Moduli spaces and tropical geometry

Lectures by Sam Payne

Spring 2020; Notes by: Jackson Van Dyke; All errors introduced are my own.

Contents

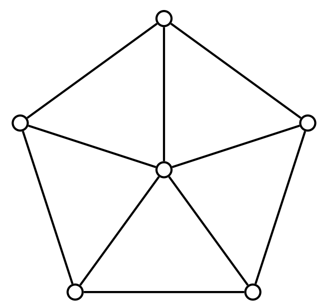


FIGURE 1. The 5-wheel.

1. Overview

Our goal is to understand the proof of the following theorem:

Lecture 1; Wednesday January 22, 2020

THEOREM 0.1. $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ *grows exponentially with* g.

REMARK 0.1. $\mathcal{M}_g$ has complex dimension $3g-3$.

This theorem defied previous expectations.

CONJECTURE 1 (Kontsevich (1993), Church-Farb-Putman (2014)). *For fixed* $k > 0$, $H^{4g-4-k}(\mathcal{M}_j, \mathbb{Q}) = 0$ *for* $g \gg 0$.

The structure of the course is as follows.

- Constructing the moduli space
  (1) Nodal curves and stable reduction theorem
  (2) Deformation theory of nodal curves
  (3) The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  (1) Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  (2) Dual complexes of normal crossings divisors (tropical geometry)
  (3) Boundary complex of $\mathcal{M}_g$ (tropical moduli space)
- Cohomology of $\mathcal{M}_h$
  (1) Stable cohomology (Madsen-Weiss 2007)
  (2) Virtual cohomological dimension of $\mathcal{M}_g$ (Harer 84) (Vanishing of H^{4g-5} (Church-Farb-Putman, Morita-Sakasai-Suzuki))
  (3) Euler characteristic of $\mathcal{M}_g$ (Harer-Zagier 86)
- Graph complexes (Kontsevich 93)
  (1) Feynman amplitudes and wheel classes. See fig. 1 for the 5-wheel.
  (2) Grothendieck-Teichmüller Lie algebra
  (3) Willwacher's theorem
- Mixed Tate motives (MTM) over $\mathbb{Z}$
  (1) Mixed Tate motives
  (2) Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements (closely related to Drinfeld's associators) generate a free Lie subalgebra."
  (3) Proof of exponential growth of H^{4g-6}.

Lecture 2; January 24, 2020

Part 1

Constructing the moduli space

CHAPTER 1

Nodal curves and stable reduction theorem

1. Nodal curves

We will work over $\mathbb{C}$. We want to show that nodal curves, and families thereof, can be written in a normal form in local coordinates. We will follow chapter X of [**1**].

DEFINITION 1.1. A *nodal curve* is a complete curve such that every singular point has a neighborhood isomorphic (analytically over $\mathbb{C}$) to a neighborhood of 0 in $(xy = 0) \subset \mathbb{C}^2$.

DEFINITION 1.2. A *family of nodal curves* over a base S is a flat proper surjective morphism $f\colon \mathcal{C} \to S$ such that every geometric fiber is a nodal curve.

Recall that a flat morphism is the agreed upon notion of a map for which the fibers form a continuously varying family of schemes (or complex analytic spaces, varieties, etc.). Properness is a relative notion of compactness; it ensures that if $\{c_i\}$ is a sequence of points with no limit in $\mathcal{C}$ then $\{f(c_i)\}$ has no limit in S.

Proposition 1.1. *Let $\pi : X \to S$ be a proper surjective morphism of $\mathbb{C}$-analytic spaces. This is a family of nodal curves if and only if at every point $p \in X$ either π is smooth at p with one-dimensional fiber, or there is a neighborhood of p that is isomorphic (over S) to a neighborhood of $(0, s)$ in $(xy = F) \subseteq \mathbb{C}^2 \times S$ where $s = \pi(p)$ and $F \in \mathfrak{m}_S \subseteq \mathcal{O}_{S,s}$.*

Lemma 1.2. *Let f be holomorphic at $0 \in \mathbb{C}^2$. Then $(f = 0)$ has a node at 0 if and only if*

$$0 = f = \frac{\partial f}{\partial x} = \frac{pf}{\partial y} \tag{1.1}$$

at 0, and the Hessian of f at 0 is non-singular.

This tells us that these nodes are the "simplest" possible singularities.

PROOF. ($\implies$): This direction is immediate.
($\Longleftarrow$): Suppose $0 = f = \partial_x f = \partial_y f$ at 0. Then

$$f = a - x^2 + 2bxy + cy^2 \tag{1.2}$$

where a, b, and c are holomorphic functions. The Hessian is

$$\begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix} \tag{1.3}$$

so being non-singular means exactly that

$$b^2 - ac \neq 0\ . \tag{1.4}$$

After a generic linear change of coordinates, we can assume $a \neq 0$. We can then change coordinates to

$$x_1 = x + \frac{b}{a}y \qquad\qquad y_1 = y\ . \tag{1.5}$$

Then we can write

$$f = a_1 x_1^2 + c_1 y_1^2 \tag{1.6}$$

where $a_1(0), c_1(0) \neq 0$. Choose square roots[1.1] α and γ of a_1 and c_1. Now replace x_1 and y_1 by $x_2 = \alpha x_1$ and $y_2 = \gamma y_1$ so that

$$f = x_2^2 + y_2^2\ . \tag{1.7}$$

Now for $x_3 = x_2 + iy_2$ and $y_3 = x_2 - iy_2$, we have $f = x_3 y_3$. □

Proof of Proposition 1.1. Let $\pi : X \to S$ be proper and surjective. Consider $x \in X$. Then either π is smooth with 1-dimensional fiber at x (nothing to show) or x is a node in $\pi^{-1}(s)$, $s = \pi(x)$. Locally near x, we have a locally closed embedding $X \subseteq \mathbb{C}^r \times S$ (working over S). Then we get a left exact sequence of tangent spaces:

$$0 \longrightarrow T_x X_s \longrightarrow T_x X \longrightarrow T_s S \tag{1.8}$$

where $\dim T_x X_s = 2$. Choose a linear projection $\mathbb{C}^r \to \mathbb{C}^2$ which is an isomorphism on $T_x X_s$. Using this projection we get:

$$\begin{tikzcd} T_x X \arrow[r, phantom, "\subseteq"] \arrow[rr, bend right] & \mathbb{C}^r \times T_s S \arrow[r] & \mathbb{C}^2 \times T_s S \end{tikzcd} \tag{1.9}$$

and the composition $T_x X \to \mathbb{C}^2 \times T_s S$ is injective. The implicit function theorem then tells us that there is a neighborhood of x which embeds in $\mathbb{C}^2 \times S$ (over S). We should think of this as a family of plane curves: each fiber has a single defining equation. More specifically we have the following.

Fact 1 (Lemma 31.18.9 (Stacks project)). *If $\mathcal{Y} \to S$ is a smooth morphism and $D \subseteq \mathcal{Y}$ is flat over S, codimension 1 in $\mathcal{Y}$, then D is a Cartier divisor.*

In particular, $X \subseteq \mathbb{C}^2 \times S$ is locally defined by a single equation $F = 0$. Now consider $\partial_x F$, $\partial_y F$, and the Hessian of F with respect to x and y. Then the proof of Lemma 1.2 shows

$$F = x_3 y_3 - f \tag{1.10}$$

where f is a function on S which vanishes at s. □

Lecture 3; January 27, 2020

2. Stability of nodal curves

The following is a corollary of Proposition 1.1.

Corollary 1.3. *A family of nodal curves $\pi : \mathcal{C} \to S$ is a local complete intersection (lci) morphism.*

This implies that there is a relative dualizing sheaf $\omega_{\mathcal{C}/S}$ which is locally free of rank 1.

[1.1] There is some subtly here since these are functions rather than scalars. Because a_1 and c_1 are nonzero at 0, we can ensure that the image of a_1 and c_1 are, say, contained in an open half space. Now we can choose a branch of log which is defined on this half space. Then multiply by 1/2 and exponentiate.

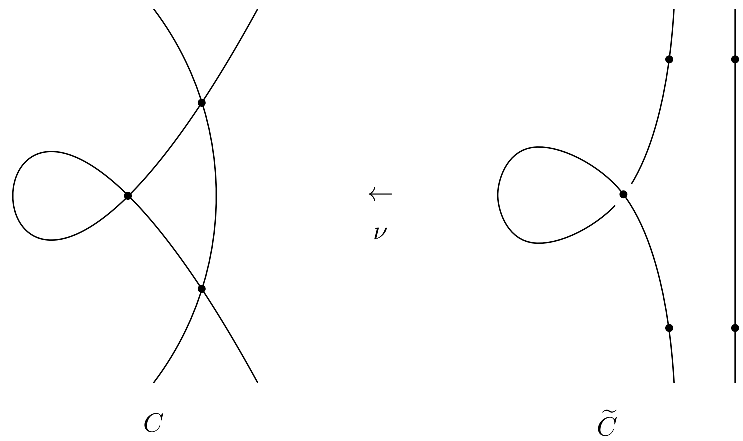


FIGURE 1. The normalization of a nodal curve. The nodal points of C each have two preimages under the normalization ν.

2.1. Serre duality. The point here is that the duality properties that we already know about for smooth curves extend naturally to nodal ones.

