Moduli spaces and tropical geometry

Lectures by Sam Payne

Notes by: Jackson Van Dyke; All errors introduced are my own.

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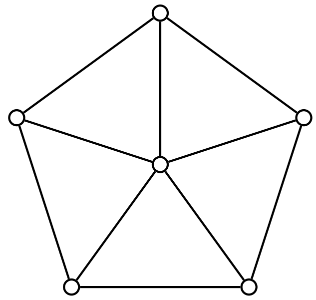


FIGURE 1. The 5-wheel.

1. Overview

Our goal is to understand the proof of the following theorem:

Lecture 1; Wednesday January 22, 2020

THEOREM 0.1. $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ *grows exponentially with* g.

REMARK 0.1. $\mathcal{M}_g$ has complex dimension $3g-3$.

This theorem defied previous expectations.

CONJECTURE 1 (Kontsevich (1993), Church-Farb-Putman (2014)). *For fixed* $k > 0$, $H^{4g-4-k}(\mathcal{M}_j, \mathbb{Q}) = 0$ *for* $g \gg 0$.

The structure of the course is as follows.

- Constructing the moduli space
  (1) Nodal curves and stable reduction theorem
  (2) Deformation theory of nodal curves
  (3) The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  (1) Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  (2) Dual complexes of normal crossings divisors (tropical geometry)
  (3) Boundary complex of $\mathcal{M}_g$ (tropical moduli space)
- Cohomology of $\mathcal{M}_h$
  (1) Stable cohomology (Madsen-Weiss 2007)
  (2) Virtual cohomological dimension of $\mathcal{M}_g$ (Harer 84) (Vanishing of H^{4g-5} (Church-Farb-Putman, Morita-Sakasai-Suzuki))
  (3) Euler characteristic of $\mathcal{M}_g$ (Harer-Zagier 86)
- Graph complexes (Kontsevich 93)
  (1) Feynman amplitudes and wheel classes. See fig. 1 for the 5-wheel.
  (2) Grothendieck-Teichmüller Lie algebra
  (3) Willwacher's theorem
- Mixed Tate motives (MTM) over $\mathbb{Z}$
  (1) Mixed Tate motives
  (2) Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements (closely related to Drinfeld's associators) generate a free Lie subalgebra."
  (3) Proof of exponential growth of H^{4g-6}.

Lecture 2; January 24, 2020

Part 1

Constructing the moduli space

CHAPTER 1

Nodal curves and stable reduction theorem

1. Nodal curves

We will work over $\mathbb{C}$. We want to show that nodal curves, and families thereof, can be written in a normal form in local coordinates. We will follow chapter X of [**1**].

Definition 1.1. A *nodal curve* is a complete curve such that every singular point has a neighborhood isomorphic (analytically over $\mathbb{C}$) to a neighborhood of 0 in $(xy = 0) \subset \mathbb{C}^2$.

Definition 1.2. A *family of nodal curves* over a base S is a flat proper surjective morphism $f : \mathcal{C} \to S$ such that every geometric fiber is a nodal curve.

Recall that a flat morphism is the agreed upon notion of a map for which the fibers form a continuously varying family of schemes (or complex analytic spaces, varieties, etc.). Properness is a relative notion of compactness; it ensures that if $\{c_i\}$ is a sequence of points with no limit in $\mathcal{C}$ then $\{f(c_i)\}$ has no limit in S.

Proposition 1.1. *Let $\pi : X \to S$ be a proper surjective morphism of $\mathbb{C}$-analytic spaces. This is a family of nodal curves if and only if at every point $p \in X$ either π is smooth at p with one-dimensional fiber, or there is a neighborhood of p that is isomorphic (over S) to a neighborhood of $(0, s)$ in $(xy = F) \subseteq \mathbb{C}^2 \times S$ where $s = \pi(p)$ and $F \in \mathfrak{m}_S \subseteq \mathcal{O}_{S,s}$.*

Lemma 1.2. *Let f be holomorphic at $0 \in \mathbb{C}^2$. Then $(f = 0)$ has a node at 0 if and only if*

$$0 = f = \frac{\partial f}{\partial x} = \frac{pf}{\partial y} \tag{1.1}$$

at 0, and the Hessian of f at 0 is non-singular.

This tells us that these nodes are the "simplest" possible singularities.

Proof. ($\Longrightarrow$): This direction is immediate.
($\Longleftarrow$): Suppose $0 = f = \partial_x f = \partial_y f$ at 0. Then

$$f = a - x^2 + 2bxy + cy^2 \tag{1.2}$$

where a, b, and c are holomorphic functions. The Hessian is

$$\begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix} \tag{1.3}$$

so being non-singular means exactly that

$$b^2 - ac \neq 0 \ . \tag{1.4}$$

After a generic linear change of coordinates, we can assume $a \neq 0$. We can then change coordinates to

$$x_1 = x + \frac{b}{a} y \qquad\qquad y_1 = y \ . \tag{1.5}$$

Then we can write

$$f = a_1 x_1^2 + c_1 y_1^2 \tag{1.6}$$

where $a_1(0), c_1(0) \neq 0$. Choose square roots[1.1] α and γ of a_1 and c_1. Now replace x_1 and y_1 by $x_2 = \alpha x_1$ and $y_2 = \gamma y_1$ so that

$$f = x_2^2 + y_2^2\ . \tag{1.7}$$

Now for $x_3 = x_2 + iy_2$ and $y_3 = x_2 - iy_2$, we have $f = x_3 y_3$. □

Proof of Proposition 1.1. Let $\pi\colon X \to S$ be proper and surjective. Consider $x \in X$. Then either π is smooth with 1-dimensional fiber at x (nothing to show) or x is a node in $\pi^{-1}(s)$, $s = \pi(x)$. Locally near x, we have a locally closed embedding $X \subseteq \mathbb{C}^r \times S$ (working over S). Then we get a left exact sequence of tangent spaces:

$$0 \longrightarrow T_x X_s \longrightarrow T_x X \longrightarrow T_s S \tag{1.8}$$

where $\dim T_x X_s = 2$. Choose a linear projection $\mathbb{C}^r \to \mathbb{C}^2$ which is an isomorphism on $T_x X_s$. Using this projection we get:

$$\begin{tikzcd} T_x X \arrow[r, phantom, "\subseteq"] \arrow[rr, bend right] & \mathbb{C}^r \times T_s S \arrow[r] & \mathbb{C}^2 \times T_s S \end{tikzcd} \tag{1.9}$$

and the composition $T_x X \to \mathbb{C}^2 \times T_s S$ is injective. The implicit function theorem then tells us that there is a neighborhood of x which embeds in $\mathbb{C}^2 \times S$ (over S). We should think of this as a family of plane curves: each fiber has a single defining equation. More specifically we have the following.

Fact 1 (Lemma 31.18.9 (Stacks project)). *If $\mathcal{Y} \to S$ is a smooth morphism and $D \subseteq \mathcal{Y}$ is flat over S, codimension 1 in $\mathcal{Y}$, then D is a Cartier divisor.*

In particular, $X \subseteq \mathbb{C}^2 \times S$ is locally defined by a single equation $F = 0$. Now consider $\partial_x F$, $\partial_y F$, and the Hessian of F with respect to x and y. Then the proof of Lemma 1.2 shows

$$F = x_3 y_3 - f \tag{1.10}$$

where f is a function on S which vanishes at s. □

Lecture 3; January 27, 2020

2. Stability of nodal curves

The following is a corollary of Proposition 1.1.

Corollary 1.3. *A family of nodal curves $\pi\colon \mathcal{C} \to S$ is a local complete intersection (lci) morphism.*

This implies that there is a relative dualizing sheaf $\omega_{\mathcal{C}/S}$ which is locally free of rank 1.

[1.1] There is some subtly here since these are functions rather than scalars. Because a_1 and c_1 are nonzero at 0, we can ensure that the image of a_1 and c_1 are, say, contained in an open half space. Now we can choose a branch of log which is defined on this half space. Then multiply by 1/2 and exponentiate.

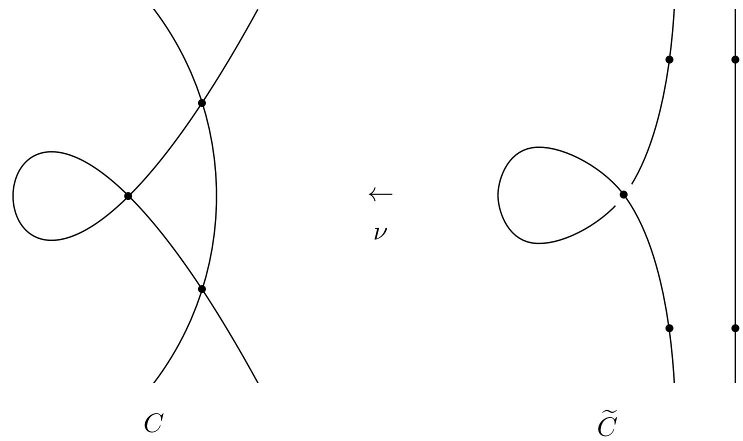


FIGURE 1. The normalization of a nodal curve. The nodal points of C each have two preimages under the normalization ν.

2.1. Serre duality. The point here is that the duality properties that we already know about for smooth curves extend naturally to nodal ones.

Let C be a nodal curve (over a point). There is a (natural) isomorphism $H^1(C, \omega_C) \cong \mathbb{C}$. Then Serre duality tells us that for any coherent sheaf $\mathcal{F}$ on C,

$$H^1(C, \mathcal{F}) \times \operatorname{Hom}(\mathcal{F}, \omega_C) \to H^1(C, \omega_C) \cong \mathbb{C} \tag{1.11}$$

is a perfect pairing, i.e.,

$$H^1(C, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{F}, \omega_C)^{\vee} \ . \tag{1.12}$$

In particular, if $\mathcal{F}$ is a vector bundle, then

$$H^1(C, \mathcal{F}) \cong H^0(C, \mathcal{F}^{\vee} \otimes \omega_C)^{\vee} \ . \tag{1.13}$$

We can form the normalization[1.2] of a nodal curve as in fig. 1.

Suppose C is nodal with components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Let $\widetilde{C} \xrightarrow{\nu} C$ be the normalization. Write $\widetilde{C}_i$ for the normalization of C_i and

$$\{p_j, q_j\} = \nu^{-1}(x_j) \tag{1.14}$$

(for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$).

A line bundle L on C has *multi-degree* $\underline{\deg}(L)$ to be

$$\underline{\deg}(L) = \left(\deg\left(L|_{C_1}\right), \ldots, \deg\left(L|_{C_s}\right)\right) \tag{1.15}$$

$$= \left(\deg\left(\nu^* L|_{\widetilde{C}_1}\right), \ldots, \deg \nu^* L|_{\widetilde{C}_s}\right) \ . \tag{1.16}$$

The following is a corollary to Serre duality.

Corollary 1.4. *If C is connected, and $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ then $H^1(C, L) = 0$.*

By $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ we mean $\deg\left(L|_{C_i}\right) \geq \deg\left(\omega_C|_{C_i}\right)$ for all i and $\underline{\deg}(L) \neq \underline{\deg}(\omega_C)$.

[1.2] Locally, the corresponding algebraic construction is taking the integral closure of the coordinate ring.

Proof. First note

$$H^1(C, L) \cong H^0\left(C, \omega_C \otimes L^{-1}\right) \ . \tag{1.17}$$

and $\underline{\deg}\left(\omega_C \otimes L^{-1}\right) < 0$.

On any connected component C_i such that $\deg \left(\omega_L \otimes L^{-1}\right)\big|_{C_i} < 0$ all sections vanish. And all sections vanish on components that meet C_i, etc. $\square$

Corollary 1.5. *L is ample if and only if* $\deg\left(L|_{C_i}\right) > 0$ *for all i.*

Proof. ($\Longrightarrow$): This direction is clear. The restriction of ample L to any component is still ample.

