Mirror Symmetry

Lectures: Professor Bernd Siebert
Notes by: Jackson Van Dyke

Overview of mirror symmetry

The course is topics in algebraic geometry. We will be doing some sort of mirror symmetry. We will start with some historical overview. Lecture 1, August 29, 2019

0.1. Enumerative mirror symmetry. Let X be a CY manifold. In particular we will focus on CY 3-folds. This means $K_X = \det T_X^* \simeq \mathcal{O}_X$ is trivial as a holomorphic line bundle. Typically this means we want $b_1 = 0$ and irreducible.

Example 0.1 (Quartic in $\mathbb{P}^4$). Take $f \in \mathbb{C}[x_0, \ldots, x_4]$ homogeneous of degree 5. If it is sufficiently general, the zero locus is smooth inside $\mathbb{P}^4$ and is an example of a CY three-fold.

We have

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{\mathbb{P}^4}|_X \to \mathcal{O}_X \to 0$$

where $\mathcal{I} = (f) \subset \mathcal{O}_{\mathbb{P}^4}$. We also have that $\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_X$ is an invertible sheaf so this first map sends $f \mapsto df$. This implies

$$K_{\mathbb{P}^4}|_X = \det \mathcal{O}_{\mathbb{P}^4}|_X = \mathcal{I}/\mathcal{I}^2 \otimes K_X \ ,$$

and

$$K_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-5) \ .$$

Then $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^4}$ which has a section with poles of order 5. The point is we can make f into a five by dividing by x_0^5, so we have that, as an abstract sheaf,

$$\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^4}(-5) \ .$$

Then we have that

$$\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X(-5)$$

and we can just take the tensor product to get

$$K_{\mathbb{P}^4}|_X \simeq \mathcal{O}_X(-5)$$

so $K_X \simeq \mathcal{O}_X$ must be trivial.

Now we want to produce a string theory out of this. This is a very delicate process. There are things called $IIA(X)$ and $IIB(X)$ theories. These are the ones relevant in mirror symmetry. These come from the super-symmetric σ-models with target some 10-dimensional space[0.1] $\mathbb{R}^{1,3} \times X$. These are the so-called super conformal field theories $SCFT_A(X)$ and $SCFT_B(X)$. These are different theories which produce observables, e.g. the Hodge number of X can be computed from these theories. In particular we can compute $h_{1,1}(X)$ and $h_{2,1}(X)$ which correspond to some physical variables. On the B-side we make the same computation but get $h_{2,1}(X)$ and $h_{1,1}(X)$. Then we postulate that there is some other X' where these

[0.1] To have an anomaly free theory.

are not flipped. In particular the observation is, for very specific X, we can find a CY Y with

$$SCFT_A(X) = SCFT_B(Y) \qquad SCFT_B(X) = SCFT_A(Y)\ .$$

Somehow then the idea is that if this is really a model for string theory, we should really be swapping

$$IIA(X) \text{``}=\text{''}\ IIB(X) \qquad IIB(X) \text{``}=\text{''}\ IIA(Y)\ .$$

But this might be a bit much to ask.

0.2. Topological twists. There is something called a topologically twisted σ-model introduced by Witten in 1988. This produces a completely different theory. We get two theories, one called $A(X)$, and one $B(X)$.

WARNING 0.1. As it turns out, $A(X)$ ends up computing things in certain limits of the $IIB(X)$ theory.

REMARK 0.1. A priori these are unrelated to $SCFT_A(X)$ and $SCFT_B(X)$.

As it turns out, if we have $SCFT_A(X) = SCFT_B(Y)$, then we have

$$A(X) = B(Y) \qquad B(X) = A(Y)\ .$$

$A(X)$ and $B(X)$ compute certain limits, called Yukawa-couplings, for $SCFT_B(Y)$ and $SCFT_A(X)$.

Note that by this twisting procedure $A(X)$ sees (X, ω) (where ω is the Kähler form) only as a symplectic manifold, and $B(X)$ depends only on the complex manifold (X, I).

0.3. Useful calculations. The reason people really got excited about mirror symmetry is that it helps us make calculations we couldn't make before.

In [**1**] the Yukawa couplings for the quintic and the mirror quintic were computed. In particular they computed F_B of the mirror quintic Y_t, and claimed this is in fact equal to F_A of the quintic. Geometrically F_A has to do with counts of (genus 0) holomorphic curves. F_B has to do with period integrals

$$\int_\alpha \Omega_{Y_t} = F_B(t)$$

for $\alpha \in H_3(Y_t)$. So they predicted some counts, then someone computed it directly and they agreed.

WARNING 0.2. This Ω_{Y_t} is only defined up to scale so really the case is that

$$F_B(t) = \exp\left(\int_{\alpha_1} \Omega_{Y_t} \Big/ \int_{\alpha_0} \Omega_{Y_t}\right)$$

for $\alpha_i \in H_3(Y_t)$.

Then Morrison/Deligne in 1992 described $F_B(Y)$ in terms of Hodge theory/more parameters for CY moduli. This is when Gromov-Witten theory entered the scene in 1993 to make $F_A(X)$ precise. So at least we had a mathematical statement.

0.4. Homological mirror symmetry. In 1994 Kontsevich gave his legendary ICM talk. This is where homological mirror symmetry took off. He said that as mathematicians we don't really know SCFTs. But what should be true is really:

$$D\,\mathrm{Fuk}\,(X) = D^b\left(\mathcal{O}_Y\right) \ .$$

This is a formulation, not an explanation.

Professor Siebert would like to convince us of a procedure to construct mirror pairs.

0.5. Proving numerical mirror symmetry. In 1996 Givental gave a proof that in the case of hypersurfaces F_A really is F_B of the mirror. This was somehow a computation showing that the sides do in fact agree. This is not very satisfying to Professor Siebert. In 1997 Lian, Liu, and Yan proved it more generally.

0.6. Proving HMS. In 2003 Paul Seidel proved HMS for the quartic in $\mathbb{P}^3$. Essentially he shows that both sides have enough rigidity to do a very minimal computation. This is also not very satisfying to Professor Siebert. It was then proved in 2011 by N. Sheridan for all CY hypersurfaces.

0.7. Modern state. There are many other manifestations of mirror symmetry. As it turns out even geometric Langlands can be viewed as some form of mirror symmetry.

As for HMS, some symplectic people are trying to prove this for so-called SYZ fibered symplectic manifolds with a rigid space as the mirror.

Then there are intrinsic constructions, things which Professor Siebert has worked on (with Mark Gross) with many applications. The idea is to use mirror symmetry as a tool in mathematics rather than just a phenomenon in physics. The point is one has to find a way of producing mirrors.

This entire story is genus 0, what physicists would call tree-level. There is also a higher genus case. From the representation theory side this has something to do with quantum groups. This is called second quantized mirror symmetry.[0.2] There is an entire field called topological recursion related to this.

0.8. Plan for the class.

1. part of the COGP computation (periods)
2. Gromov-Witten theory, virtual fundamental class/moduli stacks
3. toric degenerations and mirror constructions[0.3]
4. One strategy for proving HMS is to compute homogeneous coordinate rings of both sides. Polishchuk has shown that this ring determines $D^b\left(\mathcal{O}_Y\right)$. It would be nice to make the analogous symplectic calculation because this would be a very sneaky proof of HMS.
5. Higher genus: Donaldson-Thomas invariants play some sort of unclear role in MS because they will have something to do with the higher genus story. One can make these computations using "crystal melting". This is some kind of statistical mechanics.

[0.2]This term comes from QFT.

[0.3]This will include some introduction to toric geometry.

CHAPTER 1

Mirror symmetry for the quintic

1. The quintic threefold, its mirror, and COGP

Take some quintic CY in $\mathbb{P}^4$, i.e. $V(f) \subset \mathbb{P}^4$ for some homogeneous degree 5 f. First let's do a dimension count for homogeneous polynomials in $x_0, \ldots, x_4$ of degree 5. This is just drawing with replacement, so we have

$$\begin{array}{ccc} x_0^2 x_2 & \leftrightarrow & || \cdot \cdot | \\ x_0 x_1^2 x_2 & \leftrightarrow & | \cdot || \cdot | \end{array}$$

and we get

$$\binom{9}{5} = 126$$

which means

$$\dim_{\mathbb{C}} \left\{ Z(f) \subset \mathbb{P}^4 \right\} = 126 - 1 = 125 \ .$$

Now we mod out by $\mathrm{PGL}(5)$, which is of dimension $5 \cdot 5 - 1 = 24$. So we get

$$\dim \underbrace{\mathcal{M}_5}_{\text{moduli space of quintics}} = 101 \ .$$

Indeed: for a projective CY manifold X, the moduli space of CY manifolds deformation equivalent to X is a smooth orbifold of complex dimension $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle. We will compute this number as an exercise next time. For $V(5) \subset \mathbb{P}^4$ this is 101.

