj}\,(\mathcal{F}) \mid p_\mathcal{F}(X) = B\} \tag{2.10}$$

and morphisms

$$\{\varphi \in \mathrm{Hom}\,(\mathcal{F}) \mid p_\mathcal{F}(\varphi) = \mathrm{id}_B\} \ . \tag{2.11}$$

[2.2] Professor Siebert says it isn't a big deal since French is basically English.

(2) A category over $\mathcal{S}$ is a groupoid over $\mathcal{S}$ (or fibered groupoid) if

(a) For all $f : B' \to B$ in $\mathcal{S}$ and $X \in \mathrm{Obj}(\mathcal{F})$ there exists $\varphi : X' \to X$ in $\mathcal{F}$ with $p_{\mathcal{F}}(\varphi) = f$:

$$\begin{tikzcd} X' \arrow[r, dashed, "\varphi"] \arrow[d, dashed] & X \arrow[d] \arrow[d, bend left, "p_{\mathcal{F}}"] \\ B' \arrow[r, "f"] & B \end{tikzcd} \; . \tag{2.12}$$

(b) For all commutative diagrams

$$\begin{tikzcd} & X'' \arrow[r, "\varphi''"] \arrow[dd] & X \arrow[dd] \arrow[dd, bend left, "p_{\mathcal{F}}"] \\ X' \arrow[ur, dashed, "\exists!"] \arrow[urr, "\varphi'"'] \arrow[dd] & & \\ & B'' \arrow[r, "f''"] & B \\ B' \arrow[ur, "h"] \arrow[urr, "f'"'] & & \end{tikzcd} \tag{2.13}$$

(i.e. $p_{\mathcal{F}}(\varphi') = p_{\mathcal{F}}(\varphi'') \circ h$) there exists unique $\chi : X' \to X''$ and $\varphi' = \varphi'' \circ \chi$.

Remark 2.2. (i) (ii) implies that $\varphi : X' \to X$ is an isomorphism iff $p_{\mathcal{F}}(\varphi)$ is an isomorphism.

(ii) $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{S}$ groupoid over $\mathcal{S}$ implies $\mathcal{F}(B)$ are groupoids.

(iii) (ii) implies

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "p_{\mathcal{F}}"] \\ B' \arrow[r, "f"] & B \end{tikzcd} \tag{2.14}$$

X' is unique up to unique isomorphism. Write $f^*X \coloneqq X'$. This is called the *pull-back*. This construction is functorial in the sense that $\psi : X'' \to X'$ in $\mathcal{F}(B)$ yields a canonical morphisms $f^*\psi : f^*X'' \to f^*X'$, i.e. $f : B' \to B$ gives us

$$f^* : \mathcal{F}(B) \to \mathcal{F}(B') \; . \tag{2.15}$$

Example 2.4 (Representable functors). Let $\mathcal{F} : \mathcal{S} \to \mathbf{Set}$ be a contravariant functor. This yields a groupoid. The objects are (B, β) such that $B \in \mathrm{Obj}(\mathcal{S})$ and $\beta \in \mathcal{F}(B)$. The idea is that $p_{\mathcal{F}} : (B, \beta) \mapsto B$. Then

$$\hom((B', \beta'), (B, \beta)) = \{f : B' \to B \,|\, F(f)(\beta) = \beta'\} \; . \tag{2.16}$$

For example if X is an S-scheme then this defines (is equivalent to) the functor $F(B) \coloneqq \mathrm{Hom}_S(B, X)$ and

$$F(\varphi : B' \to B) : (f : B \to X) \mapsto (f \circ \varphi : B' \to X) \; . \tag{2.17}$$

The associated groupoid $\underline{X} = \mathcal{F}$ has objects $f : B \to X$ in $\mathcal{Y} = \mathbf{Sch}/S$ with morphisms

$$\begin{tikzcd} B' \arrow[d, "\varphi"] \arrow[dr, "f'"] & \\ B \arrow[r, "f"] & X \end{tikzcd} \tag{2.18}$$

and $p_X : (B \to X) \mapsto B$.

EXAMPLE 2.5 (Quotient stack). Let X/S be a scheme with an action of a (flat) group scheme G/S (e.g. GL_n). Then we can take the quotient $[X/G]$. The objects are diagrams

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & X \\ B & \end{tikzcd} \tag{2.19}$$

where E/B is a G-principal bundle and f is G-equivariant. The morphisms are given by

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[rr, bend left] & E \arrow[r, "f"] \arrow[d] & X \\ B' \arrow[r] & B & \end{tikzcd} \tag{2.20}$$

where the square on the left must be cartesian.

FACT 1. *G acts freely on X and X/G exists as a scheme so $[X/G] = \underline{X/G}$.*

EXAMPLE 2.6 (Classifying spaces of principal G-bundles). For $X = \mathrm{pt}$, $BG :=$ $[\mathrm{pt}/G]$.

So we have three main examples. First, if X is scheme we get Lecture 7; September 19, 2019

$$\underline{X}(S) := \mathrm{Hom}(S, X)\ . \tag{2.21}$$

(Note that $\underline{S} = \mathcal{S}$).

Then for $G \circlearrowright X$ we get $[X/G]$. For $F : \mathbf{Sch}/S \to \mathbf{Set}$ we get $\mathcal{F}$ an S-groupoid.

Third, we have the moduli groupoid $\mathcal{M}_g$. The objects are curves $X \to B$, for B any scheme, X_s non-singular for all s. The morphisms are given by cartesian diagrams:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}\ . \tag{2.22}$$

Note this is *not* the groupoid associated to F_g.

Similarly, one defines the *universal curve* $\mathcal{C}_g$ over $\mathcal{M}_g$. The objects are pairs $(X \to B, \sigma)$ where $\sigma : B \to X$ is a section.

2.1. Morphisms of groupoids.

DEFINITION 2.2. A morphism between groupoids F_1, F_2 over S is a functor

$$\begin{tikzcd} F_1 \arrow[rr, "p"] \arrow[dr, "p_{F_1}"'] & & F_2 \arrow[dl, "p_{F_2}"] \\ & \mathcal{S} & \end{tikzcd} \tag{2.23}$$

where $p_{F_1} = p_{F_2} \circ p$. Note this is *equality* of functors.

EXAMPLE 2.7. We have the forgetful functor $\mathcal{C}_g \to \mathcal{M}_g$ which simply forgets the section.

EXAMPLE 2.8. Let $f : X \to Y$ be a morphism of schemes. This is equivalent to $p : \underline{X} \to \underline{Y}$ being a morphism of the associated groupoids.

PROOF. $\Longrightarrow$: On objects, $p\left(B \xrightarrow{u} X\right) := X \xrightarrow{f\circ u} Y$. On morphisms

$$p\left(\begin{tikzcd} B \arrow[d, "s"] \arrow[dr, "u"] & \\ B' \arrow[r, "u'"] & X \end{tikzcd}\right) := \begin{tikzcd} B \arrow[d, "s"] \arrow[dr, "f\circ u"] & \\ B' \arrow[r, "f\circ u'"'] & Y \end{tikzcd} . \tag{2.24}$$

($\Longleftarrow$): If we have

$$\begin{tikzcd} \underline{X} \arrow[rr, "p"] \arrow[dr] & & \underline{Y} \arrow[dl] \\ & \mathcal{S} & \end{tikzcd} \tag{2.25}$$

then $p\left(X \xrightarrow{\mathrm{id}_X} X\right) = \left(X \xrightarrow{f} Y\right) \in \mathrm{Obj}\,\underline{Y}\,(X)$ for some f. This is exactly the setup of Yoneda's Lemma: p can be viewed as a natural transformation between

$$\mathrm{Hom}_{\mathcal{S}}\,(\cdot, X) : \mathcal{S} \to \mathbf{Set} \tag{2.26}$$

and another functor $G : \mathcal{S} \to \mathbf{Set}$. Then Yoneda says that

$$\mathrm{Nat}\,(\mathrm{Hom}_{\mathcal{S}}\,(\cdot, X)\,, G) \xrightarrow{\equiv} G\,(X) \tag{2.27}$$

where $\Phi \mapsto \Phi\,(\mathrm{id}_X)$. This shows us that p is induced by f. □

EXAMPLE 2.9. Similarly, for a scheme B and a groupoid over $\mathcal{S}$ F, we get that

$$\{p : \underline{B} \to F\} = F\,(B) \tag{2.28}$$

where $p \mapsto p\,(\mathrm{id}_B)$.

EXAMPLE 2.10. $\mathcal{S} = \underline{S}$ and for any groupoid F over S, we can view $p_F : F \to \mathcal{S}$ as a morphism of groupoids $F \to \underline{S}$.

EXAMPLE 2.11. Let X/S be a scheme with the action of a group scheme G/S. This yields a quotient morphism $q : \underline{X} \to [X/G]$.

On objects:

$$
\left(B \xrightarrow{s} X\right) \mapsto \left(\begin{tikzcd} (g,b) \arrow[r, mapsto] & g \cdot s(b) \\ G \circlearrowright G \times B \arrow[r] \arrow[d] & X \\ B & \end{tikzcd}\right). \tag{2.29}
$$

On morphisms

$$
\begin{tikzcd} B' \arrow[rr, "f"] \arrow[dr, "s'"'] & & B \arrow[dl, "s"] \\ & X & \end{tikzcd} \mapsto \left(\begin{tikzcd} & & X \\ G \times B' \arrow[r, "\mathrm{id} \times f"] \arrow[d] \arrow[urr, bend left] & G \times B \arrow[d] \arrow[ur, bend left] & \\ B' \arrow[r, "f"] & B & \end{tikzcd}\right) \tag{2.30}
$$

Remark 2.3. Isomorphisms of groupoids are given by equivalences of categories over S. In particular, $p_1 : F_1 \xrightarrow{\text{iso}} F_2$ may not have an inverse, just a quasi-inverse $q : F_2 \to F_1$ such that pq is naturally isomorphic to id_{F_2} and qp naturally isomorphic to id_{F_1}.

