

# **Orderability and 3-manifold groups**

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## CHAPTER 1

# Orders on groups; basic definitions and properties

The book for the course is [CR].

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Recall that a *strict total order* (STO) on a set  $X$  is a binary relation  $<$  which satisfies:

- (1)  $x < y$  and  $y < z$  implies  $x < z$ ;
- (2)  $\forall x, y \in X$  exactly one of:  $x < y$ ,  $y < x$ ,  $x = y$ , holds.

A *left order* (LO) on a group  $G$  is an STO such that  $g < h$  implies  $fg < fh$  for all  $f \in G$ .  $G$  is *left-orderable* (LO) if there exists an LO on  $G$ . We similarly define a *right order* (RO) and *right orderability* (RO). A *bi-order* (BO) on  $G$  is an LO on  $G$  that is also an RO.

REMARK 1.1. (1) If  $G$  is abelian,  $<$  is a LO iff  $<$  is an RO iff  $<$  is a BO.  
(2) If  $<$  is an LO on  $G$ , then  $\prec$  defined by:

$$(1.1) \quad g \prec h \iff h^{-1} < g^{-1}$$

is an RO on  $G$ . Therefore  $G$  is LO iff  $G$  is RO. We will stick to LO's.

- (3) For  $H < G$ , an LO (resp. BO) on  $G$  induces an LO (resp. BO) on  $H$ .

EXAMPLE 1.1.  $(\mathbb{R}, +)$  with the usual  $<$  is BO. The subgroups  $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$  are also BO.

**Lemma 1.1.** *Let  $<$  be an LO on  $G$ . Then*

- (1)  $g > 1, h > 1$  implies  $gh > 1$ ;
- (2)  $g > 1$  implies  $g^{-1} < 1$ ;
- (3)  $<$  is a BO iff  $(g < h \implies f^{-1}gf < f^{-1}hf \forall f \in G)$  (i.e.  $<$  is conjugation invariant).

PROOF. (1)  $h > 1$  implies  $gh > g \cdot 1g > 1$ .

(2)  $g > 1$  implies  $g^{-1}g > g^{-1}$  implies  $1 > g^{-1}$ .

(3)  $(\implies)$  is immediate.  $(\impliedby)$ : We need to show  $<$  is a RO.  $g < h$  implies  $fg < fh$  implies  $f^{-1}(fg)f < f^{-1}(fh)f$  which implies  $gf < hf$  as desired. □

**Lemma 1.2.** *If  $<$  is a BO on  $G$ , then*

- (1)  $g < h$  implies  $g^{-1} > h^{-1}$ ;
- (2)  $g_1 < h, g_2 < h_2$  implies  $g_1g_2 < h_1h_2$ .

PROOF. (1) If  $g < h$ , then  $g^{-1}g < g^{-1}h$ , which implies  $1 < g^{-1}h$ , which implies  $1 \cdot h^{-1} < g^{-1}$ , which implies  $h^{-1} < g^{-1}$ .

(2)  $g_2 < h_2$  implies  $g_1g_2 < g_1h_2 < h_1h_2$ . □

WARNING 1.1. These don't necessarily true for LO's.

**Lemma 1.3.** *If  $G$  is LO then it is torsion free.*

PROOF. Consider  $g \in G \setminus \{1\}$ . If  $g > 1$ , then  $g^2 > g > 1$ , and similarly for all  $n \geq 1$ ,  $g^n > 1$ . Similarly  $g < 1$  implies  $g^n < 1$  for all  $n \geq 1$ .  $\square$

So LO is not preserved under taking quotients (e.g.  $\mathbb{Z} \rightarrow \mathbb{Z}/n$ ).

Consider an indexed family of groups  $\{G_\lambda \mid \lambda \in \Lambda\}$ . Recall that the direct product

$$(1.2) \quad \prod_{\lambda \in \Lambda} G_\lambda = \{(g_\lambda)_{\lambda \in \Lambda}\}$$

with multiplication defined co-ordinatewise.

Recall a *well-order* (WO) on a set  $X$  is a STO  $\prec$  on  $X$  such that if  $A \subset X$  and  $A \neq \emptyset$  then there exists  $a_0 \in A$  such that  $a_0 \prec a$  for all  $a \in A \setminus \{a_0\}$ . Recall that the axiom of choice is equivalent to every set having a WO.

**THEOREM 1.4.**  $G_\lambda$  has a LO (resp. BO) for all  $\lambda \in \Lambda$  iff  $\prod_{\lambda \in \Lambda} G_\lambda$  has a LO (resp. BO).

PROOF. ( $\Leftarrow$ ):  $G_\lambda < \prod_{\lambda} G_\lambda$  so we are finished.

( $\Rightarrow$ ): Choose a WO  $\prec$  on  $\Lambda$ , and order  $\prod_{\lambda} G_\lambda$  lexicographically. Let  $g = (g_\lambda)$ ,  $h = (h_\lambda)$ ,  $g \neq h$ . Then  $\lambda_0$  be the  $\prec$ -least element of  $\Lambda$  such that  $g_{\lambda_0} \neq h_{\lambda_0}$ . Then define  $g < h$  iff  $g_{\lambda_0} < h_{\lambda_0}$  (in  $G_{\lambda_0}$ ). Then  $<$  is an LO (resp. BO) on  $\prod_{\lambda} G_\lambda$ . Left (resp. left and right) invariance is clear. Now we show transitivity. Suppose  $f < g$ ,  $g < h$ . Let  $\lambda_0$  be the  $\prec$ -least element of  $\Lambda$  such that  $f_{\lambda_0} \neq g_{\lambda_0}$ . Let  $\mu_0$  be the  $\prec$ -least element of  $\Lambda$  such that  $g_{\mu_0} \neq h_{\mu_0}$ .

- (1) ( $\lambda_0 \preccurlyeq \mu_0$ ): Then  $f_\lambda = g_\lambda = h_\lambda$  for all  $\lambda \prec \lambda_0$ . Then  $g_{\lambda_0}$  is  $<$  (resp.  $=$ )  $h_{\lambda_0}$  if  $\lambda_0 = \mu_0$  (resp.  $\lambda_0 \prec \mu_0$ ). So  $f_{\lambda_0} < g_{\lambda_0} \leq h_{\lambda_0}$ , and therefore  $f_{\lambda_0} < h_{\lambda_0}$ .
- (2) ( $\mu_0 < \lambda_0$ ): This follows similarly.

$\square$

Let  $\sum_{\lambda \in \Lambda} G_\lambda$  be the *direct sum* of  $\{G_\lambda\}$ . Recall this is the subgroup of  $\prod_{\lambda \in \Lambda} G_\lambda$  consisting of elements such that all but finitely many co-ordinates are 1.

**Corollary 1.5.**  $G_\lambda$  is LO (resp. BO) for all  $\lambda \in \Lambda$  iff  $\sum_{\lambda \in \Lambda} G_\lambda$  is LO (resp. BO).

**Corollary 1.6.** Free abelian groups are BO.

PROOF. Free abelian groups on  $\Lambda$  are  $\sum_{\lambda \in \Lambda} \mathbb{Z}$ .  $\square$

Let  $<$  be an LO on  $G$ . The *positive cone*  $P = P_{<}$  of  $<$  is  $\{g \in G \mid g > 1\}$ .

**Lemma 1.7.** Let  $P$  be as above.

- (1)  $g, h \in P$ , implies  $gh \in P$  (i.e.  $PP \subset P$ ).
- (2)  $G = P \amalg P^{-1} \amalg \{1\}$ .
- (3)  $<$  is a BO on  $G$  iff  $f^{-1}Pf \subset P$  for all  $f \in G$ .

PROOF. (1) This follows from Lemma 1.1 (1).

(2) This follows from Lemma 1.1 (2).

(3) This follows from Lemma 1.1 (3).  $\square$

We say  $P \subset G$  is a *positive cone* if  $P$  satisfies the conditions in Lemma 1.7.

**Lemma 1.8.** Let  $P \subset G$  be a positive cone. Then  $g < h$  implies  $g^{-1}h \in P$  defines a LO  $<$  on  $G$  (With  $P_{<} = P$ ).

PROOF.  $<$  is a STO, so:

- (i)  $f < g, g < h$  implies  $f^{-1}g \in P, g^{-1}h \in P$ , which implies (by the first property) that  $(f^{-1}g)(g^{-1}h) \in P$ , which implies  $f < h$ .
- (ii) By the second property, for all  $g, h \in G$  exactly one of the following holds:  $g^{-1}h \in P, g^{-1}h \in P^{-1}$ , and  $g^{-1}h = 1$ . Equivalently,  $g < h, h < g$  (since  $h^{-1}g \in P$ ), and  $g = h$ . Now we show left invariance.  $g < h$  implies  $g^{-1}h \in P$ , but  $g^{-1}h = (g^{-1}f^{-1})(fh)$  which implies  $fg < fh$ .

□

Lemmata 1.7 and 1.8 show that:

$$(1.3) \quad \{\text{LO's on } G\} \quad \leftrightarrow \quad \{\text{positive cones in } G\}$$

$$(1.4) \quad \{\text{BO's on } G\} \quad \leftrightarrow \quad \{\text{conjugacy-invariance positive cones in } G\} .$$

Consider the free group of rank  $n$ ,  $F_n$ .

THEOREM 1.9.  $F_2$  is LO.

PROOF BY SUSIC. Write  $F_2 = F(a, b)$ .  $g \in F_2$  implies we can write it as a reduced word

$$(1.5) \quad (a^{m_1})b^{n_1} \dots a^{m_k}(b^{n_k})$$

for  $k \geq 0, m_i, n_i \in \mathbb{Z} \setminus \{0\}$ . Recall 1 is the empty word,  $k = 0$ . Let  $e(g)$  be the number of syllables in  $g$  with positive exponent, minus the number of syllables in  $g$  with negative exponent. Then define  $j(g)$  so be the number of  $a^m b^n$ 's in  $f$ , minus the number of  $b^n a^m$ 's in  $G$ . So  $j(g) = 0$ , or  $\pm 1$ . For example:

$$(1.6) \quad j(a^* \dots a^*) = 0$$

$$(1.7) \quad j(b^* \dots b^*) = 0$$

$$(1.8) \quad j(a^* \dots b^*) = 1$$

$$(1.9) \quad j(b^* \dots a^*) = -1 .$$

Finally define

$$(1.10) \quad \tau(g) = e(g) + j(g) .$$

Note that

$$(1.11) \quad e(g^{-1}) = -e(g) \quad j(g^{-1}) = -j(g) .$$

**Lemma 1.10.** If  $g \neq 1$ , then  $\tau(g) \equiv 1 \pmod{2}$ .

PROOF.  $e(f)$  is congruent to the number of syllables mod 2, and  $j(g)$  is congruent to the number of syllables  $+1 \pmod{2}$ . □

**Lemma 1.11.**  $|\tau(gh) - \tau(g) - \tau(h)| \leq 1$ .

PROOF. If  $gh$  or  $g$  or  $h = 1$  we are done. So suppose  $gh, g, h \neq 1$ . Clearly  $e(gh) = e(g) + e(h) + \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$ . Similarly:

$$(1.12) \quad j(gh) = j(g) + j(h) + \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} .$$

Therefore:

$$(1.13) \quad |\tau(gh) - \tau(g) - \tau(h)| \leq 2$$

so by Lemma 1.10 we have

$$(1.14) \quad |\tau(gh) - \tau(g) - \tau(h)| \leq 1.$$

□

REMARK 1.2. Lemma 1.11 says that  $\tau : F_2 \rightarrow \mathbb{Z}(< \mathbb{R})$  is what is called a *quasi-morphism*.

Define  $P \subset F_2$  by

$$(1.15) \quad P = \{g \in F_2 \mid \tau(g) > 0\}.$$

Then  $F_2 = P \amalg P^{-1} \amalg \{1\}$  by Lemma 1.10 and that  $\tau(g^{-1}) = -\tau(g)$ . Then  $PP \subset P$  by Lemma 1.11 since

$$(1.16) \quad \tau(gh) \geq \tau(g) + \tau(h) - 1 \geq 1.$$

Therefore  $P$  is a positive cone for a LO on  $F_2$ . ■

**Corollary 1.12.** *Any countable free group is LO.*

PROOF. A countable free group is a subgroup of  $F_2$ . □

REMARK 1.3. (1)  $\tau(a^{-1}b) = 1$ , so  $a^{-1}b > 1$ , so  $b > a$ . On the other hand,  $\tau(ab^{-1}) = 1$ , so  $ab^{-1} > 1$ , so  $b^{-1} > a^{-1}$ . So  $\tau$  does not define a BO on  $F_2$ .

(2) We will see later that all free groups are LO.

(3) Even later we will see that all free groups are BO.

THEOREM 1.13. *Let  $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$  be a short-exact sequence of groups. Then*

(1)  *$H, Q$  LO implies  $G$  is LO;*

(2) *if  $Q$  is BO and  $H$  has a BO that is invariant under conjugation in  $G$  then  $G$  is BO.*

PROOF. Write  $\varphi : G \rightarrow Q$  and regard  $H$  as  $\ker \varphi < G$ . Let  $P_H$  (resp.  $P_Q$ ) be positive cones for LO's on  $H$  (resp.  $Q$ ). Define  $P = \varphi^{-1}(P_Q) \amalg P_H$ .

CLAIM 1.1.  $P$  is a positive cone for an LO on  $G$ .

PROOF. We need to check (1) and (2) from Lemma 1.7. Let  $g, h \in P$ . Then we want to show  $gh \in P$ . We have three cases.

(a)  $g, h \in \varphi^{-1}(P_Q)$ : In this case  $\varphi(g), \varphi(h) \in P_Q$ , so  $\varphi(gh) = \varphi(g)\varphi(h) \in P_Q$ . Therefore  $gh \in \varphi^{-1}(P_Q)$ .

(b)  $g, h \in P_H$ : In this case  $gh \in P_H$ .

(c)  $g \in \varphi^{-1}(P_Q), h \in P_H$ : Then  $\varphi(gh) = \varphi(g) \in P_Q$ , so  $gh \in \varphi^{-1}(P_Q)$ . Similarly  $hg \in \varphi^{-1}(P_Q)$ .

Now we need to check  $P \amalg P^{-1} \amalg \{1\}$ . But this follows from the fact that:

$$(1.17) \quad G = (H \setminus \{1\}) \amalg \varphi^{-1}(Q \setminus \{1\}) \amalg \{1\} = \varphi^{-1}(P_Q) \amalg \varphi^{-1}(P_Q^{-1})$$

since  $H \setminus \{1\} = P_H \amalg P_H^{-1}$ . □

We leave (2) as an exercise. [Hint: Recall  $P$  is a positive cone for BO on  $G$  iff it is a conjugacy invariant cone for an LO.] ■



### 1. Orderability of manifold groups

EXAMPLE 1.2. Let  $X^2$  be the Klein bottle. This has fundamental group

$$(1.18) \quad K = \pi_1(X^2) = \langle a, b \mid b^{-1}ab = a^{-1} \rangle.$$

This fits in the SES:

$$(1.19) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & K & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\ & & \parallel & & & & \\ & & \langle a \rangle & & b \longmapsto gm & & \end{array}$$

which means  $K$  is LO by Theorem 1.13.

Note that  $K$  is *not* BO. We have that  $a > 1$  iff  $b^{-1}ab > 1$ , but this is  $a^{-1}$ , so  $a^{-1} > 1$  which is a contradiction.

Notice that  $\mathbb{Z}$  has exactly two LO's. The usual one, and the opposite. Therefore, if we choose an LO on  $\langle a \rangle$  and  $K/\langle a \rangle$ , this gives 4 LO's on  $K$  determined by:

- (i)  $a > 1, b > 1$ ;
- (ii)  $a > 1, b < 1$ ;
- (iii)  $a < 1, b > 1$ ;
- (iv)  $a < 1, b < 1$ .

THEOREM 1.14. *These are the only LO's on  $K$ .*

PROOF. It suffices to show that each of these conditions determines a unique positive cone.

- (i)  $a > 1, b > 1$ :

CLAIM 1.2.  $a^k < b$  for all  $k \in \mathbb{Z}$ .

PROOF.  $b < a^k$  implies  $a^{-k}b < 1$ . But  $a^{-k}b = ba^k$  and  $b > 1$ , so  $b < a^k$  implies  $a^k > 1$ , which implies  $ba^k > 1$  which is a contradiction.  $\square$

Note that every element in  $K$  has a unique representative of the form  $a^m b^n$  for  $m, n \in \mathbb{Z}$ .

CLAIM 1.3.  $a^m b^n > 1$  iff either  $n > 0$  or  $n = 0$  and  $m > 0$ .

PROOF. If  $n = 0$ , then this is clear. If  $n > 0$ , then  $a^m b > 1$  for any  $m$  by claim 1 (for  $k = -m$ ). But we also know  $b > 1$  which implies  $b^n > 1$ , so we get  $a^m b^n > 1$  for  $n > 0$ . On the other hand, if  $m < 0$  then  $a^m b^n = b^n a^{\pm m} = (a^{\text{mpm}} b^{-n})^{-1}$ . Then we know  $a^{\text{mpm}} b^{-n} > 1$  by the case above, so its inverse is  $< 1$ .  $\square$

If  $<$  is an LO on  $G$ , and  $\alpha : G \rightarrow G$  is an automorphism, then this induces an LO  $<_\alpha$  on  $G$  given by:  $g <_\alpha h$  iff  $\alpha(g) < \alpha(h)$ . Now notice that there are automorphisms  $\alpha_1, \alpha_2$  of  $K$  such that

$$(1.20) \quad \alpha_1(a) = a, \quad \alpha_1(b) = b^{-1}$$

$$(1.21) \quad \alpha_1(a) = a^{-1}, \quad \alpha_1(b) = b.$$

In particular,  $\alpha_1$  is given by

$$(1.22) \quad \langle a, b \mid b^{-1}ab = a^{-1} \rangle \cong \langle a, b \mid bab^{-1} = a^{-1} \rangle$$

and similarly for  $\alpha_2$ .

Write  $<_{(i)}$  for the unique LO on  $K$  determined by (i). Then  $<_{(ii)}$  is induced by  $<_{(i)}$  and  $\alpha_1$ ,  $<_{(iii)}$  is induced by  $<_{(i)}$  and  $\alpha_2$ , and  $<_{(iv)}$  is induced by  $<_{(i)}$  and  $\alpha_1 \alpha_2$ .

■

FACT 1. *If  $G$  has only finitely many LO's, then the number of LO's is of the form  $2^n$ .*

EXERCISE 1.1. Show that for all  $n \geq 0$  there exists a group  $G$  with exactly  $2^n$  LO's.

**Corollary 1.15.** *For any LO on  $K$ , if  $h \in \langle a \rangle$ ,  $g \in K \setminus \langle a \rangle$ , and  $g > 1$ , then  $g > h$ .*

PROOF. It is sufficient to check this for the first LO, since the other three are determined by the above automorphisms. Let  $a > 1$ ,  $b > 1$ . By claim 2 from above, we know  $g = a^m b^n$  for  $n > 0$ . We now there is some  $k$  such that  $h = a^k$ , and therefore

$$(1.23) \quad h^{-1}g = a^{m-k}b^n > 1$$

by claim 2, so  $g > h$ . □

## 2. Three-manifold groups

Suppose  $M$  is a closed, orientable, connected three-manifold. Then we might ask if  $\pi_1(M)$  is LO? BO?

Immediately we notice that not all such groups are. If  $M$  is a lens space, then  $\pi_1(M) \cong \mathbb{Z}/n$  for  $n > 1$ , so this is not LO. More generally, for  $\pi_1(M)$  nontrivial and finite is not LO. Recall that if  $M = M_1 \# M_2$ , then this implies  $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$ . So, for example, if  $M_1 \#$  lens space, then  $\pi_1(M)$  has torsion, so not LO.

But at least some of them are. Consider  $M \cong T^3 = S^1 \times S^1 \times S^1$ . Then  $\pi_1(M) = \mathbb{Z}^3$  is of course LO. Similarly  $M = \#_n (S^1 \times S^2) \cong F_n$ , so  $\pi_1(M)$  is LO.

We will show that there exist (three-manifold) groups that are torsion-free, but not LO.

Let  $p : T^2 \rightarrow X^2$  be a two-fold covering of the Klein bottle. Recall that

$$(1.24) \quad K > p_* (\pi_1(T^2)) = \langle a, b^2 \rangle \cong \mathbb{Z} \times \mathbb{Z} .$$

Let  $N$  be the mapping cylinder of  $p$ , namely:

$$(1.25) \quad N = (T^2 \times I) \amalg X^2 / ((x, 0) \sim p(x) \forall x \in T^2) .$$

The orientation reversing curve representing  $b$  doesn't lift. So  $N$  is orientable. Note that  $\partial N \cong T^2$ . There is a strong deformation retraction  $N \rightarrow X^2$ , so  $\pi_1(N) \cong K$ . Let  $N_1, N_2$  be two copies of  $N$ . Write

$$(1.26) \quad \pi_1(N_i) = \langle a_i, b_i \mid b_i^{-1} a_i b_i = a_i^{-1} \rangle .$$

Notice that  $\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z} = \langle a_i, b_i^2 \rangle < \pi_1(N_i)$ . Let  $\varphi : \partial N_1 \rightarrow \partial N_2$  be a homeomorphism. Let  $M_\varphi = N_1 \cup_\varphi N_2$ . This is a closed, orientable three-manifold. Therefore

$$(1.27) \quad \pi_1(M_\varphi) = \pi_1(N_1) *_{\mathbb{Z} \times \mathbb{Z}} \pi_1(N_2) \cong K_1 *_{\mathbb{Z} \times \mathbb{Z}} K_2 .$$

Since  $K$  is torsion-free,  $\pi_1(M_\varphi)$  is torsion-free. But in fact we have the following theorem.

**THEOREM 1.16.** *If  $H_1(M_\varphi)$  is finite, then  $\pi_1(M_\varphi)$  is not LO.*

**REMARK 1.4.** We will see later that for  $M$  a prime three-manifold with  $H_1(M)$  infinite has  $\pi_1(M)$  LO.

PROOF.  $\varphi$  is determined up to isotopy, so the resulting manifold  $M_\varphi$  depends only on  $\varphi_* : H_1(\partial N_1) \rightarrow H_1(\partial N_2)$ . We know

$$(1.28) \quad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_1, 2b_1 \rangle \quad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_2, 2b_2 \rangle$$

so  $\varphi_*$  is given by some  $2 \times 2$  matrix with  $\mathbb{Z}$  coefficients

$$(1.29) \quad \begin{bmatrix} p & r \\ q & s \end{bmatrix}$$

with determinant  $ps - qr = \pm 1$ . Specifically we have:

$$(1.30) \quad \varphi_*(a_1) = pa_2 + 2qb_2$$

$$(1.31) \quad \varphi_*(2b_1) = ra_2 + 2sb_2 .$$

Now we have  $H_1(N_i) = \mathbb{Z} \oplus \mathbb{Z}_2$  with basis  $b_i$  and  $a_i$  respectively. Then  $H_q(M_\varphi)$  is presented by

$$(1.32) \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{bmatrix} .$$

where we order the basis as  $\{a_1, b_1, a_2, b_2\}$ . Interchanging columns 2 and 3 we get

$$(1.33) \quad \det A = 4 \left| \det \begin{bmatrix} 0 & 2q \\ -2 & 2s \end{bmatrix} \right| = 16 |q| .$$

Therefore  $H_1(M_\varphi)$  is finite iff  $q \neq 0$  iff  $\varphi_*(a_1) \neq \pm a_2$ .

Suppose  $\pi_1(M_\varphi)$  is LO. Then we would get an induced LO on the common boundary  $\partial N_1 = \partial N_2$ . But there are only 4 LO's on  $\pi_1(N_i)$  (for  $i \in \{1, 2\}$ ). By Corollary 1.15, for any LO on  $\pi_1(N)$ ,  $\langle a \rangle$  is the unique  $\mathbb{Z}$ -summand of  $\pi_1(\partial N) = \langle a, b^2 \rangle$  such that if  $h \in \langle a \rangle$  and  $g \in \pi_1(\partial N) \setminus \{1\}$ ,  $g > 1$ , then  $g > h$ . Therefore  $\varphi_*(a_1) = \pm a_2$  which is a contradiction.  $\square$

Let  $<$  be an STO on a set  $X$ . Let  $\mathcal{B}(X, <)$  be the group of  $<$ -preserving bijections  $X \rightarrow X$ .

Lecture 3; January 28, 2020

**THEOREM 1.17.**  $\mathcal{B}(X, <)$  is always LO.

**PROOF.** Let  $\prec$  be a WO on  $X$ . Let  $f, g \in \mathcal{B}(X, <)$  such that  $f \neq g$ . Write

$$(1.34) \quad [f \neq g] = \{x \in X \mid f(x) \neq g(x)\} \neq \emptyset .$$

Let  $x_0$  be the  $\prec$ -least element of  $[f \neq g]$ . Define

$$(1.35) \quad f < g \iff f(x_0) < g(x_0) .$$

Then we claim that this is an LO on  $\mathcal{B}(X, <)$ . Left-invariance is clear. To see this is a STO we need “trichotomy” and transitivity. Trichotomy is easy, and transitivity follows from the same argument as the proof of Theorem 1.4.  $\square$

**EXAMPLE 1.3.** Let  $<$  be the standard order on  $\mathbb{R}$ . Then  $\mathcal{B}(\mathbb{R}, <)$  consists of the orientation-preserving homeomorphisms  $\mathbb{R} \rightarrow \mathbb{R}$ , written  $\text{Homeo}_+(\mathbb{R})$ .

**Corollary 1.18.**  $\text{Homeo}_+(\mathbb{R})$  is LO.

**REMARK 1.5.** For  $x \in \mathbb{R}$ , let  $\prec_x$  be a WO on  $\mathbb{R}$  such that  $x$  is the  $\prec_x$ -least element of  $\mathbb{R}$ . Let  $<_x$  be the LO on  $\text{Homeo}_+(\mathbb{R})$  induced by  $\prec_x$ , as in the proof of Theorem 1.17. Given  $x \neq y \in \mathbb{R}$ , there exists  $g \in \text{Homeo}_+(\mathbb{R})$  such that  $g(x) > x$  and  $g(y) < y$ . But this means

$$g <_x 1 \qquad g <_y 1 .$$

which implies  $<_x \neq <_y$ . Therefore  $\text{Homeo}_+(\mathbb{R})$  has uncountably many LO's.

REMARK 1.6. It is a fact that the number of LO's on a group  $G$  is either finite (and of the form  $2^n$ ) or uncountable.

**Corollary 1.19.** *A group  $G$  is LO iff  $G$  acts faithfully<sup>1.1</sup> on a STO'd set  $(X, <)$ .*

PROOF. ( $\Leftarrow$ ): This follows from Theorem 1.17.

( $\Rightarrow$ ):  $G$  acts faithfully on  $(G, <)$  by left multiplication.  $\square$

Corollary 1.18 implies that any subgroup of  $\text{Homeo}_+(\mathbb{R})$  is LO. E.g. one can show that  $F_2$  (the free group of rank 2) is a subgroup of  $\text{Homeo}_+(\mathbb{R})$ . (This is another way to show that countable free groups are LO.) In fact this characterizes countable LO groups.

THEOREM 1.20. *Let  $G$  be a countable group. Then  $G$  is LO iff there exists an injective homomorphism  $G \rightarrow \text{Homeo}_+(\mathbb{R})$ .*

PROOF. ( $\Leftarrow$ ): This follows from Corollary 1.18.

( $\Rightarrow$ ): We actually prove something slightly stronger. This will follow from Theorem 1.21.  $\square$

THEOREM 1.21. *Let  $(G, <)$  be a countable group with an LO. Then there exists a LO on  $\text{Homeo}_+(\mathbb{R})$  and an order-preserving injective homomorphism  $(G, <) \rightarrow (\text{Homeo}_+(\mathbb{R}), <)$ .*

SKETCH OF PROOF. Let  $<$  be an LO on  $G$ . If  $G = \{1\}$  this is immediate, so assume  $G \neq \{1\}$ . Therefore it is infinite, since LO groups are torsion free. Let  $g_1, g_2, \dots$  be some enumeration of the elements of  $G$ .

Define an embedding  $e : G \rightarrow \mathbb{R}$  by  $e(g_1) = 0$ , and inductively by:

(i) If  $g_{n+1} \begin{Bmatrix} > \\ < \end{Bmatrix} g_i$  for all  $1 \leq i \leq n$ , then set

$$(1.36) \quad e(g_{n+1}) = \begin{Bmatrix} \max \{e(g_i) \mid 1 \leq i \leq n\} + 1 \\ \min \{e(g_i) \mid 1 \leq i \leq n\} - 1 \end{Bmatrix}.$$

(ii) Otherwise let

$$g_l = \max \{g_i \mid 1 \leq i \leq n, g_i < g_{n+1}\}$$

$$g_r = \min \{g_i \mid 1 \leq i \leq n, g_i > g_{n+1}\}$$

and set

$$e(g_{n+1}) = \frac{e(g_l) + e(g_r)}{2}.$$

REMARK 1.7. (1)  $e$  is order-preserving, i.e.  $a < b \implies e(a) < e(b)$ .

(2)  $e(g_{n+1}) \in \mathbb{Z}$  iff (i) holds.

(3) If  $g > 1$  then  $g^2 > g$  and  $g^{-1} < g$ . If  $g < 1$  then  $g^2 < g$  and  $g^{-1} > g$ , which implies  $\mathbb{Z} \subset e(G) = \Gamma$ .

(4)  $G$  acts on  $\Gamma$  by  $g(e(a)) = e(ga)$ . In fact,  $G$  acts on  $(\Gamma, <)$  (where  $<$  is the restriction of  $<$  on  $\mathbb{R}$ ) since  $e(a) < e(b)$  iff  $a < b$  iff  $ga < gb$  iff  $e(ga) < e(gb)$  iff  $g(e(a)) < g(e(b))$ .

To see that this action extends to an action of  $G$  on  $\mathbb{R}$ , we have a few steps.

Step 1: The action of  $G$  on  $\Gamma$  is continuous,

Step 2: The action of  $G$  on  $\Gamma$  extends to a continuous action of  $G$  on  $\bar{\Gamma}$ .

---

<sup>1.1</sup>Recall this means  $g(x) = x$  for all  $x \in X$  iff  $g = 1$ .

Step 3:  $\mathbb{R} \setminus \bar{\Gamma}$  is a countable  $\Pi$  of open intervals  $(a_i, b_i)$ ; the action of  $G$  is defined on  $\{a_i, b_i\}$ ; and extends to  $[a_i, b_i]$ .

Note, to ensure Step 1:, it is not enough to take  $e$  to be an order-preserving of  $G$  in  $\mathbb{R}$ . It must be continuous.

