Orderability and 3-manifold groups

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Contents

CHAPTER 1

Orders on groups; basic definitions and properties

The book for the course is [**1**]. Lecture 1; January 21, 2020

Recall that a *strict total order* (STO) on a set X is a binary relation $<$ which satisfies:

(1) $x < y$ and $y < z$ implies $x < z$;
(2) $\forall x, y \in X$ exactly one of: $x < y$, $y < x$, $x = y$, holds.

A *left order* (LO) on a group G is an STO such that $g < h$ implies $fg < fh$ for all $f \in G$. G is *left-orderable* (LO) if there exists an LO on G. We similarly define a *right order* (RO) and *right orderability* (RO). A *bi-order* (BO) on G is an LO on G that is also an RO.

Remark 1.1. (1) If G is abelian, $<$ is a LO iff $<$ is an RO iff $<$ is a BO.
(2) If $<$ is an LO on G, then $\prec$ defined by:

$$g \prec h \iff h^{-1} < g^{-1} \tag{1.1}$$

is an RO on G. Therefore G is LO iff G is RO. We will stick to LO's.
(3) For $H < G$, an LO (resp. BO) on G induces an LO (resp. BO) on H.

Example 1.1. $(\mathbb{R}, +)$ with the usual $<$ is BO. The subgroups $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$ are also BO.

Lemma 1.1. *Let $<$ be an LO on G. Then*

(1) $g > 1$, $h > 1$ implies $gh > 1$;
(2) $g > 1$ implies $g^{-1} < 1$;
(3) $<$ is a BO iff $\left(g < h \implies f^{-1}gf < f^{-1}hf \forall f \in G\right)$ (i.e. $<$ is conjugation invariant).

Proof. (1) $h > 1$ implies $gh > g \cdot 1g > 1$.
(2) $g > 1$ implies $g^{-1}g > g^{-1}$ implies $1 > g^{-1}$.
(3) $(\implies)$ is immediate. $(\impliedby)$: We need to show $<$ is a RO. $g < h$ implies $fg < fh$ implies $f^{-1}(fg)f < f^{-1}(fh)f$ which implies $gf < hf$ as desired.

$\square$

Lemma 1.2. *If $<$ is a BO on G, then*

(1) $g < h$ implies $g^{-1} > h^{-1}$;
(2) $g_1 < h$, $g_2 < h_2$ implies $g_1g_2 < h_1h_2$.

Proof. (1) If $g < h$, then $g^{-1}g < g^{-1}h$, which implies $1 < g^{-1}h$, which implies $1 \cdot h^{-1} < g^{-1}$, which implies $h^{-1} < g^{-1}$.
(2) $g_2 < h_2$ implies $g_1g_2 < g_1h_2 < h_1h_2$.

$\square$

Warning 1.1. These don't necessarily true for LO's.

Lemma 1.3. *If G is LO then it is torsion free.*

Proof. Consider $g \in G \setminus \{1\}$. If $g > 1$, then $g^2 > g > 1$, and similarly for all $n \geq 1$, $g^n > 1$. Similarly $g < 1$ implies $g^n < 1$ for all $n \geq 1$. □

So LO is not preserved under taking quotients (e.g. $\mathbb{Z} \to \mathbb{Z}/n$).

Consider an indexed family of groups $\{G_\lambda \,|\, \lambda \in \Lambda\}$. Recall that the direct product

$$\prod_{\lambda \in \Lambda} G_\lambda = \{(g_\lambda)_{\lambda \in \Lambda}\} \tag{1.2}$$

with multiplication defined co-ordinatewise.

Recall a *well-order* (WO) on a set X is a STO $\prec$ on X such that if $A \subset X$ and $A \neq \emptyset$ then there exists $a_0 \in A$ such that $a_0 \prec a$ for all $a \in A \setminus \{a_0\}$. Recall that the axiom of choice is equivalent to every set having a WO.

Theorem 1.4. *G_λ has a LO (resp. BO) for all $\lambda \in \Lambda$ iff $\prod_{\lambda \in \Lambda} G_\lambda$ has a LO (resp. BO).*

Proof. ($\Longleftarrow$): $G\lambda < \prod_\lambda G_\lambda$ so we are finished.

($\Longrightarrow$): Choose a WO $\prec$ on Λ, and order $\prod_\lambda G_\lambda$ lexicographically. Let $g = (g_\lambda)$, $h = (h_\lambda)$, $g \neq h$. Then λ_0 be the $\prec$-least element of Λ such that $g_{\lambda_0} \neq h_{\lambda_0}$. Then define $g < h$ iff $g_{\lambda_0} < h_{\lambda_0}$ (in G_{λ_0}). Then $<$ is an LO (resp. BO) on $\prod_\lambda G_\lambda$. Left (resp. left and right) invariance is clear. Now we show transitivity. Suppose $f < g$, $g < h$. Let λ_0 be the $\prec$-least element of Λ such that $f_{\lambda_0} \neq g_{\lambda_0}$. Let μ_0 be the $\prec$-least element of Λ such that $g_{\mu_0} \neq h_{\mu_0}$.

(1) ($\lambda_0 \preccurlyeq \mu_0$): Then $f_\lambda = g_\lambda = h_\lambda$ for all $\lambda \prec \lambda_0$. Then g_{λ_0} is $<$ (resp. $=$) h_{λ_0} if $\lambda_0 = \mu_0$ (resp. $\lambda_0 \prec \mu_0$). So $f_{\lambda_0} < g_{\lambda_0} \leq h_{\lambda_0}$, and therefore $f_{\lambda_0} < h_{\lambda_0}$.
(2) ($\mu_0 < \lambda_0$): This follows similarly.

□

Let $\sum_{\lambda in \Lambda} G_\lambda$ be the *direct sum of* $\{G_\lambda\}$. Recall this is the subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of elements such that all but finitely many co-ordinates are 1.

Corollary 1.5. *G_λ is LO (resp. BO) for all $\lambda \in \Lambda$ iff $\sum_{\lambda \in \Lambda} G_\lambda$ is LO (resp. BO).*

Corollary 1.6. *Free abelian groups are BO.*

Proof. Free abelian groups on Λ are $\sum_{\lambda \in \Lambda} \mathbb{Z}$. □

Let $<$ be an LO on G. The *positive cone* $P = P_<$ of $<$ is $\{g \in G \,|\, g > 1\}$.

Lemma 1.7. *Let P be as above.*

(1) $g, h \in P$, implies $gh \in P$ (i.e. $PP \subset P$).
(2) $G = P \amalg P^{-1} \amalg \{1\}$.
(3) $<$ is a BO on G iff $f^{-1}Pf \subset P$ for all $f \in G$.

Proof. (1) This follows from Lemma 1.1 (1).
(2) This follows from Lemma 1.1 (2).
(3) This follows from Lemma 1.1 (3).

□

We say $P \subset G$ is a *positive cone* if P satsfies the conditions in Lemma 1.7.

Lemma 1.8. *Let $P \subset G$ be a positive cone. Then $g < h$ implies $g^{-1}h \in P$ defines a LO $<$ on G (With $P_< = P$).*

PROOF. $<$ is a STO, so:

(i) $f<g$, $g<h$ implies $f^{-1}g\in P$, $g^{-1}h\in P$, which implies (by the first property) that $\left(f^{-1}g\right)\left(g^{-1}h\right)\in P$, which implies $f<h$.

(ii) By the second property, for all $g,h\in G$ exactly one of the following holds: $g^{-1}h\in P$, $g^{-1}h\in P^{-1}$, and $g^{-1}h=1$. Equivalently, $g<h$, $h<g$ (since $h^{-1}g\in P$), and $g=h$. Now we show left invariance. $g<h$ implies $g^{-1}h\in P$, but $g^{-1}h=\left(g^{-1}f^{-1}\right)(fh)$ which implies $fg<fh$.

□

Lemmata 1.7 and 1.8 show that:

$$\{\text{LO's on } G\} \leftrightarrow \{\text{positive cones in } G\} \tag{1.3}$$

$$\{\text{BO's on } G\} \leftrightarrow \{\text{conjugacy-invariance positive cones in } G\}\ . \tag{1.4}$$

Consider the free group of rank n, F_n.