S-groupoids in fact form a 2-category $\mathbf{Grpd}/S$. The objects are groupoids over S. The 1-morphisms are functors over S between the groupoids. But now we in fact have another kind of morphism, called a 2-morphism which are morphisms between morphisms.

Remark 2.4. We can't even define what a cartesian diagram is without talking about these 2-morphisms so it really is necessary to understand them.

Proposition 2.1. *Let X and Y be schemes. Then $X \simeq Y$ as schemes iff $\underline{X} \simeq \underline{Y}$ as groupoids over S.*

Proof. ($\Longrightarrow$): Let $f : X \to Y$ be an isomorphism. Then the induced map $p : \underline{X} \to \underline{Y}$ is a strong equivalence. Indeed, f^{-1} induces $q : \underline{Y} \to \underline{X}$ with $pq = \mathrm{id}_{\underline{Y}}$ and $qp = \mathrm{id}_{\underline{X}}$.

($\Longleftarrow$): Let $p : \underline{X} \to \underline{Y}$ be an equivalence, $q : \underline{Y} \to \underline{X}$ a quasi-inverse. As we have seen this means p, q are induced by $f : X \to Y$ and $g : Y \to X$ which implies $qp\left(X \xrightarrow{\mathrm{id}_X} X\right) = \left(X \xrightarrow{gf} X\right)$ as objects in $\underline{X}$. Hence q is a quasi-inverse of p, which implies there exists an isomorphism

$$
\begin{tikzcd} X \arrow[rr, dashed, "\simeq"] \arrow[dr, "gf"'] & & X \arrow[dl, "\mathrm{id}_X"] \\ & X & \end{tikzcd} \tag{2.31}
$$

and similarly for fg, so f is an isomorphism. $\square$

Remark 2.5. So this tells us we have some kind of subcategory of schemes inside of groupoids. In the future we will write X instead of $\underline{X}$. Similarly we will also write $\mathcal{S}$, $\underline{S}$, and S for the same thing.

2.2. Fibre products and cartesian diagrams.

DEFINITION 2.3. Consider three groupoids F, G, and H over S and morphisms $f : F \to G$, $h : H \to G$. The *fiber product* is as follows. The objects (over a base B) are triples (x, y, ψ) where $x \in F(B)$, $Y \in H(B)$, and $\psi : f(x) \to h(y)$ is an isomorphism in $G(B)$.

The morphisms (over B) $(x, y, \psi) \to (x', y', \psi')$ are pairs

$$\left(x' \xrightarrow{\alpha} x, y' \xrightarrow{\beta} Y\right) \tag{2.32}$$

such that

$$\psi \circ f(\alpha) = h(\beta) \circ \psi' \ . \tag{2.33}$$

So now this fits in the diagram

$$\begin{tikzcd} F \times_G H \arrow[r, "q"] \arrow[d, "p"] & H \arrow[d, "h"] \\ F \arrow[r, "f"] & G \end{tikzcd} \tag{2.34}$$

which only commutes up to 2-morphisms.

WARNING 2.1. This diagram does not commute in general. We have that

$$fp(x, y, \psi) = f(x) \qquad\qquad gq(x, y, \psi) = h(y) \ . \tag{2.35}$$

But there is is a natural isomorphism of functors $fp \simeq hq$, i.e. the diagram eq. (2.34) is 2-commutative. So we need this ψ twisting to get them to agree.

Then $F \times_G H$ has the universal property for 2-commutative diagrams:

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & F \times_G H \arrow[r] \arrow[d] & H \arrow[d] \\ & F \arrow[r] & G \end{tikzcd} \ . \tag{2.36}$$

EXAMPLE 2.12. For X, Y, and Z schemes we have

$$\underline{X} \times_{\underline{Z}} \underline{Y} = \underline{X \times_Z Y} \ . \tag{2.37}$$

EXAMPLE 2.13 (Base change). If $T \to S$ is a morphism of schemes then $T \times_S F$ is a groupoid over T. Actually, for all $B \to T$, $F(B)$ and $(T \times_S F)(B)$ are equivalent

$$\begin{tikzcd} X \arrow[d] & \in & F(B) \\ B \arrow[dr] \arrow[dd] & & \\ & T \arrow[dl] & \\ S & & \end{tikzcd} \ . \tag{2.38}$$

2.3. Definition of stacks. We want to get closer to something which allows us to do algebraic geometry.

Definition 2.4 (Iso-functor). Let (F, p_F) be a groupoid over S, B a scheme over S, and $X, Y \in \mathrm{Obj}(F(B))$. Then

$$\mathrm{Iso}_B(X, Y) : \mathbf{Sch}/B \to \mathbf{Set} \tag{2.39}$$

is the following contravariant functor. On objects:

$$\left(B' \xrightarrow{f} B\right) \mapsto \left\{ f^*X \xrightarrow{\varphi} f^*Y \,\middle|\, \varphi \text{ iso} \right\} . \tag{2.40}$$

On morphisms we get:

$$\begin{tikzcd} B'' \arrow[rr, "h"] \arrow[dr, "g"'] & & B' \arrow[dl, "f"] \\ & B & \end{tikzcd} \mapsto \left((f^*X \to f^*Y) \mapsto \left(\underbrace{h^*f^*X}_{=g^*X} \to \underbrace{h^*f^*Y}_{=g^*Y} \right) \right) . \tag{2.41}$$

Theorem 2.2 (Deligne-Mumford). *Take two curves X/B, Y/B of genus 2. The iso-functor* $\mathrm{Iso}_B(X, Y)$ *is represented by a scheme.*

Proof. We know we have the relative holomorphic cotangent bundles $\omega_{X/B}$ and $\omega_{Y/B}$. These are ample, so they give us an embedding into projective space over B. These are canonical bundles, so any isomorphism $f^*X \to g^*Y$ (for any $f : B' \to B$) preserves this polarization. Now we can use the relative Hilbert scheme (for the graph of $f^*X \to f^*Y$). □

Remark 2.6. $\mathrm{Iso}_B(X, Y)$ is finite and unramified over B, but not in general flat (e.g. fibre cardinalities can jump).

Lecture 8; September 24, 2019

Definition 2.5 (Stack). A groupoid $(\mathcal{F}, p_{\mathcal{F}})$ over S is a *stack* if:

(1) for any B over S and any $X, X' \in \mathrm{Obj}(\mathcal{F}(B))$, $\mathrm{Iso}_B(X, X')$ is a sheaf in the étale topology;
(2) for $\{B_i \to B\}$ an étale covering of B, $X_i \in \mathrm{Obj}(F(B_i))$, isomorphisms

$$\varphi_{ij} : X_j|_{B_i \times_B B_j} \to X_i|_{B_i \times_B B_j} \tag{2.42}$$

satisfying the cocycle condition, they glue, i.e. there exists $X \in \mathrm{Obj}(\mathcal{F}(B))$ and isomorphisms $X|_{B_i} \simeq X_i$ inducing φ_{ij}.

Remark 2.7. The first condition could be thought of as some sort of descent for morphisms, and the second can be thought of as descent for objects.

Étale topology. The authoritative reference on this subject is [**6**], [**10**] is easier to read. Replace Zariski open subsets by étale morphisms $U \to X$. The intuition is to think of these as being an unbranched morphism which is locally a diffeomorphism in U. The intersection of $U \to X$, $V \to X$ is given b y $U \times_X V \to X$. This is an example of what is called a *Grothendieck topology*.

Formally, smooth/unramified/étale morphisms $f : X \to Y$ are morphisms of schemes (of finite type). Let $T = \mathrm{Spec}\, A \hookrightarrow T' = \mathrm{Spec}\, A'$ be an infinitesimal extension, i.e. $A = A'/I$ where $I^n = 0$ for some n.

Example 2.14. $\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, k[\epsilon]/(\epsilon^n) \hookrightarrow \mathrm{Spec}\, k[\epsilon]/\left(\epsilon^{n+1}\right)$. For $n = 2$ this is a square zero extension $\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, k[\epsilon]/\left(\epsilon^2\right)$.

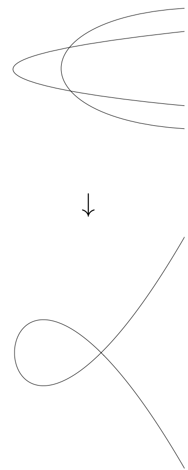

FIGURE 1. The two to one cover of the curve $V\left(y^2 - x^3 - x^2\right)$.

In this situation, we look at all diagrams:

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ T' \arrow[r, "\psi"] \arrow[ur, dashed, "\exists\tilde{\psi}?"] & Y \end{tikzcd} . \tag{2.43}$$

Then we ask for the properties:

- $\exists\tilde{\psi}$
- uniqueness of $\tilde{\psi}$.

DEFINITION 2.6. f is formally

(1) *smooth* iff this exists,
(2) *unramified* iff this is unique, and
(3) *étale* iff it exists and is unique.

EXAMPLE 2.15. The $2:1$ cover in fig. 1 is étale but is not smooth.

The smooth case should look like a projection. So we can lift a tangent vector but it won't be unique. The étale case looks like mapping two curves onto one, then we choose a point upstairs using φ and the lift is unique. Then a ramified example (which is not smooth) is the cover $\mathbb{A}^1 \to \mathbb{A}^1$ corresponding to the algebraic map $z \mapsto z^2$. The only vector with unique lift is the 0 vector, and the others can somehow lift to anything.

2.4. Examples of stacks.

EXAMPLE 2.16. Let $\mathcal{F}$ be the groupoid associated to a functor

$$F : \mathcal{S} \to \mathbf{Set} . \tag{2.44}$$

Then $\mathcal{F}$ is a stack iff F is a sheaf (of sets) in the étale topology. If we take $F = \mathrm{Hom}_{\mathcal{S}}(-, X)$ and $X \in \mathcal{S}$ then this is a stack (needs étale descent for morphisms).

