Moduli spaces and tropical geometry

Lectures by: Professor Sam Payne

Spring 2020; Notes by: Jackson Van Dyke; All errors introduced are my own.

Contents

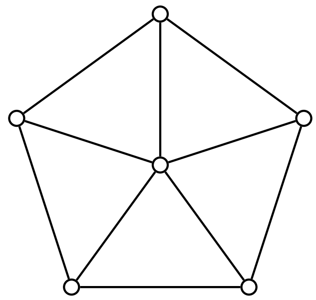

FIGURE 1. The 5-wheel.

1. Overview

Our goal is to understand the proof of the following theorem: Lecture 1; Wednesday January 22, 2020

THEOREM 0.1. $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ *grows exponentially with* g.

REMARK 0.1. $\mathcal{M}_g$ has complex dimension $3g-3$.

This theorem defied previous expectations.

CONJECTURE 1 (Kontsevich (1993), Church-Farb-Putman (2014)). *For fixed* $k > 0$, $H^{4g-4-k}(\mathcal{M}_j, \mathbb{Q}) = 0$ *for* $g \gg 0$.

The structure of the course is as follows.

- Constructing the moduli space
  (1) Nodal curves and stable reduction theorem
  (2) Deformation theory of nodal curves
  (3) The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  (1) Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  (2) Dual complexes of normal crossings divisors (tropical geometry)
  (3) Boundary complex of $\mathcal{M}_g$ (tropical moduli space)
- Cohomology of $\mathcal{M}_h$
  (1) Stable cohomology (Madsen-Weiss 2007)
  (2) Virtual cohomological dimension of $\mathcal{M}_g$ (Harer 84) (Vanishing of H^{4g-5} (Church-Farb-Putman, Morita-Sakasai-Suzuki))
  (3) Euler characteristic of $\mathcal{M}_g$ (Harer-Zagier 86)
- Graph complexes (Kontsevich 93)
  (1) Feynman amplitudes and wheel classes. See Fig. 1 for the 5-wheel.
  (2) Grothendieck-Teichmüller Lie algebra
  (3) Willwacher's theorem
- Mixed Tate motives (MTM) over $\mathbb{Z}$
  (1) Mixed Tate motives
  (2) Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements (closely related to Drinfeld's associators) generate a free Lie subalgebra."
  (3) Proof of exponential growth of H^{4g-6}.

Lecture 2; January 24, 2020

Part 1

Constructing the moduli space

CHAPTER 1

Nodal curves and stable reduction theorem

1. Nodal curves

We will work over $\mathbb{C}$. We want to show that nodal curves, and families thereof, can be written in a normal form in local coordinates. We will follow chapter X of [**1**].

DEFINITION 1.1. A *nodal curve* is a complete curve such that evert singular point has a neighborhood isomorphic (analytically over $\mathbb{C}$) to a neighborhood of 0 in $(xy = 0) \subset \mathbb{C}^2$.

DEFINITION 1.2. A *family of nodal curves* over a base S is a flat proper surjective morphism $\mathcal{C} \to S$ such that every geometric fiber is a nodal curve.

Recall that a flat morphism is the agreed upon notion of a map for which the fibers form a continuously varying family of schemes (or complex analytic spaces, varieties, etc.). Properness is saying that nothing can "disappear" as we approach any particular point in the base.

Proposition 1.1. *Let $\pi : X \to S$ be a proper surjective morphism of $\mathbb{C}$-analytic spaces. This is a family of nodal curves iff at every point $p \in X$ either π is smooth at p with one-dimensional fiber, or there is a neighborhood of p which is isomorphic (over S) to a neighborhood of $(0, s)$ in $(xy = F) \subseteq \mathbb{C}^2 \times S$ where $s = \pi(p)$ and $F \in \mathfrak{m}_S \subseteq \mathcal{O}_{S,s}$.*

Lemma 1.2. *Let f be holomorphic at $0 \in \mathbb{C}^2$. Then $(f = 0)$ has a node at 0 iff*

$$0 = f = \frac{\partial f}{\partial x} = \frac{pf}{\partial y} \tag{1.1}$$

at 0, and the Hessian of f at 0 is non-singular.

This tells us that these nodes are the "simplest" possible singularities.

