Orderability and 3-manifold groups

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Contents

CHAPTER 1

Orders on groups; basic definitions and properties

The book for the course is [**1**]. Lecture 1; January 21, 2020

Recall that a *strict total order* (STO) on a set X is a binary relation $<$ which satisfies:

(1) $x < y$ and $y < z$ implies $x < z$;
(2) $\forall x, y \in X$ exactly one of: $x < y$, $y < x$, $x = y$, holds.

A *left order* (LO) on a group G is an STO such that $g < h$ implies $fg < fh$ for all $f \in G$. G is *left-orderable* (LO) if there exists an LO on G. We similarly define a *right order* (RO) and *right orderability* (RO). A *bi-order* (BO) on G is an LO on G that is also an RO.

Remark 1.1. (1) If G is abelian, $<$ is a LO iff $<$ is an RO iff $<$ is a BO.
(2) If $<$ is an LO on G, then $\prec$ defined by:

$$g \prec h \iff h^{-1} < g^{-1} \tag{1.1}$$

is an RO on G. Therefore G is LO iff G is RO. We will stick to LO's.
(3) For $H < G$, an LO (resp. BO) on G induces an LO (resp. BO) on H.

Example 1.1. $(\mathbb{R}, +)$ with the usual $<$ is BO. The subgroups $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$ are also BO.

Lemma 1.1. *Let $<$ be an LO on G. Then*

(1) $g > 1$, $h > 1$ implies $gh > 1$;
(2) $g > 1$ implies $g^{-1} < 1$;
(3) $<$ is a BO iff $\left(g < h \implies f^{-1}gf < f^{-1}hf \forall f \in G\right)$ (i.e. $<$ is conjugation invariant).

Proof. (1) $h > 1$ implies $gh > g \cdot 1g > 1$.
(2) $g > 1$ implies $g^{-1}g > g^{-1}$ implies $1 > g^{-1}$.
(3) ($\implies$) is immediate. ($\impliedby$): We need to show $<$ is a RO. $g < h$ implies $fg < fh$ implies $f^{-1}(fg)f < f^{-1}(fh)f$ which implies $gf < hf$ as desired.

$\square$

Lemma 1.2. *If $<$ is a BO on G, then*

(1) $g < h$ implies $g^{-1} > h^{-1}$;
(2) $g_1 < h$, $g_2 < h_2$ implies $g_1 g_2 < h_1 h_2$.

Proof. (1) If $g < h$, then $g^{-1}g < g^{-1}h$, which implies $1 < g^{-1}h$, which implies $1 \cdot h^{-1} < g^{-1}$, which implies $h^{-1} < g^{-1}$.
(2) $g_2 < h_2$ implies $g_1 g_2 < g_1 h_2 < h_1 h_2$.

$\square$

Warning 1.1. These don't necessarily true for LO's.

Lemma 1.3. *If G is LO then it is torsion free.*

PROOF. Consider $g \in G \setminus \{1\}$. If $g > 1$, then $g^2 > g > 1$, and similarly for all $n \geq 1$, $g^n > 1$. Similarly $g < 1$ implies $g^n < 1$ for all $n \geq 1$. □

So LO is not preserved under taking quotients (e.g. $\mathbb{Z} \to \mathbb{Z}/n$).

Consider an indexed family of groups $\{G_\lambda \,|\, \lambda \in \Lambda\}$. Recall that the direct product

$$\prod_{\lambda \in \Lambda} G_\lambda = \left\{(g_\lambda)_{\lambda \in \Lambda}\right\} \tag{1.2}$$

with multiplication defined co-ordinatewise.

Recall a *well-order* (WO) on a set X is a STO $\prec$ on X such that if $A \subset X$ and $A \neq \emptyset$ then there exists $a_0 \in A$ such that $a_0 \prec a$ for all $a \in A \setminus \{a_0\}$. Recall that the axiom of choice is equivalent to every set having a WO.

THEOREM 1.4. *G_λ has a LO (resp. BO) for all $\lambda \in \Lambda$ iff $\prod_{\lambda \in \Lambda} G_\lambda$ has a LO (resp. BO).*

PROOF. ($\Longleftarrow$): $G\lambda < \prod_\lambda G_\lambda$ so we are finished.