THEOREM 1.9. *F_2 is LO.*

PROOF BY SUNIC. Write $F_2=F(a,b)$. $g\in F_2$ implies we can write it as a reduced word

$$\left(a^{m_1}\right)b^{n_1}\dots a^{m_k}\left(b^{n_k}\right) \tag{1.5}$$

for $k\geq 0$, $m_i,n_i\in\mathbb{Z}\setminus\{0\}$. Recall 1 is the empty word, $k=0$. Let $e(g)$ be the number of syllables in g with positive exponent, minus the number of syllables in g with negative exponent. Then define $j(g)$ so be the number of a^mb^n's in f, minus the number of b^na^ms in G. So $j(g)=0$, or ±1. For example:

$$j\left(a^*\dots a^*\right)=0 \tag{1.6}$$

$$j\left(b^*\dots b^*\right)=0 \tag{1.7}$$

$$j\left(a^*\dots b^*\right)=1 \tag{1.8}$$

$$j\left(b^*\dots a^*\right)=-1\ . \tag{1.9}$$

Finally define

$$\tau(g)=e(g)+j(g)\ . \tag{1.10}$$

Note that

$$e\left(g^{-1}\right)=-e(g) \qquad j\left(g^{-1}\right)=-j(g)\ . \tag{1.11}$$

Lemma 1.10. *If $g\neq 1$, then $\tau(g)\equiv 1 \pmod 2$.*

PROOF. $e(f)$ is congruent to the number of syllables mod 2, and $j(g)$ is congruent to the number of syllables $+1$ mod 2. □

Lemma 1.11. $|\tau(gh)-\tau(g)-\tau(h)|\leq 1$.

PROOF. If gh or g or $h=1$ we are done. So suppose $gh,g,h\neq 1$. Clearly $e(gh)=e(g)+e(h)+\begin{Bmatrix}0\\1\\-1\end{Bmatrix}$. Similarly:

$$j(gh)=j(g)+j(h)+\begin{Bmatrix}0\\1\\-1\end{Bmatrix}\ . \tag{1.12}$$

Therefore:

$$|\tau(gh) - \tau(g) - \tau(H)| \leq 2 \tag{1.13}$$

so by Lemma 1.10 we have

$$|\tau(gh) - \tau(g) - \tau(h)| \leq 1 \ . \tag{1.14}$$

□

REMARK 1.2. Lemma 1.11 says that $\tau : F_2 \to \mathbb{Z}(< \mathbb{R})$ is what is called a *quasi-morphism.*

Define $P \subset F_2$ by

$$P = \{g \in F_2 \,|\, \tau(g) > 0\} \ . \tag{1.15}$$

Then $F_2 = P \amalg P^{-1} \amalg \{1\}$ by Lemma 1.10 and that $\tau\left(g^{-1}\right) = -\tau(g)$. Then $PP \subset P$ by Lemma 1.11 since

$$\tau(gh) \geq \tau(g) + \tau(h) - 1 \geq 1 \ . \tag{1.16}$$

Therefore P is a positive cone for a LO on F_2. ■

Corollary 1.12. *Any countable free group is LO.*

PROOF. A countable free group is a subgroup of F_2. □

REMARK 1.3.
(1) $\tau\left(a^{-1}b\right) = 1$, so $a^{-1}b > 1$, so $b > a$. On the other hand, $\tau\left(ab^{-1}\right) = 1$, so $ab^{-1} > 1$, so $b^{-1} > a^{-1}$. So τ does not define a BO on F_2.
(2) We will see later that all free groups are LO.
(3) Even later we will see that all free groups are BO.

THEOREM 1.13. *Let $1 \to H \to G \to Q \to 1$ be a short-exact sequence of groups. Then*
(1) H, Q LO implies G is LO;
(2) if Q is BO and H has a BO that is invariant under conjugation in G then G is BO.

Lecture 2; January 23, 2019

PROOF. Write $\varphi : G \to Q$ and regard H as $\ker\varphi < G$. Let P_H (resp. P_Q) be positive cones for LO's on H (resp. Q). Define $P = \varphi^{-1}(P_Q) \amalg P_H$.

CLAIM 1.1. P is a positive cone for an LO on G.

PROOF. We need to check (1) and (2) from Lemma 1.7. Let $g, h \in P$. Then we want to show $gh \in P$. We have three cases.
(a) $g, h \in \varphi^{-1}(P_Q)$: In this case $\varphi(g), \varphi(h) \in P_Q$, so $\varphi(gh) = \varphi(g)\varphi(h) \in P_Q$. Therefore $gh \in \varphi^{-1}(P_Q)$.
(b) $g, h \in P_H$: In this case $gh \in P_H$.
(c) $g \in \varphi^{-1}(P_Q)$, $h \in P_H$: Then $\varphi(gh) = \varphi(g) \in P_Q$, so $gh \in \varphi^{-1}(P_Q)$. Similarly $hg \in \varphi^{-1}(P_Q)$.

Now we need to check $P \amalg P^{-1} \amalg \{1\}$. But this follows from the fact that:

$$G = (H \setminus \{1\}) \amalg \varphi^{-1}(Q \setminus \{1\}) \amalg \{1\} = \varphi^{-1}(P_Q) \amalg \varphi^{-1}\left(P_Q^{-1}\right) \tag{1.17}$$

since $H \setminus \{1\} = P_H \amalg P_H^{-1}$. □

We leave (2) as an exercise. [Hint: Recall P is a positive cone for BO on G iff it is a conjugacy invariant cone for an LO.] ■

1. Orderability of manifold groups

EXAMPLE 1.2. Let X^2 be the Klein bottle. This has fundamental group

$$K = \pi_1\left(X^2\right) = \left\langle a, b \,\middle|\, b^{-1}ab = a^{-1}\right\rangle \ . \tag{1.18}$$

This fits in the SES:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \| & & & & & & \\ & & \langle a \rangle & & b & \longmapsto & gm & & \end{array} \tag{1.19}$$

which means K is LO by Theorem 1.13.

Note that K is *not* BO. We have that $a > 1$ iff $b^{-1}ab > 1$, but this is a^{-1}, so $a^{-1} > 1$ which is a contradiction.

Notice that $\mathbb{Z}$ has exactly two LO's. The usual one, and the opposite. Therefore, if we choose an LO on $\langle a \rangle$ and $K/\langle a \rangle$, this gives 4 LO's on K determined by:

(i) $a > 1$, $b > 1$;
(ii) $a > 1$, $b < 1$;
(iii) $a < 1$, $b > 1$;
(iv) $a < 1$, $b < 1$.

THEOREM 1.14. *These are the only LO's on K.*

PROOF. It suffices to show that each of these conditions determines a unique positive cone.

(i) $a > 1$, $b > 1$:

CLAIM 1.2. $a^k < b$ for all $k \in \mathbb{Z}$.

PROOF. $b < a^k$ implies $a^{-k}b < 1$. But $a^{-k}b = ba^k$ and $b > 1$, so $b < a^k$ implies $a^k > 1$, which implies $ba^k > 1$ which is a contradiction. □

Note that every element in K has a unique representative of the form $a^m b^n$ for $m, n \in \mathbb{Z}$.

CLAIM 1.3. $a^m b^n > 1$ iff either $n > 0$ or $n = 0$ and $m > 0$.

PROOF. If $n = 0$, then this is clear. If $n > 0$, then $a^m b > 1$ for any m by claim 1 (for $k = -m$). But we also know $b > 1$ which implies $b^n > 1$, so we get $a^m b^n > 1$ for $n > 0$. On the other hand, if $m < 0$ then $a^m b^n = b^n a^{\pm m} = \left(a^{\mp m} b^{-n}\right)^{-1}$. Then we know $a^{\mp m} b^{-n} > 1$ by the case above, so its inverse is < 1. □

If $<$ is an LO on G, and $\alpha : G \to G$ is an automorphism, then this induces an LO $<_\alpha$ on G given by: $g <_\alpha h$ iff $\alpha(g) < \alpha(h)$. Now notice that there are automorphisms α_1, α_2 of K such that

$$\alpha_1(a) = a \ , \qquad \alpha_1(b) = b^{-1} \tag{1.20}$$

$$\alpha_1(a) = a^{-1} \ , \qquad \alpha_1(b) = b \ . \tag{1.21}$$

In particular, α_1 is given by

$$\left\langle a, b \,\middle|\, b^{-1}ab = a^{-1}\right\rangle \cong \left\langle a, b \,\middle|\, bab^{-1} = a^{-1}\right\rangle \tag{1.22}$$

and similarly for α_2.

Write $<_{(i)}$ for the unique LO on K determined by (i). Then $<_{(ii)}$ is induced by $<_{(i)}$ and α_1, $<_{(iii)}$ is induced by $<_{(i)}$ and α_2, and $<_{(iv)}$ is induced by $<_{(i)}$ and $\alpha_1\alpha_2$.

■

FACT 1. *If G has only finitely many LO's, then the number of LO's is of the form 2^n.*

EXERCISE 1.1. Show that for all $n \geq 0$ there exists a group G with exactly 2^n LO's.

