Orderability and 3-manifold groups

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Contents

CHAPTER 1

Orders on groups; basic definitions and properties

The book for the course is [**1**]. Lecture 1; January 21, 2020

Recall that a *strict total order* (STO) on a set X is a binary relation $<$ which satisfies:

(1) $x < y$ and $y < z$ implies $x < z$;
(2) $\forall x, y \in X$ exactly one of: $x < y$, $y < x$, $x = y$, holds.

A *left order* (LO) on a group G is an STO such that $g < h$ implies $fg < fh$ for all $f \in G$. G is *left-orderable* (LO) if there exists an LO on G. We similarly define a *right order* (RO) and *right orderability* (RO). A *bi-order* (BO) on G is an LO on G that is also an RO.

Remark 1.1. (1) If G is abelian, $<$ is a LO iff $<$ is an RO iff $<$ is a BO.
(2) If $<$ is an LO on G, then $\prec$ defined by:

$$g \prec h \iff h^{-1} < g^{-1} \tag{1.1}$$

is an RO on G. Therefore G is LO iff G is RO. We will stick to LO's.
(3) For $H < G$, an LO (resp. BO) on G induces an LO (resp. BO) on H.

Example 1.1. $(\mathbb{R}, +)$ with the usual $<$ is BO. The subgroups $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$ are also BO.

Lemma 1.1. *Let $<$ be an LO on G. Then*

(1) $g > 1$, $h > 1$ implies $gh > 1$;
(2) $g > 1$ implies $g^{-1} < 1$;
(3) $<$ is a BO iff $\left(g < h \implies f^{-1}gf < f^{-1}hf \forall f \in G\right)$ (i.e. $<$ is conjugation invariant).

Proof. (1) $h > 1$ implies $gh > g \cdot 1g > 1$.
(2) $g > 1$ implies $g^{-1}g > g^{-1}$ implies $1 > g^{-1}$.
(3) ($\implies$) is immediate. ($\impliedby$): We need to show $<$ is a RO. $g < h$ implies $fg < fh$ implies $f^{-1}(fg)f < f^{-1}(fh)f$ which implies $gf < hf$ as desired.

$\square$

Lemma 1.2. *If $<$ is a BO on G, then*

(1) $g < h$ implies $g^{-1} > h^{-1}$;
(2) $g_1 < h$, $g_2 < h_2$ implies $g_1 g_2 < h_1 h_2$.

Proof. (1) If $g < h$, then $g^{-1}g < g^{-1}h$, which implies $1 < g^{-1}h$, which implies $1 \cdot h^{-1} < g^{-1}$, which implies $h^{-1} < g^{-1}$.
(2) $g_2 < h_2$ implies $g_1 g_2 < g_1 h_2 < h_1 h_2$.

$\square$

Warning 1.1. These don't necessarily true for LO's.

Lemma 1.3. *If G is LO then it is torsion free.*

PROOF. Consider $g \in G \setminus \{1\}$. If $g > 1$, then $g^2 > g > 1$, and similarly for all $n \geq 1$, $g^n > 1$. Similarly $g < 1$ implies $g^n < 1$ for all $n \geq 1$. $\square$

So LO is not preserved under taking quotients (e.g. $\mathbb{Z} \to \mathbb{Z}/n$).

Consider an indexed family of groups $\{G_\lambda \,|\, \lambda \in \Lambda\}$. Recall that the direct product

$$\prod_{\lambda \in \Lambda} G_\lambda = \left\{(g_\lambda)_{\lambda \in \Lambda}\right\} \tag{1.2}$$

with multiplication defined co-ordinatewise.

Recall a *well-order* (WO) on a set X is a STO $\prec$ on X such that if $A \subset X$ and $A \neq \emptyset$ then there exists $a_0 \in A$ such that $a_0 \prec a$ for all $a \in A \setminus \{a_0\}$. Recall that the axiom of choice is equivalent to every set having a WO.

THEOREM 1.4. *G_λ has a LO (resp. BO) for all $\lambda \in \Lambda$ iff $\prod_{\lambda \in \Lambda} G_\lambda$ has a LO (resp. BO).*

PROOF. ($\Longleftarrow$): $G\lambda < \prod_\lambda G_\lambda$ so we are finished.

($\Longrightarrow$): Choose a WO $\prec$ on Λ, and order $\prod_\lambda G_\lambda$ lexicographically. Let $g = (g_\lambda)$, $h = (h_\lambda)$, $g \neq h$. Then λ_0 be the $\prec$-least element of Λ such that $g_{\lambda_0} \neq h_{\lambda_0}$. Then define $g < h$ iff $g_{\lambda_0} < h_{\lambda_0}$ (in G_{λ_0}). Then $<$ is an LO (resp. BO) on $\prod_\lambda G_\lambda$. Left (resp. left and right) invariance is clear. Now we show transitivity. Suppose $f < g$, $g < h$. Let λ_0 be the $\prec$-least element of Λ such that $f_{\lambda_0} \neq g_{\lambda_0}$. Let μ_0 be the $\prec$-least element of Λ such that $g_{\mu_0} \neq h_{\mu_0}$.

(1) ($\lambda_0 \preccurlyeq \mu_0$): Then $f_\lambda = g_\lambda = h_\lambda$ for all $\lambda \prec \lambda_0$. Then g_{λ_0} is $<$ (resp. $=$) h_{λ_0} if $\lambda_0 = \mu_0$ (resp. $\lambda_0 \prec \mu_0$). So $f_{\lambda_0} < g_{\lambda_0} \leq h_{\lambda_0}$, and therefore $f_{\lambda_0} < h_{\lambda_0}$.
(2) ($\mu_0 < \lambda_0$): This follows similarly.