PROOF. ($\Longrightarrow$): This direction is immediate.
($\Longleftarrow$): Suppose $0 = f = \partial_x f = \partial_y f$ at 0. Then

$$f = a - x^2 + 2bxy + cy^2 \tag{1.2}$$

where a, b, and c are holomorphic functions. The Hessian is

$$\begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix} \tag{1.3}$$

so being non-singular means exactly that

$$b^2 - ac \neq 0\ . \tag{1.4}$$

After possible making a linear change of coordinates, we can assume $a \neq 0$, and change to coordinates

$$x_1 = x + \frac{b}{a} y \qquad\qquad y_1 = y \tag{1.5}$$

so we can write

$$f = a_1 x_1^2 + c_1 y_1^2 \tag{1.6}$$

where $a_1(0), c_1(0) \neq 0$. Choose square roots[1.1] α and γ of a_1 and c_1. Now replace x_1 and y_1 by $x_2 = \alpha x_1$ and $y_2 = \gamma y_1$ so that

$$f = x_2^2 + y_2^2\ . \tag{1.7}$$

Now for $x_3 = x_2 + iy_2$ and $y_3 = x_2 - iy_2$, we have $f = x_3 y_3$. $\square$

Proof of Proposition 1.1. Let $\pi : X \to S$ be proper and surjective. Consider $x \in X$. Then either π is smooth with 1-dimensional fiber at x (nothing to show) or x is a node in $\pi^{-1}(s)$, $s = \pi(x)$. Locally near x we can embed $X \subseteq \mathbb{C}^r \times S$. This is locally closed (over S). Now we have a left exact sequence of tangent spaces:

$$0 \longrightarrow T_x X_s \longrightarrow T_x X \longrightarrow T_s S \tag{1.8}$$

where $\dim T_x X_s = 2$. Now choose a linear projection $\mathbb{C}^r \to \mathbb{C}^2$ which is an isomorphism on $T_x X_s$. Using this projection we get a composition:

$$\begin{tikzcd} T_x X \arrow[rr, bend right] & \subseteq \ \mathbb{C}^r \times T_s S \arrow[r] & \mathbb{C}^2 \times T_s S \end{tikzcd} \tag{1.9}$$

and we claim this is injective. Then the implicit function theorem tells us that there is a neighborhood of x which embeds in $\mathbb{C}^2 \times S$ (over S). We should think of this as a family of plane curves: each fiber has a single defining equation. More specifically we have the following.

Fact 1 (Lemma 31.18.9 (Stacks project)). *If $\mathcal{Y} \to S$ is a smooth morphism and $D \subseteq \mathcal{Y}$ is flat over S, codimension 1 in $\mathcal{Y}$, then D is a Cartier divisor.*

By Fact 1, $X \subseteq \mathbb{C}^2 \times S$ is locally defined by a single equation $F = 0$. Now consider $\partial_x F$, $\partial_y F$, and the Hessian of F with respect to x and y. Then the proof of Lemma 1.2 shows that

$$F = x_3 y_3 - f \tag{1.10}$$

where f is a function on S which vanishes at s. $\square$

Lecture 3; January 27, 2020

2. Stability of nodal curves

The following is a corollary of Proposition 1.1.

Corollary 1.3. *A family of nodal curves $\pi : X \to S$ is a local complete intersection (lci) morphism.*

This implies that there is a relative dualizing sheaf $\omega_{X/S}$ which is locally free of rank 1.

[1.1] There is some subtly here since these are functions rather than scalars. Because a_1 and c_1 are nonzero at 0, we can ensure that the image of a_1 and c_1 are, say, contained in an open half space. Now we can choose a branch of log which is defined on this half space. Then multiply by 1/2 and exponentiate.