($\Longrightarrow$): Choose a WO $\prec$ on Λ, and order $\prod_\lambda G_\lambda$ lexicographically. Let $g = (g_\lambda)$, $h = (h_\lambda)$, $g \neq h$. Then λ_0 be the $\prec$-least element of Λ such that $g_{\lambda_0} \neq h_{\lambda_0}$. Then define $g < h$ iff $g_{\lambda_0} < h_{\lambda_0}$ (in G_{λ_0}). Then $<$ is an LO (resp. BO) on $\prod_\lambda G_\lambda$. Left (resp. left and right) invariance is clear. Now we show transitivity. Suppose $f < g$, $g < h$. Let λ_0 be the $\prec$-least element of Λ such that $f_{\lambda_0} \neq g_{\lambda_0}$. Let μ_0 be the $\prec$-least element of Λ such that $g_{\mu_0} \neq h_{\mu_0}$.

(1) ($\lambda_0 \preccurlyeq \mu_0$): Then $f_\lambda = g_\lambda = h_\lambda$ for all $\lambda \prec \lambda_0$. Then g_{λ_0} is $<$ (resp. $=$) h_{λ_0} if $\lambda_0 = \mu_0$ (resp. $\lambda_0 \prec \mu_0$). So $f_{\lambda_0} < g_{\lambda_0} \leq h_{\lambda_0}$, and therefore $f_{\lambda_0} < h_{\lambda_0}$.
(2) ($\mu_0 < \lambda_0$): This follows similarly.

□

Let $\sum_{\lambda in \Lambda} G_\lambda$ be the *direct sum of* $\{G_\lambda\}$. Recall this is the subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of elements such that all but finitely many co-ordinates are 1.

Corollary 1.5. *G_λ is LO (resp. BO) for all $\lambda \in \Lambda$ iff $\sum_{\lambda \in \Lambda} G_\lambda$ is LO (resp. BO).*

Corollary 1.6. *Free abelian groups are BO.*

PROOF. Free abelian groups on Λ are $\sum_{\lambda \in \Lambda} \mathbb{Z}$. □

Let $<$ be an LO on G. The *positive cone* $P = P_<$ of $<$ is $\{g \in G \,|\, g > 1\}$.

Lemma 1.7. *(1) P is a subset of G, i.e. $g, h \in P$, implies $gh \in P$ (i.e. $PP \subset P$).*
(2) $G = P \amalg P^{-1} \amalg \{1\}$.
(3) $<$ is a BO on G iff $f^{-1}Pf \subset P$ for all $f \in G$.

PROOF. (1) This follows from Lemma 1.1 (1).
(2) This follows from Lemma 1.1 (2).
(3) This follows from Lemma 1.1 (3).

□

We say $P \subset G$ is a *positive cone* if P satsfies the conditions in Lemma 1.7.

Lemma 1.8. *Let $P \subset G$ be a positive cone. Then $g < h$ implies $g^{-1}h \in P$ defines a LO $<$ on G (With $P_< = P$).*

PROOF. $<$ is a STO, so:

(i) $f < g$, $g < h$ implies $f^{-1}g \in P$, $g^{-1}h \in P$, which implies (by the first property) that $\left(f^{-1}g\right)\left(g^{-1}h\right) \in P$, which implies $f < h$.
(ii) By the second property, for all $g, h \in G$ exactly one of the following holds: $g^{-1}h \in P$, $g^{-1}h \in P^{-1}$, and $g^{-1}h = 1$. Equivalently, $g < h$, $h < g$ (since $h^{-1}g \in P$), and $g = h$. Now we show left invariance. $g < h$ implies $g^{-1}h \in P$, but $g^{-1}h = \left(g^{-1}f^{-1}\right)(fh)$ which implies $fg < fh$.

□

Lemmata 1.7 and 1.8 show that:

$$\{\text{LO's on } G\} \leftrightarrow \{\text{positive cones in } G\} \tag{1.3}$$

$$\{\text{BO's on } G\} \leftrightarrow \{\text{conjugacy-invariance positive cones in } G\}\ . \tag{1.4}$$

Consider the free group of rank n, F_n.

THEOREM 1.9. *F_2 is LO.*

SUNIC. Write $F_2 = F(a, b)$. $g \in F_2$ implies we can write it as a reduced word

$$(a^{m_1})\, b^{n_1} \ldots a^{m_k}\, (b^{n_k}) \tag{1.5}$$

for $k \geq 0$, $m_i, n_i \in \mathbb{Z} \setminus \{0\}$. Recall 1 is the empty word, $k = 0$. Let $e(g)$ be the number of syllables in g with positive exponent, minus the number of syllables in g with negative exponent. Then define $j(g)$ so be the number of $a^m b^n$'s in f, minus the number of $b^n a^m$s in G. So $j(g) = 0$, or ± 1. For example:

$$j(a^* \ldots a^*) = 0 \tag{1.6}$$

$$j(b^* \ldots b^*) = 0 \tag{1.7}$$

$$j(a^* \ldots b^*) = 1 \tag{1.8}$$

$$j(b^* \ldots a^*) = -1\ . \tag{1.9}$$

Finally define

$$\tau(g) = e(g) + j(g)\ . \tag{1.10}$$

Note that

$$e\left(g^{-1}\right) = -e(g) \qquad j\left(g^{-1}\right) = -j(g)\ . \tag{1.11}$$