Corollary 1.15. *For any LO on K, if $h \in \langle a \rangle$, $g \in K \setminus \langle a \rangle$, and $g > 1$, then $g > h$.*

PROOF. It is sufficient to check this for the first LO, since the other three are determined by the above automorphisms. Let $a > 1$, $b > 1$. By claim 2 from above, we know $g = a^m b^n$ for $n > 0$. We now there is some k such that $h = a^k$, and therefore

$$h^{-1} g = a^{m-k} b^n > 1 \tag{1.23}$$

by claim 2, so $g > h$. □

2. Three-manifold groups

Suppose M is a closed, orientable, connected three-manifold. Then we might ask if $\pi_1(M)$ is LO? BO?

Immediately we notice that not all such groups are. If M is a lens space, then $\pi_1(M) \cong \mathbb{Z}/n$ for $n > 1$, so this is not LO. More generally, for $\pi_1(M)$ nontrivial and finite is not LO. Recall that if $M = M_1 \# M_2$, then this implies $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. So, for example, if $M_1 \#$ lens space, then $\pi_1(M)$ has torsion, so not LO.

But at least some of them are. Consider $M \cong T^3 = S^1 \times S^1 \times S^1$. Then $\pi_1(M) = \mathbb{Z}^3$ is of course LO. Similarly $M = \#_n \left(S^1 \times S^2\right) \cong F_n$, so $\pi_1(M)$ is LO.

We will show that there exist (three-manifold) groups that are torsion-free, but not LO.

Let $p : T^2 \to X^2$ be a two-fold covering of the Klein bottle. Recall that

$$K > p_* \left(\pi_1 \left(T^2\right)\right) = \left\langle a, b^2 \right\rangle \cong \mathbb{Z} \times \mathbb{Z} \ . \tag{1.24}$$

Let N be the mapping cylinder of p, namely:

$$N = \left(T^2 \times I\right) \amalg X^2 / \left((x, 0) \sim p(x) \, \forall x \in T^2\right) \ . \tag{1.25}$$

The orientation reversing curve representing b doesn't lift. So N is orientable. Note that $\partial N \cong T^2$. There is a strong deformation retraction $N \to X^2$, so $\pi_1(N) \cong K$. Let N_1, N_2 be two copies of N. Write

$$\pi_1(N_i) = \left\langle a_i, b_i \,\middle|\, b_i^{-1} a_i b_i = a_i^{-1} \right\rangle \ . \tag{1.26}$$

Notice that $\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z} = \left\langle a_i, b_i^2 \right\rangle < \pi_1(N_i)$. Let $\varphi : \partial N_1 \to \partial N_2$ be a homeomorphism. Let $M_\varphi = N_1 \cup_\varphi N_2$. This is a closed, orientable three-manifold. Therefore

$$\pi_1(M_\varphi) = \pi_1(N_1) *_{\mathbb{Z} \times \mathbb{Z}} \pi_1(N_2) \cong K_1 *_{\mathbb{Z} \times \mathbb{Z}} K_2 \ . \tag{1.27}$$

Since K is torsion-free, $\pi_1(M_\varphi)$ is torsion-free. But in fact we have the following theorem.

THEOREM 1.16. *If $H_1(M_\varphi)$ is finite, then $\pi_1(M_\varphi)$ is not LO.*

REMARK 1.4. We will see later that for M a prime three-manifold with $H_1(M)$ infinite has $\pi_1(M)$ LO.

PROOF. φ is determined up to isotopy, so the resulting manifold M_φ depends only on $\varphi_* : H_1(\partial N_1) \to H_1(\partial N_2)$. We know

$$\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_1, 2b_1 \rangle \qquad\qquad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_2, 2b_2 \rangle \tag{1.28}$$

so φ_* is given by some 2×2 matrix with $\mathbb{Z}$ coefficients

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \tag{1.29}$$

with determinant $ps - qr = \pm 1$. Specifically we have:

$$\varphi_* (a_1) = pa_2 + 2qb_2 \tag{1.30}$$

$$\varphi_* (2b_1) = ra_2 + 2sb_2 \ . \tag{1.31}$$

Now we have $H_1 (N_i) = \mathbb{Z} \oplus \mathbb{Z}_2$ with basis b_i and a_i respectively. Then $H_q (M_\varphi)$ is presented by

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{bmatrix} \ . \tag{1.32}$$

where we order the basis as $\{a_1, b_1, a_2, b_2\}$. Interchanging columns 2 and 3 we get

$$\det A = 4 \left| \det \begin{bmatrix} 0 & 2q \\ -2 & 2s \end{bmatrix} \right| = 16 \, |q| \ . \tag{1.33}$$

Therefore $H_1 (M_\varphi)$ is finite iff $q \neq 0$ iff $\varphi_* (a_1) \neq \pm a_2$.

Suppose $\pi_1 (M_\varphi)$ is LO. Then we would get an induced LO on the common boundary $\partial N_1 = \partial N_2$. But there are only 4 LO's on $\pi_1 (N_i)$ (for $i \in \{1, 2\}$). By Corollary 1.15, for any LO on $\pi_1 (N)$, $\langle a \rangle$ is the unique $\mathbb{Z}$-summand of $\pi_1 (\partial N) = \langle a, b^2 \rangle$ such that if $h \in \langle a \rangle$ and $g \in \pi_1 (\partial N) \setminus \{1\}$, $g > 1$, then $g > h$. Therefore $\varphi_* (a_1) = \pm a_2$ which is a contradiction. ☐

Let $<$ be an STO on a set X. Let $\mathcal{B}(X, <)$ be the group of $<$-preserving bijections $X \to X$. Lecture 3; January 28, 2020

Theorem 1.17. *$\mathcal{B}(X, <)$ is always LO.*

Proof. Let $\prec$ be a WO on X. Let $f, g \in \mathcal{B}(X, <)$ such that $f \neq g$. Write

$$[f \neq g] = \{x \in X \mid f(x) \neq g(x)\} \neq \emptyset \ . \tag{1.34}$$

Let x_0 be the $\prec$-least element of $[f \neq g]$. Define

$$f < g \iff f(x_0) < g(x_0) \ . \tag{1.35}$$

Then we claim that this is an LO on $\mathcal{B}(X, <)$. Left-invariance is clear. To see this is a STO we need "trichotomy" and transitivity. Trichotomy is easy, and transitivity follows from the same argument as the proof of Theorem 1.4. ☐

Example 1.3. Let $<$ be the standard order on $\mathbb{R}$. Then $\mathcal{B}(\mathbb{R}, <)$ consists of the orientation-preserving homeomorphisms $\mathbb{R} \to \mathbb{R}$, written $\mathrm{Homeo}_+ (\mathbb{R})$.

Corollary 1.18. $\mathrm{Homeo}_+ (\mathbb{R})$ *is LO.*

Remark 1.5. For $x \in \mathbb{R}$, let $\prec_x$ be a WO on $\mathbb{R}$ such that x is the $\prec_x$-least element of $\mathbb{R}$. Let $<_x$ be the LO on $\mathrm{Homeo}_+ (\mathbb{R})$ induced by $\prec_x$, as in the proof of Theorem 1.17. Given $x \neq y \in \mathbb{R}$, there exists $g \in \mathrm{Homeo}_+ (\mathbb{R})$ such that $g(x) > x$ and $g(y) < y$. But this means

$$g <_x 1 \qquad\qquad g <_y 1 \ .$$

which implies $<_x \neq <_y$. Therefore $\mathrm{Homeo}_+ (\mathbb{R})$ has uncountably many LO's.

REMARK 1.6. It is a fact that the number of LO's on a group G is either finite (and of the form 2^n) or uncountable.

Corollary 1.19. *A group G is LO iff G acts faithfully*[1.1] *on a STO'd set $(X, <)$.*

PROOF. ($\Longleftarrow$): This follows from Theorem 1.17.
($\Longrightarrow$): G acts faithfully on $(G, <)$ by left multiplication. □

Corollary 1.18 implies that any subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is LO. E.g. one can show that F_2 (the free group of rank 2) is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. (This is another way to show that countable free groups are LO.) In fact this characterizes countable LO groups.

THEOREM 1.20. *Let G be a countable group. Then G is LO iff there exists an injective homomorphism $G \to \mathrm{Homeo}_+(\mathbb{R})$.*

PROOF. ($\Longleftarrow$): This follows from Corollary 1.18.
($\Longrightarrow$): We actually prove something slightly stronger. This will follow from Theorem 1.21. □

THEOREM 1.21. *Let $(G, <)$ be a countable group with an LO. Then there exists a LO on $\mathrm{Homeo}_+(\mathbb{R})$ and an order-preserving injective homomorphism $(G, <) \to (\mathrm{Homeo}_+(\mathbb{R}), <)$.*

SKETCH OF PROOF. Let $<$ be an LO on G. If $G = \{1\}$ this is immediate, so assume $G \neq \{1\}$. Therefore it is infinite, since LO groups are torsion free. Let $g_1, g_2, \ldots$ be some enumeration of the elements of G.