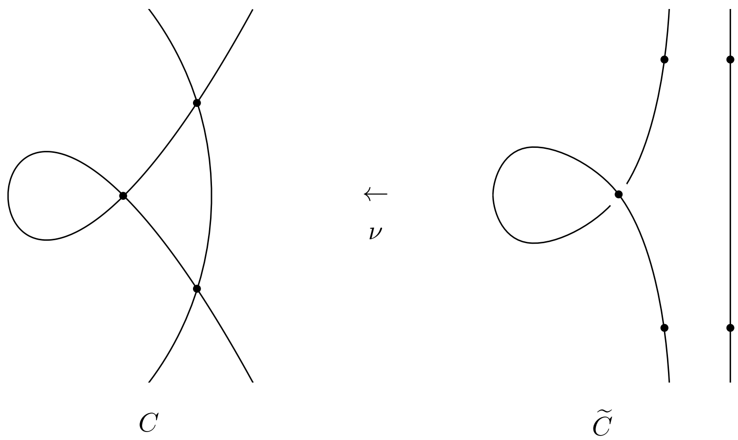

FIGURE 1. The normalization of a nodal curve. The nodal points of C each have two preimages under the normalization ν.

2.1. Serre duality. The point here is that the duality properties that we already know about for smooth curves extend naturally to nodal ones.

Let C be a nodal curve (over a point). There is a (natural) isomorphism $H^1(C, \omega_C) \cong \mathbb{C}$. Then Serre duality tells us that for any coherent sheaf $\mathcal{F}$ on C,

$$H^1(C, \mathcal{F}) \times \operatorname{Hom}(\mathcal{F}, \omega_C) \to H^1(C, \omega_C) \cong \mathbb{C} \tag{1.11}$$

is a perfect pairing, i.e.

$$H^1(C, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{F}, \omega_C)^\vee \ . \tag{1.12}$$

In particular, if $\mathcal{F}$ is a vector bundle, then

$$H^1(C, \mathcal{F}) \cong H^0(C, \mathcal{F}^\vee \otimes \omega_C)^\vee \ . \tag{1.13}$$

We can form the normalization[1.2] of a nodal curve as in Fig. 1.

Suppose C is nodal with components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Let $\widetilde{C} \xrightarrow{\nu} C$ be the normalization. Write $\widetilde{C}_i$ for the normalization of C_i and

$$\{p_j, q_j\} = \nu^{-1}(x_j) \tag{1.14}$$

(for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$).

A line bundle L on C has *multi-degree* $\underline{\deg}(L)$ to be

$$\underline{\deg}(L) = \left(\deg\left(L|_{C_1}, \ldots, \deg\left(L|_{C_s}\right)\right)\right) \tag{1.15}$$

$$= \left(\deg\left(\nu^* L|_{\widetilde{C}_1}\right), \ldots, \deg \nu^* L|_{\widetilde{C}_s}\right) \ . \tag{1.16}$$

The following is a corollary to Serre duality.

Corollary 1.4. *If C is connected, and $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ then $H^1(C, L) = 0$.*

By $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ we mean that $\deg\left(L|_{C_i}\right) \geq \deg\left(\omega_C|_{C_i}\right)$ for all i and $\underline{\deg}(L) \neq \underline{\deg}(\omega_C)$.

[1.2] Locally, the corresponding algebraic construction is taking the integral closure of the coordinate ring.

Proof. First note

$$H^1(C,L) \cong H^0\left(C, \omega_C \otimes L^{-1}\right) \ . \tag{1.17}$$

and $\underline{\deg}\left(\omega_C \otimes L^{-1}\right) < 0$.

On any connected component C_i such that $\deg\left(\omega_L \otimes L^{-1}\right)\big|_{C_i} < 0$ all sections vanish. And all sections vanish on components that meet C_i, etc. ☐

Corollary 1.5. *L is ample if and only if $\deg\left(L|_{C_i}\right) > 0$ for all i.*

Proof. ($\implies$): This direction is clear. The restriction of ample L to any component is still ample.

($\Longleftarrow$): Suppose $\deg\left(L|_{C_i}\right) > 0$. It is enough to show that $L^{\otimes N}$ is very ample for some N. Choose N sufficiently large so that

$$\deg\left(L^{\otimes N}\big|_{C_i}\right) > \deg\left(\omega_C|_{C_i}\right) + 2 \ . \tag{1.18}$$

Let $S \subseteq C$ be the union of two distinct smooth points. Then we have a short exact sequence

$$0 \to I_S \to \mathcal{O}_C \to \mathcal{O}_S \to 0 \tag{1.19}$$

which we can tensor with $L^{\otimes N}$ to get a sequence which is still exact, which gives us a long exact sequence