($\Longleftarrow$): Suppose $\deg\left(L|_{C_i}\right) > 0$. It is enough to show that $L^{\otimes N}$ is very ample for some N. Choose N sufficiently large so that

$$\deg\left(L^{\otimes N}\big|_{C_i}\right) > \deg\left(\omega_C|_{C_i}\right) + 2 \ . \tag{1.18}$$

Let $S \subseteq C$ be the union of two distinct smooth points. Then we have a short exact sequence

$$0 \to I_S \to \mathcal{O}_C \to \mathcal{O}_S \to 0 \tag{1.19}$$

which we can tensor with $L^{\otimes N}$ to get a sequence which is still exact, which gives us a long exact sequence

(1.20)

$$0 \longrightarrow H^0\left(L^{\otimes N}(-s)\right) \longrightarrow H^0\left(L^{\otimes N}\right) \longrightarrow H^0\left(L^{\otimes N}\big|_S\right) \longrightarrow H^1\left(L^{\otimes N}(-s)\right) \longrightarrow \dots$$

but $H^1(L(-s)) = 0$, so we have a surjection

$$H^0(L) \twoheadrightarrow H^0\left(L|_S\right) \ . \tag{1.21}$$

This shows that sections of $L^{\otimes N}$ separate the two points in S. Similar arguments show that sections of high tensor powers of L separate arbitrary pairs of points and tangent vectors. Therefore, high tensor powers of L are very ample, and so L is ample. $\square$

3. Description of ω_C

We now describe the canonical sheaf of a nodal curve in terms of meromorphic differential forms. See [**11**, Chapter 6] or [**8**, Chapter 3, Section A] for proofs and further details.

Proposition 1.6. *Let C be a nodal curve with nodes $x_1, \dots, x_r$, write $(p_i, q_i) = \nu^{-1}(x_i)$. Then*

$$\omega_C \cong \nu_*\left(\omega'_{\widetilde{C}}(p_1 + q_1 + \dots + p_r + q_r)\right) \tag{1.22}$$

where $\omega'_{\widetilde{C}}(p_1 + \dots + q_r) \subseteq \omega_{\widetilde{C}}(p_1, \dots, q_r)$ is the subsheaf where

$$\operatorname{res}_{p_i}(\omega) + \operatorname{res}_{q_i}(\omega) = 0 \ . \tag{1.23}$$

Remark 1.1 (Rosenlicht differentials). There is a related explicit description of $\omega_{X/S}$ for a family of nodal curves. Near a point where $X/S \cong (xf = F) \subseteq \mathbb{C}^2 \times S$ $\omega_{C/S}$ is generated by dx/x and dy/y which satisfy

$$\frac{dx}{x} + \frac{dy}{y} = 0 \ . \tag{1.24}$$

Definition 1.3. A nodal curve is *stable* if ω_C is ample.

Lecture 4; January 29, 2020

Proposition 1.7. *Let $X \to S$ be a family of nodal curves. Then*

$$\{s \in S \,|\, X_s \text{ is stable}\}$$

is Zariski open.

Proof. Let L be any line bundle on X. Then

$$\left\{s \in S \,\middle|\, L|_{X_s} \text{ is ample}\right\}$$

is Zariski-open. This is Theorem 1.2.17 of [**10**]. □

Theorem 1.8. *A nodal curve C is stable if and only if* $\operatorname{Aut}(C)$ *is finite.*

Proof. Say C has components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Write $\{p_i, q_i\} = \nu^{-1}(x_i)$ for the preimage of the nodes under the normalization ν. Write $Q = \{p_1, q_1, \ldots, p_r, q_r\}$. Notice that $\operatorname{Aut}(C)$ is finite if and only if

$$\{\sigma \in \operatorname{Aut}(C) \,|\, \sigma \text{ acts by 1 on } \{C_1, \ldots, C_s\}\}$$

is finite.

Fix C_i. Note that $\operatorname{Aut}(C_i)$ is finite if and only if there are only finitely many automorphisms of $\tilde{C}_i$ that fix $Q \cap \tilde{C}_i$. This is the case exactly when

(1) $g\left(\tilde{C}_i\right) \geq 2$;
(2) $g\left(\tilde{C}_i\right) = 1$, and $Q \cap \tilde{C}_i \neq \emptyset$; or
(3) $g\left(\tilde{C}_i\right) = 0$ and $Q \cap \tilde{C}_i \geq 3$.

By direct computation, these are precisely the cases where

$$2g\left(\tilde{C}_i\right) - 2 + \#\left(Q \cap \tilde{C}_i\right) > 0.$$

The left hand side is $\deg\left(\omega_C|_{C_i}\right)$, by our description of the dualizing sheaf in terms of meromorphic differentials.

So we have shown that $\operatorname{Aut}(C)$ is finite if and only if the degree of the dualizing sheaf is positive on every component, which is equivalent to ω_C being ample, i.e., to C being stable. □

Definition 1.4. A *graph* G is a set $X(G)$ together with an involution $i : X(G) \circlearrowleft$ and a retraction $r\colon X(G) \to X(G)^i$. The vertices $V(G)$, half edges $H(G)$, and edges $E(G)$ are defined as:

$$\begin{aligned} V(G) &= X(G)^i \\ H(G) &= X(G) \setminus V(G) \\ E(G) &= H(G)/i\ . \end{aligned}$$

We say $r(h)$ is the vertex incident to $h \in H(G)$.

The *dual graph* $G(C)$ of a nodal curve C is as follows. The vertices $\{v_1, \ldots, v_s\}$ correspond to the components $C_1, \ldots, C_s$; and the half-edges incident to v_i are given by the points of $\tilde{C}_i \cap Q$. An edge is made from a pair of half-edges corresponding to a pair $\{p_i, q_i\}$. The "genus function" assigns the genus of $\tilde{C}_i$ to the corresponding vertex v_i. See fig. 2 for examples.

We can read the stability off from the dual graph. Every vertex labelled with a 1 should have at least one incident edge, and all unlabelled vertices should have valence at least 3.

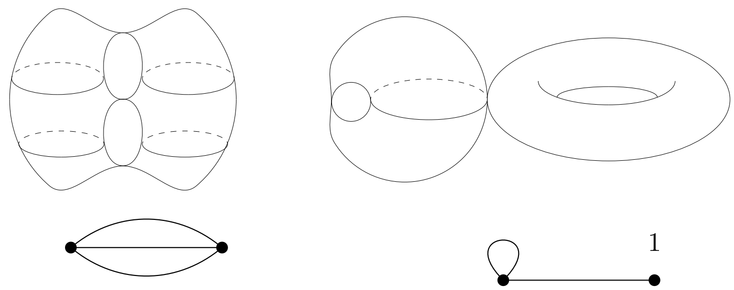


FIGURE 2. Two examples of genus 2 stable curves with their dual graphs below them. Notice we can read their stability off from the graphs. All unlabelled vertices have at least three incident edges, and the labelled one has one incident edge.

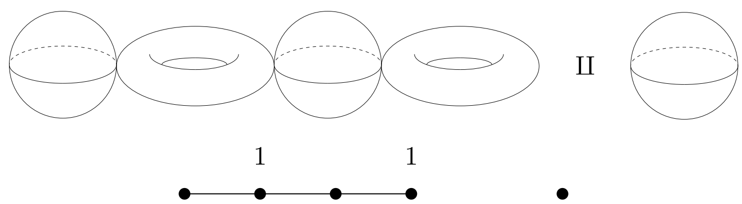


FIGURE 3. An example of an unstable genus 2 curve with its dual graph below it. Notice we can read the fact that it is unstable off of the graph. All three unlabelled vertices of valence less than 3.

Recall that the arithmetic genus of a curve C is

$$p_a(C) = 1 - \chi(\mathcal{O}_C) \ .$$

In particular, if C is connected then $p_a(C) = h^1(\mathcal{O}_C)$. Recall the Euler characteristic of a graph G is

$$\chi(G) = \#V(G) - \#E(G) \ .$$

Note if G is connected, then $h^1(G) = 1 - \chi(G)$. Also note that C is connected if and only if $G(C)$ is connected. The dual graph also detects the arithmetic genus in the following sense.

THEOREM 1.9. *Let C be a nodal curve. Then*

$$p_a(C) = 1 - \chi(G(C)) + \sum_v g(v) \ . \tag{1.25}$$

Corollary 1.10. *If C is connected then*

$$p_a(C) = \sum_v g(v) + h^1(G) \ . \tag{1.26}$$

PROOF OF THEOREM 1.9. Proceed by induction on the number of nodes $\#E(G) = \#C^{\mathrm{sing}}$. The base case is when $E(G) = \emptyset$, so the graph is just s vertices v_i with genus $g(v_i)$. Then

$$1 - \chi(\mathcal{O}_C) = 1 - s + \sum_i g(v_i) \tag{1.27}$$

as desired.

Now suppose C' is obtained from C by gluing two smooth points p, q to x. Write $\pi\colon C \to C'$. Then we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \pi_* \mathcal{O}_C \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.28}$$

which implies the Euler characteristic of the middle term is the sum of the Euler characteristics of the other two terms. Now since π is proper and finite, $\chi(\pi_*\mathcal{O}_C) = \chi(\mathcal{O}_C)$. Altogether this gives us:

$$\chi(\mathcal{O}_C) = \chi(\pi_*\mathcal{O}_C) = \chi(\mathcal{O}_{C'}) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C'}) + 1\ , \tag{1.29}$$

and the theorem follows. □

Lecture 5; January 31, 2020

4. Stable reduction

There are two statements. The first is the nodal reduction theorem (which does not involve stability) and the second is stabilization, which adds uniqueness. The reference is [**1**] Chapter X, Section 4. Write

$$\Delta = \{z \in \mathbb{C} \mid |z| < \epsilon\} \tag{1.30}$$

for a small disk. Write $\Delta^\times = \Delta \setminus \{0\}$ for the punctured disk, both viewed as having one complex dimension.

Consider a flat proper surjective map $\pi\colon X \to \Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Write $X^\times$ for the complement of the fiber over 0. Let $k > 0$ be an integer. Consider the map $\varphi_k\colon \Delta' \to \Delta$ from the disk to itself given by $z \mapsto z^k$. Note that φ_k is *not* a smooth map. Now we can construct a base change

$$\begin{tikzcd} X_k^\times \coloneqq X^\times \times_{\varphi_k} \Delta'^\times \arrow[r] \arrow[d, "\pi'"] & X^\times \arrow[d, "\pi"] \\ \Delta'^\times \arrow[r, "\varphi_k"] & \Delta^\times \end{tikzcd} \ . \tag{1.31}$$

Theorem 1.11 (Nodal reduction theorem). *Let $\pi\colon X \to \Delta$ be a flat proper surjective map such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Then there exists an integer $k > 0$ such that after a base change as above, the map π' extends to a family of nodal curves over Δ.*

Theorem 1.12 (Stable reduction). *If $\pi|_{\Delta^\times}$ is* stable*, then this extension can be chosen to be stable, and the fiber over 0 depends* only *on $\pi|_{\Delta^\times}$ up to isomorphism.*

Remark 1.2. Uniqueness is related to separatedness for moduli of stable curves; existence and uniqueness is related to properness.

Remark 1.3. The intuition is as follows. Let Σ be a class of objects with a moduli space (or stack) $\mathcal{M}$, i.e., there is a universal family $\mathcal{I} \to \mathcal{M}$ of objects in Σ such that any family $X \to S$ of objects in Σ is the pullback of the universal family under a unique morphism $S \to \mathcal{M}$. In other words,

$$\mathrm{Hom}(-, \mathcal{M}) \cong \{\text{families of } \Sigma \text{ objects over } -\}\ . \tag{1.32}$$

If $\mathcal{M}$ is separated, i.e., Hausdorff, then for $\Delta^\times \to \mathcal{M}$ there exists at most one extension $\Delta \to \mathcal{M}$. If $\mathcal{M}$ is proper, then each map $\Delta^\times \to \mathcal{M}$ extends uniquely to $\Delta \to \mathcal{M}$. Roughly speaking, when one has a large class of objects with a moduli space $\mathcal{M}'$ such that maps $\Delta^\times \to \mathcal{M}'$ extend in many different ways to $\Delta \to \mathcal{M}'$ then one naturally looks for a stability

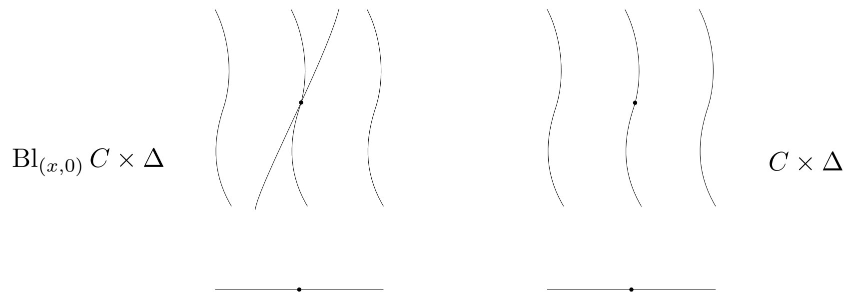


FIGURE 4. The constant family $\pi\colon C \times \Delta \to \Delta$ as well as the blowup $\mathrm{Bl}_{(x,0)}\, C \times \Delta \to \Delta$.

condition on the parametrized objects, so that the subspace $\mathcal{M} \subset \mathcal{M}'$ parametrizing stable objects is open and proper.