Lemma 1.10. *If $g \neq 1$, then $\tau(g) \equiv 1 \pmod 2$.*

PROOF. $e(f)$ is congruent to the number of syllables mod 2, and $j(g)$ is congruent to the number of syllables $+1$ mod 2. □

Lemma 1.11. $|\tau(gh) - \tau(g) - \tau(h)| \leq 1$.

PROOF. If gh or g or $h = 1$ we are done. So suppose $gh, g, h \neq 1$. Clearly $e(gh) = e(g) + e(h) + \left\{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}\right\}$. Similarly:

$$j(gh) = j(g) + j(h) + \left\{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}\right\}\ . \tag{1.12}$$

Therefore:

$$|\tau(gh) - \tau(g) - \tau(H)| \leq 2 \tag{1.13}$$

so by Lemma 1.10 we have

$$|\tau(gh) - \tau(g) - \tau(h)| \leq 1 \ . \tag{1.14}$$

□

REMARK 1.2. Lemma 1.11 says that $\tau : F_2 \to \mathbb{Z}\,(< \mathbb{R})$ is what is called a *quasi-morphism.*

Define $P \subset F_2$ by

$$P = \{g \in F_2 \,|\, \tau(g) > 0\} \ . \tag{1.15}$$

Then $F_2 = P \amalg P^{-1} \amalg \{1\}$ by Lemma 1.10 and that $\tau\left(g^{-1}\right) = -\tau(g)$. Then $PP \subset P$ by Lemma 1.11 since

$$\tau(gh) \geq \tau(g) + \tau(h) - 1 \geq 1 \ . \tag{1.16}$$

Therefore P is a positive cone for a LO on F_2. ■

Corollary 1.12. *Any countable free group is LO.*

PROOF. A countable free group is a subgroup of F_2. □

REMARK 1.3.
(1) $\tau\left(a^{-1}b\right) = 1$, so $a^{-1}b > 1$, so $b > a$. On the other hand, $\tau\left(ab^{-1}\right) = 1$, so $ab^{-1} > 1$, so $b^{-1} > a^{-1}$. So τ does not define a BO on F_2.
(2) We will see later that all free groups are LO.
(3) Even later we will see that all free groups are BO.

THEOREM 1.13. *Let $1 \to H \to G \to Q \to 1$ be a short-exact sequence of groups. Then*
(1) H, Q LO implies G is LO;
(2) if Q is BO and H has a BO that is invariant under conjugation in G then G is BO.

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PROOF. Write $\varphi : G \to Q$ and regard H as $\ker\varphi < G$. Let P_H (resp. P_Q) be positive cones for LO's on H (resp. Q). Define $P = \varphi^{-1}(P_Q) \amalg P_H$.

CLAIM 1.1. P is a positive cone for an LO on G.

PROOF. We need to check (1) and (2) from Lemma 1.7. Let $g, h \in P$. Then we want to show $gh \in P$. We have three cases.
(a) $g, h \in \varphi^{-1}(P_Q)$: In this case $\varphi(g), \varphi(h) \in P_Q$, so $\varphi(gh) = \varphi(g)\varphi(h) \in P_Q$. Therefore $gh \in \varphi^{-1}(P_Q)$.
(b) $g, h \in P_H$: In this case $gh \in P_H$.
(c) $g \in \varphi^{-1}(P_Q)$, $h \in P_H$: Then $\varphi(gh) = \varphi(g) \in P_Q$, so $gh \in \varphi^{-1}(P_Q)$. Similarly $hg \in \varphi^{-1}(P_Q)$.

Now we need to check $P \amalg P^{-1} \amalg \{1\}$. But this follows from the fact that:

$$G = (H \setminus \{1\}) \amalg \varphi^{-1}(Q \setminus \{1\}) \amalg \{1\} = \varphi^{-1}(P_Q) \amalg \varphi^{-1}\left(P_Q^{-1}\right) \tag{1.17}$$

since $H \setminus \{1\} = P_H \amalg P_H^{-1}$. □

We leave (2) as an exercise. [Hint: Recall P is a positive cone for BO on G iff it is a conjugacy invariant cone for an LO.] ■

EXAMPLE 1.2. Let X^2 be the Klein bottle. This has fundamental group

$$K=\pi_1\left(X^2\right)=\left\langle a,b\,\middle|\,b^{-1}ab=a^{-1}\right\rangle \ . \tag{1.18}$$

This fits in the SES:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & K & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \| & & & & & & \\ & & \langle a\rangle & & b & \longmapsto & gm & & \end{array} \tag{1.19}$$

which means K is LO by Theorem 1.13.