(1.20)
$$0 \longrightarrow H^0\left(L^{\otimes N}(-s)\right) \longrightarrow H^0\left(L^{\otimes N}\right) \longrightarrow H^0\left(L^{\otimes N}\big|_S\right) \longrightarrow H^1\left(L^{\otimes N}(-s)\right) \longrightarrow \dots$$

but $H^1(L(-s)) = 0$, so we have a surjection

$$H^0(L) \twoheadrightarrow H^0\left(L|_S\right) \ . \tag{1.21}$$

This shows that sections of $L^{\otimes N}$ separate the two points in S. Similar arguments show that sections of high tensor powers of L separate arbitrary pairs of points and tangent vectors. Therefore, high tensor powers of L are very ample, and so L is ample. ☐

3. Description of ω_C

We now describe the canonical sheaf of a nodal curve in terms of meromorphic differential forms. See [**4**, Chapter 6] or [**2**, Chapter 3, Section A] for proofs and further details.

Proposition 1.6. *Let C be a nodal curve with nodes $x_1, \dots, x_r$, write $(p_i, q_i) = \nu^{-1}(x_i)$. Then*

$$\omega_C \cong \nu_*\left(\omega'_{\widetilde{C}}(p_1 + q_1 + \dots + p_r + q_r)\right) \tag{1.22}$$

where $\omega'_{\widetilde{C}}(p_1 + \dots + q_r) \subseteq \omega_{\widetilde{C}}(p_1, \dots, q_r)$ is the subsheaf where

$$\mathrm{res}_{p_i}(\omega) + \mathrm{res}_{q_i}(\omega) = 0 \ . \tag{1.23}$$

Remark 1.1 (Rosenlicht differentials). There is a related explicit description of $\omega_{X/S}$ for a family of nodal curves. Near a point where $X/S \cong (xf = F) \subseteq \mathbb{C}^2 \times S$ $\omega_{C/S}$ is generated by dx/x and dy/y which satisfy

$$\frac{dx}{x} + \frac{dy}{y} = 0 \ . \tag{1.24}$$

Definition 1.3. A nodal curve is *stable* if ω_C is ample.

Lecture 4; January 29, 2020

Proposition 1.7. *Let $X \to S$ be a family of nodal curves. Then*

$$\{s \in S \mid X_s \text{ is stable}\}$$

is Zariski open.

PROOF. Let L be any line bundle on X. Then

$$\left\{s \in S \,\middle|\, L|_{X_s} \text{ is ample}\right\}$$

is Zariski-open. This is Theorem 1.2.17 of [**3**]. □

THEOREM 1.8. *A nodal curve C is stable if and only if* $\operatorname{Aut}(C)$ *is finite.*

PROOF. Say C has components $C_1, \dots, C_s$ and nodes $x_1, \dots, x_r$. Write $\{p_i, q_i\} = \nu^{-1}(x_i)$ for the preimage of the nodes under the normalization ν. Write $Q = \{p_1, q_1, \dots, p_r, q_r\}$. Notice that $\operatorname{Aut}(C)$ is finite if and only if

$$\{\sigma \in \operatorname{Aut}(C) \mid \sigma \text{ acts by 1 on } \{C_1, \dots, C_s\}\}$$

is finite.

Fix C_i. Note that $\operatorname{Aut}(C_i)$ is finite if and only if there are only finitely many automorphisms of $\tilde{C}_i$ that fix $Q \cap \tilde{C}_i$. This is the case exactly when

(1) $g\left(\tilde{C}_i\right) \geq 2$;
(2) $g\left(\tilde{C}_i\right) = 1$, and $Q \cap \tilde{C}_i \neq \emptyset$; or
(3) $g\left(\tilde{C}_i\right) = 0$ and $Q \cap \tilde{C}_i \geq 3$.

By direct computation, these are precisely the cases where

$$2g\left(\tilde{C}_i\right) - 2 + \#\left(Q \cap \tilde{C}_i\right) > 0.$$

The left hand side is $\deg\left(\omega_C|_{C_i}\right)$, by our description of the dualizing sheaf in terms of meromorphic differentials.