$\square$

Let $\sum_{\lambda in \Lambda} G_\lambda$ be the *direct sum of* $\{G_\lambda\}$. Recall this is the subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of elements such that all but finitely many co-ordinates are 1.

Corollary 1.5. *G_λ is LO (resp. BO) for all $\lambda \in \Lambda$ iff $\sum_{\lambda \in \Lambda} G_\lambda$ is LO (resp. BO).*

Corollary 1.6. *Free abelian groups are BO.*

PROOF. Free abelian groups on Λ are $\sum_{\lambda \in \Lambda} \mathbb{Z}$. $\square$

Let $<$ be an LO on G. The *positive cone* $P = P_<$ of $<$ is $\{g \in G \,|\, g > 1\}$.

Lemma 1.7. *(1) P is a subset of G, i.e. $g, h \in P$, implies $gh \in P$ (i.e. $PP \subset P$).*
(2) $G = P \amalg P^{-1} \amalg \{1\}$.
(3) $<$ is a BO on G iff $f^{-1}Pf \subset P$ for all $f \in G$.

PROOF. (1) This follows from Lemma 1.1 (1).
(2) This follows from Lemma 1.1 (2).
(3) This follows from Lemma 1.1 (3).

$\square$

We say $P \subset G$ is a *positive cone* if P satsfies the conditions in Lemma 1.7.

Lemma 1.8. *Let $P \subset G$ be a positive cone. Then $g < h$ implies $g^{-1}h \in P$ defines a LO $<$ on G (With $P_< = P$).*

PROOF. $<$ is a STO, so:

(i) $f<g$, $g<h$ implies $f^{-1}g\in P$, $g^{-1}h\in P$, which implies (by the first property) that $\left(f^{-1}g\right)\left(g^{-1}h\right)\in P$, which implies $f<h$.

(ii) By the second property, for all $g,h\in G$ exactly one of the following holds: $g^{-1}h\in P$, $g^{-1}h\in P^{-1}$, and $g^{-1}h=1$. Equivalently, $g<h$, $h<g$ (since $h^{-1}g\in P$), and $g=h$. Now we show left invariance. $g<h$ implies $g^{-1}h\in P$, but $g^{-1}h=\left(g^{-1}f^{-1}\right)(fh)$ which implies $fg<fh$.

$\square$

Lemmata 1.7 and 1.8 show that:

$$\{\text{LO's on } G\} \quad \leftrightarrow \quad \{\text{positive cones in } G\} \tag{1.3}$$

$$\{\text{BO's on } G\} \quad \leftrightarrow \quad \{\text{conjugacy-invariance positive cones in } G\}\ . \tag{1.4}$$

Consider the free group of rank n, F_n.

THEOREM 1.9. *F_2 is LO.*

SUNIC. Write $F_2=F(a,b)$. $g\in F_2$ implies we can write it as a reduced word

$$\left(a^{m_1}\right)b^{n_1}\dots a^{m_k}\left(b^{n_k}\right) \tag{1.5}$$

for $k\geq 0$, $m_i,n_i\in\mathbb{Z}\setminus\{0\}$. Recall 1 is the empty word, $k=0$. Let $e(g)$ be the number of syllables in g with positive exponent, minus the number of syllables in g with negative exponent. Then define $j(g)$ so be the number of a^mb^n's in f, minus the number of b^na^ms in G. So $j(g)=0$, or ±1. For example:

$$j\left(a^*\dots a^*\right)=0 \tag{1.6}$$

$$j\left(b^*\dots b^*\right)=0 \tag{1.7}$$

$$j\left(a^*\dots b^*\right)=1 \tag{1.8}$$

$$j\left(b^*\dots a^*\right)=-1\ . \tag{1.9}$$

Finally define

$$\tau(g)=e(g)+j(g)\ . \tag{1.10}$$

Note that

$$e\left(g^{-1}\right)=-e(g) \qquad\qquad j\left(g^{-1}\right)=-j(g)\ . \tag{1.11}$$

Lemma 1.10. *If $g\neq 1$, then $\tau(g)\equiv 1 \pmod 2$.*

PROOF. $e(f)$ is congruent to the number of syllables mod 2, and $j(g)$ is congruent to the number of syllables $+1$ mod 2. $\square$

Lemma 1.11. $|\tau(gh)-\tau(g)-\tau(h)|\leq 1$.

PROOF. If gh or g or $h=1$ we are done. So suppose $gh,g,h\neq 1$. Clearly $e(gh)=e(g)+e(h)+\left\{\begin{matrix}0\\1\\-1\end{matrix}\right\}$. Similarly:

$$j(gh)=j(g)+j(h)+\left\{\begin{matrix}0\\1\\-1\end{matrix}\right\}\ . \tag{1.12}$$

Therefore:

$$|\tau(gh)-\tau(g)-\tau(H)|\leq 2 \tag{1.13}$$

so by Lemma 1.10 we have

$$|\tau(gh) - \tau(g) - \tau(h)| \leq 1 \ . \tag{1.14}$$

□

REMARK 1.2. Lemma 1.11 says that $\tau : F_2 \to \mathbb{Z}(< \mathbb{R})$ is what is called a *quasi-morphism.*