Lecture 2;
September 3, 2019

2. Quintic 3-folds

We will be looking at quintic 3-folds $V(5) \subset \mathbb{P}^4$ given by $Z(f)$ for some homogeneous degree 5 polynomial.

Theorem 1.1. *For X a projective CY manifold then the moduli space of CY deformation-equivalent to X is a smooth space of dimension $h^1(\Theta_X)$.*

By an elementary argument we saw that $\dim \mathcal{M}_{V(5)} = 101$.

2.1. Computation of $H^1(\Theta_X)$. We will compute in the case $X = V(5) \subset \mathbb{P}^4$. We start with the Euler sequence. We have $\mathbb{P}^4 = \mathrm{Proj}\, \mathbb{C}[x_0, \ldots, x_4]$ and

$$x_i \partial_{x_i} = x_i \frac{\partial}{\partial x_i}$$

are well-define logarithmic vector fields. Then the sequence is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow 0$$

$$1 \longmapsto \sum e_i$$

$$e_i \longmapsto x_i \partial_{x_i}$$

Then we have the conormal sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{\mathbb{P}^4}\big|_X \xrightarrow{\mathrm{res}_x} \Omega^1_X \longrightarrow 0$$

where $\mathcal{I}$ is the ideal sheaf of X. This is dual to

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4}\big|_X \longrightarrow N_{X/\mathbb{P}^4} \simeq \mathcal{O}_{\mathbb{P}^4}(5) \longrightarrow 0 \tag{1.1}$$

$\mathcal{I}/\mathcal{I}^2$ can be computed because $\mathcal{I} \simeq \mathcal{O}(-5)$. The restriction sequences are:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.2}$$

$$0 \longrightarrow \Theta_{\mathbb{P}^4}(-5) \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow \Theta_{\mathbb{P}^4}\big|_X \longrightarrow 0 \tag{1.3}$$

Now (1.1) gives us

$$\begin{tikzcd} H^0(\Theta_X) \arrow[r] \arrow[d, equal] & H^0(\Theta_{\mathbb{P}^4}|_X) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \arrow[r] \arrow[d, equal] & H^1(\Theta_X) \arrow[r] & H^1(\Theta_{\mathbb{P}^4}|_X) \arrow[d, equal] \\ 0 & \mathbb{C}^{24} \arrow[u, "\simeq"] & \mathbb{C}^{125} & \mathbb{C}^{101} \arrow[u, "\simeq"] & 0 \end{tikzcd}.$$

For $H^1(\Theta_{\mathbb{P}^4}|_X)$, (1.3) give us:

$$H^1(\Theta_{\mathbb{P}^4}) \longrightarrow H^1(\Theta_{\mathbb{P}^4}|_X) \longrightarrow H^2(\Theta_{\mathbb{P}^4}(-5)) \quad .$$

The Euler sequence gives us:

$$\begin{tikzcd} H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5} \arrow[r] \arrow[d, equal] & H^1(\Theta_{\mathbb{P}^4}) \arrow[r] & H^2(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\ 0 & & 0 \end{tikzcd},$$

then we can tensor the Euler sequence with $\mathcal{O}(-5)$ to get

$$\begin{tikzcd} H^i(\mathcal{O}_{\mathbb{P}^4}(-4)) \arrow[r] \arrow[d, equal] & H^i(\Theta_{\mathbb{P}^4}(-5)) \arrow[r] \arrow[d, equal] & H^i(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\ 0 \arrow[r, Rightarrow] & 0 & 0 \arrow[l, Rightarrow] \end{tikzcd}.$$

For $H^0(\mathcal{O}_{\mathbb{P}^4}(5)|_X)$ we can tensor (1.2) with $\mathcal{O}(5)$ to get

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & H^0(\mathcal{O}_{\mathbb{P}^4}) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \arrow[r] \arrow[d, equal] & H^0(\mathcal{O}_X(5)) \arrow[r] \arrow[d, equal] & H^1(\mathcal{O}_{\mathbb{P}^4}) \\ \mathbb{C} & \mathbb{C}^{126} \arrow[u, "\simeq"] \arrow[r, Rightarrow] & \mathbb{C}^{125} & 0 & \end{tikzcd}.$$

Now for $H^0\left(\Theta_{\mathbb{P}^4}|_X\right)$ we have that (1.3) gives us

$$\begin{tikzcd}
H^0\left(\Theta_{\mathbb{P}^4}(-5)\right) \arrow[r] \arrow[d, equal] & H^0\left(\Theta_{\mathbb{P}^4}\right) \arrow[r] \arrow[d, equal] & H^0\left(\Theta_{\mathbb{P}^4}|_X\right) \arrow[r] & H^1\left(\Theta_{\mathbb{P}^4}(-5)\right) \arrow[d, equal] \\
0 & \mathbb{C}^{24} \arrow[r, Rightarrow] & \mathbb{C}^{24} \arrow[u, "\simeq"'] & 0
\end{tikzcd} .$$

Then the result is that:

$$h^1\left(\Theta_X\right) = 101 \ .$$

3. Hodge diamond

3.1. Dolbeault cohomology. Let X be a compact Kähler manifold. Recall we have the Dolbeault cohomology $H^{i,j}_{\bar{\partial}}$ is the cohomology of the sequence

$$\mathcal{A}^{i,j} \xrightarrow{\bar{\partial}} \mathcal{A}^{i,j+1}$$

where this looks like

$$\sum h_{\mu\nu}\, dz_{\mu_1} \wedge \ldots \wedge dz_{\mu_i} \wedge d\bar{z}_{\nu_1} \wedge \ldots \wedge d\bar{z}_{\nu_j}$$

where the $h_{\mu\nu} \in \mathcal{C}^\infty$.

Then we have the following facts:

(1) We have a canonical isomorphism from the Dolbeault cohomology

$$H^{i,j}_{\bar{\partial}} = H^j\left(X, \Omega^i_X\right) = \overline{H^{j,i}_{\bar{\partial}}}$$

which implies

$$\dim H^{i,j}_{\bar{\partial}} = h_{ij} = h_{ji} \ .$$

(2)

$$H^{n-i,n-j}_{\bar{\partial}} = H^{n-j}\left(X, \Omega^{n-i}_X\right) = {}^{[1.1]} H^j\left(X, K_X \otimes \left(\mathcal{O}^{n-i}_X\right)^*\right)^* H^{i,j}_{\bar{\partial}}$$

which, in particular, means $h_{ij} = h_{n-i,n-j}$.

(3)

$$H^k(X,\mathbb{C}) = H^k_{dR}(X) = \bigoplus_{i+j=k} H^{i,j}_{\bar{\partial}} \ .$$

Recall $b_i = \dim_{\mathbb{C}} H^k(X,\mathbb{C})$. For CY n-folds, $b_1 = 0$ by the definition of CY. This implies $h_{10} = h_{01} = 0$. Moreover

$$H^{n,0} = H^0\left(X, \underbrace{K_X}_{\mathcal{O}_X}\right) \simeq \mathbb{C} \ .$$

This implies that

$$h_{n,0} = h_{0,n} = 1 \ .$$

We say a CY is irreducible if the universal cover $\tilde{X} \to X$ is not a nontrivial product of CY. This is equivalent to $H^{k,0} = 0$ for $k = 1, \ldots, n-1$.

Counterexample 1. The product $K3 \times K3$ is not irreducible.

[1.1] By Serre duality

3.2. Hodge diamond. The hodge diamond is the following:

$$
\begin{array}{ccccccccc}
 & & & h_{33} & & & & & \leftarrow H^6 \\
 & & h_{23} & & h_{32} & & & & \leftarrow H^5 \\
 & h_{13} & & h_{22} & & h_{31} & & & \vdots \\
h_{01} & & h_{12} & & h_{21} & & h_{30} & & \\
 & h_{02} & & h_{11} & & h_{20} & & & \\
 & & h_{01} & & h_{10} & & & & \\
 & & & h_{00} & & & & &
\end{array}
$$

The only interesting part is the center:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & {\color{blue}h_{22}} & & 0 & \\
 & & & \Vert & & & \\
1 & & {\color{red}h_{12}} & = & {\color{red}h_{21}} & & 1 \\
 & & & \Vert & & & \\
 & 0 & & {\color{blue}h_{11}} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

since

$$H^1(\Theta_X) = H^1\left(\Theta_X \otimes \underbrace{K_X}_{\mathcal{O}_X}\right) = H^1\left(\mathcal{O}_X^{n-1}\right) = H^{n-1,1}\ .$$

So the question is reduced to making these calculations. In the $V(5)$ case $h_{21} = 101$ as we saw.

