Orderability and 3-manifold groups

Lectures by: Professor Cameron Gordon

Spring 2020; Notes by: Jackson Van Dyke; All errors introduced are my own.

Contents

CHAPTER 1

Orders on groups; basic definitions and properties

The book for the course is [**1**]. Lecture 1; January 21, 2020

Recall that a *strict total order* (STO) on a set X is a binary relation $<$ which satisfies:

(1) $x < y$ and $y < z$ implies $x < z$;
(2) $\forall x, y \in X$ exactly one of: $x < y$, $y < x$, $x = y$, holds.

A *left order* (LO) on a group G is an STO such that $g < h$ implies $fg < fh$ for all $f \in G$. G is *left-orderable* (LO) if there exists an LO on G. We similarly define a *right order* (RO) and *right orderability* (RO). A *bi-order* (BO) on G is an LO on G that is also an RO.

REMARK 1.1. (1) If G is abelian, $<$ is a LO iff $<$ is an RO iff $<$ is a BO.
(2) If $<$ is an LO on G, then $\prec$ defined by:

$$g \prec h \iff h^{-1} < g^{-1} \tag{1.1}$$

is an RO on G. Therefore G is LO iff G is RO. We will stick to LO's.
(3) For $H < G$, an LO (resp. BO) on G induces an LO (resp. BO) on H.

EXAMPLE 1.1. $(\mathbb{R}, +)$ with the usual $<$ is BO. The subgroups $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$ are also BO.

Lemma 1.1. *Let $<$ be an LO on G. Then*

(1) $g > 1$, $h > 1$ implies $gh > 1$;
(2) $g > 1$ implies $g^{-1} < 1$;
(3) $<$ is a BO iff $\left(g < h \implies f^{-1}gf < f^{-1}hf \forall f \in G\right)$ (i.e. $<$ is conjugation invariant).

PROOF. (1) $h > 1$ implies $gh > g \cdot 1g > 1$.
(2) $g > 1$ implies $g^{-1}g > g^{-1}$ implies $1 > g^{-1}$.
(3) ($\implies$) is immediate. ($\Longleftarrow$): We need to show $<$ is a RO. $g < h$ implies $fg < fh$ implies $f^{-1}(fg) f < f^{-1}(fh) f$ which implies $gf < hf$ as desired.

□

Lemma 1.2. *If $<$ is a BO on G, then*

(1) $g < h$ implies $g^{-1} > h^{-1}$;
(2) $g_1 < h$, $g_2 < h_2$ implies $g_1 g_2 < h_1 h_2$.

PROOF. (1) If $g < h$, then $g^{-1}g < g^{-1}h$, which implies $1 < g^{-1}h$, which implies $1 \cdot h^{-1} < g^{-1}$, which implies $h^{-1} < g^{-1}$.
(2) $g_2 < h_2$ implies $g_1 g_2 < g_1 h_2 < h_1 h_2$.

□

WARNING 1.1. These don't necessarily true for LO's.

Lemma 1.3. *If G is LO then it is torsion free.*

Proof. Consider $g \in G \setminus \{1\}$. If $g > 1$, then $g^2 > g > 1$, and similarly for all $n \geq 1$, $g^n > 1$. Similarly $g < 1$ implies $g^n < 1$ for all $n \geq 1$. □

So LO is not preserved under taking quotients (e.g. $\mathbb{Z} \to \mathbb{Z}/n$).

Consider an indexed family of groups $\{G_\lambda \,|\, \lambda \in \Lambda\}$. Recall that the direct product

$$\prod_{\lambda \in \Lambda} G_\lambda = \{(g_\lambda)_{\lambda \in \Lambda}\} \tag{1.2}$$

with multiplication defined co-ordinatewise.

Recall a *well-order* (WO) on a set X is a STO $\prec$ on X such that if $A \subset X$ and $A \neq \emptyset$ then there exists $a_0 \in A$ such that $a_0 \prec a$ for all $a \in A \setminus \{a_0\}$. Recall that the axiom of choice is equivalent to every set having a WO.

