Moduli spaces and tropical geometry

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Spring 2020; Notes by: Jackson Van Dyke; All errors introduced are my own.

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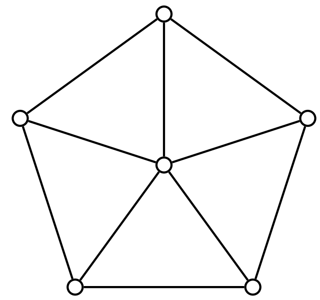

FIGURE 1. The 5-wheel.

1. Overview

Our goal is to understand the proof of the following theorem:

Lecture 1; Wednesday January 22, 2020

THEOREM 0.1. $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ *grows exponentially with* g.

REMARK 0.1. $\mathcal{M}_g$ has complex dimension $3g-3$.

This theorem defied previous expectations.

CONJECTURE 1 (Kontsevich (1993), Church-Farb-Putman (2014)). *For fixed* $k > 0$, $H^{4g-4-k}(\mathcal{M}_j, \mathbb{Q}) = 0$ *for* $g \gg 0$.

The structure of the course is as follows.

- Constructing the moduli space
  1. Nodal curves and stable reduction theorem
  2. Deformation theory of nodal curves
  3. The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  1. Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  2. Dual complexes of normal crossings divisors (tropical geometry)
  3. Boundary complex of $\mathcal{M}_g$ (tropical moduli space)
- Cohomology of $\mathcal{M}_h$
  1. Stable cohomology (Madsen-Weiss 2007)
  2. Virtual cohomological dimension of $\mathcal{M}_g$ (Harer 84) (Vanishing of H^{4g-5} (Church-Farb-Putman, Morita-Sakasai-Suzuki))
  3. Euler characteristic of $\mathcal{M}_g$ (Harer-Zagier 86)
- Graph complexes (Kontsevich 93)
  1. Feynman amplitudes and wheel classes. See Fig. 1 for the 5-wheel.
  2. Grothendieck-Teichmüller Lie algebra
  3. Willwacher's theorem
- Mixed Tate motives (MTM) over $\mathbb{Z}$
  1. Mixed Tate motives
  2. Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements/Drinfeld associators generate a free Lie subalgebra."
  3. Proof of exponential growth of H^{4g-6}.

Lecture 2; January 24, 2020

Part 1

Constructing the moduli space

CHAPTER 1

Nodal curves and stable reduction theorem

We will work over $\mathbb{C}$. We want to show that nodal curves, and families thereof, can be written in a normal form in local coordinates. We will follow chapter X of [**1**].

DEFINITION 1.1. A *nodal curve* is a complete curve such that evert singular point has a neighborhood isomorphic (analytically over $\mathbb{C}$) to a neighborhood of 0 in $(xy = 0) \subset \mathbb{C}^2$.

DEFINITION 1.2. A *family of nodal curves* over a base S is a flat proper surjective morphism $\mathcal{C} \to S$ such that every geometric fiber is a nodal curve.

Recall that a flat morphism is the agreed upon notion of a map which gives a continuously varying family of schemes (or complex analytic spaces, varieties, etc.) given by the fibers. Properness is saying that nothing can "disappear" as we approach any particular point in the base.

EXAMPLE 1.1. Let $S = \mathbb{A}^1$. Now consider $\mathbb{A}^2$ minus the x-axis and the positive y-axis living over S. Let φ be the morphism projecting to the x-coordinate. This is not proper in a neighborhood of the origin in the base.

write this better/draw picture

Proposition 1.1. *Let $\pi : X \to S$ be a proper surjective morphism of $\mathbb{C}$-analytic spaces. This is a family of nodal curves iff at every point $p \in X$ either π is smooth at p with one-dimensional fiber, or there is a neighborhood of p which is isomorphic (over S) to a neighborhood of $(0, s)$ in $(xy = F) \subseteq \mathbb{C}^2 \times S$ where $s = \pi(p)$ and $F \in \mathfrak{m}_S \subseteq \mathcal{O}_{S,s}$.*

Lemma 1.2. *Let f be holomorphic at $0 \in \mathbb{C}^2$. Then $(f = 0)$ has a node at 0 iff*

$$0 = f = \frac{\partial f}{\partial x} = \frac{pf}{\partial y} \tag{1.1}$$

at 0, and the Hessian of f at 0 is non-singular.