3.3. Lefschetz theorem on $(1,1)$-classes. Now we work at the generality of compact Kähler manifolds. The Néron-Severi group $NS(X)$, is the preimage of $H^{1,1}$ under $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$. So somehow morally $NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}$. Now we have the exponential sequence of abelian sheaves:

$$1 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^{\times} \longrightarrow 1$$

$$
\begin{array}{ccccccccc}
H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1\left(X,\mathcal{O}_X^{\times}\right) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_C) \\
 & & \Vert & & \Vert & & & & \\
 & & H^{0,1} & & \mathrm{Pic}(X) & & & &
\end{array}\ .
$$

So $c_1(\mathrm{Pic}(X)) = NS(X)$.

Now we know how to compute $\mathrm{Pic}(X)$ for CY n-folds. In particular, as long as $n \geq 3$, $h_{11} = \mathrm{rank}_{\mathbb{Z}} NS(X)$ and $\mathrm{Pic}(X) \simeq NS(X)$ since $\mathrm{Pic}^0(X) = 0$ in the CY case.

3.4. Hard Lefschetz theorem. Let $X \subset \mathbb{P}^n$ be a Kähler manifold. This tells us that

$$H^k(\mathbb{P}^n, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is an isomorphism for $k < n-1 = \dim X$ and surjective for $k = n-1$.

Now for a CY 3-fold we have $0 = H^1(\mathbb{P}^n, \mathbb{Z}) = H^1(X, \mathbb{Z})$ and $H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z} \twoheadrightarrow H^2(X, \mathbb{Z})$ which means $NS(X) \simeq \mathbb{Z}$. This is generated by the hyperplane class $c_1(\mathcal{O}(1))$ and then restricting to X gives the ample line bundle on X and this generates the group.

So today we learned that the interesting part of the diamond in the quintic case looks like

$$\begin{array}{ccc} & 1 & \\ 101 & & 101 \\ & 1 & \end{array}$$

and then as it turns out, the mirror quintic will look like

$$\begin{array}{ccc} & 101 & \\ 1 & & 1 \\ & 101 & \end{array}$$

so we have a huge Picard group.

Lecture 3;
September 5, 2019

4. Lefschetz hyperplane theorem

This is an addendum to last lecture. Let $X \subset \mathbb{P}^n$ be a hypersurface. Then

$$H^k\left(\mathbb{P}^{n+1}, \mathbb{Z}\right) \xrightarrow{\mathrm{res}} H^k(X, \mathbb{Z})$$

is an isomorphism for $k < n-1$ and surjective for $k = n-1$.

5. The mirror quintic

We should have mirrored Hodge numbers. Recall the interesting part of the Hodge diamond for the original quintic is

$$\begin{array}{ccc} & 1 & \\ 101 & & 101 \\ & 1 & \end{array}$$

and then the mirror diamond should be:

$$\begin{array}{ccc} & 101 & \\ 1 & & 1 \\ & 101 & \end{array} \cdot$$

So $h_{21}(Y) = 1$, i.e. it has a one-dimensional moduli space, but $h_{11}(Y) = 101$ so it has a big $\mathrm{Pic}(Y)$.

5.1. Construction. This is physically motivated by the orbifold of the "minimal CFT" related to the Fermat quintic, i.e. the one given by $x_0^5 + \ldots + x_4^5 = 0$. $(\mathbb{Z}/5)^5$ acts diagonally on $\mathbb{P}^4$. This gives an effective action of $(\mathbb{Z}/5)^4 = (\mathbb{Z}/5)^5 / (\mathbb{Z}/5)$ since one copy of $\mathbb{Z}/5$ acts trivially. Now we take the finite quotient

$$\bar{Y} = X/(\mathbb{Z}/5)^4 \ .$$

The action is not free, so this is not a manifold, i.e. it has some orbifold singularities. (X is smooth by Jacobi criterion.) The stabilizer $G_Y \subset (\mathbb{Z}/5)^4$ is nontrivial. There are two cases:

- $x_i = x_j = 0$ $(i \neq j)$, in which case $G_Y \simeq \mathbb{Z}/5$. This gives quintic curves $\tilde{C}_{ij} = Z(x_i, x_j) \subset X$. The local action is given by $\zeta(z_1, z_2, z_3) = (\zeta z_1, \zeta^{-1} z_2, z_3)$. This gives rise to the singularity given by $uv = w^5$, the A_4 singularity in $\mathbb{C}^3$ (with coordinates u, v, w).
- $x_i = x_j = x_k = 0$ (i, j, k pairwise disjoint), where we get $G_Y \simeq (\mathbb{Z}/5)^2$. This gives us
$$\tilde{P}_{i,j,k} \to P_{ijk} \in \bar{Y} \ .$$
Now the local action looks like
$$(\zeta, \xi) \cdot (z_1, z_2, z_3) = \left(\zeta\xi z_1, \zeta^{-1} z_2, \xi^{-1} z_3\right) \ .$$

Remark 1.1. Inside $\bar{Y}$ we have

$$C_{01} = Z\left(x_0, x_1, x_2^5 + x_3^5 + x_4^5\right) / (\mathbb{Z}/5)^3 \simeq Z(u + v + w) \simeq \mathbb{P}^1 \subset \mathbb{P}^2 \ .$$

Any C_{ij} looks like a $\mathbb{P}^1$.

We want to blow these singularities up locally, but this is delicate if we want to stay in the world of projective algebraic varieties, i.e. we might just not have an ample line bundle after blowing up. So we have to prove something extra.

Proposition 1.1. *There exists a projective resolution $Y \to \bar{Y}$.*

This is done most efficiently by toric methods, but can be done by hand.

Let's count the independent exceptional divisors in T. We have 4 over each C_{ij} and 6 over each P_{ijk}. So we have 40 from the C_{ij} and 60 from the P_{ijk} and we have 100 in total. Together with the hyperplane class they span the H^2.

Proposition 1.2. *$h_{11}(Y) = 101$, $h_{21}(Y) = 1$.*

The proof was done directly by S.S. Roan and done by by toric methods by Batyrev.

5.2. Mirror families. This mod G construction generalizes to what is called the "Dwork family". In particular we have $X_\psi = V(f_\psi)$ where

$$f_\psi = x_0^5 + \ldots + x_4^5 - 5\psi x_0 x_1 \ldots x_4$$

and ψ is a complex parameter. So we have

$$\begin{array}{ccc} (\mathbb{Z}/5)^4 = \{(\zeta_0, \ldots, \zeta_4) \mid \prod \zeta_i = 1\} & \subset & (\mathbb{Z}/5)^5 \\ \downarrow & & \downarrow \\ G = (\mathbb{Z}/5)^3 & & (\mathbb{Z}/5)^4 \end{array}$$

and

$$\begin{tikzcd} X_\psi = Z(f_\psi) \arrow[d, "/(\mathbb{Z}/5)^3"] & \\ \bar{Y}_\psi & Y_\psi \arrow[l] \end{tikzcd} .$$

Note that $Y\psi \simeq Y_{\zeta\psi}$ for $\zeta^5 = 1$. Then we have

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "/\mathbb{Z}/5"] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathbb{P}^1_\psi \arrow[r, "/\mathbb{Z}/5"] & \mathbb{P}^1_z \end{tikzcd}$$

where $z \coloneqq (5\psi)^{-5}$. This is a family of CY 3-folds which are smooth for $z \neq 0, \infty$.

The special fibers are as follows.

- $z = 0$: $x_0 \ldots x_4 = 0$ implies
$$Y_{z=0} \simeq \bigcup_5 \mathbb{P}^3 \subset \mathbb{P}^4$$
is a union of coordinate hyperplanes.
- $z = 5^{-5}$, i.e. $\psi = 1$. This corresponds to one 3-fold A_1 singularity. X_1 has 125 three dimensional A_1-singularities. Which locally look like $x^2 + y^2 + z^2 + w^2 = 0$. They all lie in one $(\mathbb{Z}/5)^3$-orbit. $Y_{5^{-1}}$ has one three dimensional A_1 singularity sometimes called the "conifold".
- $z = \infty$, i.e. $\psi = 0$: this is the Fermat quintic. This has an additional $\mathbb{Z}/5$ symmetry because we drop the condition that the product of the x_i has to be 1. So this is really an orbifold point.