So we have shown that $\operatorname{Aut}(C)$ is finite if and only if the degree of the dualizing sheaf is positive on every component, which is equivalent to ω_C being ample, i.e., to C being stable. □

DEFINITION 1.4. A *graph* G is a set $X(G)$ together with an involution $i : X(G) \circlearrowleft$ and a retraction $r : X(G) \to X(G)^i$. The vertices $V(G)$, half edges $H(G)$, and edges $E(G)$ are defined as:

$$\begin{aligned} V(G) &= X(G)^i \\ H(G) &= X(G) \setminus V(G) \\ E(G) &= H(G)/i\ . \end{aligned}$$

We say $r(h)$ is the vertex incident to $h \in H(G)$.

The *dual graph* $G(C)$ of a nodal curve C is as follows. The vertices $\{v_1, \dots, v_s\}$ correspond to the components $C_1, \dots, C_s$; and the half-edges incident to v_i are given by the points of $\tilde{C}_i \cap Q$. An edge is made from a pair of half-edges corresponding to a pair $\{p_i, q_i\}$. The "genus function" assigns the genus of $\tilde{C}_i$ to the corresponding vertex v_i. See Fig. 2 for examples.

We can read the stability off from the dual graph. Every vertex labelled with a 1 should have at least one incident edge, and all unlabelled vertices should have valence at least 3.

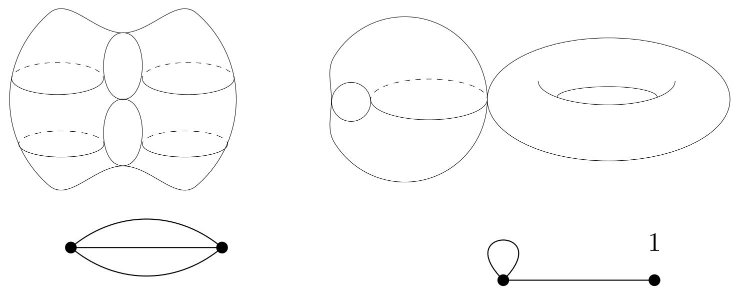

FIGURE 2. Two examples of genus 2 stable curves with their dual graphs below them. Notice we can read their stability off from the graphs. All unlabelled vertices have at least three incident edges, and the labelled one has one incident edge.

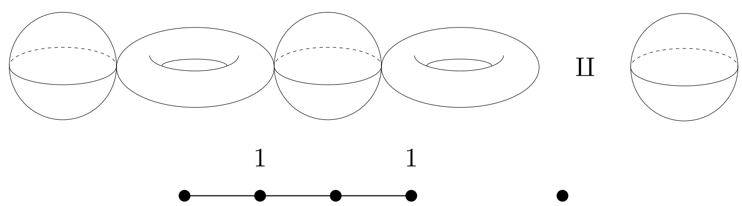

FIGURE 3. An example of an unstable genus 2 curve with its dual graph below it. Notice we can read the fact that it is unstable off of the graph. All three unlabelled vertices of valence less than 3.

Recall that the arithmetic genus of a curve C is

$$p_a(C) = 1 - \chi(\mathcal{O}_C) \ .$$

In particular, if C is connected then $p_a(C) = h^1(\mathcal{O}_C)$. Recall the Euler characteristic of a graph G is

$$\chi(G) = \#V(G) - \#E(G) \ .$$

Note if G is connected, then $h^1(G) = 1 - \chi(G)$. Also note that C is connected if and only if $G(C)$ is connected. The dual graph also detects the arithmetic genus in the following sense.

THEOREM 1.9. *Let C be a nodal curve. Then*

$$p_a(C) = 1 - \chi(G(C)) + \sum_v g(v) \ . \tag{1.25}$$

Corollary 1.10. *If C is connected then*

$$p_a(C) = \sum_v g(v) + h^1(G) \ . \tag{1.26}$$

PROOF OF THEOREM 1.9. Proceed by induction on the number of nodes $\#E(G) = \#C^{\text{sing}}$. The base case is when $E(G) = \emptyset$, so the graph is just s vertices v_i with genus $g(v_i)$. Then

$$1 - \chi(\mathcal{O}_C) = 1 - s + \sum_i g(v_i) \tag{1.27}$$

as desired.