To define an LO on  $\text{Homeo}_+(\mathbb{R})$  that restricts to the LO on  $\Gamma$  from  $G$ , first pick any  $\gamma \in \Gamma$ . Then  $g > 1$  (resp.  $< 1$ ) iff  $g(\gamma) > \gamma$  (resp.  $< \gamma$ ). Let  $\prec$  be a WO on  $\mathbb{R}$  such that  $\gamma$  is the  $\prec$ -least element of  $\mathbb{R}$ . Then let  $\leq$  be the LO on  $\text{Homeo}_+(\mathbb{R})$  induced by  $\prec$ . Then  $g > 1$  (resp.  $< 1$ ) in  $G$  iff  $g \succ 1$  (resp.  $\prec$ ) in  $\text{Homeo}_+(\mathbb{R})$ .  $\square$

### 3. Group rings

Let  $R$  be a ring (with 1).

- $a \in R$  is a *unit* if there exists  $b \in R$  such that  $ab = ba = 1$ .
- $a \in R$  is a *zero-divisor* if  $a \neq 0$  and there exists  $b \neq 0$  such that either  $ab = 0$  or  $ba = 0$ .
- $a \in R$  is a *non-trivial idempotent* if  $a^2 = a$  but  $a \neq 0$  and  $a \neq 1$ .

Let  $G$  be a group and  $R$  a ring. Then the  $R$ -group ring of  $G$  consists of formal sums:

$$(1.37) \quad RG := \left\{ \sum r_g g \mid g \in G, r_g \in R, r_g \neq 0 \forall \text{ but f'tly many } g \in G \right\}.$$

$RG$  is a ring with respect to the obvious operations. For  $g \in G$  and  $r \in R$  a unit, then  $rg$  is a unit in  $RG$ . A unit in  $RG$  is *non-trivial* if it is not of this form.

REMARK 1.8. If  $\tilde{X} \rightarrow X$  is a universal covering, then  $\pi = \pi_1(X)$  acts on  $\tilde{X}$  so  $H_*(\tilde{X}, \mathbb{Z})$  is a  $\mathbb{Z}\pi$ -module.

THEOREM 1.22. Suppose  $G$  has non-trivial torsion, and  $K$  is a field of characteristic 0.

- (1)  $KG$  has zero divisors,
- (2)  $KG$  has non-trivial units,
- (3)  $KG$  has non-trivial idempotents.

PROOF. Let  $g \in G$  have order  $n \geq 2$ . Define

$$\sigma = 1 + g + g^2 + \dots + g^{n-1} \in KG.$$

First notice that

$$(1.38) \quad g\sigma = \sigma$$

which implies  $(1 - g)\sigma = 0$  so we have zero divisors.

(1.38) also gives us that  $\sigma^2 = n\sigma$ . Therefore

$$(1 - \sigma) \left( 1 - \frac{1}{n-1} \sigma \right) = 1$$

so we have a nontrivial unit for  $n > 2$ . If  $n = 2$ ,  $1 - \sigma = -g$ , but we still have:

$$(1.39) \quad (1 - 2\sigma) \left( 1 - \frac{2}{3} \sigma \right) = 1.$$

Finally, we have that

$$(1.40) \quad \left( \frac{1}{n} \sigma \right)^2 = \left( \frac{1}{n^2} \right) \sigma^2 = \frac{1}{n} \sigma$$

so we have nontrivial idempotents.  $\square$

Note that the proof of (1) works even for  $\mathbb{Z}G$ .

REMARK 1.9. If  $n \notin \{2, 3, 4, 6\}$  then  $\mathbb{Z}G$  has nontrivial units. This is a theorem of Higman.

EXAMPLE 1.4. For  $n = 5$ ,

$$(1.41) \quad (1 - g - g^4)(1 - g^2 - g^3) = 1 .$$

But what if  $G$  is torsion free? This brings us to the famous Kaplansky conjectures.

CONJECTURE 1 (Kaplansky). *If  $G$  is torsion free and  $K$  is a field, then:*

*I (Units conjecture):  $KG$  has no non-trivial units,*

*II (Zero-divisors conjecture):  $KG$  has no zero divisors,*

*III (Idempotents conjecture):  $KG$  has no non-trivial idempotents.*

REMARK 1.10. Clearly II implies III since  $a^2 = a$  implies  $a(a - 1) = 0$ , which by II implies  $a = 0$  or  $a = 1$  which implies III. In fact they're all equivalent, but this is nontrivial to see.

Lecture 4; January  
30, 2020

REMARK 1.11. Note that if  $R$  is an integral domain (e.g.  $\mathbb{Z}$ ) then  $R$  is contained in its field of fractions. In this case items I and II and item III for its field of fractions imply the corresponding versions of items I and II and item III for  $R$ .

REMARK 1.12. We know this is true for LO groups. As we have seen, we should think of LO as being a stronger version of torsion free.

THEOREM 1.23. *If  $G$  is LO then  $KG$  satisfies items I and II and item III.*

PROOF. Since item I implies item III by the above remark we show item I and item II.  
item I: Suppose

$$(1.42) \quad \left( \sum_{i=1}^m \alpha_i g_i \right) \left( \sum_{j=1}^n \beta_j h_j \right) = 1$$

with  $m, n$  not both 1,  $\alpha_i, \beta_j \neq 0 \in K$ , distinct  $g_i \in G$ , and distinct  $h_i \in G$ . Note this product can be rewritten as the following sum with  $mn$  terms:

$$(1.43) \quad \sum_{i,j} (\alpha_i \beta_j) (g_i h_j) .$$

Assume WLOG that  $h_1 < h_2 < \dots < h_n$ . Let  $g_k h_l$  be a minimal element of

$$(1.44) \quad S = \{g_i h_j \mid 1 \leq i \leq m, 1 \leq j \leq n\} \subset G .$$

We know  $h_1 < h_j$  for  $j > 1$ , so  $g_k h_1 < g_k h_j$  for all  $j > 1$ . Therefore  $l = 1$ . Also  $gh_1 = g'h_1$  which implies  $g = g'$ . Therefore  $g_k h_1$  is the unique

$$(1.45) \quad (k, 1) \in \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

such that  $g_k h_1$  is a minimal element of  $S$ .

Similarly, there is a unique

$$(1.46) \quad (r, n) \in \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

such that  $g_r h_n$  is a maximal element of  $S$ .

CLAIM 1.4.  $g_k h_1 \neq g_r h_n$ .

If they were equal, then  $r = k$ ,  $n = 1$ , so  $m > 1$ . So  $g_k h_1 = g_r h_1$ , and therefore  $g_r = g_k$ . But this cannot be the case since they are distinct by assumption.

This implies that (1.43) has  $\geq 2$  terms after cancellation, so it cannot be 1.

item II: Now suppose

$$(1.47) \quad \left( \sum_{i=1}^m \alpha_i g_i \right) \left( \sum_{j=1}^n \beta_j h_j \right) = 0$$

for  $m, n \geq 1$ . Then there is a unique minimal element and nonzero coefficient, which means it is nonzero.  $\square$

CONJECTURE 2 (Isomorphism conjecture). *If  $G$  is torsion free, then  $\mathbb{Z}G \cong \mathbb{Z}H$  implies  $G \cong H$ .*

REMARK 1.13. In [H] a finite counterexample to the conjecture for arbitrary groups was provided, i.e. it is shown that there exists finite  $G, H$  such that  $\mathbb{Z}G \cong \mathbb{Z}H$ ,  $G \not\cong H$ .

**Corollary 1.24** ([LR]). *If  $G$  is LO, then  $G$  satisfies the isomorphism conjecture.*

PROOF. Theorem 1.23 implies that  $\mathbb{Z}G$  has no nontrivial units. Call  $\mathcal{U}_{\mathbb{Z}G}$  the group of units in  $\mathbb{Z}G = \mathbb{Z}/2 \times G$ . Suppose  $\mathbb{Z}G \cong \mathbb{Z}H$ . Theorem 1.23 says that  $\mathbb{Z}G$  has no 0-divisors. This implies  $\mathbb{Z}H$  has no 0-divisors, which means (by Theorem 1.22) that  $H$  is torsion-free. Now  $H < \mathcal{U}_{\mathbb{Z}H} \cong \mathcal{U}_{\mathbb{Z}G} \cong \mathbb{Z}/2 \times G$  which implies  $H < G$  (since  $H$  is torsion-free), which implies  $H$  is LO (since  $G$  is), which implies  $\mathcal{U}_{\mathbb{Z}H} \cong \mathbb{Z}/2 \times H$ , which implies  $\mathbb{Z}/2 \times H \cong \mathbb{Z}/2 \times G$  which implies  $H \cong G$  (since  $H, G$  are torsion free).  $\square$

REMARK 1.14. We might wonder if it is ever the case that (for  $G \neq 1$ )  $(G * \mathbb{Z}) / \langle \langle w \rangle \rangle = 1$ ? This is known for  $G$  torsion free [K1].

COUNTEREXAMPLE 1. If we consider the question of whether we can ever have  $(A * B) / \langle \langle w \rangle \rangle = 1$  for  $A, B$  nontrivial, a counterexample is given by:

$$\mathbb{Z}/2 * \mathbb{Z}/3 / (a = b) .$$

#### 4. BO's on $\mathbb{Z} \times \mathbb{Z}$

Recall we have 2 orders on  $\mathbb{Z}$ . Consider a line of slope  $\alpha$  in  $\mathbb{Z} \times \mathbb{Z}$ . Then we have two cases.

- (1)  $\alpha$  irrational: The associated positive cone is everything above the line. Specifically,  $P \subset \mathbb{Z} \times \mathbb{Z}$  is given by

$$(1.48) \quad P = \{(m, n) \mid n > m\alpha\} .$$

It is easy to check that this is a positive cone. This means there are uncountable many BO's on  $\mathbb{Z} \times \mathbb{Z}$ .

- (2)  $\alpha$  rational: Notice that now

$$(1.49) \quad \{(m, n) \mid n = m\alpha\} \cong \mathbb{Z} < \mathbb{Z} \times \mathbb{Z} .$$

Now let  $P_0$  be one of the two positive cones on  $\mathbb{Z}$ . Then we can check that

$$P = P_0 \amalg \{(m, n) \mid n > m\alpha\}$$

is a positive cone for  $\mathbb{Z} \times \mathbb{Z}$ .

REMARK 1.15. (1) (Up to reversal) these are all the BOs on  $\mathbb{Z} \times \mathbb{Z}$ . I.e. for  $\alpha$  rational we get two, and for  $\alpha$  irrational we get 4.

- (2) This generalizes in the obvious way to  $\mathbb{Z}^n$ .

**5. BO's on  $\mathbb{R}$** 

Regard  $\mathbb{R}$  as a vector space on  $\mathbb{Q}$  with uncountable bases  $\Lambda$ . Recall  $\Lambda$  exists by the axiom of choice. Therefore  $\mathbb{R} \subset \mathbb{Q}^\Lambda$ . In particular it is the elements of  $\mathbb{Q}^\Lambda$  with only finitely many non-zero coordinates. There are uncountable many WO's on  $\Lambda$ , and each gives rise to a lexicographic BO on  $\mathbb{Q}^\Lambda$ . This gives us uncountably many BOs on  $\mathbb{R}$ .



## CHAPTER 2

### The space of left-orders on a group

The basic idea is that since lefts orders are determined by positive cones, we can give this space a topology. Consider a family of sets  $\{X_\lambda \mid \lambda \in \Lambda\}$ . Then write

$$X = \prod_{\lambda \in \Lambda} X_\lambda$$

and  $\pi_\lambda : X \rightarrow X_\lambda$  for the projection. If  $X_\lambda$  is a topological space, then  $X$  can be given the product topology. This is the largest topology on  $X$  such that  $\pi_\lambda$  is continuous for all  $\lambda \in \Lambda$ . So  $X$  has subbasis

$$(2.1) \quad \left\{ \pi_\lambda^{-1}(U_\lambda) = U_\lambda \times \prod_{\mu \neq \lambda} X_\mu \mid U_\lambda \subset X_\lambda \text{ open, } \lambda \in \Lambda \right\}.$$

**THEOREM.** *If  $X_\lambda$  is compact for all  $\lambda \in \Lambda$  then  $\prod_{\lambda \in \Lambda} X_\lambda$  is compact.*

**REMARK 2.1 (Exercises).** (1)  $X_\lambda$  Hausdorff (for all  $\lambda \in \Lambda$ ) implies  $\prod_{\lambda \in \Lambda} X_\lambda$  is Hausdorff.

(2) A space  $X$  is totally disconnected if the only nonempty connected subspaces are singletons  $\{x\}$  for  $x \in X$ . This is equivalent to the connected components of  $X$  all being  $\{x\}$ . Show that  $X_\lambda$  totally disconnected (for all  $\lambda \in \Lambda$ ) implies  $\prod_{\lambda \in \Lambda} X_\lambda$  is totally disconnected.

Let  $X$  be a set, let  $\mathcal{S}(X)$  be the set of subsets of  $X$  (i.e. the power set). Then we have a correspondence:

$$\mathcal{S}(X) \quad \leftrightarrow \quad \{f : X \rightarrow \{0, 1\}\}$$

which sends:

$$A \subset X \quad \leftrightarrow \quad f_A : X \rightarrow \{0, 1\}$$

where

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Give  $\{0, 1\}$  the discrete topology, and give

$$\mathcal{S}(X) = \{0, 1\}^X = 2^X = \prod_{x \in X} \{0, 1\}$$

the product topology. Note  $\{0, 1\}$  is a compact, Hausdorff, totally-disconnected space, which means  $\mathcal{S}(X)$  is too. For  $x \in X$  let

$$\begin{aligned} U_x &= \pi_x^{-1}(1) = \{A \subset X \mid x \in A\} \\ V_x &= \pi_x^{-1}(0) = \{A \subset X \mid x \notin A\}. \end{aligned}$$

Note that  $V_x = \mathcal{S}(X) \setminus U_x$  so  $U_x$  and  $V_x$  are open and closed. Then

$$(2.2) \quad \{U_x \mid x \in X\} \cup \{V_x \mid x \in X\}$$

is a subbasis for  $\mathcal{S}(X)$ .

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4, 2020

**Lemma 2.1.** *Suppose  $B \subset X$ . Then*

$$\{A \subset X \mid B \not\subset A\} \quad \{A \subset X \mid A \cap B \neq \emptyset\}$$

*are open subsets of  $\mathcal{S}(X)$ .*

PROOF.

$$\{A \subset X \mid B \not\subset A\} = \bigcup_{b \in B} \{A \subset X \mid b \notin A\} = \bigcup_{b \in B} V_b$$

so it is open. The argument for the other set is similar.  $\square$

If  $G$  is a group, let

$$(2.3) \quad \text{LO}(G) = \{\text{positive cones } \subset G\} \subset \mathcal{S}(G)$$

and equip it with the subspace topology. We call this the *space of left-orders on  $G$* .

EXAMPLE 2.1.  $\text{LO}(\mathbb{Z}) = \text{pt} \amalg \text{pt}$ .  $\text{LO}(\mathbb{Z} \times \mathbb{Z})$  is the cantor set.

THEOREM 2.2.  $\text{LO}(G)$  is closed in  $\mathcal{S}(G)$  and hence compact.

PROOF. We show  $\mathcal{S}(G) \setminus \text{LO}(G)$  is open. Suppose  $A \in \mathcal{S}(G) \setminus \text{LO}(G)$ , i.e.  $A \subset G$  is not a positive cone. So either:

- (i)  $\exists g, h \in A$  such that  $gh \notin A$  or
- (ii)  $\exists g \in G$  such that  $g, g^{-1} \in A$  or
- (iii)  $1 \in A$  or
- (iv)  $\exists g, g \neq 1$  such that  $g \notin A$  and  $g^{-1} \notin A$ .

Now the point is that these are open conditions since we can write them in terms of the  $U_x$ 's and  $V_x$ 's. In particular:

$$\begin{aligned} (i) &\iff A \in U_g \cap U_h \cap V_{gh} & (ii) &\iff A \in U_g \cap U_{g^{-1}} \\ (iii) &\iff A \in U_1 & (iv) &\iff A \in \bigcup_{g \neq 1} (V_g \cap V_{g^{-1}}) . \end{aligned}$$

Therefore  $\text{LO}(G)$  is compact, Hausdorff, and totally disconnected.  $\square$

Similarly one can define the space of biorders on  $G$ ,  $\text{BO}(G)$ , to be the set of conjugation invariant positive cones in  $G$ .

EXERCISE 2.1. Show that  $\text{BO}(G)$  is closed inside of  $\text{LO}(G)$ .

Therefore  $\text{BO}(G)$  is compact, Hausdorff, and totally disconnected.

### 1. The cantor set

The cantor set  $C \subset I \subset \mathbb{R}$  is defined as follows. First write

$$\begin{aligned} C_1 &= [0, 1/3] \cup [2/3, 1] \\ C_2 &= ([0, 1/9] \cup [2/9, 1/3]) \cup ([2/3, 7/9] \cup [8/9, 1]) \\ &\dots \end{aligned}$$

then define

$$(2.4) \quad C = \bigcap_{n=1}^{\infty} C_n .$$

The idea is that we keep removing the middle thirds.

$C$  is uncountable, totally-disconnected, closed in  $I$ . Therefore it is also compact and Hausdorff. This is a very surprising example. We can easily write down something uncountable and totally-disconnected, such as the irrationals, but they do not form a compact set.

Any  $x \in I$  has a ternary expansion:

$$x = 0.x_1x_2\dots = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

which is unique up to:

$$\dots x_k 22 \dots = \dots (x_{k+1}) 00 \dots$$

Now notice

$$x_1 = 1 \quad \Longleftrightarrow \quad x \in (1/3, 2/3)$$

with the convention that

$$\frac{1}{3} = 0.022\dots$$

Similarly (with the same convention) we have

$$x_1 \neq 1, x_2 = 1 \quad \Longleftrightarrow \quad x \in (1/9, 2/9) \cup (7/9, 8/9)$$

and so on. Then

$$(2.5) \quad C = \{x \in I \mid x = 0.x_1x_2\dots \mid \forall n, x_n = 0 \text{ or } 2\} .$$

Now give  $\{0, 2\}^{\mathbb{N}}$  the product topology.

EXERCISE 2.2. Show that the map sending

$$(2.6) \quad 0.x_1x_2\dots \mapsto (x_1, x_2, \dots)$$

defines a homeomorphism

$$(2.7) \quad C \xrightarrow{\cong} \{0, 2\}^{\mathbb{N}} .$$

Now recall that  $\text{LO}(G)$  is compact in  $\{0, 1\}^G$ , so if  $G$  is countable, then  $\text{LO}(G)$  is homeomorphic to a subspace of  $C$ .

We say  $x \in X$  is *isolated* if  $\{x\}$  is open. We say  $X$  is *perfect* if it has no isolated points. As it turns out, the Cantor set is perfect.

**THEOREM.** *If  $X$  is a compact, totally-disconnected, and perfect metric space, then  $X \cong C$ .*

Therefore, if  $G$  is countable,  $\text{LO}(G) \neq \emptyset$ , and has no isolated points, then  $\text{LO}(G) \cong C$ .

EXAMPLE 2.2. In 2004 [S] it was shown that if  $n > 1$  then  $\text{LO}(\mathbb{Z}^n) = \text{BO}(\mathbb{Z}^n) \cong C$ .

EXAMPLE 2.3. In 1985 [M2] it was shown that  $\text{LO}(F_n) \cong C$ . It is unknown if  $\text{LO}(F_n)$  has isolated points.

REMARK 2.2. As it turns out, the braid group is LO. The first proof of this fact was not topological, so topologists started to think of a topological proof. When someone asked Thurston, he said “of course the braid group is left-orderable!”

If  $X \subset G$ , let  $S(X)$  be the semigroup generated by  $X$  in  $G$ . This is the same as the non-empty product of elements in  $X$ . There is a characterization of left orderability in terms of finite subsets of  $G$ .

THEOREM 2.3.  $G$  is LO iff for all finite  $F \subset G \setminus \{1\}$ , there exists  $\epsilon : F \rightarrow \{\pm 1\}$  such that

$$(2.8) \quad 1 \notin S\left(\left\{f^{\epsilon(f)} \mid f \in F\right\}\right) (= S(F, \epsilon)) .$$

REMARK 2.3. It follows from this that, given a solution to the word problem in  $G$ , there exists a machine such that if  $G$  is not LO, the machine will eventually tell you that. Nathan Dunfield has an explicit algorithm for three-manifold groups.

REMARK 2.4. If we take the  $n$ -fold cyclic branch cover of the knot  $5_2$ , then we can consider  $\pi_1(\Sigma_n(5_2))$ . For  $n = 2$ , this is a lens space so  $\pi_1$  is finite. It is also not LO for  $n = 3, 4$ , and  $5$ . But it is unknown for  $n = 6, 7$ , and  $8$ . (If the  $L$ -space conjecture is true,<sup>2.1</sup> then it should be LO for these values of  $n$ .) For  $n \geq 9$  it is known to be LO.

PROOF. ( $\implies$ ): Define

$$\epsilon(f) = \begin{cases} +1 & f > 1 \\ -1 & f < 1 \end{cases} .$$

( $\impliedby$ ): Let  $F \subset G \setminus \{1\}$  be finite,  $\epsilon : F \rightarrow \{\pm 1\}$ . Define

$$Q(F, \epsilon) := \left\{ Q \subset G \setminus \{1\} \mid S(F, \epsilon) \subset Q, S(F, \epsilon)^{-1} \cap Q = \emptyset \right\} .$$

Note that  $Q(F, \epsilon) \neq \emptyset$  iff (2.8) holds. Let

$$Q(F) = \cup_{\epsilon} Q(F, \epsilon) .$$

Note this is a finite union.

CLAIM 2.1.  $Q(F)$  is closed in  $S(G)$ .

PROOF. It is sufficient to show that  $Q(F, \epsilon)$  is closed, i.e.  $S(G) \setminus Q(F, \epsilon)$  is open. Suppose  $A \subset G$ ,  $A \not\subset Q(F, \epsilon)$  i.e. either  $1 \in A$ , or  $S(F, \epsilon) \not\subset A$ , or  $S(F, \epsilon)^{-1} \cap A \neq \emptyset$ . These conditions are all open by Lemma 2.1.  $\square$

Note that if  $F \subset F'$ , then

$$(2.9) \quad S(F, \epsilon'|_{F'}) \subset S(F', \epsilon')$$

and therefore

$$(2.10) \quad Q(F') \subset Q(F) .$$

---

<sup>2.1</sup>Which is looking quite likely. It has been checked for something like three-hundred thousand manifolds.

Let  $F_1, F_2, \dots, F_n$  be finite subsets of  $G \setminus \{1\}$ . Then

$$\bigcap_{i=1}^n Q(F_i) \supset Q(F_1 \cup F_2 \cup \dots \cup F_n) \neq \emptyset$$

since (2.8) holds. This means  $\{Q(F)\}$  has the *finite intersection property* (FIP) and each one is closed. Therefore, since  $\mathcal{S}(G)$  is compact,

$$\bigcap_{F \subset G \setminus \{1\} \text{ finite}} Q(F) \neq \emptyset.$$

So let  $P \in \bigcap Q(F)$ .

CLAIM 2.2.  $P$  is a positive cone for  $G$ .

PROOF. First notice  $1 \notin P$  since  $1 \notin Q(F)$  for any finite  $F \subset G \setminus \{1\}$ .

Now we show  $g, h \in P$  implies  $gh \in P$ . Let  $F = \{g, h\}$ . Then there are  $\epsilon(g), \epsilon(h) \in \{\pm 1\}$  such that

$$S(g^{\epsilon(g)}, h^{\epsilon(h)}) \subset P \quad S(g^{\epsilon(g)}, h^{\epsilon(h)})^{-1} \cap P = \emptyset.$$

Therefore  $\epsilon(g) = \epsilon(h) = +1$ , which implies  $gh \in S(g^{\epsilon(g)}, h^{\epsilon(h)}) \subset P$ .

Now we show  $P \cap P^{-1} = \emptyset$ . Let  $g \in P$ , and  $F = \{g\}$ . Therefore  $S(g) \subset P$ , which means  $S(g)^{-1} \cap P = \emptyset$ , so  $g^{-1} \notin P$ .

Finally we show  $P \cap P^{-1}G \setminus \{1\} = \emptyset$ . Take  $g \in G$  such that  $g \neq 1$ . Let  $F = \{g\}$ . Then there exists  $\epsilon = \pm 1$  such that  $S(g^\epsilon) \subset P$  (and  $S(g^{-1}) \cap P = \emptyset$ ) which implies  $g^\epsilon \in P$ .  $\square$

REMARK 2.5. There exists an analogue of this for BO.

THEOREM 2.4.  $G$  is BO if and only if for all finite  $F \subset G \setminus \{1\}$  there is some  $\epsilon : F \rightarrow \{\pm 1\}$  such that  $1 \notin T(F, \epsilon)$  where  $T(F, \epsilon)$  is the smallest semigroup which

- (i) contains  $S(F, \epsilon)$ , and
- (ii) for all  $g, h \in T(F, \epsilon)$ ,  $g, h, g^{-1}, g^{-1}hg \in T(F, \epsilon)$ .

EXERCISE 2.3. Prove Theorem 2.4.

Let  $P$  be a property of groups. A group  $G$  is *locally*  $P$  if and only if every finitely generated subgroup of  $G$  has property  $P$ . (So  $\text{loc}(\text{loc}(P)) \equiv \text{loc}(P)$ .)  $P$  is a *local property* if  $\text{loc}(P) \implies P$ .

THEOREM 2.5.  $G$  is locally LO (resp. BO) if and only if  $G$  is LO (resp. BO).

PROOF. ( $\Leftarrow$ ): LO and BO are inherited by subgroups.

( $\Rightarrow$ ): Let  $G$  be a finite set contained in  $G \setminus \{1\}$ . Then  $\langle F \rangle < G$  is finitely generated.  $G$  loc(LO) implies  $\langle F \rangle$  is LO. Therefore there exists  $\epsilon$  such that (2.8) holds (from Theorem 2.3). This is true for all  $F$ , so  $G$  is LO by Theorem 2.3. The argument for BO is similar, using Theorem 2.4 instead.  $\square$

**Corollary 2.6.** An abelian group is BO iff it is torsion free.

PROOF. ( $\Rightarrow$ ): This follows from Lemma 1.3.

( $\Leftarrow$ ):  $G$  is LO iff  $G$  is loc(LO). For  $H$  finitely generated inside of torsion free  $G$ , then  $H \cong \mathbb{Z}^n$ , so it is LO.  $\square$

**Corollary 2.7.** *An arbitrary free group is LO.*

PROOF. Let  $F$  be a free group. For  $H$  a finitely generated subgroup of  $F$ ,  $H \cong F_n$  for some  $n$ . Then  $H$  is LO by Corollary 2.7, so  $F$  is LO by Theorem 2.5.  $\square$

**THEOREM 2.8.** *Let  $\{G_\lambda\}_{\lambda \in \Lambda}$  be a collection of groups. Then  $G_\lambda$  is LO for all  $\lambda \in \Lambda$  if and only if  $\ast_{\lambda \in \Lambda} G_\lambda$  is LO.*

PROOF. ( $\Leftarrow$ ):  $G_\lambda < \ast_{\lambda \in \Lambda} G_\lambda$ .

( $\Rightarrow$ ): There exists a homomorphism

$$G = \ast_{\lambda \in \Lambda} G_\lambda \xrightarrow{\varphi} \prod_{\lambda \in \Lambda} G_\lambda$$

$$g_\lambda \longmapsto (1, \dots, 1, g_\lambda, 1, \dots)$$

So we get a SES

$$(2.11) \quad 1 \rightarrow H \rightarrow \ast_\Lambda G_\lambda \xrightarrow{\varphi} \prod_\Lambda G_\lambda \rightarrow 1$$

where  $H = \ker \varphi$ . By the Kurosh subgroup theorem

$$H = \left( \ast_\mu H_\mu \right) \ast F$$

where  $H_\mu$  is a subgroup of a conjugate of  $G_{\lambda_\mu}$  in  $G$ , and  $F$  is a free group. But  $H = \ker \varphi$ , and  $\varphi|_{G_\lambda}$  is injective for all  $\lambda \in \Lambda$ . Therefore for all  $\lambda \in \Lambda$  and  $g \in G$  we have  $H \cap g^{-1}G_\lambda g = \{1\}$ . Therefore  $H = F$ .

But now  $G_\lambda$  LO for all  $\lambda \in \Lambda$  implies  $\prod_{\lambda \in \Lambda} G_\lambda$  is LO by Theorem 1.4, and  $F = H$  is LL by Corollary 2.7, so  $G$  is LO by Theorem 1.13.  $\square$

Let  $P$  be a property of groups. A group  $G$  is residually  $P$ ,  $\text{res}(P)$ , if and only if for all  $g \in G \setminus \{1\}$  there exists an epimorphism  $\varphi : G \rightarrow H$  such that  $H$  has property  $P$ , and  $\varphi(g) \neq 1$ .

REMARK 2.6. Note that  $P$  implies  $\text{res}(P)$ , and  $\text{res}(\text{res}(P))$  implies  $\text{res}(P)$ .

We say  $P$  is a *residual property* if and only if  $\text{res}(P)$  implies  $P$ .

EXAMPLE 2.4. Finiteness is not a residual property. E.g.  $\mathbb{Z}$  is  $\text{res}(\text{finite})$ .

**Lemma 2.9.** *If  $P$  is closed under taking subgroups and direct products, then  $P$  is a residual property.*

**Corollary 2.10.** *LO and BO are residual properties.*

PROOF OF LEMMA 2.9. Suppose  $G$  is  $\text{res}(P)$ . Then for all  $g \in G \setminus \{1\}$  there is an epimorphism  $\varphi_g : G \rightarrow H_g$  such that  $H_g$  has  $P$ , and  $\varphi_g(g) \neq 1$ . The collection of these  $\{\varphi_g \mid g \in G \setminus \{1\}\}$  induces a homomorphism

$$\varphi : G \rightarrow \prod_{g \in G \setminus \{1\}} H_g .$$

Then this is injective, and  $\varphi_g(g) \neq 1$ .  $H_g$  has  $P$  for all  $g \in G \setminus \{1\}$ . Therefore  $\prod_{g \in G \setminus \{1\}} H_g$  has  $P$ . But

$$G \cong \varphi(G) < \prod_{g \in G \setminus \{1\}} H_g$$

so  $G$  has  $P$ .  $\square$

REMARK 2.7. Residual properties are related to areas of active research. For example the geometrization conjecture is related to residual finiteness of 3-manifolds.

REMARK 2.8. Let  $G$  be a group. Let  $\text{FQ}(G)$  consist of the finite quotients of  $G$ . Then the following is an open question. Let  $F_2$  be a free group of rank 2. If  $G$  is a residually finite group such that  $\text{FQ}(G) = \text{FQ}(F_2)$  is  $G \cong F_2$ ? Note that  $\text{FQ}(F_2)$  consists of the finite groups generated by two elements. So this is really quite concrete.

Another open question is if  $G_1$  and  $G_2$  are residually finitely presented, then does  $\text{FQ}(G_1) = \text{FQ}(G_2)$  imply  $G_1 \cong G_2$ ?

EXAMPLE 2.5.  $\text{LO}(\mathbb{Z}^n)$  and  $\text{LO}(F_n)$  are both the cantor set.