Let C be a nodal curve (over a point). There is a (natural) isomorphism $H^1(C, \omega_C) \cong \mathbb{C}$. Then Serre duality tells us that for any coherent sheaf $\mathcal{F}$ on C,

$$H^1(C, \mathcal{F}) \times \operatorname{Hom}(\mathcal{F}, \omega_C) \to H^1(C, \omega_C) \cong \mathbb{C} \tag{1.11}$$

is a perfect pairing, i.e.,

$$H^1(C, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{F}, \omega_C)^\vee \ . \tag{1.12}$$

In particular, if $\mathcal{F}$ is a vector bundle, then

$$H^1(C, \mathcal{F}) \cong H^0(C, \mathcal{F}^\vee \otimes \omega_C)^\vee \ . \tag{1.13}$$

We can form the normalization[1.2] of a nodal curve as in fig. 1.

Suppose C is nodal with components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Let $\widetilde{C} \xrightarrow{\nu} C$ be the normalization. Write $\widetilde{C}_i$ for the normalization of C_i and

$$\{p_j, q_j\} = \nu^{-1}(x_j) \tag{1.14}$$

(for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$).

A line bundle L on C has *multi-degree* $\underline{\deg}(L)$ to be

$$\underline{\deg}(L) = \left(\deg\left(L|_{C_1}, \ldots, \deg\left(L|_{C_s}\right)\right)\right) \tag{1.15}$$

$$= \left(\deg\left(\nu^* L|_{\widetilde{C}_1}\right), \ldots, \deg \nu^* L|_{\widetilde{C}_s}\right) \ . \tag{1.16}$$

The following is a corollary to Serre duality.

Corollary 1.4. *If C is connected, and $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ then $H^1(C, L) = 0$.*

By $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ we mean $\deg\left(L|_{C_i}\right) \geq \deg\left(\omega_C|_{C_i}\right)$ for all i and $\underline{\deg}(L) \neq \underline{\deg}(\omega_C)$.

[1.2] Locally, the corresponding algebraic construction is taking the integral closure of the coordinate ring.

Proof. First note

$$H^1(C,L) \cong H^0\left(C,\omega_C \otimes L^{-1}\right) \ . \tag{1.17}$$

and $\underline{\deg}\left(\omega_C \otimes L^{-1}\right) < 0$.

On any connected component C_i such that $\deg\left(\omega_L \otimes L^{-1}\right)\big|_{C_i} < 0$ all sections vanish. And all sections vanish on components that meet C_i, etc. □

Corollary 1.5. *L is ample if and only if $\deg\left(L|_{C_i}\right) > 0$ for all i.*

Proof. ($\implies$): This direction is clear. The restriction of ample L to any component is still ample.

($\Longleftarrow$): Suppose $\deg\left(L|_{C_i}\right) > 0$. It is enough to show that $L^{\otimes N}$ is very ample for some N. Choose N sufficiently large so that

$$\deg\left(L^{\otimes N}\big|_{C_i}\right) > \deg\left(\omega_C|_{C_i}\right) + 2 \ . \tag{1.18}$$

Let $S \subseteq C$ be the union of two distinct smooth points. Then we have a short exact sequence

$$0 \to I_S \to \mathcal{O}_C \to \mathcal{O}_S \to 0 \tag{1.19}$$

which we can tensor with $L^{\otimes N}$ to get a sequence which is still exact, which gives us a long exact sequence

$$0 \longrightarrow H^0\left(L^{\otimes N}(-s)\right) \longrightarrow H^0\left(L^{\otimes N}\right) \longrightarrow H^0\left(L^{\otimes N}\big|_S\right) \longrightarrow H^1\left(L^{\otimes N}(-s)\right) \longrightarrow \dots \tag{1.20}$$

but $H^1(L(-s)) = 0$, so we have a surjection

$$H^0(L) \twoheadrightarrow H^0\left(L|_S\right) \ . \tag{1.21}$$

This shows that sections of $L^{\otimes N}$ separate the two points in S. Similar arguments show that sections of high tensor powers of L separate arbitrary pairs of points and tangent vectors. Therefore, high tensor powers of L are very ample, and so L is ample. □

3. Description of ω_C

We now describe the canonical sheaf of a nodal curve in terms of meromorphic differential forms. See [**9**, Chapter 6] or [**6**, Chapter 3, Section A] for proofs and further details.

Proposition 1.6. *Let C be a nodal curve with nodes $x_1, \dots, x_r$, write $(p_i, q_i) = \nu^{-1}(x_i)$. Then*

$$\omega_C \cong \nu_*\left(\omega'_{\widetilde{C}}(p_1 + q_1 + \dots + p_r + q_r)\right) \tag{1.22}$$

where $\omega'_{\widetilde{C}}(p_1 + \dots + q_r) \subseteq \omega_{\widetilde{C}}(p_1, \dots, q_r)$ is the subsheaf where

$$\operatorname{res}_{p_i}(\omega) + \operatorname{res}_{q_i}(\omega) = 0 \ . \tag{1.23}$$

Remark 1.1 (Rosenlicht differentials). There is a related explicit description of $\omega_{X/S}$ for a family of nodal curves. Near a point where $X/S \cong (xf = F) \subseteq \mathbb{C}^2 \times S$ $\omega_{C/S}$ is generated by dx/x and dy/y which satisfy

$$\frac{dx}{x} + \frac{dy}{y} = 0 \ . \tag{1.24}$$

Definition 1.3. A nodal curve is *stable* if ω_C is ample.

Lecture 4; January 29, 2020

Proposition 1.7. *Let $X \to S$ be a family of nodal curves. Then*

$$\{s \in S \,|\, X_s \text{ is stable}\}$$

is Zariski open.

Proof. Let L be any line bundle on X. Then

$$\left\{s \in S \,\middle|\, L|_{X_s} \text{ is ample}\right\}$$

is Zariski-open. This is Theorem 1.2.17 of [**8**]. □

Theorem 1.8. *A nodal curve C is stable if and only if* $\operatorname{Aut}(C)$ *is finite.*

Proof. Say C has components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Write $\{p_i, q_i\} = \nu^{-1}(x_i)$ for the preimage of the nodes under the normalization ν. Write $Q = \{p_1, q_1, \ldots, p_r, q_r\}$. Notice that $\operatorname{Aut}(C)$ is finite if and only if

$$\{\sigma \in \operatorname{Aut}(C) \,|\, \sigma \text{ acts by 1 on } \{C_1, \ldots, C_s\}\}$$

is finite.

Fix C_i. Note that $\operatorname{Aut}(C_i)$ is finite if and only if there are only finitely many automorphisms of $\tilde{C}_i$ that fix $Q \cap \tilde{C}_i$. This is the case exactly when

(1) $g\left(\tilde{C}_i\right) \geq 2$;
(2) $g\left(\tilde{C}_i\right) = 1$, and $Q \cap \tilde{C}_i \neq \emptyset$; or
(3) $g\left(\tilde{C}_i\right) = 0$ and $Q \cap \tilde{C}_i \geq 3$.

By direct computation, these are precisely the cases where

$$2g\left(\tilde{C}_i\right) - 2 + \#\left(Q \cap \tilde{C}_i\right) > 0.$$

The left hand side is $\deg\left(\omega_C|_{C_i}\right)$, by our description of the dualizing sheaf in terms of meromorphic differentials.

So we have shown that $\operatorname{Aut}(C)$ is finite if and only if the degree of the dualizing sheaf is positive on every component, which is equivalent to ω_C being ample, i.e., to C being stable. □

Definition 1.4. A *graph* G is a set $X(G)$ together with an involution $i : X(G) \circlearrowleft$ and a retraction $r : X(G) \to X(G)^i$. The vertices $V(G)$, half edges $H(G)$, and edges $E(G)$ are defined as:

$$\begin{aligned} V(G) &= X(G)^i \\ H(G) &= X(G) \setminus V(G) \\ E(G) &= H(G)/i \,. \end{aligned}$$

We say $r(h)$ is the vertex incident to $h \in H(G)$.

The *dual graph* $G(C)$ of a nodal curve C is as follows. The vertices $\{v_1, \ldots, v_s\}$ correspond to the components $C_1, \ldots, C_s$; and the half-edges incident to v_i are given by the points of $\tilde{C}_i \cap Q$. An edge is made from a pair of half-edges corresponding to a pair $\{p_i, q_i\}$. The "genus function" assigns the genus of $\tilde{C}_i$ to the corresponding vertex v_i. See fig. 2 for examples.

We can read the stability off from the dual graph. Every vertex labelled with a 1 should have at least one incident edge, and all unlabelled vertices should have valence at least 3.

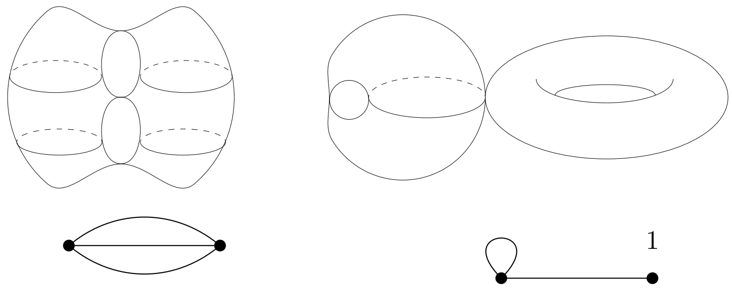


FIGURE 2. Two examples of genus 2 stable curves with their dual graphs below them. Notice we can read their stability off from the graphs. All unlabelled vertices have at least three incident edges, and the labelled one has one incident edge.

