Orderability and 3-manifold groups

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Contents

CHAPTER 1

Orders on groups; basic definitions and properties

The book for the course is [**1**]. Lecture 1; January 21, 2020

Recall that a *strict total order* (STO) on a set X is a binary relation $<$ which satisfies:

(1) $x < y$ and $y < z$ implies $x < z$;
(2) $forall x, y \in X$ exactly one of: $x < y$, $y < x$, $x = y$, holds.

A *left order* (LO) on a group G is an STO such that $g < h$ implies $fg < fh$ for all $f \in G$. G is *left-orderable* (LO) if there exists an LO on G. We similarly define a *right order* (RO) and *right orderability* (RO). A *bi-order* (BO) on G is an LO on G that is also an RO.

Remark 1.1. (1) If G is abelian, $<$ is a LO iff $<$ is an RO iff $<$ is a BO.
(2) If $<$ is an LO on G, then $\prec$ defined by:

$$g \prec h \iff h^{-1} < g^{-1} \tag{1.1}$$

is an RO on G. Therefore G is LO iff G is RO. We will stick to LO's.
(3) For $H < G$, an LO (resp. BO) on G induces an LO (resp. BO) on H.

Example 1.1. $(\mathbb{R}, +)$ with the usual $<$ is BO. The subgroups $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$ are also BO.

Lemma 1.1. *Let $<$ be an LO on G. Then*

(1) $g > 1$, $h > 1$ implies $gh > 1$;
(2) $g > 1$ implies $g^{-1} < 1$;
(3) $<$ is a BO iff $\left(g < h \implies f^{-1}gf < f^{-1}hf \forall f \in G\right)$ (i.e. $<$ is conjugation invariant).

Proof. (1) $h > 1$ implies $gh > g \cdot 1g > 1$.
(2) $g > 1$ implies $g^{-1}g > g^{-1}$ implies $1 > g^{-1}$.
(3) ($\implies$) is immediate. ($\impliedby$): We need to show $<$ is a RO. $g < h$ implies $fg < fh$ implies $f^{-1}(fg)f < f^{-1}(fh)f$ which implies $gf < hf$ as desired.

$\square$

Lemma 1.2. *If $<$ is a BO on G, then*

(1) $g < h$ implies $g^{-1} > h^{-1}$;
(2) $g_1 < h$, $g_2 < h_2$ implies $g_1g_2 < h_1h_2$.

Proof. (1) If $g < h$, then $g^{-1}g < g^{-1}h$, which implies $1 < g^{-1}h$, which implies $1 \cdot h^{-1} < g^{-1}$, which implies $h^{-1} < g^{-1}$.
(2) $g_2 < h_2$ implies $g_1g_2 < g_1h_2 < h_1h_2$.

$\square$

Warning 1.1. These don't necessarily true for LO's.

Lemma 1.3. *If G is LO then it is torsion free.*

Proof. Consider $g \in G \setminus \{1\}$. If $g > 1$, then $g^2 > g > 1$, and similarly for all $n \geq 1$, $g^n > 1$. Similarly $g < 1$ implies $g^n < 1$ for all $n \geq 1$. $\square$

So LO is not preserved under taking quotients (e.g. $\mathbb{Z} \to \mathbb{Z}/n$).

Consider an indexed family of groups $\{G_\lambda \,|\, \lambda \in \Lambda\}$. Recall that the direct product

$$\prod_{\lambda \in \Lambda} G_\lambda = \left\{(g_\lambda)_{\lambda \in \Lambda}\right\} \tag{1.2}$$

with multiplication defined co-ordinatewise.

Recall a *well-order* (WO) on a set X is a STO $\prec$ on X such that if $A \subset X$ and $A \neq \emptyset$ then there exists $a_0 \in A$ such that $a_0 \prec a$ for all $a \in A \setminus \{a_0\}$. Recall that the axiom of choice is equivalent to every set having a WO.

Theorem 1.4. *G_λ has a LO (resp. BO) for all $\lambda \in \Lambda$ iff $\prod_{\lambda \in \Lambda} G_\lambda$ has a LO (resp. BO).*

Proof. ($\Longleftarrow$): $G\lambda < \prod_\lambda G_\lambda$ so we are finished.

