

Orderability and 3-manifold groups

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CHAPTER 1

Orders on groups; basic definitions and properties

The book for the course is [1].

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Recall that a *strict total order* (STO) on a set X is a binary relation $<$ which satisfies:

- (1) $x < y$ and $y < z$ implies $x < z$;
- (2) $\forall x, y \in X$ exactly one of: $x < y$, $y < x$, $x = y$, holds.

A *left order* (LO) on a group G is an STO such that $g < h$ implies $fg < fh$ for all $f \in G$. G is *left-orderable* (LO) if there exists an LO on G . We similarly define a *right order* (RO) and *right orderability* (RO). A *bi-order* (BO) on G is an LO on G that is also an RO.

REMARK 1.1. (1) If G is abelian, $<$ is a LO iff $<$ is an RO iff $<$ is a BO.
(2) If $<$ is an LO on G , then \prec defined by:

$$(1.1) \quad g \prec h \iff h^{-1} < g^{-1}$$

is an RO on G . Therefore G is LO iff G is RO. We will stick to LO's.

- (3) For $H < G$, an LO (resp. BO) on G induces an LO (resp. BO) on H .

EXAMPLE 1.1. $(\mathbb{R}, +)$ with the usual $<$ is BO. The subgroups $\mathbb{Z} < \mathbb{Q} < \mathbb{R}$ are also BO.

Lemma 1.1. *Let $<$ be an LO on G . Then*

- (1) $g > 1, h > 1$ implies $gh > 1$;
- (2) $g > 1$ implies $g^{-1} < 1$;
- (3) $<$ is a BO iff $(g < h \implies f^{-1}gf < f^{-1}hf \forall f \in G)$ (i.e. $<$ is conjugation invariant).

PROOF. (1) $h > 1$ implies $gh > g \cdot 1g > 1$.

(2) $g > 1$ implies $g^{-1}g > g^{-1}$ implies $1 > g^{-1}$.

(3) (\implies) is immediate. (\impliedby) : We need to show $<$ is a RO. $g < h$ implies $fg < fh$ implies $f^{-1}(fg)f < f^{-1}(fh)f$ which implies $gf < hf$ as desired. □

Lemma 1.2. *If $<$ is a BO on G , then*

- (1) $g < h$ implies $g^{-1} > h^{-1}$;
- (2) $g_1 < h, g_2 < h_2$ implies $g_1g_2 < h_1h_2$.

PROOF. (1) If $g < h$, then $g^{-1}g < g^{-1}h$, which implies $1 < g^{-1}h$, which implies $1 \cdot h^{-1} < g^{-1}$, which implies $h^{-1} < g^{-1}$.

(2) $g_2 < h_2$ implies $g_1g_2 < g_1h_2 < h_1h_2$. □

WARNING 1.1. These don't necessarily true for LO's.

Lemma 1.3. *If G is LO then it is torsion free.*

PROOF. Consider $g \in G \setminus \{1\}$. If $g > 1$, then $g^2 > g > 1$, and similarly for all $n \geq 1$, $g^n > 1$. Similarly $g < 1$ implies $g^n < 1$ for all $n \geq 1$. \square

So LO is not preserved under taking quotients (e.g. $\mathbb{Z} \rightarrow \mathbb{Z}/n$).

Consider an indexed family of groups $\{G_\lambda \mid \lambda \in \Lambda\}$. Recall that the direct product

$$(1.2) \quad \prod_{\lambda \in \Lambda} G_\lambda = \{(g_\lambda)_{\lambda \in \Lambda}\}$$

with multiplication defined co-ordinatewise.

Recall a *well-order* (WO) on a set X is a STO \prec on X such that if $A \subset X$ and $A \neq \emptyset$ then there exists $a_0 \in A$ such that $a_0 \prec a$ for all $a \in A \setminus \{a_0\}$. Recall that the axiom of choice is equivalent to every set having a WO.

THEOREM 1.4. G_λ has a LO (resp. BO) for all $\lambda \in \Lambda$ iff $\prod_{\lambda \in \Lambda} G_\lambda$ has a LO (resp. BO).

PROOF. (\Leftarrow): $G_\lambda < \prod_{\lambda} G_\lambda$ so we are finished.

(\Rightarrow): Choose a WO \prec on Λ , and order $\prod_{\lambda} G_\lambda$ lexicographically. Let $g = (g_\lambda)$, $h = (h_\lambda)$, $g \neq h$. Then λ_0 be the \prec -least element of Λ such that $g_{\lambda_0} \neq h_{\lambda_0}$. Then define $g < h$ iff $g_{\lambda_0} < h_{\lambda_0}$ (in G_{λ_0}). Then $<$ is an LO (resp. BO) on $\prod_{\lambda} G_\lambda$. Left (resp. left and right) invariance is clear. Now we show transitivity. Suppose $f < g$, $g < h$. Let λ_0 be the \prec -least element of Λ such that $f_{\lambda_0} \neq g_{\lambda_0}$. Let μ_0 be the \prec -least element of Λ such that $g_{\mu_0} \neq h_{\mu_0}$.

- (1) ($\lambda_0 \preccurlyeq \mu_0$): Then $f_\lambda = g_\lambda = h_\lambda$ for all $\lambda \prec \lambda_0$. Then g_{λ_0} is $<$ (resp. $=$) h_{λ_0} if $\lambda_0 = \mu_0$ (resp. $\lambda_0 \prec \mu_0$). So $f_{\lambda_0} < g_{\lambda_0} \leq h_{\lambda_0}$, and therefore $f_{\lambda_0} < h_{\lambda_0}$.
- (2) ($\mu_0 < \lambda_0$): This follows similarly.

\square

Let $\sum_{\lambda \in \Lambda} G_\lambda$ be the *direct sum* of $\{G_\lambda\}$. Recall this is the subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ consisting of elements such that all but finitely many co-ordinates are 1.

Corollary 1.5. G_λ is LO (resp. BO) for all $\lambda \in \Lambda$ iff $\sum_{\lambda \in \Lambda} G_\lambda$ is LO (resp. BO).

Corollary 1.6. Free abelian groups are BO.

PROOF. Free abelian groups on Λ are $\sum_{\lambda \in \Lambda} \mathbb{Z}$. \square

Let $<$ be an LO on G . The *positive cone* $P = P_{<}$ of $<$ is $\{g \in G \mid g > 1\}$.

Lemma 1.7. Let P be as above.

- (1) $g, h \in P$, implies $gh \in P$ (i.e. $PP \subset P$).
- (2) $G = P \amalg P^{-1} \amalg \{1\}$.
- (3) $<$ is a BO on G iff $f^{-1}Pf \subset P$ for all $f \in G$.

PROOF. (1) This follows from Lemma 1.1 (1).

(2) This follows from Lemma 1.1 (2).

(3) This follows from Lemma 1.1 (3). \square

We say $P \subset G$ is a *positive cone* if P satisfies the conditions in Lemma 1.7.

Lemma 1.8. Let $P \subset G$ be a positive cone. Then $g < h$ implies $g^{-1}h \in P$ defines a LO $<$ on G (With $P_{<} = P$).