Now, at least from a physics point of view we are done.[1.2]

6. Yukawa couplings

6.1. A-model. The A-model (symplectic) will deal with the quintic. So we have $H^2(X, \mathbb{Z}) = \mathbb{Z} \cdot h$ for $h = PD$ (hyperplane). Then the Yukawa coupling is

$$\langle h, h, h \rangle_A = \sum_{d \in \mathbb{N}} N_d d^3 q^d \ .$$

At this point (historically) it was not clear what the N_d actually were because Gromov-Witten theory was sort of being developed in parallel. Nowadays we know

[1.2] Professor Siebert says that maybe mathematicians would be better off like this: not worrying so much and just seeing what life brings.

that the N_d are Gromov-Witten (GW) counts of rational curves ($g = 0$) of degree d. If we write n_d for the primitive counts, then

$$5 + \sum_{d>0} d^3 n_d \underbrace{\frac{q^d}{1-q^d}}_{\text{multiple cover of } \deg d \text{ curves}} \in \mathbb{Q}[\![q]\!] .$$

6.2. B-model. Now we consider the mirror quintic Y_z, $z = (5\psi)^{-5}$. Now the Yukawa coupling is given by:

$$\langle \partial_z, \partial_z, \partial_z \rangle_B := \int_{Y_z} \Omega^\nu(z) \wedge \partial_z^3 \Omega^\nu(z)$$

where Ω^ν is a "normalized" holomorphic volume form:

$$\int_{\beta_0} \Omega^\nu = \text{constant}$$

where $\beta_0 \in H_2(Y, \mathbb{Z})$.

We need some kind of mirror to the vector field h on the moduli space of symplectic structures. What we really want is actually the exponentiated thing $e^{2\pi i h}$. So as it turns out on this side of mirror symmetry this looks like $\partial/\partial w$ which corresponds to a vector field on the complex moduli space of Y.

It turns out

$$w = \int_{\beta_1} \Omega^\nu(z)$$

where $\beta_1 \in H_3(Y, \mathbb{Z})$.

6.3. Mirror symmetry. Now the actual statement of mirror symmetry is that

$$\langle h, h, h \rangle_A = \langle \partial_z, \partial_z, \partial_z \rangle_B$$

where $q = e^{2\pi i w(z)}$ ($w = c \cdot z + \mathcal{O}(z^2)$).

So now we have to:

(1) write down the holomorphic 3 form (not too bad)
(2) do the normalization period integral (not too bad)
(3) computing this second integral (more bad)
(4) computing the Yukawa coupling.

6.4. Computation of the periods. There is an account of this in lecture notes by Mark Gross (Nordfjordeid). Recall we have:

$$\begin{array}{ccc} & & Y_\psi \\ & & \downarrow \\ X_\psi & \xrightarrow{/G} & \bar{Y}_\psi \end{array} .$$

We know $H_3(Y_\psi, \mathbb{Z}) \simeq \mathbb{Z}^4$. Near $\psi = \infty$ (large complex structure limit) we have a vanishing cycle.

This looks like Professor Siebert's favorite picture of a degeneration. Consider $zw = t$. At $t = 0$ this looks like two disks meeting at a point. For $t \neq 0$ this looks like a cylinder. But now if we do a Dehn twist, we see that there is an S^1 which gets collapsed in this degeneration. A similar story holds in higher dimension.

In particular, our vanishing cycle looks like $\beta_0 = T^3$. Locally

$$u_1 \dots u_4 = z, \beta_0 = \left\{ |u_1| = \dots = |u_4| = |z|^{1/4}, \operatorname{Arg} u_1 \dots u_4 = 0 \right\}$$

where the u_i are holomorphic coordinates.

If we lift to X_ψ we get an explicit three-torus

$$T = \left\{ |x_0| = |x_1| = |x_2| = \delta \ll 1, x_3 = x_3\left(x_0, x_1, x_2\right) \text{ soln of } f_\psi\left(x_1, x_2, x_3, 1\right) = 0, \exists! \xrightarrow{z \to 0} 0 \right\}$$

Recall we have

Lecture 4; September 10, 2019

$$\begin{array}{ccc} & & Y_\psi \\ & & \downarrow \\ X_\psi & \xrightarrow{/G} & \bar{Y}_\psi \end{array}$$

where $G = (\mathbb{Z}/5\mathbb{Z})^3$. Recall last time we discussed:

(1) Vanishing cycle T^3, $\beta_0 = \in H_3\left(Y_\psi, \mathbb{Z}\right)$,
(2) Holomorphic 3-form,
(3) Normalization,
(4) Further periods, and
(5) Canonical coordinate/mirror map.

7. Holomorphic 3-form

We will construct the holomorphic 3-form as the residue of a meromorphic/rational 4-form on $\mathbb{P}^4$ with zeros along X_ψ:

$$\Omega\left(\psi\right) = 5\psi \operatorname{Res}_{X_\psi} \frac{\tilde{\Omega}}{f_\psi} \in \Gamma\left(X_\psi, \Omega^3_{X_\psi}\right)$$

where

$$\tilde{\Omega} = \sum x_i \, dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_4 \quad .$$

Locally $x_4 = 1$, $\partial_{x_3} f \neq 0$,

$$\Omega\left(\psi\right) = 5\psi \left.\frac{dx_0 \wedge dx_1 \wedge dx_2}{\partial_{x_3} f_\psi}\right|_{X_\psi} \quad .$$

8. Normalization

Now we deal with normalization. We have $\varphi_0 := \int_{\beta_0} \Omega\left(\psi\right)$, then

$$\tilde{\Omega} = \varphi_0^{-1} \Omega\left(\psi\right)$$

is normalized with resides

$$\begin{aligned}
2\pi i \int_{\beta_0} \Omega(\psi) &= \int_{T^4} 5\psi \, dx_0 \ dx_1 \ dx_2 \ dx_3 \\
&= \int_{T^4} \frac{dx_0 \ \ldots \ dx_3}{x_0 \ldots x_3} \frac{1}{\frac{\left(1+x_0^5+\ldots+x_3^5\right)}{5\psi x_0 \ldots x_3} - 1} \\
&= -\sum_{n\geq 0} \int_{T^4} \frac{dx_0 \ldots dx_3}{x_0 \ldots x_3} \frac{\left(1+x_0^5+\ldots+x_3^5\right)^n}{(5\psi)^n \left(x_0 \ldots x_3\right)^n} \\
&= -\sum_{n\geq 0} \int_{T^4} \frac{dx_0 \ldots dx_3}{x_0 \ldots x_3} \frac{\left(1+x_0^5+\ldots+x_3^5\right)^{5n}}{(5\psi)^{5n} \left(x_0 \ldots x_3\right)^{5n}}
\end{aligned}$$

where all summands in both the numerator and denominator must be 5th powers to contribute. So from some combinatorics we have

$$2\pi i \int_{\beta_0} \Omega(\psi) = -(2\pi i)^4 \sum \frac{(5n)!}{(n!)^5 (5\psi)^{5n}} =: \varphi_0(z)$$

where $z = (1/5\psi)^5$. The number $(5n)!/(n!)^5$ is the number of terms

$$x_0^{5n} \ldots x_3^{5n} \left(1+x_0^5+\ldots+x_3^5\right)^{5n} \quad .$$

9. Further periods

There is a procedure called Griffith's reduction of pole order. This involves the Picard-Fucks equation.

Locally $H^3(Y_\psi, \mathbb{C})$ is constant with dimension 4. This gives a trivial holomorphic vector bundle

$$\begin{array}{ccc} & & E = \mathcal{U} \times \mathbb{C}^4 \\ & & \downarrow \\ \mathcal{M}_Y & \hookleftarrow & \mathcal{U} \end{array} \quad .$$

This has a flat connection ∇^{GM} called the Gauß-Manin connection. Pointwise

$$E_\psi = \bigoplus_{p+q=3} H^{p,q}(X_\psi)$$

and we have seen $\Omega(\psi) \in H^{3,0}$. Now consider:

$$\Omega(z) \qquad \partial_z \Omega(z) \qquad \partial_z^2 \Omega(z) \qquad \partial_z^3 \Omega(z) \qquad \partial_z^4 \Omega(z)$$

which are related by a fourth order ODE with holomorphic coefficients called the Picard-Fuchs equation.