Note that $F = F_g$ (moduli functor) is not a stack since families might not glue, i.e. (2) from the definition might not be satisfied.

EXAMPLE 2.17. The moduli groupoid $\mathcal{M}_g$ is a stack.

Proposition 2.3. *If G/S is a flat (affine/separated) group scheme $\circlearrowright X$ then $[X/G]$ is a stack.*

PROOF. For (2) we need an étale descent for principal bundles (G-torsors).

For (1) we claim that $\underline{\mathrm{Iso}_B}(f, f')$ is represented by a scheme where

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr] & X & E' \arrow[l, "f'"'] \arrow[dl] \\ & B & \end{tikzcd} \tag{2.45}$$

It is enough to check étale locally on B, so we may assume $E = B \times G = E'$. So the situation we have $\sigma : B \to E$ and $\sigma' : B \to E'$. Then we get $f \circ \sigma$ and $f' \circ \sigma'$ are two morphisms $B \to X$, and then

$$\underline{\mathrm{Iso}_B}(f, f') = B \times_X B \tag{2.46}$$

which fits into the fiber diagram:

$$\begin{tikzcd} B \times_X B \arrow[r] \arrow[d] & B \arrow[d, "f' \circ \sigma'"] \\ B \arrow[r, "f \circ \sigma"] & X \end{tikzcd} \tag{2.47}$$

□

REMARK 2.8. The étale descent for schemes themselves (not their morphisms) leads to the notion of algebraic spaces (Artin, Knutson).

EXAMPLE 2.18. If $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are stacks with maps $\mathcal{F} \to \mathcal{G}$ and $\mathcal{H} \to \mathcal{G}$ then $\mathcal{F} \times_{\mathcal{G}} \mathcal{H}$ is a stack.

2.5. Representable morphisms. Some morphisms of stacks are "scheme-like":

DEFINITION 2.7. A morphism $f : \mathcal{F} \to \mathcal{G}$ of stacks is *representable* if for any scheme B and morphism $B \to \mathcal{G}$ the fiber product $\mathcal{F} \times_B \mathcal{G}$ is (the groupoid associated to) a scheme (better: an algebraic space).

EXAMPLE 2.19. Consider $X \to [X/G]$. E such that the following diagram is fibered:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & {[X/G]} \end{tikzcd} \tag{2.48}$$

is a scheme.

EXAMPLE 2.20. Recall we have $\mathcal{M}_g$ the moduli space of genus g curves over B, and then we have $\mathcal{C}_g \to \mathcal{M}_g$ which consists of curves along with sections. Then we have a fibered diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{C}_g \arrow[d] \\ B \arrow[r] & \mathcal{M}_g \end{tikzcd} \tag{2.49}$$

The idea is that if we have properties of morphisms of schemes stable under base change (so they are compatible with the philosophy of stacks) then we can move these properties to the world of stacks.

DEFINITION 2.8. Let $\mathcal{F} \to \mathcal{G}$ be representable. Then this has property P (of morphisms of schemes), stable under base change,[2.3] if for any $B \to \mathcal{G}$, for B a scheme, $B \times_{\mathcal{G}} \mathcal{F} \to B$ has this property.

EXAMPLE 2.21. For G smooth over S, $G \circlearrowright X$, $X \to [X/G]$ is smooth.

2.6. Definition of DM-stack.

DEFINITION 2.9. A stack $\mathcal{F}$ is a *Deligne-Mumford* stack if

(1) $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable, quasi-compact, and separated;[2.4] If $\Delta_{\mathcal{F}}$ is in addition proper, one says $\mathcal{F}$ is *separated*.
(2) There is an étale surjective morphism (an étale atlas) $\varphi : U \to \mathcal{F}$ with U a scheme.

Now we have some comments. Lecture 9; September 26, 2019

(1) The first is the representability of Δ_X.

Proposition 2.4. Δ_X *is representable iff any morphism* $B \to C$ *where* B *is a scheme and is representable.*

PROOF. ($\implies$): So we for two schemes B and B' we want to show that $B' \times_X B$ is a scheme:

$$\begin{array}{ccc} B' \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & X \end{array} . \tag{2.50}$$

But this is implied by Δ_X being representable:

$$\begin{array}{ccc} B' \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \Delta_X} \\ B' \times_S B & \xrightarrow{f \times g} & X \times_S X \end{array} . \tag{2.51}$$

∎

Another aspect is that $X, Y \in \operatorname{Obj}(\mathcal{F}(B))$ implies $\operatorname{Iso}_{\underline{B}}(X, Y)$ is representable by a scheme.

PROOF. $\operatorname{Iso}_{\underline{B}}(X, Y)$ is represented by the following. X and Y correspond to two maps $f, g : B \to X$. Then the fiber product

$$\begin{array}{ccc} B \times_{X \times_S X} X & & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{(f,g)} & X \times_S X \end{array} \tag{2.52}$$

represents the iso-functor and hence it is a scheme. ∎

(2) Δ_X is quasi-compact: we don't want $\operatorname{Iso}_{\underline{B}}(X, Y)$ to be too wild. This can be relaxed (as in the Stacks project).

[2.3] E.g. finite type, separated, flat, affine, proper, ...

[2.4] The latter two conditions are sometimes relaxed.

(3) Δ_X separated: an isomorphism is the identity if it is so generally.
(4) When is Δ_X proper? This should be related to the separatedness of the stack X. For schemes we act for $X \to X \times_S X$ to be a closed embedding.
(5) The atlas provides what are called versal deformation spaces for deformations over Artin rings. Write $\bar{A} = A/I$ such that $I^n = 0$ The idea is that we have

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, dashed] \arrow[dr, dashed] & \mathcal{U} \arrow[d, two heads] \\ \operatorname{Spec} \bar{A} \arrow[u, hook] \arrow[r] \arrow[ur] & X \end{tikzcd} \tag{2.53}$$

and we get unique maps.
(6) Atlas is étale: makes Δ_F unramified, i.e. automorphisms are somehow discrete. If we only have a smooth atlas this is what is called an *Artin stack* (or *algebraic stack*).

Example 2.22. If we take $BG = [\mathrm{pt}/G]$ for G/S flat (not étale) this is an Artin stack.

Separatedness of stacks.

Lemma 2.5. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of stacks fulfilling condition 1 Then $\Delta_{F/G} : F \to F \times_G F$ is representable.*

Proof. See [**8**]. □

A criterion for an Artin stack to be DM.

Theorem 2.6. *Let $\mathcal{F}$ be an algebraic stack over a Noetherian scheme S with a smooth atlas $U \to \mathcal{F}$ of finite type over S. Then $\mathcal{F}$ is DM iff $\Delta_{\mathcal{F}}$ is unramified.*

Proof. See [**8**] or [**4**] Theorem 2.1. □

Corollary 2.7. *Let $\mathcal{G}/S$ be a smooth affine group scheme acting on a Noetherian scheme X/S, both of finite type over S, such that the geometric points have finite and reduced stabilizers.*

Then

(i) $[X/G]$ is a DM-stack (for trivial stabilizers, an algebraic space)
(ii) $[X/G]$ is separated iff action is proper.

Proof. (1) Having finite reduced stabilizers implies that for all

$$B \to [X/G] = \left(\begin{array}{l} E \longrightarrow X \\ \downarrow \\ B \end{array} \right) \tag{2.54}$$

which means $\mathrm{Iso}_{\underline{B}}\left(E, E\right)/B$ is unramified. Then this means $\Delta_{[X/G]}$ is unramified and $X \to [X/G]$ is smooth which implies (by the above theorem) that $[X/G]$ is DM.

□

3. Stable curves

For a complete (read: compact) moduli stack of curves we need to add singular curves. The insight of Deligne-Mumford was that adding node[2.5] suffices.

DEFINITION 2.10. A (DM) *stable curve* over a scheme S is a proper flat morphism $\pi : C \to S$ such that

(i) the geometric fibers C_s are reduced, connected, one-dimensional with at most nodes as singularities,
(ii) if $E \subset C_S$ is non-singular rational, $\nu : \tilde{E} \to E$ the normalization, then the number of the preimage of singular points under ν is ≥ 3.

We also want C_s to not be smooth of genus 0 or 1.

REMARK 2.9. Condition (ii) is equivalent to $\mathrm{Aut}\,(C_S)$ being finite.

Note that the genus is $g = h^1\left(C_S, \mathcal{O}_{C_S}\right)$.
This gives us immediately that this is a groupoid of stable curves of genus g: $\bar{\mathcal{M}}_g$ which is (at this point) a stack over $\mathbb{Z}$.

3.1. $\overline{\mathcal{M}_g}$ is a DM-stack. Recall that for a stable curve, $\pi : C \to S$ is an étale locally complete intersection morphism and hence has a relative dualizing invertible sheaf $\omega_{C/S}$. Explicitly on geometric fibers we have the normalization:

$$\begin{gathered} C'_S \xrightarrow{\ \nu\ } C_S \\ x_i, y_i \longmapsto i\text{th node} \end{gathered} \quad . \tag{2.55}$$

Then we can write

$$\nu^* \omega_{C_S} = \omega_{C'_S}\left(x_1 + \ldots + x_r + y_1 + \ldots + y_r\right) \ . \tag{2.56}$$

For $\alpha \in \omega_{C_S}(U)$ (where U is a neighborhood of the ith node) we have that $\nu^*\alpha$ is a rational 1-form with

$$\boxed{\mathrm{Res}_{x_i}\left(\nu^*\alpha\right) + \mathrm{Res}_{y_i}\left(\nu^*\alpha\right) = 0} \ . \tag{2.57}$$

Now use $\omega_{C/S}^{\otimes n}$ to embed C/S into $\mathbb{P}^N_S$ for some N.