Theorem 1.4. *G_λ has a LO (resp. BO) for all $\lambda \in \Lambda$ iff $\prod_{\lambda \in \Lambda} G_\lambda$ has a LO (resp. BO).*

Proof. ($\Longleftarrow$): $G\lambda < \prod_\lambda G_\lambda$ so we are finished.

($\Longrightarrow$): Choose a WO $\prec$ on Λ, and order $\prod_\lambda G_\lambda$ lexicographically. Let $g = (g_\lambda)$, $h = (h_\lambda)$, $g \neq h$. Then λ_0 be the $\prec$-least element of Λ such that $g_{\lambda_0} \neq h_{\lambda_0}$. Then define $g < h$ iff $g_{\lambda_0} < h_{\lambda_0}$ (in G_{λ_0}). Then $<$ is an LO (resp. BO) on $\prod_\lambda G_\lambda$. Left (resp. left and right) invariance is clear. Now we show transitivity. Suppose $f < g$, $g < h$. Let λ_0 be the $\prec$-least element of Λ such that $f_{\lambda_0} \neq g_{\lambda_0}$. Let μ_0 be the $\prec$-least element of Λ such that $g_{\mu_0} \neq h_{\mu_0}$.

(1) ($\lambda_0 \preccurlyeq \mu_0$): Then $f_\lambda = g_\lambda = h_\lambda$ for all $\lambda \prec \lambda_0$. Then g_{λ_0} is $<$ (resp. $=$) h_{λ_0} if $\lambda_0 = \mu_0$ (resp. $\lambda_0 \prec \mu_0$). So $f_{\lambda_0} < g_{\lambda_0} \leq h_{\lambda_0}$, and therefore $f_{\lambda_0} < h_{\lambda_0}$.
(2) ($\mu_0 < \lambda_0$): This follows similarly.

□

Let $\sum_{\lambda in \Lambda} G_\lambda$ be the *direct sum of* $\{G_\lambda\}$. Recall this is the subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of elements such that all but finitely many co-ordinates are 1.

Corollary 1.5. *G_λ is LO (resp. BO) for all $\lambda \in \Lambda$ iff $\sum_{\lambda \in \Lambda} G_\lambda$ is LO (resp. BO).*

Corollary 1.6. *Free abelian groups are BO.*

Proof. Free abelian groups on Λ are $\sum_{\lambda \in \Lambda} \mathbb{Z}$. □

Let $<$ be an LO on G. The *positive cone* $P = P_<$ of $<$ is $\{g \in G \,|\, g > 1\}$.

Lemma 1.7. *(1) P is a subset of G, i.e. $g, h \in P$, implies $gh \in P$ (i.e. $PP \subset P$).*
(2) $G = P \amalg P^{-1} \amalg \{1\}$.
(3) $<$ is a BO on G iff $f^{-1}Pf \subset P$ for all $f \in G$.

Proof. (1) This follows from Lemma 1.1 (1).
(2) This follows from Lemma 1.1 (2).
(3) This follows from Lemma 1.1 (3).

□

We say $P \subset G$ is a *positive cone* if P satsfies the conditions in Lemma 1.7.

Lemma 1.8. *Let $P \subset G$ be a positive cone. Then $g < h$ implies $g^{-1}h \in P$ defines a LO $<$ on G (With $P_< = P$).*

Proof. $<$ is a STO, so:

(i) $f < g$, $g < h$ implies $f^{-1}g \in P$, $g^{-1}h \in P$, which implies (by the first property) that $\left(f^{-1}g\right)\left(g^{-1}h\right) \in P$, which implies $f < h$.
(ii) By the second property, for all $g, h \in G$ exactly one of the following holds: $g^{-1}h \in P$, $g^{-1}h \in P^{-1}$, and $g^{-1}h = 1$. Equivalently, $g < h$, $h < g$ (since $h^{-1}g \in P$), and $g = h$. Now we show left invariance. $g < h$ implies $g^{-1}h \in P$, but $g^{-1}h = \left(g^{-1}f^{-1}\right)(fh)$ which implies $fg < fh$.

$\square$

Lemmata 1.7 and 1.8 show that:

$$\{\text{LO's on } G\} \leftrightarrow \{\text{positive cones in } G\} \tag{1.3}$$

$$\{\text{BO's on } G\} \leftrightarrow \{\text{conjugacy-invariance positive cones in } G\}\ . \tag{1.4}$$

Consider the free group of rank n, F_n.