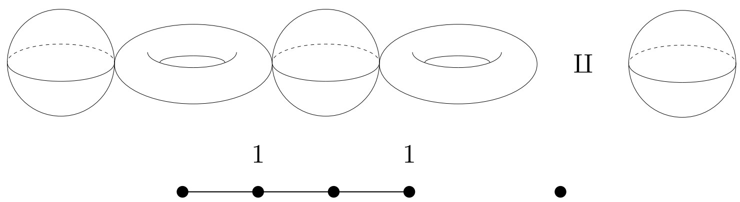


FIGURE 3. An example of an unstable genus 2 curve with its dual graph below it. Notice we can read the fact that it is unstable off of the graph. All three unlabelled vertices of valence less than 3.

Recall that the arithmetic genus of a curve C is

$$p_a(C) = 1 - \chi(\mathcal{O}_C) \ .$$

In particular, if C is connected then $p_a(C) = h^1(\mathcal{O}_C)$. Recall the Euler characteristic of a graph G is

$$\chi(G) = \#V(G) - \#E(G) \ .$$

Note if G is connected, then $h^1(G) = 1 - \chi(G)$. Also note that C is connected if and only if $G(C)$ is connected. The dual graph also detects the arithmetic genus in the following sense.

THEOREM 1.9. *Let C be a nodal curve. Then*

$$p_a(C) = 1 - \chi(G(C)) + \sum_v g(v) \ . \tag{1.25}$$

Corollary 1.10. *If C is connected then*

$$p_a(C) = \sum_v g(v) + h^1(G) \ . \tag{1.26}$$

PROOF OF THEOREM 1.9. Proceed by induction on the number of nodes $\#E(G) = \#C^{\text{sing}}$. The base case is when $E(G) = \emptyset$, so the graph is just s vertices v_i with genus $g(v_i)$. Then

$$1 - \chi(\mathcal{O}_C) = 1 - s + \sum_i g(v_i) \tag{1.27}$$

as desired.

Now suppose C' is obtained from C by gluing two smooth points p, q to x. Write $\pi : C \to C'$. Then we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \pi_*\mathcal{O}_C \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.28}$$

which implies the Euler characteristic of the middle term is the sum of the Euler characteristics of the other two terms. Now since π is proper and finite, $\chi(\pi_*\mathcal{O}_C) = \chi(\mathcal{O}_C)$. Altogether this gives us:

$$\chi(\mathcal{O}_C) = \chi(\pi_*\mathcal{O}_C) = \chi(\mathcal{O}_{C'}) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C'}) + 1 \ , \tag{1.29}$$

and the theorem follows. ☐

Lecture 5; January 31, 2020

4. Stable reduction

There are two statements. The first is the nodal reduction theorem (which does not involve stability) and the second is stabilization, which adds uniqueness. The reference is [**1**] Chapter X, Section 4. Write

$$\Delta = \{z \in \mathbb{C} \mid |z| < \epsilon\} \tag{1.30}$$

for a small disk. Write $\Delta^\times = \Delta \setminus \{0\}$ for the punctured disk, both viewed as having one complex dimension.

Consider a flat proper surjective map $\pi : X \to \Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Write $X^\times$ for the complement of the fiber over 0. Let $k > 0$ be an integer. Consider the map $\varphi_k : \Delta' \to \Delta$ from the disk to itself given by $z \mapsto z^k$. Note that φ_k is *not* a smooth map. Now we can construct a base change

$$\begin{tikzcd} X_k^\times := X^\times \times_{\varphi_k} \Delta'^\times \arrow[r] \arrow[d, "\pi'"] & X^\times \arrow[d, "\pi"] \\ \Delta'^\times \arrow[r, "\varphi_k"] & \Delta^\times \end{tikzcd} \ . \tag{1.31}$$

Theorem 1.11 (Nodal reduction theorem). *Let $\pi : X \to \Delta$ be a flat proper surjective map such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Then there exists an integer $k > 0$ such that after a base change as above, the map π' extends to a family of nodal curves over Δ.*

Theorem 1.12 (Stable reduction). *If $\pi|_{\Delta^\times}$ is* stable*, then this extension can be chosen to be stable, and the fiber over 0 depends* only *on $\pi|_{\Delta^\times}$ up to isomorphism.*

Remark 1.2. Uniqueness is related to separatedness for moduli of stable curves; existence and uniqueness is related to properness.

Remark 1.3. The intuition is as follows. Let Σ be a class of objects with a moduli space (or stack) $\mathcal{M}$, i.e., there is a universal family $\mathcal{I} \to \mathcal{M}$ of objects in Σ such that any family $X \to S$ of objects in Σ is the pullback of the universal family under a unique morphism $S \to \mathcal{M}$. In other words,

$$\mathrm{Hom}(-, \mathcal{M}) \cong \{\text{families of } \Sigma \text{ objects over } -\} \ . \tag{1.32}$$

If $\mathcal{M}$ is separated, i.e., Hausdorff, then for $\Delta^\times \to \mathcal{M}$ there exists at most one extension $\Delta \to \mathcal{M}$. If $\mathcal{M}$ is proper, then each map $\Delta^\times \to \mathcal{M}$ extends uniquely to $\Delta \to \mathcal{M}$. Roughly speaking, when one has a large class of objects with a moduli space $\mathcal{M}'$ such that maps $\Delta^\times \to \mathcal{M}'$ extend in many different ways to $\Delta \to \mathcal{M}'$ then one naturally looks for a stability

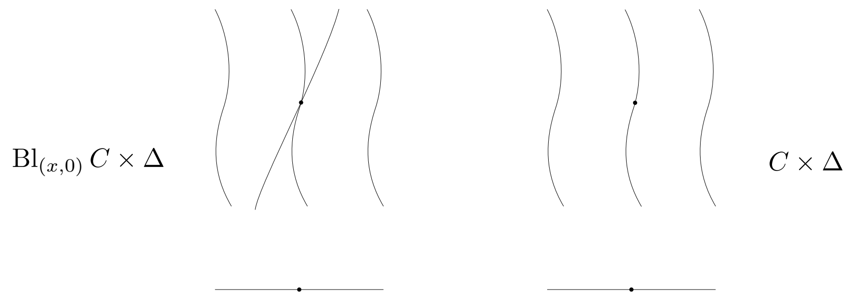


FIGURE 4. The constant family $\pi : C \times \Delta \to \Delta$ as well as the blowup $\mathrm{Bl}_{(x,0)}\, C \times \Delta \to \Delta$.

condition on the parametrized objects, so that the subspace $\mathcal{M} \subset \mathcal{M}'$ parametrizing stable objects is open and proper.

The notion of stability for nodal curves is a prototypical example. Indeed, if we don't impose stability, given a family of nodal curves $X^\times \to \Delta^\times$, then it may extend in many different ways to a nodal family $X' \to \Delta$ (and will always extend in many different ways, after a totally ramified base chance $\Delta \to \Delta$, given by $z \mapsto z^k$). So the existence and uniqueness of the special fiber in the theorem above is a special consequence of our specified stability condition.

EXAMPLE 1.1. Consider a smooth curve $C = (f = 0) \subseteq \mathbb{P}^2$. Then $C \times \Delta^\times \to \Delta^\times$ is a constant family which extends to $C \times \Delta \to \Delta$. Now for any $x \in C$, $C \times \Delta^\times$ also extends to $\mathrm{Bl}_{(x,0)}\, C \times \Delta$. We can picture this as in fig. 4.

The upshot is that moduli of nodal curves are not separated/Hausdorff. Lecture 6; February 3, 2020

Interlude: Some motivating examples.

Degeneration of a smooth curve to a nodal curve. We should think of the total space as being a surface. Consider the surface in fig. 5. This has two different rulings, as pictured in fig. 5. As in fig. 5, we can project this surface to a line by taking the intersection with parallel planes at different points of the line. Generically this gives us hyperbolas, but for two special values we get the union of two lines from the two different rulings. In particular this is given by the equation $xy = t^2 - t$. The node is exactly the point of tangency. So when we have a non-reduced curve, this is singular at every point on the curve.

Degeneration of a smooth curve to a non-reduced curve. Consider the surface defined by the equation $x^3 + t\,(x + y + 1) = 0$. At $t = 0$ we just get a line with multiplicity 3. This looks something like fig. 6.

Understanding the base change and its fibers. Again we consider a flat proper surjective map $\pi : X \to \Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. For simplicity assume that in fact $X = \mathbb{P}^1 \times \Delta$. Consider the map $\varphi_k : \Delta' \to \Delta$ from the disk to itself given by $z \mapsto z^k$. Note that φ_k is *not* a smooth map. In particular:

$$\varphi_k^{-1}(0) = \mathrm{Spec}\left(\mathbb{C}\left[\epsilon\right] / \epsilon^k\right) \ . \tag{1.33}$$

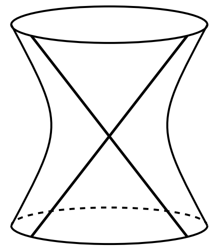


FIGURE 5. The surface given by $xy = t^2 - t$. Projection to the t-line has fibers which generically look like hyperbolas, but when the plane is tangent to the surface we get the union of two lines.