THEOREM 2.8. *For $g \geq 2$, $\omega_{C/S}^{\otimes n}$ is relatively very ample for $n \geq 3$. Moreover, $\pi_*\left(\omega_{C/S}^{\otimes n}\right)$ is locally free of rank $(2n-1)(g-1)$.*

Lecture 10; October 1, 2019

SKETCH PROOF. For C smooth, $\omega_{C/S}^{\otimes n}$ is relatively very amply for all $n \geq 2$. Let's consider the case $S = \mathrm{Spec}\, k$. By Riemann-Roch and Serre duality we can compute:

$$h^0\left(C, \omega_C^{\otimes n}\right) = h^1\left(C, \omega_C^{\otimes n}\right) + \deg\left(\omega_C^{\otimes n}\right) + 1 - g \tag{2.58}$$

$$= h^0\left(C, \left(\omega_C^{\otimes n}\right)^{-1} \otimes \omega_C\right) + n\,(2g-2) + 1 - g \ . \tag{2.59}$$

[2.5]This means that étale locally it looks like $V(zw) \subset \mathbb{A}^2$.

For C smooth we get that $\left(\omega_X^{\otimes n}\right)^{-1} \otimes \omega_C = \omega_C^{1-n}$, and the degree is $(1-n)(2g-g_2) < 0$, so this gives so $h^0\left(C, \left(\omega_C^{1-n}\right)\right) = 0$ and we get

$$h^0\left(C, \omega_C^{\otimes n}\right) = (2n-1)(g-1) \ . \tag{2.60}$$

We can also twist $h^0\left(C, \omega_C^{\otimes n}(-p_1-p_2)\right)$ and this is still zero. So we can separate points and tangents, which is a criterion for being very ample.

For C nodal, we can take the normalization $\nu : C' \to C$ where C' is smooth. Then

$$C_{\mathrm{sing}} = \{z_1, \dots, z_n\} \qquad\qquad \nu^{-1}(z_i) = \{x_i, y_i\} \ . \tag{2.61}$$

For $\alpha \in H^0(U, \omega_C)$ we have that $\nu^*\alpha$ is a rational section of $\omega_{C'} = \Omega^1_{C'}$ with simple poles at $\{x_i, y_i\}$ (with $\mathrm{Res}_{x'}(\nu^*\alpha) + \mathrm{Res}_{y_i}(\nu^*\alpha) = 0$) so we have

$$\nu^*\omega_C = \Omega_{C'}\left(\sum x_i + \sum y_i\right) \ . \tag{2.62}$$

This shows, in any case, that $\omega_C^{\otimes n}$ is very ample for $n \geq 3$. Worse case we have a rational ($g = 0$) irreducible component D of C with 3 special points which gives us

$$\nu^*\omega_C|_D \simeq \mathcal{O}_{\mathbb{P}^1}(1) \ . \tag{2.63}$$

Finally, use cohomology and base change. □

Corollary 2.9. *Every stable curve C of genus g can be (n-canonically) embedded into $\mathbb{P}^N$ for $N = (2n-2)(g-1)-1$ ($n \geq 3$) with Hilbert polynomial*

$$P_{g,n}(t) = (2nt-1)(g-1) \tag{2.64}$$

Definition 2.11. $\overline{H}_{g,n} \subset \mathrm{Hilb}_{\mathbb{P}^N}^{P_{g,n}}$ consists of n-canonically embedded nodal curves.

This is an open subscheme (nodes can at most smooth out)

Remark 2.10.
(a) Having a morphism $S \to \overline{H}_{g,n}$ is the same as for a stable curve $\pi : C \to S$ of genus g having an isomorphism $\mathbb{P}\left(\pi_*\left(\omega_{C/S}^{\otimes n}\right)\right) \xrightarrow{\simeq} S \times \mathbb{P}^N$.
(b) $\mathrm{PGL}(N+1)$ acts on $\overline{H}_{g,n}$ by its action on $\mathbb{P}^n$.

Theorem 2.10. $\overline{\mathcal{M}}_g \simeq \left[\overline{H}_{g,n}/\,\mathrm{PGL}(N+1)\right]$.

Proof. We will define a functor

$$p : \overline{\mathcal{M}}_g \to \left[\overline{H}_{g,n}/\,\mathrm{PGL}(N+1)\right] \ . \tag{2.65}$$

On objects:

$$\begin{array}{ccc} C & & E \\ \downarrow\pi & \mapsto & \downarrow \\ B & & B \end{array} \tag{2.66}$$

where E is the $\mathrm{PGL}(N+1)$-principal bundle associated to $\mathbb{P}\left(\pi_*\omega_{C/S}^{\otimes n}\right) = \mathbb{P}/B$. We still need a morphism $E \to \overline{H}_{g,n}$ which is $\mathrm{PGL}(N+1)$ equivariant. Consider

$$\begin{tikzcd} C \times_B E \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ E \arrow[r] & B \end{tikzcd} \ . \tag{2.67}$$

Now

$$E \times_B \mathbb{P} = \mathbb{P}\left(\tilde{\pi}_*\left(\omega^{\otimes n}_{C\times_B E/E}\right)\right) \tag{2.68}$$

is trivial, because it is a principal bundle with a (tautological) section.

On morphisms we have

$$\begin{CD} C' @>>> C \\ @V{\pi'}VV @VV{\pi}V \\ B' @>{\varphi}>> B \end{CD} \tag{2.69}$$

which implies

$$\pi'_*\left(\omega^{\otimes n}_{C'/B'}\right) = \varphi^*\pi_*\left(\omega^{\otimes n}_{C/B}\right) \tag{2.70}$$

which leads to a cartesian square:

$$\begin{CD} E' @>>> E \\ @VVV @VVV \\ B' @>>> B \end{CD} \quad . \tag{2.71}$$

Remark 2.11. p is fully faithful and essentially surjective. These properties have geometric meaning. The fact that p is faithful tells us that a nontrivial automorphism on C induces a non-trivial automorphism of $\mathbb{P}\left(H^0\left(\omega_C^{\otimes n}\right)\right)$. The fact that p is full tells us that if $\Phi \in \mathrm{PGL}\,(N+1)$ with $\Phi(C) = C$ then Φ is induced by an automorphism of C, i.e. an n-canonical embedding is not contained in a linear subspace.

Essential surjectivity tells us the following. Let

$$\begin{array}{l} E \longrightarrow \overline{H}_{g,n} \\ \downarrow \\ B \end{array} \in \mathrm{Obj}\left[\overline{H}_{g,n}/\,\mathrm{PGL}\,(N+1)\right] \quad . \tag{2.72}$$

Then we have $\pi_E : C_E \to E$ and an isomorphism

$$\mathbb{P}\left(\pi_{E*}\left(\omega_{C_E/E}\right)\right) \simeq E \times \mathbb{P}^n \ . \tag{2.73}$$

Then this descends to

$$B = E/\,\mathrm{PGL}\,(N+1)\,, C = C_E/\,\mathrm{PGL}\,(N+1) \quad . \tag{2.74}$$

□

Corollary 2.11. *$\overline{\mathcal{M}}_g$ is a separated Deligne-Mumford stack of finite type over S (e.g. $S = \mathbb{Z}, k$.)*

Proof. To show this we use a criterion of [**3**], and for separatedness we check finiteness of $\mathrm{Iso}_B\,(C', C) \to B$.

See [**3**] for details. □

3.2. Further properties.

Proposition 2.12. *$\overline{\mathcal{M}}_g$ is proper (over $\mathbb{Z}$ or k).*

Digression 1 (Properties of (morphisms) stacks). For those properties that are local in the smooth (D-M,étale) topology (flat, smooth, unramified, locally Noetherian, normal…) we can just check it on a smooth atlas. Properness is not of this form.

Definition 2.12. A morphism of stacks $f : X \to Y$ is *proper* if it is separated, of *finite type* and *universally closed* (for the Zariski topology[2.6] on the sets $|X|$ and $|Y|$).

We won't use this definition directly. One instead typically uses the valuative criteria for properness.

First we state the condition for schemes. Let R be a discrete valuation ring. Write K for the quotient field of R. Then we have

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "?"] & Y \end{tikzcd} . \tag{2.75}$$

Then in this setting uniqueness of such an arrow is separatedness and existence is properness.

Now in the case of stacks, we ask for a finite extension K'/K and then write R' for the normalization of R in K'. Then the question is if this exists/is unique:

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed, "?"] & \operatorname{Spec} R \arrow[r] & Y \end{tikzcd} \tag{2.76}$$

In moduli theory this goes by the name of stable reduction.

Consider the case $\overline{\mathcal{M}_g}/\operatorname{Spec}\mathbb{Z}$, $g \geq 2$. As it turns out Lecture 11; October 3, 2019

Theorem 2.13. *$\overline{\mathcal{M}_g}$ is proper.*

By the above valuative criteria it suffices to show the following:

Theorem 2.14. *Let R be a DVR, $B = \operatorname{Spec} R$, and k be the quotient field of R. Consider some stable curve $f : X_K \to \operatorname{Spec} K$. Then there is a finite extension K'/K and a* unique *stable curve $X' \to B'$, $B' = \operatorname{Spec} R'$ where R' is the normalization of R in K', with*

$$X_{K'} \simeq X_K \times_{\operatorname{Spec} R} \operatorname{Spec} K' \tag{2.77}$$

which fits into

$$\begin{tikzcd} X' \arrow[d] & \supset & X_{K'} \arrow[r] \arrow[d] & X_K \arrow[d] \\ B' & \supset & \operatorname{Spec} K' \arrow[r] & \operatorname{Spec} K \end{tikzcd} . \tag{2.78}$$

[2.6] This is generated by representable open embeddings $U \to X$ (resp. $U \to Y$).