Define $P \subset F_2$ by

$$P = \{g \in F_2 \,|\, \tau(g) > 0\} \ . \tag{1.15}$$

Then $F_2 = P \amalg P^{-1} \amalg \{1\}$ by Lemma 1.10 and that $\tau\left(g^{-1}\right) = -\tau(g)$. Then $PP \subset P$ by Lemma 1.11 since

$$\tau(gh) \geq \tau(g) + \tau(h) - 1 \geq 1 \ . \tag{1.16}$$

Therefore P is a positive cone for a LO on F_2. ■

Corollary 1.12. *Any countable free group is LO.*

PROOF. A countable free group is a subgroup of F_2. □

REMARK 1.3.
(1) $\tau\left(a^{-1}b\right) = 1$, so $a^{-1}b > 1$, so $b > a$. On the other hand, $\tau\left(ab^{-1}\right) = 1$, so $ab^{-1} > 1$, so $b^{-1} > a^{-1}$. So τ does not define a BO on F_2.
(2) We will see later that all free groups are LO.
(3) Even later we will see that all free groups are BO.

THEOREM 1.13. *Let $1 \to H \to G \to Q \to 1$ be a short-exact sequence of groups. Then*
(1) H, Q LO implies G is LO;
(2) if Q is BO and H has a BO that is invariant under conjugation in G then G is BO.

Lecture 2; January 23, 2019

PROOF. Write $\varphi : G \to Q$ and regard H as $\ker \varphi < G$. Let P_H (resp. P_Q) be positive cones for LO's on H (resp. Q). Define $P = \varphi^{-1}(P_Q) \amalg P_H$.

CLAIM 1.1. P is a positive cone for an LO on G.

PROOF. We need to check (1) and (2) from Lemma 1.7. Let $g, h \in P$. Then we want to show $gh \in P$. We have three cases.
(a) $g, h \in \varphi^{-1}(P_Q)$: In this case $\varphi(g), \varphi(h) \in P_Q$, so $\varphi(gh) = \varphi(g)\varphi(h) \in P_Q$. Therefore $gh \in \varphi^{-1}(P_Q)$.
(b) $g, h \in P_H$: In this case $gh \in P_H$.
(c) $g \in \varphi^{-1}(P_Q)$, $h \in P_H$: Then $\varphi(gh) = \varphi(g) \in P_Q$, so $gh \in \varphi^{-1}(P_Q)$. Similarly $hg \in \varphi^{-1}(P_Q)$.

Now we need to check $P \amalg P^{-1} \amalg \{1\}$. But this follows from the fact that:

$$G = (H \setminus \{1\}) \amalg \varphi^{-1}(Q \setminus \{1\}) \amalg \{1\} = \varphi^{-1}(P_Q) \amalg \varphi^{-1}\left(P_Q^{-1}\right) \tag{1.17}$$

since $H \setminus \{1\} = P_H \amalg P_H^{-1}$. □

We leave (2) as an exercise. [Hint: Recall P is a positive cone for BO on G iff it is a conjugacy invariant cone for an LO.] ■

1. Orderability of manifold groups

EXAMPLE 1.2. Let X^2 be the Klein bottle. This has fundamental group

$$K = \pi_1\left(X^2\right) = \left\langle a, b \,\middle|\, b^{-1}ab = a^{-1}\right\rangle \ . \tag{1.18}$$

This fits in the SES:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \| & & & & & & \\ & & \langle a \rangle & & b & \longmapsto & gm & & \end{array} \tag{1.19}$$

which means K is LO by Theorem 1.13.

Note that K is *not* BO. We have that $a > 1$ iff $b^{-1}ab > 1$, but this is a^{-1}, so $a^{-1} > 1$ which is a contradiction.

Notice that $\mathbb{Z}$ has exactly two LO's. The usual one, and the opposite. Therefore, if we choose an LO on $\langle a \rangle$ and $K/\langle a \rangle$, this gives 4 LO's on K determined by:

(i) $a > 1$, $b > 1$;
(ii) $a > 1$, $b < 1$;
(iii) $a < 1$, $b > 1$;
(iv) $a < 1$, $b < 1$.

THEOREM 1.14. *These are the only LO's on K.*

PROOF. It suffices to show that each of these conditions determines a unique positive cone.

(i) $a > 1$, $b > 1$:

CLAIM 1.2. $a^k < b$ for all $k \in \mathbb{Z}$.

PROOF. $b < a^k$ implies $a^{-k}b < 1$. But $a^{-k}b = ba^k$ and $b > 1$, so $b < a^k$ implies $a^k > 1$, which implies $ba^k > 1$ which is a contradiction. □

Note that every element in K has a unique representative of the form $a^m b^n$ for $m, n \in \mathbb{Z}$.

CLAIM 1.3. $a^m b^n > 1$ iff either $n > 0$ or $n = 0$ and $m > 0$.

