LECTURE 2
MIRROR SYMMETRY

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1. Quintic 3-folds

We will be looking at quintic 3-folds $V(5) \subset \mathbb{P}^4$ given by $Z(f)$ for some homogeneous degree 5 polynomial.

Theorem 1. *For X a projective CY manifold then the moduli space of CY deformation-equivalent to X is a smooth space of dimension $h^1(\Theta_X)$.*

By an elementary argument we saw that $\dim \mathcal{M}_{V(5)} = 101$.

1.1. **Computation of $H^1(\Theta_X)$.** We will compute in the case $X = V(5) \subset \mathbb{P}^4$. We start with the Euler sequence. We have $\mathbb{P}^4 = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_4]$ and

$$x_i \partial_{x_i} = x_i \frac{\partial}{\partial x_i}$$

are well-define logarithmic vector fields. Then the sequence is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow 0$$

$$1 \longmapsto \sum e_i$$

$$e_i \longmapsto x_i \partial_{x_i}$$

Then we have the conormal sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{\mathbb{P}^4}\big|_X \xrightarrow{\operatorname{res}_x} \Omega^1_X \longrightarrow 0$$

where $\mathcal{I}$ is the ideal sheaf of X. This is dual to

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4}\big|_X \longrightarrow N_{X/\mathbb{P}^4} \simeq \mathcal{O}_{\mathbb{P}^4}(5) \longrightarrow 0 \tag{1}$$

$\mathcal{I}/\mathcal{I}^2$ can be computed because $\mathcal{I} \simeq \mathcal{O}(-5)$. The restriction sequences are:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow \Theta_{\mathbb{P}^4}(-5) \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow \Theta_{\mathbb{P}^4}\big|_X \longrightarrow 0 \tag{3}$$

Date: September 3, 2019.

Now (1) gives us

$$\begin{tikzcd} H^0(\Theta_X) \arrow[r] \arrow[d, equal] & H^0(\Theta_{\mathbb{P}^4}|_X) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \arrow[r] \arrow[d, equal] & H^1(\Theta_X) \arrow[r] & H^1(\Theta_{\mathbb{P}^4}|_X) \arrow[d, equal] \\ 0 & \mathbb{C}^{24} \arrow[u, "\simeq"] & \mathbb{C}^{125} & \mathbb{C}^{101} \arrow[u, "\simeq"] & 0 \end{tikzcd} .$$

For $H^1(\Theta_{\mathbb{P}^4}|_X)$, (3) give us:

$$H^1(\Theta_{\mathbb{P}^4}) \longrightarrow H^1(\Theta_{\mathbb{P}^4}|_X) \longrightarrow H^2(\Theta_{\mathbb{P}^4}(-5)) \quad .$$

The Euler sequence gives us:

$$\begin{tikzcd} H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5} \arrow[r] \arrow[d, equal] & H^1(\Theta_{\mathbb{P}^4}) \arrow[r] & H^2(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\ 0 & & 0 \end{tikzcd} ,$$

then we can tensor the Euler sequence with $\mathcal{O}(-5)$ to get

$$\begin{tikzcd} H^i(\mathcal{O}_{\mathbb{P}^4}(-4)) \arrow[r] \arrow[d, equal] & H^i(\Theta_{\mathbb{P}^4}(-5)) \arrow[r] \arrow[d, equal] & H^i(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\ 0 \arrow[r, Rightarrow] & 0 & 0 \arrow[l, Rightarrow] \end{tikzcd} .$$

For $H^0(\mathcal{O}_{\mathbb{P}^4}(5)|_X)$ we can tensor (2) with $\mathcal{O}(5)$ to get

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & H^0(\mathcal{O}_{\mathbb{P}^4}) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \arrow[r] \arrow[d, equal] & H^0(\mathcal{O}_X(5)) \arrow[r] \arrow[d, equal] & H^1(\mathcal{O}_{\mathbb{P}^4}) \\ \mathbb{C} & \mathbb{C}^{126} \arrow[u, "\simeq"] \arrow[r, Rightarrow] & \mathbb{C}^{125} & 0 & \end{tikzcd} .$$

Now for $H^0(\Theta_{\mathbb{P}^4}|_X)$ we have that (3) gives us

$$\begin{tikzcd} H^0(\Theta_{\mathbb{P}^4}(-5)) \arrow[r] \arrow[d, equal] & H^0(\Theta_{\mathbb{P}^4}) \arrow[r] \arrow[d, equal] & H^0(\Theta_{\mathbb{P}^4}|_X) \arrow[r] & H^1(\Theta_{\mathbb{P}^4}(-5)) \arrow[d, equal] \\ 0 & \mathbb{C}^{24} \arrow[r, Rightarrow] & \mathbb{C}^{24} \arrow[u, "\simeq"] & 0 \end{tikzcd} .$$

Then the result is that:

$$h^1(\Theta_X) = 101 .$$

2. Hodge diamond

Let X be a compact Kähler manifold. Recall we have the Dolbeault cohomology $H^{i,j}_{\bar\partial}$ is the cohomology of the sequence

$$\mathcal{A}^{i,j} \xrightarrow{\bar\partial} \mathcal{A}^{i,j+1}$$

where this looks like

$$\sum h_{\mu\nu}\, dz_{\mu_1} \wedge \ldots \wedge dz_{\mu_i} \wedge d\bar z_{\nu_1} \wedge \ldots \wedge d\bar z_{\nu_j}$$

where the $h_{\mu\nu} \in \mathcal{C}^\infty$.