PROOF. $<$ is a STO, so:

- (i) $f < g, g < h$ implies $f^{-1}g \in P, g^{-1}h \in P$, which implies (by the first property) that $(f^{-1}g)(g^{-1}h) \in P$, which implies $f < h$.
- (ii) By the second property, for all $g, h \in G$ exactly one of the following holds: $g^{-1}h \in P, g^{-1}h \in P^{-1}$, and $g^{-1}h = 1$. Equivalently, $g < h, h < g$ (since $h^{-1}g \in P$), and $g = h$. Now we show left invariance. $g < h$ implies $g^{-1}h \in P$, but $g^{-1}h = (g^{-1}f^{-1})(fh)$ which implies $fg < fh$.

□

Lemmata 1.7 and 1.8 show that:

$$(1.3) \quad \{\text{LO's on } G\} \quad \leftrightarrow \quad \{\text{positive cones in } G\}$$

$$(1.4) \quad \{\text{BO's on } G\} \quad \leftrightarrow \quad \{\text{conjugacy-invariance positive cones in } G\} .$$

Consider the free group of rank n , F_n .

THEOREM 1.9. F_2 is LO.

PROOF BY SUSIC. Write $F_2 = F(a, b)$. $g \in F_2$ implies we can write it as a reduced word

$$(1.5) \quad (a^{m_1}) b^{n_1} \dots a^{m_k} (b^{n_k})$$

for $k \geq 0, m_i, n_i \in \mathbb{Z} \setminus \{0\}$. Recall 1 is the empty word, $k = 0$. Let $e(g)$ be the number of syllables in g with positive exponent, minus the number of syllables in g with negative exponent. Then define $j(g)$ so be the number of $a^m b^n$'s in f , minus the number of $b^n a^m$'s in G . So $j(g) = 0$, or ± 1 . For example:

$$(1.6) \quad j(a^* \dots a^*) = 0$$

$$(1.7) \quad j(b^* \dots b^*) = 0$$

$$(1.8) \quad j(a^* \dots b^*) = 1$$

$$(1.9) \quad j(b^* \dots a^*) = -1 .$$

Finally define

$$(1.10) \quad \tau(g) = e(g) + j(g) .$$

Note that

$$(1.11) \quad e(g^{-1}) = -e(g) \quad j(g^{-1}) = -j(g) .$$

Lemma 1.10. If $g \neq 1$, then $\tau(g) \equiv 1 \pmod{2}$.

PROOF. $e(f)$ is congruent to the number of syllables mod 2, and $j(g)$ is congruent to the number of syllables $+1 \pmod{2}$. □

Lemma 1.11. $|\tau(gh) - \tau(g) - \tau(h)| \leq 1$.

PROOF. If gh or g or $h = 1$ we are done. So suppose $gh, g, h \neq 1$. Clearly $e(gh) = e(g) + e(h) + \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$. Similarly:

$$(1.12) \quad j(gh) = j(g) + j(h) + \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} .$$

Therefore:

$$(1.13) \quad |\tau(gh) - \tau(g) - \tau(h)| \leq 2$$

so by Lemma 1.10 we have

$$(1.14) \quad |\tau(gh) - \tau(g) - \tau(h)| \leq 1.$$

□

REMARK 1.2. Lemma 1.11 says that $\tau : F_2 \rightarrow \mathbb{Z}(< \mathbb{R})$ is what is called a *quasi-morphism*.

Define $P \subset F_2$ by

$$(1.15) \quad P = \{g \in F_2 \mid \tau(g) > 0\}.$$

Then $F_2 = P \amalg P^{-1} \amalg \{1\}$ by Lemma 1.10 and that $\tau(g^{-1}) = -\tau(g)$. Then $PP \subset P$ by Lemma 1.11 since

$$(1.16) \quad \tau(gh) \geq \tau(g) + \tau(h) - 1 \geq 1.$$

Therefore P is a positive cone for a LO on F_2 . ■

Corollary 1.12. *Any countable free group is LO.*

PROOF. A countable free group is a subgroup of F_2 . □

REMARK 1.3. (1) $\tau(a^{-1}b) = 1$, so $a^{-1}b > 1$, so $b > a$. On the other hand, $\tau(ab^{-1}) = 1$, so $ab^{-1} > 1$, so $b^{-1} > a^{-1}$. So τ does not define a BO on F_2 .

(2) We will see later that all free groups are LO.

(3) Even later we will see that all free groups are BO.

THEOREM 1.13. *Let $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ be a short-exact sequence of groups. Then*

(1) *H, Q LO implies G is LO;*

(2) *if Q is BO and H has a BO that is invariant under conjugation in G then G is BO.*

PROOF. Write $\varphi : G \rightarrow Q$ and regard H as $\ker \varphi < G$. Let P_H (resp. P_Q) be positive cones for LO's on H (resp. Q). Define $P = \varphi^{-1}(P_Q) \amalg P_H$.

CLAIM 1.1. P is a positive cone for an LO on G .

PROOF. We need to check (1) and (2) from Lemma 1.7. Let $g, h \in P$. Then we want to show $gh \in P$. We have three cases.

(a) $g, h \in \varphi^{-1}(P_Q)$: In this case $\varphi(g), \varphi(h) \in P_Q$, so $\varphi(gh) = \varphi(g)\varphi(h) \in P_Q$. Therefore $gh \in \varphi^{-1}(P_Q)$.

(b) $g, h \in P_H$: In this case $gh \in P_H$.

(c) $g \in \varphi^{-1}(P_Q), h \in P_H$: Then $\varphi(gh) = \varphi(g) \in P_Q$, so $gh \in \varphi^{-1}(P_Q)$. Similarly $hg \in \varphi^{-1}(P_Q)$.

Now we need to check $P \amalg P^{-1} \amalg \{1\}$. But this follows from the fact that:

$$(1.17) \quad G = (H \setminus \{1\}) \amalg \varphi^{-1}(Q \setminus \{1\}) \amalg \{1\} = \varphi^{-1}(P_Q) \amalg \varphi^{-1}(P_Q^{-1})$$

since $H \setminus \{1\} = P_H \amalg P_H^{-1}$. □

We leave (2) as an exercise. [Hint: Recall P is a positive cone for BO on G iff it is a conjugacy invariant cone for an LO.] ■

1. Orderability of manifold groups

EXAMPLE 1.2. Let X^2 be the Klein bottle. This has fundamental group

$$(1.18) \quad K = \pi_1(X^2) = \langle a, b \mid b^{-1}ab = a^{-1} \rangle.$$

This fits in the SES:

$$(1.19) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & K & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\ & & \parallel & & & & \\ & & \langle a \rangle & & b \longmapsto gm & & \end{array}$$

which means K is LO by Theorem 1.13.

Note that K is *not* BO. We have that $a > 1$ iff $b^{-1}ab > 1$, but this is a^{-1} , so $a^{-1} > 1$ which is a contradiction.

Notice that \mathbb{Z} has exactly two LO's. The usual one, and the opposite. Therefore, if we choose an LO on $\langle a \rangle$ and $K/\langle a \rangle$, this gives 4 LO's on K determined by:

- (i) $a > 1, b > 1$;
- (ii) $a > 1, b < 1$;
- (iii) $a < 1, b > 1$;
- (iv) $a < 1, b < 1$.

THEOREM 1.14. *These are the only LO's on K .*

PROOF. It suffices to show that each of these conditions determines a unique positive cone.

- (i) $a > 1, b > 1$:

CLAIM 1.2. $a^k < b$ for all $k \in \mathbb{Z}$.

PROOF. $b < a^k$ implies $a^{-k}b < 1$. But $a^{-k}b = ba^k$ and $b > 1$, so $b < a^k$ implies $a^k > 1$, which implies $ba^k > 1$ which is a contradiction. \square

Note that every element in K has a unique representative of the form $a^m b^n$ for $m, n \in \mathbb{Z}$.