Define an embedding $e : G \to \mathbb{R}$ by $e(g_1) = 0$, and inductively by:

(i) If $g_{n+1} \begin{Bmatrix} > \\ < \end{Bmatrix} g_i$ for all $1 \leq i \leq n$, then set

$$e(g_{n+1}) = \begin{Bmatrix} \max\{e(g_i) \mid 1 \leq i \leq n\} + 1 \\ \min\{e(g_i) \mid 1 \leq i \leq n\} - 1 \end{Bmatrix} . \tag{1.36}$$

(ii) Otherwise let

$$g_l = \max\{g_i \mid 1 \leq i \leq n, g_i < g_{n+1}\}$$
$$g_r = \min\{g_i \mid 1 \leq i \leq n, g_i > g_{n+1}\}$$

and set

$$e(g_{n+1}) = \frac{e(g_l) + e(g_r)}{2} .$$

REMARK 1.7. (1) e is order-preserving, i.e. $a < b \implies e(a) < e(b)$.
(2) $e(g_{n+1}) \in \mathbb{Z}$ iff (i) holds.
(3) If $g > 1$ then $g^2 > g$ and $g^{-1} < g$. If $g < 1$ then $g^2 < g$ and $g^{-1} > g$, which implies $\mathbb{Z} \subset e(G) = \Gamma$.
(4) G acts on Γ by $g(e(a)) = e(ga)$. In fact, G acts on $(\Gamma, <)$ (where $<$ is the restriction of $<$ on $\mathbb{R}$) since $e(a) < e(b)$ iff $a < b$ iff $ga < gb$ iff $e(ga) < e(gb)$ iff $g(e(a)) < g(e(b))$.

To see that this action extends to an action of G on $\mathbb{R}$, we have a few steps.

Step 1: The action of G on Γ is continuous,
Step 2: The action of G on Γ extends to a continuous action of G on $\bar{\Gamma}$.

[1.1] Recall this means $g(x) = x$ for all $x \in X$ iff $g = 1$.

Step 3: $\mathbb{R}\backslash\bar{\Gamma}$ is a countable $\amalg$ of open intervals (a_i, b_i); the action of G is defined on $\{a_i, b_i\}$; and extends to $[a_i, b_i]$.

Note, to ensure Step 1:, it is not enough to take e to be an order-preserving of G in $\mathbb{R}$. It must be continuous.

To define an LO on $\mathrm{Homeo}_+(\mathbb{R})$ that restricts to the LO on Γ from G, first pick any $\gamma \in \Gamma$. Then $g > 1$ (resp. < 1) iff $g(\gamma) > \gamma$ (resp. $< \gamma$). Let $\prec$ be a WO on $\mathbb{R}$ such that γ is the $\prec$-least element of $\mathbb{R}$. Then let $\lessdot$ be the LO on $\mathrm{Homeo}_+(\mathbb{R})$ induced by $\prec$. Then $g > 1$ (resp. < 1) in G iff $g \gtrdot 1$ (resp. $\lessdot$) in $\mathrm{Homeo}_+(\mathbb{R})$. $\square$

3. Group rings

Let R be a ring (with 1).

- $a \in R$ is a *unit* if there exists $b \in R$ such that $ab = ba = 1$.
- $a \in R$ is a *zero-divisor* if $a \neq 0$ and there exists $b \neq 0$ such that either $ab = 0$ or $ba = 0$.
- $a \in R$ is a *non-trivial idempotent* if $a^2 = a$ but $a \neq 0$ and $a \neq 1$.

Let G be a group and R a ring. Then the *R-group ring of G* consists of formal sums:

$$RG := \left\{ \sum r_g g \,\middle|\, g \in G, r_g \in R, r_G \neq 0 \forall \text{ but f'tly many } g \in G \right\} . \tag{1.37}$$

RG is a ring with respect to the obvious operations. For $g \in G$ and $r \in R$ a unit, then rg is a unit in RG. A unit in RG is *non-trivial* if it is not of this form.

Remark 1.8. If $\tilde{X} \to X$ is a universal covering, then $\pi = \pi_1(X)$ acts on $\tilde{X}$ so $H_*\left(\tilde{X}, \mathbb{Z}\right)$ is a $\mathbb{Z}\pi$-module.

Theorem 1.22. *Suppose G has non-trivial torsion, and K is a field of characteristic 0.*

(1) KG has zero divisors,
(2) KG has non-trivial units,
(3) KG has non-trivial idempotents.

Proof. Let $g \in G$ have order $n \geq 2$. Define

$$\sigma = 1 + g + g^2 + \ldots + g^{n-1} \in KG .$$

First notice that

$$g\sigma = \sigma \tag{1.38}$$

which implies $(1 - g)\sigma = 0$ so we have zero divisors.

(1.38) also gives us that $\sigma^2 = n\sigma$. Therefore

$$(1 - \sigma)\left(1 - \frac{1}{n-1}\sigma\right) = 1$$

so we have a nontrivial unit for $n > 2$. If $n = 2$, $1 - \sigma = -g$, but we still have:

$$(1 - 2\sigma)\left(1 - \frac{2}{3}\sigma\right) = 1 . \tag{1.39}$$

Finally, we have that

$$\left(\frac{1}{n}\sigma\right)^2 = \left(\frac{1}{n^2}\right)\sigma^2 = \frac{1}{n}\sigma \tag{1.40}$$

so we have nontrivial idempotents. $\square$

Note that the proof of (1) works even for $\mathbb{Z}G$.

REMARK 1.9. If $n \notin \{2,3,4,6\}$ then $\mathbb{Z}G$ has nontrivial units. This is a theorem of Higman.

EXAMPLE 1.4. For $n = 5$,

$$\left(1 - g - g^4\right)\left(1 - g^2 - g^3\right) = 1 \ . \tag{1.41}$$

But what if G is torsion free? This brings us to the famous Kaplansky conjectures.

CONJECTURE 1 (Kaplansky). *If G is torsion free and K is a field, then:*

I (Units conjecture): KG has no non-trivial units,

II (Zero-divisors conjecture): KG has no zero divisors,

III (Idempotents conjecture): KG has no non-trivial idempotents.

REMARK 1.10. Clearly II implies III since $a^2 = a$ implies $a\,(a-1) = 0$, which by II implies $a = 0$ or $a = 1$ which implies III. In fact they're all equivalent, but this is nontrivial to see.

Lecture 4; January 30, 2020

REMARK 1.11. Note that if R is an integral domain (e.g. $\mathbb{Z}$) then R is contained in its field of fractions. In this case Items I and II and Item III for its field of fractions imply the corresponding versions of Items I and II and Item III for R.

REMARK 1.12. We know this is true for LO groups. As we have seen, we should think of LO as being a stronger version of torsion free.

THEOREM 1.23. *If G is LO then KG satisfies Items I and II and Item III.*

PROOF. Since Item I implies Item III by the above remark we show Item I and Item II.

Item I: Suppose

$$\left(\sum_{i=1}^{m} \alpha_i g_i\right)\left(\sum_{j=1}^{n} \beta_j h_j\right) = 1 \tag{1.42}$$

with m, n not both 1, $\alpha_i, \beta_j \neq 0 \in K$, distinct $g_i \in G$, and distinct $h_i \in G$. Note this product can be rewritten as the following sum with mn terms:

$$\sum_{i,j} \left(\alpha_i \beta_j\right)\left(g_i h_j\right) \ . \tag{1.43}$$

Assume WLOG that $h_1 < h_2 < \ldots < h_n$. Let $g_k h_l$ be a minimal element of

$$S = \{g_i h_j \,|\, 1 \leq i \leq m, 1 \leq j \leq n\} \subset G \ . \tag{1.44}$$

We know $h_1 < h_j$ for $j > 1$, so $g_k h_1 < g_k h_j$ for all $j > 1$. Therefore $l = 1$. Also $g h_1 = g' h_1$ which implies $g = g'$. Therefore $g_k h_1$ is the unique

$$(k, 1) \in \{(i,j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\} \tag{1.45}$$

such that $g_k h_1$ is a minimal element of S.

Similarly, there is a unique

$$(r, n) \in \{(i,j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\} \tag{1.46}$$

such that $g_r h_n$ is a maximal element of S.

CLAIM 1.4. $g_k h_1 \neq g_r h_n$.

If they were equal, then $r = k$, $n = 1$, so $m > 1$. So $g_k h_1 = g_r h_1$, and therefore $g_r = g_k$. But this cannot be the case since they are distinct by assumption.

