

# **Moduli spaces and tropical geometry**

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Spring 2020; Notes by: Jackson Van Dyke; All errors introduced are my own.

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FIGURE 1. The 5-wheel.

### 1. Overview

Our goal is to understand the proof of the following theorem:

**THEOREM 0.1.**  $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$  grows exponentially with  $g$ .

**REMARK 0.1.**  $\mathcal{M}_g$  has complex dimension  $3g - 3$ .

This theorem defied previous expectations.

**CONJECTURE 1** (Kontsevich (1993), Church-Farb-Putman (2014)). For fixed  $k > 0$ ,  $H^{4g-4-k}(\mathcal{M}_g, \mathbb{Q}) = 0$  for  $g \gg 0$ .

The structure of the course is as follows.

- Constructing the moduli space
  - (1) Nodal curves and stable reduction theorem
  - (2) Deformation theory of nodal curves
  - (3) The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  - (1) Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  - (2) Dual complexes of normal crossings divisors (tropical geometry)
  - (3) Boundary complex of  $\mathcal{M}_g$  (tropical moduli space)
- Cohomology of  $\mathcal{M}_h$ 
  - (1) Stable cohomology (Madsen-Weiss 2007)
  - (2) Virtual cohomological dimension of  $\mathcal{M}_g$  (Harer 84) (Vanishing of  $H^{4g-5}$  (Church-Farb-Putman, Morita-Sakasai-Suzuki))
  - (3) Euler characteristic of  $\mathcal{M}_g$  (Harer-Zagier 86)
- Graph complexes (Kontsevich 93)
  - (1) Feynman amplitudes and wheel classes. See Fig. 1 for the 5-wheel.
  - (2) Grothendieck-Teichmüller Lie algebra
  - (3) Willwacher's theorem
- Mixed Tate motives (MTM) over  $\mathbb{Z}$ 
  - (1) Mixed Tate motives
  - (2) Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements (closely related to Drinfeld's associators) generate a free Lie subalgebra."
  - (3) Proof of exponential growth of  $H^{4g-6}$ .

Lecture 1;  
Wednesday January  
22, 2020

Lecture 2; January  
24, 2020

## Part 1

# Constructing the moduli space

## CHAPTER 1

# Nodal curves and stable reduction theorem

### 1. Nodal curves

We will work over  $\mathbb{C}$ . We want to show that nodal curves, and families thereof, can be written in a normal form in local coordinates. We will follow chapter X of [1].

**DEFINITION 1.1.** A *nodal curve* is a complete curve such that every singular point has a neighborhood isomorphic (analytically over  $\mathbb{C}$ ) to a neighborhood of 0 in  $(xy = 0) \subset \mathbb{C}^2$ .

**DEFINITION 1.2.** A *family of nodal curves* over a base  $S$  is a flat proper surjective morphism  $\mathcal{C} \rightarrow S$  such that every geometric fiber is a nodal curve.

Recall that a flat morphism is the agreed upon notion of a map for which the fibers form a continuously varying family of schemes (or complex analytic spaces, varieties, etc.). Properness is saying that nothing can “disappear” as we approach any particular point in the base.

**Proposition 1.1.** *Let  $\pi : X \rightarrow S$  be a proper surjective morphism of  $\mathbb{C}$ -analytic spaces. This is a family of nodal curves iff at every point  $p \in X$  either  $\pi$  is smooth at  $p$  with one-dimensional fiber, or there is a neighborhood of  $p$  which is isomorphic (over  $S$ ) to a neighborhood of  $(0, s)$  in  $(xy = F) \subseteq \mathbb{C}^2 \times S$  where  $s = \pi(p)$  and  $F \in \mathfrak{m}_S \subseteq \mathcal{O}_{S,s}$ .*

**Lemma 1.2.** *Let  $f$  be holomorphic at  $0 \in \mathbb{C}^2$ . Then  $(f = 0)$  has a node at 0 iff*

$$(1.1) \quad 0 = f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

*at 0, and the Hessian of  $f$  at 0 is non-singular.*

This tells us that these nodes are the “simplest” possible singularities.