CLAIM 1.3. $a^m b^n > 1$ iff either $n > 0$ or $n = 0$ and $m > 0$.

PROOF. If $n = 0$, then this is clear. If $n > 0$, then $a^m b > 1$ for any m by claim 1 (for $k = -m$). But we also know $b > 1$ which implies $b^n > 1$, so we get $a^m b^n > 1$ for $n > 0$. On the other hand, if $m < 0$ then $a^m b^n = b^n a^{\pm m} = (a^{\mp m} b^{-n})^{-1}$. Then we know $a^{\mp m} b^{-n} > 1$ by the case above, so its inverse is < 1 . \square

If $<$ is an LO on G , and $\alpha : G \rightarrow G$ is an automorphism, then this induces an LO $<_\alpha$ on G given by: $g <_\alpha h$ iff $\alpha(g) < \alpha(h)$. Now notice that there are automorphisms α_1, α_2 of K such that

$$(1.20) \quad \alpha_1(a) = a, \quad \alpha_1(b) = b^{-1}$$

$$(1.21) \quad \alpha_1(a) = a^{-1}, \quad \alpha_1(b) = b.$$

In particular, α_1 is given by

$$(1.22) \quad \langle a, b \mid b^{-1}ab = a^{-1} \rangle \cong \langle a, b \mid bab^{-1} = a^{-1} \rangle$$

and similarly for α_2 .

Write $<_{(i)}$ for the unique LO on K determined by (i). Then $<_{(ii)}$ is induced by $<_{(i)}$ and α_1 , $<_{(iii)}$ is induced by $<_{(i)}$ and α_2 , and $<_{(iv)}$ is induced by $<_{(i)}$ and $\alpha_1 \alpha_2$.

■

FACT 1. *If G has only finitely many LO's, then the number of LO's is of the form 2^n .*

EXERCISE 1.1. Show that for all $n \geq 0$ there exists a group G with exactly 2^n LO's.

Corollary 1.15. *For any LO on K , if $h \in \langle a \rangle$, $g \in K \setminus \langle a \rangle$, and $g > 1$, then $g > h$.*

PROOF. It is sufficient to check this for the first LO, since the other three are determined by the above automorphisms. Let $a > 1$, $b > 1$. By claim 2 from above, we know $g = a^m b^n$ for $n > 0$. We now there is some k such that $h = a^k$, and therefore

$$(1.23) \quad h^{-1}g = a^{m-k}b^n > 1$$

by claim 2, so $g > h$. □

2. Three-manifold groups

Suppose M is a closed, orientable, connected three-manifold. Then we might ask if $\pi_1(M)$ is LO? BO?

Immediately we notice that not all such groups are. If M is a lens space, then $\pi_1(M) \cong \mathbb{Z}/n$ for $n > 1$, so this is not LO. More generally, for $\pi_1(M)$ nontrivial and finite is not LO. Recall that if $M = M_1 \# M_2$, then this implies $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. So, for example, if $M_1 \#$ lens space, then $\pi_1(M)$ has torsion, so not LO.

But at least some of them are. Consider $M \cong T^3 = S^1 \times S^1 \times S^1$. Then $\pi_1(M) = \mathbb{Z}^3$ is of course LO. Similarly $M = \#_n (S^1 \times S^2) \cong F_n$, so $\pi_1(M)$ is LO.

We will show that there exist (three-manifold) groups that are torsion-free, but not LO.

Let $p : T^2 \rightarrow X^2$ be a two-fold covering of the Klein bottle. Recall that

$$(1.24) \quad K > p_* (\pi_1(T^2)) = \langle a, b^2 \rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

Let N be the mapping cylinder of p , namely:

$$(1.25) \quad N = (T^2 \times I) \amalg X^2 / ((x, 0) \sim p(x) \forall x \in T^2).$$

The orientation reversing curve representing b doesn't lift. So N is orientable. Note that $\partial N \cong T^2$. There is a strong deformation retraction $N \rightarrow X^2$, so $\pi_1(N) \cong K$. Let N_1, N_2 be two copies of N . Write

$$(1.26) \quad \pi_1(N_i) = \langle a_i, b_i \mid b_i^{-1} a_i b_i = a_i^{-1} \rangle.$$

Notice that $\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z} = \langle a_i, b_i^2 \rangle < \pi_1(N_i)$. Let $\varphi : \partial N_1 \rightarrow \partial N_2$ be a homeomorphism. Let $M_\varphi = N_1 \cup_\varphi N_2$. This is a closed, orientable three-manifold. Therefore

$$(1.27) \quad \pi_1(M_\varphi) = \pi_1(N_1) *_{\mathbb{Z} \times \mathbb{Z}} \pi_1(N_2) \cong K_1 *_{\mathbb{Z} \times \mathbb{Z}} K_2.$$

Since K is torsion-free, $\pi_1(M_\varphi)$ is torsion-free. But in fact we have the following theorem.

THEOREM 1.16. *If $H_1(M_\varphi)$ is finite, then $\pi_1(M_\varphi)$ is not LO.*

REMARK 1.4. We will see later that for M a prime three-manifold with $H_1(M)$ infinite has $\pi_1(M)$ LO.

PROOF. φ is determined up to isotopy, so the resulting manifold M_φ depends only on $\varphi_* : H_1(\partial N_1) \rightarrow H_1(\partial N_2)$. We know

$$(1.28) \quad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_1, 2b_1 \rangle \quad \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} \langle a_2, 2b_2 \rangle$$

so φ_* is given by some 2×2 matrix with \mathbb{Z} coefficients

$$(1.29) \quad \begin{bmatrix} p & r \\ q & s \end{bmatrix}$$

with determinant $ps - qr = \pm 1$. Specifically we have:

$$(1.30) \quad \varphi_*(a_1) = pa_2 + 2qb_2$$

$$(1.31) \quad \varphi_*(2b_1) = ra_2 + 2sb_2 .$$

Now we have $H_1(N_i) = \mathbb{Z} \oplus \mathbb{Z}_2$ with basis b_i and a_i respectively. Then $H_q(M_\varphi)$ is presented by

$$(1.32) \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & p & 2q \\ 0 & -2 & r & 2s \end{bmatrix} .$$

where we order the basis as $\{a_1, b_1, a_2, b_2\}$. Interchanging columns 2 and 3 we get

$$(1.33) \quad \det A = 4 \left| \det \begin{bmatrix} 0 & 2q \\ -2 & 2s \end{bmatrix} \right| = 16 |q| .$$

Therefore $H_1(M_\varphi)$ is finite iff $q \neq 0$ iff $\varphi_*(a_1) \neq \pm a_2$.

Suppose $\pi_1(M_\varphi)$ is LO. Then we would get an induced LO on the common boundary $\partial N_1 = \partial N_2$. But there are only 4 LO's on $\pi_1(N_i)$ (for $i \in \{1, 2\}$). By Corollary 1.15, for any LO on $\pi_1(N)$, $\langle a \rangle$ is the unique \mathbb{Z} -summand of $\pi_1(\partial N) = \langle a, b^2 \rangle$ such that if $h \in \langle a \rangle$ and $g \in \pi_1(\partial N) \setminus \{1\}$, $g > 1$, then $g > h$. Therefore $\varphi_*(a_1) = \pm a_2$ which is a contradiction. \square

Let $<$ be an STO on a set X . Let $\mathcal{B}(X, <)$ be the group of $<$ -preserving bijections $X \rightarrow X$.

