LECTURE 4
MIRROR SYMMETRY

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Recall we have

$$\begin{tikzcd} & Y_\psi \arrow[d] \\ X_\psi \arrow[r, "/G"] & \bar{Y}_\psi \end{tikzcd}$$

where $G = (\mathbb{Z}/5\mathbb{Z})^3$. Recall last time we discussed:

(1) Vanishing cycle T^3, $\beta_0 = \in H_3(Y_\psi, \mathbb{Z})$,
(2) Holomorphic 3-form,
(3) Normalization,
(4) Further periods, and
(5) Canonical coordinate/mirror map.

1. Holomorphic 3-form

We will construct the holomorphic 3-form as the residue of a meromorphic/rational 4-form on $\mathbb{P}^4$ with zeros along X_ψ:

$$\Omega(\psi) = 5\psi \operatorname{Res}_{X_\psi} \frac{\tilde{\Omega}}{f_\psi} \in \Gamma\left(X_\psi, \Omega^3_{X_\psi}\right)$$

where

$$\tilde{\Omega} = \sum x_i \, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_4 \ .$$

Locally $x_4 = 1$, $\partial_{x_3} f \neq 0$,

$$\Omega(\psi) = 5\psi \left. \frac{dx_0 \wedge dx_1 \wedge dx_2}{\partial_{x_3} f_\psi} \right|_{X_\psi} .$$

2. Normalization

Now we deal with normalization. We have $\varphi_0 := \int_{\beta_0} \Omega(\psi)$, then

$$\tilde{\Omega} = \varphi_0^{-1} \Omega(\psi)$$

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is normalized with resides

$$\begin{aligned} 2\pi i \int_{\beta_0} \Omega(\psi) &= \int_{T^4} 5\psi \, dx_0 \; dx_1 \; dx_2 \; dx_3 \\ &= \int_{T^4} \frac{dx_0 \, \dots \, dx_3}{x_0 \dots x_3} \frac{1}{\frac{\left(1+x_0^5+\dots+x_3^5\right)}{5\psi x_0 \dots x_3} - 1} \\ &= -\sum_{n \geq 0} \int_{T^4} \frac{dx_0 \dots dx_3}{x_0 \dots x_3} \frac{\left(1+x_0^5+\dots+x_3^5\right)^n}{(5\psi)^n (x_0 \dots x_3)^n} \\ &= -\sum_{n \geq 0} \int_{T^4} \frac{dx_0 \dots dx_3}{x_0 \dots x_3} \frac{\left(1+x_0^5+\dots+x_3^5\right)^{5n}}{(5\psi)^{5n} (x_0 \dots x_3)^{5n}} \end{aligned}$$

where all summands in both the numerator and denominator must be 5th powers to contribute. So from some combinatorics we have

$$2\pi i \int_{\beta_0} \Omega(\psi) = -(2\pi i)^4 \sum \frac{(5n)!}{(n!)^5 (5\psi)^{5n}} =: \varphi_0(z)$$

where $z = (1/5\psi)^5$. The number $(5n)!/(n!)^5$ is the number of terms

$$x_0^{5n} \dots x_3^{5n} \left(1+x_0^5+\dots+x_3^5\right)^{5n} \quad .$$

3. Further periods

There is a procedure called Griffith's reduction of pole order. This involves the Picard-Fucks equation.

Locally $H^3(Y_\psi, \mathbb{C})$ is constant with dimension 4. This gives a trivial holomorphic vector bundle

$$\begin{array}{ccc} & & E = \mathcal{U} \times \mathbb{C}^4 \\ & & \downarrow \\ \mathcal{M}_Y & \hookleftarrow & \mathcal{U} \end{array} \quad .$$

This has a flat connection ∇^{GM} called the Gauß-Manin connection. Pointwise

$$E_\psi = \bigoplus_{p+q=3} H^{p,q}(X_\psi)$$

and we have seen $\Omega(\psi) \in H^{3,0}$. Now consider:

$$\Omega(z) \qquad \partial_z \Omega(z) \qquad \partial_z^2 \Omega(z) \qquad \partial_z^3 \Omega(z) \qquad \partial_z^4 \Omega(z)$$

which are related by a fourth order ODE with holomorphic coefficients called the Picard-Fuchs equation.

3.1. **Derivation of the equation.** Take $X = Z(f_\psi) \subset \mathbb{P}^4$. We can produce more 3-forms from forms with higher-order poles. Consider the long-exact sequence:

$$\begin{tikzcd}
H^4\left(\mathbb{P}^4,\mathbb{C}\right) \arrow[r] & H^4\left(\mathbb{P}^4\setminus X,\mathbb{C}\right) \arrow[r] & H^5\left(\mathbb{P}^4,\mathbb{P}^4\setminus X;\mathbb{C},\right) \arrow[r] \arrow[d, equal, "\text{excision}"'] & H^5\left(\mathbb{P}^4,\mathbb{C}\right) \\
 & & H^5\left(\mathcal{U},\mathcal{U}\setminus X;\mathbb{C}\right) \arrow[d, equal] & \\
 & & H^5\left(\mathcal{U},\partial\mathcal{U};\mathbb{C}\right) \arrow[d, equal, "LD"'] & \\
 & & H_3\left(\mathcal{U}\setminus\partial\mathcal{U};\mathbb{C}\right) \arrow[d, equal] & \\
 & & H_3\left(X,\mathbb{C}\right) = H^3\left(X,\mathbb{C}\right) &
\end{tikzcd}$$

where $X \subset \mathcal{U}$ is a tubular neighborhood and we are using the form of Lefschetz duality which states that $H^q(M,\partial M) = H_{n-q}(M\setminus\partial M)$ and in the last step we use Poincare duality. So we start with things of high pole order, this gives us some class in H^3, then in our case we take derivatives, and for certain classes we know they should be zero and this gives us some equations.