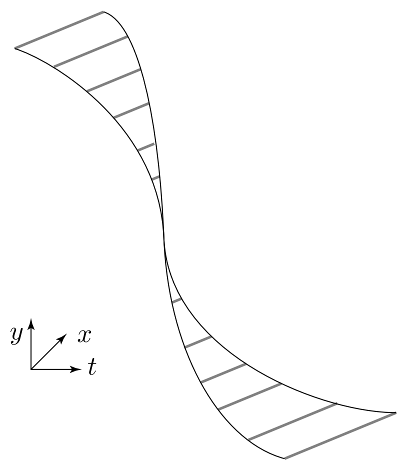


FIGURE 6. The surface $x^3 + t(x + y + 1) = 0$. Projecting to the t-line gives us smooth fibers which degenerate to a line with multiplicity 3 at $t = 0$.

Consider a base change for a family of curves

$$\begin{tikzcd} \left(\mathbb{P}^1 \times \Delta\right) \times_{\varphi_k} \Delta' \arrow[d, "\pi'"] & \mathbb{P}^1 \times \Delta \arrow[d, "\pi"] \\ \Delta' \arrow[r, "\varphi_k"] & \Delta \end{tikzcd} \quad . \tag{1.34}$$

If we think of the preimage under $\varphi_k \circ \pi'$ we have actually made things worse, since the preimage of 0 is:

$$(\varphi_k \circ \pi)^{-1}(0) \simeq \mathbb{P}^1 \times \operatorname{Spec}\left(\mathbb{C}[\epsilon]/\epsilon^k\right) \quad . \tag{1.35}$$

But in our construction we are replacing π by π', not $\varphi_k \circ \pi'$. The moral is that (at least for specific k) this replacement makes things better.

Proof of the nodal reduction theorem.

Proof of Theorem 1.11. We will operate under the simplifying assumption that $X^\times \to \Delta^\times$ is smooth. The first step is to resolve the singularities of X. This is easy since $\dim X = 2$. First we normalize to get something regular in codimension 1. Then we blowup the finitely many singular points. Then repeat, i.e., normalize and then blowup the finitely many singular points. It is a theorem (not especially difficult) that this process terminates, giving us $X' \xrightarrow{\pi'} \Delta$ where X' is smooth. However, the central fiber $(\pi')^{-1}(0) = X'_0$ might have arbitrary singularities. To deal with this, we first resolve the non-nodal singularities of $X_0'^{\mathrm{red}}$. The process is again straightforward; the reduced curve $X_0'^{\mathrm{red}}$ has finitely many singular points. We blow up the singular points that are not nodes. The resulting total space is still smooth, and we repeat, blowing up the finitely many singular points of the reduced special fiber that are not nodes. It is again a theorem (and not particularly difficult) that this process terminates. Hence we may assume that X' is smooth and $X_0'^{\mathrm{red}}$ has only nodal singularities.

Locally near each node of $X_0'^{\mathrm{red}}$, the surface X' is isomorphic (over Δ) to $z = x^a y^b$ in $\mathbb{C}^2 \times \Delta$, where x and y are the coordinates on $\mathbb{C}^2$ and z is the coordinate on Δ. Similarly, near each smooth point of ${X'}_0^{\mathrm{red}}$, the surface X' is isomorphic (over Δ) to $z = x^c$. We can then cover ${X'}_0^{\mathrm{red}}$ by finitely many open sets where we have such local charts, and set

$$k = \operatorname{lcm}\{ab, c\} \ . \tag{1.36}$$

The rough idea is that this choice of k will ensure that base change along $\varphi_k \colon \Delta \to \Delta$, given by $z \mapsto z^k$, will unwind the multiplicities of the components of X'_0.

In fact, the base change

$$X'' = \varphi_k^* X' \tag{1.37}$$

is not necessarily normal, but we claim that

Claim 1.1. $(X'')^\nu \xrightarrow{\pi'} \Delta'$ is a nodal family, where $(X'')^\nu \to X''$ is the normalization.

To prove the claim, we first consider π' near a point where $X' \cong (z = x^c)$. Write $z = \zeta^k$ and $k = ch$ so that

$$x^c - z = x^c - \zeta^{ch} = \prod_{\omega^c = 1} \left(x - \omega\zeta^k\right) \ . \tag{1.38}$$

Note that this product gives rise to c different smooth and irreducible components, which are disjoint in the general fiber but intersect in the special fiber. Normalizing pulls apart the intersections in the special fiber, giving rise to the disjoint union $\amalg_{\omega^c=1}\left(x - \omega\zeta^h\right)$, which is smooth over Δ.

It remains to consider π' near a point where $X' \cong \left(z = x^a y^b\right)$. Write $k = rsuv$ where $a = ru$, $b = su$, and $(r, s) = 1$. Write ζ for the coordinate on Δ'. Then X'' is locally given by

$$0 = x^a y^b - \zeta^k \ . \tag{1.39}$$

This need not be normal. Indeed, if $u > 1$ then $x^r y^s$ obviously satisfies a nontrivial monic polynomial. Choose ω a primitive uth root of unity, so we have a factorization

$$\left(x^a y^b - \zeta^k\right) = \prod_{i=1}^{u} \left(x^r y^s - \omega^i \zeta^{rsv}\right) \ . \tag{1.40}$$

We can again pass to the disjoint union of surfaces with local defining equations $x^r y^2 - \omega^i z^{rsv}$, but this is only a partial normalization. Indeed, these surfaces are all isomorphic, but $\zeta^{vrs} = x^r y^s$ need not be normal. Then we claim the following.

CLAIM 1.2. The normalization is locally isomorphic to the surface defined by $\zeta^v = \alpha\beta$, where ζ, α, β are coordinates on $\mathbb{C}^3$, with the normalization map given by $x = \alpha^s$, $y = \beta^r$.

To check that this is the normalization we need to check that

(1) this surface is normal,
(2) the map is generically one-to-one, and
(3) the map is surjective.

To see that this surface normal, notice that $\zeta^v = \alpha\beta$ is the toric surface corresponding to the cone spanned by $(1,0)$ and $(v,1)$ in $\mathbb{R}^2$, with respect to the standard lattice $\mathbb{Z}^2$. It is well-known and easy to prove that toric varieties are normal (see [**4**, §2.1]). We now show that the map is generically one-to-one. Given (α,β,ζ) and (α',β',ζ') so that

$$\alpha^s = (\alpha')^s \qquad \beta^r = (\beta')^r \qquad \zeta = \zeta' \ . \tag{1.41}$$

This means $\alpha' = \sigma\alpha$ for σ an sth root of unity, and similarly $\beta' = \tau\beta$ for τ an rth root of unity. But if α and β are nonzero, then $\alpha\beta = \alpha'\beta'$ implies $\sigma\tau = 1$, so $\sigma = \tau = 1$, so

$$(\alpha,\beta,\zeta) = (\alpha',\beta',\zeta') \ . \tag{1.42}$$

Since the points where α and β are nonzero form an open dense set we are done.

Now consider (x,y,ζ) such that $x^r y^s = \zeta^{vrs}$. Then we must find points (α,β,ζ) such that $\alpha\beta = \zeta^v$ and $x = \alpha^s$, and $y = \beta^r$. Choose α_0, β_0 such that $\alpha_0^s = x$ and $\beta_0^r = y$. The point being that $\alpha_0 \cdot \beta_0 = \xi\zeta^v$ where $\xi^{rs} = 1$. Now write

$$1 = mr + ns \tag{1.43}$$

so the coordinates are

$$\alpha = \alpha_0\xi^{-mr} \qquad \beta = \beta_0\xi^{-ns} \ . \tag{1.44}$$

□

Now we claim that X' can be chosen stably if $X|_{\Delta^\times}$ is stable. Lecture 7; February 5, 2020

THEOREM 1.13 (Stabilization theorem). *Let $X \xrightarrow{\pi} \Delta$ be a family of nodal curves such that $\pi|_{\Delta^\times}$ is stable. Then there is*

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr] & & X' \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{1.45}$$

such that

(1) $\psi : X|_{\Delta^\times} \to X'|_{\Delta^\times}$ is an isomorphism;
(2) for each component C_i of the central fiber $C = X_0$, ψ maps C_i either to a point, or birationally onto its image; and
(3) X' is a family of stable curves.[1.3]

Moreover, $X' \to \Delta$ is unique.

REMARK 1.4. The moral of the story is that

$$X' = \mathrm{Proj}_\Delta \left(\bigoplus_{n \geq 0} \pi_* \left(\omega_{X/\Delta}^{\otimes n} \right) \right) \ . \tag{1.46}$$

[1.3] So $X' \to \Delta$ is flat and proper with nodal stable fibers.

Recall that when we take this big direct sum we get a sheaf of graded $\mathcal{O}_\Delta$-algebras, so it makes sense to take relative Proj_Δ of this. The minimal model program deals with finite generation of things like this.