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THEOREM 1.17. $\mathcal{B}(X, <)$ is always LO.

PROOF. Let \prec be a WO on X . Let $f, g \in \mathcal{B}(X, <)$ such that $f \neq g$. Write

$$(1.34) \quad [f \neq g] = \{x \in X \mid f(x) \neq g(x)\} \neq \emptyset .$$

Let x_0 be the \prec -least element of $[f \neq g]$. Define

$$(1.35) \quad f < g \iff f(x_0) < g(x_0) .$$

Then we claim that this is an LO on $\mathcal{B}(X, <)$. Left-invariance is clear. To see this is a STO we need “trichotomy” and transitivity. Trichotomy is easy, and transitivity follows from the same argument as the proof of Theorem 1.4. \square

EXAMPLE 1.3. Let $<$ be the standard order on \mathbb{R} . Then $\mathcal{B}(\mathbb{R}, <)$ consists of the orientation-preserving homeomorphisms $\mathbb{R} \rightarrow \mathbb{R}$, written $\text{Homeo}_+(\mathbb{R})$.

Corollary 1.18. $\text{Homeo}_+(\mathbb{R})$ is LO.

REMARK 1.5. For $x \in \mathbb{R}$, let \prec_x be a WO on \mathbb{R} such that x is the \prec_x -least element of \mathbb{R} . Let $<_x$ be the LO on $\text{Homeo}_+(\mathbb{R})$ induced by \prec_x , as in the proof of Theorem 1.17. Given $x \neq y \in \mathbb{R}$, there exists $g \in \text{Homeo}_+(\mathbb{R})$ such that $g(x) > x$ and $g(y) < y$. But this means

$$g <_x 1 \qquad g <_y 1 .$$

which implies $<_x \neq <_y$. Therefore $\text{Homeo}_+(\mathbb{R})$ has uncountably many LO's.

REMARK 1.6. It is a fact that the number of LO's on a group G is either finite (and of the form 2^n) or uncountable.

Corollary 1.19. *A group G is LO iff G acts faithfully^{1.1} on a STO'd set $(X, <)$.*

PROOF. (\Leftarrow): This follows from Theorem 1.17.

(\Rightarrow): G acts faithfully on $(G, <)$ by left multiplication. \square

Corollary 1.18 implies that any subgroup of $\text{Homeo}_+(\mathbb{R})$ is LO. E.g. one can show that F_2 (the free group of rank 2) is a subgroup of $\text{Homeo}_+(\mathbb{R})$. (This is another way to show that countable free groups are LO.) In fact this characterizes countable LO groups.

THEOREM 1.20. *Let G be a countable group. Then G is LO iff there exists an injective homomorphism $G \rightarrow \text{Homeo}_+(\mathbb{R})$.*

PROOF. (\Leftarrow): This follows from Corollary 1.18.

(\Rightarrow): We actually prove something slightly stronger. This will follow from Theorem 1.21. \square

THEOREM 1.21. *Let $(G, <)$ be a countable group with an LO. Then there exists a LO on $\text{Homeo}_+(\mathbb{R})$ and an order-preserving injective homomorphism $(G, <) \rightarrow (\text{Homeo}_+(\mathbb{R}), <)$.*

SKETCH OF PROOF. Let $<$ be an LO on G . If $G = \{1\}$ this is immediate, so assume $G \neq \{1\}$. Therefore it is infinite, since LO groups are torsion free. Let g_1, g_2, \dots be some enumeration of the elements of G .

Define an embedding $e : G \rightarrow \mathbb{R}$ by $e(g_1) = 0$, and inductively by:

(i) If $g_{n+1} \begin{Bmatrix} > \\ < \end{Bmatrix} g_i$ for all $1 \leq i \leq n$, then set

$$(1.36) \quad e(g_{n+1}) = \begin{Bmatrix} \max \{e(g_i) \mid 1 \leq i \leq n\} + 1 \\ \min \{e(g_i) \mid 1 \leq i \leq n\} - 1 \end{Bmatrix}.$$

(ii) Otherwise let

$$g_l = \max \{g_i \mid 1 \leq i \leq n, g_i < g_{n+1}\}$$

$$g_r = \min \{g_i \mid 1 \leq i \leq n, g_i > g_{n+1}\}$$

and set

$$e(g_{n+1}) = \frac{e(g_l) + e(g_r)}{2}.$$

REMARK 1.7. (1) e is order-preserving, i.e. $a < b \implies e(a) < e(b)$.

(2) $e(g_{n+1}) \in \mathbb{Z}$ iff (i) holds.

(3) If $g > 1$ then $g^2 > g$ and $g^{-1} < g$. If $g < 1$ then $g^2 < g$ and $g^{-1} > g$, which implies $\mathbb{Z} \subset e(G) = \Gamma$.

(4) G acts on Γ by $g(e(a)) = e(ga)$. In fact, G acts on $(\Gamma, <)$ (where $<$ is the restriction of $<$ on \mathbb{R}) since $e(a) < e(b)$ iff $a < b$ iff $ga < gb$ iff $e(ga) < e(gb)$ iff $g(e(a)) < g(e(b))$.

To see that this action extends to an action of G on \mathbb{R} , we have a few steps.

Step 1: The action of G on Γ is continuous,

Step 2: The action of G on Γ extends to a continuous action of G on $\bar{\Gamma}$.

^{1.1}Recall this means $g(x) = x$ for all $x \in X$ iff $g = 1$.

Step 3: $\mathbb{R} \setminus \bar{\Gamma}$ is a countable Π of open intervals (a_i, b_i) ; the action of G is defined on $\{a_i, b_i\}$; and extends to $[a_i, b_i]$.

Note, to ensure Step 1:, it is not enough to take e to be an order-preserving of G in \mathbb{R} . It must be continuous.

To define an LO on $\text{Homeo}_+(\mathbb{R})$ that restricts to the LO on Γ from G , first pick any $\gamma \in \Gamma$. Then $g > 1$ (resp. < 1) iff $g(\gamma) > \gamma$ (resp. $< \gamma$). Let \prec be a WO on \mathbb{R} such that γ is the \prec -least element of \mathbb{R} . Then let \leq be the LO on $\text{Homeo}_+(\mathbb{R})$ induced by \prec . Then $g > 1$ (resp. < 1) in G iff $g \succ 1$ (resp. \prec) in $\text{Homeo}_+(\mathbb{R})$. \square

3. Group rings

Let R be a ring (with 1).

- $a \in R$ is a *unit* if there exists $b \in R$ such that $ab = ba = 1$.
- $a \in R$ is a *zero-divisor* if $a \neq 0$ and there exists $b \neq 0$ such that either $ab = 0$ or $ba = 0$.
- $a \in R$ is a *non-trivial idempotent* if $a^2 = a$ but $a \neq 0$ and $a \neq 1$.

Let G be a group and R a ring. Then the R -group ring of G consists of formal sums:

$$(1.37) \quad RG := \left\{ \sum r_g g \mid g \in G, r_g \in R, r_g \neq 0 \forall \text{ but f'tly many } g \in G \right\}.$$

RG is a ring with respect to the obvious operations. For $g \in G$ and $r \in R$ a unit, then rg is a unit in RG . A unit in RG is *non-trivial* if it is not of this form.

