Lecture 4 The Seiberg-Witten invariants.

Recall spin^c structure $\quad S = S^+ \oplus S^-$
$$\begin{array}{c} S \\ \downarrow \\ X \end{array}$$
$\rho : Cl(TX) \to S$. Clifford multiplication.

$\rho(e_0) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad \rho(e_i) = \begin{bmatrix} 0 & -\sigma_i^* \\ \sigma_i^* & 0 \end{bmatrix}$.

Clifford connection $\quad A = \nabla^S = \nabla^A$ unitary $\qquad \Gamma(S^+)$ spinors

spin^c-connection $\qquad \nabla^A_X(Y\cdot\Phi) = (\nabla_X Y)\cdot\Phi + Y\cdot(\nabla^A_X \Phi)$.

Levi-Civita.

Fact spin^c connections form an affine space over $\Omega^1(X; i\mathbb{R})$.

$\tilde{A} - A = a \otimes 1_S \qquad$ (Follows from Schur's lemma).

ρ is irreducible.

$\Rightarrow$ it's convenient to study A^t, connection induced on $\det S^+$.

$\rightsquigarrow \tilde{A}^t - A^t = 2a$.

Consider (A, Φ). How can they interact? $\boxed{D^+_A \Phi = 0}$ $\quad \Phi$ is A harmonic.

We can extend $\rho : \Lambda^* TX \otimes \mathbb{C} \to S$

$\rho(\alpha\wedge\beta) = \frac{1}{2}(\rho(\alpha)\rho(\beta) + (-1)^{|\alpha||\beta|}\rho(\beta)\rho(\alpha))$.

Exercise $\rho : \Lambda^+ \xrightarrow{\sim} su(S^+)$.

If Φ is $\Phi(S^+) \Rightarrow (\Phi\Phi^*)_0 \in isu(S^+)$. $\Phi = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \quad \Phi\Phi^* = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}(\bar\alpha\ \bar\beta) = \begin{pmatrix} |\alpha|^2 & \alpha\bar\beta \\ \beta\bar\alpha & |\beta|^2 \end{pmatrix}$

$\rightsquigarrow (\Phi\Phi^*)_0 =$

$\rightarrow \rho^{-1}((\Phi\Phi^*)_0)$ is imaginary valued self-dual 2-form.

$\rightarrow \boxed{\frac{1}{2} F^+_{A^t} = \rho^{-1}((\Phi\Phi^*)_0)}$

$\mathcal{C}(X, s) = \{(A, \Phi) \mid A \text{ spin}^c \text{ connection}, \Phi \text{ spinor}\}$.

$\mathcal{G}(X, s)$ gauge group $= \{u : X \to S^1\}$. $\quad u\cdot(A, \Phi) = (A - u^{-1}du, u\cdot\Phi)$.

The action is very nice.

If $\Phi \neq 0$, $\mathrm{Stab}(A,\Phi) = \{1\}$ irreducible

$\Phi = 0$ $\mathrm{Stab}(A,\Phi) = S^1$ reducible.

costante

Fix $\omega \in i\Omega^2(X)$.

The Seiberg-Witten equations perturbed by ω are

$$\begin{cases} D_A^+ \Phi = 0 & \in \Gamma(S^-) \\ \frac{1}{2}(F_{A^t}^+ - 4\omega^+) - (\Phi\Phi^*)_0 = 0 & \in i\mathfrak{su}(S^+). \end{cases} \qquad \mathfrak{F}_\omega(A,\Phi).$$

Exercise these are gauge-invariant

$$\mathcal{M}_{\omega,g}(X,\mathfrak{s}) = \{[A,\Phi] \mid \mathfrak{F}_\omega(A,\Phi) = 0\} / \mathcal{G}(X,\mathfrak{s}).$$

$\boxed{\text{Fact 1}}$ $\mathcal{M}_{\omega,g}(X,\mathfrak{s})$ is compact.

scalar curvature.

Key: Weitzenböck formula $D_A^- D_A^+ \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{2}\rho_X(F_{A^t}^+)\Phi + \frac{1}{4} s \Phi$

$\rightsquigarrow$ A priori C^0 bounds for solutions + elliptic regularity.