Proof. The idea is to consider $C = X_0$ with components $C_0, \ldots, C_s$. Then we will look at

$$\left\{C_i \,\middle|\, \omega_C|_{C_i} \text{ is not ample}\right\} = \left\{C_i \,\middle|\, \deg\left(\omega_C|_{C_i}\right) \leq 0\right\} \ . \tag{1.47}$$

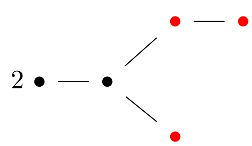
Call this set the set of unstable components. We continue with out simplifying assumption that $X|_{\Delta^\times}$ is smooth and with connected fibers. Note that this implies this has arithmetic genus $p_a(C) \geq 2$. Then the set of unstable components is:

$$\left\{C_i \,\middle|\, \omega_C|_{C_i} \text{ is not ample}\right\} = \left\{C_i \,\middle|\, c_i \cong \mathbb{P}^1, \# \left\{C_i \cap \mathrm{Cl}\left((C \setminus C_i)\right)\right\} \leq 2\right\} \ . \tag{1.48}$$

Then we have the following observation from [**1**]. The union of the unstable components is a union of rational curves that intersect the rest of the curves (the stable components) at one or two points on the ends of the chain. Let C' be the curve obtained by contracting all unstable chains. Note that $p_a(C') = p_a(C)$.

Warning 1.1. Now we reach an error in [**1**] (page 112, second sentence). In particular, they claim that by construction X' is stable. But this is false, as the following example shows.

Counterexample 1. Let the following graph be the dual graph of C:



with three unstable components (in red). Then after contracting the unstable chain, we get that C' has dual graph

$$2 \bullet \text{ --- } \bullet$$

which is not stable.

We will proceed by stabilizing the special fiber C, i.e. we consider $\varphi : C \to C'$ such that

(i) $\varphi|_{C_i}$ is either constant or birational onto its image (and an isomorphism on $C_i \cap C^{\mathrm{smooth}}$).
(ii) $p_a(C') = p_a(C)$, and
(iii) C' is stable.

Now we construct Lecture 8; February 7, 2020

$$\begin{tikzcd} X \arrow[rr, "\varphi'"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\circlearrowright"] & X' \arrow[dl, "\pi'"] \\ & \Delta & \end{tikzcd} \tag{1.49}$$

such that

(i) X' is a stable family,
(ii) φ' is an isomorphism on $\Delta^\times$,
(iii) $X'_0 \cong C'$, and
(iv) $\varphi'|_{X_0} = \varphi$.

Given $\psi : C \to C'$ as above with C' stable, let L_0 be $\varphi^* \omega_{C'}$. Let $d_i = \deg\left(L_0|_{C_i}\right)$ and $\underline{d} = \underline{\deg}\,(L_0)$. Note all $d_i \geq 0$. Now choose d_i sections of π that meet C_i at distinct smooth pojnts of C. We can do this by Hensel's lemma. Write $D_{i_1}, \ldots, D_{i_{d_i}}$ for the images of these d_i sections. Write

$$D = \sum_{D_{i_j}} \ . \tag{1.50}$$

Then $L = \mathcal{O}(D)$ is a line bundle on X, and

$$\underline{\deg}\left(L|_{X_0}\right) = \underline{\deg}\,(L_0)\ . \tag{1.51}$$

Then we make the following observations:

- L is relatively ample on $\Delta^\times$ (with degree $2g-2$),
- $L|_{X_0}$ is the pullback of an ample line bundle L' on C'.

Lemma 1.14. *For any line bundle M' on C',*

$$H^i(C', M') = H^i(C, \varphi^* M') \tag{1.52}$$

(for $i \in \{0,1\}$).

Proof. The pullback induces an isomorphism on H^0, and

$$\begin{aligned}\chi(M') &= \chi(\mathcal{O}_{C'}) + \deg(M') \\ &= \chi(\mathcal{O}_C) + \deg(\varphi^* M') \\ &= \chi(\varphi^* M')\ .\end{aligned}$$

□

The consequences are as follows. For large n, $H^1(X_0, L^{\otimes n}) = 0$ (vanishing on C' by ampleness and Lemma 1.14). This implies $h^0(X_s, L^{\otimes n})$ is a constant function of $s \in \Delta$. Therefore $\pi_* L^{\otimes n}$ is locally free by Grauert's theorem.[1.4]

Now we choose n sufficiently large such that $L^{\otimes n}$ is very ample on fibers over $\Delta^\times$, and pullbacks of very able on C' to C (over 0). Then $\pi_* L^{\otimes n}$ induces $\psi : X \to \Delta \times \mathbb{P}^N$. Then we have

$$\psi(X_0) \cong C'\ . \tag{1.53}$$

Take $X' = \operatorname{im}(\psi)$. Note that $X' \to \Delta$ is flat by the Hilbert polynomial criterion. This is the end of the proof of stable reduction. ■

Remark 1.5. When one is learning algebraic geometry, it is important to keep track of when global facts can be shown by showing things on fibers. E.g. Grauert's theorem, Hensel's lemma, and somehow Nakayama's lemma is at the heart of things.

Definition 1.5. An *n-pointed nodal curve* is a pair $(X; p_1, \ldots, p_n)$ such that X is a nodal curve, and $p_1, \ldots, p_n$ are distinct smooth points of X.

Definition 1.6. We say $(X; p_1, \ldots, p_n)$ is *stable* if and only if $\omega_X(p_1 + \ldots + p_n)$ is ample.

[1.4] Recall this says that if the dimension of H^i is constant, the sheaf is coherent, and the morphism is proper, the $R^i\pi_*$ is locally free. See [**7**].

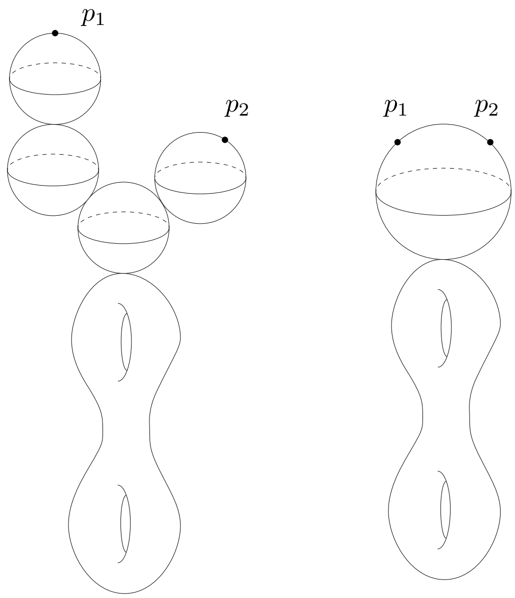


FIGURE 7. The left curve is unstable. When we stabilize, we contract to get a stable curve as on the right. Note that the marked points follow the contraction.

THEOREM 1.15. $(X; p_1, \ldots, p_n)$ *is stable if and only if*

$$\operatorname{Aut}(X; p_1, \ldots, p_n) = \{\sigma \in \operatorname{Aut}(X) \mid \sigma(p_i) = p_i\} \tag{1.54}$$

is finite.

DEFINITION 1.7. A *family of pointed nodal curves* is a family of nodal curves $\pi : X \to S$ with sections $\sigma_1, \ldots, \sigma_n$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_1"'] \arrow[u, bend right=70, "\sigma_n"'] \end{tikzcd} \tag{1.55}$$

such that $\{\sigma_i(S)\}$ are disjoint and contained in π^{smooth}.

Then there are generalizations of nodal reduction, stabilization, and stable reduction for pointed curves as well.

The idea is as follows. When we contract during the stabilization process, the marked points follow the contraction. See fig. 7.

Then there is a theorem that forgetting marked points and stabilization actually behave well in families. So if we have a functor from S to the moduli space of stable curves (of a given genus) with n-marked points, we can get the moduli space of stable curves (of the same genus) with $(n-1)$-marked points in such a way that we actually have a natural transformation between the corresponding functors. Lecture 9; February 10, 2020

CHAPTER 2

Deformation theory

Definition 2.1. A *deformation* of a proper (connected) scheme (a $\mathbb{C}$-analytic space) X is a flat and proper morphism $\mathcal{X} \xrightarrow{\varphi} S$ to a pointed scheme (S, s) together with an isomorphism $\mathcal{X}_s \xrightarrow{\sim} X$.

An *infinitesimal deformation* is a deformation over $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$.

A morphism of deformations is a cartesian square

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ (S,s) \arrow[r] & (S',s') \end{tikzcd} \tag{2.1}$$

such that the induced map

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rrr, bend right] & \mathcal{X}_s \arrow[r] & \mathcal{X}'_{s'} \arrow[r, "\sim"] & X \end{tikzcd} \tag{2.2}$$

is the identity.

Theorem 2.1. *If X is smooth then the isomorphism classes of infinitesimal deformations of X are in natural bijection with $H^1(X, T_X)$.*

Proof. The first step is to find a natural map from the isomorphism classes of infinitesimal deformations to $H^1(X, T_X)$. Since X is smooth, we have a smooth map $\mathcal{X} \to S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$ which gives rise to the short exact sequence

$$0 \to T_X \to T_{\mathcal{X}} \to \varphi^* T_S \to 0 \tag{2.3}$$

which gives us a long exact sequence on cohomology:

(2.4)

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, T_X) \arrow[r] & H^0(X, T_{\mathcal{X}}) \arrow[r] & H^0(X, \varphi^* T_S) \arrow[r, "\delta"] \arrow[d, equal] & H^1(X, T_X) \arrow[r] & \dots \\ & & & H^0(X, \mathcal{O}_X) \arrow[d, equal] & & \\ & & & \mathbb{C} & & \end{tikzcd}$$

so $1 \in \mathbb{C}$ lands in some class $\delta(1) \in H^1(X, T_X)$.