**PROOF.** ( $\implies$ ): This direction is immediate.

( $\impliedby$ ): Suppose  $0 = f = \partial_x f = \partial_y f$  at 0. Then

$$(1.2) \quad f = a - x^2 + 2bxy + cy^2$$

where  $a$ ,  $b$ , and  $c$  are holomorphic functions. The Hessian is

$$(1.3) \quad \begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix}$$

so being non-singular means exactly that

$$(1.4) \quad b^2 - ac \neq 0.$$

After possibly making a linear change of coordinates, we can assume  $a \neq 0$ , and change to coordinates

$$(1.5) \quad x_1 = x + \frac{b}{a}y \qquad y_1 = y$$

so we can write

$$(1.6) \quad f = a_1 x_1^2 + c_1 y_1^2$$

where  $a_1(0), c_1(0) \neq 0$ . Choose square roots<sup>1.1</sup>  $\alpha$  and  $\gamma$  of  $a_1$  and  $c_1$ . Now replace  $x_1$  and  $y_1$  by  $x_2 = \alpha x_1$  and  $y_2 = \gamma y_1$  so that

$$(1.7) \quad f = x_2^2 + y_2^2.$$

Now for  $x_3 = x_2 + iy_2$  and  $y_3 = x_2 - iy_2$ , we have  $f = x_3 y_3$ .  $\square$

PROOF OF PROPOSITION 1.1. Let  $\pi : X \rightarrow S$  be proper and surjective. Consider  $x \in X$ . Then either  $\pi$  is smooth with 1-dimensional fiber at  $x$  (nothing to show) or  $x$  is a node in  $\pi^{-1}(s)$ ,  $s = \pi(x)$ . Locally near  $x$  we can embed  $X \subseteq \mathbb{C}^r \times S$ . This is locally closed (over  $S$ ). Now we have a left exact sequence of tangent spaces:

$$(1.8) \quad 0 \rightarrow T_x X_s \rightarrow T_x X \rightarrow T_s S$$

where  $\dim T_x X_s = 2$ . Now choose a linear projection  $\mathbb{C}^r \rightarrow \mathbb{C}^2$  which is an isomorphism on  $T_x X_s$ . Using this projection we get a composition:

$$(1.9) \quad T_x X \subseteq \mathbb{C}^r \times T_s S \xrightarrow{\quad} \mathbb{C}^2 \times T_s S$$

and we claim this is injective. Then the implicit function theorem tells us that there is a neighborhood of  $x$  which embeds in  $\mathbb{C}^2 \times S$  (over  $S$ ). We should think of this as a family of plane curves: each fiber has a single defining equation. More specifically we have the following.

FACT 1 ([Lemma 31.18.9 \(Stacks project\)](#)). *If  $\mathcal{Y} \rightarrow S$  is a smooth morphism and  $D \subseteq \mathcal{Y}$  is flat over  $S$ , codimension 1 in  $\mathcal{Y}$ , then  $D$  is a Cartier divisor.*

By Fact 1,  $X \subseteq \mathbb{C}^2 \times S$  is locally defined by a single equation  $F = 0$ . Now consider  $\partial_x F$ ,  $\partial_y F$ , and the Hessian of  $F$  with respect to  $x$  and  $y$ . Then the proof of Lemma 1.2 shows that

$$(1.10) \quad F = x_3 y_3 - f$$

where  $f$  is a function on  $S$  which vanishes at  $s$ .  $\square$

Lecture 3; January  
27, 2020

## 2. Stability of nodal curves

The following is a corollary of Proposition 1.1.

**Corollary 1.3.** *A family of nodal curves is local complete intersection (lci).*

This means that there is a relative dualizing sheaf  $\omega_{X/S}$  which is locally free of rank 1.

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<sup>1.1</sup>There is some subtlety here since these are functions rather than scalars. Because  $a_1$  and  $c_1$  are nonzero at 0, we can ensure that the image of  $a_1$  and  $c_1$  are, say, contained in an open half space. Now we can choose a branch of log which is defined on this half space. Then multiply by 1/2 and exponentiate.

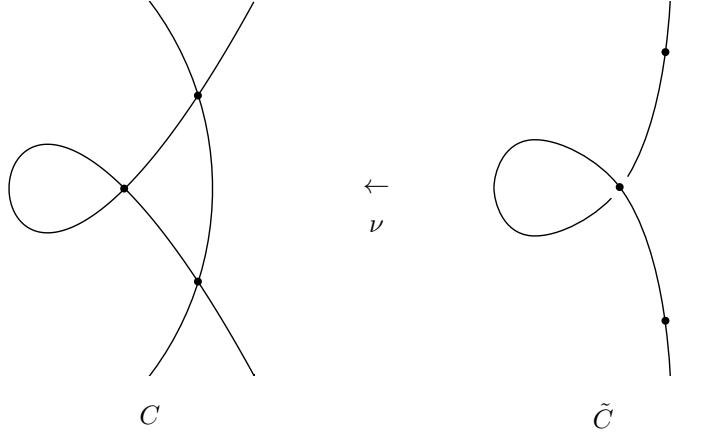


FIGURE 1. The normalization of a nodal curve. The nodal points of  $C$  each have two preimages under the normalization  $\nu$ .