This implies that (1.43) has ≥ 2 terms after cancellation, so it cannot be 1.

Item II: Now suppose

$$\left(\sum_{i=1}^{m} \alpha_i g_i\right)\left(\sum_{j=1}^{n} \beta_j h_j\right) = 0 \tag{1.47}$$

for $m, n \geq 1$. Then there is a unique minimal element and nonzero coefficient, which means it is nonzero. □

Conjecture 2 (Isomorphism conjecture). *If G is torsion free, then $\mathbb{Z}G \cong \mathbb{Z}H$ implies $G \cong H$.*

Remark 1.13. In [**3**] a finite counterexample to the conjecture for arbitrary groups was provided, i.e. it is shown that there exists finite G, H such that $\mathbb{Z}G \cong \mathbb{Z}H$, $G \ncong H$.

Corollary 1.24 ([**5**]). *If G is LO, then G satisfies the isomorphism conjecture.*

Proof. Theorem 1.23 implies that $\mathbb{Z}G$ has no nontrivial units. Call $\mathcal{U}_{\mathbb{Z}G}$ the group of units in $\mathbb{Z}G = \mathbb{Z}/2 \times G$. Suppose $\mathbb{Z}G \cong \mathbb{Z}H$. Theorem 1.23 says that $\mathbb{Z}G$ has no 0-divisors. This implies $\mathbb{Z}H$ has no 0-divisors, which means (by Theorem 1.22) that H is torsion-free. Now $H < \mathcal{U}_{\mathbb{Z}H} \cong \mathcal{U}_{\mathbb{Z}G} \cong \mathbb{Z}/2 \times G$ which implies $H < G$ (since H is torsion-free), which implies H is LO (since G is), which implies $\mathcal{U}_{\mathbb{Z}H} \cong \mathbb{Z}/2 \times H$, which implies $\mathbb{Z}/2 \times H \cong \mathbb{Z}/2 \times G$ which implies $H \cong G$ (since H, G are torsion free). □

Remark 1.14. We might wonder if it is ever the case that (for $G \neq 1$) $(G * \mathbb{Z}) / \langle\langle w \rangle\rangle = 1$? This is known for G torsion free [**4**].

Counterexample 1. If we consider the question of whether we can ever have $(A * B) / \langle\langle w \rangle\rangle = 1$ for A, B nontrivial, a counterexample is given by:

$$\mathbb{Z}/2 * \mathbb{Z}/3 / (a = b) \ .$$

4. BO's on $\mathbb{Z} \times \mathbb{Z}$

Recall we have 2 orders on $\mathbb{Z}$. Consider a line of slope α in $\mathbb{Z} \times \mathbb{Z}$. Then we have two cases.

(1) α irrational: The associated positive cone is everything above the line. Specifically, $P \subset \mathbb{Z} \times \mathbb{Z}$ is given by

$$P = \{(m, n) \,|\, n > m\alpha\} \ . \tag{1.48}$$

It is easy to check that this is a positive cone. This means there are uncountable many BO's on $\mathbb{Z} \times \mathbb{Z}$.

(2) α rational: Notice that now

$$\{(m, n) \,|\, n = m\alpha\} \cong \mathbb{Z} < \mathbb{Z} \times \mathbb{Z} \ . \tag{1.49}$$

Now let P_0 be one of the two positive cones on $\mathbb{Z}$. Then we can check that

$$P = P_0 \amalg \{(m, n) \,|\, n > m\alpha\}$$

is a positive cone for $\mathbb{Z} \times \mathbb{Z}$.

Remark 1.15. (1) (Up to reversal) these are all the BOs on $\mathbb{Z} \times \mathbb{Z}$. I.e. for α rational we get two, and for α irrational we get 4.

(2) This generalizes in the obvious way to $\mathbb{Z}^n$.

5. BO's on $\mathbb{R}$

Regard $\mathbb{R}$ as a vector space on $\mathbb{Q}$ with uncountable bases Λ. Recall Λ exists by the axiom of choice. Therefore $\mathbb{R} \subset \mathbb{Q}^{\Lambda}$. In particular it is the elements of $\mathbb{Q}^{\Lambda}$ with only finitely many non-zero coordinates. There are uncountable many WO's on Λ, and each gives rise to a lexicographic BO on $\mathbb{Q}^{\Lambda}$. This gives us uncountably many BOs on $\mathbb{R}$.

CHAPTER 2

The space of left-orders on a group

The basic idea is that since lefts orders are determined by positive cones, we can give this space a topology. Consider a family of sets $\{X_\lambda \mid \lambda \in \Lambda\}$. Then write

$$X = \prod_{\lambda \in \Lambda} X_\lambda$$

and $\pi_\lambda : X \to X_\lambda$ for the projection. If X_λ is a topological space, then X can be given the product topology. This is the largest topology on X such that π_λ is continuous for all $\lambda \in \Lambda$. So X has subbasis

$$\left\{ \pi_\lambda^{-1}(U_\lambda) = U_\lambda \times \prod_{\mu \neq \lambda X_\mu} \middle| U_\lambda \subset X_\lambda \text{ open}, \lambda \in \Lambda \right\} . \tag{2.1}$$

Theorem. *If X_λ is compact for all $\lambda \in \Lambda$ then $\prod_{\lambda \in \Lambda} X_\lambda$ is compact.*

Remark 2.1 (Exercises).
(1) X_λ Hausdorff (for all $\lambda \in \Lambda$) implies $\prod_{\lambda \in \Lambda} X_\lambda$ is Hausdorff.
(2) A space X is totally disconnected if the only nonempty connected subspaces are singletons $\{x\}$ for $x \in X$. This is equivalent to the connected components of X all being $\{x\}$. Show that X_λ totally disconnected (for all $\lambda \in \Lambda$) implies $\prod_{\lambda \in \Lambda} X_\lambda$ is totally disconnected.

Let X be a set, let $\mathcal{S}(X)$ be the set of subsets of X (i.e. the power set). Then we have a correspondence:

$$\mathcal{S}(X) \qquad \leftrightarrow \qquad \{f : X \to \{0,1\}\}$$

which sends:

$$A \subset X \qquad \leftrightarrow \qquad f_A : X \to \{0,1\}$$

where

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} .$$

Give $\{0,1\}$ the discrete topology, and give

$$\mathcal{S}(X) = \{0,1\}^X = 2^X = \prod_{x \in X} \{0,1\}$$

the product topology. Note $\{0,1\}$ is a compact, Hausdorff, totally-disconnected space, which means $\mathcal{S}(X)$ is too. For $x \in X$ let

$$U_x = \pi_x^{-1}(1) = \{A \subset X \mid x \in A\}$$
$$V_x = \pi_x^{-1}(0) = \{A \subset X \mid x \notin A\} .$$

Note that $V_x = \mathcal{S}(X) \setminus U_x$ so U_x and V_x are open and closed. Then

$$\{U_x \mid x \in X\} \cup \{V_x \mid x \in X\} \tag{2.2}$$

is a subbasis for $\mathcal{S}(X)$.

Lecture 5; February 4, 2020

Lemma 2.1. *Suppose $B \subset X$. Then*

$$\{A \subset X \mid B \not\subset A\} \qquad\qquad \{A \subset X \mid A \cap B \neq \emptyset\}$$

are open subsets of $\mathcal{S}(X)$.

Proof.

$$\{A \subset X \mid B \not\subset A\} = \bigcup_{b \in B} \{A \subset X \mid b \notin A\} = \bigcup_{b \in B} V_b$$

so it is open. The argument for the other set is similar. □

If G is a group, let

$$\mathrm{LO}(G) = \{\text{positive cones} \subset G\} \subset \mathcal{S}(G) \tag{2.3}$$

and equip it with the subspace topology. We call this the *space of left-orders on G*.

Example 2.1. $\mathrm{LO}(\mathbb{Z}) = \mathrm{pt} \amalg \mathrm{pt}$. $\mathrm{LO}(\mathbb{Z} \times \mathbb{Z})$ is the cantor set.

Theorem 2.2. $\mathrm{LO}(G)$ *is closed in* $\mathcal{S}(G)$ *and hence compact.*

Proof. We show $\mathcal{S}(G) \setminus \mathrm{LO}(G)$ is open. Suppose $A \in \mathcal{S}(G) \setminus \mathrm{LO}(G)$, i.e. $A \subset G$ is not a positive cone. So either:

(i) $\exists g, h \in A$ such that $gh \notin A$ or
(ii) $\exists g \in G$ such that $g, g^{-1} \in A$ or
(iii) $1 \in A$ or
(iv) $\exists g,\, g \neq 1$ such that $g \notin A$ and $g^{-1} \notin A$.