($\Longrightarrow$): Choose a WO $\prec$ on Λ, and order $\prod_\lambda G_\lambda$ lexicographically. Let $g = (g_\lambda)$, $h = (h_\lambda)$, $g \neq h$. Then λ_0 be the $\prec$-least element of Λ such that $g_{\lambda_0} \neq h_{\lambda_0}$. Then define $g < h$ iff $g_{\lambda_0} < h_{\lambda_0}$ (in G_{λ_0}). Then $<$ is an LO (resp. BO) on $\prod_\lambda G_\lambda$. Left (resp. left and right) invariance is clear. Now we show transitivity. Suppose $f < g$, $g < h$. Let λ_0 be the $\prec$-least element of Λ such that $f_{\lambda_0} \neq g_{\lambda_0}$. Let μ_0 be the $\prec$-least element of Λ such that $g_{\mu_0} \neq h_{\mu_0}$.

(1) ($\lambda_0 \preccurlyeq \mu_0$): Then $f_\lambda = g_\lambda = h_\lambda$ for all $\lambda \prec \lambda_0$. Then g_{λ_0} is $<$ (resp. $=$) h_{λ_0} if $\lambda_0 = \mu_0$ (resp. $\lambda_0 \prec \mu_0$). So $f_{\lambda_0} < g_{\lambda_0} \leq h_{\lambda_0}$, and therefore $f_{\lambda_0} < h_{\lambda_0}$.
(2) ($\mu_0 < \lambda_0$): This follows similarly.

$\square$

Let $\sum_{\lambda in \Lambda} G_\lambda$ be the *direct sum of* $\{G_\lambda\}$. Recall this is the subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of elements such that all but finitely many co-ordinates are 1.

Corollary 1.5. *G_λ is LO (resp. BO) for all $\lambda \in \Lambda$ iff $\sum_{\lambda \in \Lambda} G_\lambda$ is LO (resp. BO).*

Corollary 1.6. *Free abelian groups are BO.*

Proof. Free abelian groups on Λ are $\sum_{\lambda \in \Lambda} \mathbb{Z}$. $\square$

Let $<$ be an LO on G. The *positive cone* $P = P_<$ of $<$ is $\{g \in G \,|\, g > 1\}$.

Lemma 1.7. *(1) P is a subset of G, i.e. $g, h \in P$, implies $gh \in P$ (i.e. $PP \subset P$).*
(2) $G = P \amalg P^{-1} \amalg \{1\}$.
(3) $<$ is a BO on G iff $f^{-1}Pf \subset P$ for all $f \in G$.

Proof. (1) This follows from Lemma 1.1 (1).
(2) This follows from Lemma 1.1 (2).
(3) This follows from Lemma 1.1 (3).

$\square$

We say $P \subset G$ is a *positive cone* if P satsfies the two conditions in Lemma 1.7.

Lemma 1.8. *Let $P \subset G$ be a positive cone. Then $g < h$ implies $g^{-1}h \in P$ defines a LO $<$ on G (With $P_< = P$).*

Proof. $<$ is a STO, so:

(i) $f < g$, $g < h$ implies $f^{-1}g \in P$, $g^{-1}h \in P$, which implies (by the first property) that $\left(f^{-1}g\right)\left(g^{-1}h\right) \in P$, which implies $f < h$.

(ii) By the second property, for all $g, h \in G$ exactly one of the following holds: $g^{-1}h \in P$, $g^{-1}h \in P^{-1}$, and $g^{-1}h = 1$. Equivalently, $g < h$, $h < g$ (since $h^{-1}g \in P$), and $g = h$. Now we show left invariance. $g < h$ implies $g^{-1}h \in P$, but $g^{-1}h = \left(g^{-1}f^{-1}\right)(fh)$ which implies $fg < fh$.

□

Lemmata 1.7 and 1.8 show that:

$$\{\text{LO's on } G\} \quad \leftrightarrow \quad \{\text{positive cones in } G\} \tag{1.3}$$

$$\{\text{BO's on } G\} \quad \leftrightarrow \quad \{\text{conjugacy-invariance positive cones in } G\} \ . \tag{1.4}$$

Consider the free group of rank n, F_n.

