LECTURE 1
MIRROR SYMMETRY

PROFESSOR BERND SIEBERT
NOTES BY: JACKSON VAN DYKE

1. General overview of mirror symmetry

The course is topics in algebraic geometry. We will be doing some sort of mirror symmetry. We will start with some historical overview.

1.1. **Enumerative mirror symmetry.** Let X be a CY manifold. In particular let this be a CY 3-fold. This means $K_X = \det T_X^* \simeq \mathcal{O}_X$ is trivial as a holomorphic line bundle. Typically this means we want $b_1 = 0$ and irreducible.

Example 1 (Quartic in $\mathbb{P}^4$)**.** Take $f \in \mathbb{C}[x_0, \ldots, x_4]$ homogeneous of degree 5. If it is sufficiently general, the zero locus is smooth inside $\mathbb{P}^4$ and is an example of a CY three-fold.

We have

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{\mathbb{P}^2}|_X \to \mathcal{O}_X \to 0$$

where $\mathcal{I} = (f) \subset \mathcal{O}_{\mathbb{P}^4}$. We also have that $\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_X$ is an invertible sheaf so this first map sends $f \mapsto df$. This implies

$$K_{\mathbb{P}^4}|_X = \det \mathcal{O}_{\mathbb{P}^4}|_X = \mathcal{I}/\mathcal{I}^2 \otimes K_X \ ,$$

and

$$K_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-5) \ .$$

Then $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^4}$ which has a section with poles of order 5. The point is we can make f into a five by dividing by x_0^5, so we have that, as an abstract sheaf,

$$\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^4}(-5) \ .$$

Then we have that

$$\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X(-5)$$

and we can just take the tensor product to get

$$K_{\mathbb{P}^4}|_X \simeq \mathcal{O}_X(-5)$$

so $K_X \simeq \mathcal{O}_X$ must be trivial.

Now we want to produce a string theory out of this. This is a very delicate process. There are things called $IIA(X)$ and $IIB(X)$ theories. These are the ones relevant in mirror symmetry. These come from the super-symmetric σ-models with target some 10-dimensional space[1] $\mathbb{R}^{1,3} \times X$. These are the so-called super conformal field theories $SCFT_A(X)$ and $SCFT_B(X)$. These are different theories which produce observables, e.g. the Hodge number of X can be computed from these theories. In particular we can compute $h_{1,1}(X)$ and $h_{2,1}(X)$ which correspond to

Date: August 29, 2019.

[1]To have an anomaly free theory.

some physical variables. On the B-side we make the same computation but get $h_{2,1}(X)$ and $h_{1,1}(X)$. Then we postulate that there is some other X' where these are not flipped. In particular the observation is, for very specific X, we can find a CY Y with

$$SCFT_A(X) = SCFT_B(Y) \qquad SCFT_B(X) = SCFT_A(Y) \ .$$

Somehow then the idea is that if this is really a model for string theory, we should really be swapping

$$IIA(X) \text{ “}=\text{''}\ IIB(X) \qquad IIB(X) \text{ “}=\text{''}\ IIA(Y) \ .$$

But this might be a bit much to ask.

1.2. **Topological twists.** There is something called a topologically twisted σ-model introduced by Witten in 1988. This produces a completely different theory. We get two theories, one called $A(X)$, and one $B(X)$.

Warning 1. As it turns out, $A(X)$ ends up computing things in certain limits of the $IIB(X)$ theory.

Remark 1. A priori these are unrelated to $SCFT_A(X)$ and $SCFT_B(X)$.

As it turns out, if we have $SCFT_A(X) = SCFT_B(Y)$, then we have

$$A(X) = B(Y) \qquad B(X) = A(Y) \ .$$

$A(X)$ and $B(X)$ compute certain limits, called Yukawa-couplings, for $SCFT_B(A)$ and $SCFT_A(X)$.

Note that by this twisting procedure $A(X)$ sees (X, ω) (where ω is the Kähler form) only as a symplectic manifold, and $B(X)$ depends only on the complex manifold (X, I).

1.3. **Useful calculations.** The reason people really got excited about mirror symmetry is that it helps us make calculations we couldn't make before.

In 1991 Candelas, dela Ossa, Greene, and Parkes compiuted the Yukawa couplings for the quintic and the mirror quintic. In particular they computed F_B of the mirror quintic Y_t, and claimed this is in fact equal to F_A of the quintic. Geometrically F_A has to do with counts of (genus 0) holomorphic curves. F_B has to do with period integrals

$$\int_\alpha \Omega_{Y_t} = F_B(t)$$

for $\alpha \in H_3(Y_t)$. So they predicted some large number of counts, then someone computed it directly and they agreed.