Now the point is that these are open conditions since we can write them in terms of the U_x's and V_x's. In particular:

$$(i) \iff A \in U_g \cap U_h \cap V_{gh} \qquad\qquad (ii) \iff A \in U_g \cap U_{g^{-1}}$$

$$(iii) \iff A \in U_1 \qquad\qquad (iv) \iff A \in \bigcup_{g \neq 1} \left(V_g \cap V_{g^{-1}}\right) .$$

Therefore $\mathrm{LO}(G)$ is compact, Hausdorff, and totally disconnected. □

Similarly one can define the space of biorders on G, $\mathrm{BO}(G)$, to be the set of conjugation invariant positive cones in G.

Exercise 0.1. Show that $\mathrm{BO}(G)$ is closed inside of $\mathrm{LO}(G)$.

Therefore $\mathrm{BO}(G)$ is compact, Hausdorff, and totally disconnected.

1. The cantor set

The cantor set $C \subset I \subset \mathbb{R}$ is defined as follows. First write

$$C_1 = [0, 1/3] \cup [2/3, 1]$$
$$C_2 = ([0, 1/9] \cup [2/9, 1/3]) \cup ([2/3, 7/9] \cup [8/9, 1])$$
$$\ldots$$

then define

$$C = \bigcap_{n=1}^{\infty} C_n \ . \tag{2.4}$$

The idea is that we keep removing the middle thirds.

C is uncountable, totally-disconnected, closed in I. Therefore it is also compact and Hausdorff. This is a very surprising example. We can easily write down something uncountable and totally-disconnected, such as the irrationals, but they do not form a compact set.

Any $x \in I$ has a ternary expansion:

$$x = 0.\, x_1 x_2 \ldots = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

which is unique up to:

$$\ldots x_k 22 \ldots = \ldots (x_{k+1})\, 00 \ldots \ .$$

Now notice

$$x_1 = 1 \qquad \Longleftrightarrow \qquad x \in (1/3, 2/3)$$

with the convention that

$$\frac{1}{3} = 0.022 \ldots \ .$$

Similarly (with the same convention) we have

$$x_1 \neq 1, x_2 = 1 \qquad \Longleftrightarrow \qquad x \in (1/9, 2/9) \cup (7/9, 8/9)$$

and so on. Then

$$C = \{x \in I \,|\, x = 0.\, x_1 x_2 \ldots \,|\, \forall n, x_n = 0 \text{ or } 2\} \ . \tag{2.5}$$

Now give $\{0, 2\}^{\mathbb{N}}$ the product topology.

Exercise 1.1. Show that the map sending

$$0.x_1 x_2 \ldots \mapsto (x_1, x_2, \ldots) \tag{2.6}$$

defines a homeomorphism

$$C \xrightarrow{\cong} \{0, 2\}^{\mathbb{N}} \ . \tag{2.7}$$

Now recall that $\mathrm{LO}\,(G)$ is compact in $\{0, 1\}^G$, so if G is countable, then $\mathrm{LO}\,(G)$ is homeomorphic to a subspace of C.

We say $x \in X$ is *isolated* if $\{x\}$ is open. We say X is perfect if it has no isolated points. As it turns out, the Cantor set is perfect.

Theorem. *If X is a compact, totally-disconnected, and perfect metric space, then $X \cong C$.*

Therefore, if G is countable, $\mathrm{LO}\,(G) \neq \emptyset$, and has no isolated points, then $\mathrm{LO}\,(G) \cong C$.

EXAMPLE 2.2. In 2004 [**10**] it was shown that if $n > 1$ then $\mathrm{LO}(\mathbb{Z}^n) = \mathrm{BO}(\mathbb{Z}^n) \cong C$.

EXAMPLE 2.3. In 1985 [**7**] it was shown that $\mathrm{LO}(F_n) \cong C$. It is unknown if $\mathrm{LO}(F_n)$ has isolated points.

REMARK 2.2. As it turns out, the braid group is LO. The first proof of this fact was not topological, so topologists started to think of a topological proof. When someone asked Thurston, he said "of course the braid group is left-orderable!"

If $X \subset G$, let $S(X)$ be the semigroup generated by X in G. This is the same as the non-empty product of elements in X. There is a characterization of left orderability in terms of finite subsets of G.

THEOREM 2.3. *G is LO iff for all finite $F \subset G \setminus \{1\}$, there exists $\epsilon : F \to \{\pm 1\}$ such that*

$$1 \notin S\left(\left\{f^{\epsilon(f)} \,\middle|\, f \in F\right\}\right) (= S(F, \epsilon)) \ . \tag{2.8}$$

REMARK 2.3. It follows from this that, given a solution to the word problem in G, there exists a machine such that if G is not LO, the machine will eventually tell you that. Nathan Dunfield has an explicit algorithm for three-manifold groups.

REMARK 2.4. If we take the n-fold cyclic branch cover of the knot 5_2, then we can consider $\pi_1(\Sigma_n(5_2))$. For $n = 2$, this is a lens space so π_1 is finite. It is also not LO for $n = 3, 4$, and 5. But it is unknown for $n = 6, 7$, and 8. (If the L-space conjecture is true,[2.1] then it should be LO for these values of n.) For $n \geq 9$ it is known to be LO.

PROOF. ($\Longrightarrow$): Define

$$\epsilon(f) = \begin{cases} +1 & f > 1 \\ -1 & f < 1 \end{cases} .$$

($\Longleftarrow$): Let $F \subset G \setminus \{1\}$ be finite, $\epsilon : F \to \{\pm 1\}$. Define

$$Q(F, \epsilon) \coloneqq \left\{Q \subset G \setminus \{1\} \,\middle|\, S(F, \epsilon) \subset Q, S(F, \epsilon)^{-1} \cap Q = \emptyset\right\} .$$

Note that $Q(F, \epsilon) \neq \emptyset$ iff (2.8) holds. Let

$$Q(F) = \cup_\epsilon Q(F, \epsilon) \ .$$

Note this is a finite union.

CLAIM 2.1. $Q(F)$ is closed in $\mathcal{S}(G)$.

PROOF. It is sufficient to show that $Q(F, \epsilon)$ is closed, i.e. $\mathcal{S}(G) \setminus Q(F, \epsilon)$ is open. Suppose $A \subset G$, $A \notin Q(F, \epsilon)$ i.e. either $1 \in A$, or $S(F, \epsilon) \not\subset A$, or $S(F, \epsilon)^{-1} \cap A \neq \emptyset$. These conditions are all open by Lemma 2.1. □

Note that if $F \subset F'$, then

$$S(F, \epsilon'|_{F'}) \subset S(F', \epsilon') \tag{2.9}$$

and therefore

$$Q(F') \subset Q(F) \ . \tag{2.10}$$

[2.1] Which is looking quite likely. It has been checked for something like three-hundred thousand manifolds.

Let $F_1, F_2, \ldots, F_n$ be finite subsets of $G \setminus \{1\}$. Then

$$\bigcap_{i=1}^{n} Q(F_i) \supset Q(F_1 \cup F_2 \cup \ldots \cup F_n) \neq \emptyset$$

since (2.8) holds. This means $\{Q(F)\}$ has the *finite intersection property* (FIP) and each one is closed. Therefore, since $\mathcal{S}(G)$ is compact,

$$\bigcap_{F \subset G \setminus \{1\} \text{ finite}} Q(F) \neq \emptyset \ .$$

So let $P \in \bigcap Q(F)$.

Claim 2.2. P is a positive cone for G.

Lecture 6; February 6, 2020

Proof. First notice $1 \notin P$ since $1 \notin Q(F)$ for any finite $F \subset G \setminus \{1\}$.

Now we show $g, h \in P$ implies $gh \in P$. Let $F = \{g, h\}$. Then there are $\epsilon(g), \epsilon(h) \in \{\pm 1\}$ such that

$$S\left(g^{\epsilon(g)}, h^{\epsilon(h)}\right) \subset P \qquad\qquad S\left(g^{\epsilon(g)}, h^{\epsilon(h)}\right)^{-1} \cap P = \emptyset \ .$$

Therefore $\epsilon(g) = \epsilon(h) = +1$, which implies $gh \in S\left(g^{\epsilon(g)}, h^{\epsilon(h)}\right) \subset P$.

Now we show $P \cap P^{-1} = \emptyset$. Let $g \in P$, and $F = \{g\}$. Therefore $S(g) \subset P$, which means $S(g)^{-1} \cap P = \emptyset$, so $g^{-1} \notin P$.

Finally we show $P \amalg P^{-1} G \setminus \{1\}$. Take $g \in G$ such that $g \neq 1$. Let $F = \{g\}$. Then there exists $\epsilon = \pm 1$ such that $S(g^{\epsilon}) \subset P$ (and $S\left(g^{-1}\right) \cap P = \emptyset$) which implies $g^{\epsilon} \in P$. □

■

Remark 2.5. There exists an analogue of this for BO.