This tells us that these nodes are the "simplest" possible singularities.

PROOF. ($\Longrightarrow$): This direction is immediate.
($\Longleftarrow$): Suppose $0 = f = \partial_x f = \partial_y f$ at 0. Then

$$f = a - x^2 + 2bxy + cy^2 \tag{1.2}$$

where a, b, and c are holomorphic functions. The Hessian is

$$\begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix} \tag{1.3}$$

so being non-singular means exactly that

$$b^2 - ac \neq 0\ . \tag{1.4}$$

After possible making a linear change of coordinates, we can assume $a \neq 0$, and change to coordinates

$$x_1 = x + \frac{b}{a} y \qquad\qquad y_1 = y \tag{1.5}$$

so we can write

$$f = a_1 x_1^2 + c_1 y_1^2 \tag{1.6}$$

where $a_1(0), c_1(0) \neq 0$. Choose square roots[1.1] α and γ of a_1 and c_1. Now replace x_1 and y_1 by $x_2 = \alpha x_1$ and $y_2 = \gamma y_1$ so that

$$f = x_2^2 + y_2^2\ . \tag{1.7}$$

Now for $x_3 = x_2 + i y_2$ and $y_3 = x_2 - i y_2$, we have $f = x_3 y_3$. $\square$

Proof of Proposition 1.1. Let $\pi : X \to S$ be proper surjective. Consider $x \in X$. Then either π is smooth with 1-dimensional fiber at x (nothing to show) or x is a node in $\pi^{-1}(s)$, $s = \pi(x)$. Locally near x we can embed $X \subseteq \mathbb{C}^r \times S$. This is locally closed (over S). Now we have an exact sequence[1.2] of tangent spaces:

$$0 \longrightarrow T_x X_s \longrightarrow T_x X \longrightarrow T_s S \tag{1.8}$$

where $\dim T_x X_s = 2$. Now choose a linear projection $\mathbb{C}^r \to \mathbb{C}^2$ which is an isomorphism on $T_x X_s$. Using this projection we get a composition:

$$\begin{tikzcd} T_x X \arrow[r, phantom, "\subseteq"] \arrow[rr, bend right] & \mathbb{C}^r \times T_s S \arrow[r] & \mathbb{C}^2 \times T_s S \end{tikzcd} \tag{1.9}$$

and we claim this is injective. Then the implicit function theorem tells us that there is a neighborhood of x which embeds in $\mathbb{C}^2 \times S$ (over S). We should think of this as a family of plane curves: each fiber has a single defining equation. More specifically we have the following.

Fact 1 (Lemma 31.18.9 (Stacks project)). *If $\mathcal{Y} \to S$ is a smooth morphism and $D \subseteq \mathcal{Y}$ is flat over S, codimension 1 in $\mathcal{Y}$, then D is a Cartier divisor.*

By Fact 1, $X \subseteq \mathbb{C}^2 \times S$ is locally defined by a single equation $F = 0$. Now consider $\partial_x F$, $\partial_y F$, and the Hessian of F with respect to x and y. Then the proof of Lemma 1.2 shows that

$$F = x_3 y_3 - f \tag{1.10}$$

where f is a function on S which vanishes at s. $\square$

[1.1] There is some subtly here since these are functions rather than scalars. Because a_1 and c_1 are nonzero at 0, we can ensure that the image of a_1 and c_1 are, say, contained in an open half space. Now we can choose a branch of log which is defined on this half space. Then multiple by 1/2 and exponentiate.

[1.2] If x is not a singular point, then this would be an honest SES (i.e. the map $T_x X \to T_s S$ would be surjective) but in this case this is not true. In any case, all we need is the exactness of these four terms.

Bibliography

[1] E. Arbarello, J.D. Harris, M. Cornalba, and P. Griffiths, *Geometry of algebraic curves: Volume ii with a contribution by joseph daniel harris*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2011.