Moduli spaces and tropical geometry

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Spring 2020; Notes by: Jackson Van Dyke; All errors introduced are my own.

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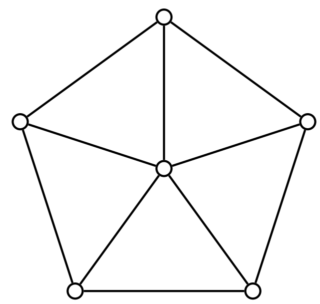

FIGURE 1. The 5-wheel.

1. Overview

Our goal is to understand the proof of the following theorem:

Lecture 1; Wednesday January 22, 2020

THEOREM 0.1. $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ *grows exponentially with* g.

REMARK 0.1. $\mathcal{M}_g$ has complex dimension $3g-3$.

This theorem defied previous expectations.

CONJECTURE 1 (Kontsevich (1993), Church-Farb-Putman (2014)). *For fixed* $k > 0$, $H^{4g-4-k}(\mathcal{M}_j, \mathbb{Q}) = 0$ *for* $g \gg 0$.

The structure of the course is as follows.

- Constructing the moduli space
  (1) Nodal curves and stable reduction theorem
  (2) Deformation theory of nodal curves
  (3) The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  (1) Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  (2) Dual complexes of normal crossings divisors (tropical geometry)
  (3) Boundary complex of $\mathcal{M}_g$ (tropical moduli space)
- Cohomology of $\mathcal{M}_h$
  (1) Stable cohomology (Madsen-Weiss 2007)
  (2) Virtual cohomological dimension of $\mathcal{M}_g$ (Harer 84) (Vanishing of H^{4g-5} (Church-Farb-Putman, Morita-Sakasai-Suzuki))
  (3) Euler characteristic of $\mathcal{M}_g$ (Harer-Zagier 86)
- Graph complexes (Kontsevich 93)
  (1) Feynman amplitudes and wheel classes. See Fig. 1 for the 5-wheel.
  (2) Grothendieck-Teichmüller Lie algebra
  (3) Willwacher's theorem
- Mixed Tate motives (MTM) over $\mathbb{Z}$
  (1) Mixed Tate motives
  (2) Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements/Drinfeld associators generate a free Lie subalgebra."
  (3) Proof of exponential growth of H^{4g-6}.

Lecture 2; January 24, 2020

Part 1

Constructing the moduli space

CHAPTER 1

Nodal curves and stable reduction theorem

1. Nodal curves

We will work over $\mathbb{C}$. We want to show that nodal curves, and families thereof, can be written in a normal form in local coordinates. We will follow chapter X of [**1**].

Definition 1.1. A *nodal curve* is a complete curve such that evert singular point has a neighborhood isomorphic (analytically over $\mathbb{C}$) to a neighborhood of 0 in $(xy = 0) \subset \mathbb{C}^2$.

Definition 1.2. A *family of nodal curves* over a base S is a flat proper surjective morphism $\mathcal{C} \to S$ such that every geometric fiber is a nodal curve.

Recall that a flat morphism is the agreed upon notion of a map which gives a continuously varying family of schemes (or complex analytic spaces, varieties, etc.) given by the fibers. Properness is saying that nothing can "disappear" as we approach any particular point in the base.

Example 1.1. Let $S = \mathbb{A}^1$. Now consider $\mathbb{A}^2$ minus the x-axis and the positive y-axis living over S. Let φ be the morphism projecting to the x-coordinate. This is not proper in a neighborhood of the origin in the base.

write this better/draw picture

Proposition 1.1. *Let $\pi : X \to S$ be a proper surjective morphism of $\mathbb{C}$-analytic spaces. This is a family of nodal curves iff at every point $p \in X$ either π is smooth at p with one-dimensional fiber, or there is a neighborhood of p which is isomorphic (over S) to a neighborhood of $(0, s)$ in $(xy = F) \subseteq \mathbb{C}^2 \times S$ where $s = \pi(p)$ and $F \in \mathfrak{m}_S \subseteq \mathcal{O}_{S,s}$.*

Lemma 1.2. *Let f be holomorphic at $0 \in \mathbb{C}^2$. Then $(f = 0)$ has a node at 0 iff*

$$0 = f = \frac{\partial f}{\partial x} = \frac{pf}{\partial y} \tag{1.1}$$

at 0, and the Hessian of f at 0 is non-singular.