Theorem 1.9. *F_2 is LO.*

Sunic. Write $F_2 = F(a, b)$. $g \in F_2$ implies we can write it as a reduced word

$$\left(a^{m_1}\right) b^{n_1} \dots a^{m_k}\left(b^{n_k}\right) \tag{1.5}$$

for $k \geq 0$, $m_i, n_i \in \mathbb{Z} \setminus \{0\}$. Recall 1 is the empty word, $k = 0$. Let $e(g)$ be the number of syllables in g with positive exponent, minus the number of syllables in g with negative exponent. Then define $j(g)$ so be the number of $a^m b^n$'s in f, minus the number of $b^n a^m$s in G. So $j(g) = 0$, or ± 1. For example:

$$j\left(a^* \dots a^*\right) = 0 \tag{1.6}$$

$$j\left(b^* \dots b^*\right) = 0 \tag{1.7}$$

$$j\left(a^* \dots b^*\right) = 1 \tag{1.8}$$

$$j\left(b^* \dots a^*\right) = -1\ . \tag{1.9}$$

Finally define

$$\tau(g) = e(g) + j(g)\ . \tag{1.10}$$

Note that

$$e\left(g^{-1}\right) = -e(g) \qquad j\left(g^{-1}\right) = -j(g)\ . \tag{1.11}$$

Lemma 1.10. *If $g \neq 1$, then $\tau(g) \equiv 1 \pmod 2$.*

Proof. $e(f)$ is congruent to the number of syllables mod 2, and $j(g)$ is congruent to the number of syllables $+1$ mod 2. $\square$

Lemma 1.11. $|\tau(gh) - \tau(g) - \tau(h)| \leq 1$.

Proof. If gh or g or $h = 1$ we are done. So suppose $gh, g, h \neq 1$. Clearly $e(gh) = e(g) + e(h) + \left\{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}\right\}$. Similarly:

$$j(gh) = j(g) + j(h) + \left\{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}\right\}\ . \tag{1.12}$$

Therefore:

$$|\tau(gh) - \tau(g) - \tau(H)| \leq 2 \tag{1.13}$$

so by Lemma 1.10 we have

$$|\tau(gh) - \tau(g) - \tau(h)| \leq 1 \ . \tag{1.14}$$

□

REMARK 1.2. Lemma 1.11 says that $\tau : F_2 \to \mathbb{Z}\,(< \mathbb{R})$ is what is called a *quasi-morphism.*

Define $P \subset F_2$ by

$$P = \{g \in F_2 \,|\, \tau(g) > 0\} \ . \tag{1.15}$$

Then $F_2 = P \amalg P^{-1} \amalg \{1\}$ by Lemma 1.10 and that $\tau\left(g^{-1}\right) = -\tau(g)$. Then $PP \subset P$ by Lemma 1.11 since

$$\tau(gh) \geq \tau(g) + \tau(h) - 1 \geq 1 \ . \tag{1.16}$$

Therefore P is a positive cone for a LO on F_2. ■

Corollary 1.12. *Any countable free group is LO.*

PROOF. A countable free group is a subgroup of F_2. □

REMARK 1.3.
(1) $\tau\left(a^{-1}b\right) = 1$, so $a^{-1}b > 1$, so $b > a$. On the other hand, $\tau\left(ab^{-1}\right) = 1$, so $ab^{-1} > 1$, so $b^{-1} > a^{-1}$. So τ does not define a BO on F_2.
(2) We will see later that all free groups are LO.
(3) Even later we will see that all free groups are BO.

THEOREM 1.13. *Let $1 \to H \to G \to Q \to 1$ be a short-exact sequence of groups. Then*
(1) H, Q LO implies G is LO;
(2) if Q is BO and H has a BO that is invariant under conjugation in G then G is BO.

Lecture 2; January 23, 2019

PROOF. Write $\varphi : G \to Q$ and regard H as $\ker\varphi < G$. Let P_H (resp. P_Q) be positive cones for LO's on H (resp. Q). Define $P = \varphi^{-1}(P_Q) \amalg P_H$.