9.1. Derivation of the equation. Take $X = Z(f_\psi) \subset \mathbb{P}^4$. We can produce more 3-forms from forms with higher-order poles. Consider the long-exact sequence:

$$\begin{tikzcd}
H^4\left(\mathbb{P}^4, \mathbb{C}\right) \arrow[r] & H^4\left(\mathbb{P}^4 \setminus X, \mathbb{C}\right) \arrow[r] & H^5\left(\mathbb{P}^4, \mathbb{P}^4 \setminus X; \mathbb{C}, \right) \arrow[r] \arrow[d, equal, "\text{excision}"'] & H^5\left(\mathbb{P}^4, \mathbb{C}\right) \\
& & H^5\left(\mathcal{U}, \mathcal{U} \setminus X; \mathbb{C}\right) \arrow[d, equal] & \\
& & H^5\left(\mathcal{U}, \partial\mathcal{U}; \mathbb{C}\right) \arrow[d, equal, "LD"'] & \\
& & H_3\left(\mathcal{U} \setminus \partial\mathcal{U}; \mathbb{C}\right) \arrow[d, equal] & \\
& & H_3\left(X, \mathbb{C}\right) = H^3\left(X, \mathbb{C}\right) &
\end{tikzcd}$$

where $X \subset \mathcal{U}$ is a tubular neighborhood and we are using the form of Lefschetz duality which states that $H^q(M, \partial M) = H_{n-q}(M \setminus \partial M)$ and in the last step we use Poincare duality. So we start with things of high pole order, this gives us some class in H^3, then in our case we take derivatives, and for certain classes we know they should be zero and this gives us some equations.

9.2. Griffiths' reduction of pole order. If we have

$$\frac{g\tilde{\Omega}}{f^l} \in H^0\left(\mathbb{P}^4, \Omega^4_{\mathbb{P}^4 \setminus X}\right)$$

then we must have $\deg g = 5l - 5$ ($l = 0$ earlier so we had no g). The exact forms look like

$$\begin{aligned} & d\left(\frac{1}{f^l}\left(\sum_{i<j}(-1)^{i+j}(x_i g_j - x_j g_i)\, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots dx_4\right)\right) \\ & \qquad = \left(l \sum g_j \partial_{x_j} f - f \sum \partial_{x_j} g_j\right) \frac{\tilde{\Omega}}{f^{l+1}}\ . \end{aligned} \tag{1.4}$$

If $l \sum g_j \partial_{x_j} f \in \mathcal{J}(f) = (\partial_{x_i} f)$ then up to an exact form, it is of lower order since one copy of f cancels. I.e. the first term over f^{l+1} is of order $l+1$, and the second term over f^{l+1} is order l. The upshot is that the numerator $g \in \mathcal{J}(f)$ can reduce l.

So the algorithm is as follows: Compute $\Omega(z)$, $\partial_z \Omega(z)$, $\partial_z^2 \Omega(z)$, …, $\partial_z^4 \Omega(z) = g\tilde{\Omega}/f_\psi^5$ where $g \in \mathcal{J}(f_\psi)$. Then we express g modulo (1.4) as a linear combination of the $\partial_z^i \Omega(z)$.

Proposition 1.3. *Any period*

$$\varphi = \int_\alpha \Omega(\psi)$$

fulfills the ODE

$$\left[\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\right]\varphi(z) = 0 \tag{1.5}$$

where $\theta = z\partial_z$.

REMARK 1.2. This is easy to check for

$$\varphi = \varphi_0 = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} z^n \ .$$

(1.5) is an ODE with a regular singular pole:

$$\theta \varphi(z) = A(z)\,\varphi(z) \tag{1.6}$$

for $\psi(z) \in \mathbb{C}^s$.

THEOREM 1.2. (1.6) *has a fundamental system of equations of the form*

$$\Phi(z) = S(z)\, z^R$$

with $S(z) \in M(s, \mathcal{O}_0)$, $R \in M(S, \mathbb{C})$, *and*

$$z^R = I + (\log z)\, R + (\log z)^2 R^2 + \dots \ .$$

If the eigenvalues do not differ by integers, we may take $R = A(0)$.

For (1.5)

$$A(0) \simeq \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

and $S = (\psi_0, \dots, \psi_3)$ where $\psi_i \in \mathcal{O}^4_{\mathbb{C},0}$. This gives us a fundamental system of solutions:

$$\begin{aligned}
\varphi_0(z) &= \psi_0 \text{ (single-valued)} \\
\varphi_1(z) &= \psi_0(z) \log z + \psi_1(z) \\
\varphi_2(z) &= \psi_0(z) (\log z)^2 + \psi_1(z) \log z + \psi_2(z) \\
\varphi_3(z) &= \psi_0(z) (\log z)^4 + \dots + \psi_3(z) \ .
\end{aligned}$$

This has something to do with monodromy. In particular, the monodromy of $z^{A(0)}$ reflects the monodromy T of $H^3(Y_z, \mathbb{C})$ about $z = 0$ (or $\psi = \infty$). In fact, one can show that there exists a symplectic basis β_0, β_1, α_1, $\alpha_0 \in H_3(Y_z, \mathbb{Q})$ with $N = T - I$. Then we have

$$\alpha_0 \mapsto \alpha_1 \mapsto \beta_1 \mapsto \beta_0 \mapsto 0$$

which means

$$\varphi_0 = \int_{\beta_0} \Omega(z)\,, \varphi_1 = \int_{\beta_1} \Omega(z)\,, \varphi_2 = \int_{\alpha_1} \Omega(z)\,, \varphi_3 = \int_{\alpha_0} \Omega(z) \ .$$

10. Canonical coordinate/mirror map

Looking at the solution set, we don't have much choice. The solution, when exponentiated should behave like z. Indeed, the canonical coordinate is

$$q = e^{2\pi i w}$$

where

$$w = \frac{\int_{\beta_1} \Omega(z)}{\int_{\beta_0} \Omega(z)} = \int_{\beta_1} \tilde{\Omega}(z) \ .$$

Then $\varphi_1(z) = \varphi_0(z)\log z + \psi_1(z)$ which is easy to obtain as series solution of (1.5).

$$\psi_1(z) = 5\sum_{n\geq 1}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)z^n$$

(up to constant c_2).

Last time we learned how to do these period calculations, get canonical equations, and reduce pole order with the Picard Fucks equation. The remaining topic is the Yukawa coupling. Lecture 5; September 12, 2019

11. Yukawa coupling

We want to compute

$$\langle\partial_z,\partial_z,\partial_z\rangle_B = \int_{Y_z}\tilde{\Omega}(z)\wedge\partial_z^3\tilde{\Omega}(z)$$

where $\tilde{\Omega}(z) = \frac{1}{\varphi_0(z)}\Omega(z)$.

We introduce the auxiliary terms

$$W_k = \int_{Y_z}\Omega(z)\wedge\partial_2^k\Omega(z)$$

for $k = 0,\ldots,4$. So really we just want W_3. Rewrite the PF equation as

$$\left(\frac{d^4}{dz^4}+\sum_{k=0}^{3}c_k\frac{d^k}{dz^k}\right)\Omega(z) = 0\ .$$

This gives us

$$W_4 + \sum_{k=0}^{3}c_kW_k = 0\ . \tag{1.7}$$

11.1. Griffiths-transversality.

Now we need to put some important information in called Griffiths-transvr. This has to do with how one defines the variation of Hodge structures. Let $\mathcal{U}$ be open inside the moduli space. Now define a decreasing filtration

$$\mathcal{F} = \underbrace{H^3(Y_z,\mathbb{C})}_{\mathbb{C}^4}\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{U}} = \mathcal{F}^0\supset\mathcal{F}^1\supset\mathcal{F}^2\supset\mathcal{F}^3$$

where

$$\mathcal{F}^k = \bigoplus_{q\geq k}R^q\pi_*\Omega^{3-q}\ .$$

This is the Hodge filtration. Note that $\Omega\in\Gamma\left(\mathcal{U},\underbrace{R^3\pi_*\Omega^0}_{=\mathcal{F}^3\simeq\mathcal{O}}\right)$. So why do we write it this way instead of using the direct sum decomposition we seem to have? Abstractly we have that

$$\nabla^{GM}\mathcal{F}^k\subseteq\mathcal{F}^{k-1}\otimes\Omega^1_{\mathcal{M}}\ .$$

This inclusion comes from the definition/construction of ∇^{GM} and Hodge/Dolbeault theory. Moreover $H^{p,q}\perp H^{p',q'}$ unless $p+p'=3=q+q'$ (from $\int_Y\alpha\wedge\beta = 0$).