EXAMPLE 2.6. Let  $B_n$  denote the braid group. As it turns out  $\text{LO}(B_n)$  has isolated points [DD].

The following is a strengthening of the fact that LO is a local property.

WARNING 2.1. At this point it is convenient to make the convention that  $\{1\}$  is *not* LO.

THEOREM 2.11 (Burns-Hale).  $G$  is LO iff every non-trivial finitely generated subgroup  $H < G$  has an LO quotient.

PROOF. ( $\implies$ ):  $G$  is LO implies  $H$  is LO.

( $\impliedby$ ):  $F = \{g_1, \dots, g_n\} \subset G \setminus \{1\}$  for  $n \geq 1$ . We show by induction on  $n$  that the condition on  $F$  in Theorem 2.3 holds. Let  $n = 1$ . Then  $\langle g_1 \rangle$  has an LO quotient by assumption. Therefore  $g_1$  has infinite order, so  $1 \notin S(g_1)$ . Now suppose  $n > 1$ . By assumption, there exists a nontrivial homomorphism  $\varphi : \langle g_1, \dots, g_n \rangle \rightarrow L$  where  $L$  is LO. For some  $m$  there exists

$$\varphi(g_i) = \begin{cases} +1 & 1 \leq i \leq m \\ -1 & m < i \leq n \end{cases}$$

By the induction hypothesis there exists  $\epsilon_1, \dots, \epsilon_m \in \{\pm 1\}$  such that  $1 \notin S(\{g_i^{\epsilon_i} \mid 1 \leq i \leq m\})$ . Let  $<$  be an LO on  $L$ . Define  $\epsilon_i \in \{\pm 1\}$  ( $m < i \leq n$ ) so that

$$(2.12) \quad \varphi(g_i^{\epsilon_i}) > 1$$

Then  $1 \notin S(\{g_i^{\epsilon_i} \mid 1 \leq i \leq n\})$ . □

A group  $G$  is *indicible* if either  $G = \{1\}$  or there is an epimorphism  $G \rightarrow \mathbb{Z}$ .

**Corollary 2.12.**  $G$  is locally indicible implies  $G$  is LO.

REMARK 2.9. Free groups are loc (indicible) so this gives another proof that free groups are LO.

REMARK 2.10. Note that  $G$  having an LO quotient does not imply  $G$  is LO.

COUNTEREXAMPLE 2.  $\mathbb{Z} * \mathbb{Z}/2$  has LO quotient, but is not LO.

We do however have:

THEOREM 2.13. Let  $G$  be a group such that every finitely generated subgroup of infinite index is indicible. Then  $G$  is LO if and only if  $G$  has an LO quotient.

PROOF. ( $\implies$ ): This direction is immediate.

( $\impliedby$ ): Apply Theorem 2.11. Let  $H < G$ ,  $H \neq \{1\}$ , finitely generated.

- Case 1:  $[G : H] = \infty$ . By hypothesis,  $H$  is indicable, so therefore (since  $H$  is nontrivial)  $G$  has quotient  $\mathbb{Z}$ .
- Case 2:  $[G : H]$  finite. By hypothesis there exists an epimorphism  $\varphi : G \rightarrow Q$  where  $Q$  is LO. Therefore  $Q$  is infinite, so  $\varphi(H) \neq \{1\}$ , (since  $[Q : \varphi(H)]$  is finite) and therefore  $H$  has LO quotient  $\varphi(H)$ .

□

REMARK 2.11. It turns out that  $G$  BO implies  $G$  is locally indicable.

REMARK 2.12. We will eventually apply Theorem 2.13 to three-manifold groups. But first we look at surfaces.

## 2. Surface groups

An  $n$ -manifold is a second-countable Hausdorff space  $M$  such that for all  $x \in M$   $x$  has a neighborhood  $U$  such that either

$$(U, x) \cong (\mathbb{R}^n, 0) \quad \text{or} \quad (U, x) \cong (\mathbb{R}_+^n, 0) .$$

Define the interior and boundary as:

$$\begin{aligned} \text{int}(M) &= \{x \in M \mid x \text{ has a neighborhood of the first type}\} \\ \partial M &= \{x \in M \mid x \text{ has a neighborhood of the second type}\} . \end{aligned}$$

Note that  $(\text{int}(M)) \cap \partial M = \emptyset$ . Also note that  $\text{int}(M)$  is an  $n$ -manifold with empty boundary, and  $\partial M$  is an  $(n-1)$ -manifold with empty boundary.  $M$  is *closed* if  $M$  is compact and  $\partial M = \emptyset$ .

A *triangulation* of  $M$  is a homeomorphism  $M \cong |K|$ , where  $K$  is a locally finite simplicial complex. Whether or not a manifold has a triangulation is a subtle question which wasn't settled until recently [M1].

FACT 2. *Every  $n$ -manifold has a triangulation for  $n \leq 3$ .*

This was shown for  $n = 2$  in [R] and for  $n = 3$  in [M5].

For us, a *surface* is a 2-manifold. There is the well-known classification of closed surfaces. In particular, they all either look like  $S^2$ ,  $T^2$ , a connect sum of copies of  $T^2$ , the projective plane  $\mathbb{P}^2$ , or connect sums of copies of  $\mathbb{P}^2$ .

There is also a classification of non-compact surfaces.

EXAMPLE 2.7. Consider the plane. Now attach handles as in fig. 1. This is an infinite genus non-compact surface. Now consider the infinite genus surface in fig. 2. Are these homeomorphic? See remark 2.13 for the answer.

Now we consider the following question.

QUESTION 1. Which surface groups  $\pi_1(S)$  are LO?

We want to use Theorem 2.11, so we will consider finitely generated subgroups of surface groups. First, recall the following.

**Lemma 2.14.** *If  $M$  is a closed  $n$ -manifold,  $N$  is a connected  $n$ -manifold, and  $f : M \rightarrow N$  is an injective map, then  $f$  is a homeomorphism.*

This uses the Jordan-Brouwer theorem for  $S^{n-1}$ s in  $S^n$ . For  $M$  compact,  $N$  Hausdorff, it is enough to show  $f$  is onto.

**Lemma 2.15.** *Let  $S$  be a non-compact surface. Then  $H_2(S) = 0$ .*



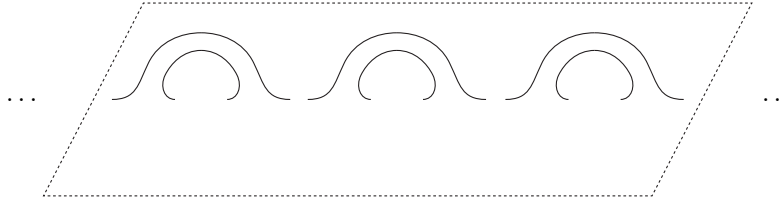


FIGURE 1. The Loch-Ness monster surface obtained by attached infinitely many handles to the plane.

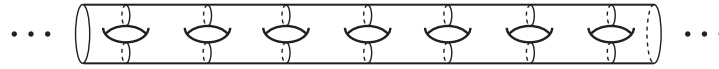


FIGURE 2. The Jacob's ladder surface.

PROOF. Triangulate  $S$ . Then we can get compact surfaces  $S_1 \subset S_2 \dots \subset S$  such that

$$S = \bigcup_{i=1}^n S_i .$$

$\partial S_i \neq \emptyset$  by Lemma 2.14, so  $S_i \simeq$  some 1-complex. Therefore  $H_2(S_i) = 0$ , for all  $i$ . And every 2-cycle in  $S$  is contained in some  $S_i$ . Therefore  $H_2(S) = 0$ .  $\square$

**Lemma 2.16.** *Let  $S$  be a surface,  $\delta$  a circle component of  $\partial S$  such that  $\pi_1(\delta) \rightarrow \pi_1(S)$  is not injective. Then  $S \cong D^2$ .*

PROOF. For  $S$  compact, this is true by the classification. So let  $S$  be non-compact. Let  $S^* = S \cup D^2$  glued along  $\delta$ . Then we have that the following commutes

$$\begin{array}{ccc} \pi_1(\delta) & \longrightarrow & \pi_1(S) \\ \downarrow \cong & & \downarrow \\ H_1(\delta) & \longrightarrow & H_1(S) \end{array} .$$

But now since  $\pi_1(\delta) \rightarrow \pi_1(S)$  is not injective,  $H_1(\delta) \rightarrow H_1(S)$  cannot be injective either. So now applying Mayer-Vietoris, we get

$$(2.13) \quad H_2(S^*) \cong \ker(H_1(\delta) \rightarrow H_1(S)) ,$$

so by definition this is nonzero. But  $S^*$  is noncompact, so this contradicts Lemma 2.15.  $\square$

REMARK 2.13. Have you answered the question from example 2.7 yet? The answer has to do with the number of *ends*, which is defined as follows. Remove compact subsets and count the remaining components. If we minimize the number of components, then this is the number of ends. This is clearly a topological invariant. The loch-ness monster has 1, and Jacob's ladder has 2.

We can also define the notion of the number of ends of a group. As it turns out,  $e(G) = 0$  iff  $G$  is finite. Then, for example, we have

$$\begin{aligned} e(\mathbb{Z}) &= 2 \\ e(\mathbb{Z}^n) &= 1 \quad (n \geq 2) \\ e(F_n) &= \infty. \end{aligned}$$

Then it turns out that for all  $G$ ,  $e(G) = 0, 1, 2$ , or  $\infty$ .

**THEOREM 2.17** (Compact core theorem for surfaces). *Let  $S$  be a connected surface with  $\pi_1(S)$  finitely generated. Then there exists a compact connected  $S_0 \xrightarrow{i} S$  such that  $i_* : \pi_1(S_0) \rightarrow \pi_1(S)$  is an isomorphism. We call  $S_0$  a compact core of  $S$ .*

**PROOF.** Triangulate  $S$ . Let  $\gamma_1, \dots, \gamma_n$  be simplicial loops in  $S$  such that  $\{[\gamma_1], \dots, [\gamma_n]\}$  are generators of  $\pi_1(S)$ . Let  $N$  be a regular neighborhood of  $\bigcup_{i=1}^n \gamma_i$  in  $S$ .  $N$  is a compact surface with  $\partial N \neq \emptyset$  (and we can in fact assume it is connected) and  $\pi_1(N) \rightarrow \pi_1(S)$  is onto.

Let  $S_0$  be  $N$  union with any disk components of  $S$  cut along  $\partial N$ .  $S_0$  is a compact surface, and  $\pi_1(S_0) \rightarrow \pi_1(S)$  is onto. If  $\partial S_0 = \emptyset$  then we are done since  $S_0 = S$ .

So suppose  $\partial S_0 \neq \emptyset$ . Let  $\delta$  be a component of  $\partial S_0$ . Since  $\pi_1(S_0) \rightarrow \pi_1(S)$  is onto,  $\delta$  separates  $S$ . (If not, there exists a loop  $\gamma \subset S$  such that  $\gamma \cap \delta$  is a single point. Therefore  $\gamma$  cannot be in  $S_0$  but  $\pi_1(S_0) \rightarrow \pi_1(S)$  is onto.)

Let  $S_1$  be the component of  $S$  cut along  $\delta$  such that  $S_0 \not\subset S_1$ . By definition of  $S_0$   $S_1$  is not a disk. Therefore by Lemma 2.16  $\pi_1(\delta) \rightarrow \pi_1(S_1)$  is one-to-one. If  $S_0$  is a disk, then  $\pi_1(S) = \{1\}$  and we are done. So assume  $S_0$  is not a disk. Then  $\pi_1(\delta) \rightarrow \pi_1(S_0)$  is injective. So do this for all the boundary components  $\delta$  of  $S_0$ . Then we see by Van-Kampen that this is just a big free product:

$$\pi_1(S) \cong \operatorname{colim} \left( \begin{array}{ccccccc} \pi_1(S_1) & & \pi_1(S_2) & & \pi_1(S_3) & \dots & \pi_1(S_k) \\ & \searrow & & \searrow & \downarrow & & \swarrow \\ & & & & \pi_1(S_0) & & \end{array} \right)$$

but by definition this means  $\pi_1(S_0) \rightarrow \pi_1(S)$  is injective. □

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**REMARK 2.14.** There is an analogue of this theorem for three-manifolds as well. This is related to the the *Whitehead manifold*, which is a contractible three-manifold not homeomorphic to  $\mathbb{R}^3$ . Whitehead invented this as a counterexample to his own theorem. Professor Cameron says this tells us it is okay to make mistakes as long as you're the one to find the counterexample.

**REMARK 2.15.** Now Theorem 2.11 implies that if  $G$  is locally indicable (and nontrivial) then  $G$  is LO.

**THEOREM 2.18.** *Let  $S$  be a surface not homeomorphic to  $\mathbb{RP}^2$ . Then  $\pi_1(S)$  is locally indicable.*

**PROOF.** Let  $H < \pi_1(S)$ ,  $H$  finitely generated and nontrivial. Then we want to show it maps to  $\mathbb{Z}$ . The point is that there exists a connected covering space  $\tilde{S} \rightarrow S$  such that  $\pi_1(\tilde{S}) \cong H$ . By Theorem 2.17,  $H \cong \pi_1(S_0)$  for  $S_0$  a compact surface. Of course  $\pi_1(S_0) \neq \{1\}$  (since  $H$  was).

Now we claim  $S_0 \not\cong \mathbb{RP}^2$ . If it was, then  $\tilde{S} \cong \mathbb{RP}^2$ , so  $S \cong \mathbb{RP}^2$ , which is a contradiction. Not by the classification of compact surfaces, there exists an epimorphism  $H_1(S_0) \twoheadrightarrow \mathbb{Z}$ , so we can just pre-compose with the map  $\pi_1(S_0) \twoheadrightarrow H_1(S_0)$ , so we get an epimorphism  $H \twoheadrightarrow \mathbb{Z}$ .  $\square$

**Corollary 2.19.** *Let  $S$  be a surface. Then  $\pi_1(S)$  is LO if and only if  $\pi_1(S) \neq \{1\}$  and  $S \not\cong \mathbb{RP}^2$ .*

REMARK 2.16. (1) If  $S$  is the Klein bottle then  $\pi_1(S)$  is locally indicable. But  $\pi_1(S)$  is not BO (there exists  $a \in \pi_1(S)$  such that  $a$  is conjugate to  $a^{-1}$ ). This shows:

- (a) locally indicable and nontrivial does not imply BO, and
- (b) there is no analog of Burns Hale for BO.
- (2) Locally indicable (and nontrivial) implies LO, but the converse is false. We will see that there are three manifolds  $M$  with  $H_1(M)$  finite<sup>2.2</sup> and  $\pi_1(M)$  LO.
- (3) It can be shown that if  $S$  is a non-compact surface, then  $\pi_1(S)$  is free. For example,  $\mathbb{R} \setminus$  a cantor set has  $\pi_1$  isomorphic to a free group on a countably infinite number of generators.
- (4) It can be shown that  $\pi_1(S) = 1$  if and only if  $S \cong S^2$  or  $D^2 \setminus X$  for  $X$  a closed subgroup of  $S^1$ .

REMARK 2.17. Colin Adams is a knot theorist who gives lectures in different personas. E.g. a sleazy real-estate agent selling property in hyperbolic space. Once he attended a class posing as a student. He started heckling the lecturer, and eventually the lecturer said “well if you know so much, you come teach the class!” so he did. Some of the students were responding to his heckling, saying “shut up man, he’s doing a great job!” so they were in for surprise when he revealed who he is.

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<sup>2.2</sup>So in particular  $\pi_1(M)$  is not locally indicable.

## CHAPTER 3

### Three-manifolds

Our three-manifolds will always be connected, orientable. They may have boundary and may be non-compact. We will always be working in the PL or smooth category.

Let  $M_1$  and  $M_2$  be oriented 3-manifolds with balls  $B_i \subset \text{int}(M_i)$ ,  $B_i \cong B^3$  for  $i = 1, 2$ . The *connect sum* of  $M_1$  and  $M_2$  is the oriented manifold

$$M_1 \# M_2 = (M_1 \setminus \text{int}(B_1)) \cup_h (M_2 \setminus \text{int}(B_2))$$

for  $h : \partial B_1 \rightarrow \partial B_2$  an orientation-reversing homeomorphism. It turns out that  $M_1 \# M_2$  is well-defined (up to orientation-preserving homeomorphism). The operation  $\#$  is associative, and commutative. Note that  $M \# S^3 \cong M$  for all  $M$ . Also note that

$$\pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2) .$$

We say  $M$  is *prime* if  $M \cong M_1 \# M_2$  implies  $M_1$  or  $M_2 \cong S^3$ .

**THEOREM** (Kneser[K2], Milnor[M3]). *Let  $M$  be a compact, oriented 3-manifold. Then*

$$M \cong \#_{i=1}^n M_i$$

*(orientation preserving (op)) where  $M_i$  is prime (and not  $\cong S^3$ ) for  $1 \leq i \leq n$ . Moreover the  $M_i$  are unique up to order and orientation-preserving homeomorphism.*

Note  $S^3$  corresponds to  $n = 0$ .

**REMARK 3.1.** In the same paper [K2] Kneser proved some other things which relied on Dehn's lemma. So he was looking closer at Dehn's proof, and found some holes. He wrote to Dehn who was on vacation to find out that he agreed there was something fishy. Thus ensued a great correspondence between the two trying to fix it. It was eventually fixed in [P1].

For  $M$  compact, and

$$M \cong \#_{i=1}^n M_i$$

where the  $M_i$  are prime, we have

$$\pi_1(M) \cong \bigast_{i=1}^n \pi_1(M_i) .$$

So  $\pi_1(M)$  is LO iff  $\pi_1(M_i)$  is LO (for  $1 \leq i \leq n$ ). This is also true for BO.

**EXERCISE 3.1.** Show  $\pi_1(M)$  is locally indicable iff  $\pi_1(M_i)$  is locally indicable for all  $1 \leq i \leq n$ . [Hint: Use the Kurosh subgroup theorem.]

The upshot is that for  $M$  compact, to answer LO, BO, or locally indicable, we may assume  $M$  is prime.

**REMARK 3.2.** There are noncompact three manifolds that cannot be expressed as  $\#$  of prime manifolds.

$M$  is irreducible if every  $S^2 \subset M$  bounds a  $B^3 \subset M$ .

FACT 3.  $M$  is irreducible iff  $M$  is prime and not homeomorphic (op) to  $S^1 \times S^2$ .

The point being that  $S^1 \times S^2$  is prime.

THEOREM (Perelman[**P2, P4, P3**]). Let  $M$  be a closed 3-manifold with universal cover  $\tilde{M}$ .

- (1) If  $\pi_1(M)$  is finite, then  $\tilde{M} \cong S^3$  and the action of  $\pi_1(M)$  on  $S^3$  is as a subgroup of  $\mathrm{SO}(4)$ .
- (2) If  $\pi_1(M)$  is infinite and  $M$  is irreducible, then  $\tilde{M} \cong \mathbb{R}^3$ .

COROLLARY (Poincaré conjecture). If  $M$  is closed and  $\pi_1(M) = 1$ ,  $M \cong S^3$ .

REMARK 3.3. We know  $\pi_1(M)$  infinite implies  $\tilde{M}$  is noncompact. Then  $M$  irreducible implies  $\pi_2(M) = 0$  (as we will see soon) so by standard stuff,  $\tilde{M}$  is contractible. But, there are contractible non-compact 3-manifolds without boundary which are not homeomorphic to  $\mathbb{R}^3$ .

The 3-manifolds with  $\pi_1$  finite can be completely described. They're all Seifert fiber spaces.

EXAMPLE 3.1. Let  $p, q \in \mathbb{Z}$  such that  $p \geq 2$   $(p, q) = 1$ . Recall we have a  $\mathbb{Z}/p$  action on  $\mathbb{C}^2$  by

$$(z, w) \mapsto (e^{2\pi i/p} z, e^{2\pi q i/p} w)$$

Now the restriction of this action to  $S^3$  is free, so we can quotient by it to get the lens space  $L(p, q)$ . Then

$$\pi_1(L(p, q)) = \mathbb{Z}/p.$$

Nonetheless, Alexander showed that  $L(5, 1) \not\cong L(5, 2)$ .

THEOREM (Redemeister).  $L(p, q)$  is homeomorphic to  $L(p, q')$  iff either  $q \cong q' \pmod{p}$  or  $qq' \cong 1 \pmod{p}$ .

The  $\Leftarrow$  direction is easy.

THEOREM (Perelman[**P2, P4, P3**]). For  $M$  and  $M'$  closed three-manifolds,  $M$  prime and not a lens space, then  $\pi_1(M) \cong \pi_1(M')$  implies  $M' \cong M$ .

So “prime three-manifolds are pretty much determined by their fundamental group”.

REMARK 3.4. The restriction to prime is necessary here. Let  $M$  be an oriented three-manifold such that  $M$  is not homeomorphic (op) to  $-M$ . For example,  $M = L(3, 1)$  or the Poincaré homology sphere.

Then  $\pi_1(M \# M) \cong \pi_1(M \# (-M))$ , but by prime decomposition,  $M \# M \not\cong M \# (-M)$ .

Lecture 11;  
February 25, 2020

## 1. Higher homotopy groups

We will give a basic overview of higher homotopy groups. Define

$$(3.1) \quad \pi_n(X, x_0) = \left\{ \text{homotopy classes of maps } (S^n, s_0) \xrightarrow{f} (X, x_0) \right\}.$$

As it turns out, we can define a composition on this set with respect to which this becomes a group. This is called the  $n$ th homotopy group of  $(X, x_0)$ . As soon as  $n \geq 2$ , this group

is abelian. Note that for  $f : (X, x_0) \rightarrow (Y, y_0)$ , we get an induced homomorphism  $f_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$ . I.e. the  $\pi_n$  are covariant functors. If  $X$  is path-connected, this implies that for all  $x_0, x_1 \in X$ ,

$$(3.2) \quad \pi_n(X, x_0) \cong \pi_n(X, x_1) .$$

Let  $f : S^n \rightarrow X$ . This induces a homomorphism

$$(3.3) \quad f_* : \underbrace{H_n(S^n)}_{\cong \mathbb{Z}} \rightarrow H_n(X) .$$

Now  $f \simeq g$  implies  $f_* = g_*$ , so we can define

$$(3.4) \quad h : \pi_n(X) \rightarrow \pi_n(X)$$

by sending  $h([f]) = f_*(1)$ . This is well-defined, and it can be shown that this is in fact a homomorphism. This is called the Hurewicz homomorphism. For  $n = 1$ ,  $h$  is just the abelianization.

Any covering projection  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  induces an isomorphism  $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$  for  $n \geq 2$ . This follows from the fact that  $\pi_1(S^n) = 1$  if  $n \geq 2$ , and the lifting criterion.

## 2. Back to three-manifolds

Let  $M$  be a connected, orientable (sometimes oriented) 3-manifold, possibly non-compact and possibly with boundary. Let  $S \subset M$  be an embedded 2-sphere. We say  $S$  is *essential*, if  $S \not\simeq \text{pt}$ . I.e. the homotopy class  $[S] \neq 0 \in \pi_2(M)$ . We say  $S$  is *incompressible* if  $S$  does not bound a 3-ball in  $M$ . Recall a manifold  $M$  is *irreducible* if and only if every 2-sphere bounds a ball, i.e. every 2-sphere is compressible.

**THEOREM (Sphere theorem [P1], [W1]).** *Let  $M$  be a 3-manifold with  $\pi_2(M) \neq 0$ . Then there exists an embedded essential 2-sphere in  $M$ .*

This implies the asphericity of knots: if  $K$  is a knot in  $S^3$ , then  $\pi_2(S^3 \setminus K) = 0$ . The proof uses the “tower construction”.

**THEOREM 3.1.** *Let  $M$  be a three-manifold and  $S$  a 2-sphere in  $M$ . Then  $S$  is incompressible if and only if  $S$  is essential.*

**PROOF.** ( $\Leftarrow$ ): If  $S$  bounds a  $B^3 \subset M$  then  $S \simeq \text{pt}$ .

( $\Rightarrow$ ): Let  $S$  be incompressible. Suppose (for a contradiction) that  $S \simeq \text{pt}$ , i.e.  $[S] = 0 \in \pi_2(M)$ , so  $[S] = 0 \in H_2(M)$  (after applying  $h$  from above). Therefore  $S$  separates  $M$ . [Recall for  $S$  non-separating, then there is a loop  $\gamma \subset M$  such that  $\gamma$  meets  $S$  transversely in 1 point, so  $[S] \neq 0 \in H_2(M)$ .] So this separates  $M$  into  $M_1$  and  $M_2$ . So  $M = M_1 \cup_S M_2$ . Now  $[S] = 0 \in H_2(M)$  implies  $S$  bounds a 3-chain in  $M$ . Therefore  $M_1$  (say) is compact and  $\partial M_1 = S$ .

If  $\pi_1(M_1) = 1$ , then  $M_1 \cong B^3$  (by the Poincaré conjecture). Therefore  $S$  is compressible, which is a contradiction. So  $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$ .

Case (1)  $\pi_1(M_2) \neq 1$ : Let  $\tilde{M}$  be the universal covering of  $M$ . Then  $\pi_1(M_1) \neq 1$ , so  $\tilde{M}_1$  has more than one boundary component. Let  $\tilde{S} \subset \tilde{M}$  be a lift of  $S$ . Then every component of  $\tilde{M}$  cut along  $\tilde{S}$  is non-compact. Therefore  $\tilde{S}$  does not bound a (finite) 3-chain in  $\tilde{M}$ , so  $[\tilde{S}] \neq 0 \in H_2(\tilde{M})$  so  $[\tilde{S}] \neq 0 \in \pi_2(\tilde{M})$ . Therefore

$$(3.5) \quad [S] = p_*([\tilde{S}]) \neq 0 \in \pi_2(M) .$$

Case (2)  $\pi_1(M_2) = 1$ : Then  $\tilde{M}$  is  $\tilde{M}_1$  with some copies of noncompact  $M_2$  attached. By hypothesis  $S$  is incompressible so  $M_2 \not\cong B^3$ . Therefore  $\pi_1(M_2) = 1$ , so either  $M_2$  is noncompact or  $\partial M_2 \setminus S \neq \emptyset$ . Now pick some lift  $\tilde{S}$ . It cannot bound a finite chain in either direction since both either have nontrivial boundary, or are noncompact. So as before,  $[\tilde{S}] \neq 0 \in H_2(\tilde{M})$ , so nonzero in  $\pi_1(\tilde{M})$ , so  $[S] \neq 0 \in \pi_1(M)$ .  $\square$

**Corollary 3.2.** *If  $M$  is a 3-manifold, then  $M$  is irreducible if and only if  $\pi_2(M) = 0$ .*

PROOF. Combine Theorem 3.1 with the sphere theorem.  $\square$

**Corollary 3.3.** *For  $\tilde{M} \rightarrow M$  a covering,  $M$  is irreducible iff  $\tilde{M}$  is irreducible.*

PROOF. This follows from Theorem 3.1 since  $\pi_2(\tilde{M}) \cong \pi_2(M)$ .  $\square$

REMARK 3.5. This falls out for nothing by Theorem 3.1, but we did use the Poincaré conjecture in our proof. This was actually known before the Poincaré conjecture. The converse is easy, but the forward implication is hard.

THEOREM. *Let  $M$  be a three-manifold with  $\pi_1(M)$  finitely generated. Then there exists a compact three-manifold  $N \xrightarrow{i} M$  such that  $i_* : \pi_1(N) \rightarrow \pi_1(M)$  is an isomorphism.*

Recall a group  $G$  is *coherent* if  $H < G$  finitely generated implies it is finitely presentable.

COROLLARY. *For  $M$  a three-manifold,  $\pi_1(M)$  is coherent.*

PROOF. For  $H$  a finitely generated subgroup of  $\pi_1(M)$ ,  $H \cong \pi_1(\tilde{M})$  for  $\tilde{M} \rightarrow M$  a covering. Then apply the Scott core theorem to  $\tilde{M}$ .  $\square$

REMARK 3.6.  $\mathrm{SL}_2(\mathbb{Z})$  is *virtually* free. (It has a free subgroup of finite index.)

Free groups are coherent, so  $\mathrm{SL}(2, \mathbb{Z})$  is coherent. We know  $\mathrm{SL}_n(\mathbb{Z})$  is incoherent for  $n \geq 4$ .

QUESTION 2 (Serre). Is  $\mathrm{SL}_3(\mathbb{Z})$  coherent?

REMARK 3.7. Once at a conference in the UK, Professor Gordon was playing table tennis. He asked a guy standing nearby if he wanted to play, and the guy beat him. He asked for his name and he replied “Jean-Pierre Serre”.

**Lemma 3.4.** *Let  $M$  be a closed orientable  $n$ -manifold,  $n$  odd. Then  $\chi(M) = 0$ .*

PROOF. Let  $\mathbb{F}$  be a field. By Poincaré duality,

$$(3.6) \quad H_i(M; \mathbb{F}) \cong H^{n-i}(M; \mathbb{F}) \cong H_{n-i}(M; \mathbb{F})$$

where the second equality follows from the universal coefficient theorem. Therefore

$$(3.7) \quad \chi(F) = \sum_{i=0}^n (-1)^i \dim H_i(M; \mathbb{F}) = 0$$

if  $n$  is odd.  $\square$

**Lemma 3.5.** *Let  $M$  be a compact three-manifold and  $\mathbb{F}$  a field. Then*

$$\dim H_1(M; \mathbb{F}) \geq \frac{1}{2} \dim H_1(\partial M; \mathbb{F}) .$$

PROOF. Let  $2M$  be the *double* of  $M$ . This is the union of  $M$  with  $M$  (i.e.  $M$  with the opposite orientation) glued along the boundary. Let  $n$  be the number of  $\partial$ -components of  $M$ . Then by Lemma 3.4,

$$(3.8) \quad 0 = \chi(2M) = 2\chi(M) - \chi(\partial M)$$

which means  $\chi(M) = \chi(\partial M)/2$ .