PROOF. If $n = 0$, then this is clear. If $n > 0$, then $a^m b > 1$ for any m by claim 1 (for $k = -m$). But we also know $b > 1$ which implies $b^n > 1$, so we get $a^m b^n > 1$ for $n > 0$. On the other hand, if $m < 0$ then $a^m b^n = b^n a^{\pm m} = \left(a^{\mp m} b^{-n}\right)^{-1}$. Then we know $a^{\mp m} b^{-n} > 1$ by the case above, so its inverse is < 1. □

If $<$ is an LO on G, and $\alpha : G \to G$ is an automorphism, then this induces an LO $<_\alpha$ on G given by: $g <_\alpha h$ iff $\alpha(g) < \alpha(h)$. Now notice that there are automorphisms α_1, α_2 of K such that

$$\alpha_1(a) = a \ , \qquad \alpha_1(b) = b^{-1} \tag{1.20}$$

$$\alpha_1(a) = a^{-1} \ , \qquad \alpha_1(b) = b \ . \tag{1.21}$$

In particular, α_1 is given by

$$\left\langle a, b \,\middle|\, b^{-1}ab = a^{-1}\right\rangle \cong \left\langle a, b \,\middle|\, bab^{-1} = a^{-1}\right\rangle \tag{1.22}$$

and similarly for α_2.

Write $<_{(i)}$ for the unique LO on K determined by (i). Then $<_{(ii)}$ is induced by $<_{(i)}$ and α_1, $<_{(iii)}$ is induced by $<_{(i)}$ and α_2, and $<_{(iv)}$ is induced by $<_{(i)}$ and $\alpha_1\alpha_2$.

■

FACT 1. *If G has only finitely many LO's, then the number of LO's is of the form 2^n.*

EXERCISE 1.1. Show that for all $n \geq 0$ there exists a group G with exactly 2^n LO's.

Corollary 1.15. *For any LO on K, if $h \in \langle a \rangle$, $g \in K \setminus \langle a \rangle$, and $g > 1$, then $g > h$.*

PROOF. It is sufficient to check this for the first LO, since the other three are determined by the above automorphisms. Let $a > 1$, $b > 1$. By claim 2 from above, we know $g = a^m b^n$ for $n > 0$. We now there is some k such that $h = a^k$, and therefore

$$h^{-1} g = a^{m-k} b^n > 1 \tag{1.23}$$

by claim 2, so $g > h$. □

2. Three-manifold groups

Suppose M is a closed, orientable, connected three-manifold. Then we might ask if $\pi_1(M)$ is LO? BO?

Immediately we notice that not all such groups are. If M is a lens space, then $\pi_1(M) \cong \mathbb{Z}/n$ for $n > 1$, so this is not LO. More generally, for $\pi_1(M)$ nontrivial and finite is not LO. Recall that if $M = M_1 \# M_2$, then this implies $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. So, for example, if $M_1 \#$ lens space, then $\pi_1(M)$ has torsion, so not LO.

But at least some of them are. Consider $M \cong T^3 = S^1 \times S^1 \times S^1$. Then $\pi_1(M) = \mathbb{Z}^3$ is of course LO. Similarly $M = \#_n \left(S^1 \times S^2\right) \cong F_n$, so $\pi_1(M)$ is LO.

We will show that there exist (three-manifold) groups that are torsion-free, but not LO.

Let $p : T^2 \to X^2$ be a two-fold covering of the Klein bottle. Recall that

$$K > p_* \left(\pi_1 \left(T^2\right)\right) = \left\langle a, b^2 \right\rangle \cong \mathbb{Z} \times \mathbb{Z} \ . \tag{1.24}$$

Let N be the mapping cylinder of p, namely:

$$N = \left(T^2 \times I\right) \amalg X^2 / \left((x, 0) \sim p(x) \, \forall x \in T^2\right) \ . \tag{1.25}$$

The orientation reversing curve representing b doesn't lift. So N is orientable. Note that $\partial N \cong T^2$. There is a strong deformation retraction $N \to X^2$, so $\pi_1(N) \cong K$. Let N_1, N_2 be two copies of N. Write

$$\pi_1(N_i) = \left\langle a_i, b_i \,\middle|\, b_i^{-1} a_i b_i = a_i^{-1} \right\rangle \ . \tag{1.26}$$

Notice that $\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z} = \left\langle a_i, b_i^2 \right\rangle < \pi_1(N_i)$. Let $\varphi : \partial N_1 \to \partial N_2$ be a homeomorphism. Let $M_\varphi = N_1 \cup_\varphi N_2$. This is a closed, orientable three-manifold. Therefore

$$\pi_1(M_\varphi) = \pi_1(N_1) *_{\mathbb{Z} \times \mathbb{Z}} \pi_1(N_2) \cong K_1 *_{\mathbb{Z} \times \mathbb{Z}} K_2 \ . \tag{1.27}$$

Since K is torsion-free, $\pi_1(M_\varphi)$ is torsion-free. But in fact we have the following theorem.

THEOREM 1.16. *If $H_1(M_\varphi)$ is finite, then $\pi_1(M_\varphi)$ is not LO.*

REMARK 1.4. We will see later that for M a prime three-manifold with $H_1(M)$ infinite has $\pi_1(M)$ LO.

PROOF. φ is determined up to isotopy, so the resulting manifold M_φ depends only on $\varphi_* : H_1(\partial N_1) \to H_1(\partial N_2)$. We know

$$\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_1, 2b_1 \rangle \qquad\qquad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_2, 2b_2 \rangle \tag{1.28}$$

so φ_* is given by some 2×2 matrix with $\mathbb{Z}$ coefficients

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \tag{1.29}$$

with determinant $ps - qr = \pm 1$. Specifically we have:

$$\varphi_* (a_1) = pa_2 + 2qb_2 \tag{1.30}$$

$$\varphi_* (2b_1) = ra_2 + 2sb_2 \ . \tag{1.31}$$

Now we have $H_1 (N_i) = \mathbb{Z} \oplus \mathbb{Z}_2$ with basis b_i and a_i respectively. Then $H_q (M_\varphi)$ is presented by

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{bmatrix} \ . \tag{1.32}$$

where we order the basis as $\{a_1, b_1, a_2, b_2\}$. Interchanging columns 2 and 3 we get

$$\det A = 4 \left| \det \begin{bmatrix} 0 & 2q \\ -2 & 2s \end{bmatrix} \right| = 16 \, |q| \ . \tag{1.33}$$

Therefore $H_1 (M_\varphi)$ is finite iff $q \neq 0$ iff $\varphi_* (a_1) \neq \pm a_2$.