CLAIM 1.1. P is a positive cone for an LO on G.

PROOF. We need to check (1) and (2) from Lemma 1.7. Let $g, h \in P$. Then we want to show $gh \in P$. We have three cases.
(a) $g, h \in \varphi^{-1}(P_Q)$: In this case $\varphi(g), \varphi(h) \in P_Q$, so $\varphi(gh) = \varphi(g)\varphi(h) \in P_Q$. Therefore $gh \in \varphi^{-1}(P_Q)$.
(b) $g, h \in P_H$: In this case $gh \in P_H$.
(c) $g \in \varphi^{-1}(P_Q)$, $h \in P_H$: Then $\varphi(gh) = \varphi(g) \in P_Q$, so $gh \in \varphi^{-1}(P_Q)$. Similarly $hg \in \varphi^{-1}(P_Q)$.

Now we need to check $P \amalg P^{-1} \amalg \{1\}$. But this follows from the fact that:

$$G = (H \setminus \{1\}) \amalg \varphi^{-1}(Q \setminus \{1\}) \amalg \{1\} = \varphi^{-1}(P_Q) \amalg \varphi^{-1}\left(P_Q^{-1}\right) \tag{1.17}$$

since $H \setminus \{1\} = P_H \amalg P_H^{-1}$. □

We leave (2) as an exercise. [Hint: Recall P is a positive cone for BO on G iff it is a conjugacy invariant cone for an LO.] ■

1. Orderability of manifold groups

EXAMPLE 1.2. Let X^2 be the Klein bottle. This has fundamental group

$$K = \pi_1 \left(X^2\right) = \left\langle a, b \,\middle|\, b^{-1}ab = a^{-1} \right\rangle \ . \tag{1.18}$$

This fits in the SES:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \| & & & & & & \\ & & \langle a \rangle & & b & \longmapsto & gm & & \end{array} \tag{1.19}$$

which means K is LO by Theorem 1.13.

Note that K is *not* BO. We have that $a > 1$ iff $b^{-1}ab > 1$, but this is a^{-1}, so $a^{-1} > 1$ which is a contradiction.

Notice that $\mathbb{Z}$ has exactly two LO's. The usual one, and the opposite. Therefore, if we choose an LO on $\langle a \rangle$ and $K/\langle a \rangle$, this gives 4 LO's on K determined by:

(i) $a > 1$, $b > 1$;
(ii) $a > 1$, $b < 1$;
(iii) $a < 1$, $b > 1$;
(iv) $a < 1$, $b < 1$.

THEOREM 1.14. *These are the only LO's on K.*

PROOF. It suffices to show that each of these conditions determines a unique positive cone.

(i) $a > 1$, $b > 1$:

CLAIM 1.2. $a^k < b$ for all $k \in \mathbb{Z}$.

PROOF. $b < a^k$ implies $a^{-k}b < 1$. But $a^{-k}b = ba^k$ and $b > 1$, so $b < a^k$ implies $a^k > 1$, which implies $ba^k > 1$ which is a contradiction. □

Note that every element in K has a unique representative of the form $a^m b^n$ for $m, n \in \mathbb{Z}$.

CLAIM 1.3. $a^m b^n > 1$ iff either $n > 0$ or $n = 0$ and $m > 0$.

PROOF. If $n = 0$, then this is clear. If $n > 0$, then $a^m b > 1$ for any m by claim 1 (for $k = -m$). But we also know $b > 1$ which implies $b^n > 1$, so we get $a^m b^n > 1$ for $n > 0$. On the other hand, if $m < 0$ then $a^m b^n = b^n a^{\pm m} = \left(a^{\mp m} b^{-n}\right)^{-1}$. Then we know $a^{\mp m} b^{-n} > 1$ by the case above, so its inverse is < 1. □

If $<$ is an LO on G, and $\alpha : G \to G$ is an automorphism, then this induces an LO $<_\alpha$ on G given by: $g <_\alpha h$ iff $\alpha(g) < \alpha(h)$. Now notice that there are automorphisms α_1, α_2 of K such that

$$\alpha_1(a) = a \ , \qquad \alpha_1(b) = b^{-1} \tag{1.20}$$

$$\alpha_1(a) = a^{-1} \ , \qquad \alpha_1(b) = b \ . \tag{1.21}$$

In particular, α_1 is given by

$$\left\langle a, b \,\middle|\, b^{-1}ab = a^{-1} \right\rangle \cong \left\langle a, b \,\middle|\, bab^{-1} = a^{-1} \right\rangle \tag{1.22}$$

and similarly for α_2.