This tells us that these nodes are the "simplest" possible singularities.

Proof. ($\implies$): This direction is immediate.
($\impliedby$): Suppose $0 = f = \partial_x f = \partial_y f$ at 0. Then

$$f = a - x^2 + 2bxy + cy^2 \tag{1.2}$$

where a, b, and c are holomorphic functions. The Hessian is

$$\begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix} \tag{1.3}$$

so being non-singular means exactly that

$$b^2 - ac \neq 0 \ . \tag{1.4}$$

After possible making a linear change of coordinates, we can assume $a \neq 0$, and change to coordinates

$$x_1 = x + \frac{b}{a} y \qquad\qquad y_1 = y \tag{1.5}$$

so we can write

$$f = a_1 x_1^2 + c_1 y_1^2 \tag{1.6}$$

where $a_1(0), c_1(0) \neq 0$. Choose square roots[1.1] α and γ of a_1 and c_1. Now replace x_1 and y_1 by $x_2 = \alpha x_1$ and $y_2 = \gamma y_1$ so that

$$f = x_2^2 + y_2^2\ . \tag{1.7}$$

Now for $x_3 = x_2 + iy_2$ and $y_3 = x_2 - iy_2$, we have $f = x_3 y_3$. ☐

Proof of Proposition 1.1. Let $\pi : X \to S$ be proper surjective. Consider $x \in X$. Then either π is smooth with 1-dimensional fiber at x (nothing to show) or x is a node in $\pi^{-1}(s)$, $s = \pi(x)$. Locally near x we can embed $X \subseteq \mathbb{C}^r \times S$. This is locally closed (over S). Now we have an exact sequence[1.2] of tangent spaces:

$$0 \longrightarrow T_x X_s \longrightarrow T_x X \longrightarrow T_s S \tag{1.8}$$

where $\dim T_x X_s = 2$. Now choose a linear projection $\mathbb{C}^r \to \mathbb{C}^2$ which is an isomorphism on $T_x X_s$. Using this projection we get a composition:

$$\begin{tikzcd} T_x X \arrow[r, phantom, "\subseteq"] \arrow[rr, bend right] & \mathbb{C}^r \times T_s S \arrow[r] & \mathbb{C}^2 \times T_s S \end{tikzcd} \tag{1.9}$$

and we claim this is injective. Then the implicit function theorem tells us that there is a neighborhood of x which embeds in $\mathbb{C}^2 \times S$ (over S). We should think of this as a family of plane curves: each fiber has a single defining equation. More specifically we have the following.

Fact 1 (Lemma 31.18.9 (Stacks project)). *If $\mathcal{Y} \to S$ is a smooth morphism and $D \subseteq \mathcal{Y}$ is flat over S, codimension 1 in $\mathcal{Y}$, then D is a Cartier divisor.*

By Fact 1, $X \subseteq \mathbb{C}^2 \times S$ is locally defined by a single equation $F = 0$. Now consider $\partial_x F$, $\partial_y F$, and the Hessian of F with respect to x and y. Then the proof of Lemma 1.2 shows that

$$F = x_3 y_3 - f \tag{1.10}$$

where f is a function on S which vanishes at s. ☐

Lecture 3; January 27, 2020

2. Stability of nodal curves

The following is a corollary of Proposition 1.1.

Corollary 1.3. *A family of nodal curves is local complete intersection (lci).*

This means that there is a relative dualizing sheaf $\omega_{X/S}$ which is locally free of rank 1.

[1.1]There is some subtly here since these are functions rather than scalars. Because a_1 and c_1 are nonzero at 0, we can ensure that the image of a_1 and c_1 are, say, contained in an open half space. Now we can choose a branch of log which is defined on this half space. Then multiple by 1/2 and exponentiate.