We will use the Betti number notation:

$$(3.9) \quad \beta_2(M) = \dim H_2(M; \mathbb{F}) .$$

Then

$$(3.10) \quad \chi(M) = 1 - \beta_1(M) + \beta_2(M)$$

Then

$$(3.11) \quad \chi(\partial M) = n - \beta_1(\partial M) + n .$$

Now by the universal coefficient theorem and Poincaré-Lefschetz duality we get

$$(3.12) \quad H_2(M; \mathbb{F}) \cong H^2(M; \mathbb{F}) \cong H_1(M, \partial M; \mathbb{F}) .$$

So now we have the exact sequence of the pair  $(M, \partial M)$ :

$$(3.13) \quad \dots \longrightarrow H_1(M, \partial M; \mathbb{F}) \longrightarrow \underbrace{H_0(\partial M; \mathbb{F})}_{\mathbb{F}^n} \longrightarrow \underbrace{H_0(M; \mathbb{F})}_{\mathbb{F}} \longrightarrow 0$$

which means

$$(3.14) \quad \beta_2(M) = \beta_1(M, \partial M) \geq n - 1 .$$

Therefore

$$(3.15) \quad 1 - \beta_1(M) + n - 1 \leq \chi(M) = \frac{1}{2}\chi(\partial M) = n - \frac{1}{2}\beta_1(\partial M)$$

so

$$(3.16) \quad \beta_1(M) \geq \frac{1}{2}\beta_1(\partial M)$$

as desired.  $\square$

**Corollary 3.6.** *Let  $M$  be a compact 3-manifold with a boundary component not homeomorphic to  $S^2$ . Then  $H_1(M)$  is infinite.*

PROOF.  $\dim H_1(M; \mathbb{R}) \geq 1$  and by the universal coefficient theorem, this is  $H_1(M) \otimes \mathbb{R} \cong \mathbb{R}^n$ . Then this is equivalent to

$$(3.17) \quad H_1(M) \cong \mathbb{Z}^n \oplus A$$

for  $A$  some finitely abelian group.  $\square$

**Lemma 3.7.** *Let  $M$  be a prime three-manifold,  $H$  a finitely generated subgroup of  $\pi_1(M)$  such that either*

- (1)  *$H$  has infinite index in  $\pi_1(M)$ , or*
- (2)  *$M$  is not closed.*

*Then  $H$  is indicable (i.e. either  $H = \{1\}$  or there is some epimorphism  $H \rightarrow \mathbb{Z}$ ).*



PROOF. If  $M = S^1 \times S^2$  this is clear, since  $\pi_1(M) \cong \mathbb{Z}$ . So we may assume  $M$  is irreducible. In case (2), replacing  $M$  by  $M \setminus \partial M$ , we may assume  $M$  is noncompact. There exists a covering  $\tilde{M} \rightarrow M$  with  $\pi_1(\tilde{M}) \cong H$ . In both cases,  $\tilde{M}$  is noncompact. Now we can use the Scott Core theorem to show that there is some compact submanifold  $N$  of  $\tilde{M}$  such that the inclusion  $\pi_1(N) \xrightarrow{\cong} \pi_1(\tilde{M})$  is an isomorphism. Let  $S$  be a 2-sphere component of  $\partial N$ . Corollary 3.3 implies  $\tilde{M}$  is irreducible. Therefore  $S$  bounds a 3-ball  $B \subset \tilde{M}$ . If  $N \subset B$ , then  $\pi_1(N) \rightarrow \pi_1(\tilde{M})$  factors through  $\pi_1(B) = 1$ . So  $\pi_1(\tilde{M}) \cong H = \{1\}$ . So  $H$  is indicable. So assume  $H \neq \{1\}$ . Then we have  $N \not\subset B$ . So replace  $N$  with  $N \cup B$ , so  $\pi_1(N \cup B) \cong \pi_1(N)$ . So now we can assume  $N$  has no 2-sphere boundary components. If  $N$  is closed, then  $N = \tilde{M}$ , but this contradicts the fact that  $\tilde{M}$  is noncompact. Therefore  $\partial N \neq \emptyset$ , so by Corollary 3.6  $H_1(N)$  is infinite, so  $\pi_1(N) \cong H$  maps onto  $\mathbb{Z}$ , so  $H$  is indicable.  $\square$

**Corollary 3.8.** *For  $M$  a prime 3-manifold,  $\pi_1(M)$  is infinite. Then  $\pi_1(M)$  is torsion-free.*

PROOF. Suppose we have an element  $g \in \pi_1(M)$  of finite order  $n > 1$ . Therefore  $\langle g \rangle \cong \mathbb{Z}/n$ . But this has infinite index in  $\pi_1(M)$ , but by Lemma 3.7 this means it is indicable. But of course it is nontrivial and does not map onto  $\mathbb{Z}$ , so this is a contradiction.  $\square$

Let  $M$  be a closed three-manifold with  $H_1(M; \mathbb{Z}) = 0$ . Then

$$(3.18) \quad H_2(M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}) = 0$$

where the first isomorphism follows from Poincaré duality, and the second follows from the universal coefficient theorem. Now note that  $H_3(M; \mathbb{Z}) \cong H^0(M; \mathbb{Z}) \cong \mathbb{Z}$ , and  $H_q(M; \mathbb{Z}) = 0$  for  $q \geq 4$ , so

$$(3.19) \quad H_*(M; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z}) .$$

In this case we say  $M$  is an integral homology sphere, or  $\mathbb{Z}$ HS. Similarly, if  $H_1(M; \mathbb{Q}) = 0$  (equivalently  $H_1(M; \mathbb{Z})$  finite) then

$$(3.20) \quad H_*(M; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q}) .$$

In this case we say  $M$  is a rational homology sphere, or  $\mathbb{Q}$ HS. As it turns out, there are infinitely many  $\mathbb{Z}$ HS's  $M$  with  $\pi_1(M) \neq 1$ . There are also infinitely many  $\mathbb{Q}$ HS's that are not  $\mathbb{Z}$ HS's. For example, lens spaces are  $\mathbb{Q}$ HS's.

Let  $K \subset S^3$  be a knot in  $S^3$ . Then the tubular neighborhood is  $N(K) \cong S^2 \times D^2$ . Now take

$$(3.21) \quad X = \text{Cl}(S^3 \setminus N(K)) .$$

Note that  $\partial X \cong T^2$ . Take  $\mu$  to be some meridional curve in the boundary, and  $\lambda$  some longitudinal one. There is (up to isotopy) a unique such  $\lambda$  in  $X$  such that  $\lambda \sim 0$ . Now for any  $\alpha$  an essential simply closed curve in  $\partial X$ ,  $\alpha \sim p\mu = q\lambda$  for  $(p, q) = 1$ . Now consider

$$(3.22) \quad X \cup_h (S^1 \times D^2)$$

where  $h : S^1 \times D^2 \rightarrow \partial X$  such that

$$h(\text{pt} \times \partial D^2) = \alpha .$$

So this gives a closed three-manifold  $K(\alpha) = K(p/q)$ . This is called the  $\alpha$  (or  $p/q$ ) Dehn surgery on  $K$ . Recall  $H_1(X) \cong \mathbb{Z}$  given by  $\mu$ . So when we add in this solid torus, we're killing  $\alpha$ .  $\lambda$  was dead anyway, so we're really just killing the  $p$ th power of the meridian. So

$$(3.23) \quad H_1(K(p/q)) \cong \mathbb{Z}/p.$$

For  $p = 1$  we get an ZHS, and for  $p > 1$  we get a QHS. The following is an open question:

QUESTION 3. Does a prime ZHS (which is not  $S^3$  or the Poincaré homology sphere) have left-orderable fundamental group?

This is related to the so-called  $L$ -space conjecture.

THEOREM 3.9. *Let  $M$  be a prime 3-manifold. Then  $\pi_1(M)$  is locally indicable. Equivalently,  $M$  is  $S^3$ , or  $M$  is not a QHS.*

PROOF. ( $\implies$ ): Suppose  $M$  is a QHS. Then  $\pi_1(M)$  is finitely generated. Therefore  $\pi_1(M)$  LI implies  $\pi_1(M)$  is indicable, so  $\pi_1(M) = 1$  or  $H_1(M)$  is infinite.

( $\impliedby$ ): Suppose  $M$  is not a QHS. Then either

- (i)  $M$  is closed and  $H_1(M)$  is infinite, or
- (ii)  $M$  is not closed.

Let  $H$  be a finitely generated subgroup of  $\pi_1(M)$ . In the first case,  $[\pi_1(M) : H] = \infty$  so  $H$  is indicable by Lemma 3.7. If  $[\pi_1(M) : H]$  is finite, then  $[H_1(M) : h(H)]$  is finite. Therefore  $h(H)$  is infinite, and so  $H$  maps onto  $\mathbb{Z}$ . Case (ii) follows from Lemma 3.7.  $\square$

Since (for  $G \neq \{1\}$ ) locally indicable implies LO (from Corollary 2.12).

**Corollary 3.10.** *If  $M$  is a prime 3-manifold which is not a QHS, then  $\pi_1(M)$  is LO.*

In fact, we can do better, using Theorem 2.13.

THEOREM 3.11 ([BRW]). *Let  $M$  be a prime three-manifold. Then  $\pi_1(M)$  is LO iff  $\pi_1(M)$  has a LO quotient.*

PROOF. ( $\implies$ ): this direction is trivial.

( $\impliedby$ ): this follows from Lemma 3.7 and Theorem 2.13.  $\square$

## CHAPTER 4

### Seifert fiber(ed) spaces

DEFINITION 4.1. A *Seifert fiber space* (SFS) is a compact (orientable) 3-manifold  $M$  that is a disjoint union of circles (called the *fibers*) such that each circle has a neighborhood which is a union of fibers and is isomorphic to a *fibered solid torus*. This is

$$(4.1) \quad (D^2 \times I) / ((x, 1) \sim (h(x), 0))$$

where  $h : D^2 \rightarrow D^2$  is rotation through  $2\pi q/p$  for  $p > 0$  and  $(p, q) = 1$ . The fibers are the images of  $0 \times I$  (the central fiber) and

$$(4.2) \quad (x \times I) \cup (h(x) \times I) \cup \dots \cup (h^{p-1}(x) \times I)$$

for  $x \neq 0$ . If  $p > 1$ , the central fiber is *exceptional*: other fibers are *ordinary*.

$M$  compact implies there are only finitely many exceptional fibers. Let  $\pi : M \rightarrow F$  be the quotient map defined by identifying each fiber to a point. For a fibered solid torus, the quotient  $\cong D^2$ . So  $F$  is a surface (called the base surface) and  $\pi : \partial M \rightarrow \partial F$  is an  $S^1$ -bundle projection. Therefore  $\partial M$  is the disjoint union of finitely many tori (possibly empty). If  $M$  has no exceptional fibers, then  $\pi : M \rightarrow F$  is an  $S^1$ -bundle projection. Conversely, an orientable  $S^1$ -bundle over  $F$  is a SFS.

Let  $N_0 \subset \text{int}(M)$  be a fibered solid torus neighborhood of an ordinary fiber, and (disjoint)  $N_i \subset \text{int}(M)$  solid tori neighborhood of the exceptional fibers with parameters  $p_i, q_i$ .

Let  $D_i = \pi(N_i)$ . These are disks in  $F$ . Then the restriction

$$(4.3) \quad \pi : \underbrace{(M \setminus \bigcup_{i=1}^n N_i)}_{M_0} \rightarrow \underbrace{F \setminus \bigcup_{i=1}^n D_i}_{F_0}$$

is an  $S^1$ -bundle projection. Let  $\alpha_1, \dots, \alpha_k \subset F_0$  be disjoint properly embedded arcs such that if we cut  $F_0$  along  $\bigcup_{i=1}^k \alpha_i$ , to get  $F_0|_{\bigcup_{i=1}^k \alpha_i}$ , this is a disk  $B$ . Then  $\pi^{-1}\alpha_i$  is an  $S^1$  bundle over  $\alpha_i$ , i.e. an annulus  $A_i \subset M_0$ . So we have an  $S^1$ -bundle over  $B$  given by

$$(4.4) \quad M_0|_{\bigcup_{i=1}^k A_i} \cong B \times S^1$$

with copies  $A_i^\pm$  of  $A_i$  inside  $\partial(B \times S^1)$ . Therefore we can recover  $M_0$  by taking the quotient:

$$M_0 \cong (B \times S^1) / (A_i^+ \sim A_i^-, 1 \leq i \leq k) .$$

We can isotope these identifications so that  $\alpha_i^- \sim \alpha_i^+$ . So we actually get a copy of  $F_0 \subset M_0$  which is a section of the  $S^1$ -bundle:

$$(4.5) \quad \begin{array}{c} F_0 \subset M_0 \xrightarrow{\pi} F_0 \\ \quad \quad \quad \searrow \text{id} \nearrow \\ \quad \quad \quad F_0 \end{array} .$$

Note that  $\pi_1(B \times S^1) \cong \mathbb{Z}$ , generated by the class of the ordinary fiber, written  $h$ . Now there are two cases:

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- (O)  $F$  orientable:  $g(F) = g \geq 0$ ,  $|\partial F| = m \geq 0$ . Now  $\pi_1(F_0)$  has generators  $a_1, b_1, \dots, a_g, b_g$  corresponding to the genus;  $d_1, \dots, d_m$  corresponding to the boundary components; and then  $c_1, \dots, c_n$  for the exceptional fibers. Then since  $M_0 \cong F_0 \times S^1$ , we have that

$$(4.6) \quad \pi_1(M_0) \cong \pi_1(F_0) \times \mathbb{Z}$$

where this copy of  $\mathbb{Z}$  is generated by  $h$ .

- (N)  $F$  non-orientable: Now  $\pi_1(F_0)$  has generators  $a_1, \dots, a_g$  (where these now come from twisted strips);  $d_1, \dots, d_m$ ; and  $c_1, \dots, c_n$ . Now  $M_0$  is a twisted  $S^1$ -bundle over  $F_0$ . The presentation is explicitly:

$$(4.7) \quad \pi_1(M_0) \cong \left\langle a_1, \dots, a_g, d_1, \dots, d_m, c_1, \dots, c_n, h \left| \begin{array}{l} a_i^{-1} h a_i = h^{-1}, 1 \leq i \leq g, \\ h \leftrightarrow d_i, h \leftrightarrow c_i \end{array} \right. \right\rangle.$$

In both cases,  $\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z}$  generated by  $h$  and  $C_i$  for  $0 \leq i \leq n$ . Now let  $\lambda_i$  and  $\mu_i$  be a longitude, meridian pair for  $N_i$ ,  $0 \leq i \leq n$ . Then  $\pi_1(\partial N_i) \cong \mathbb{Z}^2$  is generated by  $\{\lambda_i, \mu_i\}$ . In terms of this basis, an ordinary fiber is:

$$(4.8) \quad h = \lambda_i^{p_i} \mu_i^{q_i}$$

where the  $(p_i, q_i)$  are the Seifert invariants of the fibered solid torus  $N_i$ . Then we can write the  $c_i$ 's as:

$$(4.9) \quad c_i = \lambda_i^{r_i} \mu_i^{s_i}$$

where  $p_i s_i - q_i r_i = 1$ . Therefore we get the relation

$$(4.10) \quad \mu_i = c_i^{p_i} h^{-r_i}.$$

Therefore by van Kampen

$$(4.11) \quad \pi_1(M) \cong \pi_1(M_0) / (c_i^{p_i} = h^{r_i}, 0 \leq i \leq n).$$

For  $p_0 = 1$  let  $b = -r_0$ . Note that (in  $\pi_1(F)$ )

$$(4.12) \quad c_0^{-1} = \begin{cases} \prod_{i=1}^g [a_i, b_i] \prod_{i=1}^m d_i \prod_{i=1}^n c_i & \text{Case (O)} \\ \prod_{i=1}^g a_i^2 \prod_{i=1}^m d_i \prod_{i=1}^n c_i & \text{Case (N)} \end{cases}$$

THEOREM 4.1. *Let  $M$  be a SFS. Then in case (O),  $\pi_1(M)$  has presentation:*

$$(4.13) \quad \left\langle a_1, b_1, \dots, a_g, b_g, d_1, \dots, d_m, c_1, \dots, c_n, h \left| \begin{array}{l} h \leftrightarrow a_i, b_i, d_i, c_i; c_i^{p_i} = h^{r_i}; \\ \prod [a_i, b_i] \prod d_i \prod c_i = h^b \end{array} \right. \right\rangle.$$

*In case (N),  $\pi_1(M)$  has presentation:*

$$(4.14) \quad \left\langle a_1, \dots, a_g, d_1, \dots, d_m, c_1, \dots, c_n, h \left| \begin{array}{l} a_i^{-1} h a_i = h^{-1}; h \leftrightarrow c_i, d_i; c_i^{p_i} = h^{r_i}; \\ \prod a_i^2 \prod d_i \prod c_i = h^b \end{array} \right. \right\rangle.$$

REMARK 4.1. Recall the  $a_i$  (and  $b_i$ ) come from the base surface, (in the (N) case these are orientation reversing) the  $d_i$  come from the boundary components, the  $c_n$  come from the singular fibers.

REMARK 4.2. (1)  $\langle h \rangle$  is central in  $\pi_1(M)$  in case (O), and normal in case (N).

- (2) The fundamental group of  $F$  is

$$\pi_1(F) \cong \pi_1(M) / (h = 1, c_i = 1, 1 \leq i \leq n)$$

so

$$\pi_* : \pi_1(M) \rightarrow \pi_1(F)$$

is onto.

- (3) Suppose  $n > 0$ , i.e. we have at least one singular fiber. Let  $\alpha$  be a properly embedded arc in  $F_0$  joining  $\partial$ -components  $C_i$  and  $C_j$  for  $0 \leq i, j \leq n, i \neq j$ . Then  $\pi^{-1}(\alpha)$  is an annulus  $H$  in  $M_0$ , with boundary components in  $\partial N_i$  and  $\partial N_j$ . Now by Dehn twisting  $M_0$  along  $A$ ,  $k$  times gives a homeomorphism  $M_0 \rightarrow M_0$ . This is the identity outside of a neighborhood of  $A$ , and

$$\begin{array}{lll} \text{on } \partial N_i & h \mapsto h & c_i \mapsto c_i h^k = c'_i \\ \text{on } \partial N_j & h \mapsto h & c_j \mapsto c_j h^{-k} = c'_j \end{array}$$

DIGRESSION 1 (Dehn twists). Let  $A$  be a properly embedded<sup>4.1</sup> annulus inside of a 3-manifold  $M$ .  $A$  has a neighborhood  $N(A) \cong A \times I \subset M$ . Then Dehn twist along  $A$  is the homeomorphism  $h : M \rightarrow M$  such that  $h|_{\text{Cl}(M \setminus A)} = \text{id}$ , and  $h|_{A \times I} : A \times I \rightarrow A \times I$  sends

$$(4.15) \quad ((\theta, s), t) \mapsto ((\theta = 2\pi t, s)) .$$

So the meridians go to:

$$\mu_i = c_i^{p_i} h^{-r_i} \mapsto c_i'^{p_i} h^{-r_i'} \quad \mu_j = c_j^{p_j} h^{-r_j} \mapsto c_j'^{p_j} h^{-r_j'}$$

where  $r_i' = r_i + kp_i$  and  $r_j' = r_j - kp_j$ . Therefore

$$(4.16) \quad \sum_{i=0}^n \frac{r_i}{p_i}$$

is unchanged. Now recall  $p_0 = 1, r_0 = -b$  so

$$(4.17) \quad -b + \sum_{i=1}^n \frac{r_i}{p_i}$$

is also unchanged. This is called the Euler number of the Seifert structure.

When  $n = 0$ , we just have a circle bundle, and this gives us the Euler number of this circle bundle. Using this, we can either normalize the  $r_i$  so that  $0 < r_i < p_i$ , or if  $n > 0$ , we can take  $b = 0$ .

- (4) One can show that  $M$  is irreducible unless  $M \cong S^1 \times S^2$  or  $\mathbb{RP}^3 \# \mathbb{RP}^3$ .  
 (5) For most SFS's, the SF structure is unique.

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<sup>4.1</sup>Recall this means  $A \cap \partial M = \partial A$ .

- (6) It follows from Perelman that if  $M$  is a closed three-manifold with  $\pi_1(M)$  finite, then  $M$  is a SFS. Those with finite  $\pi_1$  are

$$\begin{array}{llll} F = S^2 & n \leq 2 & M \not\cong S^1 \times S^2 & (S^2, \text{lens spaces}) \\ F = S^2 & n = 3 & \sum_{i=1}^3 \frac{1}{p_i} > 1 & \left( \begin{array}{l} \text{platonic triples : } (2, 2, p), \\ (2, 3, 3), (2, 3, 4), (2, 3, 5) \end{array} \right) \\ F = \mathbb{RP}^2 & n = 0, b \neq 0 & & (n = b = 0 \leadsto \mathbb{RP}^3 \# \mathbb{RP}^3) \\ & n = 1 & & \end{array}$$

- (7) SFS's are one of the building blocks of the JSJ decomposition of three-manifolds. The other pieces are hyperbolic three-manifolds.

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**THEOREM 4.2.** *Let  $M$  be a SFS. Then  $\pi_1(M)$  is BO if and only if  $M$  is either  $S^1 \times S^2$  or an  $S^1$ -bundle over an orientable surface not homeomorphic to  $S^2$ .*

**REMARK 4.3.** The  $S^1$  bundles over  $S^2$  are  $S^1 \times S^2 = L(0, 1)$ ,  $S^3 = L(1, 1)$ , and  $L(p, 1)$  for  $p \geq 2$ .

**PROOF.** ( $\Leftarrow$ ):  $S^1 \times S^2$  is immediate. Let  $M$  be an  $S^1$  bundle over an orientable surface  $F \not\cong S^2$ . For any bundle we have a homotopy exact sequence:

$$(4.18) \quad \underbrace{1}_{\pi_2(F)} \rightarrow \underbrace{\pi_1(S^1)}_{=\mathbb{Z}} \rightarrow \pi_1(M) \rightarrow \pi_1(F) \rightarrow 1.$$

**Lemma 4.3.**  $\pi_1(F) \neq 1$  implies  $\pi_1(M)$  is BO.

We will prove this later.  $M$  orientable implies the conjugation action of  $\pi_1(F)$  on  $\mathbb{Z}$  is by the identity, so the BO on  $\mathbb{Z}$  is conjugation invariant, so  $\pi_1(M)$  is BO by Theorem 1.13.

( $\Rightarrow$ ): We first need the following lemma.

**Lemma 4.4.** *In case (N),  $h \neq 1 \in \pi_1(M)$ .*

**PROOF.** First take  $n = 0$ . Then we can kill the  $d_i$ 's and  $a_i$ 's for  $i > 1$  to get the quotient of  $\pi_1(M)$ :

$$(4.19) \quad \langle a, h \mid a^{-1}ha = h^{-1}, a^2 = h^b \rangle$$

For  $b$  even, this has quotient

$$(4.20) \quad \langle a, h \mid a^2 = h^2 = (ah)^2 = 1 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

For  $b$  odd, this has quotient

$$(4.21) \quad \langle a, h \mid a^{-1}ha = h^{-1}, h^2 = 1, a^2 = h \rangle \cong \langle a \mid a^4 = 1 \rangle \cong \mathbb{Z}/4.$$

For  $n > 0$ , we also kill the  $c_i$  for  $i \geq 2$ , so we get a quotient

$$(4.22) \quad \langle a, c, h \mid h^2 = 1, a \leftrightarrow h, c^p = h^r, a^2c = 1 \rangle \cong \langle a, h \mid h^2 = 1, a \leftrightarrow h, a^{2p} = h^{-1} \rangle$$

$$(4.23) \quad \cong \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}/2p & r \text{ even} \\ \mathbb{Z}_{4p} & r \text{ odd} \end{cases}.$$

□

**Lemma 4.5.** *In case (N),  $\pi_1(M)$  is not BO.*

PROOF.  $a^{-1}ha = h^{-1}$ , and  $h \neq 1$  by Lemma 4.4, so  $\pi_1(M)$  is not BO.  $\square$

**Lemma 4.6.** *Suppose  $G$  is BO. If  $h^m \leftrightarrow g$  for some  $n \neq 0$ , then  $h \leftrightarrow g$ .*

PROOF. Since  $x \leftrightarrow g$  iff  $x^{-1} \leftrightarrow g$ , we may assume  $n > 0$ . Suppose  $h \not\leftrightarrow g$ , i.e.  $ghg^{-1}h^{-1} \neq 1$ . WLOG this means  $ghg^{-1}h^{-1} > 1$ , so  $ghg^{-1} > h$ . Therefore  $(ghg^{-1})^n = gh^n g^{-1} > h^n$ , so  $gh^n g^{-1}h^{-n} > 1$  which is a contradiction.  $\square$

**Lemma 4.7.** *In case (O), if  $g > 0$  and  $n > 0$  then  $\pi_1(M)$  is not BO.*

PROOF. Assume  $\pi_1(M)$  is BO. We have the relation

$$(4.24) \quad c_1^{p_1} = h^{r_1}.$$

Now recall in case (O),  $h$  is central, so  $c_1^{p_1}$  is central, so  $c_1$  is central (by Lemma 4.6). Now kill the  $a_i$ 's,  $b_i$ 's,  $d_i$ 's and  $c_i$ 's for  $i > 1$ , and  $h$ . So we get a quotient

$$(4.25) \quad \langle a, b, c \mid c^p = 1, c = [a, b] \rangle \cong \langle a, b \mid [a, b]^p = 1 \rangle$$

for  $p \geq 2$ . Now we have the following lemma:

**Lemma 4.8.**  *$[a, b]$  is not central in  $G_p = \langle a, b \mid [a, b]^p = 1 \rangle$  (for  $p \geq 2$ ).*

PROOF. Define a representation  $G_p \rightarrow S_{2p}$  by

$$(4.26) \quad a \longmapsto (1\ 2)(3\ 4)\dots(2p-1\ 2p)$$

$$b \longmapsto (1\ 2\ 3\ \dots\ 2p)$$

so we have that

$$(4.27) \quad b^{-1}ab = (2\ 3)(4\ 5)\dots(2p\ 1).$$

Therefore

$$(4.28) \quad [a, b] = a^{-1}b^{-1}ab = (2\ 4\ 6\dots 2p)(1\ 2p-1\ \dots\ 3)$$

is of order  $p$ . Furthermore,  $a^2 = 1$  in  $S_{2p}$ , so we have

$$(4.29) \quad a^{-1}[a, b]a = a^{-1}a^{-1}b^{-1}aba = b^{-1}aba = (a^{-1}b^{-1}ab)^{-1} = [a, b]^{-1}$$

so  $[a, b]$  is not central if  $p \geq 3$ . In the case  $p = 2$  we can instead send  $a \mapsto (1\ 2\ 3)$   $b \mapsto (2\ 3\ 4)$ .  $\square$

This lemma completes the proof.  $\square$

**Lemma 4.9.** *If case (O), if  $g = 0$ ,  $n > 0$ , and  $m + n \geq 3$ , then  $\pi_1(M)$  is not BO.*

PROOF. Recall  $c_i^{p_i} = h^{r_i}$ . This implies  $c_i^{p_i}$  is central, which implies  $c_i$  is central (if  $\pi_1(M)$  is BO). Then we have three-subcases:

- (a)  $m \geq 2$ : Kill  $h$ , and all the  $c_i$ 's except 1, use least to eliminate  $d_m$ , kill all remaining  $d_i$ 's except  $d_1$ . So we get a quotient

$$(4.30) \quad \langle c, d \mid c^p = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}/p$$

so  $c$  is not central.

- (b)  $m = 1$ ,  $n \geq 2$ : Kill  $h$  and eliminate  $d_1$ . Now kill all but 2  $c_i$ 's to get a quotient

$$(4.31) \quad \langle c_1, c_2 \mid c_1^{p_1} = c_2^{p_2} = 1 \rangle \cong \mathbb{Z}/p_1 * \mathbb{Z}/p_2$$

which is non-abelian.

(c)  $m = 0, n \geq 3$ : Kill  $h$  and all but 3  $c_i$ 's. So we get a quotient

$$(4.32) \quad \langle c_1, c_2, c_3 \mid c_i^{p_i} = 1, 1 \leq i \leq 3, c_1 c_2 c_3 = 1 \rangle = T(p_1, p_2, p_3) .$$

Then  $c_1, c_2$ , and  $c_3$  central implies  $T(p_1, p_2, p_3)$  is abelian.

**Lemma 4.10.**  $T(p_1, p_2, p_3)$  is nonabelian unless  $p_1 = p_2 = p_3$ , in which case it is  $\mathbb{Z}/2 \times \mathbb{Z}/2$ .

PROOF. Let  $A_1 A_2 A_3$  be a geodesic triangle with angles  $\pi/p_1, \pi/p_2, \pi/p_3$ . If the sum

$$(4.33) \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$$

is  $> 1$  this is in the sphere  $\mathbb{S}^2$ , if it is  $= 1$  this is in Euclidean space  $\mathbb{E}^2$ , and if this is  $< 1$ , this is the hyperbolic plane  $\mathbb{H}^2$ . Let  $\rho_{ij}$  be reflection of the plane ( $\mathbb{S}^2, \mathbb{E}^2$ , or  $\mathbb{H}^2$ ) over  $A_i A_j$ . Then

$$\gamma_1 = \rho_{13} \rho_{12} \quad \gamma_2 = \rho_{21} \rho_{23} \quad \gamma_3 = \rho_{23} \rho_{12} = (\gamma_1 \gamma_2)^{-1} .$$

We can visualize  $\gamma_i$  as rotation about  $A_i$  through  $2\pi/p_i$ . Let  $\Gamma(p_1, p_2, p_3)$  be the subgroup of  $\text{Isom}^+$  (of whatever plane we are in) generated by  $\gamma_1, \gamma_2$ . Since  $\gamma_i^{p_i} = 1$ ,  $1 \leq i \leq 3$  and  $\gamma_1 \gamma_2 \gamma_3 = 1$ . Therefore  $\Gamma(p_1, p_2, p_3)$  is a quotient of  $T(p_1, p_2, p_3)$ . (In fact,  $\Gamma = T$ .) Clearly  $\Gamma(p_1, p_2, p_3)$  is non-abelian unless  $p_1 = p_2 = p_3$ . Therefore  $T(p_1, p_2, p_3)$  is nonabelian unless  $p_1 = p_2 = p_3$ .  $\square$

$\square$  Lecture 15; March 10, 2020

**Lemma 4.11.** In case (O),  $g = 0, m = 0, n \geq 3, p_i = 2$  for all  $i$ ,  $\pi_1(M)$  is nonabelian and hence not BO.

PROOF. It is enough to do  $n = 3$ . (Otherwise just take the quotient.) Now we will kill  $h^2$  instead of  $h$  (we can assume  $b = 0$ ). So since  $p_1 = p_2 = p_3 = 2$ , we have that  $r_1, r_2$ , and  $r_3$  are odd. So we get a quotient:

$$(4.34) \quad \langle c_1, c_2, c_3, h \mid h^2 = 1, c_1^2 = c_2^2 = h, c_1 c_2 c_3 = 1 \rangle \cong \langle c_1, c_2, h \mid h^2 = 1, c_1^2 = c_2^2 = (c_1 c_2)^2 = h \rangle .$$

Now sending  $x_1 \mapsto i, c_2 \mapsto j$  (so  $(c_1 c_2) \mapsto k$ ), and  $h \mapsto -1$  we get that the unit quaternion group is a quotient of this, which is non-abelian.  $\square$

( $\implies$ ): Now the only cases left after all of these lemmas are  $g = 0, m = 0$ , and  $n = 0, 1, 2$ ; and  $g = 0, m = 1, n = 0, 1$ . If  $n = 0$ , then we just have an  $S^1$  bundle over  $S^2$  or  $D^2$ . In the second case we are just a solid torus, and in the first case, it must either be  $S^3$ ,  $L(p, 1)$ , or  $S^1 \times S^2$  and only the last one is BO.