Suppose $\pi_1 (M_\varphi)$ is LO. Then we would get an induced LO on the common boundary $\partial N_1 = \partial N_2$. But there are only 4 LO's on $\pi_1 (N_i)$ (for $i \in \{1, 2\}$). By Corollary 1.15, for any LO on $\pi_1 (N)$, $\langle a \rangle$ is the unique $\mathbb{Z}$-summand of $\pi_1 (\partial N) = \langle a, b^2 \rangle$ such that if $h \in \langle a \rangle$ and $g \in \pi_1 (\partial N) \setminus \{1\}$, $g > 1$, then $g > h$. Therefore $\varphi_* (a_1) = \pm a_2$ which is a contradiction. ☐

Let $<$ be an STO on a set X. Let $\mathcal{B} (X, <)$ be the group of $<$-preserving bijections $X \to X$. Lecture 3; January 28, 2020

Theorem 1.17. *$\mathcal{B} (X, <)$ is always LO.*

Proof. Let $\prec$ be a WO on X. Let $f, g \in \mathcal{B} (X, <)$ such that $f \neq g$. Write

$$[f \neq g] = \{x \in X \mid f (x) \neq g (x)\} \neq \emptyset \ . \tag{1.34}$$

Let x_0 be the $\prec$-least element of $[f \neq g]$. Define

$$f < g \iff f (x_0) < g (x_0) \ . \tag{1.35}$$

Then we claim that this is an LO on $\mathcal{B} (X, <)$. Left-invariance is clear. To see this is a STO we need "trichotomy" and transitivity. Trichotomy is easy, and transitivity follows from the same argument as the proof of Theorem 1.4. ☐

Example 1.3. Let $<$ be the standard order on $\mathbb{R}$. Then $\mathcal{B} (\mathbb{R}, <)$ consists of the orientation-preserving homeomorphisms $\mathbb{R} \to \mathbb{R}$, written $\mathrm{Homeo}_+ (\mathbb{R})$.

Corollary 1.18. $\mathrm{Homeo}_+ (\mathbb{R})$ *is LO.*

Remark 1.5. For $x \in \mathbb{R}$, let $\prec_x$ be a WO on $\mathbb{R}$ such that x is the $\prec_x$-least element of $\mathbb{R}$. Let $<_x$ be the LO on $\mathrm{Homeo}_+ (\mathbb{R})$ induced by $\prec_x$, as in the proof of Theorem 1.17. Given $x \neq y \in \mathbb{R}$, there exists $g \in \mathrm{Homeo}_+ (\mathbb{R})$ such that $g (x) > x$ and $g (y) < y$. But this means

$$g <_x 1 \qquad\qquad g <_y 1 \ .$$

which implies $<_x \neq <_y$. Therefore $\mathrm{Homeo}_+ (\mathbb{R})$ has uncountably many LO's.

REMARK 1.6. It is a fact that the number of LO's on a group G is either finite (and of the form 2^n) or uncountable.

Corollary 1.19. *A group G is LO iff G acts faithfully*[1.1] *on a STO'd set $(X, <)$.*

PROOF. ($\Longleftarrow$): This follows from Theorem 1.17.
($\Longrightarrow$): G acts faithfully on $(G, <)$ by left multiplication. □

Corollary 1.18 implies that any subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is LO. E.g. one can show that F_2 (the free group of rank 2) is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. (This is another way to show that countable free groups are LO.) In fact this characterizes countable LO groups.

THEOREM 1.20. *Let G be a countable group. Then G is LO iff there exists an injective homomorphism $G \to \mathrm{Homeo}_+(\mathbb{R})$.*

PROOF. ($\Longleftarrow$): This follows from Corollary 1.18.
($\Longrightarrow$): We actually prove something slightly stronger. This will follow from Theorem 1.21. □

THEOREM 1.21. *Let $(G, <)$ be a countable group with an LO. Then there exists a LO on $\mathrm{Homeo}_+(\mathbb{R})$ and an order-preserving injective homomorphism $(G, <) \to (\mathrm{Homeo}_+(\mathbb{R}), <)$.*

SKETCH OF PROOF. Let $<$ be an LO on G. If $G = \{1\}$ this is immediate, so assume $G \neq \{1\}$. Therefore it is infinite, since LO groups are torsion free. Let $g_1, g_2, \ldots$ be some enumeration of the elements of G.