[1.2]If x is not a singular point, then this would be an honest SES (i.e. the map $T_x X \to T_s S$ would be surjective) but in this case this is not true. In any case, all we need is the exactness of these four terms.

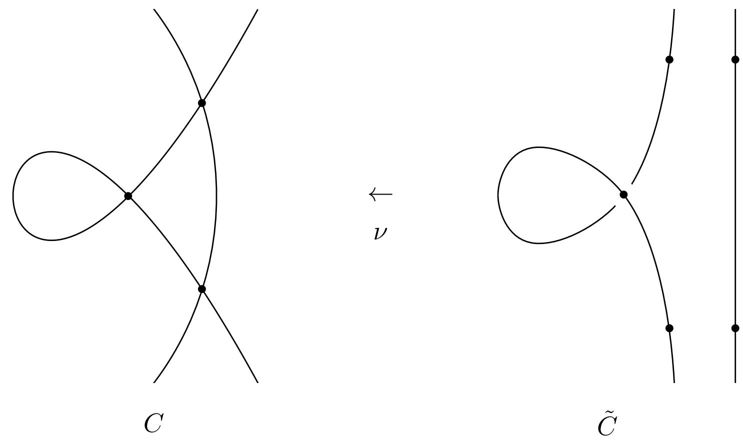

FIGURE 1. The normalization of a nodal curve. The nodal points of C each have two preimages under the normalization ν.

2.1. Serre duality. The point here is that the things we already know about smooth curves are also true about nodal ones.

Let C be a nodal curve (over a point). There is a (natural) isomorphism $H^1(C,\omega_C) \cong \mathbb{C}$. Then Serre duality tells us that for any coherent sheaf $\mathcal{F}$ on C,

$$H^1(C,\mathcal{F}) \times \mathrm{Hom}(\mathcal{F},\omega_C) \to H^1(C,\omega_C) \cong \mathbb{C} \tag{1.11}$$

is a perfect pairing, i.e.

$$H^1(C,\mathcal{F}) \cong \mathrm{Hom}(\mathcal{F},\omega_C)^\vee \ . \tag{1.12}$$

In particular, if $\mathcal{F}$ is a vector bundle, then

$$H^1(C,\mathcal{F}) \cong H^0(C,\mathcal{F}^\vee \otimes \omega_C)^\vee \ . \tag{1.13}$$

We can form the normalization[1.3] of a nodal curve as in Fig. 1.

Suppose C is nodal with components $C_1,\ldots,C_s$ and nodes $x_1,\ldots,x_r$. Let $\tilde{C} \xrightarrow{\nu} C$ be the normalization. Write $\tilde{C}_i$ for the normalization of C_i and

$$\{p_j,q_j\} = \nu^{-1}(x_j) \tag{1.14}$$

(for $i \in \{1,\ldots,s\}$ and $j \in \{1,\ldots,r\}$).

A line bundle L on C has *multi-degree* $\underline{\deg}(L)$ to be

$$\underline{\deg}(L) = \left(\deg\left(L|_{C_1}\right,\ldots,\deg\left(L|_{C_s}\right)\right)) \tag{1.15}$$

$$= \left(\deg\left(\nu^* L|_{\tilde{C}_1}\right),\ldots,\deg \nu^* L|_{\tilde{C}_s}\right) \ . \tag{1.16}$$

The following is a corollary to Serre duality.

Corollary 1.4. *If C is connected, and $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ then $H^1(C,L) = 0$.*

By $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ we mean that for all i $\deg\left(L|_{C_i}\right) \geq \deg\left(\omega_C|_{C_i}\right)$ for all i and $\underline{\deg}(L) \neq \underline{\deg}(\omega_C)$.

[1.3] The algebraic construction is taking the integral closure in the fraction field.

Proof. First note

$$H^1(C, L) \cong H^0\left(C, \omega_C \otimes L^{-1}\right) \ . \tag{1.17}$$

and $\underline{\deg}\left(\omega_C \otimes L^{-1}\right) < 0$.