Let $\mathcal{X} \to S$ be an infinitesimal deformation, $\mathcal{X}_0 \xrightarrow{\sim} X$. Note that $\mathcal{O}_{\mathcal{X}}$ is locally free (of rank 2) as an $\mathcal{O}_X$-module. Now we cover $\mathcal{X}$ by finitely many open U_α such that $\mathcal{O}_X|_{U_\alpha}$ is free. Let $z_{\alpha_1}, \dots, z_{\alpha_n}$ be local coordinates on these $U_{\alpha_i} \subseteq X$, and let $f_{\alpha\beta}$ be the transition functions, i.e. $z_\alpha = f_{\alpha\beta} z_\beta$. These functions satisfy

$$f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) = f_{\alpha\gamma}(z_\gamma) \ . \tag{2.5}$$

Now consider $\mathcal{X}$ as being glued from the $U_\alpha \times S$. In particular $U_\alpha \times S$ is glued to $U_\beta \times S$ along $(U_\alpha \cap U_\beta) \times S$. So we have z_α and ϵz_α, and

$$z_\alpha = \underbrace{f_{\alpha\beta}(z_\beta) + \epsilon g_{\alpha\beta}(z_\beta)}_{\tilde{f}_{\alpha b}(z_\beta)}, \tag{2.6}$$

i.e. we write $\tilde{f}_{\alpha\beta}$ for the new transition functions, and moreover, the new transition functions agree with the old ones up to ϵ. This is the gluing data describing the construction of $\mathcal{X}$ from the charts $U_\alpha \times S$.

REMARK 2.1. The geometric picture is that we start with some X, then we spread this out into a higher-dimensional fibration. So assuming we've shrunk U_α sufficiently, it has no interesting topology, and if we look at it inside of the fibers all at once, this is just a cylinder $U_\alpha \times S$. So then the total space is glued out of these cylinders.

Then these have to satisfy the gluing condition

$$\tilde{f}_\alpha\left(\tilde{f}_{\beta\gamma}(z_\gamma)\right) = \tilde{f}_{\alpha\gamma}(z_\gamma) \tag{2.7}$$

$$= \underbrace{f_{\alpha\beta}(f_{\beta\gamma})}_{f_{\alpha\gamma}} + f_{\alpha\beta}(\epsilon g_{\beta\gamma}) + \epsilon g_{\alpha\beta}(f_{\beta\gamma}) \ . \tag{2.8}$$

The first term just comes from gluing on X, and the second term can be thought of as a version of Leibniz' rule:

$$\frac{\partial f_{\alpha\beta}}{\partial z_\beta} g_{\beta\gamma} + g_{\alpha\beta} = g_{\alpha\gamma} \ . \tag{2.9}$$

Another way of writing this is that:

$$\Theta_{\alpha\beta} = (g_{\alpha_i\beta_i}) \begin{pmatrix} \partial/\partial z_{\alpha_1} \\ \vdots \\ \partial/\partial z_{\alpha_n} \end{pmatrix} \in H^0\left(U_{\alpha\beta}, T_X|_{U_{\alpha\beta}}\right) \tag{2.10}$$

is a cocycle, so it defines a class:

$$[\Theta_{\alpha\beta}] \in H^1(X, T_X) \tag{2.11}$$

which is the image of 1 in $H^1(X, T_X)$.

The point of this is that the deformation $\varphi : X \to S$ goes to the coboundary $\delta(\partial/\partial\epsilon)$ where we regard $\partial/\partial\epsilon \in H^0(\varphi^* T_S)$.

So by direct calculation, every 1-cocycle on T_X comes from a deformation. The derived functor approach tells us that the map

$$\{\text{isomorphism classes of deformations}\} \twoheadrightarrow H^1(X, T_X) \tag{2.12}$$

is well-defined, and the cocycle approach tells us that it is surjective. Then cohomologous cocycles give rise to isomorphic deformations. □

The reference for this is [**1**, Chapter XI, section 2]. Lecture 10;
In general, for a deformation February 12, 2020

$$\begin{array}{c} X \\ \downarrow\varphi \\ (B, b_0) \end{array} \tag{2.13}$$

we get that T_{B,b_0} corresponds to $\{S \to (B, b_0)\}$. So we have a map $\rho : T_{B,b_0} \to H^1(X, T_X)$. This is called the *Kodaira-Spencer map.*

In our case, $X = C$ is a curve of genus[2.1] $g(X) \geq 2$. So now we want to understand But by Serre duality, we get a canonical isomorphism

$$H^1(C, T_C) \cong H^0\left(C, T_C^\vee \otimes \omega_C\right)^\vee \cong H^0\left(C, \omega_C^{\otimes 2}\right)^\vee \ . \tag{2.14}$$

Sometimes $H^0\left(C, \omega^{\otimes 2}\right)$ is referred to as the *quadratic differentials*. Since $\deg\left(\omega_C^{\otimes 2}\right) = 4g - 4$ and $g \geq 2$, Riemann-Roch tells us that

$$h^0\left(\omega_C^{\otimes 2}\right) = 3g - 3 \ . \tag{2.15}$$

Another special feature of a curve, is that the ideal sheaf of a point I_p is locally free,[2.2], so we have a short exact sequence

$$0 \longrightarrow I_p \cong \mathcal{O}(-p) \qquad \mathcal{O}_C \longrightarrow \mathcal{O}_p \longrightarrow 0 \ . \tag{2.16}$$

Tensoring with $T_C(p)$ gives us the short exact sequence

$$0 \to T_C \to T_C(p) \to T_C(p)|_p \to 0 \ . \tag{2.17}$$

This gives the long exact sequence

$$\begin{array}{ccccccc} H^0(C, T_C) & \longrightarrow & H^0(C, T_C(p)) & \longrightarrow & H^0(p, T_C(p)) & \xrightarrow{\delta} & H^1(C, T_C) \\ & & & & \| & & \\ & & & & \mathbb{C} & & \end{array} \ . \tag{2.18}$$

So $p \in C$ gives rise to $\delta(C) \subseteq H^1(C, T_C)$ which is an infinitesimal deformation well-defined up to $\mathbb{C}^\times$. These are called *Schiffer deformations* and are written as δ_p.

An alternative construction is as follows. We have

$$C \to \mathbb{P}\left(H^0\left(C, \omega_C^{\otimes 2}\right)^\vee\right) \ , \tag{2.19}$$

so $p \in C$ maps to some δ_p in this projective space.

Fact 2 (Important fact). *Schiffer deformations are integrable, i.e. they come from deformations over a small disk $\Delta = \{z \,|\, |z| < b\}$.*

The idea is as follows. Let $p \in C$ be a point in our curve. Then let U be a neighborhood of p with a local coordinate $z : U \xrightarrow{\sim} \Delta$ which maps U isomorphically to the disk Δ. Then define:

$$U' = \{z \in U \,|\, |z| < b/3\} \qquad\qquad U'' = \{w \in U \,|\, |w| < 2b/3\} \ . \tag{2.20}$$

That is $U' \subset U'' \subset U$. Then we can think of C as being obtained by gluing

$$C = (C \setminus U') \cup U'' \ . \tag{2.21}$$

In particular, for t sufficiently small, consider the space C_t obtained by gluing $C \setminus U'$ to U'' along $w = z + t/z$.

Claim 2.1 ([**1**, XI, §2]). *δ_p is the infinitesimal deformation associated to the family $\{C_t\}$.*

[2.1] To be stable it must have genus $g(C) \geq 2$.

[2.2] If we are instead on a surface, for example, then the ideal sheaf of a point will have rank 1 everywhere away from the point, but you need two generators at the point itself.

In fact we get something even better. Choose multiple points $p_1, \dots, p_s \in C$, so we get multiple Schiffer deformations. Now by choosing disjoint coordinate patches at the points we can simultaneously integrate all δ_{p_i} to get

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \Delta^s \end{array} . \tag{2.22}$$

Note that

$$f = f_{|\omega_C^{\otimes 2}|} : C \otimes \mathbb{P}\left(H^0\left(C, \omega^{\otimes 2}\right)^\vee\right) \tag{2.23}$$

is nondegenerate, i.e. not contained in a hyperplane, so the Schiffer deformations span $H^1(C, T_C)$. In particular, for $s = 3g-3$, $p_1, \dots, p_s$ general, $\{\delta_{p_1}, \dots, \delta_{p_s}\}$ is a basis for $H^1(C, T_C)$. Therefore for $\varphi : \mathcal{C} \to \Delta^s$ the Kodaira-Spencer map

$$\rho : T_{\Delta^s, 0} \xrightarrow{\sim} H^1(C, T_C) \tag{2.24}$$

is an isomorphism. The paper [**11**] somehow shows that moduli spaces of arbitrarily nice objects are arbitrarily bad. This is all a way of saying that this is a very special feature of curves.

Definition 2.2. A deformation

$$\begin{array}{c} \mathcal{C} \\ \downarrow{\scriptstyle\varphi} \\ (B, b_0) \end{array} \tag{2.25}$$

$(C_{b_0} \xrightarrow{\sim} C)$ is a Kuranishi family if for any deformation $\mathcal{D} \xrightarrow{\varphi} (E, e_0)$ of C, and any sufficiently small neighborhood U of e_0, there is a unique morphism of deformations

$$\varphi'|_U \to \varphi . \tag{2.26}$$

These can be thought of as *local moduli spaces*. We will now study these for nodal curves. Lecture 11; February 14, 2020

1. Deformations of nodal curves

Let C be a nodal curve.