REMARK 1.8. If $\tilde{X} \rightarrow X$ is a universal covering, then $\pi = \pi_1(X)$ acts on \tilde{X} so $H_*(\tilde{X}, \mathbb{Z})$ is a $\mathbb{Z}\pi$ -module.

THEOREM 1.22. Suppose G has non-trivial torsion, and K is a field of characteristic 0.

- (1) KG has zero divisors,
- (2) KG has non-trivial units,
- (3) KG has non-trivial idempotents.

PROOF. Let $g \in G$ have order $n \geq 2$. Define

$$\sigma = 1 + g + g^2 + \dots + g^{n-1} \in KG.$$

First notice that

$$(1.38) \quad g\sigma = \sigma$$

which implies $(1 - g)\sigma = 0$ so we have zero divisors.

(1.38) also gives us that $\sigma^2 = n\sigma$. Therefore

$$(1 - \sigma) \left(1 - \frac{1}{n-1} \sigma \right) = 1$$

so we have a nontrivial unit for $n > 2$. If $n = 2$, $1 - \sigma = -g$, but we still have:

$$(1.39) \quad (1 - 2\sigma) \left(1 - \frac{2}{3} \sigma \right) = 1.$$

Finally, we have that

$$(1.40) \quad \left(\frac{1}{n} \sigma \right)^2 = \left(\frac{1}{n^2} \right) \sigma^2 = \frac{1}{n} \sigma$$

so we have nontrivial idempotents. \square

Note that the proof of (1) works even for $\mathbb{Z}G$.

REMARK 1.9. If $n \notin \{2, 3, 4, 6\}$ then $\mathbb{Z}G$ has nontrivial units. This is a theorem of Higman.

EXAMPLE 1.4. For $n = 5$,

$$(1.41) \quad (1 - g - g^4)(1 - g^2 - g^3) = 1 .$$

But what if G is torsion free? This brings us to the famous Kaplansky conjectures.

CONJECTURE 1 (Kaplansky). *If G is torsion free and K is a field, then:*

I (Units conjecture): KG has no non-trivial units,

II (Zero-divisors conjecture): KG has no zero divisors,

III (Idempotents conjecture): KG has no non-trivial idempotents.

REMARK 1.10. Clearly II implies III since $a^2 = a$ implies $a(a - 1) = 0$, which by II implies $a = 0$ or $a = 1$ which implies III. In fact they're all equivalent, but this is nontrivial to see.

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REMARK 1.11. Note that if R is an integral domain (e.g. \mathbb{Z}) then R is contained in its field of fractions. In this case Items I and II and Item III for its field of fractions imply the corresponding versions of Items I and II and Item III for R .

REMARK 1.12. We know this is true for LO groups. As we have seen, we should think of LO as being a stronger version of torsion free.

THEOREM 1.23. *If G is LO then KG satisfies Items I and II and Item III.*

PROOF. Since Item I implies Item III by the above remark we show Item I and Item II.
Item I: Suppose

$$(1.42) \quad \left(\sum_{i=1}^m \alpha_i g_i \right) \left(\sum_{j=1}^n \beta_j h_j \right) = 1$$

with m, n not both 1, $\alpha_i, \beta_j \neq 0 \in K$, distinct $g_i \in G$, and distinct $h_i \in G$. Note this product can be rewritten as the following sum with mn terms:

$$(1.43) \quad \sum_{i,j} (\alpha_i \beta_j) (g_i h_j) .$$

Assume WLOG that $h_1 < h_2 < \dots < h_n$. Let $g_k h_l$ be a minimal element of

$$(1.44) \quad S = \{g_i h_j \mid 1 \leq i \leq m, 1 \leq j \leq n\} \subset G .$$

We know $h_1 < h_j$ for $j > 1$, so $g_k h_1 < g_k h_j$ for all $j > 1$. Therefore $l = 1$. Also $gh_1 = g'h_1$ which implies $g = g'$. Therefore $g_k h_1$ is the unique

$$(1.45) \quad (k, 1) \in \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

such that $g_k h_1$ is a minimal element of S .

Similarly, there is a unique

$$(1.46) \quad (r, n) \in \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

such that $g_r h_n$ is a maximal element of S .

CLAIM 1.4. $g_k h_1 \neq g_r h_n$.

If they were equal, then $r = k$, $n = 1$, so $m > 1$. So $g_k h_1 = g_r h_1$, and therefore $g_r = g_k$. But this cannot be the case since they are distinct by assumption.

This implies that (1.43) has ≥ 2 terms after cancellation, so it cannot be 1.

Item II: Now suppose

$$(1.47) \quad \left(\sum_{i=1}^m \alpha_i g_i \right) \left(\sum_{j=1}^n \beta_j h_j \right) = 0$$

for $m, n \geq 1$. Then there is a unique minimal element and nonzero coefficient, which means it is nonzero. \square

CONJECTURE 2 (Isomorphism conjecture). *If G is torsion free, then $\mathbb{Z}G \cong \mathbb{Z}H$ implies $G \cong H$.*

REMARK 1.13. In [3] a finite counterexample to the conjecture for arbitrary groups was provided, i.e. it is shown that there exists finite G, H such that $\mathbb{Z}G \cong \mathbb{Z}H$, $G \not\cong H$.

Corollary 1.24 ([5]). *If G is LO, then G satisfies the isomorphism conjecture.*

PROOF. Theorem 1.23 implies that $\mathbb{Z}G$ has no nontrivial units. Call $\mathcal{U}_{\mathbb{Z}G}$ the group of units in $\mathbb{Z}G = \mathbb{Z}/2 \times G$. Suppose $\mathbb{Z}G \cong \mathbb{Z}H$. Theorem 1.23 says that $\mathbb{Z}G$ has no 0-divisors. This implies $\mathbb{Z}H$ has no 0-divisors, which means (by Theorem 1.22) that H is torsion-free. Now $H < \mathcal{U}_{\mathbb{Z}H} \cong \mathcal{U}_{\mathbb{Z}G} \cong \mathbb{Z}/2 \times G$ which implies $H < G$ (since H is torsion-free), which implies H is LO (since G is), which implies $\mathcal{U}_{\mathbb{Z}H} \cong \mathbb{Z}/2 \times H$, which implies $\mathbb{Z}/2 \times H \cong \mathbb{Z}/2 \times G$ which implies $H \cong G$ (since H, G are torsion free). \square

REMARK 1.14. We might wonder if it is ever the case that (for $G \neq 1$) $(G * \mathbb{Z}) / \langle \langle w \rangle \rangle = 1$? This is known for G torsion free [4].

COUNTEREXAMPLE 1. If we consider the question of whether we can ever have $(A * B) / \langle \langle w \rangle \rangle = 1$ for A, B nontrivial, a counterexample is given by:

$$\mathbb{Z}/2 * \mathbb{Z}/3 / (a = b) .$$

4. BO's on $\mathbb{Z} \times \mathbb{Z}$

Recall we have 2 orders on \mathbb{Z} . Consider a line of slope α in $\mathbb{Z} \times \mathbb{Z}$. Then we have two cases.

- (1) α irrational: The associated positive cone is everything above the line. Specifically, $P \subset \mathbb{Z} \times \mathbb{Z}$ is given by

$$(1.48) \quad P = \{(m, n) \mid n > m\alpha\} .$$

It is easy to check that this is a positive cone. This means there are uncountable many BO's on $\mathbb{Z} \times \mathbb{Z}$.