On any connected component C_i such that $\deg\left(\omega_L \otimes L^{-1}\right)\big|_{C_i} < 0$ all sections vanish. And all sections vanish on components that C_i, etc. ☐

Corollary 1.5. *L is ample iff $\deg\left(L|_{C_i}\right) > 0$ for all i.*

Proof. ($\Longrightarrow$): This direction is clear. The restriction of ample L to any component is still ample.

($\Longleftarrow$): Suppose $\deg\left(L|_{C_i}\right) > 0$. It is enough to show that $L^{\otimes N}$ is very ample for some N. Choose N sufficiently large so that

$$\deg\left(L^{\otimes N}\big|_{C_i}\right) > \deg\left(\omega_C|_{C_i}\right) + 2 \ . \tag{1.18}$$

Let $S \subseteq C$ be a 0-dimensional subscheme of length 2. Then we have a short exact sequence

$$0 \to I_S \to \mathcal{O}_C \to \mathcal{O}_S \to 0 \tag{1.19}$$

which we can tensor with $L^{\otimes N}$ to get a sequence which is still exact, which gives us a long exact sequence

(1.20)

$$0 \longrightarrow H^0\left(L^{\otimes N}(-s)\right) \longrightarrow H^0\left(L^{\otimes N}\right) \longrightarrow H^0\left(L^{\otimes N}\big|_S\right) \longrightarrow H^1\left(L^{\otimes N}(-s)\right) \longrightarrow \dots$$

but $H^1(L(-s)) = 0$, so we have a surjection

$$H^0(L) \twoheadrightarrow H^0\left(L|_S\right) \ . \tag{1.21}$$

This is the kind of argument to show separation of points and tangent vectors, which is a criterion for very ampleness. ☐

3. Description of ω_C

See chapter 6 of [**4**] chapter 3 §A from [**2**] for references.

Proposition 1.6. *Let C be a nodal curve with nodes $x_1, \dots, x_r$, write $(p_i, q_i) = \nu^{-1}(x_i)$. Then*

$$\omega_C \cong \nu_*\left(\omega'_{\tilde{C}}(p_1 + q_1 + \ldots + p_r + q_r)\right) \tag{1.22}$$

where $\omega'_{\tilde{C}}(p_1 + \ldots + q_r) \subseteq \omega_{\tilde{C}}(p_1, \dots, q_r)$ is the subsheaf where

$$\mathrm{res}_{p_i}(\omega) + \mathrm{res}_{q_i}(\omega) = 0 \ . \tag{1.23}$$

Remark 1.1 (Rosenlicht differentials). There is a related explicit description of $\omega_{X/S}$ for a family of nodal curves. Near a point where $X/S \cong (xf = F) \subseteq \mathbb{C}^2 \times S$ $\omega_{C/S}$ is generated by dx/x and dy/y which satisfy

$$\frac{dx}{x} + \frac{dy}{y} = 0 \ . \tag{1.24}$$

Definition 1.3. A nodal curve is *stable* if ω_C is ample.

Lecture 4; January 29, 2020

FIGURE 2. Examples of stable and unstable curves.

Proposition 1.7. *Let $X \to S$ be a family of nodal curves. Then*

$$\{s \in S \mid X_s \text{ is stable}\}$$

is Zariski open.

PROOF. Let L be any line bundle on X. Then

$$\left\{s \in S \,\middle|\, L|_{X_s} \text{ is ample}\right\}$$

is Zariski-open. This is Theorem 1.2.17 of [**3**]. □

THEOREM 1.8. *A nodal curve C is stable iff* $\operatorname{Aut}(C)$ *is finite.*

PROOF. Say C has components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Write $\{p_i, q_i\} = \nu^{-1}(x_i)$ for the preimage of the nodes under the normalization ν. Write $Q = \{p_1, q_1, \ldots, p_r, q_r\}$. Notice that $\operatorname{Aut}(C)$ is finite iff

$$\{\sigma \in \operatorname{Aut}(C) \mid \sigma \text{ acts by 1 on } \{C_1, \ldots, C_s\}\}$$

is finite.