If  $n = 1$  or  $2$ , then if  $m = 0$  we have the union of two solid tori, so therefore  $S^3, S^1 \times S^2$ , or a lens space. If  $m = 1$ , then  $M \cong S^1 \times S^2$ .  $\blacksquare$

### 1. Left-orderability of $\pi_1$ SFS's

For  $M$  a SFS, then  $M$  is prime iff  $M$  is not  $\mathbb{RP}^3 \# \mathbb{RP}^3$ . In this case  $\pi_1 \cong \mathbb{Z}/2 * \mathbb{Z}/2$  which is certainly not LO. Therefore by Corollary 3.10, unless  $H_1(M)$  is a prime QHS (equivalently,  $H_1(M)$  is finite),  $\pi_1(M)$  is LO.

We know  $\pi_* : \pi_1(M) \rightarrow \pi_1(F)$  is onto, so

$$(4.35) \quad \pi_* : H_1(M) \rightarrow H_1(F)$$

is onto. Therefore for  $M$  a QHS,  $F \cong S^2$  or  $\mathbb{RP}^2$ .



THEOREM 4.12. *If  $M$  is a SFS with base surface  $\mathbb{RP}^2$ , then  $\pi_1(M)$  is not LO.*

PROOF. Recall we have the presentation:

$$(4.36) \quad \pi_1(M) = \left\langle a, c_1, \dots, c_n, h \mid a^{-1}ha = h^{-1}, h \leftrightarrow c_i, c_i^{p_i} = h^{r_i}, a^2 \prod c_i = h^b \right\rangle.$$

Suppose  $\pi_1(M)$  is LO.  $a \neq 1 \in \pi_1(M)$  (since  $\pi_*(a)$  is a generator of  $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$ ). Then  $h \neq 1 \in \pi_1(M)$  by Lemma 4.4. So we may assume  $h > 1$ . Otherwise, reverse the order.

**Lemma 4.13.** (1) *If  $a > 1$ , then  $a > h^k$  for all  $k \in \mathbb{Z}$ .*

(2) *If  $a < 1$ , then  $a < h^k$  for all  $k \in \mathbb{Z}$ .*

PROOF. (1)  $h > 1$  implies  $h^{-1} < 1$ , so if  $k \leq 0$  we are done. So assume  $k > 0$ .

Then  $h^k > 1 > a^{-1}$ , so  $1 > h^{-k}a^{-1} = a^{-1}h^k$ , so  $a > h^k$ .

(2) This is similar. □

Now we treat the following cases.

(a)  $n = 0$ :

(a)  $a > 1$ :  $h^b = a^2$  from the last relation, but this means  $h^b > a$  which contradicts Lemma 4.13.

(b)  $a < 1$ :  $h^b = a^2 < 1$ , which also contradicts Lemma 4.13.

(b)  $n > 0$ :

(a)  $a > 1$ : From the discussion after Theorem 4.1, we can assume  $r_i > 0$  for  $1 \leq i \leq n$ . Then  $c_i^{p_i} = h^{r_i} > 1$ , so therefore  $c_i > 1$ , so  $\prod c_i > 1$ , so  $h^b = a^2 \prod c_i a^2 > a$  which contradicts Lemma 4.13.

(b)  $a < 1$ : Choose  $r_i < 0$ . Then this contradicts Lemma 4.13. ■

Say a SFS is of type  $F(p_1, \dots, p_n)$  if base surface is  $F$  and there are  $n$  exception fiber with  $p_i \geq 2$ . So we are left with

$$(4.37) \quad M = S^2(p_1, \dots, p_n)$$

for  $n \geq 3$ . (If  $n \leq 2$ , we have  $S^1 \times S^2$ ,  $S^3$ , or a lens space.)

REMARK 4.4. Conjecturally, any prime ZHS  $M$  (except  $S^3$ , and the Poincaré homology sphere) has  $\pi_1(M)$  LO. But this is wide open in general. It is known for graph manifolds, but whenever there are hyperbolic pieces it is wide open.

THEOREM 4.14. (1) *Let  $M$  be a SFS ZHS <sup>4.2</sup> Then either  $M \cong S^3$  or  $M$  is of type  $S^2(p_1, \dots, p_n)$  where  $n \geq 3$  and the  $p_i$  are pairwise coprime.*

(2) *Let  $p_1, \dots, p_n$ ,  $n \geq 3$  pairwise coprime,  $p_i \geq 2$ . Then there is a unique (up to homeomorphism) SFS ZHS  $M$  of type  $S^2(p_1, \dots, p_n)$  (called  $\Sigma(p_1, \dots, p_n)$ ).*

---

<sup>4.2</sup>“There are a lot of TLA's in this world.”

PROOF. (1) Since  $H_1(M) = 0$ ,  $H_1(F) = 0$ , we have  $F \cong S^2$ . If  $n \leq 2$ , then  $g(M) \leq 1$  and therefore  $M \cong S^3$ . For  $n \geq 3$ ,  $H_1(M)$  is presented by the matrix:

$$(4.38) \quad A = \begin{bmatrix} 1 & 1 & \cdots & 1 & b \\ p_1 & 0 & \cdots & 0 & r_1 \\ 0 & p_2 & \cdots & 0 & r_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_n & r_n \end{bmatrix}.$$

Then

$$(4.39) \quad |\det A| = \prod_{i=1}^n p_i \left( -b + \sum_{i=1}^n \frac{r_i}{p_i} \right).$$

Write

$$(4.40) \quad \Pi = \prod_{i=1}^n p_i$$

and define  $s_i = \Pi/p_i$  for  $0 \leq i \leq n$  (note  $s_0 = \Pi$ ,  $r_0 = -b$ ). Then

$$(4.41) \quad |\det A| = \sum_{i=0}^n r_i s_i.$$

If  $d$  divides  $p_i$  for two distinct indices  $i$ , then  $d|s_i$ ,  $0 \leq i \leq n$ , so therefore if  $M$  is a  $\mathbb{Z}\text{HS}$ , then  $|\det A| = 1$ , which implies that the  $p_i$ 's are pairwise coprime.

- (2) Suppose  $p_1, \dots, p_n$  are pairwise coprime,  $\geq 2$  and that  $n \geq 3$ . Then  $\gcd(s_0, \dots, s_n) = 1$ . Therefore

$$(4.42) \quad \sum_{i=0}^n r_i s_i = 1$$

has a solution  $(r_i)_{i=0}^n$ . Also

$$(4.43) \quad \text{lcm}(s_0, \dots, s_n) = \Pi$$

so if  $(r'_i)_{i=0}^n$  is another solution, then there are some integers  $k_i$  such that  $\sum k_i = 0$ , and

$$(4.44) \quad r'_i = r_i + k_i \left( \frac{\Pi}{s_i} \right) = r_i + k_i p_i$$

for  $0 \leq i \leq n$ . So by the discussion after Theorem 4.1, the  $r_i$ 's and  $r'_i$ 's give the same SFS. Finally

$$(4.45) \quad \sum_{i=0}^n (-r_i) s_i = -1$$

but we can replace  $r_i$  by  $-r_i$  which just corresponds to reversing orientation, i.e. taking  $M \rightarrow -M$ . Therefore the  $r_i$ 's define a SFS  $\mathbb{Z}\text{HSof}$  type  $S^2(p_1, \dots, p_n)$  which is unique up to homeomorphism.

□

The proof of the second part tells us the following. First,  $|H_1(M)|$  is relatively prime to the  $p_i$ 's, and conversely, given  $d$  such that  $(d, p_i) = 1$ ,  $1 \leq i \leq n$ , then there exists a (unique up to homeomorphism) SFS of type  $S^2(p_1, \dots, p_n)$ ,  $M$ , with  $|H_1(M)| = d$ .

Recall we showed that if  $M$  is of type  $S^1(p_1, \dots, p_n)$  as above, then  $\pi_1(M)$  maps onto  $T(p_1, p_2, p_3)$ , so it is infinite unless  $\{p_1, p_2, p_3\} = \{2, 3, 5\}$ . Therefore:

THEOREM 4.15. *A SFS  $\mathbb{Z}HSM$  has  $\pi_1(M)$  infinite unless  $M \cong S^3$  or  $\Sigma(2, 3, 5)$ .*

THEOREM 4.16. *Let  $M$  be a SFS. Then  $\pi_1(M)$  is LO iff  $\pi_1(M)$  is infinite. (i.e.  $M \not\cong S^3$  or  $\Sigma(2, 3, 5)$ .)*

PROOF. ( $\implies$ ): This is clear.

( $\impliedby$ ): By Theorem 4.14,  $M$  is of type  $S^2(p_1, \dots, p_n)$ ,  $p_i$ 's pairwise coprime  $n \geq 3$ , and not  $n = 3$ ,  $\{p_1, p_2, p_3\} = \{2, 3, 5\}$ . As before, we get an epimorphism

$$(4.46) \quad \pi_1(M) \rightarrow \Gamma(p_1, p_2, p_3) < \text{Isom}_+(\mathbb{H}^2) .$$

Recall  $\mathbb{S}^2$  corresponded to  $(2, 3, 5)$ , and Euclidean space corresponded to

$$(4.47) \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

which gave us  $(3, 3, 3)$ ,  $(2, 3, 6)$ , and  $(2, 4, 4)$ . So we in the smallest case we are already in isometries of the hyperbolic plane  $\mathbb{H}^2$ .

$$(4.48) \quad \text{SL}_2(\mathbb{R}) = \{A \in M_{2 \times 2}\} .$$

Note the center is:

$$(4.49) \quad Z(\text{SL}_2(\mathbb{R})) = \{\pm I\} \cong \mathbb{Z}/2 .$$

The quotient is:

$$(4.50) \quad \text{SL}_2(\mathbb{R}) / \{\pm I\} = \text{PSL}_2(\mathbb{R}) .$$

$\text{PSL}_2(\mathbb{R})$  is a three-dimensional Lie group. Recall the hyperbolic plane  $\mathbb{H}^2$  can be thought of as the interior of the unit disk, where geodesics are given by circles orthogonal to the boundary. We can also think of it as the upper half plane

$$(4.51) \quad \mathbb{H}^2 = \mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

where now the geodesics are circle orthogonal to the  $x$ -axis, the limiting case being a vertical line.

$\text{PSL}_2(\mathbb{R})$  acts on  $\mathbb{C}$  by linear fractional transformations. Explicitly for

$$(4.52) \quad \text{SL}_2(\mathbb{R}) \ni A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the class  $[A] \in \text{PSL}_2(\mathbb{R})$  sends

$$(4.53) \quad z \mapsto \frac{az + b}{cz + d} .$$

This action preserves  $\mathbb{R}_+^2$ . It turns out that this gives an isomorphism

$$(4.54) \quad \text{Isom}_+(\mathbb{H}^2) \simeq \text{PSL}_2(\mathbb{R}) .$$

In the disk model, this extends to an action to the boundary  $\partial D^2 = S^1 = S_\infty^1$ . So

$$(4.55) \quad \text{PSL}_2(\mathbb{R}) < \text{Homeo}_+(S^1) .$$

$\text{PSL}_2(\mathbb{R})$  acts transitively on  $\mathbb{H}^2$ , and for  $h \in \mathbb{H}^2$ ,  $\text{Stab}(x) \cong S^1$ . We can think of this as rotating around  $x$ . Therefore we get that:

$$(4.56) \quad \text{PSL}_2(\mathbb{R}) \cong \mathbb{H}^2 \times S^1 .$$

Therefore  $\pi_1(\mathrm{PSL}_2(\mathbb{R})) \cong \mathbb{Z}$ . Write  $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$  for the universal cover of  $\mathrm{PSL}_2(\mathbb{R})$ .

Let  $G$  be a connected Lie group. Consider its universal covering  $p : \widetilde{G} \rightarrow G$ . This is a Lie group as well. Then we have the following two facts.

(1) We get a central extension:

$$(4.57) \quad 1 \rightarrow \pi_1(G) \rightarrow \widetilde{G} \rightarrow G \rightarrow 1 .$$

In particular,  $\pi_1(G)$  is abelian.

(2) If  $G$  acts on  $X$ , then  $\widetilde{G}$  acts on the universal cover  $\widetilde{X}$ .

So we get a central extension

$$(4.58) \quad 1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{PSL}_2(\mathbb{R})} \rightarrow \mathrm{PSL}_2(\mathbb{R}) \rightarrow 1 .$$

And  $\mathrm{PSL}_2(\mathbb{R})$  acts on  $S^1$ , so  $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$  acts on  $\mathbb{R}$ , so we can think of it as a subgroup  $\mathrm{Homeo}_+(\mathbb{R})$ .

REMARK 4.5. This is this nice little matrix group, but there is a more general thing happening here with the much more unruly infinite-dimensional  $\mathrm{Homeo}_+(S^1)$  sitting inside

$$(4.59) \quad 1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{Homeo}_+(S^1)} \rightarrow \mathrm{Homeo}_+(S^1) \rightarrow 1 .$$

So  $\mathrm{PSL}_2(\mathbb{R}) < \mathrm{Homeo}_+(S^1)$  and  $\widetilde{\mathrm{PSL}_2(\mathbb{R})} < \mathrm{Homeo}_+(\mathbb{R})$ , so what we would really like is to lift:

$$(4.60) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathrm{PSL}_2(\mathbb{R})} & \longrightarrow & \mathrm{PSL}_2(\mathbb{R}) \longrightarrow 1 \\ & & & & \nwarrow \tilde{\rho} & & \uparrow \rho \\ & & & & & & \pi_1(M) \end{array} .$$

DIGRESSION 2 (Group (co)homology). Let  $G$  be a group. A  $K(G, 1)$  is a connected CW-complex  $X$  such that  $\pi_1(X) \cong G$  and  $\pi_i(X) = 0$ ,  $i \geq 2$ . Such a group exists and is unique up to homotopy equivalence.  $X$  has the (universal) property that given a homomorphism  $\varphi : G \rightarrow H$ , and given a space  $Y$  with  $\pi_1(Y) \cong H$ , there exists a map  $f : X \rightarrow Y$  such that  $f_* = \varphi$ .

Now we can define

$$(4.61) \quad H_*(G; A) = H_*(X, A) \quad H^*(G; A) = H^*(X, A) .$$

There are algebraic definitions as well.

FACT 4. For  $A$  an abelian group, the central extensions

$$(4.62) \quad 1 \rightarrow A \rightarrow H \rightarrow G \rightarrow 1$$

are classified by  $H^2(G; A)$ .

See appendix A for more.

By this fact, if we have

$$(4.63) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & H & \longrightarrow & G \longrightarrow 1 \\ & & & & \uparrow \tilde{\rho} & \nearrow \rho & \\ & & & & \pi & & \end{array}$$

for some group  $\pi$ , the central extension corresponds to some element  $e \in H^2(G; \mathbb{Z})$  and  $\rho^*(e) \in H^2(\pi, \mathbb{Z})$ . Then  $\rho^*e = 0$  iff the corresponding central extension  $1 \rightarrow \mathbb{Z} \rightarrow \tilde{\pi} \rightarrow$

$\pi \rightarrow 1$  is  $\tilde{\pi} = \pi \times \mathbb{Z}$ . This is equivalent to  $\rho$  lifting to  $\tilde{\rho}$ . So here the lift  $\tilde{\rho}$  exists iff  $\rho^*e = 0 \in H^2(\pi_1(M); \mathbb{Z})$ .

THEOREM (Hurewicz). *Let  $X$  be a space with  $\pi_0(X) = 0$  for  $0 \leq i < n$  where  $n \geq 2$ . Then  $H_i(X) = 0$  for  $0 \leq i < n$ , and the Hurewicz map*

$$(4.64) \quad h : \pi_n(X) \rightarrow H_n(X)$$

*is an isomorphism.*

THEOREM 4.17. *Let  $M$  be an irreducible three-manifold with infinite  $\pi_1$ . Then  $M$  is a  $K(\pi_1(M), 1)$  (i.e.  $\pi_i(M) = 0$  for  $i \geq 2$ ).*

PROOF. Let  $\tilde{M} \rightarrow M$  be the universal covering.  $\pi_1(\tilde{M}) = 0$ , and  $\pi_2(\tilde{M}) \cong \pi_2(M)$ , but by Corollary 3.2,  $\pi_2(M) = 0$  since  $M$  is irreducible. So therefore  $H_2(\tilde{M}) = 0$ , but  $\tilde{M}$  is noncompact since  $\pi_1(M)$  is infinite, so therefore  $H_i(\tilde{M}) = 0$  for  $i \geq 3$ . So we have a simply-connected space with trivial higher homology groups, so the higher homotopy groups  $\pi_i(\tilde{M}) = 0$  for  $i \geq 2$  from the Hurewicz theorem, but these are the same as the ones for  $M$ , so these vanish too.  $\square$

Now return to our problem. We have  $M$  a SFS  $\mathbb{Z}$ HS,  $\pi_1(M)$  infinite. So this means  $M$  is a  $K(\pi_1(M), 1)$ . Then we wanted to lift

$$(4.65) \quad 1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{PSL}}_2(\mathbb{R}) \rightarrow \mathrm{SL}_2(\mathbb{R}) \rightarrow 1$$

and the obstruction was exactly:

$$(4.66) \quad \rho^*e \in H^2(\pi_1(M); \mathbb{Z}) = H^2(M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) = 0$$

by Poincaré duality and the fact that  $M$  is a  $\mathbb{Z}$ HS. So then  $\pi_1(M)$  has a nontrivial homomorphism into  $\widetilde{\mathrm{PSL}}_2(\mathbb{R}) < \mathrm{Homeo}_+(\mathbb{R})$ , so by the Boyer-Rolfson-Wiest theorem  $\pi_1(M)$  is LO. This completes the proof of Theorem 4.16.  $\blacksquare$

Lecture 17; March 31, 2020

- REMARK 4.6. (1)  $H_1(M) = 0$ , so  $\pi_1(M)$  is certainly not locally indicable. Recall this implies left orderable. At one point people wondered if it was equivalence.  $\Sigma(2, 3, 7)$  was the first example of a group which was LO but not locally indicable.
- (2) Conjecturally, this should hold for any prime  $\mathbb{Z}$ HS with  $\pi_1$  infinite, i.e. not  $S^3$  or  $\Sigma(2, 3, 5)$ , has  $\pi_1(M)$  LO. This is the  $L$ -space conjecture. Which we will presumably state at some point.<sup>4.3</sup>

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<sup>4.3</sup>This course is a bit like one of Cameron's favorite books, "The Life and Opinions of Tristram Shandy, Gentleman" by Laurence Sterne. It is meant to be a biography, but the author keeps getting distracted. Much like we keep missing the  $L$  space conjecture.

## CHAPTER 5

# Foliations

We now shift focus to foliations. There is some connection to left-orderability, and the  $L$ -space conjecture suggests a very strong connection in dimension 3.

Lecture 19; April 7, 2020

### 1. Definition and examples

A *foliation*  $\mathcal{F}$  on an  $n$ -manifold  $M$  ( $\partial M = \emptyset$ ) is a disjoint union

$$(5.1) \quad \coprod_{\lambda \in \Lambda} L_{\lambda}$$

of connected  $k$ -manifolds, for some  $0 \leq k < n$ ; and a continuous bijection

$$(5.2) \quad f: \coprod_{\lambda \in \Lambda} L_{\lambda} \rightarrow M$$

where  $M$  is covered by coordinate charts  $\varphi: \overset{\sim}{\rightarrow} \mathbb{R}^n$  such that for all  $\lambda \in \Lambda$

$$(5.3) \quad \varphi(f(L_{\lambda}) \cap U) = \mathbb{R}^k \times X_{\lambda}$$

for some  $X_{\lambda} \subset \mathbb{R}^{n-k}$ . The picture is as in fig. 1.

The *codimension* of  $\mathcal{F}$  is  $n - k$ . The  $L_{\lambda}$ , or (by abuse of notation)  $f(L_{\lambda})$ , are the *leaves* of  $\mathcal{F}$ .

- REMARK 5.1. (1) Sometimes one imposes various smoothness conditions on  $\mathcal{F}$ .  
 (2) There extensions to manifolds with boundary. For example, in codimension 1, we might insist that the leaves are transverse to the boundary, or that you want the boundary component to actually be a leaf.

EXAMPLE 5.1. A fiber bundle  $F \rightarrow M \rightarrow B$  gives us a foliation of  $M$  where the leaves are the fibers.

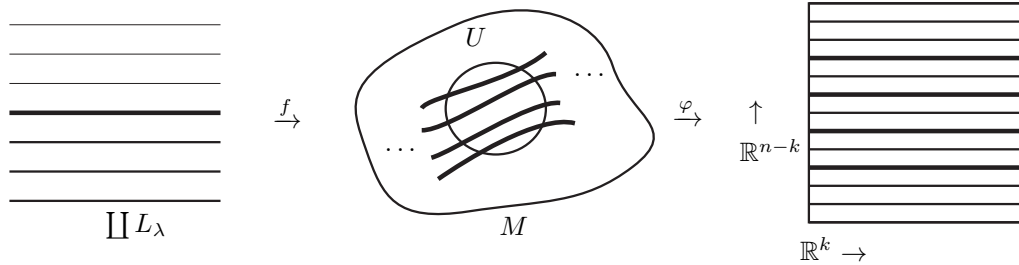


FIGURE 1. Cartoon of a foliation.

EXAMPLE 5.2. A SFS has a codimension 2 foliation where the leaves are circles (the Seifert fibers). As it turns out, we have the following.

THEOREM (Epstein). *If  $M$  is a compact 3-manifold with a foliation whose leaves are circles, then  $M$  is a SFS (and the foliation is a Seifert fibration).*

EXAMPLE 5.3. Let  $\widetilde{M} \rightarrow M$  be a covering space. If  $\mathcal{F}$  is a foliation on  $M$ , then it lifts to a foliation  $\widetilde{\mathcal{F}}$  on  $\widetilde{M}$ .

Conversely, if  $\widetilde{M} \rightarrow M$  is a regular  $G$ -covering space, and  $\widetilde{\mathcal{G}}$  is a  $G$ -invariant foliation on  $\widetilde{M}$ , then  $\widetilde{\mathcal{F}}/G$  is a foliation on  $\widetilde{M}/G = M$ .

EXAMPLE 5.4. Consider the universal cover  $\mathbb{R}^2 \rightarrow T^2$ . This is an example of example 5.3 for  $G = \mathbb{Z} \times \mathbb{Z}$ . For  $\alpha \in \mathbb{R}$ ,  $\mathbb{R}^2$  has a foliation  $\widetilde{\mathcal{F}}_\alpha$  by fibers of slope  $\alpha$ . This is  $\mathbb{Z} \times \mathbb{Z}$ -invariant, so it induces a foliation  $\mathcal{F}_\alpha$  on  $T^2$ .

- (i) If  $\alpha \in \mathbb{Q}$ , then the leaves of  $\mathcal{F}_\alpha$  are circles, so  $T^2 \cong S^1 \times S^1$  has leaves  $S^1 \times \{\lambda\}$  for  $\lambda \in S^1$ .
- (ii) If  $\alpha \notin \mathbb{Q}$ , then the leaves of  $\mathcal{F}_\alpha$  are  $\mathbb{R}$ . Every leaf is dense in  $T^2$ .

EXAMPLE 5.5. This is similar to the second case of example 5.4. As it turns out, there is a foliation of  $T^3$  with leaves homeomorphic to  $\mathbb{R}^2$ .

THEOREM (Rosenberg-Sondow). *If  $M$  is a closed 3-manifold with a foliation with leaves homeomorphic to  $\mathbb{R}^2$ , then  $M \cong T^3$ .*

EXAMPLE 5.6. Consider  $[-1, 1] \times \mathbb{R}$ . This has a foliation  $\mathcal{F}$  with leaves homeomorphic to  $\mathbb{R}$ . See fig. 2. The leaves are the graphs of  $y = f(x) + c$  for  $c \in \mathbb{R}$  and suitable  $f$ , along with  $\{\pm 1\} \times \mathbb{R}$ .  $\mathcal{F}$  is invariant under the shift  $(x, y) \mapsto (x, y + 1)$ , so we get a foliation on the quotient by this shift, which is just an annulus  $[-1, 1] \times S^1$ . The leaves are mostly  $\mathbb{R}$ , except  $\{\pm 1\} \times S^1$ .

If we rotate  $[-1, 1] \times \mathbb{R}$  about the  $y$ -axis, this foliation induces a foliation on  $D^2 \times \mathbb{R}$ , which induces the Reeb foliation on  $D^2 \times S^1$ . The leaves are all  $\mathbb{R}^2$  except  $\partial(D^2 \times S^1)$ .

From now on, we will only consider foliations of codimension 1. The Reeb foliation is an example of a codimension 1 foliation on the torus.

REMARK 5.2. Reeb's advisor Ehresmann told him to try to prove that  $S^3$  doesn't have a codimension 1 foliation. Reeb came back and said, well here's a foliation of the solid torus, the leaves are all  $\mathbb{R}^2$  except the boundary, so it's codimension 1. So now glue two of these together to get one on  $S^3$ . The moral of the story is not to believe what your advisor asks you to prove. It's probably rubbish.

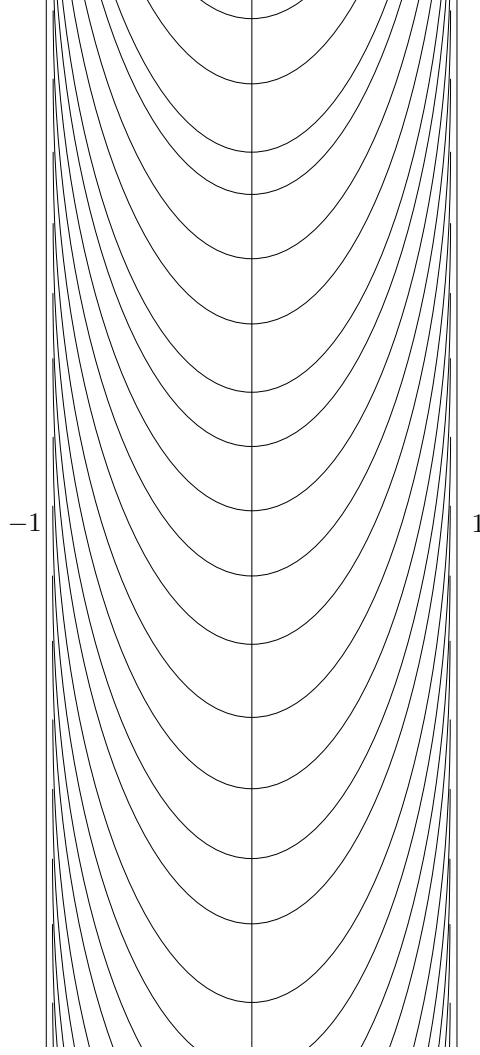
## 2. Codimension one foliations of three-manifolds

THEOREM 5.1 (Lickorish; Zieschang). *Every closed 3-manifold has a codimension 1 foliation.*

This proof uses two classical theorems in 3-dimensional topology.

THEOREM (LickorishWallace). *Every closed 3-manifold can be obtained by Dehn surgery on a link in  $S^3$ .*

THEOREM (Alexander). *Every link in  $S^3$  is the closure of a braid.*

FIGURE 2. Foliation of  $[-1, 1] \times \mathbb{R}$  with leaves homeomorphic to  $\mathbb{R}$ .

PROOF OF THEOREM 5.1.  $M$  is a surgery on a link  $L$ , and  $L$  is the closure of some braid  $\beta$ . So write  $L'$  for the union of  $L$  and the braid axis as in fig. 3.

Now let  $X = S^3 \setminus \text{int}(N(L'))$ . Notice that  $X$  is an  $F$ -bundle over  $S^1$ , where  $F$  is an  $n$ -punctured disk, where  $n$  is the number of strands of  $\beta$ . Now there is a codimension 1 foliation on  $X$ , where the leaves are copies of this punctured disk. The boundary is given by:

$$(5.4) \quad \partial X = \prod_{i=0}^n T_i$$



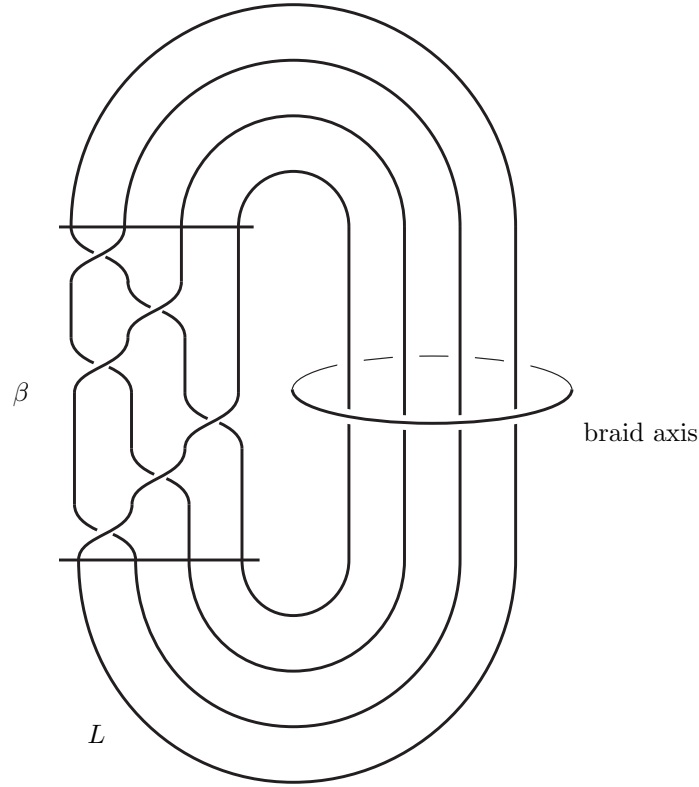


FIGURE 3. The link  $L'$  obtained by taking the union of  $L$  and the braid axis.

where each  $T_i \cong T^2$ , and  $1 \leq m \leq n$  is the number of components of  $L$ . Near each  $T_i$ , we will perturb this foliation  $\mathcal{F}$  by *spinning* the leaves around the  $T_i$  to get a new foliation  $\mathcal{F}'$ . The leaves of  $\mathcal{F}'$  are  $S^2 \setminus \{n+1 \text{ points}\}$ , plus the  $T_i$ .

Now our manifold is

$$(5.5) \quad M = X \cup \left( \coprod_{i=0}^m V_i \right)$$

where the  $V_i$  are solid tori, glued to  $X$  along  $T_i$  along  $T_i \leftrightarrow \partial V_i$ . But it doesn't matter how these are glued in, because  $\mathcal{F}'$  extends to a foliation of  $M$  by putting Reeb foliations on each  $V_i$ .  $\square$

REMARK 5.3. So this theorem says there are no restrictions on having a codimension 1 foliation, so maybe it isn't really that interesting. But as we saw, the proof relies heavily on Reeb foliations.

REMARK 5.4. If a manifold  $M$  has a codimension 1 foliation (which is cooriented<sup>5.1</sup>) then we can define a map from the manifold to itself by pushing off from these leaves. This doesn't have any fixed points, and is homotopic to the identity, so by the Lefschetz fixed point theorem, the Euler characteristic of the manifold is 0. So there was a whole business of

<sup>5.1</sup>We can assume this without much loss of generality.

finding these foliations. But then Thurston came along and wiped the field out by proving the following theorem.