The notion of stability for nodal curves is a prototypical example. Indeed, if we don't impose stability, given a family of nodal curves $X^\times \to \Delta^\times$, then it may extend in many different ways to a nodal family $X' \to \Delta$ (and will always extend in many different ways, after a totally ramified base chance $\Delta \to \Delta$, given by $z \mapsto z^k$). So the existence and uniqueness of the special fiber in the theorem above is a special consequence of our specified stability condition.

EXAMPLE 1.1. Consider a smooth curve $C = (f = 0) \subseteq \mathbb{P}^2$. Then $C \times \Delta^\times \to \Delta^\times$ is a constant family which extends to $C \times \Delta \to \Delta$. Now for any $x \in C$, $C \times \Delta^\times$ also extends to $\mathrm{Bl}_{(x,0)}\, C \times \Delta$. We can picture this as in fig. 4.

The upshot is that moduli of nodal curves are not separated/Hausdorff. Lecture 6; February 3, 2020

Interlude: Some motivating examples.

Degeneration of a smooth curve to a nodal curve. We should think of the total space as being a surface. Consider the surface in fig. 5. This has two different rulings, as pictured in fig. 5. As in fig. 5, we can project this surface to a line by taking the intersection with parallel planes at different points of the line. Generically this gives us hyperbolas, but for two special values we get the union of two lines from the two different rulings. In particular this is given by the equation $xy = t^2 - t$. The node is exactly the point of tangency. So when we have a non-reduced curve, this is singular at every point on the curve.

Degeneration of a smooth curve to a non-reduced curve. Consider the surface defined by the equation $x^3 + t\,(x + y + 1) = 0$. At $t = 0$ we just get a line with multiplicity 3. This looks something like fig. 6.

Understanding the base change and its fibers. Again we consider a flat proper surjective map $\pi\colon X \to \Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. For simplicity assume that in fact $X = \mathbb{P}^1 \times \Delta$. Consider the map $\varphi_k\colon \Delta' \to \Delta$ from the disk to itself given by $z \mapsto z^k$. Note that φ_k is *not* a smooth map. In particular:

$$\varphi_k^{-1}(0) = \operatorname{Spec}\left(\mathbb{C}\,[\epsilon]\,/\epsilon^k\right) \ . \tag{1.33}$$

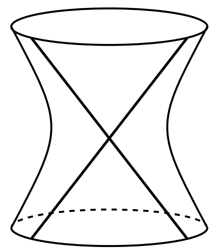


FIGURE 5. The surface given by $xy = t^2 - t$. Projection to the t-line has fibers which generically look like hyperbolas, but when the plane is tangent to the surface we get the union of two lines.

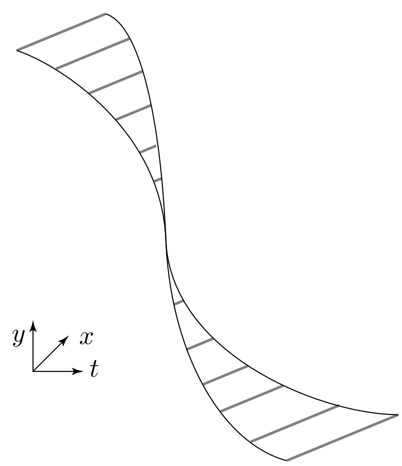


FIGURE 6. The surface $x^3+t\,(x+y+1) = 0$. Projecting to the t-line gives us smooth fibers which degenerate to a line with multiplicity 3 at $t = 0$.

Consider a base change for a family of curves

$$\begin{tikzcd} \left(\mathbb{P}^1 \times \Delta\right) \times_{\varphi_k} \Delta' \arrow[d, "\pi'"] & \mathbb{P}^1 \times \Delta \arrow[d, "\pi"] \\ \Delta' \arrow[r, "\varphi_k"] & \Delta \end{tikzcd} \quad . \tag{1.34}$$

If we think of the preimage under $\varphi_k \circ \pi'$ we have actually made things worse, since the preimage of 0 is:

$$(\varphi_k \circ \pi)^{-1}(0) \simeq \mathbb{P}^1 \times \mathrm{Spec}\left(\mathbb{C}[\epsilon]/\epsilon^k\right) \ . \tag{1.35}$$

But in our construction we are replacing π by π', not $\varphi_k \circ \pi'$. The moral is that (at least for specific k) this replacement makes things better.

Proof of the nodal reduction theorem.

Proof of Theorem 1.11. We will operate under the simplifying assumption that $X^\times \to \Delta^\times$ is smooth. The first step is to resolve the singularities of X. This is easy since $\dim X = 2$. First we normalize to get something regular in codimension 1. Then we blowup the finitely many singular points. Then repeat, i.e., normalize and then blowup the finitely many singular points. It is a theorem (not especially difficult) that this process terminates, giving us $X' \xrightarrow{\pi'} \Delta$ where X' is smooth. However, the central fiber $(\pi')^{-1}(0) = X'_0$ might have arbitrary singularities. To deal with this, we first resolve the non-nodal singularities of $X_0'^{\mathrm{red}}$. The process is again straightforward; the reduced curve $X_0'^{\mathrm{red}}$ has finitely many singular points. We blow up the singular points that are not nodes. The resulting total space is still smooth, and we repeat, blowing up the finitely many singular points of the reduced special fiber that are not nodes. It is again a theorem (and not particularly difficult) that this process terminates. Hence we may assume that X' is smooth and $X_0'^{\mathrm{red}}$ has only nodal singularities.

Locally near each node of $X_0'^{\mathrm{red}}$, the surface X' is isomorphic (over Δ) to $z = x^a y^b$ in $\mathbb{C}^2 \times \Delta$, where x and y are the coordinates on $\mathbb{C}^2$ and z is the coordinate on Δ. Similarly, near each smooth point of ${X'}_0^{\mathrm{red}}$, the surface X' is isomorphic (over Δ) to $z = x^c$. We can then cover ${X'}_0^{\mathrm{red}}$ by finitely many open sets where we have such local charts, and set

$$k = \operatorname{lcm}\{ab, c\} \ . \tag{1.36}$$

The rough idea is that this choice of k will ensure that base change along $\varphi_k\colon \Delta \to \Delta$, given by $z \mapsto z^k$, will unwind the multiplicities of the components of X'_0.

In fact, the base change

$$X'' = \varphi_k^* X' \tag{1.37}$$

is not necessarily normal, but we claim that

Claim 1.1. $(X'')^\nu \xrightarrow{\pi'} \Delta'$ is a nodal family, where $(X'')^\nu \to X''$ is the normalization.

To prove the claim, we first consider π' near a point where $X' \cong (z = x^c)$. Write $z = \zeta^k$ and $k = ch$ so that

$$x^c - z = x^c - \zeta^{ch} = \prod_{\omega^c = 1} \left(x - \omega\zeta^k\right) \ . \tag{1.38}$$

Note that this product gives rise to c different smooth and irreducible components, which are disjoint in the general fiber but intersect in the special fiber. Normalizing pulls apart the intersections in the special fiber, giving rise to the disjoint union $\amalg_{\omega^c=1}\left(x - \omega\zeta^h\right)$, which is smooth over Δ.

It remains to consider π' near a point where $X' \cong \left(z = x^a y^b\right)$. Write $k = rsuv$ where $a = ru$, $b = su$, and $(r, s) = 1$. Write ζ for the coordinate on Δ'. Then X'' is locally given by

$$0 = x^a y^b - \zeta^k \ . \tag{1.39}$$

This need not be normal. Indeed, if $u > 1$ then $x^r y^s$ obviously satisfies a nontrivial monic polynomial. Choose ω a primitive uth root of unity, so we have a factorization

$$\left(x^a y^b - \zeta^k\right) = \prod_{i=1}^{u} \left(x^r y^s - \omega^i \zeta^{rsv}\right) \ . \tag{1.40}$$

We can again pass to the disjoint union of surfaces with local defining equations $x^r y^2 - \omega^i z^{rsv}$, but this is only a partial normalization. Indeed, these surfaces are all isomorphic, but $\zeta^{vrs} = x^r y^s$ need not be normal. Then we claim the following.

Claim 1.2. The normalization is locally isomorphic to the surface defined by $\zeta^v = \alpha\beta$, where ζ, α, β are coordinates on $\mathbb{C}^3$, with the normalization map given by $x = \alpha^s$, $y = \beta^r$.

To check that this is the normalization we need to check that

(1) this surface is normal,
(2) the map is generically one-to-one, and
(3) the map is surjective.

To see that this surface normal, notice that $\zeta^v = \alpha\beta$ is the toric surface corresponding to the cone spanned by $(1,0)$ and $(v,1)$ in $\mathbb{R}^2$, with respect to the standard lattice $\mathbb{Z}^2$. It is well-known and easy to prove that toric varieties are normal (see [**6**, §2.1]). We now show that the map is generically one-to-one. Given (α, β, ζ) and $(\alpha', \beta', \zeta')$ so that

$$\alpha^s = (\alpha')^s \qquad \beta^r = (\beta')^r \qquad \zeta = \zeta' \ . \tag{1.41}$$

This means $\alpha' = \sigma\alpha$ for σ an sth root of unity, and similarly $\beta' = \tau\beta$ for τ an rth root of unity. But if α and β are nonzero, then $\alpha\beta = \alpha'\beta'$ implies $\sigma\tau = 1$, so $\sigma = \tau = 1$, so

$$(\alpha, \beta, \zeta) = (\alpha', \beta', \zeta') \ . \tag{1.42}$$

Since the points where α and β are nonzero form an open dense set we are done.

Now consider (x, y, ζ) such that $x^r y^s = \zeta^{vrs}$. Then we must find points (α, β, ζ) such that $\alpha\beta = \zeta^v$ and $x = \alpha^s$, and $y = \beta^r$. Choose α_0, β_0 such that $\alpha_0^s = x$ and $\beta_0^r = y$. The point being that $\alpha_0 \cdot \beta_0 = \xi\zeta^v$ where $\xi^{rs} = 1$. Now write

$$1 = mr + ns \tag{1.43}$$

so the coordinates are

$$\alpha = \alpha_0 \xi^{-mr} \qquad \beta = \beta_0 \xi^{-ns} \ . \tag{1.44}$$

□

Now we claim that X' in the nodal reduction theorem can be chosen to be stable if $X|_{\Delta^\times}$ is stable. Lecture 7; February 5, 2020

Theorem 1.13 (Stabilization theorem). *Let $X \xrightarrow{\pi} \Delta$ be a family of nodal curves such that $\pi|_{\Delta^\times}$ is stable. Then there is*

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr] & & X' \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{1.45}$$

such that

(1) $\psi\colon X|_{\Delta^\times} \to X'|_{\Delta^\times}$ is an isomorphism;
(2) for each component C_i of the central fiber $C = X_0$, ψ maps C_i either to a point, or birationally onto its image; and
(3) X' is a family of stable curves.[1.3]

Moreover, $X' \to \Delta$ is unique.