Note that K is *not* BO. We have that $a>1$ iff $b^{-1}ab>1$, but this is a^{-1}, so $a^{-1}>1$ which is a contradiction.

Notice that $\mathbb{Z}$ has exactly two LO's. The usual one, and the opposite. Therefore, if we choose an LO on $\langle a\rangle$ and $K/\langle a\rangle$, this gives 4 LO's on K determined by:

(i) $a>1$, $b>1$;
(ii) $a>1$, $b<1$;
(iii) $a<1$, $b>1$;
(iv) $a<1$, $b<1$.

THEOREM 1.14. *These are the only LO's on K.*

PROOF. It suffices to show that each of these conditions determines a unique positive cone.

(i) $a>1$, $b>1$:

CLAIM 1.2. $a^k<b$ for all $k\in\mathbb{Z}$.

PROOF. $b<a^k$ implies $a^{-k}b<1$. But $a^{-k}b=ba^k$ and $b>1$, so $b<a^k$ implies $a^k>1$, which implies $ba^k>1$ which is a contradiction. □

Note that every element in K has a unique representative of the form a^mb^n for $m,n\in\mathbb{Z}$.

CLAIM 1.3. $a^mb^n>1$ iff either $n>0$ or $n=0$ and $m>0$.

PROOF. If $n=0$, then this is clear. If $n>0$, then $a^mb>1$ for any m by claim 1 (for $k=-m$). But we also know $b>1$ which implies $b^n>1$, so we get $a^mb^n>1$ for $n>0$. On the other hand, if $m<0$ then $a^mb^n=b^na^{\pm m}=\left(a^{\mp m}b^{-n}\right)^{-1}$. Then we know $a^{\mp m}b^{-n}>1$ by the case above, so its inverse is <1. □

If $<$ is an LO on G, and $\alpha:G\to G$ is an automorphism, then this induces an LO $<_\alpha$ on G given by: $g<_\alpha h$ iff $\alpha(g)<\alpha(h)$. Now notice that there are automorphisms α_1,α_2 of K such that

$$\alpha_1(a)=a\ , \qquad \alpha_1(b)=b^{-1} \tag{1.20}$$

$$\alpha_1(a)=a^{-1}\ , \qquad \alpha_1(b)=b\ . \tag{1.21}$$

In particular, α_1 is given by

$$\left\langle a,b\,\middle|\,b^{-1}ab=a^{-1}\right\rangle\cong\left\langle a,b\,\middle|\,bab^{-1}=a^{-1}\right\rangle \tag{1.22}$$

and similarly for α_2.

Write $<_{(i)}$ for the unique LO on K determined by (i). Then $<_{(ii)}$ is induced by $<_{(i)}$ and α_1, $<_{(iii)}$ is induced by $<_{(i)}$ and α_2, and $<_{(iv)}$ is induced by $<_{(i)}$ and $\alpha_1\alpha_2$.

■

FACT 1. *If G has only finitely many LO's, then the number of LO's is of the form 2^n.*

EXERCISE 0.1. Show that for all $n \geq 0$ there exists a group G with exactly 2^n LO's.

Corollary 1.15. *For any LO on K, if $h \in \langle a \rangle$, $g \in K \setminus \langle a \rangle$, and $g > 1$, then $g > h$.*

PROOF. It is sufficient to check this for the first LO, since the other three are determined by the above automorphisms. Let $a > 1$, $b > 1$. By claim 2 from above, we know $g = a^m b^n$ for $n > 0$. We now there is some k such that $h = a^k$, and therefore

$$h^{-1} g = a^{m-k} b^n > 1 \tag{1.23}$$

by claim 2, so $g > h$. ☐

Suppose M is a closed, orientable, connected three-manifold. Then we might ask if $\pi_1(M)$ is LO? BO?

Immediately we notice that not all such groups are. If M is a lens space, then $\pi_1(M) \cong \mathbb{Z}/n$ for $n > 1$, so this is not LO. More generally, for $\pi_1(M)$ nontrivial and finite is not LO. Recall that if $M = M_1 \# M_2$, then this implies $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. So, for example, if $M_1 \#$ lens space, then $\pi_1(M)$ has torsion, so not LO.