**THEOREM.** *A closed  $n$ -manifold  $M$  has a codimension 1 foliation iff  $\chi(M) = 0$ .*

Consider

$$(5.6) \quad \mathbb{RP}_0^3 = \overline{\mathbb{RP}^3 \setminus 3\text{-ball}} .$$

This is a twisted  $I$ -bundle over  $\mathbb{RP}^2$ . This has 2-fold cover  $S^2 \times [-1, 1]$ . Write  $\tau: S^2 \times [-1, 1] \rightarrow S^2 \times [-1, 1]$  for the covering transformation

$$(5.7) \quad \tau(x, t) = (-x, -t) .$$

Therefore  $\mathbb{RP}^3 \# \mathbb{RP}^3$  has a 2-fold cover  $S^2 \times S^1$ . So  $\mathbb{RP}_0^3$  has a foliation  $\mathcal{F}$  with leaves given by copies of  $S^2$ , except one  $\mathbb{RP}^2$ . Then  $\tilde{\mathcal{F}}$  is the standard foliation on  $S^2 \times S^1$  with leaves given by  $S^2 \times \{\text{pt}\}$ .

**REMARK 5.5.** This is like how there is a foliation of the Möbius band where each leaf is a circle, all of which wrap around twice except one.

### 3. Reeb stability, transverse loops, and Novikov's theorem

The following is a special case of Reeb stability.

**THEOREM** (Special case of Reeb stability). *Let  $M$  be a closed 3-manifold with a foliation  $\mathcal{F}$  with a leaf homeomorphic to  $S^2$ , or  $\mathbb{RP}^2$ . Then  $M \cong S^2 \times S^1$ , or  $\mathbb{RP}^3 \# \mathbb{RP}^3$ , and  $\mathcal{F}$  is as above.*

Let  $\mathcal{F}$  be a codimension 1 foliation on  $M$ . A *transverse loop* in  $M$  is a loop which is transverse to  $\mathcal{F}$ .

**Lemma 5.2.** *If  $M$  is compact then there is a transverse loop in  $M$ .*

**PROOF.** Start at some  $x_0 \in L$  in some coordinate neighborhood. Proceed transversely.  $M$  is compact, so we eventually return to some previously visited coordinate neighborhood. Now we join them up. If we come back in the wrong direction, then we proceed until we come back a second time, and join up with the appropriate one.  $\square$

**THEOREM** (Novikov's Theorem). *Let  $\mathcal{F}$  be a Reebless foliation  $\mathcal{F}$  on a closed three-manifold not homeomorphic to  $S^2 \times S^1$  or  $\mathbb{RP}^3 \# \mathbb{RP}^3$ . Then*

- (1) *for any leaf  $L$  of  $\mathcal{F}$ ,  $\pi_1(L) \rightarrow \pi_1(M)$  is injective,*
- (2) *every transverse loop is essential.*

**REMARK 5.6.** Recall we say that every 3-manifold has a codimension 1 foliation. So there are no restrictions. But the construction relied heavily on the Reeb foliation. So this says that when we have Reebless ones we do get restrictions, so it's much more interesting.

By Lemma 5.2,  $\mathcal{F}$  has a transverse loop  $\gamma$ . Then  $\gamma^n$  is also transverse for all  $n \geq 1$ . Therefore, by (2) of Novikov's Theorem,  $[\gamma]$  has infinite order in  $\pi_1(M)$ . Therefore  $\pi_1(M)$  is infinite. So the universal cover  $\tilde{M}$  is noncompact. We can lift the foliation to  $\tilde{\mathcal{F}}$ . Then by (1) of Novikov's Theorem  $\pi_1(\tilde{L}) = 1$  for all leaves  $\tilde{L}$  of  $\tilde{\mathcal{F}}$ . So by Reeb stability, and by our assumption on  $M$ , no leaf  $L$  is  $S^2$  or  $\mathbb{RP}^2$ , so every leaf  $\tilde{L}$  of  $\tilde{\mathcal{F}}$  is just

$$(5.8) \quad \tilde{L} = \mathbb{R}^2 .$$

Therefore the following applies to  $(\widetilde{M}, \widetilde{\mathcal{F}})$ .

**THEOREM** (Palmeira's Theorem). *Let  $\mathcal{F}$  be a foliation with leaves  $\cong \mathbb{R}^2$  of a simply connected 3-manifold  $M$ . Then*

$$(5.9) \quad (M, \mathcal{F}) \cong (\mathbb{R}^2, \mathcal{L}) \times \mathbb{R}$$

where  $\mathcal{L}$  is a foliation of  $\mathbb{R}^2$  with leaves  $\cong \mathbb{R}^1$ .

**REMARK 5.7.** Palmeira proved the analogue in all dimensions.

**Corollary 5.3.** *Under the hypothesis of Novikov's Theorem,  $\widetilde{M} \cong \mathbb{R}^3$ . Equivalently,  $M$  is irreducible and  $\pi_1(M)$  is infinite.*

#### 4. Taut foliations

**DEFINITION 5.1.** Let  $\mathcal{F}$  be a foliation of a closed 3-manifold  $M$ .  $\mathcal{F}$  is *taut* if  $\mathcal{F}$  has a transverse loop  $\gamma$  such that for all leaves  $L$ ,  $L \cap \gamma \neq \emptyset$ .

If we have Reeb foliations as in fig. 2, we can't have such a transverse loop as in definition 5.1. So

$$\text{taut} \implies \text{Reebless}.$$

The converse is false.

**COUNTEREXAMPLE 3.** Let  $T_0$  be a once-punctured torus, to  $\partial T_0 = S^1$ . Then take  $X = T_0 \times S^1$ . Spinning the  $T_0 \times \{\text{pt}\}$ 's around  $\partial X$  gives a foliation  $\mathcal{F}$  of  $X$ . Let  $M = X' \cup_{\partial} X$  where  $X'$  is just a copy of  $X$ . Then  $\mathcal{F}$  and  $\mathcal{F}'$  give a foliation  $\mathcal{F}^*$  on  $M$ . The boundary  $T = \partial X' = \partial X' \subset M$  is Reebless, but not taut. G

But they are almost equivalent. When we play a game such as counterexample 3, we are forced to have a torus leaf, and we have the following theorem.

**THEOREM** (Goodman). *If a foliation  $\mathcal{F}$  on a closed 3-manifold  $M$  is not taut, then  $\mathcal{F}$  has a torus leaf.*

#### 5. Coorientable foliations

**DEFINITION 5.2.** If  $\mathcal{F}$  is a codimension 1 foliation on a closed  $n$ -manifold  $M$ , we say  $\mathcal{F}$  is *co-orientable* if there is a consistent transverse orientation to the leaves of  $\mathcal{F}$ .

- REMARK 5.8.**
- (1) If  $M$  is orientable and  $\mathcal{F}$  has a nonorientable leaf, then  $\mathcal{F}$  is *not* coorientable.
  - (2) The foliations on 3-manifolds constructed in Theorem 5.1 are coorientable.
  - (3) Take the Reeb foliation on  $[-1, 1] \times S^1$ . If we identify  $\{\pm 1\} \times S^1$ , we get a foliation on  $T^2$ . One leaf is  $S^1$ , and the rest are  $\mathbb{R}$ . This is not co-orientable.
  - (4) Every codimension 1 foliation  $(M, \mathcal{F})$  has a 2-fold cover  $(\widetilde{M}, \widetilde{\mathcal{F}})$  such that  $\widetilde{\mathcal{F}}$  is coorientable. So  $H_1(M; \mathbb{Z}/2) = 0$  implies  $\mathcal{F}$  is coorientable.

**THEOREM 5.4.** *Let  $M$  be a closed  $n$ -manifold. If  $M$  has a codimension 1 foliation  $\mathcal{F}$ , then  $\chi(M) = 0$ .*

**PROOF.** From the above remark, there is a 2-fold cover  $\widetilde{M} \rightarrow M$  such that  $\widetilde{M}$  has a foliation  $\widetilde{\mathcal{F}}$  which is coorientable. This implies there is a nowhere vanishing vector field on  $\widetilde{M}$ . This implies  $\chi(\widetilde{M}) = 0$ , but  $\chi(\widetilde{M}) = 2\chi(M)$ , so  $\chi(M) = 0$ .  $\square$

CONJECTURE 3 (Half of the  $L$ -space conjecture). *If  $M$  is a closed prime 3-manifold, then  $M$  has a coorientable taut foliation (CTF) if and only if  $\pi_1(M)$  is LO.*

THEOREM (Gabai). *If  $M$  is a closed prime 3-manifold, with  $H_1(M)$  infinite, then  $M$  has a CTF.*

Gabai's theorem and Corollary 3.10 imply that this conjecture is true for  $H_1(M)$  infinite. Recall Theorem 3.9 said that for  $M$  a prime closed 3-manifold, then  $\pi_1(M)$  is locally indicable iff  $H_1(M)$  is infinite. Again being locally indicable implies LO. So the other option is that  $M$  is a QHS. We will see this conjecture is true for SFS QHS this conjecture is true.

## 6. The leaf space

Let  $\mathcal{F}$  be a codimension 1 foliation on closed  $n$ -manifold  $M$ . The leaf space  $\Lambda = \Lambda(\mathcal{F})$  of  $\mathcal{F}$  is the quotient space of  $M$  by identifying each leaf to a point.

EXAMPLE 5.7. Recall from example 5.4 the torus gets a foliation induced by a line in  $\mathbb{R}^2$  of rational slope. Then the leaf space is just the transverse meridian.

If we take the foliation we get from a line of irrational slope from example 5.4, every leaf is dense, so the leaf space is uncountable, but every point is dense. So it is not even  $T_1$ . So as a topological space this is really bad.

From now on, a 1-manifold will be a second countable topological space (possibly non-Hausdorff) such that every point has a neighborhood  $\cong \mathbb{R}$ . So  $\Lambda$  for the second part of the previous example is not a 1-manifold.

EXAMPLE 5.8. Consider a foliation of  $\mathbb{R}^2$  with leaves  $\mathbb{R}$  given as follows. Outside of the strip  $[-1, 1] \times \mathbb{R}$  foliate by vertical lines, and inside the strip foliate as in fig. 2. Call the leaves inside the strip  $L_t$ , indexed by their intersection with the axis  $\{0\} \times \mathbb{R}$ . Then  $\{\pm 1\} \times \mathbb{R}$  are leaves we will write as  $L_{\pm}$ .

Every neighborhood of  $L_{\pm}$  meets

$$(5.10) \quad \bigcup_{t \leq t_0} L_t$$

for some  $t_0$ . The leaf space looks like

$$(5.11) \quad \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array}$$

where the black points are the the images  $q(L_-)$  and  $q(L_+)$ , and they don't have disjoint neighborhoods. So it is locally Euclidean, but not Hausdorff.

THEOREM 5.5. *Let  $\mathcal{L}$  be a codimension 1 foliation of  $\mathbb{R}^2$ . Then*

- (1) *every leaf  $\cong \mathbb{R}$ ,*
- (2)  *$\Lambda(\mathcal{L})$  is a simply-connected 1-manifold.*

- PROOF. (1) If not, then  $\mathcal{F}$  has a leaf  $L \cong S^1$ .  $L$  bounds a disk  $D \subset \mathbb{R}^2$ , and  $\mathcal{L}|_D$  is a foliation of  $D$ , with  $\partial D$  a leaf. Therefore we get a foliation on  $D \cup_{\partial} D \cong S^1$ , and  $\chi(S^2) \neq 0$  which is the desired contradiction.
- (2) We will prove this later. □

### 6.1. $\mathbb{R}$ -covered foliations.

DEFINITION 5.3. Let  $\mathcal{F}$  be a foliation on a closed 3-manifold  $M$ . Then  $\mathcal{F}$  is  $\mathbb{R}$ -covered if  $\Lambda(\tilde{\mathcal{F}}) \cong \mathbb{R}$  where  $\tilde{\mathcal{F}}$  is the lift of  $\mathcal{F}$  to the universal cover  $\tilde{M}$ .

EXAMPLE 5.9. Let  $M$  be an  $F$ -bundle over  $S^1$ , where  $F$  is a closed orientable surface. Then if  $\mathcal{F}$  is the foliation of  $M$  with leaves  $\cong F$ , then  $\tilde{M} \cong \mathbb{R}^3$ , and  $\Lambda(\tilde{\mathcal{F}}) \cong \mathbb{R}$ .

THEOREM (Brittenham; Goodman-Shields). *Let  $\mathcal{F}$  be an  $\mathbb{R}$ -covered foliation on a closed 3-manifold  $M \not\cong S^2 \times S^1$  or  $\mathbb{RP}^3 \# \mathbb{RP}^3$ . Then  $\mathcal{F}$  is taut.*

So the point is that by Novikov's Theorem and Palmeira's Theorem,

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$$\begin{aligned}
 (5.12) \quad & \mathbb{R}\text{-covered} \implies \text{taut} \\
 (5.13) \quad & \implies \text{Reebless} \\
 (5.14) \quad & \implies (\tilde{M}, \tilde{\mathcal{F}}) \cong (\mathbb{R}^2, \mathcal{L}) \times \mathbb{R}
 \end{aligned}$$

where  $\mathcal{L}$  is a foliation of  $\mathbb{R}^2$  with leaves  $\cong \mathbb{R}$ . But then we have

$$(5.15) \quad \Lambda(\tilde{\mathcal{F}}) \cong \Lambda(\mathcal{L})$$

so  $\mathcal{F}$  is  $\mathbb{R}$ -covered iff  $\Lambda(\mathcal{L}) \cong \mathbb{R}$ . But this is equivalent to:

$$(5.16) \quad (\mathbb{R}^2, \mathcal{L}) \cong (\mathbb{R}, \{t\}) \times \mathbb{R}^2$$

with the product foliation with leaves  $\cong \mathbb{R}^2$ .

There are 3-manifolds with foliations but no  $\mathbb{R}$ -covered foliations. In [B2], Brittenham gave some examples which are graph manifolds. In [F] Fenley gave some examples which are hyperbolic.

THEOREM 5.6. *Let  $M$  be a closed 3-manifold with a co-orientable  $\mathbb{R}$ -covered foliation  $\mathcal{F}$ . Then  $\pi_1(M)$  is LO.*

PROOF.  $\pi_1(M)$  acts on  $(\tilde{M}, \tilde{\mathcal{F}})$ . Hence on  $\Lambda(\tilde{\mathcal{F}}) \cong \mathbb{R}$ .  $\mathcal{F}$  is co-orientable, which implies  $\tilde{\mathcal{F}}$  is co-orientable, and the action of  $\pi_1(M)$  preserves the transverse orientation, so the action on  $\mathbb{R}$  is by orientation preserving homeomorphisms.

Since  $M$  is compact, there is some compact  $C \subset \tilde{M}$  such that for all  $x \in \tilde{M}$ , there is  $g \in \pi_1(M)$  such that  $g(x) \in C$ . Then for all  $\lambda \in \Lambda(\tilde{\mathcal{F}})$ , there is  $g \in \pi_1(M)$  such that  $g(\lambda) \in q(X)$  where

$$(5.17) \quad q: \tilde{M} \rightarrow \Lambda(\tilde{\mathcal{F}}) \cong \mathbb{R}$$

is the quotient map. Therefore the action of  $\pi_1(M)$  on  $\mathbb{R}$  is nontrivial, so we get a nontrivial homomorphism  $\pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$ . If  $M \cong S^1 \times S^2$ , then  $\pi_1(M)$  is LO. And  $M \not\cong \mathbb{RP}^3 \# \mathbb{RP}^3$ , since  $\pi_1 \cong \mathbb{Z}/2 * \mathbb{Z}/2$ . In all other cases,  $\tilde{M} \cong \mathbb{R}^3$ , so  $M$  is irreducible, therefore prime, so by Theorem 3.11,  $\pi_1(M)$  is LO. □

### 7. Back to SFS's

A good source of  $\mathbb{R}$ -covered foliations is the following. Let  $M$  be a (closed) SFS. A foliation  $\mathcal{F}$  on  $M$  is *horizontal* if each Seifert fiber is transverse to  $\mathcal{F}$ .

NOTE. Horizontal implies taut.

This is because every leaf meets *some* Seifert fiber. Therefore  $\widetilde{M} \cong \mathbb{R}^3$  or  $S^2 \times \mathbb{R}$  by Novikov's Theorem. Assume  $\widetilde{M} \cong \mathbb{R}^3$ . Therefore there is some foliation  $\mathcal{L}$  of  $\mathbb{R}^2$  such that

$$(5.18) \quad (\widetilde{M}, \widetilde{\mathcal{F}}) \cong (\mathbb{R}^2, \mathcal{L}) \times \mathbb{R}$$

so the leaves of  $\widetilde{\mathcal{F}}$  are  $\cong \mathbb{R}^2$ . The codimension 2 foliation of  $M$  by circles, namely by the Seifert fibers lifts to a foliation of  $\widetilde{M}$  with leaves homeomorphic to  $\mathbb{R}$ . So  $\widetilde{M}$  is a product  $\mathbb{R}$ -bundle over  $\mathbb{R}^2$ .

Every Seifert fiber in  $M$  has a Seifert fibered neighborhood

$$(5.19) \quad \cong S^1 \times D^2 = q^{-1}(D)$$

for  $D \subset F$  where  $F$  is the base surface. So we have a diagram

$$(5.20) \quad \begin{array}{ccc} \widetilde{M} & \longrightarrow & M \\ \downarrow p & & \downarrow q \\ \mathbb{R}^2 & \longrightarrow & F \end{array}$$

such that

$$(5.21) \quad \widetilde{F} \cap (S^1 \times D^2) = \{\{t\} \times D^2 \mid t \in S^1\} .$$

Up in  $\widetilde{M} \cong \mathbb{R}^3$ , every  $\mathbb{R} \times \{x\}$  has a neighborhood homeomorphic to  $\mathbb{R} \times D_x$ , where  $D_x$  is a disk in  $\mathbb{R}^2$ , and

$$(5.22) \quad \widetilde{\mathcal{F}} \cap (\mathbb{R} \times D_x) = (\mathbb{R}, \{t\}) \times D_x .$$

So the leaves are all just  $\mathbb{R}^2$ , and they just intersect these vertical infinite cylinders in these meridian disks.

If  $L$  is a leaf of the foliation of  $\widetilde{M}$ , consider the restriction of  $p|_L : L \rightarrow \mathbb{R}^2$ . By (5.22),  $p(L)$  is open in  $\mathbb{R}^2$ . Also by eq. (5.22), if  $x \in \mathbb{R}^2 \setminus p(L)$ , then

$$(5.23) \quad D_x \cap p(L) = \emptyset ,$$

so  $p(L)$  is closed in  $\mathbb{R}^2$  as well. Therefore it is all of  $\mathbb{R}^2$ . Now the inverse of one of these disks is:

$$(5.24) \quad (p|_L)^{-1}(D_x) = (L \cap \mathbb{R} \times \{x\}) \times D_x ,$$

i.e. just a disjoint union of disks which each map homeomorphically onto  $D_x$ . Therefore this is a covering projection. Since  $L$  is connected,  $p|_L : L \rightarrow \mathbb{R}^2$  is a homeomorphism.

The point is the following theorem.

**THEOREM 5.7.** *A horizontal foliation on a SFS is  $\mathbb{R}$ -covered.*

**PROOF.** Let  $z \in \mathbb{R}^3$ . There is some leaf  $L$  of  $\widetilde{\mathcal{F}}$  such that  $z \in L$ . This leaf  $L$  has to meet  $\mathbb{R}_0 = \mathbb{R} \times \{0\}$  in a unique point  $t$ . Let  $(t, 0) = L \cap \mathbb{R}_0$  be this point. Then define  $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$  by

$$(5.25) \quad \alpha(z) = t .$$

Note  $\alpha$  is continuous, onto, and  $\alpha|_{\mathbb{R} \times \{x\}}$  is one-to-one. So now we can make the leaves “horizontal”. Define  $H: \mathbb{R}^3 \rightarrow \mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$  by sending

$$(5.26) \quad H(z) = (\alpha(z), p(z)) .$$

Then  $H$  is a continuous bijection, so it is a homeomorphism by invariance of domain.  $H$  takes  $\tilde{\mathcal{F}}$  to a foliation of  $\mathbb{R}^3$  by  $\{t\} \times \mathbb{R}^2$ , for  $t \in \mathbb{R}$ , so  $\Lambda(\tilde{\mathcal{F}}) \cong \mathbb{R}$ .  $\square$

**COROLLARY.** *If  $M$  is a closed SFS with a coorientable horizontal foliation, then  $\pi_1(M)$  is LO.*

**THEOREM 5.8.** *Let  $M$  be a SFS QHS. If  $\pi_1(M)$  is LO, then  $M$  has a coorientable horizontal foliation.*

We know

$$\text{horizontal} \implies \mathbb{R}\text{-covered} \implies \text{taut}$$

so this theorem says it is in fact  $\mathbb{R}$ -covered, so taut.

**PROOF.** Recall from Theorem 1.20  $\pi_1(M)$  LO implies there is a monomorphism

$$(5.27) \quad \varphi: \pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R}) .$$

So we have an action on  $\mathbb{R}$ , and then we want to construct an action on  $\mathbb{R}^2$ , and fit them together to act on  $\mathbb{R}^2 \times \mathbb{R}$ . Then our manifold will be quotient of this. The idea is that these first copies of  $\mathbb{R}$  are lifts of the Seifert fibers.

Recall  $\text{Fix}(G)$  is the fixed point set of  $G$ :

$$(5.28) \quad \text{Fix}(G) = \{x \in \mathbb{R} \mid g(x) = x \forall g \in G\} .$$

The action is *fixed-point free* if  $\text{Fix}(G) = \emptyset$ .

**Lemma 5.9.** *If there is a nontrivial homomorphism  $G \rightarrow \text{Homeo}_+(\mathbb{R})$ , then there exists one such that the corresponding action is fixed-point free.*

**PROOF.** This is nontrivial, so  $\text{Fix}(G) \subsetneq \mathbb{R}$  is proper and closed. So its complement is a nonempty disjoint union of open intervals. Each interval is  $G$ -invariant, so just restrict the action to some interval, and reparameterize as  $\mathbb{R}$ .  $\square$

Let  $M$  be a SFS QHS with  $\pi_1(M)$  LO. Recall this means  $M$  is of type  $S^2(p_1, \dots, p_n)$  for  $n \geq 3$ , and  $\pi_1$  is explicitly:

$$(5.29) \quad \pi_1(M) = \left\langle c_1, \dots, c_k, h \mid h \leftrightarrow c_i, c_i^{p_i} = h^{r_i}, \prod_{i=1}^n c_i = h^b \right\rangle .$$

Then  $\pi_1(M)$  LO implies there is a homomorphism  $\pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$  which has corresponding action which is fixed-point free.

**Lemma 5.10.** *Let  $g \in \text{Homeo}_+(\mathbb{R})$ . If  $g^m(x) = x$  for some  $m \neq 0$  then  $g(x) = x$ .*

**PROOF.** Suppose  $g(x) \neq x$ , say  $g(x) > x$ . Then  $g$  is order preserving, so  $g^2(x) > g(x) > x$ , etc.  $\square$

**Lemma 5.11.**  $\text{Fix}(h) = \emptyset$ .

PROOF. Suppose  $h(x) = x$ . Then  $c_i^{p_i} = h^{r_i}$  so

$$(5.30) \quad c_i^{p_i}(x) = x$$

so by Lemma 5.10,  $c_i(x) = x$ . But these generate the group, so for all  $g \in \pi_1(M)$   $g(x) = x$ .  $\square$

**Lemma 5.12.** *The action of  $\langle h \rangle$  on  $\mathbb{R}$  is free and properly discontinuous. So it is a covering space action.*

PROOF. The action is free by Lemmata 5.10 and 5.11. Recall the definition of a *properly discontinuous action* of  $G$  on  $X$  is that for any compact  $C \subset X$ ,  $g(C) \cap C = \emptyset$  for all but finitely many  $g \in G$ .

WLOG,  $h(x) > x$  for all  $x \in \mathbb{R}$ . Therefore there is some neighborhood  $U$  of  $x$  such that

$$(5.31) \quad h^r(U) \cap U = \emptyset$$

for all  $r \neq 0$ .  $\square$

REMARK 5.9. This implies that  $h$  is conjugate (in  $\text{Homeo}_+(\mathbb{R})$ ) to translation  $\tau: \mathbb{R} \rightarrow \mathbb{R}$  where  $\tau(x) = x + 1$ .

Since Lemma 5.12 implies this is a covering space action, we have two covering spaces  $\mathbb{R} \rightarrow \mathbb{R}/\langle h \rangle$  and  $\mathbb{R} \rightarrow \mathbb{R}/\langle \tau \rangle$ , but in both cases the quotient is  $S^1$ . So just write down any homeomorphism  $f: S^1 \rightarrow S^1$ , and it lifts:

$$(5.32) \quad \begin{array}{ccc} \mathbb{R} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R}/\langle h \rangle \cong S^1 & \xrightarrow{f} & \mathbb{R}/\langle \tau \rangle \cong S^1 \end{array}$$

and then  $\tilde{f}h\tilde{f}^{-1} = \tau$ .

EXERCISE 5.1. Show that this implies  $h$  is conjugate to  $\tau$ .

Now

$$(5.33) \quad \pi_1(M)/\langle h \rangle = \left\langle c_1, \dots, c_n \left| c_i^{p_i} = 1, \prod_i c_i = 1 \right. \right\rangle$$

$$(5.34) \quad = T(p_1, \dots, p_n)$$

where  $n \geq 3$  and  $p_i \geq 2$ . Recall for a Euclidean  $n$ -gon, with angles  $\alpha_i$ ,  $1 \leq i \leq n$ . Then the exterior angles are the complements  $\pi - \alpha_i$ , and the sum is

$$(5.35) \quad \sum_{i=1}^n (\pi - \alpha_i) = 2\pi.$$

There exists an  $n$ -gon  $P = P(p_1, \dots, p_n)$  in  $\left\{ \begin{array}{c} \mathbb{S}^2 \\ \mathbb{E}^2 \\ \mathbb{H}^2 \end{array} \right\}$  with angles  $\pi/p_i$  if

$$(5.36) \quad \sum_{i=1}^n \frac{1}{p_i} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} (n-2).$$



We treated the  $n = 3$  case in Lemma 4.10. So let  $n \geq 4$ , where we have

$$(5.37) \quad \sum_{i=1}^n \frac{1}{p_i} \leq \frac{n}{2} \leq n - 2$$

with equality iff  $n = 4$  and

$$(5.38) \quad p_1 = p_2 = p_3 = p_4 = 2 .$$

In the Euclidean plane, we just have a square. So now assume  $P \subset \mathbb{H}^2$ .

Let  $\tilde{\Gamma}$  denote the subgroup of  $\text{Isom}(\mathbb{H}^2)$  generated by  $p_1, \dots, p_n$ . This has an index 2 subgroup  $\Gamma < \tilde{\Gamma}$  generated by the rotations  $\gamma_1, \dots, \gamma_n$ .

LEMMA (Poincaré's lemma). (1) *The images of the geodesic rays containing  $A_i A_{i+1}$  give a tiling of  $\mathbb{H}^2$  by the translations of  $P$  under  $\tilde{\Gamma}$ .  $P$  is a fundamental domain for the action of  $\tilde{\Gamma}$ .*

*Similarly,  $P \cup p_1(P)$  is a fundamental domain for  $\Gamma$ .*

(2) *The quotient map  $T(p_1, \dots, p_n) \rightarrow \Gamma$  is an isomorphism. I.e.*

$$(5.39) \quad \Gamma = \left\langle \gamma_1, \dots, \gamma_n \mid \gamma_i^{p_i} = 1, \prod_i \gamma_i = 1 \right\rangle .$$

The upshot is that

$$(5.40) \quad \Gamma \cong \pi_1(M) / \langle h \rangle .$$

**Lemma 5.13.** *If  $G \in \Gamma$  has a fixed point ( $\in \mathbb{H}^2$ ), then  $g$  is conjugate to some  $\gamma_i$ .*

PROOF. If  $g$  fixes a vertex in the tiling, then a conjugate of  $g$  fixes some  $A_i$ . Recall an orientation preserving isometry of  $\mathbb{H}^2$  is either

- elliptic: one fixed point in  $\mathbb{H}^2$ ,
- parabolic: one fixed point in the circle at infinity,
- hyperbolic: two fixed points in the circle at infinity.

Since  $P \cup p_1(P)$  is a fundamental domain for  $\Gamma$ ,  $g$  can't fix any point in the interior.  $\square$

So now we have this quotient

$$(5.41) \quad \pi_1(M) \rightarrow \pi_1(M) / \langle h \rangle = \Gamma \hookrightarrow \text{Isom}_+(\mathbb{H}^2)$$

which gives an action of  $\pi_1(M)$  on  $\mathbb{H}^2$ . Then

$$(5.42) \quad \pi_1(M) \hookrightarrow \text{Homeo}_+(\mathbb{R})$$

gives an action of  $\pi_1(M)$  on  $\mathbb{R}$ . Now define the diagonal action of  $\pi_1(M)$  on  $\mathbb{H}^2 \times \mathbb{R}$  by  $g(x, t)(g(x), g(t))$ . The point will be that this is a covering space action, and the quotient is just  $M$ .

**Lemma 5.14.** *This action of  $\pi_1(M)$  on  $\mathbb{H}^2 \times \mathbb{R}$  is*

- (1) *free, and*
- (2) *properly discontinuous.*

PROOF. (1) Suppose we have  $g$  such that  $g(x, t) = (x, t)$ . Let  $g \mapsto \bar{g} \in \Gamma$  (under the quotient map). So  $\bar{g}(x) = x$ . Therefore  $\bar{g}$  is conjugate to  $\gamma_i^k$ . Then  $g$  is conjugate to  $c_i^k h^\ell$  so there is  $w$  such that

$$(5.43) \quad g = w^{-1} (c_i^k h^\ell) w .$$

Now  $g(t) = t$ , so

$$(5.44) \quad x_i^k h^\ell(e) = s$$

where  $s = w(t)$ . Therefore

$$(5.45) \quad c_i^{kp_i} h^{lp_i}(s) = s$$

but

$$(5.46) \quad c_i^{kp_i} h^{lp_i} = h^{kr_i + \ell p_i}$$

but  $h$  acts freely on  $\mathbb{R}$  by Lemma 5.12, so  $kr_i + \ell p_i = 0$ . Since  $p_i$  and  $r_i$  are relatively prime, this means  $k = ap_i$  for some  $a \in \mathbb{Z}$ . So  $g$  is conjugate to

$$(5.47) \quad c_i^{ap_i} h^\ell = h^{ar_i + \ell}$$

so this acts freely by Lemma 5.12.

(2) Now we show this action is properly discontinuous.

LEMMA. *Let  $H \triangleleft G$ . Suppose that  $G/H$  acts properly discontinuously on  $X$ , and  $G$  acts on  $Y$  such that the action of  $G$  on  $Y$  is properly discontinuous. Then the diagonal action of  $G$  on  $X \times Y$  is properly discontinuous.*

EXERCISE 5.2. Prove this.