Remark 1.4. The moral of the story is that

$$X' = \mathrm{Proj}_\Delta \left(\bigoplus_{n \geq 0} \pi_* \left(\omega_{X/\Delta}^{\otimes n} \right) \right) \ . \tag{1.46}$$

[1.3] This means that $X' \to \Delta$ is flat and proper, and its geometric fibers are stable nodal curves.

Recall that when we take this big direct sum we get a sheaf of graded $\mathcal{O}_\Delta$-algebras, so it makes sense to take relative Proj_Δ, provided that these graded $\mathcal{O}_\Delta$-algebras are finitely generated. The minimal model program deals with finite generation of things like this.

Proof. Suppose $C = X_0$ with components $C_1, \ldots, C_s$. Consider

$$\left\{C_i \,\middle|\, \omega_C|_{C_i} \text{ is not ample}\right\} = \left\{C_i \,\middle|\, \deg\left(\omega_C|_{C_i}\right) \leq 0\right\} \ . \tag{1.47}$$

Call this set the set of unstable components. We continue with our simplifying assumption that $X|_{\Delta^\times}$ is smooth and with connected fibers. Note that stability of the general fiber (in the absence of marked points) implies that $p_a(C) \geq 2$. Then the set of unstable components is:

$$\left\{C_i \,\middle|\, \omega_C|_{C_i} \text{ is not ample}\right\} = \left\{C_i \cong \mathbb{P}^1 \,\middle|\, \#\left\{C_i \cap \mathrm{Cl}\left(\left(C \setminus C_i\right)\right)\right\} \leq 2\right\} \ . \tag{1.48}$$

Then we have the following observation from [**1**]. Each connected component in the union of the unstable components is a chain of rational curves that intersects the union of the stable components at one or two points, on either or both ends of the chain. Let C' be the curve obtained by contracting all unstable chains. Note that $p_a(C') = p_a(C)$.

Warning 1.1. Now we encounter a minor error in [**1**] (page 112, second sentence), where it is claimed that C' is stable. The following is a counterexample to that claim.

Counterexample 1. Suppose C has the following dual graph:

$$\begin{tikzcd} & & \bullet \arrow[r, dash] & \bullet \\ 2\,\bullet \arrow[r, dash] & \bullet \arrow[ur, dash] \arrow[dr, dash] & & \\ & & \bullet & \end{tikzcd}$$

with three unstable components (in red), that form two chains. After contracting both unstable chains, we get C' with dual graph

$$2\,\bullet \text{ --- } \bullet$$

which is not stable.

Nevertheless, the argument in [**1**] is easily salvaged. By iterating the procedure of contracting chains of unstable rational curves, one eventually does obtain a map $\varphi\colon C \to C'$ such that

(i) $\varphi|_{C_i}$ is either constant or birational onto its image (and an isomorphism on $C_i \cap C^{\text{smooth}}$).
(ii) $p_a(C') = p_a(C)$, and
(iii) C' is stable.

Now, given $\varphi\colon C \to C'$ as above, with C' stable, we follow the arguments in [**1**] to construct Lecture 8; February 7, 2020

$$\begin{tikzcd} X \arrow[rr, "\varphi'"] \arrow[dr, "\varphi"'] & \circlearrowleft & X' \arrow[dl, "\pi'"] \\ & \Delta & \end{tikzcd} \tag{1.49}$$

such that

(i) $\pi'\colon X' \to \Delta$ is a family of stable nodal curves,
(ii) φ' is an isomorphism over $\Delta^\times$,
(iii) $X'_0 \cong C'$, and

(iv) $\varphi'|_{X_0} = \varphi$.

Let L_0 be $\varphi^* \omega_{C'}$ and $\underline{d} = \underline{\deg}\,(L_0)$, i.e., $\underline{d} = (d_1, \ldots, d_s)$, where $d_i = \deg\left(L_0|_{C_i}\right)$. Note all $d_i \geq 0$. Choose d_i sections of π that meet C_i at distinct smooth points of C. (We can find a section through an arbitrary smooth point of C, using Hensel's lemma.) Let D be the divisor on X given by the union of these sections. Then $L = \mathcal{O}\,(D)$ is a line bundle on X, and

$$\underline{\deg}\left(L|_{X_0}\right) = \underline{\deg}\,(L_0)\ . \tag{1.50}$$

Then we make the following observations:

- L is relatively ample over $\Delta^{\times}$,
- $L|_{X_0}$ is the pullback of an ample line bundle L' on C'.

Lemma 1.14. *For any line bundle M' on C',*

$$H^i\left(C', M'\right) = H^i\left(C, \varphi^* M'\right) \tag{1.51}$$

(for $i \in \{0, 1\}$).

Proof. The pullback induces an isomorphism on H^0, and

$$\begin{aligned}\chi\,(M') &= \chi\,(\mathcal{O}_{C'}) + \deg\,(M') \\ &= \chi\,(\mathcal{O}_{C}) + \deg\,(\varphi^* M') \\ &= \chi\,(\varphi^* M')\ .\end{aligned}$$

□

The consequences are as follows. For large n, $H^1\left(X_0, L^{\otimes n}\right) = 0$ (vanishing on C' by ampleness and Lemma 1.14). This implies $h^0\left(X_s, L^{\otimes n}\right)$ is a constant function of $s \in \Delta$. Therefore $\pi_* L^{\otimes n}$ is locally free by Grauert's theorem.[1.4]

Now we choose n sufficiently large such that $L^{\otimes n}$ is very ample on fibers over $\Delta^{\times}$, and the restriction of $L^{\otimes n}$ to C is the pullbacks of a very ample line bundle on C'. Then $\pi_* L^{\otimes n}$ induces $\psi\colon X \to \Delta \times \mathbb{P}^N$, and $\psi|_C$ agrees with $\varphi\colon C \to C'$. Take $X' = \mathrm{im}\,(\psi)$. Note that $X' \to \Delta$ is flat by the Hilbert polynomial criterion, and hence is the required family of stable nodal curves. ■

Definition 1.5. An *n-pointed nodal curve* is a pair $(X; p_1, \ldots, p_n)$ such that X is a nodal curve, and $p_1, \ldots, p_n$ are distinct smooth points of X.

Definition 1.6. We say $(X; p_1, \ldots, p_n)$ is *stable* if and only if $\omega_X\,(p_1 + \ldots + p_n)$ is ample.

Theorem 1.15. *$(X; p_1, \ldots, p_n)$ is stable if and only if*

$$\mathrm{Aut}\,(X; p_1, \ldots, p_n) = \{\sigma \in \mathrm{Aut}\,(X) \mid \sigma\,(p_i) = p_i\} \tag{1.52}$$

is finite.

Definition 1.7. A *family of pointed nodal curves* is a family of nodal curves $\pi\colon X \to S$ with sections $\sigma_1, \ldots, \sigma_n$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_1"'] \arrow[u, bend right=60, "\sigma_n"'] \end{tikzcd} \tag{1.53}$$

[1.4] Recall this says that if the dimension of H^i is constant, the sheaf is coherent, and the morphism is proper, the $R^i \pi_*$ is locally free. See Chapter III of [**9**].

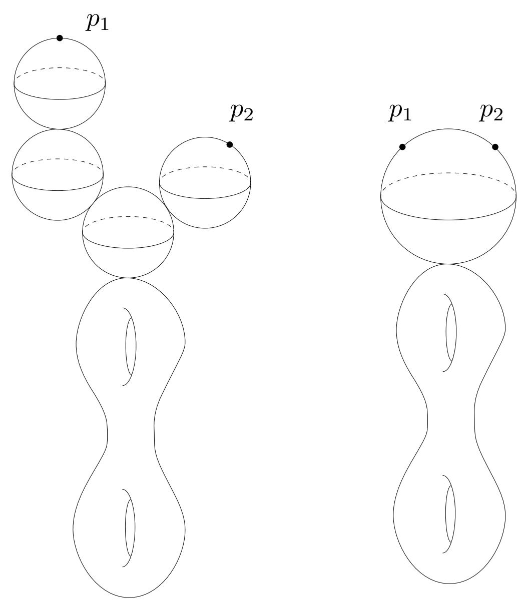


FIGURE 7. The left curve is unstable. When we stabilize, we contract to get a stable curve as on the right. Note that the marked points follow the contraction.

such that $\{\sigma_i(S)\}$ are disjoint and contained in π^{smooth}.

Then there are generalizations of nodal reduction, stabilization, and stable reduction for pointed curves as well. Note that, when we contract during the stabilization process, the marked points follow the contraction. See fig. 7.

An argument similar to the construction of X' above shows that stabilization of nodal curves behaves well in families, i.e., given a family of nodal curves $\mathcal{C} \to S$ there is morphism of families of nodal curves $\mathcal{C} \to \mathcal{C}'$ over S such that $\mathcal{C}'$ is a family of stable nodal curves and the restriction to a fiber C is the stabilization map $\varphi\colon C \to C'$ obtained by contracting chains of unstable rational curves, and then iterating.

Lecture 9; February 10, 2020

CHAPTER 2

Deformation theory

The reference for today's material is [**1**, Chapter XI, section 2].

Definition 2.1. A *deformation* of a proper (connected) scheme X is a flat and proper morphism $\mathcal{X} \xrightarrow{\varphi} S$ to a pointed scheme (S, s) together with an isomorphism $\mathcal{X}_s \xrightarrow{\sim} X$.

An *infinitesimal deformation* is a deformation over $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$.

Sometimes these infinitesimal deformations are referred to as *first order deformations.*

A morphism of deformations is a cartesian square

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ (S,s) \arrow[r] & (S',s') \end{tikzcd} \tag{2.1}$$

such that the induced map

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rrr, bend right] & \mathcal{X}_s \arrow[r] & \mathcal{X}'_{s'} \arrow[r, "\sim"] & X \end{tikzcd} \tag{2.2}$$

is the identity.

Theorem 2.1. *If X is smooth then the isomorphism classes of infinitesimal deformations of X are in natural bijection with $H^1(X, T_X)$.*

Proof. The first step is to find a natural map from the isomorphism classes of infinitesimal deformations to $H^1(X, T_X)$. Let $\mathcal{X} \to S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$ be an infinitesimal deformation. Since X is smooth and smoothness is open in families, the morphism $\mathcal{X} \to S$ is smooth, and gives rise to the short exact sequence

$$0 \to T_X \to T_{\mathcal{X}} \to \varphi^* T_S \to 0\ . \tag{2.3}$$

This induces a long exact sequence on cohomology:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, T_X) \arrow[r] & H^0(X, T_{\mathcal{X}}) \arrow[r] & H^0(X, \varphi^* T_S) \arrow[r, "\delta"] \arrow[d, equal] & H^1(X, T_X) \arrow[r] & \cdots \\ & & & \mathbb{C} \cdot d\epsilon & & \end{tikzcd} \tag{2.4}$$

so $d\epsilon \in H^0(X, \varphi^* T_S)$ maps to some class $\delta(d\epsilon) \in H^1(X, T_X)$. We claim that the map taking an infinitesimal deformation $\mathcal{X} \to S$ to $\delta(d\epsilon)$ gives the required bijection.

Let $\mathcal{X} \to S$ be an infinitesimal deformation, $\mathcal{X}_0 \xrightarrow{\sim} X$. Note that $\mathcal{O}_{\mathcal{X}}$ is locally free (of rank 2) as an $\mathcal{O}_X$-module. Now we cover $\mathcal{X}$ by finitely many open U_α such that $\mathcal{O}_{\mathcal{X}}|_{U_\alpha}$ is free. Let $z_{\alpha_1}, \ldots, z_{\alpha_n}$ be local coordinates on these $U_{\alpha_i} \subseteq X$, and let $f_{\alpha\beta}$ be the transition functions, i.e., $z_\alpha = f_{\alpha\beta} z_\beta$. These functions satisfy

$$f_{\alpha\beta}\left(f_{\beta\gamma}\left(z_\gamma\right)\right) = f_{\alpha\gamma}\left(z_\gamma\right) \ . \tag{2.5}$$

Now consider $\mathcal{X}$ as being glued from the $U_\alpha \times S$. In particular $U_\alpha \times S$ is glued to $U_\beta \times S$ along $(U_\alpha \cap U_\beta) \times S$. So we have z_α and ϵz_α, and

$$z_\alpha = \underbrace{f_{\alpha\beta}\left(z_\beta\right) + \epsilon g_{\alpha\beta}\left(z_\beta\right)}_{\tilde{f}_{\alpha b}(z_\beta)} \ , \tag{2.6}$$

i.e., we write $\tilde{f}_{\alpha\beta}$ for the new transition functions, and moreover, the new transition functions agree with the old ones modulo ϵ. This is the gluing data describing the construction of $\mathcal{X}$ from the charts $U_\alpha \times S$.