- (2) α rational: Notice that now

$$(1.49) \quad \{(m, n) \mid n = m\alpha\} \cong \mathbb{Z} < \mathbb{Z} \times \mathbb{Z} .$$

Now let P_0 be one of the two positive cones on \mathbb{Z} . Then we can check that

$$P = P_0 \amalg \{(m, n) \mid n > m\alpha\}$$

is a positive cone for $\mathbb{Z} \times \mathbb{Z}$.

REMARK 1.15. (1) (Up to reversal) these are all the BOs on $\mathbb{Z} \times \mathbb{Z}$. I.e. for α rational we get two, and for α irrational we get 4.

- (2) This generalizes in the obvious way to \mathbb{Z}^n .

5. BO's on \mathbb{R}

Regard \mathbb{R} as a vector space on \mathbb{Q} with uncountable bases Λ . Recall Λ exists by the axiom of choice. Therefore $\mathbb{R} \subset \mathbb{Q}^\Lambda$. In particular it is the elements of \mathbb{Q}^Λ with only finitely many non-zero coordinates. There are uncountable many WO's on Λ , and each gives rise to a lexicographic BO on \mathbb{Q}^Λ . This gives us uncountably many BOs on \mathbb{R} .

CHAPTER 2

The space of left-orders on a group

The basic idea is that since lefts orders are determined by positive cones, we can give this space a topology. Consider a family of sets $\{X_\lambda \mid \lambda \in \Lambda\}$. Then write

$$X = \prod_{\lambda \in \Lambda} X_\lambda$$

and $\pi_\lambda : X \rightarrow X_\lambda$ for the projection. If X_λ is a topological space, then X can be given the product topology. This is the largest topology on X such that π_λ is continuous for all $\lambda \in \Lambda$. So X has subbasis

$$(2.1) \quad \left\{ \pi_\lambda^{-1}(U_\lambda) = U_\lambda \times \prod_{\mu \neq \lambda} X_\mu \mid U_\lambda \subset X_\lambda \text{ open, } \lambda \in \Lambda \right\}.$$

THEOREM. *If X_λ is compact for all $\lambda \in \Lambda$ then $\prod_{\lambda \in \Lambda} X_\lambda$ is compact.*

REMARK 2.1 (Exercises). (1) X_λ Hausdorff (for all $\lambda \in \Lambda$) implies $\prod_{\lambda \in \Lambda} X_\lambda$ is Hausdorff.

(2) A space X is totally disconnected if the only nonempty connected subspaces are singletons $\{x\}$ for $x \in X$. This is equivalent to the connected components of X all being $\{x\}$. Show that X_λ totally disconnected (for all $\lambda \in \Lambda$) implies $\prod_{\lambda \in \Lambda} X_\lambda$ is totally disconnected.

Let X be a set, let $\mathcal{S}(X)$ be the set of subsets of X (i.e. the power set). Then we have a correspondence:

$$\mathcal{S}(X) \quad \leftrightarrow \quad \{f : X \rightarrow \{0, 1\}\}$$

which sends:

$$A \subset X \quad \leftrightarrow \quad f_A : X \rightarrow \{0, 1\}$$

where

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Give $\{0, 1\}$ the discrete topology, and give

$$\mathcal{S}(X) = \{0, 1\}^X = 2^X = \prod_{x \in X} \{0, 1\}$$

the product topology. Note $\{0, 1\}$ is a compact, Hausdorff, totally-disconnected space, which means $\mathcal{S}(X)$ is too. For $x \in X$ let

$$\begin{aligned} U_x &= \pi_x^{-1}(1) = \{A \subset X \mid x \in A\} \\ V_x &= \pi_x^{-1}(0) = \{A \subset X \mid x \notin A\}. \end{aligned}$$

Note that $V_x = \mathcal{S}(X) \setminus U_x$ so U_x and V_x are open and closed. Then

$$(2.2) \quad \{U_x \mid x \in X\} \cup \{V_x \mid x \in X\}$$

is a subbasis for $\mathcal{S}(X)$.

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4, 2020

Lemma 2.1. *Suppose $B \subset X$. Then*

$$\{A \subset X \mid B \not\subset A\} \quad \{A \subset X \mid A \cap B \neq \emptyset\}$$

are open subsets of $\mathcal{S}(X)$.

PROOF.

$$\{A \subset X \mid B \not\subset A\} = \bigcup_{b \in B} \{A \subset X \mid b \notin A\} = \bigcup_{b \in B} V_b$$

so it is open. The argument for the other set is similar. \square

If G is a group, let

$$(2.3) \quad \text{LO}(G) = \{\text{positive cones } \subset G\} \subset \mathcal{S}(G)$$

and equip it with the subspace topology. We call this the *space of left-orders on G* .

EXAMPLE 2.1. $\text{LO}(\mathbb{Z}) = \text{pt} \amalg \text{pt}$. $\text{LO}(\mathbb{Z} \times \mathbb{Z})$ is the cantor set.

THEOREM 2.2. $\text{LO}(G)$ is closed in $\mathcal{S}(G)$ and hence compact.

PROOF. We show $\mathcal{S}(G) \setminus \text{LO}(G)$ is open. Suppose $A \in \mathcal{S}(G) \setminus \text{LO}(G)$, i.e. $A \subset G$ is not a positive cone. So either:

- (i) $\exists g, h \in A$ such that $gh \notin A$ or
- (ii) $\exists g \in G$ such that $g, g^{-1} \in A$ or
- (iii) $1 \in A$ or
- (iv) $\exists g, g \neq 1$ such that $g \notin A$ and $g^{-1} \notin A$.

Now the point is that these are open conditions since we can write them in terms of the U_x 's and V_x 's. In particular:

$$\begin{aligned} (i) &\iff A \in U_g \cap U_h \cap V_{gh} & (ii) &\iff A \in U_g \cap U_{g^{-1}} \\ (iii) &\iff A \in U_1 & (iv) &\iff A \in \bigcup_{g \neq 1} (V_g \cap V_{g^{-1}}) . \end{aligned}$$

Therefore $\text{LO}(G)$ is compact, Hausdorff, and totally disconnected. \square

Similarly one can define the space of biorders on G , $\text{BO}(G)$, to be the set of conjugation invariant positive cones in G .

EXERCISE 0.1. Show that $\text{BO}(G)$ is closed inside of $\text{LO}(G)$.

Therefore $\text{BO}(G)$ is compact, Hausdorff, and totally disconnected.

1. The cantor set

The cantor set $C \subset I \subset \mathbb{R}$ is defined as follows. First write

$$\begin{aligned} C_1 &= [0, 1/3] \cup [2/3, 1] \\ C_2 &= ([0, 1/9] \cup [2/9, 1/3]) \cup ([2/3, 7/9] \cup [8/9, 1]) \\ &\dots \end{aligned}$$

then define

$$(2.4) \quad C = \bigcap_{n=1}^{\infty} C_n .$$

The idea is that we keep removing the middle thirds.

C is uncountable, totally-disconnected, closed in I . Therefore it is also compact and Hausdorff. This is a very surprising example. We can easily write down something uncountable and totally-disconnected, such as the irrationals, but they do not form a compact set.