Theorem 2.2. *There is a natural bijection between isomorphism classes of infinitesimal deformations of C and* $\mathrm{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right)$.

Remark 2.2. If C is in fact smooth, then the sheaf of Kähler differentials Ω^1_C is the dualizing sheaf $\Omega^1_C \cong \omega_C$. So

$$\mathrm{Ext}^1(\omega_C, \mathcal{O}_C) \cong \mathrm{Ext}^1\left(\omega_C^{\otimes 2}, \omega_C\right) \tag{2.27}$$

$$\cong H^0\left(C, \omega_C^{\otimes 2}\right) \tag{2.28}$$

$$\cong H^1\left(C, \left(\omega_C^{\otimes 2}\right)^\vee \otimes \omega_C\right) \tag{2.29}$$

$$\cong H^1(C, T_C) \tag{2.30}$$

where the second and third equalities come from (the appropriate version of) Serre duality. So we do obtain our old result from this.

PROOF. Let

$$\begin{array}{c} \mathcal{C} \\ \downarrow\varphi \\ S = \operatorname{Spec}\mathbb{C}\left[\epsilon\right]/\epsilon^2 \end{array} \tag{2.31}$$

be an infinitesimal deformation of C. Then we get an exact sequence

$$\varphi^*\Omega^1_S \to \Omega^1_{\mathcal{C}} \to \Omega^1_{\mathcal{C}/S} \to 0\ . \tag{2.32}$$

Now tensoring is right-exact, so we can tensor with $\mathcal{O}_C$ to get:

$$\mathcal{O}_C \to \Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C \to \Omega^1_C \to 0\ . \tag{2.33}$$

Now this looks almost like an extension of Ω^1_C by $\mathcal{O}_C$, except it isn't left exact.

CLAIM 2.2.1. $\mathcal{O}_C \to \Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C$ is injective.

PROOF. Note $\mathcal{O}_C = \varphi^*\Omega^1_S \otimes \mathcal{O}_C$ is generated by $d\epsilon$. At a smooth point of C, $\mathcal{C}$ is locally $C \times S$, which implies $d\epsilon$ maps to something nontrivial, which is sufficient to show injectivity.

Therefore (2.33) is a short exact sequence in $\operatorname{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right)$. $\square$

CLAIM 2.2.2. This assignment of deformations to extensions is injective.

PROOF. Suppose

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{C}' \\ \downarrow & & \downarrow \\ S & & S \end{array} \tag{2.34}$$

give rise to the same extension class. Then we have a map γ such that the following diagram commutes:

$$\begin{tikzcd} & \Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C \arrow[dd, "\sim"', "\gamma"] \arrow[dr] & \\ \mathcal{O}_C \arrow[ur] \arrow[dr] & & \mathcal{O}^1_C \\ & \mathcal{O}^1_{\mathcal{C}'} \otimes \mathcal{O}_C \arrow[ur] & \end{tikzcd} . \tag{2.35}$$

So we need to show that there exists a morphism $\beta : \mathcal{O}_{\mathcal{C}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{C}'}$ (over S) which restricts to the identity on $\mathcal{O}_C$. $\square$

CLAIM 2.2.2'. There exists a unique $\beta\left(h\right) \in \mathcal{O}_{\mathcal{C}'}$ such that

$$\beta\left(h\right)\big|_C = h\big|_C \tag{2.36}$$

and

$$d\beta\ \left(h\right)\big|_C = \gamma\left(\ dh\,\big|_C\right) \tag{2.37}$$

where we write $d\beta\ \left(h\right)\big|_C$ for the image of $d\beta\ \left(h\right)$ in $\Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C$.

PROOF. First we show local uniqueness. If $f|_C = 0$, then locally $f = \epsilon g$. This implies $df = g\, d\epsilon\,|_C$. If, in addition, $df\,|_C = 0$ then $f = 0$. This implies uniqueness, use $f = h_1 - h_2$.
Now local uniqueness means that it is enough to construct $\beta\,(h)$ locally.

First, $h|_C$ extends to $\tilde{h}$ on $\mathcal{C}'$. The difference between $d\tilde{h}\Big|_C$ and $\gamma\,(\,dh\,|_C)$ is of the form $g\,d\epsilon$. Set

$$\beta\,(H) = \tilde{h} - \epsilon g\ . \tag{2.38}$$

This gives rise to a canonical set theoretic map

$$\beta : \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}'} \tag{2.39}$$

which is a priori only a map of sheaves of sets, but in fact it is a map of sheaves of rings. This follows from the Leibniz rule. This proves claim 2.2.2', which implies claim 2.2.2. □

CLAIM 2.2.3. The map from deformations to extensions is surjective.

PROOF. Now we have the following exact sequence, called the local-to-global sequence. In our case it collapses to:

$$0 \to H^1\left(C, \mathcal{H}\mathrm{om}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \to \mathrm{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right) \to H^0\left(C, \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_C}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \to 0\ . \tag{2.40}$$

The point is that the sheaf $\mathcal{E}\mathrm{xt}$ is only given by local extensions. In particular, it vanishes for vector bundles. Note that $\mathcal{E}\mathrm{xt}^1$ is supported in C^{sing}:

$$H^0\left(\mathcal{E}\mathrm{xt}^1\left(\Omega^1_C, \mathcal{O}_C\right)\right) = \bigoplus_{p \in C^{\mathrm{sing}}} \mathrm{Ext}^1\left(\Omega^1_{C,p}, \mathcal{O}_{C,p}\right)\ . \tag{2.41}$$

Lecture 12; February 17, 2020

The Wikipedia page and this Stack Exchange post are quite good references for the general local-global Ext sequence:

$$E_2^{pq} = H^p\left(\mathcal{E}\mathrm{xt}^q\right) \Rightarrow \mathrm{Ext}^{p+q}\ . \tag{2.42}$$

This is an example of a Grothendieck spectral sequence for the composition of two functors.
Note that

$$H^1\left(\mathcal{H}\mathrm{om}\left(\mathcal{O}^1_C, \mathcal{O}_C\right)\right) = \{\text{locally trivial extensions}\} = \left\{\mathcal{O}_X \to \mathcal{F} \to \Omega^1_C\right\}\ .$$

So we have an open cover $\{U_\alpha\}$ and isomorphisms

$$\mathcal{F}|_{U_\alpha} \xrightarrow{\varphi_\alpha} \mathcal{O}_C \oplus \Omega^1_C \tag{2.43}$$

so that on $U_\alpha \cap U_\beta$

$$\begin{tikzcd} \mathcal{O} \oplus \Omega^1 \arrow[r, "\varphi_\alpha^{-1}"] \arrow[rr, bend right, "f_{\alpha\beta}"'] & \mathcal{F}\left(U_\alpha \cap U_\beta\right) \arrow[r, "\varphi_b"] & \mathcal{O} \oplus \Omega^1_C \end{tikzcd}\ . \tag{2.44}$$

So we get $\{f_{\alpha\beta}\}$ is a 1-cocycle for $\mathcal{H}\mathrm{om}\left(\Omega^1, \mathcal{O}\right)$.
Locally, nontrivial extensions correspond to "smoothings of nodes". Locally near $p \in C^{\mathrm{sing}}$ we have

$$\begin{tikzcd} (C,p) \arrow[r, phantom, "\cong"] \arrow[dr] & ((xy=0),0) \arrow[d, "\subseteq"] \\ & \left(\mathbb{C}^2, 0\right) \end{tikzcd}\ . \tag{2.45}$$

Recall the conormal exact sequence is

$$I_C/I_C^2 \to \mathcal{O}^1_{(\mathbb{C}^2,0)}/\left(xy\right) \to \Omega^1_{C,p} \to 0\ . \tag{2.46}$$

A basis for $(xy)/(xy)^2$ is given by monomials of the form

$$xy^i \qquad\qquad yx^i\ . \tag{2.47}$$

This is locally free of rank 1 on C, i.e.

$$\mathcal{O}_{\mathbb{C}^2}\left(-C\right)\big|_C\ . \tag{2.48}$$

Then the functor $\mathrm{RHom}\left(-,\mathcal{O}_{C,p}\right)$ gives us the long-exact sequence:

$$\begin{array}{ccc} \mathrm{Hom}\left(\Omega^1_{\mathbb{C}^2/(xy)},\mathcal{O}\right) \xrightarrow{\eta} \mathrm{Hom}\left(I_C/I_C^2,\mathcal{O}_C\right) & \longrightarrow & \mathrm{Ext}^1\left(\Omega^1_{C,p},\mathcal{O}_{C,p}\right) \\ & & \downarrow \\ & & \mathrm{Ext}^1\left(\Omega^1_{\mathbb{C}^2}\big|_C,\mathcal{O}_C\right) \simeq 0 \end{array} \tag{2.49}$$

where the last term vanishes since

$$\mathrm{Ext}^1\left(\Omega^1_{\mathbb{C}^2}\big|_C,\mathcal{O}_C\right) \simeq \mathrm{Ext}^1\left(\mathcal{O}_C^{\oplus 2},\mathcal{O}_C\right) = 0\ . \tag{2.50}$$

The image of η is

$$\mathfrak{m}_p\,\mathrm{Hom}\left(I_C/I_C^2,\mathcal{O}_{C,p}\right) \cong \mathfrak{m}_p \tag{2.51}$$

so we get a non-canonical isomorphism

$$\mathrm{Ext}^1\left(\mathcal{O}^1_{C,p},\mathcal{O}_{C,p}\right) \cong \mathcal{O}_{C,p}/\mathfrak{m}_p \cong \mathbb{C}\ . \tag{2.52}$$

If we followed the monomials carefully, we would get the canonical isomorphism

$$\mathrm{Ext}^1\left(\mathcal{O}^1_{C,p},\mathcal{O}_{C,p}\right) \cong T_{\tilde{C},p_1}\otimes T_{\tilde{C},q_1} \tag{2.53}$$

where $\{p_1,q_1\} = \nu^{-1}(p) \subseteq \tilde{C}$. See [**1**, XI, §3].