Together this gives us that $W_0 = W_1 = W_2 = 0$. In particular

$$0 = \frac{d^2W_2}{dz^2} = \ldots = 2W_3' - W_4$$

and (1.7) tells us that

$$W_3' + \frac{1}{2}c_3 W_3 = 0 \ .$$

Now we compute

$$c_3(z) = \frac{6}{2} - \frac{25^5}{1-5^5 z}$$

which by separation of variables gives us

$$W_3 = \frac{c_1}{(2\pi i)^3 z^3 (5^5 z - 1)}$$

for c_1 some integration constant. Finally, reexpress in $q = e^{2\pi i w}$ where $w = \varphi_1(z)/\varphi_0(z)$. When we expand we get

$$\begin{aligned}\langle \partial_w, \partial_w, \partial_w \rangle_B &= -c_1 - 575\frac{c_1}{c_2}q - \frac{1950750}{2}\frac{c_1}{c_2^2}q^3 - \frac{10277490000}{6}\frac{c_1}{c_2^3}q^3 + \dots \\ &\stackrel{!}{=} 5 + \sum_{d\geq 1} d^3 n_d \frac{q^d}{1-q^d} \\ &= 5 + n_1 q + (8n_2 + n_1)\, q^2 + (27 n_3 + n_1)\, q^3 + \dots \ .\end{aligned}$$

This predicts that $n_1 = 2875$ (which was classically known) and $n_2 = 6092500$ was also correct (as was shown just a few years earlier than this development by Katz in 1986). Then $n_3 = 317206375$ which originally disagreed with the result, but they found out there was an error in the computation so it also agreed. The first proof of this was in 1996 by Givental.

Lessons learned:[1.3]

(1) The prediction depends on the large complex structure limit because this has something to do with the Kähler cone:

$$\mathcal{K}_X = d\left\{[\omega] \in H^2_{dR}(X) \,\middle|\, \omega \text{ Kähler}\right\} \ .$$

In particular the monodromy in $H^3(Y) = \bigoplus_{p=0}^{3} H^{p,3-p}$ corresponds to $\smile [\omega_X]$ on $\bigoplus_{p=0}^{3} H^{p,p}(X)$. Note however that $\langle\rangle_B$ is defined on all of $\mathcal{M}_B$!

(2) Orbifolding construction of the mirror is special to the quintic. Batyrev/Borisov consider a mirror for complete intersections in toric varieties.

[1.3] Besides that computations are hard. . .

CHAPTER 2

Gromov-Witten theory

We will first learn some things about moduli spaces and moduli stacks to motivate a discussion of Gromov-Witten theory. Hopefully also the logarithmic version and Donaldson-Thomas theory.

The general task is to make sense of curve counting, e.g. the number of genus 0 holomorphic curves in a quintic. Then we have the following problems:

- Classical enumerative algebraic geometry: "general position arguments" needed to make counts work. Transversaily can be very difficult.
- Translate into problem of topology, e.g. intersection theory in Grassmannian $\mathrm{Gr}\,(k, n)$ (Schubert calculus).
- Generally, spaces of curves, e.g. on a given quintic don't have the right dimension.

    Example 2.1. Consider the Dwork family $f_\psi = 0$. There are 375 isolated lines ($\simeq \mathbb{P}^1$), e.g. $\left(u, v, -\zeta^k u, -\zeta^k v, 0\right)$ for $u, v \in \mathbb{P}^1$, $\zeta^5 = 1$, $0 \leq k,\ l \leq 4$ and then two irreducible families. In degree > 1 we also always have multiple covers that come in families.

    So how do we count in the absence of general deformations? The solution is that there is a virtual formalism. This is exactly what Gromov-Witten theory does. Invariants are constant in families of targets.

1. Moduli spaces

So we are interested in a set of closed points. In particular, this this consists of isomorphism classes of certain algebraic geometric objects e.g. varieties, subvarieties. Then we want so somehow give it some extra structure. The best scenario would be to view it as a variety.

Let $T \to \mathcal{M}$ be the structure sheaf. Then holomorphic maps correspond to families of objects over T.

Example 2.2. Fix some N. Consider the Hilbert scheme (of fixed Hilbert polynomial). $Z \subset \mathbb{P}^N$ leads to $\mathrm{Hilb}\left(\mathbb{P}^N\right)$ which satisfies

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d] & \mathrm{Hilb}\left(\mathbb{P}^N\right) \times \mathbb{P}^n \arrow[dl] \\ \mathrm{Hilb}\left(\mathbb{P}^N\right) & \end{tikzcd}.$$

Call this the universal family $\mathcal{U}$. This has the unviersal property that for

$$\begin{array}{l} Z \ \subset \ T \times \mathbb{P}^N \\ \downarrow^{\text{flat, proper}} \\ T \end{array}$$

we have a unique map φ such that

$$\begin{array}{ccc} Z = \mathcal{Z}_T & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ T \times \mathbb{P}^n & \longrightarrow & \operatorname{Hilb}\left(\mathbb{P}^N\right) \times \mathbb{P}^N \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & \operatorname{Hilb}\left(\mathbb{P}^n\right) \end{array} .$$

In categorical terms this says that we have a functor

$$\mathbf{Sch} \xrightarrow{F} \mathbf{Set}$$

$$T \longmapsto \left\{Z \to T, Z \hookrightarrow T \times \mathbb{P}^N \text{ flat, proper}\right\}$$

which is corepresented by $\operatorname{Hilb}\left(\mathbb{P}^N\right)$. This means we have a natural isomorphism $F \to \hom\left(\cdot, \operatorname{Hilb}\left(\mathbb{P}^N\right)\right)$. In particular we get that $\mathrm{id}_{\operatorname{Hilb}(\mathbb{P}^n)} \in \hom\left(\operatorname{Hilb}\left(\mathbb{P}^N\right), \operatorname{Hilb}\left(\mathbb{P}^N\right)\right)$ corresponds to $F\left(\operatorname{Hilb}\left(\mathbb{P}^N\right)\right)$, e.g. the universal family $\mathcal{U}$.

Discouraging observation: For families of curves[2.1] we cannot have (co-)representability.

The reason is that there are families of curves where all of the fibers are isomorphic but they are not globally a product. So if we had such a corepresentation then this can't pull back to the identity.

Lecture 6; September 17, 2019

1.1. The problem of moduli for curves. We will kind of follow [**3**]. [**2**] is where stacks were really worked out.

We want to repeat the story of Hilb for complete curves of genus g.

Recall the notion of families. If S is a scheme (think of this as some sort of parameter space), then a *curve* of genus g over S is a morphism $\pi : C \to S$ such that:

(1) π is proper, flat;
(2) recall the geometric fibers

$$C_S = \operatorname{Spec} K \times_S C$$

for K algebraically closed, fit into the diagram

$$\begin{array}{ccc} C_S & \longrightarrow & C \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & S \end{array} .$$

Then we ask that:
(a) C_S is reduced, connected, $\dim C_S = 1$,
(b) $h^1\left(C_S, \mathcal{O}_C\right) = g$ (arithmetic genus).

[2.1] or any other moduli proble with objects with automorphisms

One can also restrict the allowed singularities of these fibers, but this is not necessary for us at the moment. E.g. one might ask for C_S to be non-singular.

Now we have a fundamental problem. The functor

$$\mathbf{Sch} \xrightarrow{\quad F_g \quad} \mathbf{Set}$$

$$S \longmapsto \{X \to S \mid \text{ non-singular curve of genus } g\}$$

cannot be representable since there are nontrivial isotrivial families of curves, i.e. $\pi : X \to S$ which becomes trivial only after (finite) base change. Suppose it is representable by some $\mathcal{M}_g$, then

$$\begin{tikzcd}
T_X C_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & C_g \arrow[d] \\
T \arrow[r, "\text{finite}"] \arrow[rr, bend right, "\text{constant}"'] & S \arrow[r, "\varphi"] & \mathcal{M}_g
\end{tikzcd}$$

but this map being constant implies φ is constant which is a contradiction.

Example 2.3. Let C_0 be a curve with $\mathrm{Aut}\,(C_0) \neq \{1\}$, e.g. $C_0 \to \mathbb{P}^1$ a two-to-one hyperelliptic curve, e.g. the projective closure of

$$\left(y^2 - (x-g-1)(x-g)\dots(x-1)(x+1)\dots(x+g+1) = 0\right) \ .$$

We have one automorphism which swaps the two branches and one which sends $x \to -x$ so we have $(\mathbb{Z}/2)^2$ symmetry. Take φ to be any automorphism such that $\varphi^a = \mathrm{id}$. Recall $\mathbb{G}_M = \mathrm{Spec}\,\mathbb{C}\left[x, x^{-1}\right]$. Take $C_0 \times \mathbb{G}_m / (\mathbb{Z}/a)$. The action is as follows. For $\mathbb{C}^\times \ni \zeta \neq 1$, $\zeta^a = 1$ we have

$$(\varphi, \zeta) : (z, t) \mapsto (\varphi(z), \zeta \cdot t) \ .$$

So this is a nontrivial bundle over $\mathbb{G}_m$.

Remark 2.1. If automorphisms are the problem, then why not just stick to ones without them. As it turns out, restricting to C_S with $\mathrm{Aut}\,(C_S) = \{1\}$ would indeed make F_g representable, but it is not very useful.