Remark 2.1. The geometric picture is that we start with some X, then we spread this out into a higher-dimensional fibration. So assuming we've shrunk U_α sufficiently, it has no interesting topology, and if we look at it inside of the fibers all at once, this is just a cylinder $U_\alpha \times S$. So then the total space is glued out of these cylinders.

These transition functions satisfy the gluing condition

$$\tilde{f}_\alpha\left(\tilde{f}_{\beta\gamma}\left(z_\gamma\right)\right) = \tilde{f}_{\alpha\gamma}\left(z_\gamma\right) \tag{2.7}$$

$$= \underbrace{f_{\alpha\beta}\left(f_{\beta\gamma}\right)}_{f_\alpha\gamma} + f_{\alpha\beta}\left(\epsilon g_{\beta\gamma}\right) + \epsilon g_{\alpha\beta}\left(f_{\beta\gamma}\right) \ . \tag{2.8}$$

The first term just comes from gluing on X, and the second term can be thought of as a version of Leibniz's rule:

$$\frac{\partial f_{\alpha\beta}}{\partial z_\beta} g_{\beta\gamma} + g_{\alpha\beta} = g_{\alpha\gamma} \ . \tag{2.9}$$

Another way of writing this is that:

$$\Theta_{\alpha\beta} = \left(g_{\alpha_i\beta_i}\right)\begin{pmatrix}\partial/\partial z_{\alpha_1} \\ \vdots \\ \partial/\partial z_{\alpha_n}\end{pmatrix} \in H^0\left(U_{\alpha\beta}, T_X|_{U_{\alpha\beta}}\right) \tag{2.10}$$

is a cocycle, so it defines a class:

$$[\Theta_{\alpha\beta}] \in H^1\left(X, T_X\right) \tag{2.11}$$

which is the image of 1 in $H^1(X, T_X)$.

The point of this is that the deformation $\varphi\colon X \to S$ goes to the coboundary $\delta(\partial/\partial\epsilon)$ where we regard $\partial/\partial\epsilon \in H^0(\varphi^* T_S)$.

Moreover, we can reverse engineer the argument above, i.e., given a 1-cocycle with coefficients in T_X we can construct a deformation $\mathcal{X} \to S$. One also checks, by direct computation with cocycles, that cohomologous cocycles give rise to isomorphic deformations, and hence one gets a well-defined inverse to the map

$$\{\text{isomorphism classes of deformations}\} \twoheadrightarrow H^1\left(X, T_X\right) \ . \tag{2.12}$$

$\square$

Let $\mathcal{X} \to (B, b_0)$ be a deformation. Recall that elements of T_{B,b_0} (i.e., "tangent vectors") correspond to morphisms $S \to B$ that send the underlying point of $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$ to b_0. Pulling back $\mathcal{X}$ along such a tangent vector $S \to B$ gives an infinitesimal deformation of the special fiber $X = \mathcal{X}_{b_0}$. The natural bijection between isomorphism classes of infinitesimal deformations of X and $H^1(X, T_X)$ therefore gives rise to the *Kodaira-Spencer map* $\rho\colon T_{B,b_0} \to H^1(X, T_X)$. (We have constructed this map set-theoretically, but it is a linear map of vector spaces.)

Lecture 10; February 12, 2020

Let us now consider the case where $X = C$ is a smooth and stable curve, i.e., a smooth curve of genus $g(C) \geq 2$. By Serre duality, we have a canonical isomorphism

$$H^1(C, T_C) \cong H^0\left(C, T_C^\vee \otimes \omega_C\right)^\vee \cong H^0\left(C, \omega_C^{\otimes 2}\right)^\vee \ . \tag{2.13}$$

Note that sections of $\omega^{\otimes 2}$ are sometimes referred to as the *quadratic differentials*. Since $\deg\left(\omega_C^{\otimes 2}\right) = 4g - 4$ and $g \geq 2$, Riemann-Roch tells us that

$$h^0\left(\omega_C^{\otimes 2}\right) = 3g - 3 \ . \tag{2.14}$$

Hence the space of infinitesimal deformations of C has dimension $3g - 3$.

The ideal sheaf of a point $p \in C$ is locally free,[2.1], so we have a short exact sequence

$$0 \longrightarrow I_p \cong \mathcal{O}(-p) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_p \longrightarrow 0 \ . \tag{2.15}$$

Tensoring with $T_C(p)$ gives us the short exact sequence

$$0 \to T_C \to T_C(p) \to T_C(p)|_p \to 0 \ . \tag{2.16}$$

This induces a long exact sequence

$$H^0(C, T_C) \longrightarrow H^0(C, T_C(p)) \longrightarrow H^0(p, T_C(p)) \xrightarrow{\delta} H^1(C, T_C) \longrightarrow \cdots \ . \tag{2.17}$$

Note that $H^0(C, T_C(p))$ vanishes, since $g(C) \geq 2$, and the vector space $H^0(p, T_C(p))$ is 1-dimensional. Hence the choice of p gives rise to a 1-dimensional subspace $\delta_p \subseteq H^1(C, T_C)$ which is an infinitesimal deformation well-defined up to $\mathbb{C}^\times$. These are called *Schiffer deformations*.

An alternative construction is as follows. The complete linear series of quadratic differentials gives a map

$$C \to \mathbb{P}\left(H^0\left(C, \omega_C^{\otimes 2}\right)^\vee\right) \ , \tag{2.18}$$

and $p \in C$ maps the point δ_p in this projective space.

Fact 2 (Important fact). *Schiffer deformations are integrable, i.e., they come from deformations over a small disk $\Delta = \{z \mid |z| < b\}$.*

The idea is as follows. Let $p \in C$ be a point in our curve. Then let U be a neighborhood of p with a local coordinate $z\colon U \xrightarrow{\sim} \Delta$ which maps U isomorphically to the disk Δ. Then define:

$$U' = \{z \in U \mid |z| < b/3\} \qquad\qquad U'' = \{w \in U \mid |w| < 2b/3\} \ . \tag{2.19}$$

That is $U' \subset U'' \subset U$. Then we can think of C as being obtained by gluing

$$C = (C \setminus U') \cup U'' \ . \tag{2.20}$$

[2.1] This is using the fact that C is smooth and 1-dimensional. If instead p were a point on a smooth surface, or a node on a singular curve, for example, then the ideal sheaf of p will have rank 1 everywhere away from the point, but the fiber over p has rank 2.

In particular, for t sufficiently small, consider the space C_t obtained by gluing $C \setminus U'$ to U'' along $w = z + t/z$.

CLAIM 2.1 ([**1**, XI, §2]). δ_p is the infinitesimal deformation associated to the family $\{C_t\}$.

Moreover, if we choose multiple distinct points $p_1, \ldots, p_s \in C$, then we get multiple Schiffer deformations that are simultaneously integrable. Indeed, by choosing disjoint coordinate patches at the points and performing the construction above on each patch, we can simultaneously integrate all δ_{p_i} to get

$$\begin{matrix} \mathcal{C} \\ \downarrow \\ \Delta^s \end{matrix} \quad . \tag{2.21}$$

Note that

$$f = f_{|\omega_C^{\otimes 2}|} \colon C \otimes \mathbb{P}\left(H^0\left(C, \omega^{\otimes 2}\right)^\vee\right) \tag{2.22}$$

is nondegenerate, i.e., the image is not contained in a hyperplane, so the Schiffer deformations span $H^1(C, T_C)$. In particular, if we choose $s = 3g - 3$ general points $p_1, \ldots, p_s$ in C, then representatives of $\{\delta_{p_1}, \ldots, \delta_{p_s}\}$ form a basis for $H^1(C, T_C)$. Hence the Kodaira-Spencer map for $\varphi \colon \mathcal{C} \to \Delta^s$

$$\rho \colon T_{\Delta^s, 0} \xrightarrow{\sim} H^1(C, T_C) \tag{2.23}$$

is an isomorphism.

The existence of such a family, over a smooth base, for which the Kodaira-Spencer map is an isomorphism is a very special feature of the geometry and deformation theory of curves. It is related to the existence of Kuranishi families and smoothness of moduli spaces (stacks) of curves, as we will discuss in the coming lectures. The paper [**13**] shows that moduli spaces (stacks) of smooth projective surfaces with very ample canonical bundle exhibit arbitrarily bad singularities, so the pleasantness of this situation for curves must not be taken for granted.

DEFINITION 2.2. A deformation

$$\begin{matrix} \mathcal{C} \\ \downarrow\varphi \\ (B, b_0) \end{matrix} \tag{2.24}$$

($C_{b_0} \xrightarrow{\sim} C$) is a Kuranishi family if for any deformation $\mathcal{D} \xrightarrow{\varphi} (E, e_0)$ of C, and any sufficiently small neighborhood U of e_0, there is a unique morphism of deformations

$$\varphi'|_U \to \varphi \ . \tag{2.25}$$

These can be thought of as *local moduli spaces*. We will study Kuranishi families not only for smooth curves, but also for nodal curves.

Lecture 11; February 14, 2020

1. Deformations of nodal curves

Happy Valentine's Day. Let C be a nodal curve.

THEOREM 2.2. *There is a natural bijection between isomorphism classes of infinitesimal deformations of C and* $\operatorname{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right)$.

REMARK 2.2. If C is in fact smooth, then the sheaf of Kähler differentials Ω^1_C is the dualizing sheaf $\Omega^1_C \cong \omega_C$. So

$$\operatorname{Ext}^1(\omega_C, \mathcal{O}_C) \cong \operatorname{Ext}^1\left(\omega_C^{\otimes 2}, \omega_C\right) \tag{2.26}$$
$$\cong H^0\left(C, \omega_C^{\otimes 2}\right) \tag{2.27}$$
$$\cong H^1\left(C, \left(\omega_C^{\otimes 2}\right)^\vee \otimes \omega_C\right) \tag{2.28}$$
$$\cong H^1(C, T_C) \tag{2.29}$$

where the second and third isomorphisms come from (appropriate versions of) Serre duality. So, in the special case where C is smooth, we recover our previous identification of infinitesimal deformations with $H^1(C, T_C)$.

PROOF. Let $S = \operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$ and let $\mathcal{C} \to S$ be an infinitesimal deformation of C. Then we get an exact sequence of sheaves on $\mathcal{C}$:

$$\varphi^*\Omega^1_S \to \Omega^1_{\mathcal{C}} \to \Omega^1_{\mathcal{C}/S} \to 0\ . \tag{2.30}$$

Now tensoring is right-exact, so we can tensor with $\mathcal{O}_C$ to get:

$$\varphi^*\Omega^1_S \otimes \mathcal{O}_C \to \Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C \to \Omega^1_C \to 0\ . \tag{2.31}$$

CLAIM 2.2.1. $\varphi^*\Omega^1_S \otimes \mathcal{O}_C \to \Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C$ is injective.

PROOF. The sheaf $\varphi^*\Omega^1_S \otimes \mathcal{O}_C$ is trivial of rank 1, generated by $d\epsilon \otimes 1$. At a smooth point of C, $\mathcal{C}$ is locally $S \times \mathcal{C}$, and hence the image of $d\epsilon \otimes 1$ is nonzero near this point. Since the smooth points are open and dense in C, this is enough to prove the claimed injectivity. □

The claim shows that (2.31) is short exact. Using the identification $\mathcal{O}_C \xrightarrow{\cong} \varphi^*\Omega^1_S \otimes \mathcal{O}_C$ given by $1 \mapsto d\epsilon \otimes 1$, we can then view $\Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C$ as an extension of Ω^1_C by $\mathcal{O}_C$. We thus get a map from isomorphism classes of infinitesimal deformations of C to extension classes in $\operatorname{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right)$.

