Lecture 1 Introduction & 4-manifolds

X compact & oriented 4-manifold.
smooth

Key invariant Q_X intersection form. $H^2(X;\mathbb{Z})/\text{tors} \otimes \to \mathbb{Z}$, $\quad H^2(X;\mathbb{Z})/\text{tors} \cong \mathbb{Z}^{b_2(X)}$

$\alpha\otimes\alpha' \longmapsto (\alpha\cup\alpha')[X]$, $\quad (\alpha\cup\alpha') \in H^4(X;\mathbb{Z})$.

↳ properties 1) Q_X is unimodular (by Poincaré duality).

2) If $[\Sigma] \overset{PD}{=} [\alpha]$, $[\Sigma'] \overset{PD}{=} [\alpha']$.
$\Rightarrow Q_X(\alpha,\alpha') = \#\Sigma\cap\Sigma'$.

Exercise every class in H^2 is dual to a surface. Hint $H^2(X;\mathbb{Z}) = [X; K(\mathbb{Z},2)]$.

Basic examples

1) S^4 $\quad H^2 = 0$.

2) $S^2\times S^2$ $\quad H^2 = \mathbb{Z}\oplus\mathbb{Z}$, generated by $S^2\times pt$ and $S^2\times pt$.

$Q_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3) $\mathbb{CP}^2$ $\quad H^2 = \mathbb{Z} \leftarrow \mathbb{CP}^1$

$[1]$.

4) $\overline{\mathbb{CP}}^2$ $\quad H^2 = \mathbb{Z}$ $\quad [-1]$.

5) $Q_{X\# X'} = Q_X \oplus Q_{X'}$, e.g. $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
↳ blow-up. → show!

Thm (Freedman) X, X' s.c. 4-manifolds. If $Q_X \cong Q_{X'} \Rightarrow X$ and X' are homeomorphic.

Invariants of Q_X 1) rank.

2) even/odd $\quad Q_X(\alpha,\alpha)\in 2\mathbb{Z}$ for every α.

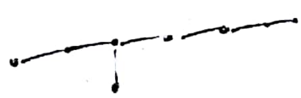
3) signature σ. (σ of $Q_X\otimes\mathbb{R}$).

Q_X is ~~positive~~ negative definite if $\sigma(X) = -b_2$. e.g.

$$-E_8 = \begin{bmatrix} +2 & 1 & & & & & & \\ 1 & +2 & 1 & & & & & \\ & 1 & +2 & 1 & & & & \\ & & 1 & +2 & 1 & & & \\ & & & 1 & +2 & 1 & & 1 \\ & & & & 1 & +2 & 1 & \\ & & & & & 1 & +2 & \\ & & & & 1 & & & -2 \end{bmatrix}$$

there is plenty of definite intersection forms!

rk $= 8$,



Thm (Donaldson) If X is smooth, $Q_X > 0$
$\Rightarrow Q_X \cong [1]^n$.

we'll prove.

Rmk There exist topological mflds with $Q_X \cong E_8$.

An interesting example

$\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subseteq \mathbb{C}P^3$

$\leadsto$ K3 surface

- simply connected.
- $Q_{K3} \cong -2E_8 \oplus H^3$ $\to$ • even • $rk = 22$ • $\sigma = 16$.

to visualize: elliptic fibration $E(2)$.

$\boxed{E(1)}$: there is $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2 \longrightarrow \mathbb{C}P^1$, generic fiber a torus.
$\leadsto$ blow up at 9 points.

Pick degree 3 polynomials p_0, p_1.

$\{p_0 = 0\}$

$\{p_1 = 0\}$

9 intersection points by Bezout.

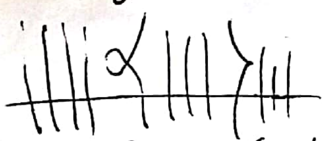
For each $p \neq p_1 \dots p_9$, there exists exactly one $[x_0 : x_1] \in \mathbb{C}P^1$ s.t. $p \in \{x_0 p_0 + x_1 p_1 = 0\}$.

$\Rightarrow$ get $\mathbb{C}P^2 \setminus \{p_1 \dots p_9\} \to \mathbb{C}P^1$.

$\Downarrow$ blow-up

$\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2 \to \mathbb{C}P^1$. generic fiber is a torus.

nodal curve



$y^2 = x^3 + x^2$ cusp $x^2 = y^3$

blow up.

$\mathbb{C}P^1$

Exercise 1) show that $\chi(\text{nbhd}(\text{nodal})) = 1$

2) Show that if there are only nodal curves $\Rightarrow$ there $= 12$ of them.

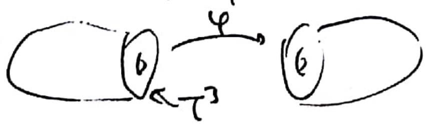
Fiber sum $X_0 \to \mathbb{C}P^1$, $X_1 \to \mathbb{C}P^1$ elliptic fibrations.

$F_i \subset X_i$ regular fiber

nbhd $\cong F_i \times D^2$
$\downarrow$
D.

F_i

$X_0 \#_F X_1 = X_0 \setminus \text{nbhd}(F_0) \cup_\varphi X_1 \setminus \text{nbhd}(F_1)$



φ T^3

Fact $E(2) = E(1) \#_F E(1)$.
$\uparrow$
K3-surface.

$\Rightarrow$ In general, can define
$E(n) = \#_F^n E(1)$.

Multiple fibers

Hopf fibration : $S^3 \longrightarrow S^2$

$\rotatebox{90}{$\subset$}$ $\{z_0, z_1 \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$.

is given by the S^1 action: $\lambda \cdot (z_0, z_1) = (\lambda z_0, \lambda^{n} z_1)$.

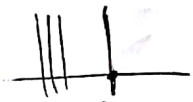
→ torus fibration $S^3 \times S^1 \to S^3 \xrightarrow{Hopf} S^2$.

Consider $S^3 \xrightarrow{f_m} S^2$ $\lambda \cdot (z_0, z_1) = (\lambda z_0, \lambda^m z_1)$

→ $(0,1)$ is a point of order m.

singular point

$Q_m : S^3 \times S^1 \longrightarrow S^2$



$m^{-1}(F)$ → homotopy class.

Exercise
$T^4 \#_F Q_m$ has odd b_1
($\Rightarrow$ no Kähler structure).

Log transform $E(n)_{p_1, \dots p_{m_k}} = E(n) \#_F Q_{p_1} \dots \#_F Q_{m_k}$.

Thm Fix $n \geq 1$, Consider $p > q \geq 2$, $(p,q) = 1$ $E(n)_{p,q}$.

1) They are s.c $\Rightarrow$ all homeomorphic.

2) Pairwise non diffeomorphic.

[n even $E(n)_{p,q}$ — p,q both odd $\Rightarrow$ ~~diff~~ homeo.
[n odd $E(n)_{p,q}$ all ~~diffeo~~ homeo

→ very subtle difference between topological / smooth classification.

We will distinguish them by means of the Seiberg-Witten invariants.

$$SW : Spin^c(X) \longrightarrow \mathbb{Z}.$$

count solution to certain non-linear PDEs.

→ need to understand some DG before!