Then we have the following facts:

(1) We have a canonical isomorphism from the Dolbeault cohomology
$$H_{\bar{\partial}}^{i,j} = H^j\left(X, \Omega_X^i\right) = \overline{H_{\bar{\partial}}^{j,i}}$$
which implies
$$\dim H_{\bar{\partial}}^{i,j} = h_{ij} = h_{ji} \ .$$

(2)
$$H_{\bar{\partial}}^{n-i,n-j} = H^{n-j}\left(X, \Omega_X^{n-i}\right) = {}^{1}H^j\left(X, K_X \otimes \left(\mathcal{O}_X^{n-i}\right)^*\right)^* H_{\bar{\partial}}^{i,j}$$
which, in particular, means $h_{ij} = h_{n-i,n-j}$.

(3)
$$H^k(X,\mathbb{C}) = H_{dR}^k(X) = \bigoplus_{i+j=k} H_{\bar{\partial}}^{i,j} \ .$$

Recall $b_i = \dim_{\mathbb{C}} H^k(X,\mathbb{C})$. For CY n-folds, $b_1 = 0$ by the definition of CY. This implies $h_{10} = h_{01} = 0$. Moreover
$$H^{n,0} = H^0\left(X, \underbrace{K_X}_{\mathcal{O}_X}\right) \simeq \mathbb{C} \ .$$
This implies that
$$h_{n,0} = h_{0,n} = 1 \ .$$

We say a CY is irreducible if the universal cover $\tilde{X} \to X$ is not a nontrivial product of CY. This is equivalent to $H^{k,0} = 0$ for $k = 1, \ldots, n-1$.

Counterexample 1. The product $K3 \times K3$ is not irreducible.

3. Hodge diamond

The hodge diamond is the following:

$$\begin{array}{ccccccccc}
 & & & h_{33} & & & & & \leftarrow H^6 \\
 & & h_{23} & & h_{32} & & & & \leftarrow H^5 \\
 & h_{13} & & h_{22} & & h_{31} & & & \vdots \\
h_{01} & & h_{12} & & h_{21} & & h_{30} & & \\
 & h_{02} & & h_{11} & & h_{20} & & & \\
 & & h_{01} & & h_{10} & & & & \\
 & & & h_{00} & & & & &
\end{array}$$

The only interesting part is the center:

[1]By Serre duality

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h_{22} & & 0 & \\
1 & & h_{12} & & h_{21} & & 1 \\
 & 0 & & h_{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

since

$$H^1\left(\Theta_X\right) = H^1\left(\Theta_X \otimes \underbrace{K_X}_{\mathcal{O}_X}\right) = H^1\left(\mathcal{O}_X^{n-1}\right) = H^{n-1,1} \ .$$

So the question is reduced to making these calculations. In the $V(5)$ case $h_{21} = 101$ as we saw.

3.1. **Lefschetz theorem on $(1,1)$-classes.** Now we work at the generality of compact Kähler manifolds. The Néron-Severi group $NS(X)$, is the preimage of $H^{1,1}$ under $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$. So somehow morally $NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}$. Now we have the exponential sequence of abelian sheaves:

$$1 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1$$

$$
\begin{array}{ccccccccc}
H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1\left(X,\mathcal{O}_X^\times\right) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_C) \\
 & & \| & & \| & & & & \\
 & & H^{0,1} & & \operatorname{Pic}(X) & & & &
\end{array} .
$$

So $c_1(\operatorname{Pic}(X)) = NS(X)$.

Now we know how to compute $\operatorname{Pic}(X)$ for CY n-folds. In particular, as long as $n \geq 3$, $h_{11} = \operatorname{rank}_{\mathbb{Z}} NS(X)$ and $\operatorname{Pic}(X) \simeq NS(X)$ since $\operatorname{Pic}^0(X) = 0$ in the CY case.

3.2. **Hard Lefschetz theorem.** Let $X \subset \mathbb{P}^n$ be a Kähler manifold. This tells us that

$$H^k(\mathbb{P}^n,\mathbb{Z}) \to H^k(X,\mathbb{Z})$$

is an isomorphism for $k < n-1 = \dim X$ and surjective for $k = n-1$.

Now for a CY 3-fold we have $0 = H^1(\mathbb{P}^n,\mathbb{Z}) = H^1(X,\mathbb{Z})$ and $H^2(\mathbb{P}^n,\mathbb{Z}) \simeq \mathbb{Z} \twoheadrightarrow H^2(X,\mathbb{Z})$ which means $NS(X) \simeq \mathbb{Z}$. This is generated by the hyperplane class $c_1(\mathcal{O}(1))$ and then restricting to X gives the ample line bundle on X and this generates the group.

So today we learned that the interesting part of the diamond in the quintic case looks like

$$\begin{array}{ccc} & 1 & \\ 101 & & 101 \\ & 1 & \end{array}$$

and then as it turns out, the mirror quintic will look like

$$\begin{array}{ccc} & 101 & \\ 1 & & 1 \\ & 101 & \end{array}$$

so we have a huge Picard group.