CLAIM 2.2.2. This assignment of extension classes to isomorphism classes of infinitesimal deformations of C is injective.

PROOF. Suppose $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are infinitesimal deformations giving rise to the same extension class. We must show that these infinitesimal deformations are isomorphic.

Since the induced extensions of Ω^1_C by $\mathcal{O}_C$ are isomorphic, we have a sheaf isomorphism γ such that the following diagram commutes:

$$\begin{tikzcd} & \Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C \arrow[dd, "\cong"', "\gamma"] \arrow[dr] & \\ \mathcal{O}_C \arrow[ur] \arrow[dr] & & \Omega^1_C \\ & \Omega^1_{\mathcal{C}'} \otimes \mathcal{O}_C \arrow[ur] & \end{tikzcd}\ . \tag{2.32}$$

To prove the claim, we must show that there is an isomorphism $\beta\colon \mathcal{O}_{\mathcal{C}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{C}'}$ (over S) which restricts to the identity on $\mathcal{O}_C$. □

CLAIM 2.2.2'. For each $h \in \mathcal{O}_{\mathcal{C}}$, there is a unique $\beta(h) \in \mathcal{O}_{\mathcal{C}'}$ such that

$$\beta(h)|_C = h|_C \tag{2.33}$$

and

$$d\beta\,(h)|_C = \gamma\,(\,dh\,|_C) \ . \tag{2.34}$$

Here, we write $d\beta\,(h)|_C$ for the image of $d\beta\,(h)$ in $\Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C$.

PROOF. First we show that uniqueness holds, even locally. If $f|_C = 0$, then locally $f = \epsilon g$. This implies $df = g\,d\epsilon\,|_C$. If, in addition, $df\,|_C = 0$ then $f = 0$. Local uniqueness follows.

Given local uniqueness and the basic properties of sheaves, it is enough to prove the existence of $\beta(h)$ locally. First, extend $h|_C$ to some $\tilde{h}$ on $\mathcal{C}'$. The difference between $d\tilde{h}\Big|_C$ and $\gamma\,(\,dh\,|_C)$ is of the form $g\,d\epsilon$. Set

$$\beta(h) = \tilde{h} - \epsilon g \ . \tag{2.35}$$

This gives rise to a canonical set theoretic map

$$\beta\colon \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}'} \ . \tag{2.36}$$

This is a priori only a map of sheaves of sets, but in fact it is a map of sheaves of rings, as can be seen using the Leibniz rule. This proves claim 2.2.2', which implies claim 2.2.2. $\square$

CLAIM 2.2.3. The map from deformations to extensions is surjective.

PROOF. Now we have the following local-to-global exact sequence for Ext:

$$0 \to H^1\left(C, \mathcal{H}\mathrm{om}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \to \mathrm{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right) \to H^0\left(C, \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_C}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \to 0 \ . \tag{2.37}$$

Recall that the sheaf $\mathcal{E}$xt encodes information about local extensions, i.e., the stalk of $\mathcal{E}\mathrm{xt}^1$ at p classifies extensions of $\Omega^1_{\mathcal{C},p}$ by $\mathcal{O}_{\mathcal{C},p}$. In particular, it vanishes at points where $\Omega^1_{\mathcal{C}}$ is locally free, and hence is supported on C^{sing}:

$$H^0\left(\mathcal{E}\mathrm{xt}^1\left(\Omega^1_C, \mathcal{O}_C\right)\right) = \bigoplus_{p \in C^{\mathrm{sing}}} \mathrm{Ext}^1\left(\Omega^1_{C,p}, \mathcal{O}_{C,p}\right) \ . \tag{2.38}$$

Lecture 12; February 17, 2020

The local-to-global exact sequence is a consequence of the local-global Ext spectral sequence:

$$E_2^{pq} = H^p\left(\mathcal{E}\mathrm{xt}^q\right) \Rightarrow \mathrm{Ext}^{p+q} \ . \tag{2.39}$$

This is an example of a Grothendieck spectral sequence for the composition of two functors. See Wikipedia page and this Stack Exchange post for a more detailed discussion and further references.

The lefthand term in (2.37) is naturally identified with the set of isomorphism classes of locally trivial extensions of Ω^1_C by $\mathcal{O}_C$, as follows.

An extension $\mathcal{O}_C \to \mathcal{F} \to \Omega^1_C$ is locally trivial if there is an open cover $\{U_\alpha\}$ of C, such that the extension splits on each U_α via isomorphisms

$$\mathcal{F}|_{U_\alpha} \xrightarrow{\varphi_\alpha} \mathcal{O}_C \oplus \Omega^1_C \ . \tag{2.40}$$

On $U_\alpha \cap U_\beta$, we then have transition functions

$$\mathcal{O}_C \oplus \Omega^1 \xrightarrow{\varphi_\alpha^{-1}} \mathcal{F}\left(U_\alpha \cap U_\beta\right) \xrightarrow{\varphi_\beta} \mathcal{O}_C \oplus \Omega^1_C \ . \tag{2.41}$$

The maps $\mathcal{O}_C \to \mathcal{O}_C$ and $\Omega^1_C \to \Omega^1_C$ induced by $\varphi_\beta \circ \varphi_\alpha^{-1}$ are the identity, and the map $\mathcal{O}_C \to \Omega^1_C$ is zero. Let $f_{\alpha\beta}$ be the induced map $\Omega^1_C(U_\alpha \cap U_\beta) \to \mathcal{O}_C(U_\alpha \cap U_\beta)$. Then $\{f_{\alpha\beta}\}$ is a 1-cocycle for $\mathcal{H}\mathrm{om}\left(\Omega^1, \mathcal{O}\right)$. Different choices of trivialization give rise to cohomologous cocycles. Conversely, a 1-cocycle for $\mathcal{H}om\left(\Omega^1, \mathcal{O}\right)$ gives rise to a locally trivial extension, and cohomologous cocycles give rise to isomorphic extensions. In this way $H^1\left(C, \mathcal{H}\mathrm{om}\left(\mathcal{O}^1_C, \mathcal{O}_C\right)\right)$ classifies locally trivial extensions of Ω^1_C by $\mathcal{O}_C$.

Let us now turn attention to extensions of Ω^1_C by $\mathcal{O}_C$ that are not locally trivial. Roughly speaking, such extensions correspond geometrically to "smoothings of nodes". Near a node $p \in C^{\mathrm{sing}}$, the curve C is locally isomorphic to $(xy = 0) \subset \mathbb{C}^2$. The conormal exact sequence for this inclusion is

$$I_C/I_C^2 \to \Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}_C \to \Omega^1_C \to 0 \ . \tag{2.42}$$

Note that I_C/I_C^2 is locally free of rank 1; it is the line bundle $\mathcal{O}_{\mathbb{C}^2}(-C)\big|_C$.

Localizing the conormal exact sequence at p and deriving the functor $\mathrm{Hom}\left(-, \mathcal{O}_{C,p}\right)$ gives us the long-exact sequence:

$$\mathrm{Hom}\left(\Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}_{C,p}, \mathcal{O}_{C,p}\right) \xrightarrow{\eta} \mathrm{Hom}\left((I_C/I_C^2)_p, \mathcal{O}_{C,p}\right) \longrightarrow \mathrm{Ext}^1\left(\Omega^1_{C,p}, \mathcal{O}_{C,p}\right) \longrightarrow 0 \ . \tag{2.43}$$

The last term is 0 because

$$\mathrm{Ext}^1\left(\Omega^1_{\mathbb{C}^2}\big|_{C,p}, \mathcal{O}_{C,p}\right) \simeq \mathrm{Ext}^1\left(\mathcal{O}^{\oplus 2}_{C,p}, \mathcal{O}_{C,p}\right) = 0 \ . \tag{2.44}$$

The image of η is

$$\mathfrak{m}_p \,\mathrm{Hom}\left((I_C/I_C^2)_p, \mathcal{O}_{C,p}\right) \cong \mathfrak{m}_p \tag{2.45}$$

so we get a non-canonical isomorphism

$$\mathrm{Ext}^1\left(\mathcal{O}^1_{C,p}, \mathcal{O}_{C,p}\right) \cong \mathcal{O}_{C,p}/\mathfrak{m}_p \cong \mathbb{C} \ . \tag{2.46}$$

Carrying through the computations more carefully, we would get a canonical isomorphism

$$\mathrm{Ext}^1\left(\mathcal{O}^1_{C,p}, \mathcal{O}_{C,p}\right) \cong T_{\widetilde{C},p_1} \otimes T_{\widetilde{C},q_1} \tag{2.47}$$

where $\{p_1, q_1\} = \nu^{-1}(p) \subseteq \widetilde{C}$. See [**1**, XI, §3].

Example 2.1. For $C = (xy = 0)$ and $a \in \mathbb{C}$, we have the deformation $xy = a\epsilon$. So we get a Kodaira-Spencer class

$$\rho\,(xy = a\epsilon) \in \mathrm{Ext}^1\left(\mathcal{O}^1_{C,p}, \mathcal{O}_{C,p}\right) \ . \tag{2.48}$$

A direct computation/diagram chase yields

$$\rho\,(xy = a\epsilon) = a\rho\,(xy = \epsilon)\,, \ . \tag{2.49}$$

and $\rho(xy = \epsilon) \neq 0$. Putting this together with the calculation showing Ext^1 is 1-dimensional, we see that all isomorphism classes of infinitesimal deformations are of this form.

This concludes the proof of claim 2.2.3, □

which completes the proof of Theorem 2.2. ■

Proposition 2.3. *$H^1\left(C, \mathcal{H}\mathrm{om}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \cong H^1\left(\widetilde{C}, T_{\widetilde{C}}\left(-p_1 - q_1 - \ldots - p_r - q_r\right)\right)$ where the p_i, q_i are the preimages of the node $x_i \in C^{sing} = \{x_1, \ldots, x_r\}$. The RHS classifies deformations of $\left(\widetilde{C}, p_1, q_1, \ldots, p_r, q_r\right)$.*

PROOF. It is enough to show that

$$\mathcal{H}\mathrm{om}\left(\Omega^1_C, \mathcal{O}_C\right) \cong T_{\widetilde{C}}\left(-p_1 - \ldots - q_r\right) \ . \tag{2.50}$$

The idea is that $\Omega^1_C = \mathcal{I}\omega_C$ where $\mathcal{I}$ is the ideal sheaf of C^{sing}. Locally near x_j,

$$\mathcal{I}\omega \cong \mathcal{I}\omega_{\widetilde{C}_1}\left(-p_j\right) \oplus \mathcal{I}_{\widetilde{C}_2}\left(-q_j\right) \tag{2.51}$$

and

$$\mathcal{I}_{x_j} = \nu_* \mathcal{I}_{(p_j \cup q_j)} \ . \tag{2.52}$$

□

THEOREM 2.4. *Let C be a nodal curve. Then there is a deformation $\mathcal{C} \to (\Delta^s, 0)$ such that the Kodaira-Spencer map $\rho\colon T_0\left(\Delta^s\right) \to \mathrm{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right)$ is an isomorphism. From our short exact sequences we explicitly get that*

$$\begin{aligned} s &= 3g - 3 + \dim \mathrm{Hom}\left(\Omega^1_C, \mathcal{O}_C\right) && (2.53)\\ &= 3g - 3 + h^0\left(\widetilde{C}, T_{\widetilde{C}}\left(-p_1 - \ldots - q_r\right)\right) && (2.54)\\ &= 3g - 3 && (2.55)\end{aligned}$$

where the h^0 vanishes since C is stable.

PROOF. Glue Schiffer deformations at smooth points to the $(xy = a\epsilon)$ deformations at the nodes. □

Lecture 13;
February 19, 2020

2. Kuranishi families

We will follow [**1**, XI, §§4-6]. Recall the following definition.