Instead, we will construct the moduli space as an *algebraic stack*, which is a generalization of the notion of a scheme accomodating automorphisms from the beginning.

2. Stacks

Another good reference (which is unfortunately only in French[2.2]) is [**4**].

The idea here is to formalize the notion of a "family of objects parameterized by a scheme along fibrewise automorphisms".

Fix a base scheme S (think $\mathbb{C}$). Write $\mathcal{S} = \mathbf{Sch}/S$ for the category of schemes over S.

Definition 2.1. (1) A *category over* $\mathcal{S}$ is a category $\mathcal{F}$ together with a functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{S}$. For $B \in \mathrm{Obj}\,(\mathcal{S})$, we have a fibre category $\mathcal{F}(B)$ which is a subcategory of $\mathcal{F}$ with objects

$$\{X \in \mathrm{Obj}\,(\mathcal{F}) \mid p_{\mathcal{F}}(X) = B\}$$

[2.2] Professor Siebert says it isn't a big deal since French is basically English.

and morphisms

$$\{\varphi \in \mathrm{Hom}\,(\mathcal{F}) \,|\, p_{\mathcal{F}}\,(\varphi) = \mathrm{id}_B\} \ .$$

(2) A category over $\mathcal{S}$ is a groupoid over $\mathcal{S}$ (or fibered groupoid) if
   (a) For all $f : B' \to B$ in $\mathcal{S}$ and $X \in \mathrm{Obj}\,(\mathcal{F})$ there exists $\varphi : X' \to X$ in $\mathcal{F}$ with $p_{\mathcal{F}}\,(\varphi) = f$:

$$\begin{tikzcd} X' \arrow[r, dashed, "\varphi"] \arrow[d, dashed] & X \arrow[d] \arrow[d, bend left, "p_{\mathcal{F}}"] \\ B' \arrow[r, "f"] & B \end{tikzcd} \ .$$

   (b) For all commutative diagrams

$$\begin{tikzcd} & X'' \arrow[r, "\varphi''"] \arrow[dd] & X \arrow[dd] \arrow[dd, bend left, "p_{\mathcal{F}}"] \\ X' \arrow[ur, dashed, "\exists!"] \arrow[urr, "\varphi'"] \arrow[dd] & & \\ & B'' \arrow[r, "f''"] & B \\ B' \arrow[ur, "h"] \arrow[urr, "f'"] & & \end{tikzcd}$$

   (i.e. $p_{\mathcal{F}}\,(\varphi') = p_{\mathcal{F}}\,(\varphi'') \circ h$) there exists unique $\chi : X' \to X''$ and $\varphi' = \varphi'' \circ \chi$.

Remark 2.2. (i) (ii) implies that $\varphi : X' \to X$ is an isomorphism iff $p_{\mathcal{F}}\,(\varphi)$ is an isomorphism.
(ii) $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{S}$ groupoid over $\mathcal{S}$ implies $\mathcal{F}\,(B)$ are groupoids.
(iii) (ii) implies

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "p_{\mathcal{F}}"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

X' is unique up to unique isomorphism. Write $f^*X \coloneqq X'$. This is called the *pull-back*. This construction is functorial in the sense that $\psi : X'' \to X'$ in $\mathcal{F}\,(B)$ yields a canonical morphisms $f^*\psi : f^*X'' \to f^*X'$, i.e. $f : B' \to B$ gives us

$$f^* : \mathcal{F}\,(B) \to \mathcal{F}\,(B') \ .$$

Example 2.4 (Representable functors). Let $\mathcal{F} : \mathcal{S} \to \mathbf{Set}$ be a contravariant functor. This yields a groupoid over $\mathcal{F}$ The objects are (B, β) such that $B \in \mathrm{Obj}\,(\mathcal{S})$ and $\beta \in \mathcal{F}\,(B)$. The idea is that $p_{\mathcal{F}} : (B, \beta) \mapsto B$. Then

$$\mathrm{hom}\,((B', \beta'), (B, \beta)) = \{f : B' \to B \,|\, F\,(f)\,(\beta) = \beta'\} \ .$$

Note that $\mathcal{F}\,(B) = F\,(B)$ (trivial morphisms category).

For example if X is an S-scheme then this defines (is equivalent to) the functor $F\,(B) \coloneqq \mathrm{Hom}_S\,(B, X)$ and

$$F\,(\varphi : B' \to B) : (f : B \to X) \mapsto (f \circ \varphi : B' \to X) \ .$$

The associated groupoid $\underline{X} = \mathcal{F}$ has objects $f : B \to X$ in $\mathcal{Y} = \mathbf{Sch}/S$ with morphisms

$$\begin{tikzcd} B' \arrow[d, "\varphi"] \arrow[dr, "f'"] & \\ B \arrow[r, "f"] & X \end{tikzcd}$$

and $p_X : (B \to X) \mapsto B$.

Example 2.5 (Quotient stack). Let X/S be a scheme with an action of a (flat) group scheme G/S (e.g. GL_n). Then we can take the quotient $[X/G]$. The objects are diagrams

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & X \\ B & \end{tikzcd}$$

where E/B is a G-principal bundle and f is G-equivariant. The morphisms are given by

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[rr, bend left] & E \arrow[r, "f"] \arrow[d] & X \\ B' \arrow[r] & B & \end{tikzcd}$$

where the square on the left must be cartesian.

Fact 1. *G acts freely on X and X/G exists as a scheme so $[X/G] = \underline{X/G}$.*

Example 2.6 (Classifying spaces of principal G-bundles). For $X = \mathrm{pt}$, $BG \coloneqq [\mathrm{pt}/G]$.

So we have three main examples. First, if X is scheme we get Lecture 7; September 19, 2019

$$\underline{X}(S) \coloneqq \mathrm{Hom}(S, X) \ .$$

(Note that $\underline{S} = \mathcal{S}$).

Then for $G \circlearrowright X$ we get $[X/G]$. For $F : \mathbf{Sch}/S \to \mathbf{Set}$ we get $\mathcal{F}$ an S-groupoid.

Third, we have the moduli groupoid $\mathcal{M}_g$. The objects are curves $X \to B$, for B any scheme, X_s non-singular for all s. The morphisms are given by cartesian diagrams:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} .$$

Note this is *not* the groupoid associated to F_g.

Similarly, one defines the *universal curve* $\mathcal{C}_g$ over $\mathcal{M}_g$. The objects are pairs $(X \to B, \sigma)$ where $\sigma : B \to X$ is a section.

2.1. Morphisms of groupoids.

Definition 2.2. A morphism between groupoids F_1, F_2 over S is a functor

$$\begin{tikzcd} F_1 \arrow[rr, "p"] \arrow[dr, "p_{F_1}"'] & & F_2 \arrow[dl, "p_{F_2}"] \\ & \mathcal{S} & \end{tikzcd}$$

where $p_{F_1} = p_{F_2} \circ p$. Note this is *equality* of functors.

EXAMPLE 2.7. We have the forgetful functor $\mathcal{C}_g \to \mathcal{M}_g$ which simply forgets the section.

EXAMPLE 2.8. Let $f : X \to Y$ be a morphism of schemes. This is equivalent to $p : \underline{X} \to \underline{Y}$ being a morphism of the associated groupoids.

PROOF. $\Longrightarrow$: On objects, $p\left(B \xrightarrow{u} X\right) := X \xrightarrow{f\circ u} Y$. On morphisms

$$p\left(\begin{tikzcd} B \arrow[d, "s"'] \arrow[dr, "u"] & \\ B' \arrow[r, "u'"'] & X \end{tikzcd}\right) := \begin{tikzcd} B \arrow[d, "s"'] \arrow[dr, "f\circ u"] & \\ B' \arrow[r, "f\circ u'"'] & Y \end{tikzcd}.$$

($\Longleftarrow$): If we have

$$\begin{tikzcd} \underline{X} \arrow[rr, "p"] \arrow[dr] & & \underline{Y} \arrow[dl] \\ & \mathcal{S} & \end{tikzcd}$$

then $p\left(X \xrightarrow{\mathrm{id}_X} X\right) = \left(X \xrightarrow{f} Y\right) \in \mathrm{Obj}\,\underline{Y}\,(X)$ for some f. This is exactly the setup of Yoneda's Lemma: p can be viewed as a natural transformation between

$$\mathrm{Hom}_{\mathcal{S}}\,(\cdot, X) : \mathcal{S} \to \mathbf{Set}$$

and another functor $G : \mathcal{S} \to \mathbf{Set}$. Then Yoneda says that

$$\mathrm{Nat}\,(\mathrm{Hom}_{\mathcal{S}}\,(\cdot, X)\,, G) \xrightarrow{=} G\,(X)$$

where $\Phi \mapsto \Phi\,(\mathrm{id}_X)$. This shows us that p is induced by f. $\square$

EXAMPLE 2.9. Similarly, for a scheme B and a groupoid over $\mathcal{S}$ F, we get that

$$\{p : \underline{B} \to F\} = F\,(B)$$

where $p \mapsto p\,(\mathrm{id}_B)$.