Proof. (In characteristic 0). First extend arbitrary by projectivity as a reduced scheme:

$$\begin{tikzcd} X_K \arrow[r, hook] \arrow[d] & \overline{X} \arrow[d] & \subset & \mathbb{P}^N_R \\ \operatorname{Spec} K \arrow[r, hook] & B \arrow[r, equal] & \operatorname{Spec} R \end{tikzcd} \tag{2.79}$$

Note that $\overline{X}$ is a surface. Now we desingularize $X \to \overline{X}$. We can do this explicitly by repeatedly blowing up. Taking the fiber at 0 gives us a nodal curve X_0, but this may not be reduced.

The DVR R has a uniformizing parameter $t \in R$, so we have $m = (t)$ maximal. Now complete a base change[2.7] $t \mapsto t^l$ where l is the least common multiple of multiplicities of X_0. This gives us $\tilde{X}' \to B = \operatorname{Spec} R'$ and $\tilde{X}'_0$ is a reduced nodal curve. But this might not be stable. But now we can contract the nonstable cusps. □

Theorem 2.15. *$\overline{\mathcal{M}}_g$ is smooth, connected, and $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (where $\mathcal{M}_g$ parameterizes smooth curves) is a normal crossings divisor with $\left\lfloor \frac{g+1}{2} \right\rfloor$ irreducible components.*

Proof. Smoothness: We can check this on a smooth atlas $\overline{H}_{g,n} \twoheadrightarrow \overline{\mathcal{M}}_g$. Checking this is really just deformation theory.

Connectedness: We can define the Hurwitz space to be:

$$\mathrm{Hur}_{k,b} = \left\{ C \to \mathbb{P}^1 \;\middle|\; \begin{array}{c} \text{simply branched, } \deg = k, \\ \text{number of branch points } = b, \\ g\,(c) = b/2 - k + 1 \end{array} \right\} . \tag{2.80}$$

Hurwitz proved that $\mathrm{Hur}_{k,b}$ is always connected, and for $k > g + 1$ we have a surjection $\mathrm{Hur}_{k,b} \twoheadrightarrow \mathcal{M}_g$, so $\mathcal{M}_g$ is connected. □

There are variations of this thing.

(a) $\overline{\mathcal{M}}_{g,k}$ consists of the stable curves of genus g with k marked points. Then the finiteness condition on the automorphisms becomes:

$$\operatorname{Aut}(C, \underline{x}) < \infty \tag{2.81}$$

where $\underline{x} = (x_1, \dots, x_k)$. In this case $2g - 2 + k > 0$, so we don't allow:

$$\cancel{\mathcal{M}_{0,0}, \dots, \mathcal{M}_{0,2}}, \mathcal{M}_{0,3}, \mathcal{M}_{0,5}, \dots \tag{2.82}$$

$$\cancel{\mathcal{M}_{1,0}}, \mathcal{M}_{1,1}, \dots \tag{2.83}$$

$$\mathcal{M}_{2,0}, \dots . \tag{2.84}$$

(b) We can also consider pre-stable curves. This just means we drop the stability condition. Fortunately the stacks project comes to our rescue here. Here everything happens in much greater generality. For example, we ask for representability of the Iso functor by algebraic spaces, and this holds very generally. This only forms an Artin stack.

[2.7] This is where we use characteristic 0. If p divided l, then this would be very nasty.

CHAPTER 3

Gromov-Witten theory and virtual techniques

1. Stable maps

Let X be a scheme ($/\mathbb{C}$ or some other field of char $= 0$).

Definition 3.1.

$$\mathcal{M}_{g,k}(X) = \left\{ \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ B \arrow[u, bend left, "\underline{x}"] & \end{tikzcd} \;\middle|\; \begin{array}{c} (C,\underline{x}) \in \tilde{\mathcal{M}}_{g,k}(B) \\ \forall \text{ geom. pts } s \to B : \\ \#\operatorname{Aut}(C_s, \underline{x}_s, f_s) < \infty \end{array} \right\} \tag{3.1}$$

where $\tilde{\mathcal{M}}_{g,k}(B)$ is the stack of nodal curves, and

$$\operatorname{Aut}(C_s, \underline{x}_s, f_s) = \{\varphi \in \operatorname{Aut}(X) \,|\, \varphi(x_i) = x_i \forall i, f \circ \varphi = f\} \ . \tag{3.2}$$

The class of f_s for $s = \operatorname{Spec}\mathbb{C}$ is $f_*[c] \in H_2(X,\mathbb{Z})$ or $H_2(X,\mathbb{Q})$ or $A_1(X)$ where $[c] \in H_2(C,\mathbb{Z})$. For fixed $\beta \in H_2(X,\mathbb{Z})$ (the "degree") we get

$$\mathcal{M}_{g,k}(X,\beta) \subset \mathcal{M}_{g,k}(X) \tag{3.3}$$

is open.

Now we provide an explanation for this definition. Let $f : (C,\underline{x}) \to X/\operatorname{Spec}\mathbb{C}$. Then for any irreducible component $C' \subset C$ we get a corresponding $f : C' \to X$. There are only two possibilities, either this map is either constant or finite. If it is finite, then already $\#\operatorname{Aut}(C', f) < \infty$. So really the contracted components (where f is constant) are the interesting pieces.

For $g = 1$ smooth, everything gets mapped to a point, so we can rule this case out. Besides this, the stability condition is equivalent to saying that any contracted component $C' \subset C$ has at least three special points. Note that a node is two special points on the normalization. For $d = 3$ in $\mathbb{P}^2$ fig. 1 shows a degeneration as embedded curves to something like $y^2 - x^3 = 0$.

Motivation. For $\mathbb{P}^n$ $(n \geq 3)$ the space of embedded curves of fixed arithmetic genus is highly singular, but from the stable maps point of view, at least for the genus 0 case:

Theorem 3.1. *$\mathcal{M}_{0,k}(\mathbb{P}^n)$ is smooth.*

Proof. First consider the stack of nodal curves (pre-stable) $\tilde{\mathcal{M}}(0,k)$. This is smooth, so it suffices to prove that the forgetful map $\mathcal{M}_{0,k}(\mathbb{P}^n) \to \tilde{\mathcal{M}}_{0,k}$ (which sends $(C,\underline{x},f) \mapsto (C,\underline{x})$) is a smooth morphism of stacks.

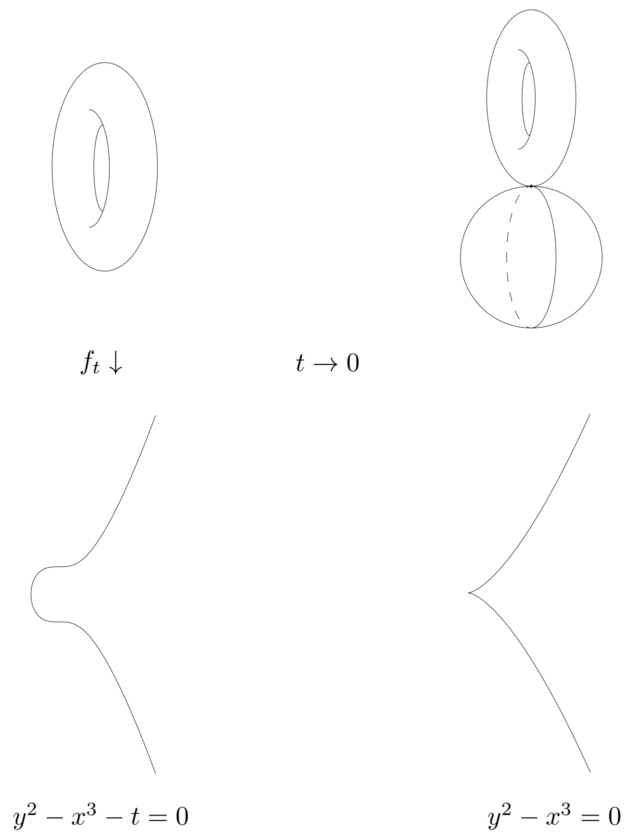

FIGURE 1. Family f_t of stable maps into $\mathbb{P}^1$. When we send $t \to 0$ we are forced to contract the genus one part of the curve which gives us the cusp.

This amounts to checking the following. For $\bar{I} = A/I$ (where $I^2 = 0$) then we have

$$\begin{tikzcd} \operatorname{Spec} \bar{A} \arrow[r] \arrow[d, hook] & \mathcal{M}_{0,k}(\mathbb{P}^n) \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \tilde{\mathcal{M}}_{0,k} \end{tikzcd} \tag{3.4}$$

and we want to find the dashed arrow. In other words we want to find:

$$\begin{tikzcd} \bar{C} \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "\bar{f}"] & C \arrow[d] \arrow[r, dashed, "\exists f?"] & X \\ \operatorname{Spec} \bar{A} \arrow[r] & \operatorname{Spec} A & \end{tikzcd} \quad . \tag{3.5}$$

I.e. given $\bar{f}$, we want to find f. The obstruction class is in $H^1\left(\bar{C}, \bar{f}^*\Theta_X\right)$. Stability from before gives us $\bar{f}^*\Theta_X$ is globally generated on each irreducible component $\bar{C}' \subset \bar{C}$. So we have

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\bar{C}'}^{\oplus l} \longrightarrow \bar{f}^*\Theta_{\mathbb{P}^n}\big|_{\bar{C}'} \longrightarrow 0 \quad . \tag{3.6}$$

For $g = 0$ we have

$$0 = H^1\left(\mathcal{O}_{\bar{C}'}\right)^{\oplus l} \longrightarrow H^1\left(\bar{f}^*\Theta_{\mathbb{P}^n}\big|_{\bar{C}'}\right) \longrightarrow H^2 = 0 \quad . \tag{3.7}$$

□

Lecture 12; October 8, 2019

THEOREM 3.2. *$\mathcal{M}_{g,k}(X, \beta)$ is a proper, separated Deligne-Mumford stack.*

PROOF. Write $\mathfrak{m}_{g,k}$ for the Artin stack of nodal curves with k marked points. If we have a diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \\ T \arrow[u, bend left, "\underline{x}"] & \end{tikzcd} \tag{3.8}$$

this is really the same data as

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"] & X \times T \arrow[r] \arrow[d, "pr_2"] & X \arrow[d] \\ T \arrow[r] \arrow[u, bend left, "\underline{x}"] & T \arrow[r] & \mathrm{pt} \end{tikzcd} \quad . \tag{3.9}$$

This is really a morphism over T, so this tells us the following. We have that

$$\begin{tikzcd} \mathcal{C}_g \simeq \mathfrak{m}_{g,1} \arrow[d] \\ \mathfrak{m}_g \end{tikzcd} \tag{3.10}$$

is a universal curve, so we have

$$\mathcal{M}_g \hookrightarrow \operatorname{Hom}_{\mathfrak{m}_g}\left(\mathcal{C}_g, \mathfrak{m}_G \times X\right) \tag{3.11}$$

is an Artin stack, locally of finite type.