THEOREM 1.9. *F_2 is LO.*

SUNIC. Write $F_2 = F(a,b)$. $g \in F_2$ implies we can write it as a reduced word

$$(a^{m_1})\, b^{n_1} \dots a^{m_k}\, (b^{n_k}) \tag{1.5}$$

for $k \geq 0$, $m_i, n_i \in \mathbb{Z} \setminus \{0\}$. Recall 1 is the empty word, $k = 0$. Let $e(g)$ be the number of syllables in g with positive exponent, minus the number of syllables in g with negative exponent. Then define $j(g)$ so be the number of $a^m b^n$'s in f, minus the number of $b^n a^m$s in G. So $j(g) = 0$, or ± 1. For example:

$$j(a^* \dots a^*) = 0 \tag{1.6}$$

$$j(b^* \dots b^*) = 0 \tag{1.7}$$

$$j(a^* \dots b^*) = 1 \tag{1.8}$$

$$j(b^* \dots a^*) = -1 \ . \tag{1.9}$$

Finally define

$$\tau(g) = e(g) + j(g) \ . \tag{1.10}$$

Note that

$$e\left(g^{-1}\right) = -e(g) \qquad j\left(g^{-1}\right) = -j(g) \ . \tag{1.11}$$

Lemma 1.10. *If $g \neq 1$, then $\tau(g) \equiv 1 \pmod 2$.*

PROOF. $e(f)$ is congruent to the number of syllables mod 2, and $j(g)$ is congruent to the number of syllables $+1$ mod 2. □

Lemma 1.11. $|\tau(gh) - \tau(g) - \tau(h)| \leq 1$.

PROOF. If gh or g or $h = 1$ we are done. So suppose $gh, g, h \neq 1$. Clearly $e(gh) = e(g) + e(h) + \left\{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}\right\}$. Similarly:

$$j(gh) = j(g) + j(h) + \left\{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}\right\} \ . \tag{1.12}$$

Therefore:

$$|\tau(gh) - \tau(g) - \tau(H)| \leq 2 \tag{1.13}$$

so by Lemma 1.10 we have

$$|\tau(gh) - \tau(g) - \tau(h)| \leq 1 \ . \tag{1.14}$$

□

REMARK 1.2. Lemma 1.11 says that $\tau : F_2 \to \mathbb{Z}\,(< \mathbb{R})$ is what is called a *quasi-morphism*.

Define $P \subset F_2$ by

$$P = \{g \in F_2 \,|\, \tau(g) > 0\} \ . \tag{1.15}$$

Then $F_2 = P \amalg P^{-1} \amalg \{1\}$ by Lemma 1.10 and that $\tau\left(g^{-1}\right) = -\tau(g)$. Then $PP \subset P$ by Lemma 1.11 since

$$\tau(gh) \geq \tau(g) + \tau(h) - 1 \geq 1 \ . \tag{1.16}$$

Therefore P is a positive cone for a LO on F_2. ■

Corollary 1.12. *Any countable free group is LO.*

PROOF. A countable free group is a subgroup of F_2. □

REMARK 1.3.
(1) $\tau\left(a^{-1}b\right) = 1$, so $a^{-1}b > 1$, so $b > a$. On the other hand, $\tau\left(ab^{-1}\right) = 1$, so $ab^{-1} > 1$, so $b^{-1} > a^{-1}$. So τ does not define a BO on F_2.
(2) We will see later that all free groups are LO.
(3) Even later we will see that all free groups are BO.

THEOREM 1.13. *Let $1 \to H \to G \to Q \to 1$ be a short-exact sequence of groups. Then*
(1) H, Q LO implies G is LO;
(2) if Q is BO and G has a BO that is invariant under conjugation in G then G is BO.

Bibliography

[1] Adam Clay and Dale Rolfsen, *Ordered groups and topology*, Graduate Studies in Mathematics, vol. 176, American Mathematical Society, Providence, RI, 2016.