EXAMPLE 2.10. $\mathcal{S} = \underline{S}$ and for any groupoid F over S, we can view $p_F : F \to \mathcal{S}$ as a morphism of groupoids $F \to \underline{S}$.

EXAMPLE 2.11. Let X/S be a scheme with the action of a group scheme G/S. This yields a quotient morphism $q : \underline{X} \to [X/G]$.

On objects:

$$\left(B \xrightarrow{s} X\right) \mapsto \left(\begin{tikzcd} (g, b) \arrow[r, mapsto] & g \cdot s\,(b) \\ G \circlearrowright G \times B \arrow[r] \arrow[d] & X \\ B & \end{tikzcd}\right).$$

On morphisms

$$\begin{tikzcd} B' \arrow[rr, "f"] \arrow[dr, "s'"'] & & B \arrow[dl, "s"] \\ & X & \end{tikzcd} \quad \mapsto \quad \left(\begin{tikzcd} & & X \\ G \times B' \arrow[r, "\mathrm{id} \times f"] \arrow[d] \arrow[urr, bend left] & G \times B \arrow[d] \arrow[ur, bend left] & \\ B' \arrow[r, "f"] & B & \end{tikzcd} \right)$$

REMARK 2.3. Isomorphisms of groupoids are given by equivalences of categories over S. In particular, $p_1 : F_1 \xrightarrow{\text{iso}} F_2$ may not have an inverse, just a quasi-inverse $q : F_2 \to F_1$ such that pq is naturally isomorphic to id_{F_2} and qp naturally isomorphic to id_{F_1}.

S-groupoids in fact form a 2-category $\mathbf{Grpd}/S$. The objects are groupoids over S. The 1-morphisms are functors over S between the groupoids. But now we in fact have another kind of morphism, called a 2-morphism which are morphisms between morphisms.

REMARK 2.4. We can't even define what a cartesian diagram is without talking about these 2-morphisms so it really is necessary to understand them.

PROPOSITION 2.1. *Let X and Y be schemes. Then $X \simeq Y$ as schemes iff $\underline{X} \simeq \underline{Y}$ as groupoids over S.*

PROOF. ($\Longrightarrow$): Let $f : X \to Y$ be an isomorphism. Then the induced map $p : \underline{X} \to \underline{Y}$ is a strong equivalence. Indeed, f^{-1} induces $q : \underline{Y} \to \underline{X}$ with $pq = \mathrm{id}_{\underline{Y}}$ and $qp = \mathrm{id}_{\underline{X}}$.

($\Longleftarrow$): Let $p : \underline{X} \to \underline{Y}$ be an equivalence, $q : \underline{Y} \to \underline{X}$ a quasi-inverse. As we have seen this means p, q are induced by $f : X \to Y$ and $g : Y \to X$ which implies $qp\left(X \xrightarrow{\mathrm{id}_X} X\right) = \left(X \xrightarrow{gf} X\right)$ as objects in $\underline{X}$. Hence q is a quasi-inverse of p, which implies there exists an isomorphism

$$\begin{tikzcd} X \arrow[rr, dashed, "\simeq"] \arrow[dr, "gf"'] & & X \arrow[dl, "\mathrm{id}_X"] \\ & X & \end{tikzcd}$$

and similarly for fg, so f is an isomorphism. □

REMARK 2.5. So this tells us we have some kind of subcategory of schemes inside of groupoids. In the future we will write X instead of $\underline{X}$. Similarly we will also write $\mathcal{S}$, $\underline{S}$, and S for the same thing.

2.2. Fibre products and cartesian diagrams.

DEFINITION 2.3. Consider three groupoids F, G, and H over S and morphisms $f : F \to G$, $h : H \to G$. The *fiber product* is as follows. The objects (over a base B) are triples (x, y, ψ) where $x \in F(B)$, $Y \in H(B)$, and $\psi : f(x) \to h(y)$ is an isomorphism in $G(B)$.

The morphisms (over B) $(x, y, \psi) \to (x', y', \psi')$ are pairs

$$\left(x' \xrightarrow{\alpha} x, y' \xrightarrow{\beta} Y\right)$$

such that

$$\psi \circ f(\alpha) = h(\beta) \circ \psi' \ .$$

So now this fits in the diagram

$$\begin{tikzcd} F \times_G H \arrow[r, "q"] \arrow[d, "p"] & H \arrow[d, "h"] \\ F \arrow[r, "f"] & G \end{tikzcd} \tag{2.1}$$

which only commutes up to 2-morphisms.

WARNING 2.1. This diagram does not commute in general. We have that

$$fp\,(x, y, \psi) = f\,(x) \qquad\qquad gq\,(x, y, \psi) = h\,(y)\ .$$

But there is is a natural isomorphism of functors $fp \simeq hq$, i.e. the diagram eq. (2.1) is 2-commutative. So we need this ψ twisting to get them to agree.

Then $F \times_G H$ has the universal property for 2-commutative diagrams:

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & F \times_G H \arrow[r] \arrow[d] & H \arrow[d] \\ & F \arrow[r] & G \end{tikzcd} .$$

EXAMPLE 2.12. For X, Y, and Z schemes we have

$$\underline{X} \times_{\underline{Z}} \underline{Y} = \underline{X \times_Z Y}\ .$$

EXAMPLE 2.13 (Base change). If $T \to S$ is a morphism of schemes then $T \times_S F$ is a groupoid over T. Actually, for all $B \to T$, $F\,(B)$ and $(T \times_S F)\,(B)$ are equivalent

$$\begin{tikzcd} X \arrow[d] & \in & F\,(B) \\ B \arrow[dr] \arrow[dd] & & \\ & T \arrow[dl] & \\ S & & \end{tikzcd} .$$

2.3. Definition of stacks. We want to get closer to something which allows us to do algebraic geometry.

DEFINITION 2.4 (Iso-functor). Let (F, p_F) be a groupoid over S, B a scheme over S, and $X, Y \in \mathrm{Obj}\,(F\,(B))$. Then

$$\mathrm{Iso}_B\,(X, Y) : \mathbf{Sch}/B \to \mathbf{Set}$$

is the following contravariant functor. On objects:

$$\left(B' \xrightarrow{f} B\right) \mapsto \left\{ f^*X \xrightarrow{\varphi} f^*Y \,\middle|\, \varphi \text{ iso} \right\}\ .$$

On morphisms we get:

$$\begin{tikzcd} B'' \arrow[rr, "h"] \arrow[dr, "g"'] & & B' \arrow[dl, "f"] \\ & B & \end{tikzcd} \mapsto \left((f^*X \to f^*Y) \mapsto \left(\underbrace{h^*f^*X}_{=g^*X} \to \underbrace{h^*f^*Y}_{=g^*Y} \right) \right) .$$

THEOREM 2.1 (Deligne-Mumford). *Take two curves X/B, Y/B of genus 2. The iso-functor* $\mathrm{Iso}_B(X,Y)$ *is represented by a scheme.*

PROOF. We know we have the relative holomorphic cotangent bundles $\omega_{X/B}$ and $\omega_{Y/B}$. These are ample, so they give us an embedding into projective space over B. These are canonical bundles, so any isomorphism $f^*X \to g^*Y$ (for any $f : B' \to B$) preserves this polarization. Now we can use the relative Hilbert scheme (for the graph of $f^*X \to f^*Y$). $\square$

REMARK 2.6. $\mathrm{Iso}_B(X,Y)$ is finite and unramified over B, but not in general flat (e.g. fibre cardinalities can jump).

DEFINITION 2.5 (Stack). Iso must be a sheaf, and some gluing condition must hold.

Bibliography

[1] Philip Candelas, Xenia C. De La Ossa, Paul S. Green, and Linda Parkes, *A pair of calabi-yau manifolds as an exactly soluble superconformal theory*, Nuclear Physics B **359** (1991), no. 1, 21 –74.

[2] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publications Mathématiques de l'IHÉS **36** (1969), 75–109 (en). MR41#6850

[3] Dan Edidin, *Notes on the construction of the moduli space of curves*, arXiv Mathematics e-prints (1998May), math/9805101, available at arXiv:math/9805101.

[4] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, Springer Berlin Heidelberg, 1999.