□

Hence

$$(5.48) \quad \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{H}^2 \times \mathbb{R} / \pi_1(M)$$

is a covering projection with Hausdorff quotient. Therefore the quotient is a closed Hausdorff 3-manifold. It is irreducible, and  $\pi_1(N) \cong \pi_1(M)$ , so  $N \cong M$ . E.g. by Perelman.

Now the lines  $\{x\} \times \mathbb{R}$  foliate  $\mathbb{H}^2 \times \mathbb{R}$ , so we get a foliation of  $M$  with leaves  $S^1$ . Hence by Epstein, this is a Seifert fibration of  $M$ . The planes  $\mathbb{H}^2 \times \{t\}$  give a horizontal foliation  $\mathcal{F}$  on  $M$ . Then the action of  $\pi_1(M)$  on  $\mathbb{R}$  is orientation preserving so  $\mathcal{F}$  is coorientable. ■

REMARK 5.10. Theorem 5.8 is false without the assumption that  $M$  is a QHS. For example, let  $M$  be an  $S^1$ -bundle over a closed orientable surface of genus  $g \geq 2$  with Euler number  $e = -b$ .

THEOREM (Milnor-Wood [M4, W2]).  *$M$  has a horizontal foliation iff  $|e| \leq 2g - 2$ .*

**Corollary 5.15.** *Let  $M$  be a SFS ZHS which is not  $S^3$  or  $\Sigma(2, 3, 5)$ . Then  $M$  has a coorientable horizontal foliation ( $\pi_1(M)$  is LO, by Theorem 5.6).*

CONJECTURE 4. *Every ZHS besides  $S^3$  or  $\Sigma(2, 3, 5)$  has a coorientable taut foliation.*

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REMARK 5.11. The Milnor-Wood theorem was generalized by Eisenbud-Hirsch-Neumann [EHN], Jenkins-Neumann [JN], and Naimi [N] to say exactly when a SFS has a coorientable horizontal foliation. For a SFS QHS with  $\pi_1$  infinite, infinitely many do, and infinitely many don't.

REMARK 5.12. The restriction to horizontal foliations is not necessary.

THEOREM 5.16 (Brittenham [B1]; Claus [C]). *Let  $M$  be a SFS. If  $\mathcal{F}$  is a coorientable taut foliation with no compact leaves, then  $\mathcal{F}$  is isotopic to a horizontal foliation.*

For  $M$  a QHS, any compact orientable surface separates  $M$ . So no compact leaf can have a transverse loop. So

$$\text{taut} \implies \text{no compact leaves.}$$

So combining this with the  $H_1$  infinite case we get the following.

**THEOREM 5.17.** *Let  $M$  be a closed SFS. Then  $M$  has a coorientable taut foliation iff  $\pi_1(M)$  is LO.*

## CHAPTER 6

### Biorderability

We will show that free groups (and closed orientable surface groups) and right-angled Artin groups are biorderable.

#### 1. Residual nilpotence

Let  $H < G$ . The *commutator subgroup*  $[G, H]$  is the subgroup generated by elements of the form  $g^{-1}h^{-1}gh$ . The *lower central series* of  $G$  is

$$(6.1) \quad G = G_0 > G_1 > G_2 > \dots$$

where

$$(6.2) \quad G_{n+1} = [G_n, G]$$

for  $n \geq 0$ . Note that for  $\varphi: G \rightarrow H$  a homomorphism we have

$$(6.3) \quad \varphi(G_n) < H_n.$$

So  $G_n$  is a fully invariant subgroup of  $G$ . In particular,  $G_n \triangleleft G$ , so (6.1) is a *central series*, meaning the successive quotients are central.

**Lemma 6.1.**

$$(6.4) \quad 1 \rightarrow G_n/G_{n+1} \rightarrow G/G_{n+1} \rightarrow G/G_n \rightarrow 1$$

is a central extension. In particular  $G_n/G_{n+1}$  is abelian.

PROOF. If  $x \in G_n$ ,  $g \in G$ , then  $[x, g] \in G_{n+1}$ . Therefore the images  $\bar{x} \in G_n/G_{n+1}$ ,  $\bar{g} \in G/G_{n+1}$  satisfy  $[\bar{x}, \bar{g}] = 1$ , so they commute.  $\square$

$G$  is *nilpotent* if  $G_n = \{1\}$  for some  $n$ . The least such  $n$  is the *nilpotence class* of  $G$ .

EXAMPLE 6.1. If  $n = 0$ ,  $G = \{1\}$ .

EXAMPLE 6.2. If  $n = 1$ ,  $G$  is abelian.

Recall the *derived series* of  $G$  is

$$(6.5) \quad G = G^{(0)} > G^{(1)} > \dots$$

where

$$(6.6) \quad G^{(n+1)} = [G^{(n)}, G^{(n)}]$$

for  $n \geq 0$ . For  $\varphi: G \rightarrow H$  a homomorphism, we get

$$(6.7) \quad \varphi(G^{(n)}) < H^{(n)}$$

so  $G^{(n)} \triangleleft G$ .  $G$  is *solvable* if  $G^{(n)} = \{1\}$  for some  $n$ .

Note that  $G^{(n)} < G_n$  so we have

abelian  $\implies$  nilpotent  $\implies$  solvable.

EXAMPLE 6.3.  $S_3$  is solvable and not nilpotent. We have

$$(6.8) \quad 1 \rightarrow \underbrace{\mathbb{Z}/3}_{\langle (123) \rangle} \rightarrow S_3 \rightarrow \mathbb{Z}/2 \rightarrow 1 .$$

The derived series terminates since  $G^{(1)} = \mathbb{Z}/2$ , and  $G^{(2)} = \{1\}$ . But

$$(6.9) \quad (123) = [(123), (12)] \in G_2$$

so therefore

$$(6.10) \quad \mathbb{Z}/3 = G_1 = G_2 = \dots$$

so the lower central series never terminates.

Note that the properties of being abelian, nilpotent, and solvable are closed under subgroups and quotients.

**Lemma 6.2.** *Suppose  $G$  is  $n$ -nilpotent and  $x \in G$ . Then  $H = \langle x, G_1 \rangle \triangleleft G$  and is  $(n-1)$ -nilpotent.*

PROOF. Note that

$$(6.11) \quad H = \{x^r c \mid t \in \mathbb{Z}, c \in G_1\}$$

since

$$(6.12) \quad x^r c \cdot x^s d \equiv x^{r+s} cd \pmod{G_2} .$$

(1) If  $g \in G$ , then

$$(6.13) \quad g^{-1} (x^r c) g \equiv x^r \pmod{G_1}$$

so  $H \triangleleft G$ .

(2) We claim  $H_1 < G_2$ .

$$(6.14) \quad [x^r, x^s d] = c^{-1} x^{-r} d^{-1} x^{-s} x^r c x^s d$$

$$(6.15) \quad = c^{-1} x^{-r} d^{-1} x^r x^{-s} c x^s d$$

$$(6.16) \quad \equiv x^{-1} x^{-s} c x^s x^{-r} d^{-1} x^r d \pmod{G_2}$$

$$(6.17) \quad = [c, x^r] [x^r, d]$$

so  $H_{m-1} < G_m$  for all  $m \geq 2$ . Therefore  $H_{n-1} < G_n = \{1\}$ .

□

For any group  $G$ , define

$$(6.18) \quad \text{Tor}(G) = \{g \in G \mid \exists k \neq 0 \text{ s.t. } g^k = 1\} \subset G .$$

THEOREM 6.3. *If  $G$  is nilpotent then  $\text{Tor}(G)$  is a (characteristic) subgroup of  $G$ .*

PROOF. We will induct on the nilpotence class of  $G$ . For  $n = 1$ ,  $G$  is abelian. For  $n > 1$ , let  $a, b \in \text{Tor}(G)$ . We must show

$$(6.19) \quad ab \in \text{Tor}(G) .$$

Let  $H = \langle b, G_1 \rangle$ . By the inductive hypothesis and Lemma 6.2,  $\text{Tor}(H)$  is a characteristic subgroup of  $G$ . By Lemma 6.2,  $H \triangleleft G$ , so  $\text{Tor}(H) \triangleleft G$ .

Suppose  $a^k = 1$  for some  $k \neq 0$ . Then

$$(6.20) \quad (ab)^k = (aba^{-1}) a^2 b a^{-2} (a^k b a^{-k}) a^k$$

$$(6.21) \quad \in \text{Tor}(H)$$

so therefore  $(ab)^k$  has finite order, so  $(ab)$  has finite order.  $\square$

Later we will need the following.

**THEOREM 6.4** (Unique extraction of roots). *Let  $G$  be torsion free and nilpotent, and  $a, b \in G$ . If  $a^k = b^k$  for  $k \neq 0$ , then  $a = b$ .*

**PROOF.** Induct on nilpotence class  $n$  of  $G$ . For  $n = 1$ ,  $G$  is abelian so

$$(6.22) \quad a^k = b^k \implies (a^{-1}b)^k = 1 \implies a^{-1}b = 1 \implies a = b$$

as desired.

For  $n > 1$

$$(6.23) \quad (b^{-1}ab)^k = b^{-1}a^k b = a^k .$$

On the other hand,

$$(6.24) \quad b^{-1}ab = a[ab]$$

so therefore  $a$  and  $b^{-1}ab$  are both elements of  $\langle a, G_1 \rangle$ . But this group has nilpotence class  $(n-1)$  by Lemma 6.2. So by the inductive hypothesis  $b^{-1}ab = a$ . So

$$(6.25) \quad 1 = a^{-k}b^k = (a^{-1}b)^k$$

so  $a^{-1}b = 1$ , and  $a = b$ .  $\square$

Let  $\varphi: G \rightarrow G_n$  denote the quotient. Define

$$(6.26) \quad G(n) := \varphi^{-1}(\text{Tor}(G/G_n))$$

$$(6.27) \quad = \{g \in G \mid \exists k \neq 0 \text{ s.t. } g^k \in G_n\} .$$

By Lemma 6.2, since  $G/G_n$  is nilpotent we have  $G(n) \triangleleft G$ . Then we have a central series:

$$(6.28) \quad G = G(0) > G(1) > G(2) > \dots$$

called the *rational lower central series* of  $G$ .

For  $\varphi: G \rightarrow H$  a homomorphism we get

$$(6.29) \quad \varphi(G(n)) < H(n) .$$

**Lemma 6.5.**  *$G/G(n)$  is torsion-free nilpotent.*

**PROOF.** Suppose  $g \in G$ , and  $g^m \in G(n)$  for some  $m \neq 0$ . Then

$$(6.30) \quad (g^m)^k \in G_n$$

for some  $k \neq 0$ . Therefore  $g \in G(n)$ , so  $G/G(n)$  is torsion free.

Now notice  $G_n < G(n)$ , so therefore  $G/G(n)$  is a quotient of  $G/G_n$ , so nilpotent.  $\square$

**Lemma 6.6.**  *$[G(n), G] < G(n+1)$  so  $\{G(n)\}$  is a central series.*

PROOF. Let  $x \in G(n)$ ,  $g \in G$ ,  $x^k \in G_n$  for some  $k \neq 0$ . Then

$$(6.31) \quad [x^k, g] \in G_{n+1} < G(n+1)$$

so therefore

$$(6.32) \quad x^{-k} g^{-1} x^k g \equiv 1 \pmod{G(n+1)}$$

so therefore

$$(6.33) \quad g^{-1} x^k g \equiv x^k \pmod{G(n+1)}$$

so by Lemma 6.5 and Theorem 6.4,

$$(6.34) \quad g^{-1} x g \equiv x \pmod{G(n+1)}$$

so

$$(6.35) \quad [x, g] \in G(n+1) .$$

□

REMARK 6.1.  $\{G(n)\}$  is the most rapidly decreasing central series with  $G/G(n)$  torsion-free.

**Lemma 6.7.**  *$G$  is torsion-free nilpotent iff  $G(n) = 1$  for some  $n$ .*

EXERCISE 6.1. Prove this.

The *torsion-free nilpotence class* is the least such  $n$ .

THEOREM 6.8. *If  $G$  is torsion-free nilpotent, then  $G$  is BO.*

PROOF. Induct on the torsion-free nilpotence class  $n$ . For  $n = 0$ ,  $G = \{1\}$ . So assume it is true for some  $n$ . By Lemma 6.6 we have a central extension

$$(6.36) \quad 1 \rightarrow G(n)/G(n+1) \rightarrow G/G(n+1) \rightarrow G/G(n) \rightarrow 1 .$$

$G/G(n)$  is of torsion-free nilpotence class  $(n-1)$ , so the inductive hypothesis applies to it. Then  $G(n)/G(n+1)$  is torsion-free abelian, so it has a conjugacy invariant BO. (Recall for  $G$  abelian,  $G$  is BO iff  $G$  is torsion-free.) The upshot is, that

$$(6.37) \quad G = G/G(n+1)$$

is biorderable by Theorem 1.13.

□

By Corollary 2.10 residually BO implies BO.

**Corollary 6.9.** *If  $G$  is residually torsion-free nilpotent, then  $G$  is BO.*

**Lemma 6.10.** (1)  *$G$  being residually nilpotent if and only if*

$$(6.38) \quad \bigcap_{n=0}^{\infty} G_n = \{1\} .$$

(2)  *$G$  is residually torsion-free nilpotent if and only if*

$$(6.39) \quad \bigcap_{n=1}^{\infty} G(n) = \{1\} .$$

EXERCISE 6.2. Prove this.

## 2. Free groups

Now we will work towards proving the following.

**THEOREM 6.11.** *Free groups are residually torsion-free nilpotent.*

**Corollary 6.12.** *Free groups are BO.*

**WARNING 6.1.** Torsion free and residually nilpotent does not imply residually torsion-free nilpotent.

**EXAMPLE 6.4.** For

$$(6.40) \quad G = \langle a, b, c \mid a^2, b^2 \text{ central}; a^2 b^2 = c^2 \rangle$$

one can show that  $G$  is torsion-free free and residually nilpotent. But it is not BO, so not residually torsion-free nilpotent.

This is not BO because it has *generalized torsion*. For example, let  $g = abc^{-1}$ . This is a nontrivial element, but

$$(6.41) \quad (ac)^{-1} g (ac) \cdot c^{-1} g c \cdot a^{-1} g a \cdot g = 1 .$$

I.e. we have a product

$$(6.42) \quad \prod_{i=1}^k x_i^{-1} g x_i = 1$$

for some  $x_i$ .

**QUESTION 4** (Motegi-Teragaito [MT]). Let  $M$  be a 3-manifold, possibly with boundary, with  $H_1(M)$  infinite. Does  $\pi_1(M)$  not BO imply  $\pi_1(M)$  has generalized torsion?

**REMARK 6.2.** There exists a group  $G$  which is not BO and has no generalized torsion.



## Homology and cohomology of groups

Lecture 17; March  
31, 2020

### 1. Topological point of view

Let  $G$  be a group. Then we have two facts.

- (i) There exists a CW-complex  $X$  such that  $\pi_1(X) \cong G$ , and  $\pi_i(X) = 0$  for  $i \geq 2$ .
- (ii) Any two such complexes are homotopy equivalent.

Then we define

$$(A.1) \quad H_*(X) = H_*(X) \quad H^*(G) = H^*(X) .$$

But we want (co)homology with coefficients. So for  $A$  a  $\mathbb{Z}G$ -modules, we define

$$(A.2) \quad H_*(G; A) = H_*(X; A) \quad H^*(G; A) = H^*(X; A) .$$

Let  $\tilde{X} \rightarrow X$  be the universal cover. Then  $\pi_1(\tilde{X}) = 1$ , and  $\pi_i(\tilde{X}) = \pi_i(X) = 0$  for  $i \geq 2$ .

The point being that  $\tilde{X}$  is contractible. Recall by the general theory of covering spaces that  $G$  acts freely on  $\tilde{X}$ , and

$$(A.3) \quad \tilde{X}/G = X .$$

Write  $C_q(\tilde{X})$  for the cellular  $q$ -chains in  $\tilde{X}$ , i.e. the free  $\mathbb{Z}G$ -module on the set of  $q$ -cells in  $\tilde{X}$ . We get an augmented chain complex

$$(A.4) \quad \cdots \rightarrow C_q(\tilde{X}) \xrightarrow{\partial_q} C_{q-1}(\tilde{X}) \rightarrow \cdots \rightarrow C_1(\tilde{X}) \rightarrow C_0(\tilde{X}) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

where  $\epsilon$  is induced by  $\tilde{X} \rightarrow \text{pt}$ . Since  $\tilde{X}$  is contractible,  $\tilde{H}_*(\tilde{X}) = 0$ . Therefore eq. (A.4) is exact. Let  $\mathbb{Z}$  have the trivial  $\mathbb{Z}G$ -module structure. I.e.  $g \cdot a = a$  for all  $g \in G$  and all  $a \in \mathbb{Z}$ . Then

$$(A.5) \quad C_q(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z} \cong C_q(X)$$

is just the cellular  $q$ -chains in  $X$ . Therefore

$$(A.6) \quad H_*(X) = H_*(X; \mathbb{Z}) \cong H_*(C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z}) .$$

So now for  $A$  any left  $\mathbb{Z}G$ -module, we define

$$(A.7) \quad H_*(G; A) = H_*(X; A) = H_*(C_*(\tilde{X}) \otimes_{\mathbb{Z}G} A) .$$

REMARK A.1. Strictly speaking,  $C_*(\tilde{X})$  should be a right  $\mathbb{Z}G$ -module. But any left  $\mathbb{Z}G$ -module is a right  $\mathbb{Z}G$ -module by letting the inverse act on the left:

$$(A.8) \quad a \cdot g := g^{-1} a .$$

Similarly, we define the cohomology to be:

$$(A.9) \quad H^*(G; A) = H^*(X; A) = H^*\left(\operatorname{Hom}_{\mathbb{Z}G}\left(C_*\left(\tilde{X}\right), A\right)\right) .$$

## 2. Algebraic point of view

Let  $\mathbb{Z}$  be the trivial  $\mathbb{Z}G$ -module as before. A *free  $\mathbb{Z}G$ -resolution of  $\mathbb{Z}$*  is an exact sequence

$$(A.10) \quad \cdots \rightarrow F_q \rightarrow F_{q-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{Z} \rightarrow 0$$

where  $F_q$  is a free  $\mathbb{Z}G$ -module for all  $q \geq 0$ . An example of this is eq. (A.4) from above.

FACT 5. *Any two free  $\mathbb{Z}G$ -resolutions are chain homotopy equivalent. Therefore*

$$(A.11) \quad H_*(G; A) = H_*(F \otimes_{\mathbb{Z}G} A)$$

and

$$(A.12) \quad H^*(G; A) = H^*(\operatorname{Hom}_{\mathbb{Z}G}(F, A)) .$$

Let

$$(A.13) \quad C_q = \mathbb{Z}[G^{q+1}]$$

where  $q \geq 0$ , with  $\mathbb{Z}G$ -module defined by

$$(A.14) \quad g \cdot (g_0, \dots, g_q) = (gg_0, \dots, gg_q) .$$

Then define

$$(A.15) \quad \partial_q: C_q \rightarrow C_{q-1}$$

by

$$(A.16) \quad \partial_q(g_0, \dots, g_q) = \sum_{i=0}^q (-1)^i (g_0, \dots, \widehat{g_i}, \dots, g_q) .$$

This is like the boundary map for any kind of homology theory, it ranges over the omission of some “face”.

Then we get an augmented chain complex:

$$(A.17) \quad \cdots \rightarrow C_q \rightarrow C_{q-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

where  $\epsilon(g) = 1$  for all  $g \in G$ .

**Lemma A.1.** *The sequence (A.17) is exact.*

SKETCH PROOF. Let  $X^0 = G$ . Suppose we have constructed  $X^{n-1}$ . Then to construct  $X^n$ , for each  $(n+1)$ -tuple,  $(g_0, \dots, g_n) \in G^{n+1}$ , take a standard  $n$ -simplex, and attach to  $X^{n-1}$  along the faces. Then we get a CW-complex

$$(A.18) \quad X = \bigcup_{n=0}^{\infty} X^n .$$

Note that  $G$  acts freely on  $X$ . Then the key point is that  $X$  is contractible. The idea is that we have inclusions:

$$(A.19) \quad (g_0, \dots, g_n) \subset (1, g_0, \dots, g_n) .$$

So we just strong deformation retract each cell to the vertex 1. This gives a strong deformation retraction from  $X$  to the vertex 1. Therefore we get a strong deformation retraction of all of  $X$  to the vertex 1.  $\square$

**Lemma A.2.**  $C_q$  is a free  $\mathbb{Z}G$ -module with basis

$$(A.20) \quad \{(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_q) \mid g_i \in G\} .$$

We will write these tuples as:

$$(A.21) \quad [g_1 \mid \dots \mid g_q] .$$

PROOF.  $G$  acts freely on  $G^{q+1}$ ,  $\mathbb{Z}[G^{q+1}]$  is a free  $\mathbb{Z}$ -module on  $G^{q+1}$ , so  $\mathbb{Z}[G^{q+1}]$  is a free  $\mathbb{Z}G$ -module on the set of orbits  $X/G$ . Then there is a one-to-one correspondence between the orbits and this basis.  $\square$

So Lemma A.1 and Lemma A.2 imply that (A.17) is a free  $\mathbb{Z}G$ -resolution of  $\mathbb{Z}$ . This is called the *standard* or *bar* resolution.

**Lemma A.3.**

$$(A.22) \quad \begin{aligned} \partial [g_1 \mid \dots \mid g_q] &= g_1 [g_2 \mid \dots \mid g_q] \\ &+ \sum_{i=1}^{q-1} (-1)^i [g_1 \mid \dots \mid g_{i-1} \mid g_i g_{i+1} \mid g_{i+2} \mid \dots \mid g_q] + (-1)^q [g_1 \mid \dots \mid g_{q-1}] . \end{aligned}$$

Now we can identify  $[g_1 \mid \dots \mid g_q]$  with  $(g_1, \dots, g_q) \in G^q$ .

For  $A$  a  $\mathbb{Z}G$ -module, we have a one-to-one correspondence

$$(A.23) \quad \text{Hom}_{\mathbb{Z}G}(C_q, A) \quad \leftrightarrow \quad \{\text{functions } G^q \rightarrow A\}$$

by Lemma A.2.

Recall the definition of the coboundary map. For an element  $u \in \text{Hom}_{\mathbb{Z}G}(C_{q-1}, A)$ , we have

$$(A.24) \quad \delta u \in \text{Hom}_{\mathbb{Z}G}(C_q, A)$$

is defined by

$$(A.25) \quad (\delta u)(c) = u(\delta c) .$$

Therefore, for  $f : G^{q-1} \rightarrow A$ , by Lemma A.3 we have that

$$(A.26) \quad \begin{aligned} (\delta f)(g_1, \dots, g_q) &= g_1 f(g_2, \dots, g_q) \\ &+ \sum_{i=1}^{q-1} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_q) + (-1)^q f(g_1, \dots, g_{q-1}) g_q . \end{aligned}$$

**2.1. Examples in low-dimensions.** Recall  $C_0$  was the free  $\mathbb{Z}G$ -module on the 0-tuple  $( )$ . Let  $C_0 \cong \mathbb{Z}G$  is the free abelian group on  $( )$ . Then the augmentation map  $\epsilon : C_0 \rightarrow \mathbb{Z}$  is defined by  $\epsilon(g) = 1$  for all  $g \in \mathbb{Z}$ . Then

$$(A.27) \quad \partial_1 : C_1 \rightarrow C_0$$

is defined by

$$(A.28) \quad \partial_1(g) = g( ) - ( ) = (g-1)( ) ,$$

or just  $\partial_1 = g - 1$ . Then

$$(A.29) \quad \partial_2 : C_2 \rightarrow C_1$$

is defined by

$$(A.30) \quad \partial_2(g, h) = g(h) = (gh) + (g)$$

and

$$(A.31) \quad \partial_3(g, h, k) = g(h, k) - (gh, k) + (g, hk) - (g, h) .$$

Dually,

$$(A.32) \quad \delta_0: \text{Hom}(C_0, A) \rightarrow \text{Hom}(C_1, A)$$

takes  $f: ( ) \rightarrow A$  to

$$(A.33) \quad (\partial_0 f)(g) = f(\partial_1(g)) = f((g-1)( ))$$

$$(A.34) \quad = (g-1)f(( ))$$

Then

$$(A.35) \quad (\delta_1 f)(g, h) = f(\partial_2(g, h))$$

$$(A.36) \quad = gf(h) - f(gh) + f(g)$$

for  $f: G \rightarrow A$ . Now for  $f: G \times G \rightarrow A$  we get

$$(A.37) \quad (\delta_2 f)(g, h, k) = gf(h, k) - f(gh, k) + f(g, hk) - f(g, h) .$$

So this tells us that  $f$  is a 2-cocycle exactly when

$$(A.38) \quad 0 = gf(h, k) - f(gh, k) + f(g, hk) - f(g, h) .$$

This is the 2-cocycle condition.

Lecture 18; 2020

### 3. Group extensions

Let  $G$  and  $A$  be groups. A group extension of  $G$  by  $A$  is a group  $E$  which fits into the short exact sequence

$$(A.39) \quad 1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1 .$$

Two such extensions are *equivalent* if and only if there exists a group homomorphism  $\psi: E \rightarrow E$  (which is necessarily an isomorphism) such that

$$(A.40) \quad \begin{array}{ccccc} & & E & & \\ & \nearrow & \downarrow \psi & \searrow \pi & \\ 1 & \longrightarrow & A & & G \longrightarrow 1 \\ & \searrow & \downarrow \psi & \nearrow \pi' & \\ & & E & & \end{array}$$

commutes. Write  $\mathcal{E}(G, A)$  for the equivalence classes of extensions of  $G$  by  $A$ . From now on, A will be abelian.

Then (A.39) makes  $A$  a (left)  $\mathbb{Z}G$ -module. For  $g \in G$ , let  $e \in E$  such that  $\pi(e) = g$ . For  $a \in A$ , define

$$(A.41) \quad g \cdot a = eae^{-1} .$$

Note that

$$(A.42) \quad A < Z(E) ,$$

where  $Z(E)$  is the center of  $E$ , if and only if  $A$  is a trivial  $\mathbb{Z}G$ -module. In this case we say (A.39) is a *central extension*.

We say (A.39) *splits* iff there is some homomorphism  $\sigma: G \rightarrow E$  such that

$$(A.43) \quad \pi\sigma = \text{id}_G .$$

This is equivalent to  $E$  be the semidirect product

$$(A.44) \quad E = A \rtimes G .$$

Set theoretically this is the cartesian product, and the group structure is given by:

$$(A.45) \quad (a, g) \cdot (b, h) = (a + gb, gh) .$$

If  $A$  is a trivial  $\mathbb{Z}G$ -module, then  $A \rtimes G$  is just the direct product  $A \times G$ .

For any extension eq. (A.39), we can define a set-theoretic section  $s: G \rightarrow E$ . Then we can define

$$(A.46) \quad f: G \times G \rightarrow A$$

by

$$(A.47) \quad f(g, h) = s(g) s(h) s(gh)^{-1} \in A .$$

Then  $s$  is a homomorphism if and only if  $f(g, h) = 0$  for all  $g, h \in G$ .

**Lemma A.4.**  *$f$  determines the extension (A.39) (up to equivalence).*

PROOF. First, note that, as a set,  $E$  is just  $A \times G$ . In particular:

$$(A.48) \quad \begin{array}{ccc} A \times G & \longrightarrow & E \\ (a, g) & \longmapsto & a \cdot s(g) \end{array} .$$

So we just need to know the multiplication. Pulling back the multiplication in  $E$ , we get

$$(A.49) \quad (a, g) \cdot (b, h) = a \cdot s(g) \cdot b \cdot s(h)$$

$$(A.50) \quad = a \cdot s(g) b s(g)^{-1} \cdot s(g) s(h)$$

$$(A.51) \quad = \underbrace{a \cdot s(g) b s(g)^{-1}}_{\in A} f(g, h) s(g, h)$$

$$(A.52) \quad = (a + gb + f(g, h), gh)$$

so the multiplication is determined by  $f$ . □

**Lemma A.5.**  *$f: G \times G \rightarrow A$  is a 2-cocycle in  $\text{Hom}_{\mathbb{Z}G}(C_q, A)$ .*

PROOF. Multiplication in  $E$  is associative:

$$(A.53) \quad ((a, g)(b, h))(c, k) = (a, g)((b, h)(c, k)) .$$

By Lemma A.4, we have

$$(A.54) \quad \text{LHS} = (a + gb + f(g, h), gh)(c, k)$$

$$(A.55) \quad = (a + gb + f(g, h) + (gh)c + f(gh, k), ghk) .$$

Then we have

$$(A.56) \quad \text{RHS} = (a, g)(b + hc + f(h, k), hk)$$

$$(A.57) \quad = (A + gb + ghc + gf(h, k) + f(g, hk), ghk) .$$

Therefore:

$$(A.58) \quad gf(h, k) - f(gh, k) + f(g, hk) - f(g, h) = 0 .$$

But this is exactly the 2-cycle condition from (A.38). But this implies

$$(A.59) \quad (\delta_2 f)(g, h, k) = 0$$

for all  $g, h, k$ , so  $\delta_2 f = 0 \in \text{Hom}_{\mathbb{Z}G}(C_3, A)$ .  $\square$

One can show that:

- (1) the class  $[f] \in H^2(G; A)$  is independent of  $s$ ;
- (2) sending extensions (A.39) to  $[f] \in H^2(G; A)$  gives a bijection:

$$(A.60) \quad \mathcal{E}(G, A) \leftrightarrow H^2(G; A) .$$

A special case of this bijection is the correspondence between equivalence classes of central extensions of  $G$  by  $\mathbb{Z}$ , and  $H^2(G; \mathbb{Z})$ .

**3.1. Central extensions.** Given an extension eq. (A.39) and a homomorphism  $\varphi: G' \rightarrow G$ , we get an extension of  $g'$  by  $A$  given by the pullback of eq. (A.39) by  $\varphi$ . Explicitly this is given by:

$$(A.61) \quad E' = \{(g', e) \in G' \times E \mid \varphi(g') = \pi(e)\} .$$

We get a commutative diagram

$$(A.62) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G \longrightarrow 1 \\ & & & & \uparrow \psi & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \xrightarrow{\pi'} & G' \longrightarrow 1 \end{array} .$$

Then we define  $\pi'(g', e) = g'$ , and  $\psi(g', e) = e$ .

EXERCISE A.1. Check that this commutes.