Define an embedding $e : G \to \mathbb{R}$ by $e(g_1) = 0$, and inductively by:

(i) If $g_{n+1} \begin{Bmatrix} > \\ < \end{Bmatrix} g_i$ for all $1 \leq i \leq n$, then set

$$e(g_{n+1}) = \begin{Bmatrix} \max\{e(g_i) \mid 1 \leq i \leq n\} + 1 \\ \min\{e(g_i) \mid 1 \leq i \leq n\} - 1 \end{Bmatrix} . \tag{1.36}$$

(ii) Otherwise let

$$g_l = \max\{g_i \mid 1 \leq i \leq n, g_i < g_{n+1}\}$$
$$g_r = \min\{g_i \mid 1 \leq i \leq n, g_i > g_{n+1}\}$$

and set

$$e(g_{n+1}) = \frac{e(g_l) + e(g_r)}{2} .$$

REMARK 1.7. (1) e is order-preserving, i.e. $a < b \implies e(a) < e(b)$.
(2) $e(g_{n+1}) \in \mathbb{Z}$ iff (i) holds.
(3) If $g > 1$ then $g^2 > g$ and $g^{-1} < g$. If $g < 1$ then $g^2 < g$ and $g^{-1} > g$, which implies $\mathbb{Z} \subset e(G) = \Gamma$.
(4) G acts on Γ by $g(e(a)) = e(ga)$. In fact, G acts on $(\Gamma, <)$ (where $<$ is the restriction of $<$ on $\mathbb{R}$) since $e(a) < e(b)$ iff $a < b$ iff $ga < gb$ iff $e(ga) < e(gb)$ iff $g(e(a)) < g(e(b))$.

To see that this action extends to an action of G on $\mathbb{R}$, we have a few steps.

Step 1: The action of G on Γ is continuous,
Step 2: The action of G on Γ extends to a continuous action of G on $\bar{\Gamma}$.

[1.1] Recall this means $g(x) = x$ for all $x \in X$ iff $g = 1$.

Step 3: $\mathbb{R}\backslash\bar{\Gamma}$ is a countable II of open intervals (a_i, b_i); the action of G is defined on $\{a_i, b_i\}$; and extends to $[a_i, b_i]$.

Note, to ensure Step 1:, it is not enough to take e to be an order-preserving of G in $\mathbb{R}$. It must be continuous.

To define an LO on $\mathrm{Homeo}_+(\mathbb{R})$ that restricts to the LO on Γ from G, first pick any $\gamma \in \Gamma$. Then $g > 1$ (resp. < 1) iff $g(\gamma) > \gamma$ (resp. $< \gamma$). Let $\prec$ be a WO on $\mathbb{R}$ such that γ is the $\prec$-least element of $\mathbb{R}$. Then let $\lessdot$ be the LO on $\mathrm{Homeo}_+(\mathbb{R})$ induced by $\prec$. Then $g > 1$ (resp. < 1) in G iff $g \gtrdot 1$ (resp. $\lessdot$) in $\mathrm{Homeo}_+(\mathbb{R})$. $\square$

3. Group rings

Let R be a ring (with 1).

- $a \in R$ is a *unit* if there exists $b \in R$ such that $ab = ba = 1$.
- $a \in R$ is a *zero-divisor* if $a \neq 0$ and there exists $b \neq 0$ such that either $ab = 0$ or $ba = 0$.
- $a \in R$ is a *non-trivial idempotent* if $a^2 = a$ but $a \neq 0$ and $a \neq 1$.

Let G be a group and R a ring. Then the *R-group ring of G* consists of formal sums:

$$RG := \left\{ \sum r_g g \,\middle|\, g \in G, r_g \in R, r_G \neq 0 \forall \text{ but f'tly many } g \in G \right\} . \tag{1.37}$$

RG is a ring with respect to the obvious operations. For $g \in G$ and $r \in R$ a unit, then rg is a unit in RG. A unit in RG is *non-trivial* if it is not of this form.

Remark 1.8. If $\tilde{X} \to X$ is a universal covering, then $\pi = \pi_1(X)$ acts on $\tilde{X}$ so $H_*\left(\tilde{X}, \mathbb{Z}\right)$ is a $\mathbb{Z}\pi$-module.

Theorem 1.22. *Suppose G has non-trivial torsion, and K is a field of characteristic 0.*

(1) KG has zero divisors,
(2) KG has non-trivial units,
(3) KG has non-trivial idempotents.

Proof. Let $g \in G$ have order $n \geq 2$. Define

$$\sigma = 1 + g + g^2 + \ldots + g^{n-1} \in KG .$$

First notice that

$$g\sigma = \sigma \tag{1.38}$$

which implies $(1 - g)\sigma = 0$ so we have zero divisors.

(1.38) also gives us that $\sigma^2 = n\sigma$. Therefore

$$(1 - \sigma)\left(1 - \frac{1}{n-1}\sigma\right) = 1$$

so we have a nontrivial unit for $n > 2$. If $n = 2$, $1 - \sigma = -g$, but we still have:

$$(1 - 2\sigma)\left(1 - \frac{2}{3}\sigma\right) = 1 . \tag{1.39}$$

Finally, we have that

$$\left(\frac{1}{n}\sigma\right)^2 = \left(\frac{1}{n^2}\right)\sigma^2 = \frac{1}{n}\sigma \tag{1.40}$$

so we have nontrivial idempotents. $\square$

Note that the proof of (1) works even for $\mathbb{Z}G$.

Remark 1.9. If $n \notin \{2,3,4,6\}$ then $\mathbb{Z}G$ has nontrivial units. This is a theorem of Higman.

Example 1.4. For $n = 5$,

$$\left(1 - g - g^4\right)\left(1 - g^2 - g^3\right) = 1 \ . \tag{1.41}$$

But what if G is torsion free? This brings us to the famous Kaplansky conjectures.

Conjecture 1 (Kaplansky). *If G is torsion free and K is a field, then:*

I (Units conjecture): KG has no non-trivial units,
II (Zero-divisors conjecture): KG has no zero divisors,
III (Idempotents conjecture): KG has no non-trivial idempotents.