Now suppose C' is obtained from C by gluing two smooth points p, q to x. Write $\pi : C \to C'$. Then we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \pi_* \mathcal{O}_C \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.28}$$

which implies the Euler characteristic of the middle term is the Euler characteristic of the other two terms. Now since this gluing is proper and finite, it doesn't change the Euler characteristic. Altogether this gives us:

$$\chi(\mathcal{O}_C) = \chi(\pi_* \mathcal{O}_C) = \chi(\mathcal{O}_{C'}) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C'}) + 1 \ . \tag{1.29}$$

□

Lecture 5; January 31, 2020

4. Stable reduction

There are two statements. The first is the nodal reduction theorem (which does not involve stability) and the second is stabilization, which adds uniqueness. The reference is [**1**] chapter X, section 4. Write

$$\Delta = \{z \in \mathbb{C} \mid |z| < \epsilon\} \tag{1.30}$$

for a small disk. Write $\Delta^\times = \Delta \setminus \{0\}$ for the punctured disk, both viewed as having one complex dimension.

Consider a flat proper surjective map $\pi : X \to \Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Write $X^\bullet$ for the complement of the fiber over 0. Let $k > 0$ be an integer. Consider the map $\varphi_k : \Delta' \to \Delta$ from the disk to itself given by $z \mapsto z^k$. Note that φ_k is *not* a smooth map. Now we can construct a base change

$$\begin{tikzcd} X_k^\bullet \coloneqq X^\bullet \times_{\varphi_k} \Delta'^\times \arrow[r] \arrow[d, "\pi'"] & X^\bullet \arrow[d, "\pi"] \\ \Delta'^\times \arrow[r, "\varphi_k"] & \Delta^\times \end{tikzcd} \ . \tag{1.31}$$

Theorem 1.11 (Nodal reduction theorem). *Let $\pi : X \to \Delta$ be a flat proper surjective map such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Then there exists an integer $k > 0$ such that after a base change as above, the map π' extends to a family of nodal curves over Δ.*

Theorem 1.12 (Stable reduction). *If $\pi|_{\Delta^\times}$ is* stable*, then this extension can be chosen to be stable, and depends* only *on $\pi|_{\Delta^\times}$ up to isomorphism.*

Remark 1.2. This is related to separatedness of moduli spaces.

Remark 1.3. The intuition is as follows. Let Σ be a class of objects with a moduli space $\mathcal{M}$, so that there is a universal family $\mathcal{I} \to \mathcal{M}$ of objects in Σ such that any family of objects $X \to S$ in Σ, is the pullback of the universal family under a unique morphism $S \to \Omega$. I.e.

$$\operatorname{Hom}(-, \Omega) \cong \{\text{families of } \Sigma \text{ objects over } -\} \ . \tag{1.32}$$

If $\mathcal{M}$ is separated, Hausdorff, then for $\Delta^\times \to \mathcal{M}$ there exists at most one extension $\Delta \to \mathcal{M}$. If $\mathcal{M}$ is compact and proper, then for $\Delta^\times \to \mathcal{M}$ there exists a unique extension $\Delta \to \mathcal{M}$. The idea is to find open separated moduli subspace via stability condition.

Note that if we don't impose stability, given a family of nodal curves $X^\bullet \to \Delta^\times$, then it may extend in many different ways to a nodal family $X' \to \Delta$.

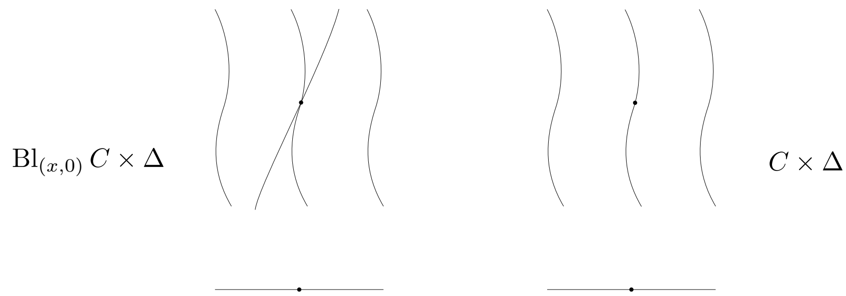

FIGURE 4. The constant family $\pi : C \times \Delta \to \Delta$ as well as the blowup $\mathrm{Bl}_{(x,0)}\, C \times \Delta \to \Delta$.