Write $<_{(i)}$ for the unique LO on K determined by (i). Then $<_{(ii)}$ is induced by $<_{(i)}$ and α_1, $<_{(iii)}$ is induced by $<_{(i)}$ and α_2, and $<_{(iv)}$ is induced by $<_{(i)}$ and $\alpha_1\alpha_2$.

■

FACT 1. *If G has only finitely many LO's, then the number of LO's is of the form 2^n.*

EXERCISE 1.1. Show that for all $n \geq 0$ there exists a group G with exactly 2^n LO's.

Corollary 1.15. *For any LO on K, if $h \in \langle a \rangle$, $g \in K \setminus \langle a \rangle$, and $g > 1$, then $g > h$.*

PROOF. It is sufficient to check this for the first LO, since the other three are determined by the above automorphisms. Let $a > 1$, $b > 1$. By claim 2 from above, we know $g = a^m b^n$ for $n > 0$. We now there is some k such that $h = a^k$, and therefore

$$h^{-1} g = a^{m-k} b^n > 1 \tag{1.23}$$

by claim 2, so $g > h$. □

2. Three-manifold groups

Suppose M is a closed, orientable, connected three-manifold. Then we might ask if $\pi_1(M)$ is LO? BO?

Immediately we notice that not all such groups are. If M is a lens space, then $\pi_1(M) \cong \mathbb{Z}/n$ for $n > 1$, so this is not LO. More generally, for $\pi_1(M)$ nontrivial and finite is not LO. Recall that if $M = M_1 \# M_2$, then this implies $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. So, for example, if $M_1\#$ lens space, then $\pi_1(M)$ has torsion, so not LO.

But at least some of them are. Consider $M \cong T^3 = S^1 \times S^1 \times S^1$. Then $\pi_1(M) = \mathbb{Z}^3$ is of course LO. Similarly $M = \#_n \left(S^1 \times S^2\right) \cong F_n$, so $\pi_1(M)$ is LO.

We will show that there exist (three-manifold) groups that are torsion-free, but not LO.

Let $p : T^2 \to X^2$ be a two-fold covering of the Klein bottle. Recall that

$$K > p_* \left(\pi_1 \left(T^2\right)\right) = \left\langle a, b^2 \right\rangle \cong \mathbb{Z} \times \mathbb{Z} . \tag{1.24}$$

Let N be the mapping cylinder of p, namely:

$$N = \left(T^2 \times I\right) \amalg X^2 / \left((x, 0) \sim p(x) \, \forall x \in T^2\right) . \tag{1.25}$$

The orientation reversing curve representing b doesn't lift. So N is orientable. Note that $\partial N \cong T^2$. There is a strong deformation retraction $N \to X^2$, so $\pi_1(N) \cong K$. Let N_1, N_2 be two copies of N. Write

$$\pi_1(N_i) = \left\langle a_i, b_i \,\middle|\, b_i^{-1} a_i b_i = a_i^{-1} \right\rangle . \tag{1.26}$$

Notice that $\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z} = \left\langle a_i, b_i^2 \right\rangle < \pi_1(N_i)$. Let $\varphi : \partial N_1 \to \partial N_2$ be a homeomorphism. Let $M_\varphi = N_1 \cup_\varphi N_2$. This is a closed, orientable three-manifold. Therefore

$$\pi_1(M_\varphi) = \pi_1(N_1) *_{\mathbb{Z} \times \mathbb{Z}} \pi_1(N_2) \cong K_1 *_{\mathbb{Z} \times \mathbb{Z}} K_2 . \tag{1.27}$$

Since K is torsion-free, $\pi_1(M_\varphi)$ is torsion-free. But in fact we have the following theorem.