But at least some of them are. Consider $M \cong T^3 = S^1 \times S^1 \times S^1$. Then $\pi_1(M) = \mathbb{Z}^3$ is of course LO. Similarly $M = \#_n \left(S^1 \times S^2\right) \cong F_n$, so $\pi_1(M)$ is LO.

We will show that there exist (three-manifold) groups that are torsion-free, but not LO.

Let $p : T^2 \to X^2$ be a two-fold covering of the Klein bottle. Recall that

$$K > p_* \left(\pi_1 \left(T^2\right)\right) = \left\langle a, b^2 \right\rangle \cong \mathbb{Z} \times \mathbb{Z} \ . \tag{1.24}$$

Let N be the mapping cylinder of p, namely:

$$N = \left(T^2 \times I\right) \amalg X^2 / \left((x, 0) \sim p(x) \, \forall x \in T^2\right) \ . \tag{1.25}$$

The orientation reversing curve representing b doesn't lift. So N is orientable. Note that $\partial N \cong T^2$. There is a strong deformation retraction $N \to X^2$, so $\pi_1(N) \cong K$. Let N_1, N_2 be two copies of N. Write

$$\pi_1(N_i) = \left\langle a_i, b_i \,\middle|\, b_i^{-1} a_i b_i = a_i^{-1} \right\rangle \ . \tag{1.26}$$

Notice that $\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z} = \left\langle a_i, b_i^2 \right\rangle < \pi_1(N_i)$. Let $\varphi : \partial N_1 \to \partial N_2$ be a homeomorphism. Let $M_\varphi = N_1 \cup_\varphi N_2$. This is a closed, orientable three-manifold. Therefore

$$\pi_1(M_\varphi) = \pi_1(N_1) *_{\mathbb{Z} \times \mathbb{Z}} \pi_1(N_2) \cong K_1 *_{\mathbb{Z} \times \mathbb{Z}} K_2 \ . \tag{1.27}$$

Since K is torsion-free, $\pi_1(M_\varphi)$ is torsion-free. But in fact we have the following theorem.

THEOREM 1.16. *If $H_1(M_\varphi)$ is finite, then $\pi_1(M_\varphi)$ is not LO.*

REMARK 1.4. We will see later that for M a prime three-manifold with $H_1(M)$ infinite has $\pi_1(M)$ LO.

PROOF. φ is determined up to isotopy, so the resulting manifold M_φ depends only on $\varphi_* : H_1(\partial N_1) \to H_1(\partial N_2)$. We know

$$\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_1, 2b_1 \rangle \qquad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_2, 2b_2 \rangle \tag{1.28}$$

so φ_* is given by some 2×2 matrix with $\mathbb{Z}$ coefficients

$$\begin{bmatrix} p & r \\ q & s \end{bmatrix} \tag{1.29}$$

with determinant $ps - qr = \pm 1$. Specifically we have:

$$\varphi_*(a_1) = pa_2 + 2qb_2 \varphi_*(2b_1) = ra_2 + 2sb_2 \ . \tag{1.30}$$

Now we have $H_1(N_i) = \mathbb{Z} \oplus \mathbb{Z}_2$ with basis b_i and a_i respectively. Then $H_q(M_\varphi)$ is presented by

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 * p & 2q & \\ 0 & -2 & r & 2s \end{bmatrix} . \tag{1.31}$$

where we order the basis as $\{a_1, b_1, a_2, b_2\}$. Interchanging columns 2 and 3 we get

$$\det A = 4 \left| \det \begin{bmatrix} 0 & 2q \\ -2 & 2s \end{bmatrix} \right| = 16 |q| \ . \tag{1.32}$$

Therefore $H_1(M_\varphi)$ is finite iff $q \neq 0$ iff $\varphi_*(a_1) \neq \pm a_2$.

Suppose $\pi_1(M_\varphi)$ is LO. Then we would get an induced LO on the common boundary $\partial N_1 = \partial N_2$. But there are only 4 LO's on $\pi_1(N_i)$ (for $i \in \{1, 2\}$). By Corollary 1.15, for any LO on $\pi_1(N)$, $\langle a \rangle$ is the unique $\mathbb{Z}$-summand of $\pi_1(\partial N) = \langle a, b^2 \rangle$ such that if $h \in \langle a \rangle$ and $g \in \pi_1(\partial N) \setminus \{1\}$, $g > 1$, then $g > h$. Therefore $\varphi_*(a_1) = \pm a_2$ which is a contradiction. $\square$

Bibliography

[1] Adam Clay and Dale Rolfsen, *Ordered groups and topology*, Graduate Studies in Mathematics, vol. 176, American Mathematical Society, Providence, RI, 2016.