Theorem 2.4. *G is BO if and only if for all finite $F \subset G \setminus \{1\}$ there is some $\epsilon : F \to \{\pm 1\}$ such that $1 \notin T(F, \epsilon)$ where $T(F, \epsilon)$ is the smallest semigroup which*

(i) contains $S(F, \epsilon)$, and
(ii) for all $g, h \in T(F, \epsilon)$, $g, h, g^{-1}, g^{-1}hg \in T(F, \epsilon)$.

Exercise 1.2. Prove Theorem 2.4.

Let P be a property of groups. A group G is *locally P* if and only if every finitely generated subgroup of G has property P. (So $\operatorname{loc}(\operatorname{loc}(P)) \equiv \operatorname{loc}(P)$.) P is a *local property* if $\operatorname{loc}(P) \implies P$.

Theorem 2.5. *G is locally LO (resp. BO) if and only if G is LO (resp. BO).*

Proof. ($\Longleftarrow$): LO and BO are inherited by subgroups.

($\implies$): Let G be a finite set contained in $G \setminus \{1\}$. Then $\langle F \rangle < G$ is finitely generated. G loc (LO) implies $\langle F \rangle$ is LO. Therefore there exists ϵ such that (2.8) holds (from Theorem 2.3). This is true for all F, so G is LO by Theorem 2.3. The argument for BO is similar, using Theorem 2.4 instead. □

Corollary 2.6. *An abelian group is BO iff it is torsion free.*

Proof. ($\implies$): This follows from Lemma 1.3.

($\Longleftarrow$) G is LO iff G is loc (LO). For H finitely generated inside of torsion free G, then $H \cong \mathbb{Z}^n$, so it is LO. □

Corollary 2.7. *An arbitrary free group is LO.*

PROOF. Let F be a free group. For H a finitely generated subgroup of F, $H \cong F_n$ for some n. Then H is LO by Corollary 2.7, so F is LO by Theorem 2.5. $\square$

THEOREM 2.8. *Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a collection of groups. Then G_λ is LO for all $\lambda \in \Lambda$ if and only if $*_{\lambda\in\Lambda} G_\lambda$ is LO.*

PROOF. ($\Longleftarrow$): $G_\lambda < *_{\lambda\in\Lambda} G_\lambda$.
($\Longrightarrow$): There exists a homomorphism

$$\begin{aligned} G = *_{\lambda\in\Lambda} G_\lambda &\xrightarrow{\quad\varphi\quad} \textstyle\prod_{\lambda\in\Lambda} G_\lambda \\ g_\lambda &\longmapsto (1, \ldots, 1, g_\lambda, 1, \ldots) \end{aligned} .$$

So we get a SES

$$1 \longrightarrow H \longrightarrow *_\Lambda G_\lambda \xrightarrow{\varphi} \textstyle\prod_\Lambda G_\lambda \longrightarrow 1 \tag{2.11}$$

where $H = \ker\varphi$. By the Kurosh subgroup theorem

$$H = \left(\mathop{*}_{\mu} H_\mu \right) * F$$

where H_μ is a subgroup of a conjugate of G_{λ_μ} in G, and F is a free group. But $H = \ker\varphi$, and $\varphi|_{G_\lambda}$ is injective for all $\lambda \in \Lambda$. Therefore for all $\lambda \in \Lambda$ and $g \in G$ we have $H \cap g^{-1} G_\lambda g = \{1\}$. Therefore $H = F$.

But now G_λ LO for all $\lambda \in \Lambda$ implies $\prod_{\lambda\in\Lambda} G_\lambda$ is LO by Theorem 1.4, and $F = H$ is LL by Corollary 2.7, so G is LO by Theorem 1.13. $\square$

Let P be a property of groups. A group G is residually P, $\operatorname{res}(P)$, if and only if for all $g \in G \setminus \{1\}$ there exists an epimorphism $\varphi : G \to H$ such that H has property P, and $\varphi(g) \neq 1$.

REMARK 2.6. Note that P implies $\operatorname{res}(P)$, and $\operatorname{res}(\operatorname{res}(P))$ implies $\operatorname{res}(P)$.

We say P is a *residual property* if and only if $\operatorname{res}(P)$ implies P.

EXAMPLE 2.4. Finiteness is not a residual property. E.g. $\mathbb{Z}$ is $\operatorname{res}(\text{finite})$.

Lemma 2.9. *If P is closed under taking subgroups and direct products, then P is a residual property.*

Corollary 2.10. *LO and BO are residual properties.*

PROOF OF LEMMA 2.9. Suppose G is $\operatorname{res}(P)$. Then for all $g \in G \setminus \{1\}$ there is an epimorphism $\varphi_g : G \to H_g$ such that H_g has P, and $\varphi_g(g) \neq 1$. The collection of these $\{\varphi_g \mid g \in G \setminus \{1\}\}$ induces a homomorphism

$$\varphi : G \to \prod_{g\in G\setminus\{1\}} H_g \ .$$

Then this is injective, and $\varphi_g(g) \neq 1$. H_g has P for all $g \in G \setminus \{1\}$. Therefore $\prod_{g\in G\setminus\{1\}} H_g$ has P. But

$$G \cong \varphi(G) < \prod_{g\in G\setminus\{1\}} H_g$$

so G has P. $\square$

REMARK 2.7. Residual properties are related to areas of active research. For example the geometrization conjecture is related to residual finiteness of 3-manifolds.

REMARK 2.8. Let G be a group. Let $\mathrm{FQ}\,(G)$ consist of the finite quotients of G. Then the following is an open question. Let F_2 be a free group of rank 2. If G is a residually finite group such that $\mathrm{FQ}\,(G) = \mathrm{FQ}\,(F_2)$ is $G \cong F_2$? Note that $\mathrm{FQ}\,(F_2)$ consists of the finite groups generated by two elements. So this is really quite concrete.

Another open question is if G_1 and G_2 are residually finitely presented, then does $\mathrm{FQ}\,(G_1) = \mathrm{FQ}\,(G_2)$ imply $G_1 \cong G_2$?

EXAMPLE 2.5. $\mathrm{LO}\,(\mathbb{Z}^n)$ and $\mathrm{LO}\,(F_n)$ are both the cantor set.

EXAMPLE 2.6. Let B_n denote the braid group. As it turns out $\mathrm{LO}\,(B_n)$ has isolated points [**2**].

The following is a strengthening of the fact that LO is a local property.

WARNING 2.1. At this point it is convenient to make the convention that $\{1\}$ is *not* LO.

THEOREM 2.11 (Burns-Hale). *G is LO iff every non-trivial finitely generated subgroup $H < G$ has an LO quotient.*

PROOF. ($\Longrightarrow$): G is LO implies H is LO.
($\Longleftarrow$): $F = \{g_1, \ldots, g_n\} \subset G \setminus \{1\}$ for $n \geq 1$. We show by induction on n that the condition on F in Theorem 2.3 holds. Let $n = 1$. Then $\langle g_1 \rangle$ has an LO quotient by assumption. Therefore g_1 has infinite order, so $1 \notin S\,(g_1)$. Now suppose $n > 1$. By assumption, there exists a nontrivial homomorphism $\varphi : \langle g_1, \ldots, g_n \rangle \to L$ where L is LO. For some m there exists

$$\varphi\,(g_i) = \begin{cases} +1 & 1 \leq i \leq m \\ -1 & m < i \leq n \end{cases}$$

By the induction hypothesis there exists $\epsilon_1, \ldots, \epsilon_m \in \{\pm 1\}$ such that $1 \notin S\,(\{g_i^{\epsilon_i} \,|\, 1 \leq i \leq m\})$. Let $<$ be an LO on L. Define $\epsilon_i \in \{\pm 1\}$ $(m < i \leq n)$ so that

$$\varphi\,(g_i^{\epsilon_i}) > 1 \tag{2.12}$$

Then $1 \notin S\,(\{g_i^{\epsilon_i} \,|\, 1 \leq i \leq n\})$. □

A group G is *indicable* if either $G = \{1\}$ or there is an epimorphism $G \to \mathbb{Z}$.

Corollary 2.12. *G is locally indicable implies G is LO.*

REMARK 2.9. Free groups are loc (indicable) so this gives another proof that free groups are LO.

REMARK 2.10. Note that G having an LO quotient does not imply G is LO.

COUNTEREXAMPLE 2. $\mathbb{Z} * \mathbb{Z}/2$ has LO quotient, but is not LO.