THEOREM 1.16. *If $H_1(M_\varphi)$ is finite, then $\pi_1(M_\varphi)$ is not LO.*

REMARK 1.4. We will see later that for M a prime three-manifold with $H_1(M)$ infinite has $\pi_1(M)$ LO.

PROOF. φ is determined up to isotopy, so the resulting manifold M_φ depends only on $\varphi_* : H_1(\partial N_1) \to H_1(\partial N_2)$. We know

$$\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_1, 2b_1 \rangle \qquad\qquad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_2, 2b_2 \rangle \tag{1.28}$$

so φ_* is given by some 2×2 matrix with $\mathbb{Z}$ coefficients

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \tag{1.29}$$

with determinant $ps - qr = \pm 1$. Specifically we have:

$$\varphi_*(a_1) = pa_2 + 2qb_2 \tag{1.30}$$

$$\varphi_*(2b_1) = ra_2 + 2sb_2 \ . \tag{1.31}$$

Now we have $H_1(N_i) = \mathbb{Z} \oplus \mathbb{Z}_2$ with basis b_i and a_i respectively. Then $H_q(M_\varphi)$ is presented by

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{bmatrix} \ . \tag{1.32}$$

where we order the basis as $\{a_1, b_1, a_2, b_2\}$. Interchanging columns 2 and 3 we get

$$\det A = 4 \left| \det \begin{bmatrix} 0 & 2q \\ -2 & 2s \end{bmatrix} \right| = 16\,|q| \ . \tag{1.33}$$

Therefore $H_1(M_\varphi)$ is finite iff $q \neq 0$ iff $\varphi_*(a_1) \neq \pm a_2$.

Suppose $\pi_1(M_\varphi)$ is LO. Then we would get an induced LO on the common boundary $\partial N_1 = \partial N_2$. But there are only 4 LO's on $\pi_1(N_i)$ (for $i \in \{1,2\}$). By Corollary 1.15, for any LO on $\pi_1(N)$, $\langle a \rangle$ is the unique $\mathbb{Z}$-summand of $\pi_1(\partial N) = \langle a, b^2 \rangle$ such that if $h \in \langle a \rangle$ and $g \in \pi_1(\partial N) \setminus \{1\}$, $g > 1$, then $g > h$. Therefore $\varphi_*(a_1) = \pm a_2$ which is a contradiction. ☐

Let $<$ be an STO on a set X. Let $\mathcal{B}(X, <)$ be the group of $<$-preserving bijections $X \to X$. Lecture 3; January 28, 2020

Theorem 1.17. *$\mathcal{B}(X, <)$ is always LO.*

Proof. Let $\prec$ be a WO on X. Let $f, g \in \mathcal{B}(X, <)$ such that $f \neq g$. Write

$$[f \neq g] = \{x \in X \mid f(x) \neq g(x)\} \neq \emptyset \ . \tag{1.34}$$

Let x_0 be the $\prec$-least element of $[f \neq g]$. Define

$$f < g \iff f(x_0) < g(x_0) \ . \tag{1.35}$$

Then we claim that this is an LO on $\mathcal{B}(X, <)$. Left-invariance is clear. To see this is a STO we need "trichotomy" and transitivity. Trichotomy is easy, and transitivity follows from the same argument as the proof of Theorem 1.4. ☐

Example 1.3. Let $<$ be the standard order on $\mathbb{R}$. Then $\mathcal{B}(\mathbb{R}, <)$ consists of the orientation-preserving homeomorphisms $\mathbb{R} \to \mathbb{R}$, written $\mathrm{Homeo}_+(\mathbb{R})$.

Corollary 1.18. $\mathrm{Homeo}_+(\mathbb{R})$ *is LO.*

Remark 1.5. For $x \in \mathbb{R}$, let $\prec_x$ be a WO on $\mathbb{R}$ such that x is the $\prec_x$-least element of $\mathbb{R}$. Let $<_x$ be the LO on $\mathrm{Homeo}_+(\mathbb{R})$ induced by $\prec_x$, as in the proof of Theorem 1.17. Given $x \neq y \in \mathbb{R}$, there exists $g \in \mathrm{Homeo}_+(\mathbb{R})$ such that $g(x) > x$ and $g(y) < y$. But this means

$$g <_x 1 \qquad\qquad g <_y 1 \ .$$

which implies $<_x \neq <_y$. Therefore $\mathrm{Homeo}_+(\mathbb{R})$ has uncountably many LO's.