Remark 1.10. Clearly II implies III since $a^2 = a$ implies $a\,(a-1) = 0$, which by II implies $a = 0$ or $a = 1$ which implies III. In fact they're all equivalent, but this is nontrivial to see.

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Remark 1.11. Note that if R is an integral domain (e.g. $\mathbb{Z}$) then R is contained in its field of fractions. In this case Items I and II and Item III for its field of fractions imply the corresponding versions of Items I and II and Item III for R.

Remark 1.12. We know this is true for LO groups. As we have seen, we should think of LO as being a stronger version of torsion free.

Theorem 1.23. *If G is LO then KG satisfies Items I and II and Item III.*

Proof. Since Item I implies Item III by the above remark we show Item I and Item II.

Item I: Suppose

$$\left(\sum_{i=1}^{m} \alpha_i g_i\right)\left(\sum_{j=1}^{n} \beta_j h_j\right) = 1 \tag{1.42}$$

with m, n not both 1, $\alpha_i, \beta_j \neq 0 \in K$, distinct $g_i \in G$, and distinct $h_i \in G$. Note this product can be rewritten as the following sum with mn terms:

$$\sum_{i,j} (\alpha_i \beta_j)(g_i h_j) \ . \tag{1.43}$$

Assume WLOG that $h_1 < h_2 < \ldots < h_n$. Let $g_k h_l$ be a minimal element of

$$S = \{g_i h_j \,|\, 1 \le i \le m, 1 \le j \le n\} \subset G \ . \tag{1.44}$$

We know $h_1 < h_j$ for $j > 1$, so $g_k h_1 < g_k h_j$ for all $j > 1$. Therefore $l = 1$. Also $g h_1 = g' h_1$ which implies $g = g'$. Therefore $g_k h_1$ is the unique

$$(k, 1) \in \{(i,j) \,|\, 1 \le i \le m, 1 \le j \le n\} \tag{1.45}$$

such that $g_k h_1$ is a minimal element of S.

Similarly, there is a unique

$$(r, n) \in \{(i,j) \,|\, 1 \le i \le m, 1 \le j \le n\} \tag{1.46}$$

such that $g_r h_n$ is a maximal element of S.

Claim 1.4. $g_k h_1 \neq g_r h_n$.

If they were equal, then $r = k$, $n = 1$, so $m > 1$. So $g_k h_1 = g_r h_1$, and therefore $g_r = g_k$. But this cannot be the case since they are distinct by assumption.

This implies that (1.43) has ≥ 2 terms after cancellation, so it cannot be 1.

Item II: Now suppose

$$\left(\sum_{i=1}^{m} \alpha_i g_i\right)\left(\sum_{j=1}^{n} \beta_j h_j\right) = 0 \tag{1.47}$$

for $m, n \geq 1$. Then there is a unique minimal element and nonzero coefficient, which means it is nonzero. $\square$

CONJECTURE 2 (Isomorphism conjecture). *If G is torsion free, then $\mathbb{Z}G \cong \mathbb{Z}H$ implies $G \cong H$.*

REMARK 1.13. In [**2**] a finite counterexample to the conjecture for arbitrary groups was provided, i.e. it is shown that there exists finite G, H such that $\mathbb{Z}G \cong \mathbb{Z}H$, $G \ncong H$.

Corollary 1.24 ([**4**]). *If G is LO, then G satisfies the isomorphism conjecture.*

PROOF. Theorem 1.23 implies that $\mathbb{Z}G$ has no nontrivial units. Call $\mathcal{U}_{\mathbb{Z}G}$ the group of units in $\mathbb{Z}G = \mathbb{Z}/2 \times G$. Suppose $\mathbb{Z}G \cong \mathbb{Z}H$. Theorem 1.23 says that $\mathbb{Z}G$ has no 0-divisors. This implies $\mathbb{Z}H$ has no 0-divisors, which means (by Theorem 1.22) that H is torsion-free. Now $H < \mathcal{U}_{\mathbb{Z}H} \cong \mathcal{U}_{\mathbb{Z}G} \cong \mathbb{Z}/2 \times G$ which implies $H < G$ (since H is torsion-free), which implies H is LO (since G is), which implies $\mathcal{U}_{\mathbb{Z}H} \cong \mathbb{Z}/2 \times H$, which implies $\mathbb{Z}/2 \times H \cong \mathbb{Z}/2 \times G$ which implies $H \cong G$ (since H, G are torsion free). $\square$

REMARK 1.14. We might wonder if it is ever the case that (for $G \neq 1$) $(G * \mathbb{Z}) / \langle\langle w \rangle\rangle = 1$? This is known for G torsion free [**3**].

COUNTEREXAMPLE 1. If we consider the question of whether we can ever have $(A * B) / \langle\langle w \rangle\rangle = 1$ for A, B nontrivial, a counterexample is given by:

$$\mathbb{Z}/2 * \mathbb{Z}/3 / (a = b) \ .$$

4. BO's on $\mathbb{Z} \times \mathbb{Z}$

Recall we have 2 orders on $\mathbb{Z}$. Consider a line of slope α in $\mathbb{Z} \times \mathbb{Z}$. Then we have two cases.

(1) α irrational: The associated positive cone is everything above the line. Specifically, $P \subset \mathbb{Z} \times \mathbb{Z}$ is given by

$$P = \{(m, n) \mid n > m\alpha\} \ . \tag{1.48}$$

It is easy to check that this is a positive cone. This means there are uncountable many BO's on $\mathbb{Z} \times \mathbb{Z}$.