This is true iff for all C_i, there are only finitely many automorphisms of $\tilde{C}_i$ that fix $Q \cap \tilde{C}_i$. And this is equivalent to it being the case that for all C_i either:

(1) $g\left(\tilde{C}_i\right) \geq 2$;
(2) $g\left(\tilde{C}_i\right) = 1$, and $Q \cap \tilde{C}_i \neq \emptyset$; or
(3) $g\left(\tilde{C}_i\right) = 0$ and $Q \cap \tilde{C}_i \geq 3$.

This is equivalent to

$$2g\left(\tilde{C}_i\right) - 2 + \#\left(Q \cap \tilde{C}_i\right) > 0 ,$$

which is equivalent to $\deg\left(\omega_C|_{C_i}\right) > 0$ for all i. So a curve is stable iff the degree of the dualizing sheaf is positive on every component, which is equivalent to ω_C being ample. □

DEFINITION 1.4. A *graph* G is a set $X(G)$ together with an involution $i : X(G) \circlearrowleft$ and a retraction $r : X(G) \to X(G)^i$. The vertices $V(G)$, half edges $H(G)$, and edges $E(G)$ are defined as:

$$\begin{aligned} V(G) &= X(G)^i \\ H(G) &= X(G) \setminus V(G) \\ E(G) &= H(G)/i \ . \end{aligned}$$

We say $r(h)$ is the vertex incident to $h \in H(G)$.

The *dual graph* $G(C)$ of a nodal curve C is as follows. The vertices $\{v_1, \ldots, v_s\}$ correspond to the components $C_1, \ldots, C_s$; and the half-edges incident to v_i are given by the points of $\tilde{C}_i \cap Q$. An edge is made from a pair of half-edges corresponding to a pair $\{p_i, q_i\}$. The "genus function" assigns the genus of $\tilde{C}_i$ to the corresponding vertex v_i. See Fig. 2 for examples.

finish figure

We can read the stability off from the dual graph. Every vertex labelled with a 1 should have at least one incident edge, and all unlabelled should have valence at least 3.

Recall that the arithmetic genus of a curve C is

$$p_a(C) = 1 - \chi(\mathcal{O}_C) \ .$$

In particular, if C is connected then $p_a(C) = h^1(\mathcal{O}_C)$. Recall the Euler characteristic of a graph G is

$$\chi(G) = \#V(G) - \#E(G) \ .$$

Note if G is connected, then $h^1(G) = 1 - \chi(G)$. Also note that C is connected iff $G(C)$ is connected. The dual graph also detects the arithmetic genus in the following sense.

THEOREM 1.9. *Let C be a nodal curve. Then*

$$p_a(C) = 1 - \chi(G(C)) + \sum_v g(v) \ . \tag{1.25}$$

Corollary 1.10. *If C is connected then*

$$p_a(C) = \sum_v g(v) + h^1(G) \ . \tag{1.26}$$

PROOF OF THEOREM 1.9. Proceed by induction on the number of nodes $\#E(G) = \#C^{\mathrm{sing}}$. The base case is when $E(G) = \emptyset$, so the graph is just s vertices v_i with genus $g(v_i)$. Then

$$1 - \chi(\mathcal{O}_C) = 1 - s + \sum_i g(v_i) \tag{1.27}$$

as desired.

Now suppose C' is obtained from C by gluing two smooth points p, q to x. Write $\pi : C \to C'$. Then we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \pi_* \mathcal{O}_C \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.28}$$

which implies the Euler characteristic of the middle term is the Euler characteristic of the other two terms. Now since this gluing is proper and finite, it doesn't change the Euler characteristic. All together this gives us:

$$\chi(\mathcal{O}_C) = \chi(\pi_* \mathcal{O}_C) = \chi(\mathcal{O}_{C'}) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C'}) + 1 \ . \tag{1.29}$$

□

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