Remark 1.6. It is a fact that the number of LO's on a group G is either finite (and of the form 2^n) or uncountable.

Corollary 1.19. *A group G is LO iff G acts faithfully*[1.1] *on a STO'd set $(X,<)$.*

Proof. ($\Longleftarrow$): This follows from Theorem 1.17.
($\Longrightarrow$): G acts faithfully on $(G,<)$ by left multiplication. □

Corollary 1.18 implies that any subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is LO. E.g. one can show that F_2 (the free group of rank 2) is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. (This is another way to show that countable free groups are LO.) In fact this characterizes countable LO groups.

Theorem 1.20. *Let G be a countable group. Then G is LO iff there exists an injective homomorphism $G \to \mathrm{Homeo}_+(\mathbb{R})$.*

Proof. ($\Longleftarrow$): This follows from Corollary 1.18.
($\Longrightarrow$): We actually prove something slightly stronger. This will follow from Theorem 1.21. □

Theorem 1.21. *Let $(G,<)$ be a countable group with an LO. Then there exists a LO on $\mathrm{Homeo}_+(\mathbb{R})$ and an order-preserving injective homomorphism $(G,<) \to (\mathrm{Homeo}_+(\mathbb{R}),<)$.*

Sketch of proof. Let $<$ be an LO on G. If $G=\{1\}$ this is immediate, so assume $G \neq \{1\}$. Therefore it is infinite, since LO groups are torsion free. Let $g_1, g_2, \ldots$ be some enumeration of the elements of G.

Define an embedding $e: G \to \mathbb{R}$ by $e(g_1)=0$, and inductively by:

(i) If $g_{n+1} \begin{Bmatrix} > \\ < \end{Bmatrix} g_i$ for all $1 \leq i \leq n$, then set

$$e(g_{n+1}) = \begin{Bmatrix} \max\{e(g_i) \mid 1 \leq i \leq n\}+1 \\ \min\{e(g_i) \mid 1 \leq i \leq n\}-1 \end{Bmatrix} . \tag{1.36}$$

(ii) Otherwise let

$$g_l = \max\{g_i \mid 1 \leq i \leq n, g_i < g_{n+1}\}$$
$$g_r = \min\{g_i \mid 1 \leq i \leq n, g_i > g_{n+1}\}$$

and set

$$e(g_{n+1}) = \frac{e(g_l)+e(g_r)}{2} .$$

Remark 1.7. (1) e is order-preserving, i.e. $a<b \implies e(a)<e(b)$.

(2) $e(g_{n+1}) \in \mathbb{Z}$ iff (i) holds.

(3) If $g>1$ then $g^2>g$ and $g^{-1}<g$. If $g<1$ then $g^2<g$ and $g^{-1}>g$, which implies $\mathbb{Z} \subset e(G) = \Gamma$.

(4) G acts on Γ by $g(e(a)) = e(ga)$. In fact, G acts on $(\Gamma,<)$ (where $<$ is the restriction of $<$ on $\mathbb{R}$) since $e(a)<e(b)$ iff $a<b$ iff $ga<gb$ iff $e(ga)<e(gb)$ iff $g(e(a))<g(e(b))$.

To see that this action extends to an action of G on $\mathbb{R}$, we have a few steps.

Step 1: The action of G on Γ is continuous,

Step 2: The action of G on Γ extends to a continuous action of G on $\bar{\Gamma}$.

[1.1] Recall this means $g(x)=x$ for all $x \in X$ iff $g=1$.

Step 3: $\mathbb{R}\backslash\bar{\Gamma}$ is a countable II of open intervals (a_i, b_i); the action of G is defined on $\{a_i, b_i\}$; and extends to $[a_i, b_i]$.

Note, to ensure Step 1:, it is not enough to take e to be an order-preserving of G in $\mathbb{R}$. It must be continuous.