Now since stability is open we want to show:

Lemma 3.3. *This is an open substack.*

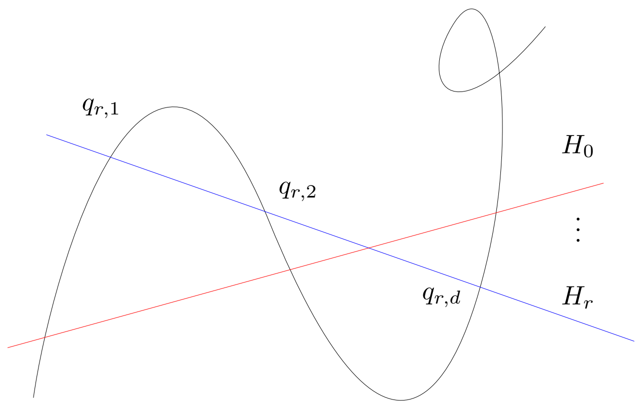

FIGURE 2. $r+1$ hyperplanes in $\mathbb{P}^r$ transversely intersection C_s.

Once we fix $\beta \in H_2(X,\mathbb{Z})$ there are only finitely many "combinatorial types"[3.1] of stable maps. This tells us that $\mathcal{M}_{g,k}(X,\beta)$ is quasi-compact.

The harder part is the stable reduction which tells that this is separated and proper. The idea (due to Fulton-Pandharipande) is that for $X \subset \mathbb{P}^r$ gives us the closed embedding $\mathcal{M}(X,\beta) \hookrightarrow \mathcal{M}(\mathbb{P}^r,d)$, so WLOG $X = \mathbb{P}^r$. The point is that $\mathcal{L} = \omega_c \otimes f^*\mathcal{O}_{\mathbb{P}^r}(1)$ is relatively ample for C/T. Then there exists some number l depending on g, k, and the degree d such that $\mathcal{L}^{\otimes l}$ is relatively very ample. $\square$

Consider the picture in fig. 2. These hyperplanes $H_i = (t_i = 0)$, for $t_i \in \Gamma(\mathbb{P}^r, \mathcal{O}(1))$ give rise to additional marked points on C. Then locally $\mathcal{M}(\mathbb{P}^r,d)$ is isomorphic to the rigidified space R with objects

$$\begin{tikzcd} C \arrow[d, "\pi"] \\ T \arrow[u, bend left, "q=(q_{ij})"] \end{tikzcd} \tag{3.12}$$

such that we have a bunch of properties. Define

$$\mathcal{H}_i = \mathcal{O}_C(q_{i,1} + \ldots + q_{i,d}) \ . \tag{3.13}$$

[3.1] By this we mean the intersection pattern of irreducible cusps, generate, and classes β_i of each component with $\beta = \sum_i \beta_i$.

Then for all i we require

$$\pi^*\pi_*\left(\mathcal{H}_i^{-1}\otimes\mathcal{H}_0\right)\to\mathcal{H}_i^{-1}\otimes\mathcal{H}_0 \tag{3.14}$$

is an isomorphism (and for all i, j $\mathcal{H}_i \simeq \mathcal{H}_j$). We also require $\lambda_0,\dots\lambda_r : T\to\mathbb{G}_m$ to scale the canonical section s_i of $\mathcal{H}_i$.

Now the idea is to do stable reduction on the rigidified level.

2. GW-invariants for hypersurfaces, $g = 0$

Let $X\subset\mathbb{P}^{n+1}$ be a hypersurface of degree l, $X = Z(F)$. Then $\mathcal{M}_0(X,d)\subset\mathcal{M}_0\left(\mathbb{P}^{n+1},d\right)$ is defined by the zero locus of a section of a vector bundle:

$$\begin{array}{ccc} \mathcal{C}_0\left(\mathbb{P}^{n+1}\right) & \xrightarrow{f} & \mathbb{P}^{n+1} \\ \big\downarrow{\scriptstyle\pi} & & \\ \mathcal{M}_0\left(\mathbb{P}^{n+1}\right) & & \end{array} . \tag{3.15}$$

On a geometric fiber $f^*\mathcal{O}(l)$ has degree ≥ 0 on each irreducible component. $g=0$ implies $H^1\left(f_s^*\mathcal{O}(l)\right)=0$. Then base change gives us

$$R^1\pi_*f^*\mathcal{O}(l)=0\ , \tag{3.16}$$

$\mathcal{E}\coloneqq\pi_*f^*\mathcal{O}(l)$ is locally free. Then

$$\operatorname{rank}\mathcal{E}=h^0\left(C_s,f_s^*\mathcal{O}(l)\right)=\deg f_s^*\mathcal{O}(l)+(1-g)=dl+1 \tag{3.17}$$

by Riemann-Roch. Then F defines $\sigma\in\Gamma\left(\mathcal{M}_0\left(\mathbb{P}^{n+1}\right),\mathcal{E}\right)$ with $\sigma((C_s,f_s))=0$, i.e. $f_s(C_s)\subset X$.

Definition 3.2 (Kontsevich).

$$[\mathcal{M}_0(X,d)]_{\mathrm{virt}}\coloneqq c_{dl+1}(\mathcal{E})\cap\left[\mathcal{M}_0\left(\mathbb{P}^{n+1},d\right)\right]\ . \tag{3.18}$$

Note that by definition

$$[\mathcal{M}_0(X,d)]_{\mathrm{virt}}\in A_*\left(\mathcal{M}\left(\mathbb{P}^{N+1},d\right)\right)\to H_{2*}\left(\mathcal{M}_0\left(\mathbb{P}^{n+1},d\right)\right)\ . \tag{3.19}$$

First note that

$$\dim\mathcal{M}_0\left(\mathbb{P}^{n+1},d\right)=\deg f_s^*\Theta_{\mathbb{P}^{n+1}}+(n+1)(1-g)-\underbrace{3}_{\dim\operatorname{Aut}(\mathbb{P}^1)} \tag{3.20}$$

$$=d(n+1)+n-2\ . \tag{3.21}$$

For $n=3$, $l=5$ we have $\operatorname{rank}\mathcal{E}=5d+1$ and by the above computation,

$$\dim\mathcal{M}_0\left(\mathbb{P}^4,d\right)=5d+1\ . \tag{3.22}$$

3. Digression on intersection theory

The book [**5**] is a good reference. Let X be a scheme over a field. We say finite sums

$$\sum n_i\left[V_i\right] \tag{3.23}$$

for $V_i \subset C$ irreducible, reduced, closed, subschemes. A rational equivalence is generated by $W \subset X$ an irreducible subvariety, $f \in K(X) \setminus \{0\}$ a rational function. Then we can define the divisor of f to be the sum over $V \subset W$ which is a codimension 1 irreducible subvariety:

$$[\operatorname{div}(f)] \coloneqq \sum_{V \subset W} \operatorname{ord}_V f \cdot [V] \ . \tag{3.24}$$

We now define the fundamental class of X. Write

$$|X| = \cup_i X_i \tag{3.25}$$

for a decomposition into irreducible components. Define

$$[X] = \sum m_i [X_i] \tag{3.26}$$

with m_i the length of $\mathcal{O}_{X,X_i}$ over $\mathcal{O}_{X,X_i}$.

Example 3.1. Take $X = \operatorname{Spec} A$ to be a thick point where $\dim_k A < \infty$. Then for $M \in A$-**Mod** we want to know the length l. By definition maximal sequence of submodules $M_1 \subset \ldots M_l = M$ such that the quotients are just quotients by prime ideals, i.e. $M_{i+1}/M_i \simeq A/\mathfrak{p}_i$ for primes $\mathfrak{p}_i$.

For example we can take $A = k[\epsilon] / \left(\epsilon^{k+1}\right)$. Then the length of A as and A-module is $k+1$ which comes from the length of the sequence:

$$\left(\epsilon^{k+1}\right) = 0 \subset \left(\epsilon^k\right) \subset \ldots \subset (\epsilon) \subset A \ . \tag{3.27}$$

In general, for an artinian local ring we should obtain the dimension.

Then the Chow groups are

$$A_*(X) = Z_*(X) / \sim_{\text{rat'l}} \tag{3.28}$$

for $* = 0, 1, 2, \ldots$.

Example 3.2. The Chow groups vanish for affine space: $A_*(\mathbb{A}^n) = 0$.

For projective space we have

$$A_8(\mathbb{P}^n) = \begin{cases} \mathbb{Z} & i = 0, \ldots, n \\ 0 & \text{o/w} \end{cases} \ . \tag{3.29}$$

Lecture 13; October 9, 2019

Proper push-forward and flat pull-back. We will define a notion of a proper push-forward. Let $f : X \to Y$ be a proper morphism. The definition is very simple. For $V \subset X$ a subvariety we insist that $W = f(Y)_{\text{red}} \subset Y$ is also a subvariety. This gives us a field extension $K(W) \hookrightarrow f^{\#}K(V)$. Then we define:

$$f_*[V] \coloneqq \begin{cases} 0 & \dim W < \dim V \\ [K(V) : K(W)] \cdot [W] & \dim W = \dim V \end{cases} \ . \tag{3.30}$$

Now we have to prove

Proposition 3.4. *$f_*[V]$ descends to $f_* : A_*(X) \to A_*(Y)$.*

Now we need flat pullback. Let $f : X \to Y$ be a flat morphism of relative dimension n. Then for $V \subset Y$ a subvariety of dimension k, then we can pullback to get $f^{-1}(V)_{\text{red}} \subset X$ which is a union of subvarieties.