Any $x \in I$ has a ternary expansion:

$$x = 0.x_1x_2\dots = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

which is unique up to:

$$\dots x_k 22 \dots = \dots (x_{k+1}) 00 \dots$$

Now notice

$$x_1 = 1 \quad \Longleftrightarrow \quad x \in (1/3, 2/3)$$

with the convention that

$$\frac{1}{3} = 0.022\dots$$

Similarly (with the same convention) we have

$$x_1 \neq 1, x_2 = 1 \quad \Longleftrightarrow \quad x \in (1/9, 2/9) \cup (7/9, 8/9)$$

and so on. Then

$$(2.5) \quad C = \{x \in I \mid x = 0.x_1x_2\dots \mid \forall n, x_n = 0 \text{ or } 2\} .$$

Now give $\{0, 2\}^{\mathbb{N}}$ the product topology.

EXERCISE 1.1. Show that the map sending

$$(2.6) \quad 0.x_1x_2\dots \mapsto (x_1, x_2, \dots)$$

defines a homeomorphism

$$(2.7) \quad C \xrightarrow{\cong} \{0, 2\}^{\mathbb{N}} .$$

Now recall that $\text{LO}(G)$ is compact in $\{0, 1\}^G$, so if G is countable, then $\text{LO}(G)$ is homeomorphic to a subspace of C .

We say $x \in X$ is *isolated* if $\{x\}$ is open. We say X is perfect if it has no isolated points. As it turns out, the Cantor set is perfect.

THEOREM. *If X is a compact, totally-disconnected, and perfect metric space, then $X \cong C$.*

Therefore, if G is countable, $\text{LO}(G) \neq \emptyset$, and has no isolated points, then $\text{LO}(G) \cong C$.

EXAMPLE 2.2. In 2004 [7] it was shown that if $n > 1$ then $\text{LO}(\mathbb{Z}^n) = \text{BO}(\mathbb{Z}^n) \cong C$.

EXAMPLE 2.3. In 1985 [6] it was shown that $\text{LO}(F_n) \cong C$. It is unknown if $\text{LO}(F_n)$ has isolated points.

REMARK 2.2. As it turns out, the braid group is LO. The first proof of this fact was not topological, so topologists started to think of a topological proof. When someone asked Thurston, he said “of course the braid group is left-orderable!”

If $X \subset G$, let $S(X)$ be the semigroup generated by X in G . This is the same as the non-empty product of elements in X . There is a characterization of left orderability in terms of finite subsets of G .

THEOREM 2.3. G is LO iff for all finite $F \subset G \setminus \{1\}$, there exists $\epsilon : F \rightarrow \{\pm 1\}$ such that

$$(2.8) \quad 1 \notin S\left(\left\{f^{\epsilon(f)} \mid f \in F\right\}\right) (= S(F, \epsilon)) .$$

REMARK 2.3. It follows from this that, given a solution to the word problem in G , there exists a machine such that if G is not LO, the machine will eventually tell you that. Nathan Dunfield has an explicit algorithm for three-manifold groups.

REMARK 2.4. If we take the n -fold cyclic branch cover of the knot 5_2 , then we can consider $\pi_1(\Sigma_n(5_2))$. For $n = 2$, this is a lens space so π_1 is finite. It is also not LO for $n = 3, 4$, and 5 . But it is unknown for $n = 6, 7$, and 8 . (If the L -space conjecture is true,^{2.1} then it should be LO for these values of n .) For $n \geq 9$ it is known to be LO.

PROOF. (\implies): Define

$$\epsilon(f) = \begin{cases} +1 & f > 1 \\ -1 & f < 1 \end{cases} .$$

(\impliedby): Let $F \subset G \setminus \{1\}$ be finite, $\epsilon : F \rightarrow \{\pm 1\}$. Define

$$Q(F, \epsilon) := \left\{ Q \subset G \setminus \{1\} \mid S(F, \epsilon) \subset Q, S(F, \epsilon)^{-1} \cap Q = \emptyset \right\} .$$

Note that $Q(F, \epsilon) \neq \emptyset$ iff (2.8) holds. Let

$$Q(F) = \cup_{\epsilon} Q(F, \epsilon) .$$

Note this is a finite union.

CLAIM 2.1. $Q(F)$ is closed in $S(G)$.

PROOF. It is sufficient to show that $Q(F, \epsilon)$ is closed, i.e. $S(G) \setminus Q(F, \epsilon)$ is open. Suppose $A \subset G$, $A \not\subset Q(F, \epsilon)$ i.e. either $1 \in A$, or $S(F, \epsilon) \not\subset A$, or $S(F, \epsilon)^{-1} \cap A \neq \emptyset$. These conditions are all open by Lemma 2.1. \square

Note that if $F \subset F'$, then

$$(2.9) \quad S(F, \epsilon'|_{F'}) \subset S(F', \epsilon')$$

and therefore

$$(2.10) \quad Q(F') \subset Q(F) .$$

^{2.1}Which is looking quite likely. It has been checked for something like three-hundred thousand manifolds.

Let F_1, F_2, \dots, F_n be finite subsets of $G \setminus \{1\}$. Then

$$\bigcap_{i=1}^n Q(F_i) \supset Q(F_1 \cup F_2 \cup \dots \cup F_n) \neq \emptyset$$

since (2.8) holds. This means $\{Q(F)\}$ has the *finite intersection property* (FIP) and each one is closed. Therefore, since $\mathcal{S}(G)$ is compact,

$$\bigcap_{F \subset G \setminus \{1\} \text{ finite}} Q(F) \neq \emptyset.$$

So let $P \in \bigcap Q(F)$.

CLAIM 2.2. P is a positive cone for G .

PROOF. First notice $1 \notin P$ since $1 \notin Q(F)$ for any finite $F \subset G \setminus \{1\}$.

Now we show $g, h \in P$ implies $gh \in P$. Let $F = \{g, h\}$. Then there are $\epsilon(g), \epsilon(h) \in \{\pm 1\}$ such that

$$S(g^{\epsilon(g)}, h^{\epsilon(h)}) \subset P \quad S(g^{\epsilon(g)}, h^{\epsilon(h)})^{-1} \cap P = \emptyset.$$

Therefore $\epsilon(g) = \epsilon(h) = +1$, which implies $gh \in S(g^{\epsilon(g)}, h^{\epsilon(h)}) \subset P$.

Now we show $P \cap P^{-1} = \emptyset$. Let $g \in P$, and $F = \{g\}$. Therefore $S(g) \subset P$, which means $S(g)^{-1} \cap P = \emptyset$, so $g^{-1} \notin P$.

Finally we show $P \amalg P^{-1}G \setminus \{1\}$. Take $g \in G$ such that $g \neq 1$. Let $F = \{g\}$. Then there exists $\epsilon = \pm 1$ such that $S(g^\epsilon) \subset P$ (and $S(g^{-1}) \cap P = \emptyset$) which implies $g^\epsilon \in P$. \square

REMARK 2.5. There exists an analogue of this for BO.

THEOREM 2.4. G is BO if and only if for all finite $F \subset G \setminus \{1\}$ there is some $\epsilon : F \rightarrow \{\pm 1\}$ such that $1 \notin T(F, \epsilon)$ where $T(F, \epsilon)$ is the smallest semigroup which

- (i) contains $S(F, \epsilon)$, and
- (ii) for all $g, h \in T(F, \epsilon)$, $g, h, g^{-1}, g^{-1}hg \in T(F, \epsilon)$.

EXERCISE 1.2. Prove Theorem 2.4.

Let P be a property of groups. A group G is *locally* P if and only if every finitely generated subgroup of G has property P . (So $\text{loc}(\text{loc}(P)) \equiv \text{loc}(P)$.) P is a *local property* if $\text{loc}(P) \implies P$.

THEOREM 2.5. G is locally LO (resp. BO) if and only if G is LO (resp. BO).