We do however have:

THEOREM 2.13. *Let G be a group such that every finitely generated subgroup of infinite index is indicable. Then G is LO if and only if G has an LO quotient.*

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PROOF. ($\Longrightarrow$): This direction is immediate.
($\Longleftarrow$): Apply Theorem 2.11. Let $H < G$, $H \neq \{1\}$, finitely generated.

Case 1: $[G : H] = \infty$. By hypothesis, H is indicable, so therefore (since H is nontrivial) G has quotient $\mathbb{Z}$.

Case 2: $[G : H]$ finite. By hypothesis there exists an epimorphism $\varphi : G \to Q$ where Q is LO. Therefore Q is infinite, so $\varphi(H) \neq \{1\}$, (since $[Q : \varphi(H)]$ is finite) and therefore H has LO quotient $\varphi(H)$.

□

REMARK 2.11. It turns out that G BO implies G is locally indicable.

REMARK 2.12. We will eventually apply Theorem 2.13 to three-manifold groups. But first we look at surfaces.

2. Surface groups

An n-manifold is a second-countable Hausdorff space M such that for all $x \in M$ x has a neighborhood U such that either

$$(U, x) \cong (\mathbb{R}^n, 0) \qquad \text{or} \qquad (U, x) \cong \left(\mathbb{R}^n_+, 0\right) .$$

Define the interior and boundary as:

$$\begin{aligned} \operatorname{int}(M) &= \{x \in M \mid x \text{ has a neighborhood of the first type}\} \\ \partial M &= \{x \in M \mid x \text{ has a neighborhood of the second type}\} . \end{aligned}$$

Note that $(\operatorname{int}(M)) \cap \partial M = \emptyset$. Also note that $\operatorname{int}(M)$ is an n-manifold with empty boundary, and ∂M is an $(n-1)$-manifold with empty boundary. M is *closed* if M is compact and $\partial M = \emptyset$.

A *triangulation* of M is a homeomorphism $M \cong |K|$, where K is a locally finite simplicial complex. Whether or not a manifold has a triangulation is a subtle question which wasn't settled until recently [**6**].

FACT 2. *Every n-manifold has a triangulation for $n \leq 3$.*

This was shown for $n = 2$ in [**9**] and for $n = 3$ in [**8**].

For us, a *surface* is a 2-manifold. There is the well-known classification of closed surfaces. In particular, they all either look like S^2, T^2, a connect sum of copies of T^2, the projective plane $\mathbb{P}^2$, or connect sums of copies of $\mathbb{P}^2$.

There is also a classification of non-compact surfaces.

EXAMPLE 2.7. Consider the plane. Now attach handles as in Fig. 1. This is an infinite genus non-compact surface. Now consider the infinite genus surface in Fig. 2. Are these homeomorphic? See Remark 2.13 for the answer.

Now we consider the following question.

QUESTION 1. Which surface groups $\pi_1(S)$ are LO?

We want to use Theorem 2.11, so we will consider finitely generated subgroups of surface groups. First, recall the following.

Lemma 2.14. *If M is a closed n-manifold, N is a connected n-manifold, and $f : M \to N$ is an injective map, then f is a homeomorphism.*

This uses the Jordan-Brouwer theorem for S^{n-1}s in S^n. For M compact, N Hausdorff, it is enough to show f is onto.

Lemma 2.15. *Let S be a non-compact surface. Then $H_2(S) = 0$.*

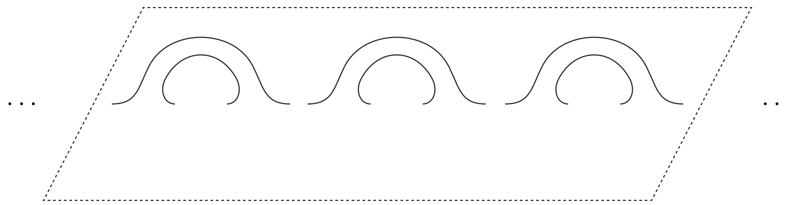

FIGURE 1. The Loch-Ness monster surface obtained by attached infinitely many handles to the plane.

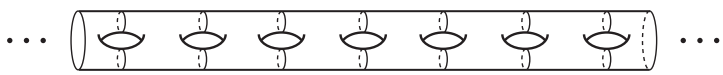

FIGURE 2. The Jacob's ladder surface.

PROOF. Triangulate S. Then we can get compact surfaces $S_1 \subset S_2 \ldots \subset S$ such that

$$S = \bigcup_{i=1}^{n} S_i \ .$$

$\partial S_i \neq \emptyset$ by Lemma 2.14, so $S_i \simeq$ some 1-complex. Therefore $H_2(S_i) = 0$, for all i. And every 2-cycle in S is contained in some S_i. Therefore $H_2(S) = 0$. $\square$

Lemma 2.16. *Let S be a surface, δ a circle component of ∂S such that $\pi_1(\delta) \to \pi_1(S)$ is not injective. Then $S \cong D^2$.*

PROOF. For S compact, this is true by the classification. So let S be non-compact. Let $S^* = S \cup D^2$ glued along δ. Then we have that the following commutes

$$\begin{tikzcd} \pi_1(\delta) \arrow[r] \arrow[d, "\cong"] & \pi_1(S) \arrow[d] \\ H_1(\delta) \arrow[r] & H_1(S) \end{tikzcd} .$$

But now since $\pi_1(\delta) \to \pi_1(S)$ is not injective, $H_1(\delta) \to H_1(S)$ cannot be injective either. So now applying Mayer-Vietoris, we get

$$H_2(S^*) \cong \ker(H_1(\delta) \to H_1(S)) \ , \tag{2.13}$$

so by definition this is nonzero. But S^* is noncompact, so this contradicts Lemma 2.15. $\square$

REMARK 2.13. Have you answered the question from Example 2.7 yet? The answer has to do with the number of *ends*, which is defined as follows. Remove compact subsets and count the remaining components. If we minimize the number of components, then this is the number of ends. This is clearly a topological invariant. The loch-ness monster has 1, and Jacob's ladder has 2.

We can also define the notion of the number of ends of a group. As it turns out, $e(G) = 0$ iff G is finite. Then, for example, we have

$$\begin{aligned} e(\mathbb{Z}) &= 2 \\ e(\mathbb{Z}^n) &= 1 \qquad (n \geq 2) \\ e(F_n) &= \infty\ . \end{aligned}$$

Then it turns out that for all G, $e(G) = 0, 1, 2$, or ∞.

Theorem 2.17 (Compact core theorem for surfaces). *Let S be a connected surface with $\pi_1(S)$ finitely generated. Then there exists a compact connected $S_0 \overset{i}{\hookrightarrow} S$ such that $i_* : \pi_1(S_0) \to \pi_1(S)$ is an isomorphism. We call S_0 a* compact core *of S.*

Proof. Triangulate S. Let $\gamma_1, \ldots, \gamma_n$ be simplicial loops in S such that $\{[\gamma_1], \ldots, [\gamma_n]\}$ are generators of $\pi_1(S)$. Let N be a regular neighborhood of $\bigcup_{i=1}^n \gamma_i$ in S. N is a compact surface with $\partial N \neq \emptyset$ (and we can in fact assume it is connected) and $\pi_1(N) \to \pi_1(S)$ is onto.

Let S_0 be N union with any disk components of S cut along ∂N. S_0 is a compact surface, and $\pi_1(S_0) \to \pi_1(S)$ is onto. If $\partial S_0 = \emptyset$ then we are done since $S_0 = S$.

So suppose $\partial S_0 \neq \emptyset$. Let δ be a component of ∂S_0. Since $\pi_1(S_0) \to \pi_1(S)$ is onto, δ separates S. (If not, there exists a loop $\gamma \subset S$ such that $\gamma \pitchfork \delta$ is a single point. Therefore γ cannot be in S_0 but $\pi_1(S_0) \to \pi_1(S)$ is onto.)

Let S_1 be the component of S cut along δ such that $S_0 \not\subset S_1$. By definition of S_0 S_1 is not a disk. Therefore by Lemma 2.16 $\pi_1(\delta) \to \pi_1(S_1)$ is one-to-one. If S_0 is a disk, then $\pi_1(S) = \{1\}$ and we are done. So assume S_0 is not a disk. Then $\pi_1(\delta) \to \pi_1(S_0)$ is injective. So do this for all the boundary components δ of S_0. Then we see by Van-Kampen that this is just a big free product:

$$\pi_1(S) \cong \operatorname{colim} \left(\begin{tikzcd} \pi_1(S_1) \arrow[drr] & \pi_1(S_2) \arrow[dr] & \pi_1(S_3) \arrow[d] & \ldots & \pi_1(S_k) \arrow[dll] \\ & & \pi_1(S_0) & & \end{tikzcd} \right)$$

but by definition this means $\pi_1(S_0) \to \pi_1(S)$ is injective. $\square$

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