# LECTURE 3
# MIRROR SYMMETRY

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## 1. Lefschetz hyperplane theorem

This is an addendum to last lecture. Let $X \subset \mathbb{P}^n$ be a hypersurface. Then

$$H^k\left(\mathbb{P}^{n+1}, \mathbb{Z}\right) \xrightarrow{\text{res}} H^k(X, \mathbb{Z})$$

is an isomorphism for $k < n-1$ and surjective for $k = n-1$.

## 2. The mirror quintic

We should have mirrored Hodge numbers. Recall the interesting part of the Hodge diamond for the original quintic is

$$\begin{matrix} & 1 & \\ 101 & & 101 \\ & 1 & \end{matrix}$$

and then the mirror diamond should be:

$$\begin{matrix} & 101 & \\ 1 & & 1 \\ & 101 & \end{matrix} \cdot$$

So $h_{21}(Y) = 1$, i.e. it has a one-dimensional moduli space, but $h_{11}(Y) = 101$ so it has a big $\operatorname{Pic}(Y)$.

2.1. **Construction.** This is physically motivated by the orbifold of the "minimal CFT" related to the Fermat quintic, i.e. the one given by $x_0^5 + \ldots + x_4^5 = 0$. $(\mathbb{Z}/5)^5$ acts diagonally on $\mathbb{P}^4$. This gives an effective action of $(\mathbb{Z}/5)^4 = (\mathbb{Z}/5)^5 / (\mathbb{Z}/5)$ since one copy of $\mathbb{Z}/5$ acts trivially. Now we take the finite quotient

$$\bar{Y} = X / (\mathbb{Z}/5)^4 \ .$$

The action is not free, so this is not a manifold, i.e. it has some orbifold singularities. ($X$ is smooth by Jacobi criterion.) The stabilizer $G_Y \subset (\mathbb{Z}/5)^4$ is nontrivial. There are two cases:

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- $x_i = x_j = 0$ ($i \neq j$), in which case $G_Y \simeq \mathbb{Z}/5$. This gives quintic curves $\tilde{C}_{ij} = Z(x_i, x_j) \subset X$. The local action is given by $\zeta(z_1, z_2, z_3) = \left(\zeta z_1, \zeta^{-1} z_2, z_3\right)$. This gives rise to the singularity given by $uv = w^5$, the $A_4$ singularity in $\mathbb{C}^3$ (with coordinates $u$, $v$, $w$).
- $x_i = x_j = x_k = 0$ ($i$, $j$, $k$ pairwise disjoint), where we get $G_Y \simeq (\mathbb{Z}/5)^2$. This gives us
$$\tilde{P}_{i,j,k} \to P_{ijk} \in \bar{Y} \ .$$
Now the local action looks like
$$(\zeta, \xi) \cdot (z_1, z_2, z_3) = \left(\zeta\xi z_1, \zeta^{-1} z_2, \xi^{-1} z_3\right) \ .$$

*Remark* 1. Inside $\bar{Y}$ we have
$$C_{01} = Z\left(x_0, x_1, x_2^5 + x_3^5 + x_4^5\right) / (\mathbb{Z}/5)^3 \simeq Z(u + v + w) \simeq \mathbb{P}^1 \subset \mathbb{P}^2 \ .$$
Any $C_{ij}$ looks like a $\mathbb{P}^1$.

We want to blow these singularities up locally, but this is delicate if we want to stay in the world of projective algebraic varieties, i.e. we might just not have an ample line bundle after blowing up. So we have to prove something extra.

**Proposition 1.** *There exists a projective resolution $Y \to \bar{Y}$.*

This is done most efficiently by toric methods, but can be done by hand.

Let's count the independent exceptional divisors in $T$. We have 4 over each $C_{ij}$ and 6 over each $P_{ijk}$. So we have 40 from the $C_{ij}$ and 60 from the $P_{ijk}$ and we have 100 in total. Together with the hyperplane class they span the $H^2$.

**Proposition 2.** *$h_{11}(Y) = 101$, $h_{21}(Y) = 1$.*

The proof was done directly by S.S. Roan and done by by toric methods by Batyrev.

2.2. **Mirror families.** This $\operatorname{mod} G$ construction generalizes to what is called the "Dwork family". In particular we have $X_\psi = V(f_\psi)$ where
$$f_\psi = x_0^5 + \ldots + x_4^5 - 5\psi x_0 x_1 \ldots x_4$$
and $\psi$ is a complex parameter. So we have
$$\begin{array}{ccc} (\mathbb{Z}/5)^4 = \{(\zeta_0, \ldots, \zeta_4) \mid \prod \zeta_i = 1\} & & \subset (\mathbb{Z}/5)^5 \\ \downarrow & & \downarrow \\ G = (\mathbb{Z}/5)^3 & & (\mathbb{Z}/5)^4 \end{array}$$
and
$$\begin{tikzcd} X_\psi = Z(f_\psi) \arrow[d, "/(\mathbb{Z}/5)^3"] & \\ \bar{Y}_\psi & Y_\psi \arrow[l] \end{tikzcd} .$$
Note that $Y\psi \simeq Y_{\zeta\psi}$ for $\zeta^5 = 1$. Then we have
$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "/\mathbb{Z}/5"] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathbb{P}^1_\psi \arrow[r, "/\mathbb{Z}/5"] & \mathbb{P}^1_z \end{tikzcd}$$

where $z := (5\psi)^{-5}$. This is a family of CY 3-folds which are smooth for $z \neq 0, \infty$.