PROOF. (\Leftarrow): LO and BO are inherited by subgroups.

(\Rightarrow): Let G be a finite set contained in $G \setminus \{1\}$. Then $\langle F \rangle < G$ is finitely generated. $G \text{ loc(LO)}$ implies $\langle F \rangle$ is LO. Therefore there exists ϵ such that (2.8) holds (from Theorem 2.3). This is true for all F , so G is LO by Theorem 2.3. The argument for BO is similar, using Theorem 2.4 instead. \square

Corollary 2.6. An abelian group is BO iff it is torsion free.

PROOF. (\Rightarrow): This follows from Lemma 1.3.

(\Leftarrow): G is LO iff G is loc(LO) . For H finitely generated inside of torsion free G , then $H \cong \mathbb{Z}^n$, so it is LO. \square

Corollary 2.7. *An arbitrary free group is LO.*

PROOF. Let F be a free group. For H a finitely generated subgroup of F , $H \cong F_n$ for some n . Then H is LO by Corollary 2.7, so F is LO by Theorem 2.5. \square

THEOREM 2.8. *Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a collection of groups. Then G_λ is LO for all $\lambda \in \Lambda$ if and only if $\ast_{\lambda \in \Lambda} G_\lambda$ is LO.*

PROOF. (\Leftarrow): $G_\lambda < \ast_{\lambda \in \Lambda} G_\lambda$.

(\Rightarrow): There exists a homomorphism

$$G = \ast_{\lambda \in \Lambda} G_\lambda \xrightarrow{\varphi} \prod_{\lambda \in \Lambda} G_\lambda$$

$$g_\lambda \longmapsto (1, \dots, 1, g_\lambda, 1, \dots)$$

So we get a SES

$$(2.11) \quad 1 \rightarrow H \rightarrow \ast_\Lambda G_\lambda \xrightarrow{\varphi} \prod_\Lambda G_\lambda \rightarrow 1$$

where $H = \ker \varphi$. By the Kurosh subgroup theorem

$$H = \left(\ast_\mu H_\mu \right) \ast F$$

where H_μ is a subgroup of a conjugate of G_{λ_μ} in G , and F is a free group. But $H = \ker \varphi$, and $\varphi|_{G_\lambda}$ is injective for all $\lambda \in \Lambda$. Therefore for all $\lambda \in \Lambda$ and $g \in G$ we have $H \cap g^{-1}G_\lambda g = \{1\}$. Therefore $H = F$.

But now G_λ LO for all $\lambda \in \Lambda$ implies $\prod_{\lambda \in \Lambda} G_\lambda$ is LO by Theorem 1.4, and $F = H$ is LL by Corollary 2.7, so G is LO by Theorem 1.13. \square

Let P be a property of groups. A group G is residually P , $\text{res}(P)$, if and only if for all $g \in G \setminus \{1\}$ there exists an epimorphism $\varphi : G \rightarrow H$ such that H has property P , and $\varphi(g) \neq 1$.

REMARK 2.6. Note that P implies $\text{res}(P)$, and $\text{res}(\text{res}(P))$ implies $\text{res}(P)$.

We say P is a *residual property* if and only if $\text{res}(P)$ implies P .

EXAMPLE 2.4. Finiteness is not a residual property. E.g. \mathbb{Z} is $\text{res}(\text{finite})$.

Lemma 2.9. *If P is closed under taking subgroups and direct products, then P is a residual property.*

Corollary 2.10. *LO and BO are residual properties.*

PROOF OF LEMMA 2.9. Suppose G is $\text{res}(P)$. Then for all $g \in G \setminus \{1\}$ there is an epimorphism $\varphi_g : G \rightarrow H_g$ such that H_g has P , and $\varphi_g(g) \neq 1$. The collection of these $\{\varphi_g \mid g \in G \setminus \{1\}\}$ induces a homomorphism

$$\varphi : G \rightarrow \prod_{g \in G \setminus \{1\}} H_g .$$

Then this is injective, and $\varphi_g(g) \neq 1$. H_g has P for all $g \in G \setminus \{1\}$. Therefore $\prod_{g \in G \setminus \{1\}} H_g$ has P . But

$$G \cong \varphi(G) < \prod_{g \in G \setminus \{1\}} H_g$$

so G has P . \square

REMARK 2.7. Residual properties are related to areas of active research. For example the geometrization conjecture is related to residual finiteness of 3-manifolds.

REMARK 2.8. Let G be a group. Let $\text{FQ}(G)$ consist of the finite quotients of G . Then the following is an open question. Let F_2 be a free group of rank 2. If G is a residually finite group such that $\text{FQ}(G) = \text{FQ}(F_2)$ is $G \cong F_2$? Note that $\text{FQ}(F_2)$ consists of the finite groups generated by two elements. So this is really quite concrete.

Another open question is if G_1 and G_2 are residually finitely presented, then does $\text{FQ}(G_1) = \text{FQ}(G_2)$ imply $G_1 \cong G_2$?

EXAMPLE 2.5. $\text{LO}(\mathbb{Z}^n)$ and $\text{LO}(F_n)$ are both the cantor set.

EXAMPLE 2.6. Let B_n denote the braid group. As it turns out $\text{LO}(B_n)$ has isolated points [2].

The following is a strengthening of the fact that LO is a local property.

WARNING 2.1. At this point it is convenient to make the convention that $\{1\}$ is *not* LO .

THEOREM 2.11 (Burns-Hale). *G is LO iff every non-trivial finitely generated subgroup $H < G$ has an LO quotient.*

PROOF. (\implies): G is LO implies H is LO .

(\impliedby): $F = \{g_1, \dots, g_n\} \subset G \setminus \{1\}$ for $n \geq 1$. We show by induction on n that the condition on F in Theorem 2.3 holds. Let $n = 1$. Then $\langle g_1 \rangle$ has an LO quotient by assumption. Therefore g_1 has infinite order, so $1 \notin S(g_1)$. Now suppose $n > 1$. By assumption, there exists a nontrivial homomorphism $\varphi : \langle g_1, \dots, g_n \rangle \rightarrow L$ where L is LO . For some m there exists

$$\varphi(g_i) = \begin{cases} +1 & 1 \leq i \leq m \\ -1 & m < i \leq n \end{cases}$$

By the induction hypothesis there exists $\epsilon_1, \dots, \epsilon_m \in \{\pm 1\}$ such that $1 \notin S(\{g_i^{\epsilon_i} \mid 1 \leq i \leq m\})$. Let $<$ be an LO on L . Define $\epsilon_i \in \{\pm 1\}$ ($m < i \leq n$) so that

$$(2.12) \quad \varphi(g_i^{\epsilon_i}) > 1$$

Then $1 \notin S(\{g_i^{\epsilon_i} \mid 1 \leq i \leq n\})$. □

A group G is *indicable* if either $G = \{1\}$ or there is an epimorphism $G \rightarrow \mathbb{Z}$.

Corollary 2.12. *G is locally indicable implies G is LO.*

REMARK 2.9. Free groups are loc (indicable) so this gives another proof that free groups are LO .

REMARK 2.10. Note that G having an LO quotient does not imply G is LO .

COUNTEREXAMPLE 2. $\mathbb{Z} * \mathbb{Z}/2$ has LO quotient, but is not LO .

We do however have:

THEOREM 2.13. *Let G be a group such that every finitely generated subgroup of infinite index is indicable. Then G is LO if and only if G has an LO quotient.*

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