(2) α rational: Notice that now

$$\{(m, n) \mid n = m\alpha\} \cong \mathbb{Z} < \mathbb{Z} \times \mathbb{Z} \ . \tag{1.49}$$

Now let P_0 be one of the two positive cones on $\mathbb{Z}$. Then we can check that

$$P = P_0 \amalg \{(m, n) \mid n > m\alpha\}$$

is a positive cone for $\mathbb{Z} \times \mathbb{Z}$.

REMARK 1.15. (1) (Up to reversal) these are all the BOs on $\mathbb{Z} \times \mathbb{Z}$. I.e. for α rational we get two, and for α irrational we get 4.

(2) This generalizes in the obvious way to $\mathbb{Z}^n$.

5. BO's on $\mathbb{R}$

Regard $\mathbb{R}$ as a vector space on $\mathbb{Q}$ with uncountable bases Λ. Recall Λ exists by the axiom of choice. Therefore $\mathbb{R} \subset \mathbb{Q}^{\Lambda}$. In particular it is the elements of $\mathbb{Q}^{\Lambda}$ with only finitely many non-zero coordinates. There are uncountable many WO's on Λ, and each gives rise to a lexicographic BO on $\mathbb{Q}^{\Lambda}$. This gives us uncountably many BOs on $\mathbb{R}$.

CHAPTER 2

The space of left-orders on a group

The basic idea is that since lefts orders are determined by positive cones, we can give this space a topology. Consider a family of sets $\{X_\lambda \mid \lambda \in \Lambda\}$. Then write

$$X = \prod_{\lambda \in \Lambda} X_\lambda$$

and $\pi_\lambda : X \to X_\lambda$ for the projection. If X_λ is a topological space, then X can be given the product topology. This is the largest topology on X such that π_λ is continuous for all $\lambda \in \Lambda$. So X has subbasis

$$\left\{ \pi_\lambda^{-1} \left(U_\lambda\right) = U_\lambda \times \prod_{\mu \neq \lambda X_\mu} \middle| U_\lambda \subset X_\lambda \text{ open}, \lambda \in \Lambda \right\} . \tag{2.1}$$

THEOREM 2.1 (Tychonoff). *If X_λ is compact for all $\lambda \in \Lambda$, then $\prod_{\lambda \in \Lambda} X_\lambda$ is compact.*

REMARK 2.1 (Exercises).

(1) X_λ Hausdorff (for all $\lambda \in \Lambda$) implies $\prod_{\lambda \in \Lambda} X_\lambda$ is Hausdorff.
(2) A space X is totally disconnected if the only nonempty connected subspaces are singletons $\{x\}$ for $x \in X$. This is equivalent to the connected components of X all being $\{x\}$. Show that X_λ totally disconnected (for all $\lambda \in \Lambda$) implies $\prod_{\lambda \in \Lambda} X_\lambda$ is totally disconnected.

Let X be a set, let $\mathcal{S}(X)$ be the set of subsets of X (i.e. the power set). Then we have a correspondence:

$$\mathcal{S}(X) \quad \leftrightarrow \quad \{f : X \to \{0,1\}\}$$

which sends:

$$A \subset X \quad \leftrightarrow \quad f_A : X \to \{0,1\}$$

where

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} .$$

Give $\{0,1\}$ the discrete topology, and give

$$\mathcal{S}(X) = \{0,1\}^X = 2^X = \prod_{x \in X} \{0,1\}$$

the product topology. Note $\{0,1\}$ is a compact, Hausdorff, totally-disconnected space, which means $\mathcal{S}(X)$ is too. For $x \in X$ let

$$U_x = \pi_x^{-1}(1) = \{A \subset X \mid x \in A\}$$
$$V_x = \pi_x^{-1}(0) = \{A \subset X \mid x \notin A\} .$$

Note that $V_x = \mathcal{S}(X) \setminus U_x$. Then

$$\{U_x \mid x \in X\} \cup \{V_x \mid x \in X\} \tag{2.2}$$

is a subbasis for $\mathcal{S}(X)$.

If G is a group, let

$$\mathrm{LO}(G) = \{\text{positive cones} \subset G\} \subset \mathcal{S}(G) \tag{2.3}$$

equipped with the subspace topology. We call this the *space of left-orders on G*.

THEOREM 2.2. $\mathrm{LO}(G)$ *is closed in* $\mathcal{S}(G)$ *and hence compact.*

EXAMPLE 2.1. $\mathrm{LO}(\mathbb{Z}) = \mathrm{pt} \amalg \mathrm{pt}$. $\mathrm{LO}(\mathbb{Z} \times \mathbb{Z})$ is the cantor set.

Bibliography

[1] Adam Clay and Dale Rolfsen, *Ordered groups and topology*, Graduate Studies in Mathematics, vol. 176, American Mathematical Society, Providence, RI, 2016.

[2] Martin Hertweck, *A counterexample to the isomorphism problem for integral group rings*, Annals of Mathematics. Second Series **1** (200107).

[3] Anton A. Klyachko, *The structure of one-relator relative presentations and their centres*, Journal of Group Theory **12** (2009Jan), no. 6.

[4] R.H. LaGrange and A.H. Rhemtulla, *A remark on the group rings of order preserving permutation groups*, Canadian Mathematical Bulletin **11** (1968), no. 5, 679?680.