Definition 3.3. $f^*[V] \coloneqq \left[f^{-1}(V)\right]$.

Then we show this commutes with rational equivalence.

Excision. We now consider the form of excision we have in this "homology" theory. Consider a closed embedding $i : Y \hookrightarrow X$. Then we get an open embedding $j : X \setminus Y \hookrightarrow X$. Then we get $A_*(X) \xrightarrow{j_*} A_*(X \setminus Y)$, and this actually turns out to be surjective. Then we have kernel $A_*(Y)$, but this might not be injective. So we get an exact sequence:

$$A_*(Y) \longrightarrow A_*(X) \xrightarrow{j_*} A_*(X \setminus Y) \longrightarrow 0 \quad . \tag{3.31}$$

PROOF.

$$Z_k(Y) \to Z_k(X) \to Z_k(X \setminus Y) \to 0 \tag{3.32}$$

□

Weil divisors. Now we have the notion of Weil divisors. Take X be to be n-dimensional. Then the Weil divisors are

$$D = \sum m_i D_i \in Z_{n-1}(X) \quad . \tag{3.33}$$

What we really want to intersect with are not these "homology" classes, but rather some "cohomology" classes called the Cartier divisors. For X any scheme, we first cover X by open subsets U_α. Then in the total ring of quotients we have

$$f_\alpha \in R(U_\alpha) \setminus \{0\} \tag{3.34}$$

for each U_α. These look like g/h for h a non-zero divisor. Then for all α and β we want that $f_\alpha / f_\beta \in \mathcal{O}^\times (U_\alpha \cap U_\beta)$, i.e. it invertible. Then a Cartier divisor is $\{(U_\alpha, f_\alpha)\}$ which gives us an invertible sheaf $\mathcal{O}(D)$.

Let X be pure n-dimensional. We have some kind of map from Cartier divisors to $A_{n-1}(X)$ sending

$$\{(U_\alpha, f_\alpha)\} \mapsto U[\operatorname{div}(f_\alpha)] \quad . \tag{3.35}$$

We will use the notation that $D \mapsto [D]$.

Cartier divisors. Now we study intersections with Cartier divisors. So we have $D = \{(U_\alpha, f_\alpha)\}$ and some subvariety $j : V \hookrightarrow X$ of pure dimension k. Then we define

$$D \cdot [V] \coloneqq \begin{cases} [j^{-1} D] & V \not\subset D \\ [C] & C \subset V \text{ Cartier div. s.t. } \mathcal{O}_V(C) \simeq \mathcal{O}_X(D)|_V \end{cases} \tag{3.36}$$

This defines:

$$D \cdot - : A_k(X) \to A_{k-1}(X) \quad . \tag{3.37}$$

First Chern class and Segre classes. Define the first Chern class as follows. Let L be a line bundle. This implies $L = \mathcal{O}_X(D)$ for D some Cartier divisor. Then we define

$$\begin{array}{ccc} A_*(X) & \xrightarrow{c_1(L)} & A_{*-1}(X) \\ z = \sum_i m_i [z_i] & \longmapsto & D \cdot z \end{array} \quad . \tag{3.38}$$

To get higher Chern classes we will define the Segre class of a vector bundle $E \downarrow X$. Write $\mathcal{E}$ for the sheaf of section of E. Write r for the rank. Then

$$E = \underline{\operatorname{Spec}} \operatorname{Sym}^\bullet \mathcal{E}^\vee \tag{3.39}$$

where

$$\mathcal{E}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \ . \tag{3.40}$$

Then the lines in E form the projective bundle

$$\mathbb{P}(E) = \underline{\mathrm{Proj}}\,\mathrm{Sym}^\bullet(E^\vee) \ . \tag{3.41}$$

This comes with $\mathcal{O}_{\mathbb{P}(E)}(1)$. The morphism

$$p : \mathbb{P}(E) \to X \tag{3.42}$$

is proper and flat, so we have the above pullback and pushforward for this morphism. Then the Segre classes (for $i \geq 0$) are

$$s_i(E) : A_k(X) \to A_{k-i}(X) \tag{3.43}$$

and we define

$$s_i(E) \frown \alpha \coloneqq p_*\left(c_1\left(\mathcal{O}_{\mathbb{P}(E)}(1)\right)^{r+i-1} \frown p^*\alpha\right) \ . \tag{3.44}$$

Now we want a projection formula for Segre classes, but this just follows from the formula for Cartier divisors. Let $f : X \to Y$ be proper. For $E \downarrow Y$ a vector bundle we have

$$f_*(s_i(f^*E) \frown \alpha) = s_i(E) \frown f_*(\alpha) \ . \tag{3.45}$$

Corollary 3.5. $s_0(E) = \mathrm{id}$.

Proof. We will use the projection formula for the inclusion of a subvariety $i : V \hookrightarrow X$. It is sufficient to show this for one variety, i.e. WLOG let $V = X$. We can compute

$$f_*(s_i(f^*\alpha) \frown p^*\alpha) = m_i[X] \tag{3.46}$$

since there are no other classes in $A_k(X) = \mathbb{Z}\cdot[X]$. So now we just have to compute the m_i. But this can be done locally, where we have that

$$\mathbb{P}(E) = X \times \mathbb{P}^{r-1} \tag{3.47}$$

and $\mathcal{O}_{\mathbb{P}(E)}(1)$ is just pulled back from $\mathbb{P}^{r-1}$. Then we have

$$c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1))^{k-1} \frown \left[\mathbb{P}^{r-1}\right] = [\mathrm{pt}] \in A_0\left(\mathbb{P}^{r-1}\right) \tag{3.48}$$

which means $m = 1$. $\square$

Chern classes. Now we get Chern classes from Segre classes. We have

$$s_t(E) = \sum s_i(E)\, t^i \in \mathrm{End}(A_*(x))[\![t]\!] \tag{3.49}$$

$$c_t(E) = s_t(E)^{-1} \ . \tag{3.50}$$

This really needs that $s_i s_j = s_j s_i$ for all i and j.

Then we have the sum formula which tells us that whenever we have

$$0 \to E' \to E \to E'' \to 0 \tag{3.51}$$

we get

$$c_t(E) = c_t(E') - c_t(E'') \ . \tag{3.52}$$

Gysin pullback. Let V be a pure k-dimensional variety, ι a regular embedding of codimension c. Then this is equivalent to being locally given by $I = (x_1, \ldots, a_d) \subset A$. Note each a_i is *not* a zero-divisor in $A/(a_1, \ldots, a_{i-1})$ for $i = 1, \ldots, d$. So we have a cartesian square:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ X \arrow[r, hook, "\iota"] & Y \end{tikzcd} . \tag{3.53}$$

For an ideal sheaf $\mathcal{I}$ we have $\mathcal{I}/\mathcal{I}^2$ is locally free of rank d, so a vector bundle of rank d. Define the normal bundle

$$N_{X/Y} = \underline{\operatorname{Spec}} \operatorname{Sym}^{\bullet} \mathcal{I}/\mathcal{I}^2 \ . \tag{3.54}$$

Similarly, for $\mathcal{I}$ an ideal sheaf of $W \hookrightarrow V$ ($\mathcal{I}/\mathcal{I}^2$ need not be locally free in this case!) we define the normal cone:

$$\begin{tikzcd} C_{W/V} \arrow[d, hook] & \coloneqq & \underline{\operatorname{Spec}}_W \bigoplus_{d \geq 0} \mathcal{I}^d/\mathcal{I}^{d+1} \\ N_{W/V} \arrow[rr, equal] & & \operatorname{Sym}^d \mathcal{I}/\mathcal{I}^2 \arrow[u, two heads] \end{tikzcd} . \tag{3.55}$$

Proposition 3.6. *For V pure k dimensional, $C_{W/V}$ is purely k-dimensional.*

Now comes a basic construction. First write

$$s_0 : W \to C_{W/V} \tag{3.56}$$

for the zero section. We know we have

$$C_{W/V} \hookrightarrow N_{W/V} \hookrightarrow g^* N_{X/Y} \tag{3.57}$$

so we can define

Definition 3.4.

$$\iota^! [V] \coloneqq s_0^* \left[C_{W/V}\right] \tag{3.58}$$

where $s_0^* = (\pi^*)^{-1}$ and

$$\pi^* : A_{k-d}(X) \to A_k(E) \ . \tag{3.59}$$

Remark 3.1. Geometrically we should think of this as intersection with the zero section.

Then this makes sense since we have:

Proposition 3.7. *π^* is an isomorphism.*

Proof. Surjectivity is easy: it follows from excision. Injectivity is harder. ☐

Slogan: Take $C_{W/V}$, embed in $g^* \left(N_{X/Y}\right)$, and finally intersect with the zero-section.

This was all for one variety V.