DEFINITION 2.3. A deformation $\mathcal{X} \to (B, b_0)$ of X is a *Kuranishi family* if for any other deformation $\mathcal{X}' \to (B', b_0')$ and any sufficiently small neighborhood U of b_0', there is a unique morphism of deformations:

$$\begin{array}{ccc} \mathcal{X}'_U & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \\ (U, b_0') & \longrightarrow & (B, b_0) \end{array} \ . \tag{2.56}$$

By definition, a morphism of deformations is a cartesian square, so $\mathcal{X}'_U$ is the fiber product of $\mathcal{X}$ and U over B, i.e., the deformation $\mathcal{X}'_U \to (U, b_0')$ is just $\mathcal{X} \to (B, b_0)$ pulled back along the map $(U, b_0) \to (U', b_0')$. In this sense, a Kuranishi family is a moduli space for deformations.

We can then make the following observations.

1. When a Kuranishi family exists, it is locally unique up to unique isomorphism, i.e., if $\begin{array}{c}\mathcal{X}\\ \downarrow\\ (B, b_0)\end{array}$ and $\begin{array}{c}\mathcal{X}'\\ \downarrow\\ (B', b_0')\end{array}$ are Kuranishi families, then for every sufficiently

small neighborhood U of b_0 there is a unique neighborhood U' of b_0' and a unique isomorphism of deformations:

$$\begin{array}{ccc} \mathcal{X}_U & \xrightarrow{\simeq} & \mathcal{X}_{U'} \\ \downarrow & & \downarrow \\ (U, b_0) & \xrightarrow{\simeq} & (U', b_0') \end{array} \quad . \tag{2.57}$$

2. The Kodaira Spencer map of any Kuranishi family

$$\rho\colon T_{B,b_0} \xrightarrow{\cong} \{\text{isomorphism classes of infinitesimal deformations of } X\} \tag{2.58}$$

is an isomorphism.
3. Suppose a Kuranishi family exists. Let $\mathcal{X} \to (B, b_0)$ be a deformation such that B is smooth at b_0, and such that the Kodaira Spencer map ρ is an isomorphism, then $\mathcal{X} \to (B, b_0)$ is Kuranishi. This follows from the universal property and some version of the implicit function theorem.
4. If $\mathcal{X} \to (B, b_0)$ is Kuranishi family for X and $\mathrm{Aut}\,(X)$ is finite, then $\mathrm{Aut}\,(X)$ acts on $\mathcal{X}_U \to (U, b_0)$ for a basis of neighborhoods U of b_0.

THEOREM 2.5. *Let C be a nodal curve. Then a Kuranishi family for C exists if and only if C is stable.*

REMARK 2.3. The analogous statement holds for nodal curves with marked points, but we will just go through the construction for unmarked curves.

Corollary 2.6. *The base of a Kuranishi family for a stable curve C of genus $p_a(C) = g$ has local dimension $3g - 3$.*

Corollary 2.7. *If $\mathcal{C} \to (B, b_0)$ is Kuranishi for a nodal curve C then there is a neighborhood of b_0 such that $\mathcal{C}_U \to (U, x)$ is Kuranishi for all $x \in U$.*

The picture to have in mind here is that $B/\,\mathrm{Aut}(C)$ looks like an open patch in the moduli space of curves.

One key technical input in the proof of Theorem 2.5 is the existence and projectivity of the Hilbert scheme, which is is the moduli space of subschemes of $\mathbb{P}^N$ with fixed Hilbert polynomial. This is one small piece of the important foundational work of Grothendieck [**7**]. See [**5**], especially Part 2 (by Nitsure) and Part 3, §6 (by Fantechi) for further reading.

PROOF OF THEOREM 2.5. First choose N such that $\omega_C^{\otimes N}$ is very ample for all stable curves C of genus g, (e.g. $N \geq 3$). Then notice that $\left|\omega_C^{\otimes N}\right|$ embeds C in $\mathbb{P}^{N'}$ with Hilbert polynomial p independent of C. Then we have open $U \subset \mathrm{Hilb}\left(\mathbb{P}^{N'}, p\right)$ parametrizing stable curves embedded by $\left(\omega_C^{\otimes N}\right)$. Notice that the group $\mathrm{PGL} = \mathrm{PGL}_{N'+1}$ acts on U.

FACT 3. *The stabilizer of a point x corresponding to the N-canonical embedding of a stable curve C is canonically isomorphic to* $\mathrm{Aut}\,(C)$.

Consider the PGL orbit through x. This is smooth of the same dimension as PGL. Write $G = \mathrm{Aut}\,(C)$. Then $G \subseteq \mathrm{PGL}$ acts as $\mathrm{Stab}\,(C)$, and $T_X\,(\mathrm{PGL} \cdot X)$ is G-invariant. Let $L \subseteq \mathbb{P}^K$ be a complementary G invariant linear space (where K is the dimension of the projective space which $\mathrm{Hilb}\left(\mathbb{P}^{N'}, p\right)$ lives).

The universal family of subschemes of $\mathbb{P}^{N'}$ over $U \cap L$ is Kuranishi for C. The picture is that Hilb might have some additional pieces (including higher-dimensional pieces) that parameterize unstable curves, but we just want to intersect with U. The fact that this has the Kuranishi property is deduced from the universal property of the Hilbert scheme. □

A consequence of the above is the following. Let C be any stable curve. Then there is an algebraic deformation $\mathcal{C} \to (X, x)$ such that Lecture 14; February 21, 2020

(1) X is affine;
(2) $\mathcal{C} \to X$ is Kuranishi at every point $x' \in X$;
(3) $G = \mathrm{Aut}\,(C)$ acts on $(\mathcal{C} \to X)$, and the induced map

$$\{g \in G \,|\, gx' = x'\} \xrightarrow{\sim} \mathrm{Aut}\,(\mathcal{C}_{x'}) \tag{2.59}$$

is an isomorphism
(4) any isomorphism $\mathcal{C}_{x_1} \xrightarrow{\sim} \mathcal{C}_{x_2}$ is induced by some g such that $gx_1 = x_2$.

Remark 2.4. X/G (at least set-theoretically) parameterizes

$$\{\mathcal{C}_x \,|\, x \in X\} \,/ \cong \ . \tag{2.60}$$

Lemma 2.8. *Let $X = \mathrm{Spec}\, A$ be an affine variety (scheme of finite type over $\mathbb{C}$) with the action of a finite group G. Then the ring of invariants*

$$A^G = \{a \in A \,|\, ga = a \text{ for all } g \in G\} \tag{2.61}$$

is finitely generated and

$$X/G = \mathrm{Spec}\, A^G \ . \tag{2.62}$$

Moreover, if X is normal then so is X/G.

We omit the proof, which is give in [**1**].

Definition 2.4. Write $\overline{M}_g$ for the collection of isomorphism classes of stable curves of genus g. For each curve C, we can build a Kuranishi family X_C, with the action of $G_C = \mathrm{Aut}\,(C)$, so we have a cover of this set by algebraic varieties:

$$\overline{M}_g = \bigcup_C X_C/G_C \ . \tag{2.63}$$

Claim 2.2. The "gluing" maps are holomorphic, so $\overline{M}_g$ is a complex analytic space.

Suppose

$$U = X_C/G_C \qquad\qquad U' = X_{C'}/G_{C'} \ . \tag{2.64}$$

The universal property of Kuranishi families implies that $U \cap U'$ is open in both U and U'. Indeed, if we have a point in the intersection, then we lift it to $x \in X$ and $x' \in X'$, and a sufficiently small neighborhood of x' is uniquely biholomorphic to a unique neighborhood of x in X. It follows that the inclusion $U' \to U$ is holomorphic away from the branch locus B' of $X' \to U'$. Covering U by bounded domains, using the normality of U', and applying Riemann Existence Theorem, it follows that the holomorphic inclusion $U' \smallsetminus B' \to U$ extends to a holomorphic map $U' \to U$, as required.

Modulo the definitions of orbifolds and Deligne-Mumford stacks, which are technical and omitted, the construction above has the following consequences:

Theorem 2.9. *$\overline{M}_g$ is the coarse space of a smooth complex-analytic orbifold $\overline{\mathcal{M}}_g$ that represents the moduli functor for stable curves of genus g:*

$$\mathfrak{M}_g \colon \mathbf{Spaces} \to \mathbf{Sets} \tag{2.65}$$

which maps a space S to families of stable nodal curves over S up to isomorphism.

With more care, we can get the following:

THEOREM 2.10. *$\overline{\mathcal{M}}_g$ is a smooth algebraic (Deligne-Mumford) stack with coarse space $\overline{M}_g$. Moreover, $\overline{M}_g$ is an irreducible projective algebraic variety.*

The analogous statements also hold with marked points, i.e., for $\overline{\mathcal{M}}_{g,n}$ and $\overline{M}_{g,n}$.

We now briefly sketch a proof of the irreducibility of $\overline{M}_{g,n}$ over $\mathbb{C}$, since this fact (and especially Corollary 2.11, below), will be important for our approach to studying the top weight cohomology of M_g. We begin by considering the case where there are no marked points. From our study of deformation theory of stable curves, we know that the subspace M_g parameterizing smooth curves is open and dense, so it is enough to show that this is irreducible. Moreover, since $\mathcal{M}_g$ is smooth, it is enough to show that M_g connected. Now M_g is the quotient of Teichmüller space (a contractible domain) by the mapping class group:

$$M_g = \mathcal{T}_g / \operatorname{Mod}(S_g) \ . \tag{2.66}$$

In particular, as a quotient of a connected space, it is connected.

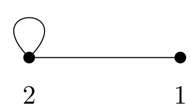
For the more general statement with marked points, we proceed by induction on the number of marked points (using irreducibility of M_g as the base case. Consider the forgetful map $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ given by forgetting the nth marked point and stabilizing if necessary, Then this map is the universal curve $\mathcal{C}_{g,n-1} \to \mathcal{M}_{g,n-1}$. Since it is a fiber bundle whose base is irreducible, by the induction hypothesis, and whose fiber is irreducible, its total space is irreducible. And therefore the coarse space $M_{g,n}$ is irreducible as well.

Corollary 2.11. *There is a stratification*

$$\overline{M}_g = \coprod_G M_G \tag{2.67}$$

where M_G is the space of stable curves with dual graph G. In particular, each M_G is irreducible.

For example, let G be the following graph.



Then

$$M_G = M_{2,3} \times M_{1,1} / \operatorname{Aut}(G) \tag{2.68}$$

so it is the quotient of an irreducible space by a finite group, and hence irreducible. Furthermore, $M_G \subset \overline{M}_{G'}$ if and only if G' is obtained from G by (weighted) edge contractions. Note that the codimension of M_G is the number of edges:

$$\operatorname{Codim} M_G = \# E(G) \ . \tag{2.69}$$

So we have a combinatorial stratification of $\overline{M}_g$ into irreducible pieces indexed by dual graphs of stable curves, with containments encoded by weighted edge contractions. This will be essential input when we study the top weight cohomology of M_g. Lecture 15; February 24, 2020

3. Boundary complexes and weight filtrations

See [**2–4**], [**12**], and [**14, 15**] for references.

Let X be an algebraic variety of dimension $\dim_{\mathbb{C}} X = n$. We will study the singular cohomology with coefficients in some ring $H^*(X, A)$. Most often we will consider $A = \mathbb{Q}$ or $\mathbb{C}$. The rational cohomology $H^*(X, \mathbb{Q})$ carries a canonical increasing weight filtration $W_\bullet$. By extending scalars, we also get a weight filtration on $H^*(X, \mathbb{C})$ which carries, in addition, a decreasing Hodge filtration $F^\bullet$. Together, these two filtrations form a mixed Hodge structure, in the sense of Deligne.

We will focus primarily on the weight filtration from Deligne's theory, listing some of its essential properties that we will use repeatedly (in the spirit of Grothendieck's "yoga of weights"). Note that the proofs of these properties (which we omit) rely on properties of the Hodge filtration.