To define an LO on $\mathrm{Homeo}_+(\mathbb{R})$ that restricts to the LO on Γ from G, first pick any $\gamma \in \Gamma$. Then $g > 1$ (resp. < 1) iff $g(\gamma) > \gamma$ (resp. $< \gamma$). Let $\prec$ be a WO on $\mathbb{R}$ such that γ is the $\prec$-least element of $\mathbb{R}$. Then let $\lessdot$ be the LO on $\mathrm{Homeo}_+(\mathbb{R})$ induced by $\prec$. Then $g > 1$ (resp. < 1) in G iff $g \gtrdot 1$ (resp. $\lessdot$) in $\mathrm{Homeo}_+(\mathbb{R})$. $\square$

3. Group rings

Let R be a ring (with 1).

- $a \in R$ is a *unit* if there exists $b \in R$ such that $ab = ba = 1$.
- $a \in R$ is a *zero-divisor* if $a \neq 0$ and there exists $b \neq 0$ such that either $ab = 0$ or $ba = 0$.
- $a \in R$ is a *non-trivial idempotent* if $a^2 = a$ but $a \neq 0$ and $a \neq 1$.

Let G be a group and R a ring. Then the *R-group ring of G* consists of formal sums:

$$RG := \left\{ \sum r_g g \,\middle|\, g \in G, r_g \in R, r_G \neq 0 \forall \text{ but f'tly many } g \in G \right\} . \tag{1.37}$$

RG is a ring with respect to the obvious operations. For $g \in G$ and $r \in R$ a unit, then rg is a unit in RG. A unit in RG is *non-trivial* if it is not of this form.

Remark 1.8. If $\tilde{X} \to X$ is a universal covering, then $\pi = \pi_1(X)$ acts on $\tilde{X}$ so $H_*\left(\tilde{X}, \mathbb{Z}\right)$ is a $\mathbb{Z}\pi$-module.

Theorem 1.22. *Suppose G has non-trivial torsion, and K is a field of characteristic 0.*
(1) KG has zero divisors,
(2) KG has non-trivial units,
(3) KG has non-trivial idempotents.

Proof. Let $g \in G$ have order $n \geq 2$. Define

$$\sigma = 1 + g + g^2 + \ldots + g^{n-1} \in KG .$$

First notice that

$$g\sigma = \sigma \tag{1.38}$$

which implies $(1-g)\sigma = 0$ so we have zero divisors.

(1.38) also gives us that $\sigma^2 = n\sigma$. Therefore

$$(1-\sigma)\left(1 - \frac{1}{n-1}\sigma\right) = 1$$

so we have a nontrivial unit for $n > 2$. If $n = 2$, $1 - \sigma = -g$, but we still have:

$$(1-2\sigma)\left(1 - \frac{2}{3}\sigma\right) = 1 . \tag{1.39}$$

Finally, we have that

$$\left(\frac{1}{n}\sigma\right)^2 = \left(\frac{1}{n^2}\right)\sigma^2 = \frac{1}{n}\sigma \tag{1.40}$$

so we have nontrivial idempotents. $\square$

Note that the proof of (1) works even for $\mathbb{Z}G$.

REMARK 1.9. If $n \notin \{2,3,4,6\}$ then $\mathbb{Z}G$ has nontrivial units. This is a theorem of Higman.

EXAMPLE 1.4. For $n = 5$,

$$\left(1 - g - g^4\right)\left(1 - g^2 - g^3\right) = 1\ . \tag{1.41}$$

But what if G is torsion free? This brings us to the famous Kaplansky conjectures.

CONJECTURE 1 (Kaplansky). *If G is torsion free and K is a field, then:*

I (Units conjecture): KG has no non-trivial units,
II (Zero-divisors conjecture): KG has no zero divisors,
III (Idempotents conjecture): KG has no non-trivial idempotents.

REMARK 1.10. Clearly II implies III since $a^2 = a$ implies $a\,(a-1) = 0$, which by II implies $a = 0$ or $a = 1$ which implies III. In fact they're all equivalent, but this is nontrivial to see.

REMARK 1.11. We know this is true for LO groups. As we have seen, we should think of LO as being a stronger version of torsion free.

Bibliography

[1] Adam Clay and Dale Rolfsen, *Ordered groups and topology*, Graduate Studies in Mathematics, vol. 176, American Mathematical Society, Providence, RI, 2016.