Consider a diagram

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$$Z \overset{\iota}{\hookrightarrow} Y \overset{f}{\leftarrow} X \tag{3.60}$$

where ι is lci. Let $W \subset X$ be a subvariety of dimension d. Then we get

$$
\begin{tikzcd}
W_Z \arrow[r] \arrow[d] & W \arrow[d, hook, "X"] \\
X_Z \arrow[r, hook] \arrow[d, "g"] & X \arrow[d, "f"] \\
Z \arrow[r, hook, "\iota"] & Y
\end{tikzcd} \tag{3.61}
$$

Then we have

$$C_{W_Z/W} = \operatorname{Spec}_W \bigoplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1} \tag{3.62}$$

pure of dimension d. Then we can pull the normal bundle back and we have:

$$C_{W_Z/W} \hookrightarrow g^* N_{Z/W} \ . \tag{3.63}$$

Then

$$\iota^! [W] \coloneqq s_0^* \left[C_{W_Z/W}\right] \in A_k (W_Z) \to A_* [X] \ . \ . \tag{3.64}$$

But why the normal cone? We have something called the deformation to the normal cone.[3.2] Consider $M \coloneqq \mathrm{Bl}_{X \times \{0\}} Y \times \mathbb{A}^1$. Over $t \in \mathbb{A}^1 \setminus \{0\}$ we just have $X \hookrightarrow Y$. Over $t = 0$

$$X \hookrightarrow C_{X/Y} \hookrightarrow \mathbb{P}\left(C_{X/Y} \oplus \mathbb{A}^1\right) \cup \mathrm{Bl}_X Y \ . \tag{3.65}$$

The point is that for problems that vary nicely in flat families and are local near X, such as intersection multiplicities, $X \hookrightarrow Y$ is as good as $X \hookrightarrow C_{X/Y}$.

Now we have the following application. We want to calculate the virtual fundamental class (VFC) for $\mathcal{M}_0(X)$ where $X = Z(f) \subset \mathbb{P}^{n+1}$ where $\deg f = l$. Recall we have $\mathcal{M}_0\left(\mathbb{P}^{n+1}\right) \xleftarrow{\pi} \mathcal{C}\varphi\mathbb{P}^{n+1}$ and then we have a vector bundle of rank r

$$
\begin{array}{c}
E = \pi_* \varphi^* \mathcal{O}(l) \\
\downarrow \\
\mathcal{M}_0\left(\mathbb{P}^{n+1}\right) = M
\end{array} \tag{3.66}
$$

called the deformation bundle. Then we have a section $s \in \Gamma(E)$ defined by f with $(s = 0) = \mathcal{M}_0(X) \subset \mathcal{M}_0\left(\mathbb{P}^{n+1}\right)$. Now intersect (s) with the zero section:

$$
\begin{tikzcd}
\mathcal{M}_0(X) \arrow[r, leftrightarrow] \arrow[d] & M \arrow[d, "s"] \\
M \arrow[r, hook, "s_0"] & E
\end{tikzcd} \ . \tag{3.67}
$$

Then we get:

$$s_0^! [M] \in A_{\dim \mathcal{M}_0(X) - r} \left(\mathcal{M}_0(X)\right) \ . \tag{3.68}$$

[3.2] This should really be called a degeneration, but everyone says deformation.

4. Obstruction theories and VFC

4.1. The idea. This idea is from [**9**]. Locally, say on $U \subset \mathcal{M}$, embed U in a smooth space V. This corresponds to an ideal sheaf $\mathcal{I}$ on V. Then we have the diagram:

$$\begin{tikzcd} C_{U/V} = \operatorname{Spec}_U \left(\bigoplus \mathcal{I}^d / \mathcal{I}^{d+1} \right) \arrow[r, hook] & N_{U/V} = \operatorname{Spec}_U \left(\operatorname{Sym}^\bullet \mathcal{I}/\mathcal{I}^2 \right) \\ & T_V|_U = \operatorname{Spec}_U \left(\operatorname{Sym}^\bullet \Omega_V|_U \right) \arrow[u, two heads] \end{tikzcd} \tag{3.69}$$

Note that the additive action of $T_V|_U$ leaves $C_{U/V}$ invariant.

Now we want to embed $N_{U/V} \hookrightarrow E_1$ into some vector bundle and globalize. (This is the same as a $T_V|_U$ action on $C_{U/V}$.)

The patching data is as follows. Whenever we have

$$\begin{tikzcd} T_V|_U \arrow[r] \arrow[d] & E_0 \\ N_{U/V} \arrow[ur, hook, "\times" marking] & \end{tikzcd} \tag{3.70}$$

we get a sequence

$$T_V|_U \to E_0 \oplus N_{U/V} \to E_1 \to 0 \tag{3.71}$$

and we ask for it to be exact.

There are two main ways to achieve compatibility. The first is with Artin style obstruction theory as in [**9**]. Alternatively we could use the cotangent complex/derived categories as in [**1**]. We will do this second option.

4.2. Digression on the cotangent complex. For $f : X \to Y$ we have an exact sequence

$$f^* \Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0\ . \tag{3.72}$$

For f smooth the arrow $f^*\Omega_Y \to \Omega_X$ is injective. In general, we need to replace $\Omega_{X/Y}$, Ω_X, ... by a complex $L^\bullet_{X/Y}$, $L^\bullet_Y$, ... of quasi-coherent sheaves. Then in the derived category we have an exact sequence

$$\ldots \to Lf^* L^\bullet_Y \to L^\bullet_X \to L^\bullet_{X/Y} \to Lf^* L_Y[1] \to L^\bullet_X[1] \to \ldots \tag{3.73}$$

which means this is an exact triangle. This is an extension of what we started in the sense that

$$h^{-1}\left(L^\bullet_{X/Y} \right) \to \underbrace{h^0\left(Lf^* L^\bullet_Y \right)}_{=\Omega_Y} \to \underbrace{h^0\left(L^\bullet_X \right)}_{=\Omega_X} \to \underbrace{\Omega_{X/Y}}_{h^0\left(L^\bullet_{X/Y} \right)} \to 0 = h^1\left(Lf^* L^\bullet_Y \right)\ . \tag{3.74}$$

Now we define the category $D\mathbf{QCoh}\left(\mathcal{O}_X \right)$. The objects are complexes

$$\ldots \to \mathcal{F}^{-2} \xrightarrow{d^{-2}_{\mathcal{F}}} \mathcal{F}^{-1} \xrightarrow{d^{-1}_{\mathcal{F}}} \mathcal{F}^0 \to \ldots \tag{3.75}$$

where

$$d^\bullet = d^\bullet_{\mathcal{F}} : \mathcal{F}^\bullet \to \mathcal{F}^\bullet[1]\ . \tag{3.76}$$

The morphisms $\varphi^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ are Hom spaces as complexes moduli homotopy, i.e. $\varphi^\bullet \simeq \psi^\bullet$ iff there exists $h^\bullet : \mathcal{F}^\bullet : \mathcal{G}^\bullet [-1]$ such that

$$\varphi^\bullet - \psi^\bullet = h^\bullet \circ d^\bullet_{\mathcal{F}} - d_{\mathcal{G}} \circ h^\bullet \ . \tag{3.77}$$

Then we localize (make invertible) quasi-isomorphisms.

Therefore a morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ in $D\mathbf{QCoh}(\mathcal{O}_X)$ is

$$\begin{tikzcd} & \tilde{\mathcal{F}}^\bullet \arrow[dl, "\psi^\bullet"'] \arrow[dr, "\tilde{\varphi}^\bullet"] & \\ \mathcal{F}^\bullet \arrow[rr, "\varphi^\bullet"] & & \mathcal{G}^\bullet \end{tikzcd} \ . \tag{3.78}$$

Now consider derived functors. Let $f : X \to Y$. Then we want to define Rf_*. Replace $\mathcal{F}^\bullet$ by quasi-isomorphic complex $\mathcal{I}^\bullet_{\mathcal{F}}$ of injective sheaves [choose one for each $\mathcal{F}^\bullet$]. Then define

$$Rf_*\mathcal{F}^\bullet \coloneqq f_*\mathcal{I}^\bullet_{\mathcal{F}} \ . \tag{3.79}$$

Remark 3.2. The left-derived tensor product $\otimes^L$ and pullback $Lf^* = f^{-1} \otimes^L \mathcal{O}_X$ are more subtle, but they work.

Now the cotangent complex is as follows. First consider the affine case. For a ring map $\varphi : A \to B$, we can resolve B freely as an A algebra. We can even do this canonically. This means we have $p_\bullet \to B \to 0$ with

$$P_\bullet = [\ldots \to A[A[B]] \to A[B]] \ . \tag{3.80}$$

Then in this case:

$$L^\bullet_{B/A} = \Omega^\bullet_{P_\bullet/A} \otimes_{P_\bullet} B \ , \tag{3.81}$$

i.e.

$$L^{-n}_{B/A} = \Omega_{P_n/A} \otimes_{P_n} B \ . \tag{3.82}$$

Let $X \to Y$ be a morphism of algebraic stacks. Then there is a similar "simplicial" resolution for $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. See [**7**] or the stacks project for more information.

The following is important. We can always embed in something smooth:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & Z \arrow[d, "\text{smooth}"] \\ & Y \end{tikzcd} \ . \tag{3.83}$$

Now writing $\mathcal{I} = \mathcal{I}_{X/Z}$ we have

$$\tau_{\geq -1} L^\bullet_{X/Y} = \left[\mathcal{I}/\mathcal{I}^2 \to \underbrace{\mathcal{O}_{Z/Y}\big|_X}_{\iota^*\Omega_{Z/Y}} \right] \ . \tag{3.84}$$

I.e. at the level of linear fibre spaces:

$$\left[T_{Z/Y}\big|_X \to N_{X/Z} \right] \tag{3.85}$$

as needed for the globalization of $C_{X/Z} \hookrightarrow N_{X/Z}$.

4.3. Behrend-Fantechi definition.

DEFINITION 3.5. Let $f : X \to Y$ be a morphism of algebraic stacks. Assume X is Deligne-Mumford. A *perfect obstruction theory* on X/Y is a 2-term complex $\mathcal{F}^\bullet = \left[\mathcal{F}^{-1} \to \mathcal{F}^0\right]$ of locally free coherent sheaves, together with a morphism (in $D\mathbf{QCoh}\left(\mathcal{O}_X\right)$)

$$\varphi^\bullet : \mathcal{F}^\bullet \to L^\bullet_{X/Y} \tag{3.86}$$

such that

(1) $h^0\left(\varphi^\bullet\right)$ is an isomorphism,
(2) $h^{-1}\left(\varphi^\bullet\right)$ is an epimorphism.

The point is...

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