Moduli spaces and tropical geometry

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Spring 2020; Notes by: Jackson Van Dyke; All errors introduced are my own.

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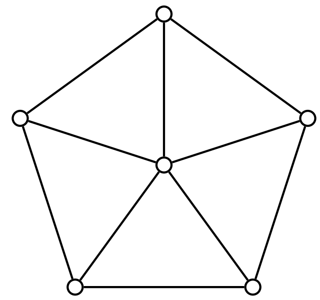

FIGURE 1. The 5-wheel.

1. Overview

Our goal is to understand the proof of the following theorem:

Lecture 1; Wednesday January 22, 2020

THEOREM 0.1. $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ *grows exponentially with* g.

REMARK 0.1. $\mathcal{M}_g$ has complex dimension $3g-3$.

This theorem defied previous expectations.

CONJECTURE 1 (Kontsevich (1993), Church-Farb-Putman (2014)). *For fixed* $k > 0$, $H^{4g-4-k}(\mathcal{M}_j, \mathbb{Q}) = 0$ *for* $g \gg 0$.

The structure of the course is as follows.

- Constructing the moduli space
  (1) Nodal curves and stable reduction theorem
  (2) Deformation theory of nodal curves
  (3) The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  (1) Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  (2) Dual complexes of normal crossings divisors (tropical geometry)
  (3) Boundary complex of $\mathcal{M}_g$ (tropical moduli space)
- Cohomology of $\mathcal{M}_h$
  (1) Stable cohomology (Madsen-Weiss 2007)
  (2) Virtual cohomological dimension of $\mathcal{M}_g$ (Harer 84) (Vanishing of H^{4g-5} (Church-Farb-Putman, Morita-Sakasai-Suzuki))
  (3) Euler characteristic of $\mathcal{M}_g$ (Harer-Zagier 86)
- Graph complexes (Kontsevich 93)
  (1) Feynman amplitudes and wheel classes. See Fig. 1 for the 5-wheel.
  (2) Grothendieck-Teichmüller Lie algebra
  (3) Willwacher's theorem
- Mixed Tate motives (MTM) over $\mathbb{Z}$
  (1) Mixed Tate motives
  (2) Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements/Drinfeld associators generate a free Lie subalgebra."
  (3) Proof of exponential growth of H^{4g-6}.