**2.1. Serre duality.** The point here is that the things we already know about smooth curves are also true about nodal ones.

Let  $C$  be a nodal curve (over a point). There is a (natural) isomorphism  $H^1(C, \omega_C) \cong \mathbb{C}$ . Then Serre duality tells us that for any coherent sheaf  $\mathcal{F}$  on  $C$ ,

$$(1.11) \quad H^1(C, \mathcal{F}) \times \text{Hom}(\mathcal{F}, \omega_C) \rightarrow H^1(C, \omega_C) \cong \mathbb{C}$$

is a perfect pairing, i.e.

$$(1.12) \quad H^1(C, \mathcal{F}) \cong \text{Hom}(\mathcal{F}, \omega_C)^\vee.$$

In particular, if  $\mathcal{F}$  is a vector bundle, then

$$(1.13) \quad H^1(C, \mathcal{F}) \cong H^0(C, \mathcal{F}^\vee \otimes \omega_C)^\vee.$$

We can form the normalization<sup>1,2</sup> of a nodal curve as in Fig. 1.

Suppose  $C$  is nodal with components  $C_1, \dots, C_s$  and nodes  $x_1, \dots, x_r$ . Let  $\tilde{C} \xrightarrow{\nu} C$  be the normalization. Write  $\tilde{C}_i$  for the normalization of  $C_i$  and

$$(1.14) \quad \{p_j, q_j\} = \nu^{-1}(x_j)$$

(for  $i \in \{1, \dots, s\}$  and  $j \in \{1, \dots, r\}$ ).

A line bundle  $L$  on  $C$  has *multi-degree*  $\underline{\deg}(L)$  to be

$$(1.15) \quad \underline{\deg}(L) = (\deg(L|_{C_1}), \dots, \deg(L|_{C_s}))$$

$$(1.16) \quad = (\deg(\nu^*L|_{\tilde{C}_1}), \dots, \deg(\nu^*L|_{\tilde{C}_s})).$$

The following is a corollary to Serre duality.

**Corollary 1.4.** *If  $C$  is connected, and  $\underline{\deg}(L) > \underline{\deg}(\omega_C)$  then  $H^1(C, L) = 0$ .*

By  $\underline{\deg}(L) > \underline{\deg}(\omega_C)$  we mean that for all  $i$   $\deg(L|_{C_i}) \geq \deg(\omega_C|_{C_i})$  for all  $i$  and  $\underline{\deg}(L) \neq \underline{\deg}(\omega_C)$ .

<sup>1,2</sup>The algebraic construction is taking the integral closure in the fraction field.



PROOF. First note

$$(1.17) \quad H^1(C, L) \cong H^0(C, \omega_C \otimes L^{-1}) .$$

and  $\deg(\omega_C \otimes L^{-1}) < 0$ .

On any connected component  $C_i$  such that  $\deg(\omega_L \otimes L^{-1})|_{C_i} < 0$  all sections vanish. And all sections vanish on components that  $C_i$ , etc.  $\square$

**Corollary 1.5.**  *$L$  is ample iff  $\deg(L|_{C_i}) > 0$  for all  $i$ .*

PROOF. ( $\implies$ ): This direction is clear. The restriction of ample  $L$  to any component is still ample.

( $\impliedby$ ): Suppose  $\deg(L|_{C_i}) > 0$ . It is enough to show that  $L^{\otimes N}$  is very ample for some  $N$ . Choose  $N$  sufficiently large so that

$$(1.18) \quad \deg(L^{\otimes N}|_{C_i}) > \deg(\omega_C|_{C_i}) + 2 .$$

Let  $S \subseteq C$  be a 0-dimensional subscheme of length 2. Then we have a short exact sequence

$$(1.19) \quad 0 \rightarrow I_S \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_S \rightarrow 0$$

which we can tensor with  $L^{\otimes N}$  to get a sequence which is still exact, which gives us a long exact sequence

$$(1.20) \quad 0 \rightarrow H^0(L^{\otimes N}(-s)) \rightarrow H^0(L^{\otimes N}) \rightarrow H^0(L^{\otimes N}|_S) \rightarrow H^1(L^{\otimes N}(-s)) \rightarrow \dots$$

but  $H^1(L(-s)) = 0$ , so we have a surjection

$$(1.21) \quad H^0(L) \rightarrow H^0(L|_S) .$$

This is the kind of argument to show separation of points and tangent vectors, which is a criterion for very ampleness.  $\square$

### 3. Description of $\omega_C$

See chapter 6 of [4] chapter 3 §A from [2] for references.