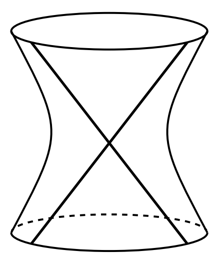

FIGURE 5. The surface given by $xy = t^2 - t$. Projection to the t-line has fibers which generically look like hyperbolas, but when the plane is tangent to the surface we get the union of two lines.

EXAMPLE 1.1. Consider a smooth curve $C = (f = 0) \subseteq \mathbb{P}^2$. Then $C \times \Delta^\times \to \Delta^\times$ is a constant family which extends to $C \times \Delta \to \Delta$. Now for any $x \in C$, $C \times \Delta^\times$ also extends to $\mathrm{Bl}_{(x,0)}\, C \times \Delta$. We can picture this as in Fig. 4.

The upshot is that moduli of nodal curves are not separated/Hausdorff. Lecture 6; February 3, 2020

Interlude: Some motivating examples.

Degeneration of a smooth curve to a nodal curve. We should think of the total space as being a surface. Consider the surface in Fig. 5. This has two different rulings, as pictured in Fig. 5. As in Fig. 5, we can project this surface to a line by taking the intersection with parallel planes at different points of the line. Generically this gives us hyperbolas, but for two special values we get the union of two lines from the two different rulings. In particular this is given by the equation $xy = t^2 - t$. The node is exactly the point of tangency. So when we have a non-reduced curve, this is singular at every point on the curve.

Degeneration of a smooth curve to a non-reduced curve. Consider the surface defined by the equation $x^3 + t\,(x + y + 1) = 0$. At $t = 0$ we just get a line with multiplicity 3. This looks something like Fig. 6.

Understanding the base change and its fibers. Again we consider a flat proper surjective map $\pi : X \to \Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. For simplicity assume that in fact $X = \mathbb{P}^1 \times \Delta$. Consider the map $\varphi_k : \Delta' \to \Delta$ from the disk to itself given by $z \mapsto z^k$.

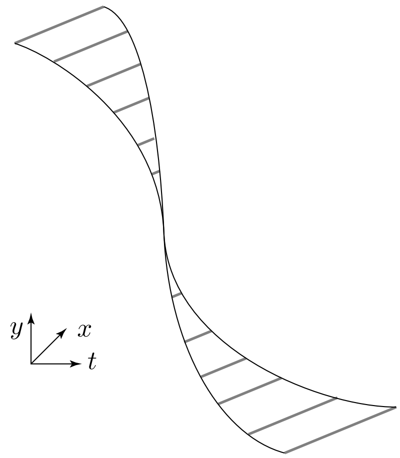

FIGURE 6. The surface $x^3 + t\,(x + y + 1) = 0$. Projecting to the t-line gives us smooth fibers which degenerate to a line with multiplicity 3 at $t = 0$.

Note that φ_k is *not* a smooth map. In particular:

$$\varphi_k^{-1}(0) = \operatorname{Spec}\left(\mathbb{C}[\epsilon]/\epsilon^k\right) \ . \tag{1.33}$$

Consider a base change for a family of curves

$$\begin{array}{ccc} \left(\mathbb{P}^1 \times \Delta\right) \times_{\varphi_k} \Delta' & & \mathbb{P}^1 \times \Delta \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ \Delta' & \xrightarrow{\varphi_k} & \Delta \end{array} \ . \tag{1.34}$$

If we think of the preimage under $\varphi_k \circ \pi'$ we have actually made things worse, since the preimage of 0 is:

$$(\varphi_k \circ \pi)^{-1}(0) \simeq \mathbb{P}^1 \times \operatorname{Spec}\left(\mathbb{C}[\epsilon]/\epsilon^k\right) \ . \tag{1.35}$$

But in our construction we are replacing π by π', not $\varphi_k \circ \pi'$. The moral is that (at least for specific k) this replacement makes things better.

Proof of the nodal reduction theorem.