Warning 2. This Ω_{Y_t} is only defined up to scale so really the case is that

$$F_B(t) = \exp\left(\int_{\alpha_1} \Omega_{Y_t} / \int_{\alpha_0} \Omega_{Y_t}\right)$$

for $\alpha_i \in H_3(Y_t)$.

Then Morrison/Deligne in 1992 described $F_B(Y)$ in terms of Hodge theory/more parameters for CY moduli. This is when Gromov-Witten theory entered the scene in 1993 to make $F_A(X)$ precise. So at least we had a mathematical statement.

1.4. **Homological mirror symmetry.** In 1994 Kontsevich gave his legendary ICM talk. This is where homological mirror symmetry took off. He said that as mathematicians we don't really know SCFTs. But what should be true is really:

$$D\,\mathrm{Fuk}\,(X) = D^b\,(\mathcal{O}_Y)\ .$$

This is a formulation, not an explanation.

Professor Siebert would like to convince us of a procedure to construct mirror pairs.

1.5. **Proving numerical mirror symmetry.** In 1996 Givental gave a proof that in the case of hypersurfaces F_A really is F_B of the mirror. This was somehow a computation showing that the sides do in fact agree. This is not very satisfying to Professor Siebert. In 1997 Lian, Liu, and Yan proved it more generally.

1.6. **Proving HMS.** In 2003 Paul Seidel proved HMS for the quartic in $\mathbb{P}^3$. Essentially he shows that both sides have enough rigidity to do a very minimal computation. This is also not very satisfying to Professor Siebert. It was then proved in 2011 by N. Sheridan for all CY hypersurfaces.

1.7. **Modern state.** There are many other manifestations of mirror symmetry. As it turns out even geometric Langlands can be viewed as some form of mirror symmetry.

As for HMS, some symplectic people are trying to prove this for so-called SYZ fibered symplectic manifolds and a rigid space as the mirror.

Then there are intrinsic constructions, things wwhich Professor Siebert has worked on (with Mark Gross) with many applications. The idea is to use mirror symmetry as a tool in mathematics rather than just a phenomenon in physics. The point is one has to find a way of producing mirrors.

This entire story is genus 0, what physicists would call tree-level. There is also a higher genus case. From the representation theory side this has something to do with quantum groups. This is called second quantized mirror symmetry.[2] There is an entire field called topological recursion related to this.

1.8. **Plan for the class.**

(1) part of the COGP computation (periods)
(2) Gromov-Witten theory, virtual fundamental class/moduli stacks
(3) toric degenerations and mirror constructions[3]
(4) As it turns out, we can compute homogeneous coordinate rings of both sides. Polishchuk has shown that this ring determines $D^b\,(\mathcal{O}_Y)$. It would be nice to make the analogous symplectic calculation, because this would be a very sneaky proof of HMS.
(5) Higher genus. Donaldson Thomas invariants play some sort of unclear role in MS because they will have something to do with the higher genus story. One can make these computations using "crystal melting". This is some kind of statistical mechanics.

[2]This term comes from QFT.

[3]This will include some introduction to toric geometry.

2. The quintic threefold, its mirror, and COGP

Take some quintic CY in $\mathbb{P}^4$, i.e. $V(f) \subset \mathbb{P}^4$ for some homogeneous degree 5 f. First let's do a dimension count for homogeneous polynomials in $x_0, \ldots, x_4$ of degree 5. This is just drawing with replacement, so we have

$$\begin{aligned} x_0^2 x_2 &\quad \leftrightarrow \quad && || \cdot \cdot | \\ x_0 x_1^2 x_2 &\quad \leftrightarrow \quad && | \cdot || \cdot | \end{aligned}$$

and we get

$$\binom{9}{5} = 126$$

which means

$$\dim_{\mathbb{C}} \left\{ Z(f) \subset \mathbb{P}^4 \right\} = 126 - 1 = 125 .$$

Now we mod out by $\mathrm{PGL}(5)$, which is of dimension $5 \cdot 5 - 1 = 24$. So we get

$$\dim \underbrace{\mathcal{M}_5}_{\text{moduli space of quintics}} = 101 .$$

Indeed: for a projective CY manifold X, the moduli space of CY manifolds deformation equivalent to X is a smooth orbifold of complex dimension $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle. We will compute this number as an exercise next time. For $V(5) \subset \mathbb{P}^4$ this is 101.