**Proposition 1.6.** *Let  $C$  be a nodal curve with nodes  $x_1, \dots, x_r$ , write  $(p_i, q_i) = \nu^{-1}(x_i)$ . Then*

$$(1.22) \quad \omega_C \cong \nu_* (\omega'_{\tilde{C}}(p_1 + q_1 + \dots + p_r + q_r))$$

where  $\omega'_{\tilde{C}}(p_1 + \dots + q_r) \subseteq \omega_{\tilde{C}}(p_1, \dots, q_r)$  is the subsheaf where

$$(1.23) \quad \text{res}_{p_i}(\omega) + \text{res}_{q_i}(\omega) = 0 .$$

REMARK 1.1 (Rosenlicht differentials). There is a related explicit description of  $\omega_{X/S}$  for a family of nodal curves. Near a point where  $X/S \cong (xf = F) \subseteq \mathbb{C}^2 \times S$   $\omega_{C/S}$  is generated by  $dx/x$  and  $dy/y$  which satisfy

$$(1.24) \quad \frac{dx}{x} + \frac{dy}{y} = 0 .$$

DEFINITION 1.3. A nodal curve is *stable* if  $\omega_C$  is ample.

**Proposition 1.7.** *Let  $X \rightarrow S$  be a family of nodal curves. Then*

$$\{s \in S \mid X_s \text{ is stable}\}$$

*is Zariski open.*

PROOF. Let  $L$  be any line bundle on  $X$ . Then

$$\{s \in S \mid L|_{X_s} \text{ is ample}\}$$

is Zariski-open. This is Theorem 1.2.17 of [3].  $\square$

**THEOREM 1.8.** *A nodal curve  $C$  is stable iff  $\text{Aut}(C)$  is finite.*

PROOF. Say  $C$  has components  $C_1, \dots, C_s$  and nodes  $x_1, \dots, x_r$ . Write  $\{p_i, q_i\} = \nu^{-1}(x_i)$  for the preimage of the nodes under the normalization  $\nu$ . Write  $Q = \{p_1, q_1, \dots, p_r, q_r\}$ . Notice that  $\text{Aut}(C)$  is finite iff

$$\{\sigma \in \text{Aut}(C) \mid \sigma \text{ acts by 1 on } \{C_1, \dots, C_s\}\}$$

is finite.

This is true iff for all  $C_i$ , there are only finitely many automorphisms of  $\tilde{C}_i$  that fix  $Q \cap \tilde{C}_i$ . And this is equivalent to it being the case that for all  $C_i$  either:

- (1)  $g(\tilde{C}_i) \geq 2$ ;
- (2)  $g(\tilde{C}_i) = 1$ , and  $Q \cap \tilde{C}_i \neq \emptyset$ ; or
- (3)  $g(\tilde{C}_i) = 0$  and  $Q \cap \tilde{C}_i \geq 3$ .

This is equivalent to

$$2g(\tilde{C}_i) - 2 + \#(Q \cap \tilde{C}_i) > 0,$$

which is equivalent to  $\deg(\omega_C|_{C_i}) > 0$  for all  $i$ . So a curve is stable iff the degree of the dualizing sheaf is positive on every component, which is equivalent to  $\omega_C$  being ample.  $\square$

**DEFINITION 1.4.** A *graph*  $G$  is a set  $X(G)$  together with an involution  $i : X(G) \rightarrow X(G)$  and a retraction  $r : X(G) \rightarrow X(G)^i$ . The vertices  $V(G)$ , half edges  $H(G)$ , and edges  $E(G)$  are defined as:

$$\begin{aligned} V(G) &= X(G)^i \\ H(G) &= X(G) \setminus V(G) \\ E(G) &= H(G)/i. \end{aligned}$$

We say  $r(h)$  is the vertex incident to  $h \in H(G)$ .

The *dual graph*  $G(C)$  of a nodal curve  $C$  is as follows. The vertices  $\{v_1, \dots, v_s\}$  correspond to the components  $C_1, \dots, C_s$ ; and the half-edges incident to  $v_i$  are given by the points of  $\tilde{C}_i \cap Q$ . An edge is made from a pair of half-edges corresponding to a pair  $\{p_i, q_i\}$ . The “genus function” assigns the genus of  $\tilde{C}_i$  to the corresponding vertex  $v_i$ . See Fig. 2 for examples.