PROOF OF THEOREM 1.11. We will operate under the simplifying assumption that $X^\bullet \to \Delta^\times$ is smooth. The first step is to resolve the singularities of X. This is easy since $\dim X = 2$. First we normalize to get something regular in codimension 1. Then we blowup the finitely many singular points, and normalize again if needed. This will give us $X' \xrightarrow{\pi'} \Delta$ where X' is smooth, but the central fiber $(\pi')^{-1}(0) = X'_0$ might haver arbitrary singularities. To deal with this, we resolve the non-nodal singularities of $X_0'^{\mathrm{red}}$ so it is nodal.

Then locally X' is isomorphic to $z = x^a y^b$ (i.e. a node) or $z = x^c$ (i.e. a point of X_0^{red}). Then we cover by finitely many such local charts and take

$$\mathcal{K} = \operatorname{lcm}\{ab, c\} \ . \tag{1.36}$$

The idea is to enable us to unwind the maximal multiplicity crossing. Then

$$X'' = \varphi_k^* X' \tag{1.37}$$

is not necessarily normal, but we claim:

CLAIM 1.1. $(X'')^\nu \xrightarrow{\pi'} \Delta'$ is a nodal family, where $(X'')^\nu$ is the normalization.

First consider π' near a point where $X' \cong (z = x^c)$. Write $z = \zeta^k$ and $k = ch$ so that

$$x^c - z = x^c - \zeta^{ch} = \prod_{\omega^c=1} \left(x - \omega\zeta^k\right) \ . \tag{1.38}$$

Note that when $z \neq 0$ this gives a disjoint union. Now we normalize to get $\amalg_{\omega^c=1}\left(x - \omega\zeta^h\right)$, a nodal family with smooth fibers.

Now consider π' near a point where $X' \cong \left(z = x^a y^b\right)$. Write $k = rsuv$ where $a = ru$, $b = su$, and $(r, s) = 1$. Write ζ for the coordinate on Δ'. Then X'' is locally given by

$$0 = x^a y^b - \zeta^k \ . \tag{1.39}$$

This is obviously not normal since we can choose elements here which satisfy a monic polynomial. In particular we can choose ω a primitive uth root of unity, so we can factor this as:

$$\left(x^a y^b - \zeta^k\right) = \prod_{i=1}^{u} \left(x^r y^s - \omega^i \zeta^{rsv}\right) \ . \tag{1.40}$$

Now $\omega\zeta^{vrs} = x^r y^s$ is a local equation on one branch, but is still not normal. Then we claim the following.

CLAIM 1.2. The normalization is given by a surface in three-space with local coordinates ζ, α, β with normalization map given by $x = \alpha^s$, $y = \beta^r$. The surface is locally given by $\zeta^v = \alpha\beta$.

To check that this is the normalization we need to check that

(1) it is normal,
(2) generically one-to-one, and
(3) surjective.

To see it is normal, notice that $\zeta^v = \alpha\beta$ is a toric surface. It is s standard fact that toric surfaces are normal. We now show it is generically one-to-one. Given (α, β, ζ) and $(\alpha', \beta', \zeta')$ so that

$$\alpha^s = (\alpha')^s \qquad \beta^r = (\beta')^r \qquad \zeta = \zeta' \ . \tag{1.41}$$

This means $\alpha' = \sigma\alpha$ for σ an sth root of unity, and similarly $\beta' = \tau\beta$ for τ an rth root of unity. But if α and β are nonzero, then $\alpha\beta = \alpha'\beta'$ implies $\sigma\tau = 1$, so $\sigma = \tau = 1$, so

$$(\alpha, \beta, \zeta) = (\alpha', \beta', \zeta') \ . \tag{1.42}$$

Sine the points where α and β are nonzero form an open dense set we are done.

Now consider (x, y, ζ) such that $x^r y^s = \zeta^{vrs}$. Then we must find points (α, β, ζ) such that $\alpha\beta = \zeta^v$ and $x = \alpha^s$, and $y = \beta^r$. Choose α_0, β_0 such that $\alpha_0^s = x$ and $\beta_0^r = y$. The point being that $\alpha_0 \cdot \beta_0 = \xi\zeta^v$ where $\xi^{rs} = 1$. Now write

$$1 = mr + ns \tag{1.43}$$

so the coordinates are

$$\alpha = \alpha_0 \xi^{-mr} \qquad \beta = \beta_0 \xi^{-ns} \ . \tag{1.44}$$

□

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