3.2. **Griffiths' reduction of pole order.** If we have

$$\frac{g\tilde{\Omega}}{f^l} \in H^0\left(\mathbb{P}^4, \Omega^4_{\mathbb{P}^4\setminus X}\right)$$

then we must have $\deg g = 5l - 5$ ($l = 0$ earlier so we had no g). The exact forms look like

$$d\left(\frac{1}{f^l}\left(\sum_{i<j}(-1)^{i+j}\left(x_i g_j - x_j g_i\right) dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots dx_4\right)\right)$$

$$= \left(l\sum g_j \partial_{x_j} f - f\sum \partial_{x_j} g_j\right)\frac{\tilde{\Omega}}{f^{l+1}}\ . \tag{1}$$

If $l\sum g_j\partial_{x_j} f \in \mathcal{J}(f) = (\partial_{x_i} f)$ then up to an exact form, it is of lower order since one copy of f cancels. I.e. the first term over f^{l+1} is of order $l+1$, and the second term over f^{l+1} is order l. The upshot is that the numerator $g \in \mathcal{J}(f)$ can reduce l.

So the algorithm is as follows: Compute $\Omega(z)$, $\partial_z\Omega(z)$, $\partial_z^2\Omega(z)$, …, $\partial_z^4\Omega(z) = g\tilde{\Omega}/f_\psi^5$ where $g \in \mathcal{J}(f_\psi)$. Then we express g modulo (1) as a linear combination of the $\partial_z^i\Omega(z)$.

Proposition 1. *Any period*

$$\varphi = \int_\alpha \Omega(\psi)$$

fulfills the ODE

$$\left[\theta^4 - 5z\left(5\theta+1\right)\left(5\theta+2\right)\left(5\theta+3\right)\left(5\theta+4\right)\right]\varphi(z) = 0 \tag{2}$$

where $\theta = z\partial_z$.

Remark 1. This is easy to check for

$$\varphi = \varphi_0 = \sum_{n\geq 0} \frac{(5n)!}{(n!)^5} z^n \ .$$

(2) is an ODE with a regular singular pole:

$$\theta\varphi(z) = A(z)\varphi(z) \tag{3}$$

for $\psi(z) \in \mathbb{C}^s$.

Theorem 1. (3) *has a fundamental system of equations of the form*

$$\Phi(z) = S(z) z^R$$

with $S(z) \in M(s, \mathcal{O}_0)$*,* $R \in M(S, \mathbb{C})$*, and*

$$z^R = I + (\log z) R + (\log z)^2 R^2 + \dots \ .$$

If the eigenvalues do not differ by integers, we may take $R = A(0)$*.*

For (2)

$$A(0) \simeq \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

and $S = (\psi_0, \dots, \psi_3)$ where $\psi_i \in \mathcal{O}^4_{\mathbb{C},0}$. This gives us a fundamental system of solutions:

$$\begin{aligned} \varphi_0(z) &= \psi_0 \text{ (single-valued)} \\ \varphi_1(z) &= \psi_0(z)\log z + \psi_1(z) \\ \varphi_2(z) &= \psi_0(z)(\log z)^2 + \psi_1(z)\log z + \psi_2(z) \\ \varphi_3(z) &= \psi_0(z)(\log z)^4 + \dots + \psi_3(z) \ . \end{aligned}$$

This has something to do with monodromy. In particular, the monodromy of $z^{A(0)}$ reflects the monodromy T of $H^3(Y_z, \mathbb{C})$ about $z = 0$ (or $\psi = \infty$). In fact, one can show that there exists a symplectic basis β_0, β_1, α_1, $\alpha_0 \in H_3(Y_z, \mathbb{Q})$ with $N = T - I$. Then we have

$$\alpha_0 \mapsto \alpha_1 \mapsto \beta_1 \mapsto \beta_0 \mapsto 0$$

which means

$$\varphi_0 = \int_{\beta_0} \Omega(z), \varphi_1 = \int_{\beta_1} \Omega(z), \varphi_2 = \int_{\alpha_1} \Omega(z), \varphi_3 = \int_{\alpha_0} \Omega(z) \ .$$

4. Canonical coordinate/mirror map

Looking at the solution set, we don't have much choice. The solution, when exponentiated should behave like z. Indeed, the canonical coordinate is

$$q = e^{2\pi i w}$$

where

$$w = \frac{\int_{\beta_1} \Omega(z)}{\int_{\beta_0} \Omega(z)} = \int_{\beta_1} \tilde{\Omega}(z) \ .$$

Then $\varphi_1(z) = \varphi_0(z) \log z + \psi_1(z)$ which is easy to obtain as series solution of (2).

$$\psi_1(z) = 5 \sum_{n \geq 1} \frac{(5n)!}{(n!)^5} \left(\sum_{j=n+1}^{5n} \frac{1}{j} \right) z^n$$

(up to constant c_2).