We can read the stability off from the dual graph. Every vertex labelled with a 1 should have at least one incident edge, and all unlabelled should have valence at least 3.

Recall that the arithmetic genus of a curve  $C$  is

$$p_a(C) = 1 - \chi(\mathcal{O}_C).$$

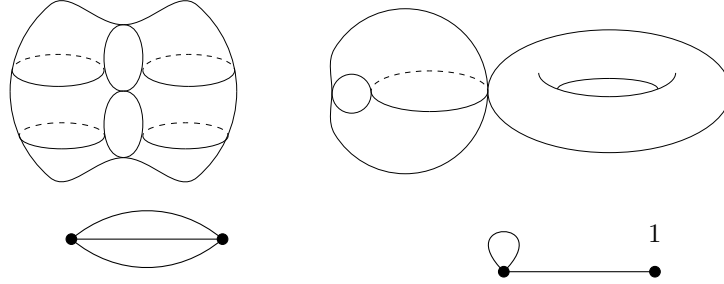


FIGURE 2. Two examples of genus 2 stable curves with their dual graphs below them. Notice we can read their stability off from the graphs. All unlabelled vertices have at least three incident edges, and the labelled one has one incident edge.

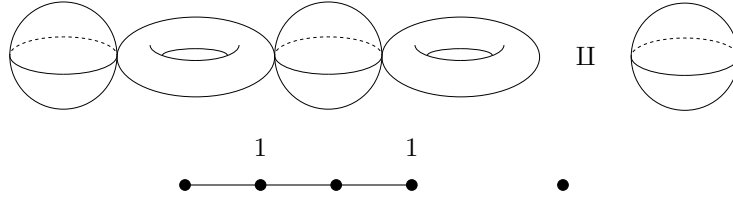


FIGURE 3. An example of an unstable genus 2 curve with its dual graph below it. Notice we can read the fact that it is unstable off of the graph. All three unlabelled vertices of valence less than 3.

In particular, if  $C$  is connected then  $p_a(C) = h^1(\mathcal{O}_C)$ . Recall the Euler characteristic of a graph  $G$  is

$$\chi(G) = \#V(G) - \#E(G) .$$

Note if  $G$  is connected, then  $h^1(G) = 1 - \chi(G)$ . Also note that  $C$  is connected iff  $G(C)$  is connected. The dual graph also detects the arithmetic genus in the following sense.

**THEOREM 1.9.** *Let  $C$  be a nodal curve. Then*

$$(1.25) \quad p_a(C) = 1 - \chi(G(C)) + \sum_v g(v) .$$

**Corollary 1.10.** *If  $C$  is connected then*

$$(1.26) \quad p_a(C) = \sum_v g(v) + h^1(G) .$$

**PROOF OF THEOREM 1.9.** Proceed by induction on the number of nodes  $\#E(G) = \#C^{\text{sing}}$ . The base case is when  $E(G) = \emptyset$ , so the graph is just  $s$  vertices  $v_i$  with genus  $g(v_i)$ . Then

$$(1.27) \quad 1 - \chi(\mathcal{O}_C) = 1 - s + \sum_i g(v_i)$$

as desired.

Now suppose  $C'$  is obtained from  $C$  by gluing two smooth points  $p, q$  to  $x$ . Write  $\pi : C \rightarrow C'$ . Then we have an exact sequence of sheaves

$$(1.28) \quad 0 \rightarrow \mathcal{O}_{C'} \rightarrow \pi_* \mathcal{O}_C \rightarrow \mathcal{O}_X \rightarrow 0$$

which implies the Euler characteristic of the middle term is the Euler characteristic of the other two terms. Now since this gluing is proper and finite, it doesn't change the Euler characteristic. All together this gives us:

$$(1.29) \quad \chi(\mathcal{O}_C) = \chi(\pi_* \mathcal{O}_C) = \chi(\mathcal{O}_{C'}) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C'}) + 1 .$$

□

## Bibliography

- [1] E. Arbarello, J.D. Harris, M. Cornalba, and P. Griffiths, *Geometry of algebraic curves: Volume ii with a contribution by joseph daniel harris*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2011.
- [2] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, Springer New York, 1998.
- [3] R.K. Lazarsfeld, *Positivity in algebraic geometry ii: Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, Springer Berlin Heidelberg, 2017.
- [4] Q. Liu and R. Erne, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics (0-19-961947-6), Oxford University Press, 2006.