The special fibers are as follows.

- $z = 0$: $x_0 \dots x_4 = 0$ implies
$$Y_{z=0} \simeq \bigcup_5 \mathbb{P}^3 \subset \mathbb{P}^4$$
is a union of coordinate hyperplanes.
- $z = 5^{-5}$, i.e. $\psi = 1$. This corresponds to one 3-fold $A_1$ singularity. $X_1$ has 125 three dimensional $A_1$-singularities. Which locally look like $x^2 + y^2 + z^2 + w^2 = 0$. They all lie in one $(\mathbb{Z}/5)^3$-orbit. $Y_{5^{-1}}$ has one three dimensional $A_1$ singularity sometimes called the "conifold".
- $z = \infty$, i.e. $\psi = 0$: this is the Fermat quintic. This has an additional $\mathbb{Z}/5$ symmetry because we drop the condition that the product of the $x_i$ has to be 1. So this is really an orbifold point.

Now, at least from a physics point of view we are done.[1]

## 3. Yukawa couplings

**3.1. A-model.** The $A$-model (symplectic) will deal with the quintic. So we have $H^2(X, \mathbb{Z}) = \mathbb{Z} \cdot h$ for $h = PD$ (hyperplane). Then the Yukawa coupling is
$$\langle h, h, h \rangle_A = \sum_{d \in \mathbb{N}} N_d d^3 q^d \ .$$
At this point (historically) it was not clear what the $N_d$ actually were because Gromov-Witten theory was sort of being developed in parallel. Nowadays we know that the $N_d$ are Gromov-Witten (GW) counts of rational curves ($g = 0$) of degree $d$. If we write $n_d$ for the primitive counts, then
$$5 + \sum_{d>0} d^3 n_d \qquad \underbrace{\frac{q^d}{1-q^d}}_{\text{multiple cover of } \deg d \text{ curves}} \qquad \in \mathbb{Q}[\![q]\!] \ .$$

**3.2. B-model.** Now we consider the mirror quintic $Y_z$, $z = (5\psi)^{-5}$. Now the Yukawa coupling is given by:
$$\langle \partial_z, \partial_z, \partial_z \rangle_B := \int_{Y_z} \Omega^\nu(z) \wedge \partial_z^3 \Omega^\nu(z)$$
where $\Omega^\nu$ is a "normalized" holomorphic volume form:
$$\int_{\beta_0} \Omega^\nu = \text{constant}$$
where $\beta_0 \in H_2(Y, \mathbb{Z})$.

We need some kind of mirror to the vector field $h$ on the moduli space of symplectic structures. What we really want is actually the exponentiated thing $e^{2\pi i h}$. So as it turns out on this side of mirror symmetry this looks like $\partial/\partial w$ which corresponds to a vector field on the complex moduli space of $Y$.

It turns out
$$w = \int_{\beta_1} \Omega^\nu(z)$$

[1]Professor Siebert says that maybe mathematicians would be better off like this: not worrying so much and just seeing what life brings.

where $\beta_1 \in H_3(Y, \mathbb{Z})$.

3.3. **Mirror symmetry.** Now the actual statement of mirror symmetry is that

$$\langle h, h, h \rangle_A = \langle \partial_z, \partial_z, \partial_z \rangle_B$$

where $q = e^{2\pi i w(z)}$ ($w = c \cdot z + \mathcal{O}(z^2)$).

So now we have to:

(1) write down the holomorphic 3 form (not too bad)
(2) do the normalization period integral (not too bad)
(3) computing this second integral (more bad)
(4) computing the Yukawa coupling.

3.4. **Computation of the periods.** There is an account of this in lecture notes by Mark Gross (Nordfjordeid). Recall we have:

$$\begin{array}{ccc} & & Y_\psi \\ & & \downarrow \\ X_\psi & \xrightarrow{/G} & \bar{Y}_\psi \end{array}.$$

We know $H_3(Y_\psi, \mathbb{Z}) \simeq \mathbb{Z}^4$. Near $\psi = \infty$ (large complex structure limit) we have a vanishing cycle.

This looks like Professor Siebert's favorite picture of a degeneration. Consider $zw = t$. At $t = 0$ this looks like two disks meeting at a point. For $t \neq 0$ this looks like a cylinder. But now if we do a Dehn twist, we see that there is an $S^1$ which gets collapsed in this degeneration. A similar story holds in higher dimension.

In particular, our vanishing cycle looks like $\beta_0 = T^3$. Locally

$$u_1 \dots u_4 = z, \beta_0 = \left\{ |u_1| = \dots = |u_4| = |z|^{1/4}, \operatorname{Arg} u_1 \dots u_4 = 0 \right\}$$

where the $u_i$ are holomorphic coordinates.

If we lift to $X_\psi$ we get an explicit three-torus

$$T = \left\{ |x_0| = |x_1| = |x_2| = \delta \ll 1, x_3 = x_3(x_0, x_1, x_2) \text{ soln of } f_\psi(x_1, x_2, x_3, 1) = 0, \exists! \xrightarrow{z \to 0} 0 \right\}$$