Question when do there exist reducible solutions?

Need $F_{A^t}^+ = 4\omega^+$.

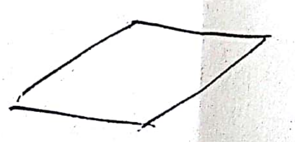
If now k is self-dual and closed ($\Rightarrow k \in \mathcal{H}^+$).

$$\int_X 4\omega \wedge k = \int 4\omega^+ \wedge k = \int F_{A^t}^+ \wedge k = \int F_{A^t} \wedge k =$$

$$= \frac{2\pi}{i}(c_1(S^+) \cup [k])[X]. \Rightarrow \text{non-trivial constraint.}$$

$\boxed{\text{Fact 2}}$ If $b_2^+ \geq 1$, for $\omega \notin$ inside a codim b_2^+ ~~hyperplane~~ in $i\Omega^2$, no reducible solutions.

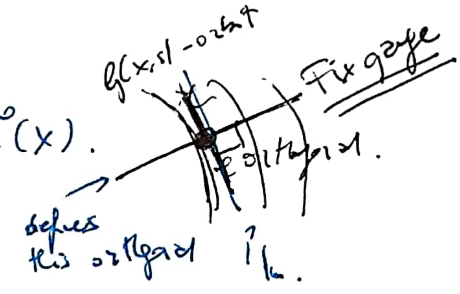
ω



$i\Omega^2$

$\boxed{\text{Fact 3}}$ For generic ω as above, $\mathcal{M}_{g,\omega}(X,s)$ is a smooth manifold of dimension $d=\frac{1}{4}(c_1^2(s^+)[X]-2\chi-3\sigma)$.

→ Map $\mathcal{F}_\omega : \mathcal{C}(X,s) \longrightarrow i\,\mathfrak{su}(S^+)\oplus\Gamma(S^-)$, $\mathcal{C}(X,s) \xrightarrow{d^*a + i\,\mathrm{Im}\langle\phi,\Phi\rangle} i\Omega^0(X)$.



$\mathcal{M}$ is locally the zero set of this map.

Linearization is $D_A \oplus (d^*+d^+)$ + lower order.

→ think of finite dimensional case $f:\mathbb{R}^n\to\mathbb{R}^m$. $f^{-1}(0)$ is generically a $n-m$ dimensional manifold.
index(df).

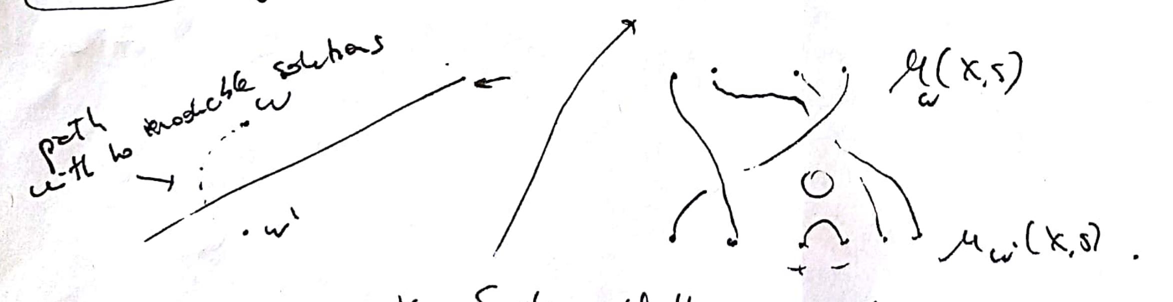
In the infinite dimensional case

→ $\frac{1}{4}(c_1^2(s^+)-\sigma(X))+1-b_1(X)+b_2^+(X)$. Exercise (check it's the formula above).

Suppose $d=0$. ⇒ $\mathcal{M}_{g,\omega}(X,s)$ is a compact 0-manifold ⇒ finite number of points (with signs).

$$SW_{g,\omega}(X,s) = \#\mathcal{M}_{g,\omega}(X,s).$$

$\boxed{\text{Fact 4}}$ If $b_2^+(X)\geq 2$, $SW_{g,\omega}(X,s)$ is independent of g,ω.



the Seiberg-Witten invariants..