Example 2.1. For $C = (xy=0)$, we get the deformation $xy = a\epsilon$ for $a \in \mathbb{C}$. So we get a Kodaira Spencer class

$$\rho\left(xy = a\epsilon\right) \in \mathrm{Ext}^1\left(\mathcal{O}^1_{C,p},\mathcal{O}_{C,p}\right)\ . \tag{2.54}$$

Then a direct computation/diagram chase yields

$$\rho\left(xy = a\epsilon\right) = a\rho\left(xy=\epsilon\right)\ . \tag{2.55}$$

So in particular, $xy = a\epsilon$ and $xy = \epsilon$ are non-isomorphic for $a \neq 1$.

The picture here is that we have some loop on the general fiber of your deformation which collapses down to your node. So this parameter a controls how this cusp is formed. If you prefer to think metrically, this a is a scaling factor and rotation factor, i.e. $a \in \mathbb{C}^\times$. Putting this together with the calculation showing Ext^1 is 1-dimensional, we get that all isomorphism classes of infinitesimal deformations are of this form.

This concludes the proof of claim 2.2.3, □

which completes the proof of Theorem 2.2. ■

Proposition 2.3. *$H^1\left(C,\mathcal{H}\mathrm{om}\left(\Omega^1_C,\mathcal{O}_C\right)\right) \cong H^1\left(\tilde{C},T_{\tilde{C}}\left(-p_1-q_1-\ldots-p_r-q_r\right)\right)$ where the p_i,q_i are the preimages of the nodes $C^{sing} = \{x_1,\ldots,x_r\}$. The RHS classifies deformations of $\left(\tilde{C},p_1,q_1,\ldots,p_r,q_r\right)$.*

PROOF. It is enough to show that

$$\mathcal{H}\mathrm{om}\left(\Omega_C^1, \mathcal{O}_C\right) \cong T_{\tilde{C}}\left(-p_1 - \ldots - q_r\right) \ . \tag{2.56}$$

The idea is that $\Omega_C^1 = \mathcal{I}\omega_C$ where $\mathcal{I}$ is the ideal sheaf of C^{sing}. Locally near x_j,

$$\mathcal{I}\omega \cong \mathcal{I}\omega_{\tilde{C}_1}\left(-p_j\right) \oplus \mathcal{I}_{\tilde{C}_2}\left(-q_j\right) \ . \tag{2.57}$$

Then

$$\mathcal{I}_{x_j} = \nu_* \mathcal{I}_{(p_j \cup q_j)} \tag{2.58}$$

and we invoke duality on Hom. □

THEOREM 2.4. *Let C be a nodal curve. Then there is a deformation $\mathcal{C} \to (\Delta^s, 0)$ such that the Kodaira-Spencer map $\rho : T_0\left(\Delta^s\right) \to \mathrm{Ext}^1\left(\Omega_C^1, \mathcal{O}_C\right)$ is an isomorphism. From our short exact sequences we explicitly get that*

$$s = 3g - 3 + \dim \mathrm{Hom}\left(\Omega_C^1, \mathcal{O}_C\right) \tag{2.59}$$

$$= 3g - 3 + h^0\left(\tilde{C}, T_{\tilde{C}}\left(-p_1 - \ldots - q_r\right)\right) \tag{2.60}$$

$$= 3g - 3 \tag{2.61}$$

where the h^0 vanishes since C is stable.

PROOF. Glue Schiffer deformations at smooth points to $xy = a\epsilon$ deformations at the nodes. □

Lecture 13; February 19, 2020

2. Kuranishi families

We will follow [**1**, XI, §§4-6]. Recall the following definition.

DEFINITION 2.3. A deformation $\mathcal{X} \to (B, b_0)$ of X is a *Kuranishi family* if for any other deformation $\mathcal{X}' \to (B', b_0')$ and any sufficiently small neighborhood U of b_0', there is a unique morphism of deformations:

$$\begin{array}{ccc} \mathcal{X}'_U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ (U, b_0') & \longrightarrow & (B, b_0) \end{array} \ . \tag{2.62}$$

In particular this $\mathcal{X}'_U$ is the fiber product of $\mathcal{X}$ and U over B, so the deformation $\mathcal{X}'_U \to (U, b_0')$ is just $\mathcal{X} \to (B, b_0)$ pulled back along the map $(U, b_0) \to (U', b_0')$.

We can then make the following observations.

1. When a Kuranishi family exists then it is essentially unique (up to unique isomorphism). I.e. if $\begin{array}{c}\mathcal{X}\\ \downarrow \\ (B, b_0)\end{array}$ and $\begin{array}{c}\mathcal{X}'\\ \downarrow \\ (B', b_0')\end{array}$ are Kuranishi families, then for every sufficiently small neighborhood U of b_0 there is a unique neighborhood U' of b_0' and a unique isomorphism of deformations:

$$\begin{array}{ccc} \mathcal{X}_U & \xrightarrow{\simeq} & \mathcal{X}_{U'} \\ \downarrow & & \downarrow \\ (U, b_0) & \xrightarrow{\simeq} & (U', b_0') \end{array} \ . \tag{2.63}$$

2. The Kodaira Spencer map of any Kuranishi family

$$\rho : T_{B,b_0} \xrightarrow{\cong} \{\text{isomorphism classes of infinitesimal deformations of } X\} \tag{2.64}$$

is an isomorphism.
3. Suppose a Kuranishi family exists. Let $\mathcal{X} \to (B, b_0)$ be a deformation such that B is smooth at b_0, and such that the Kodaira Spencer map ρ is an isomorphism, then $\mathcal{X} \to (B, b_0)$ is Kuranishi. This follows from the universal property and some version of the implicit function theorem.
4. If $\mathcal{X} \to (B, b_0)$ is Kuranishi family for X and $\mathrm{Aut}\,(X)$ is finite, then $\mathrm{Aut}\,(X)$ acts on $\mathcal{X}_U \to (U, b_0)$ for a basis of neighborhoods U of b_0.

THEOREM 2.5. *Let C be a nodal curve. Then a Kuranishi family for C exists if and only if C is stable.*

REMARK 2.3. The analogous statement holds for nodal curves with marked points, but we will just go through the construction for unmarked curves.

Corollary 2.6. *The base of a Kuranishi family has local dimension $3g - 3$ (for a stable curve C with $p_a(C) = g$).*

Corollary 2.7. *If $\mathcal{C} \to (B, b_0)$ is Kuranishi for a nodal curve C then there is a neighborhood of b_0 such that $\mathcal{C}_U \to (U, x)$ is Kuranishi for all $x \in U$.*

The picture to have in mind here is that a Kuranishi family for such a curve C looks like an open patch in the moduli space of curves.

The main technical input of the proof of Theorem 2.5 is the notion of the Hilbert scheme of projective space $\mathbb{P}^n$ and Riemann-Roch. The Hilbert scheme is the moduli space of subschemes of $\mathbb{P}^N$ with fixed Hilbert polynomial. If we fix the Hilbert polynomial then it is a projective scheme. This was some of the early and very important work of Grothendieck [**5**]. See [**3**], [**10**, Part 2], and [**2**, Part 3, §6] for further reading.

PROOF OF THEOREM 2.5. First choose N such that $\omega_C^{\otimes N}$ is very ample for all stable curves C of genus g, (e.g. $N \geq 3$). Then notice that $\left|\omega_C^{\otimes N}\right|$ embeds C in $\mathbb{P}^{N'}$ with Hilbert polynomial p independent of C. Then we have open $U \subset \mathrm{Hilb}\left(\mathbb{P}^N, p\right)$ parametrizing stable curves embedded by $\left(\omega_C^{\otimes N}\right)$. Notice that the group $\mathrm{PGL}\,(N' + 1)$ acts on U.

FACT 3. *The stabilizer of a point x corresponding to the N-canonical embedding of a stable curve C is canonically isomorphic to* $\mathrm{Aut}\,(C)$.

Consider the PGL orbit through x. This is smooth of the same dimension as PGL. Write $G = \mathrm{Aut}\,(C)$. Then $G \subseteq \mathrm{PGL}$ acts as $\mathrm{Stab}\,(c)$, and $T_X\,(\mathrm{PGL} \cdot X)$ is G-invariant. Let $L \subseteq \mathbb{P}^K$ be a complementary G invariant linear space (where K is the dimension of the projective space which $\mathrm{Hilb}\left(\mathbb{P}^{N'}, p\right)$ lives).

The universal family of subschemes of $\mathbb{P}^{N'}$ over $U \cap L$ is Kuranishi for C. The picture is that $\mathrm{Hilb} \setminus U$ might have some higher-dimensional pieces parameterizing unstable curves, but we just want to intersection with U. The fact that this is Kuranishi comes from the universal property of the Hilbert scheme. $\square$

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