The weight filtration on $H^k(X, \mathbb{Q})$ is an increasing filtration

$$0 \subseteq W_0 H^k(X, \mathbb{Q}) \subseteq W_1 H^k(X, \mathbb{Q}) \subseteq \ldots \subseteq W_{2k} H^k(X, \mathbb{Q}) = H^k(X, \mathbb{Q}), \tag{2.70}$$

whose associated graded pieces are

$$\operatorname{Gr}_j^W H^k(X, \mathbb{Q}) = W_j H^k(X, \mathbb{Q}) / W_{j-1} H^k(X, \mathbb{Q}) \tag{2.71}$$

Note that $\operatorname{Gr}_j^W H^*(X, \mathbb{Q})$ is sometimes informally referred to as "weight j cohomology" or the "weight j part of cohomology," even though it is a subquotient, not a subspace. We say that $H^k(X, \mathbb{Q})$ *has weights in* $I \subseteq \{0, \ldots, 2k\}$ if

$$\operatorname{Gr}_j^W H^k(X, \mathbb{Q}) = 0 \tag{2.72}$$

for $j \notin I$.

The weight filtration satisfies the following properties:

- If X is compact, then $H^k(X, \mathbb{Q})$ has weight in $\{0, \ldots, k\}$.
- If X is smooth, then $H^k(X, \mathbb{Q})$ has weights in $\{k, \ldots, 2k\}$.
- For all k, $H^k(X, \mathbb{Q})$ has weights in $\{0, \ldots, 2n\}$.

The last condition is meaningful when $k > n$. A key special case is when X is smooth and compact (e.g., smooth and projective). In this case $\operatorname{Gr}_j^W H^k(X, \mathbb{Q})$ vanishes for $j \neq k$, and we say that $H^k(X, \mathbb{Q})$ has (pure) weight k.

Remark 2.5. There are similar filtrations on $H_k(X, \mathbb{Q})$ and $H_c^k(X, \mathbb{Q})$.

Important. *All* natural maps between cohomology groups of algebraic varieties strictly respect weight filtrations.

Example 2.2. If $f\colon X \to Y$ is a morphism, then

$$f^* W_j H^k(Y, \mathbb{Q}) \subseteq W_j H^k(X, \mathbb{Q}) \ . \tag{2.73}$$

Moreover,

$$f^* H^k(Y, \mathbb{Q}) \cap W_j H^k(X, \mathbb{Q}) = f^* W_j H^k(Y, \mathbb{Q}) \ . \tag{2.74}$$

I.e., f^* induces

$$\operatorname{Gr}_j^W H^k(Y, \mathbb{Q}) \to \operatorname{Gr}_j^W H^k(X, \mathbb{Q}) \tag{2.75}$$

and

$$\operatorname{rank} f^*|_{H^k} = \sum_j \operatorname{rank} f^*|_{\operatorname{Gr}_j^W H^k} \ . \tag{2.76}$$

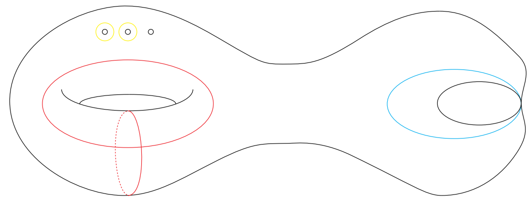


FIGURE 1. A triple punctured curve C of geometric genus 1 with 1 node.

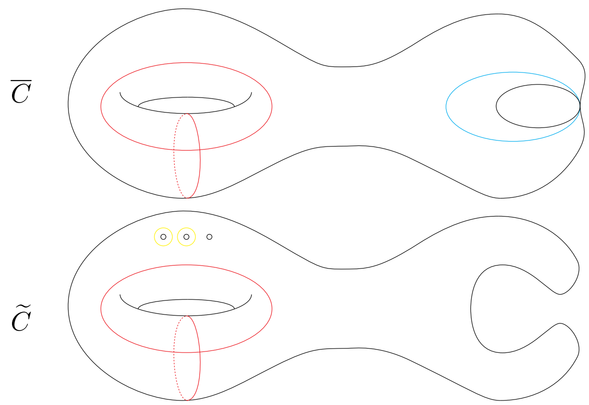


FIGURE 2. A compactification of our curve on top, and the normalization on the bottom.

EXAMPLE 2.3. Consider the nodal curve in fig. 1. Note that $H_1(C, \mathbb{Q}) \cong \mathbb{Q}^5$ has a basis given by the classes of the red, yellow, and blue curves. We consider also the dual basis for $H^1(C, \mathbb{Q})$.

We consider the compactification $i \colon C \hookrightarrow \bar{C}$, obtained by adding three smooth points (this makes $\bar{C}$ unique), as shown in fig. 2. This has a basis for homology given by the classes of the blue and red curves; the yellow curves are homologous to zero in $\bar{C}$.

Dualizing to cohomology we get

$$H^1(C, \mathbb{Q}) / i^* H^1(\bar{C}, \mathbb{Q}) = \mathrm{Gr}_2^W H^1(C, \mathbb{Q}) \tag{2.77}$$

Hence the dual basis elements corresponding to the yellow curves freely generate $\mathrm{Gr}_2^W H^1(C, \mathbb{Q})$.

On the other hand, we can normalize to get $\nu \colon \widetilde{C} \to C$. See fig. 2. On $\widetilde{C}$, the classes of the yellow and red curves give a basis for H_1. Passing to cohomology, we have

$$\ker\left(\nu^* \colon H^1(C, \mathbb{Q}) \to H^1\left(\widetilde{C}, \mathbb{Q}\right)\right) = W_0 H^1(C, \mathbb{Q}) \ . \tag{2.78}$$

So, the dual basis element corresponding to the blue curve generates $W_0 H^1(C, \mathbb{Q})$.

Dual basis elements corresponding to the red curves are generators of $f_* H^1\left(\overline{\widetilde{C}}, \mathbb{Q}\right)$, and these freely generate $\mathrm{Gr}_1^W H^1(C, \mathbb{Q})$. .

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February 26, 2020

4. Poincaré duality

Let X be an irreducible variety of dimension $\dim_{\mathbb{C}} X = n$. If X is smooth then Poincaré duality tells us that the natural map

$$H^k(X,\mathbb{Q}) \times H_c^{2n-k}(X,\mathbb{Q}) \xrightarrow{\smile} H_c^{2n}(X,\mathbb{Q}) \xrightarrow{\int_X} \mathbb{Q} \tag{2.79}$$

is a perfect pairing. The basic properties of the weight filtration (i.e., the facts that weights are additive under tensor product, and that there are no nontrivial natural maps between cohomology groups of different weights, as discussed below) ensure that this induces perfect pairings

$$\mathrm{Gr}_j^W H^k(X,\mathbb{Q}) \times \mathrm{Gr}_{2n-j}^W H_c^{2n-k}(X,\mathbb{Q}) \to \mathrm{Gr}_{2n}^W H_c^{2n}(X,\mathbb{Q}) = H_c^{2n}(X,\mathbb{Q}) \ . \tag{2.80}$$

For arbitrary X, the idea behind the weight filtration is that

$$\mathrm{Gr}_j^W H^k(X,\mathbb{Q}) \tag{2.81}$$

carries a (pure) Hodge structure of weight j. In other words,

$$\mathrm{Gr}_j^W H^k(X,\mathbb{C}) = \mathrm{Gr}_j^W H^k(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{2.82}$$

$$= \bigoplus_{p+q=j} H^{p,q}\left(\mathrm{Gr}_j^W H^k(X,\mathbb{C})\right) \tag{2.83}$$

and $H^{p,q} = \overline{H^{q,p}}$.

Important. All natural maps between cohomology groups of algebraic varieties respect Hodge structures and the p,q decomposition but not necessarily cohomological degree. In particular, there are no nontrivial maps between Gr_j^W and $\mathrm{Gr}_{j'}^W$ for $j \neq j'$.

Example 2.4. Let X be an algebraic variety with Zariski closed subset $V \subset X$. Then we have a long exact sequence

$$\cdots \longrightarrow H^k(X,\mathbb{Q}) \longrightarrow H^k(V,\mathbb{Q}) \xrightarrow{\delta} H_c^{k+1}(X \setminus V,\mathbb{Q}) \longrightarrow H^{k+1}(X,\mathbb{Q}) \longrightarrow \cdots \ . \tag{2.84}$$

In particular, let X be $X = \tilde{\tilde{C}}$ from example 2.3. Take V to be the three points which were initially punctures. Not take $k = 0$. Then we get:

$$0 \to H^0(X,\mathbb{Q}) \to H^0(V,\mathbb{Q}) \xrightarrow{\delta} W_0 H_c^1(X \setminus V,\mathbb{Q}) \to W_0 H^1(X,\mathbb{Q}) = 0 \ . \tag{2.85}$$

Here, the first term is zero because $X \setminus V$ is not compact, and hence $H_c^0(X \setminus V,\mathbb{Q}) = 0$. The last term is zero because X is smooth and projective, and hence $H^1(X,\mathbb{Q})$ has pure weight 1.

Then

$$W_0 H_C^1(X \setminus V,\mathbb{Q}) \cong H^0(V,\mathbb{Q}) / H^0(X,\mathbb{Q}) \cong \widetilde{H}^0(V,\mathbb{Q}) \tag{2.86}$$

and applying Poincaré duality gives us

$$\mathrm{Gr}_2^W H^1(X \setminus V,\mathbb{Q}) \cong \widetilde{H}_0(V,\mathbb{Q}) \ . \tag{2.87}$$

4.1. Mayer-Vietoris. Recall the Mayer-Vietoris sequence. Let $X = U_1 \cup U_2$ for U_i open. Then we get a long exact sequence

(2.88)

$$\cdots \to H^k(X,\mathbb{Q}) \to H^k(U_1,\mathbb{Q}) \oplus H^k(U_2,\mathbb{Q}) \to H^n(U_1 \cap U_2,\mathbb{Q}) \xrightarrow{\delta} H^{n+1}(X,\mathbb{Q}) \to \cdots .$$

We will be especially interested in cases where U_1, U_2, and $U_1 \cap U_2$ are open subvarieties (or open tubular neighborhoods of closed subvarieties).

Now we have a Mayer-Vietoris spectral sequence. Assume

$$X = U_1 \cup \ldots \cup U_r \tag{2.89}$$

for open U_i. This gives us a spectral sequence. The E_0 page is.

$$E_0^{p,q} = \bigoplus_{\substack{I \subseteq \{1,\ldots,r\} \\ |I = q+1|}} \left(C^p \left(\bigcap_{i \in I} U_i \right), \mathbb{Q} \right) . \tag{2.90}$$

Two things are going on here. First, we are intersecting increasing numbers of open sets, which should remind you of Čech cohomology. We also have cochains of different degrees. So on this direct sum there are two differentials. One increases p (this is just the ordinary differential on cochains) and the other one is the combinatorial differential which increases the number of open sets being intersected. Then the E_1 page is given by:

$$E_1^{p,q} = \bigoplus_{\substack{I \subseteq \{1,\ldots,r\} \\ |I| = q+1}} \left(H^p \left(\bigcap_{i \in I} U_i \right), \mathbb{Q} \right) . \tag{2.91}$$

If each U_i deformation retracts to a smooth projective variety, then the yoga of weights implies the spectral sequence collapses at E_2. This is because past the E_2 page, we will be looking at maps between things of different weight, and therefore they are zero, so the sequence collapses.

Bibliography

[1] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, *Geometry of algebraic curves. Volume II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris. MR2807457

[2] Pierre Deligne, *Théorie de Hodge. I*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, 1971, pp. 425–430. MR0441965

[3] ———, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57. MR498551

[4] ———, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77. MR498552

[5] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained. MR2222646

[6] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. MR1234037

[7] Alexander Grothendieck, *Fondements de la géométrie algébrique*, Séminaire Bourbaki, Vol. 7, 1995, pp. 297–307. MR1611235

[8] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR1631825

[9] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. MR0463157

[10] Robert Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. MR2095472

[11] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications. MR1917232

[12] Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008. MR2393625

[13] Ravi Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces*, Invent. Math. **164** (2006), no. 3, 569–590. MR2227692

[14] Claire Voisin, *Hodge theory and complex algebraic geometry. I*, English, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2007. Translated from the French by Leila Schneps. MR2451566

[15] ———, *Hodge theory and complex algebraic geometry. II*, English, Cambridge Studies in Advanced Mathematics, vol. 77, Cambridge University Press, Cambridge, 2007. Translated from the French by Leila Schneps. MR2449178