So we get an extension:

$$(A.63) \quad 0 \rightarrow A \rightarrow E' \rightarrow G' \rightarrow 0 .$$

**Lemma A.6.** *If (A.39) corresponds to  $\alpha \in H^2(G; A)$ , then (A.63) corresponds to  $\varphi^*(\alpha) \in H^2(G'; A)$ .*

PROOF. Recall the definition of  $\varphi^*: H^q(G, A) \rightarrow H^q(G'; A)$ . Let  $f: G^q \rightarrow A$  be a  $q$ -cocycle such that  $[f] = \alpha \in H^q(G; A)$ . Then we define

$$(A.64) \quad f': (G')^q \rightarrow A$$

by

$$(A.65) \quad f'(g'_1, \dots, g'_q) = f(\varphi(g'_1), \dots, \varphi(g'_q)) .$$

Then

$$(A.66) \quad \varphi^*(\alpha) = [f'] \in H^q(G'; A) .$$

Now we check that this holds for (A.39) and (A.63) as above. So we have the diagram

$$(A.67) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G \longrightarrow 0 \\ & & & & \uparrow \psi & \xleftarrow{s} & \uparrow \varphi \\ 0 & \longrightarrow & A & \longrightarrow & E' & \xrightarrow{\pi} & G' \longrightarrow 0 \\ & & & & & \xleftarrow{s'} & \end{array} .$$

Define  $s' : G' \rightarrow E'$  by

$$(A.68) \quad s'(g') = (g', e)$$

for any  $e$  such that  $\varphi(g') = \pi(e)$ . Then define  $s$  such that

$$(A.69) \quad s(\varphi(g')) = e$$

as above. So we have

$$(A.70) \quad \psi s' = s\varphi \quad \text{and} \quad \psi|_A = \text{id} .$$

Now

$$(A.71) \quad f(\varphi(g'), \varphi(h')) = s\varphi(g') s\varphi(h') (s\varphi(g'h'))^{-1}$$

$$(A.72) \quad = \psi s'(g') \psi s'(h') (\psi s'(g'h'))^{-1}$$

$$(A.73) \quad = \psi \left( s'(g') s'(h') s'(g'h')^{-1} \right)$$

$$(A.74) \quad = s'(g') s'(h') s'(g'h') .$$

□

**THEOREM A.7.** *Let the extension (A.39) correspond to  $\alpha \in H^2(G; A)$ . Then  $\varphi : G' \rightarrow G$  lifts to  $\tilde{\varphi} : G' \rightarrow E$  (i.e.  $\pi\tilde{\varphi} = \varphi$ ) if and only if*

$$(A.75) \quad \varphi^* \alpha = 0 \in H^2(G'; A) .$$

**PROOF.** Recall that (A.63) splits iff  $\varphi^*(\alpha) = 0$ . ( $\Leftarrow$ ):  $\varphi^*(\alpha) = 0$  implies (A.63) splits. Let  $\sigma' : G' \rightarrow E'$  be a splitting homomorphism. Then

$$(A.76) \quad \psi\sigma' = \tilde{\varphi}$$

is a lift of  $\varphi$ .

( $\Rightarrow$ ): Recall

$$(A.77) \quad E' = \{(g', e) \in G' \times E \mid \varphi(g') = \pi(e)\} .$$

Then define  $\sigma' : G' \rightarrow E'$  by

$$(A.78) \quad \sigma'(g') = (g', \tilde{\varphi}(g')) .$$

This is clearly a homomorphism and

$$(A.79) \quad \pi'\sigma' = \text{id}_{G'}$$

so this is a splitting, and  $\varphi^* \alpha = 0$ .

□

## APPENDIX B

### Orderings of the braid group

We will follow [DDRW].

Let  $z_1, \dots, z_n \in \mathbb{D}^2$ . A *braid on  $n$  strands* is a subset  $\beta \subset \mathbb{D}^2 \times I$  such that  $\beta$  is a union of smoothly embedded intervals (called *strands*) in  $\mathbb{D}^2 \times I$  such that

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- (1)  $\beta \cap (D^2 \times \{1\}) = \{(z_1, 1), \dots, (z_n, 1)\}$ ,
- (2)  $\beta \cap (D^2 \times \{0\}) = \{(z_1, 0), \dots, (z_n, 0)\}$ ,
- (3)  $\beta \cap (\mathbb{D}^2 \times \{t\})$  in  $n$  points.

We should think of braids as these strands weaving around one another as in fig. 1.

We say two braids are equivalent if there is a deformation from one to the other through braids. There is an operation on braids called *stacking*. This takes two braids and stacks them to make a new braid.

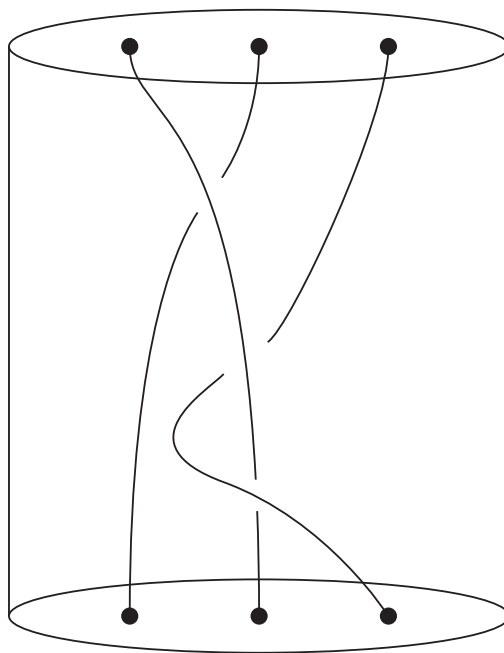
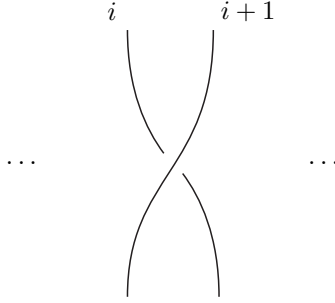
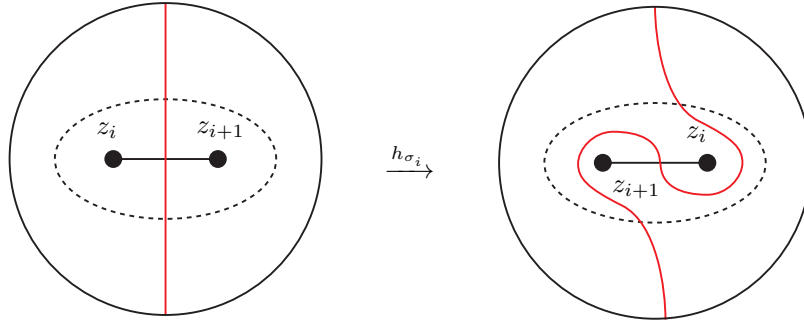


FIGURE 1. A braid on 3 strands.



FIGURE 2. The generator  $\sigma_i$  of  $B_n$ .FIGURE 3. The half-Dehn twist about the straight arc connecting  $z_i$  and  $z_{i+1}$ .

THEOREM B.1 (Artin). *The set of  $n$ -strand braids form a group  $B_n$  with group operation given by stacking. In particular, it has the following presentation:*

$$(B.1) \quad B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{l} |i-j| > 1 \implies \sigma_i \sigma_j = \sigma_j \sigma_i, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right. \right\rangle.$$

Geometrically, the generators  $\sigma_i$  correspond to braids as in fig. 2. Now a braid  $\beta$  is an equivalence class of words in the  $\sigma_i$ .

There is a map  $B_n \rightarrow \text{MCG}(D_n)$  from the braid group to the mapping class group of  $D_n$ , i.e. the group of orientation preserving homeomorphisms of  $\mathbb{D}^2$  with  $n$  punctures such that the punctures are fixed setwise, and  $\partial\mathbb{D}^2$  is fixed pointwise. The map sends the generators

$$\sigma_i \mapsto h_{\sigma_i} : D_n \curvearrowright$$

to half-Dehn twists about the straight arc connecting  $z_i$  and  $z_{i+1}$ . See fig. 3.

CLAIM B.1. This map is an isomorphism.

### 1. Dehornoy's ordering

DEFINITION B.1. A braid word  $w$  is said to be  $\sigma$ -positive (resp.  $\sigma$ -negative) if, among the letters  $\sigma_i^{\pm 1}$  that occur in  $w$ , the one with lowest index occurs with only positive (resp. negative) exponent, i.e.  $\sigma_i$  occurs but not  $\sigma_i^{-1}$ . In this case we say  $w$  is  $\sigma_i$  positive.

REMARK B.1. Usually we don't care for which  $i$  the word is  $\sigma_i$  positive. In this scenario we just say  $\omega$  is  $\sigma$ -positive.

EXAMPLE B.1.  $\sigma_1\sigma_2$  and  $\sigma_1\sigma_2^{-1}$  are both  $\sigma_1$  positive.  $\sigma_1^{-1}\sigma_2$  is  $\sigma_1$ -negative.

WARNING B.1. Some braids are neither, e.g.  $\sigma_2^{-1}\sigma_3\sigma_2$ .

DEFINITION B.2. We say  $1 <_{Deh} \beta$  if  $\beta$  is  $\sigma$ -positive.

Note  $\beta_1 <_{Deh} \beta_2$  iff  $1 <_{Deh} \beta_1\beta_2$ .

THEOREM B.2 (Dehornoy). *The above definition for  $<_{Deh}$  defines an LO on  $B_n$ .*

PROOF IDEA. We use the following properties to prove the theorem.

- Property A (Acyclicity): a  $\sigma$ -positive word is always nontrivial.
- Property C (Comparison): Every nontrivial braid of  $B_n$  admits an  $n$ -strand representative word that is  $\sigma$ -positive or  $\sigma$ -negative.

Write  $P_n$  for the positive braids on  $n$ -strands. We will show that  $P_n$  is a positive cone.

- (1)  $P_n$  is closed: let  $\beta_1, \beta_2 \in P_n$ . If  $\beta_1$  is  $\sigma_i$ -positive,  $\beta_2$  is  $\sigma_j$  positive for  $i \leq j$ . Then  $\beta_1\beta_2$  is  $\sigma_i$  positive. For example:

$$(B.2) \quad \beta_1 = \sigma_1\sigma_2\sigma_3\sigma_2^{-1}$$

$$(B.3) \quad \beta_2 = \sigma_2\sigma_3\sigma_2\sigma_3^{-1}$$

$$(B.4) \quad \beta_1\beta_2 = \sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_3^{-1}.$$

- (2)  $B_n \setminus \{1\} = P_n \cup P_n^{-1}$ : property A implies  $1 \notin P_n$  and then property C implies this.

- (3) Disjoint union: Suppose  $\beta \in P_n \cap P_n^{-1}$ . Then  $\beta^{-1} \in P_n$ , so  $\beta\beta^{-1} = 1 \in P_n$  which is a contradiction.

□

**Proposition B.3.**  $B_n$  for  $n \geq 3$  is not BO.

PROOF. Define

$$\Delta_n = (\sigma_1 \dots \sigma_{n-1}) (\sigma_1 \dots \sigma_{n-2}) \dots (\sigma_1 \sigma_2) \sigma_1.$$

For example, see fig. 4 for  $\Delta_4$ .

CLAIM B.2.  $\Delta_n \sigma_i = \sigma_{n-i} \Delta_n$ .

Now suppose  $\prec$  is a BO on  $B_n$ . WLOG  $\sigma_1 \prec \sigma_{n-1}$  implies

$$\underbrace{\Delta_n \sigma_1 \Delta_n^{-1}}_{\sigma_{n-1}} \prec \underbrace{\Delta_n \sigma_{n-1} \Delta_n^{-1}}_{\sigma_1}$$

so  $\sigma_{n-1} \prec \sigma_1$ , so

$$\Delta_n \sigma_i \Delta_n^{-1} = \sigma_{n-i} \Delta_n \Delta_n^{-1} = \sigma_{n-i}$$

which is a contradiction. □

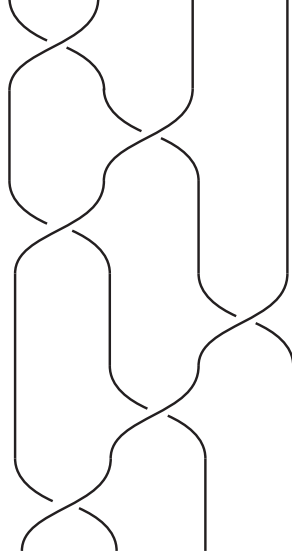
REMARK B.2. (1) For each  $n$ , two elements of  $(B_n, <_{Deh})$  can be compared in polynomial time (in the length of words).

- (2) This ordering has applications to knot theory. If  $\beta \in B_n$  and  $\beta < \Delta_n^{-6}$  or  $\beta > \Delta_n^6$ , then its closure  $\hat{\beta}$  is prime.

DEFINITION B.3. (1) An LO group  $(G, <)$  is *Conradian* if for all  $g, h > 1$ , there is some  $p \in \mathbb{Z}^+$  with  $h < gh^p$ .

- (2)  $(G, <)$  is *Archimedean* if for all  $g, h > 1$ , there is  $p \in \mathbb{Z}^+$  with  $g < h^p$ .

**Proposition B.4.**  $(B_n, <_{Deh})$  is not Conradian nor Archimedean.

FIGURE 4. The braid  $\Delta_4$ .

## 2. Nielsen-Thurston orderings on $B_n$

DEFINITION B.4. Suppose  $G \curvearrowright \mathbb{R}$  by orientation preserving homeomorphisms and there is  $x \in \mathbb{R}$  with  $\text{Stab}_G(x) = \{1\}$ . Then  $(G, <_x)$  is defined by declaring  $g <_x g'$  iff  $g(x) <_{\mathbb{R}} g'(x)$ .

REMARK B.3. (1) This is an LO since  $G < \text{Homeo}^+(\mathbb{R})$ .  
 (2) Using  $y \in \mathbb{R}$ ,  $y \neq x$  could give a different ordering.

The goal is to get an action  $B_n \curvearrowright \mathbb{R}$ .

We can give  $D_n$  a hyperbolic metric.  $\widetilde{D}_n$  is a subset of  $\mathbb{H}^2$ . Now compactify  $\mathbb{H}^2$  by adding  $S^1_{\infty}$ . Compactify  $\widetilde{D}_n$  by adding in limit points of lifts of  $\partial D_n$ . This is a closed disk  $\widetilde{D}_n$ .  $\partial \widetilde{D}_n$  has two types of points:

- (1) limit points, and
- (2) arcs which cover  $\partial D_n$ .

Now pick a basepoint  $\star$ . For each  $b \in B_n$ , take  $\beta \mapsto h_{\beta} : D_n \curvearrowright$ . Note that  $h_{\beta}$  has many lifts in  $\widetilde{D}_n$ . Pick one  $\tilde{h}_b$  that fixes the basepoint. Now since  $\partial \widetilde{D}_n \setminus \{\star\} \cong \mathbb{R}$ , we can restrict  $\tilde{h}_{\beta}$  to  $\partial \widetilde{D}_n \setminus \{\star\}$  to get an action on  $\mathbb{R}$ . Then it turns out this is all well-defined.

DEFINITION B.5. An LO  $<$  on  $B_n$  is of *Nielsen-Thurston type* if there is some  $x \in \mathbb{R}$  such that for all  $\beta, \beta' \in B_n$   $\beta < \beta'$  iff  $\beta(x) <_{\mathbb{R}} \beta'(x)$ .

FACT 6. (1) Some choices  $x \in \mathbb{R}$  have non-trivial stabilizer. These cannot give an ordering.  
 (2) Some choices  $x \neq y \in \mathbb{R}$  give the same ordering.  
 (3) Uncountably many of them are distinct.

## 3. Isolated orderings

Recall LO's on  $G$  correspond to positive cones.

DEFINITION B.6. An ordering  $<$  in  $\text{LO}(G)$  is *finitely determined* if there is a finite subset  $S = \{g_1, \dots, g_k\} \subset G$  such that  $<$  is the unique LO on  $G$  such that  $S$  is positive.

- EXAMPLE B.2. (1)  $(\mathbb{Z}, <)$  is determined by choosing  $\{1\} \subset P$ .  
 (2) If  $P \subset G$  is finitely generated as a semi-group then the order  $<$  determined by  $P$  is finitely determined.  
 (3)  $K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$  is determined by  $\{a, b\}$ .

**Proposition B.5.** *A points in  $\text{LO}(G)$  is isolated iff  $<$  is finitely determined.*

PROOF. ( $\Leftarrow$ ): Suppose that  $< \in \text{LO}(G)$  is finitely determined by  $f_1, \dots, f_m$ . Recall  $\text{LO}(G) \subset \{0, 1\}^G$ . A basis for the topology is given by sets of the form:

$$(B.5) \quad B = \left\{ \left( \underbrace{g_1, \dots, g_k}_{\text{yes}}, \underbrace{h_1, \dots, h_l}_{\text{no}}, \underbrace{\dots}_{\text{whatever}} \right) \right\} \cap \text{LO}(G) .$$

Now we can impose that

- (1) The set of  $g \in G$  which we say “yes” to is closed,
- (2) never say “yes” to both  $g$  and  $g^{-1}$
- (3) never say “no” to  $g$  and  $g^{-1}$ .

Then for

$$(B.6) \quad U = \{(f_1, \dots, f_m, f_1^{-1}, \dots, f_m, \dots)\}$$

there is no other order inside  $U$ , so  $<$  is isolated.

( $\Rightarrow$ ): Assume  $< \in \text{LO}(G)$  is isolated. There is an open set  $U$  such that  $<$  is the only element of  $\text{LO}(G)$ . Write  $< \in B \subset U$  where  $B$  is of the form (B.5). Then

$$(B.7) \quad P \supset \{g_1, \dots, g_k, h_1^{-1}, \dots, h_l^{-1}\}$$

so  $<$  is finitely determined.  $\square$

DEFINITION B.7 ([DD]). Let  $P_{DD}$  be the set of  $\beta \in B_3$  such that  $\beta$  is  $\sigma_1$ -positive or  $\sigma_2$ -negative.

THEOREM B.6.  $P_{DD}$  is a positive cone, and is generated as a semigroup by  $\sigma_1\sigma_2$  and  $\sigma_2^{-1}$ .

PROOF. We will assume that a  $\sigma_i$ -positive word is never trivial. We will also assume that either  $\beta$  is  $\sigma_1$ -positive or  $\sigma_1$ -negative or  $\sigma_1$ -free. Note that this implies  $\sigma_1$ -free braids are always  $\sigma_2$ -positive or  $\sigma_2$ -negative.

Now we show  $P_{DD}$  is a positive cone. Write  $Q = \langle \sigma_1\sigma_2, \sigma_2^{-1} \rangle$ . This is a semigroup. Write  $\beta_1 = \sigma_1\sigma_2$  and  $\beta_2 = \sigma_2^{-1}$ . It is immediate that  $Q \subset P_{DD}$ . Now we show the opposite. We have two cases:

Case 1.  $\beta$  or  $\beta^{-1}$  is  $\sigma_2$ -positive: Then  $\beta = \sigma_2^p$  for some  $p \in \mathbb{Z} \setminus \{0\}$ . For  $p > 0$  we have  $\beta^{-1} \in Q$ , and for  $p < 0$  we have  $\beta \in Q^{-1}$ .

Case 2.  $\beta$  is  $\sigma_1$ -positive: then there are  $m_i \in \mathbb{Z}$ ,  $1 \leq i \leq k$ , such that

$$(B.8) \quad \beta = \sigma_2^{m_1} \sigma_1 \sigma_2^{m_2} \sigma_1 \dots \sigma_1 \sigma_2^{m_k}$$

$$(B.9) \quad = \beta_2^{P_1} \beta_1 \beta_2^{P_2} \beta_1 \dots \beta_1 \beta_2^{P_k}$$

for some  $P_i \in \mathbb{Z}$ . Then we have

$$(B.10) \quad \beta_2 \beta_1^2 \beta_2 = \beta_1$$

so we can cancel things and keep replacing  $\beta_1$  by this, until all exponents of  $\beta_2$  are positive, so  $\beta \in Q$ .

Case 3.  $\beta$  is  $\sigma_1$ -negative: so  $\beta^{-1}$  is  $\sigma_1$ -positive, so  $\beta^{-1} \in Q$  by case 2.

Then this means  $<_{DD}$  is an ordering on  $B_n$ , so it is isolated in  $\text{LO}(G)$ .  $\square$

## APPENDIX C

### Orderability and knot groups

A *smooth knot* in  $S^3$  is a (smooth) embedding  $K : S^1 \hookrightarrow S^3$ . The *knot complement* of  $K$  is

$$(C.1) \quad X_K := S^3 \setminus \text{int}(\nu(K)) .$$

The *knot group* of  $K$  is  $\pi_1(X_K) =: \pi_1(K)$ .

**Proposition C.1.**  $H_1(X_K) \cong \mathbb{Z}$ .

PROOF. The idea is to use Mayer-Vietoris with  $\nu(K)$  and  $X_K$ . This gives us the sequence

$$\underbrace{H_2(S^3)}_{=0} \rightarrow \underbrace{H_1(\nu(K) \cap X_K)}_{=\mathbb{Z} \oplus \mathbb{Z}} \rightarrow \underbrace{H_1(\nu(K))}_{=\mathbb{Z}} \oplus H_1(X_K) \rightarrow \underbrace{H_1(S^3)}_{=0}$$

so the result follows from exactness. □

**THEOREM C.2.** *Suppose  $M$  is a prime orientable three-manifold with  $\pi_1(M)$  finitely generated. Then  $\pi_1(M)$  is locally indicable iff  $\text{rank } H_1(M) \geq 1$ .*

Recall if  $\pi_1(M)$  is BO then  $\pi_1(M)$  is locally indicable, which implies  $\pi_1(M)$  is LO.

**Corollary C.3.** *If  $\text{rank } H_1(M) \geq 1$  then  $\pi_1(M)$  is LO.*

**Corollary C.4.** *Knot groups are LO.*

#### 1. Generalized torsion

An element  $g$  in a group  $G$  is a *generalized torsion element* if and only if

$$\alpha_1^{-1} g^{n_1} \alpha_1 \alpha_2^{-1} g^{n_2} \alpha_2 \dots \alpha_k^{-1} g^{n_k} \alpha_k = 1$$

for some  $\alpha_1, \dots, \alpha_n \in G$  and  $n_1, \dots, n_k \in \mathbb{Z}^+$ . As it turns out, if  $G$  has a generalized torsion element, then  $G$  is not BO.

**EXAMPLE C.1.** Consider the Klein bottle group  $\langle a, b \mid a^{-1}bab = 1 \rangle$ . The element  $b$  is a generalized torsion element.

**REMARK C.1.** There are non BO groups without generalized torsion.

The following is open:  $\pi_1(M)$  is BO iff  $\pi_1(M)$  has no generalized torsion.

A *torus knot* is a knot in  $S^3$  which embeds in a Heegaard torus. So this is some simple closed curve on the torus. We know that these are parameterized by some rational number

$$\frac{p}{q} \in \mathbb{Q} \cup \left\{ \frac{1}{0} \right\} .$$

Write  $T_{p,q}$  for the associated knot. Note that  $T_{p,q}$  is the unknot iff  $|p| = 1$  or  $|q| = 1$ .

**EXERCISE C.5.**  $\pi_1(T_{p,q}) = \langle a, b \mid a^p = b^q \rangle$ .

**Proposition C.6.** *If  $T_{p,q}$  is nontrivial, then  $\pi_1(T_{p,q})$  has generalized torsion.*

PROOF. Assume  $p, q > 1$ . Write  $[x, y] = x^{-1}y^{-1}xy$ . Note the following identities:

$$\begin{aligned} (C.2) \quad & [x^n, y] = x^{-1} [x^{n-1}, y] x [x, y] \\ (C.3) \quad & [x, y^n] = [x, y] y^{-1} [x, y^{n-1}] y . \end{aligned}$$

□

EXERCISE C.7.  $[a^p, b^q]$  is a product of conjugates of  $[a, b]$ .

$[a, b] \neq 1$ , but  $[a^p, b^q] = 1$  so  $[a, b]$  is a generalized torsion element.

**Corollary C.8.**  $\pi_1(T_{p,q})$  is not BO.

**Corollary C.9.**  $G$  locally indicable does not imply  $G$  is BO.

## 2. Knot groups as extensions

Let  $Y := [\pi_1(K), \pi_1(K)]$ . Since  $H_1(X_K) \cong \mathbb{Z}$  we have a short exact sequence

$$(C.4) \quad 1 \otimes Y \rightarrow \pi_1(K) \xrightarrow{\rho} \mathbb{Z} \rightarrow 1 .$$

Let  $\mu \in \rho^{-1}(1)$ . Define

$$\varphi_\mu \in \text{Aut}(Y)$$

by

$$y \mapsto \mu^{-1}y\mu .$$

EXERCISE C.10.  $\pi_1(K)$  is BO iff there is an order on  $Y$  invariant under  $\varphi_\mu$ .

**Proposition C.11.**  $\pi_1(K_1 \# K_2)$  is BO iff  $\pi_1(K_1)$  is BO and  $\pi_1(K)$  is BO.

The lower central series is as follows. Define  $Y_1 = Y$ , and

$$(C.5) \quad Y_n = [Y_{n-1}, Y]$$

for  $n > 1$ . Notice that  $Y_n/Y_{n+1}$  is abelian. Define  $\overline{Y_n}$  to be the preimage of  $\text{Tor}(Y/Y_{n+1})$  under the quotient. Write  $A_n := \overline{Y_n}/\overline{Y_{n+1}}$ .

FACT 7. (1)  $\overline{Y_b}/\overline{Y_{n+1}}$  is a torsion free abelian group.  
(2)  $\overline{Y_n}$  are characteristic.

This implies that  $\varphi_M$  induces a well-defined

$$\varphi_n \in \text{Aut}(\overline{Y_n}/\overline{Y_{n+1}}) .$$

A group  $G$  is *nilpotent* if  $G_n = \{1\}$  for some  $n$ .

EXERCISE C.12.  $Y$  is residually torsion-free nilpotent if and only if

$$\bigcap_n \overline{Y_n} = \{1\} .$$

**Proposition C.13.** *If  $Y$  is residually torsion-free nilpotent and there are orders  $<_n$  on each quotient  $A_n$  invariant under  $\varphi_n$  then  $\pi_1(K)$  is BO.*

PROOF. We know

$$(C.6) \quad \bigcap_n \overline{Y_n} = \{1\}$$

so for  $y \in Y \setminus \{1\}$  there is a unique  $n(y)$  such that  $y \in \overline{Y_n}$  and  $y \notin \overline{Y_{n+1}}$ , so

$$[y]_{n(y)} \in A_n$$

is not 0. We have positive cones  $P_n \subset A_n$  invariant under  $\varphi_n$ . Now write

$$(C.7) \quad P = \left\{ y \in Y \mid y \neq 1, [y]_{n(y)} \in P_{n(y)} \right\}.$$

(1)  $Y = P \amalg P^{-1} \amalg \{1\}$  is clear.

(2) Let  $y_1, y_2 \in P$ ,  $n_i := n(y_i)$ . If  $n_1 < n_2$  then  $y_1, y_2 \in \overline{Y_{n_1}}$ . Then

$$[y_1, y_2]_{n_1} = [y_1]_{n_1} + \cancel{[y_2]_{n_2}} = [y_1]_{n_1} \in P_{n_1}.$$

The case that  $n_1 > n_2$  is similar. If  $n_1 = n_2$ , then

$$[y_1, y_2]_{n_1} = [y_1]_{n_1} + [y_2]_{n_1=n_2} \in P_n$$

Therefore  $y_1 y_2 \in P$ . So this shows us that this is an LO. Now we want to see it is a BO.

(3) Let  $p \in P$ ,  $y \in Y$ . Since  $p \in \overline{Y_{n(p)}}$  we have that  $[p, q] \in \overline{Y_{n(p)+1}}$ . Now

$$y^{-1} p y = p p^{-1} y^{-1} p y = p [p, y]$$

so

$$[y^{-1} p y]_{n(p)} = [p]_{n(p)} + \cancel{[p^{-1} y^{-1} p y]}$$

so this is in  $P_{n(p)}$ , so  $y^{-1} p y \in P$ .

(4) We know  $n(\varphi_\mu(p)) = n(p)$ , so

$$[\varphi_\mu(p)]_{n(p)} = \varphi_n([p]_{n(p)}) \in P_n$$

so  $\varphi_n(p) \in P$ .

Therefore by an earlier proposition  $\pi_1(K)$  is BO. □

So we have orders  $A_n$  invariant under  $\varphi_n$ . Now define

$$(C.8) \quad V_n := \mathbb{Q} \otimes_{\mathbb{Z}} A_n \quad L_n := \text{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \varphi_n.$$

Notice that this is a vector space and a linear map on it.<sup>C.1</sup>

**Lemma C.14.**  *$L_n$  preserves an order on  $V_n$  if and only if every irreducible factor of the characteristic polynomial  $\text{ch}(L_n)$  has a real positive root.*

**Lemma C.15.** *There is an embedding  $V_n \hookrightarrow V_1^{\otimes n}$  such that*

$$(C.9) \quad \begin{array}{ccc} V_n & \hookrightarrow & V_1^{\otimes n} \\ \downarrow L_n & & \downarrow L_1^{\otimes n} \\ V_n & \xrightarrow{\iota} & V_1^{\otimes n} \end{array}.$$

**Proposition C.16.** *If  $\text{ch}(L_1)$  has all real positive roots then there are orders on the  $A_n$  invariant under the  $\varphi_n$ .*

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<sup>C.1</sup>Which is of course a mathematician's bread and butter.



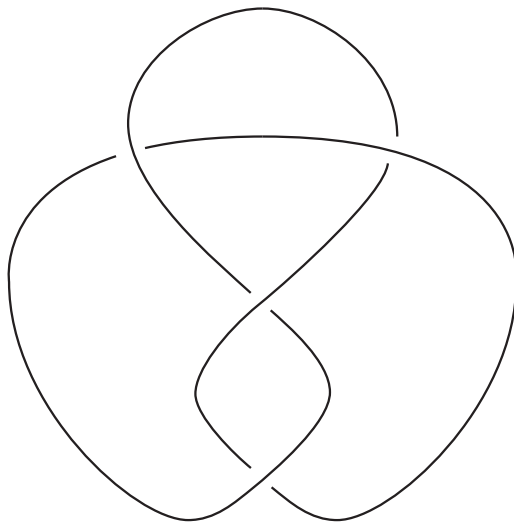


FIGURE 1. The figure-eight knot.

So by definition  $L_1 = \text{id}_{\mathbb{Q}} \otimes \varphi_1$ , where  $\varphi_1$  is the automorphism of  $Y/Y_2$  induced by  $\varphi_\mu$  which is conjugate to a scalar multiple of the action on  $H_1(\tilde{X}, \mathbb{Q})$  induced by the meridian. Then  $\text{ch}(L_1)$  is a scalar multiple of the Alexander polynomial  $\Delta_K(t)$ .

**THEOREM C.17.** *If  $Y$  is residually torsion-free nilpotent and  $\Delta_K(t)$  has all real positive roots then  $\pi_1(K)$  is BO.*

**EXAMPLE C.2.** Consider the figure-eight knot as in fig. 1. This has Alexander polynomial

$$(C.10) \quad \Delta_K(t) = t^2 - 3t + 1.$$

Then  $Y_K \cong F_2$ , and free groups are residually torsion-free nilpotent. So  $\pi_1(K)$  is BO.

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