Lecture 3 Dirac operators & Spinors.

Last time we saw $(d+d^*)^2 = \Delta$ ↳ Hodge Laplacian on forms has a square root.

$\downarrow$

a special Dirac operator.

Bochner Laplacian: $\nabla : \Omega^k(X) \to \Omega^{k+1}(X, \Omega^k)$, ∇^*

is $\nabla^*\nabla$.

Simplest case $-\frac{d}{dx_1^2} - \frac{d}{dx_2^2} + \dots - \frac{d}{dx_n^2} = \Delta$ (− sign so it is negative definite!).

$$\left(a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} + \dots + a_n \frac{d}{dx_n}\right)^2 = \Delta$$

need ↳ $\begin{cases} a_i^2 = -1 \\ a_1 a_2 + a_2 a_1 = 0 \end{cases}$ ⇝ relations defining Clifford algebra.

Example $\boxed{n=1}$ ⇝ $a_1^2 = -1$ ⇝ $\mathbb{C}$.
 $\overset{\shortparallel}{i}$

$\boxed{n=2}$ $a_1^2 = a_2^2 = -1$ $\quad$ $a_1 = i$, $a_2 = j$, ~~$a_1 a_2 = k$~~ ⇝ $\mathbb{H}$ quaternions.

set $a_1 a_2 = k$

Exercise what is $Cl(\mathbb{R}^3)$?

Def

In general (V, q) euclidean space.

$$Cl(V,q) = \frac{T(V)}{\{v \otimes v = -q(v)\}} \qquad T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

(This implies $v \otimes w + w \otimes v = -2q(v,w)$.)

Rmk 1 $Cl(V)$ is closely related to the exterior algebra!

$\frac{T(V)}{\{v \otimes v = 0\}}$ ↳ $v \otimes w + w \otimes v = 0$.

$Cl(V,q)$ has natural filtration $\mathcal{F}$ coming $k \subseteq k \oplus V \subseteq k \oplus V \oplus V \otimes V \dots$

↳ $Gr(Cl(V,q)) \cong \Lambda^* V$. as algebras

Can think of this as a new product structure on $\Lambda^* V$.

$$v \cdot (v_1 \wedge v_2 \dots \wedge v_k) = v \wedge v_1 \wedge \dots \wedge v_k - v \lrcorner (v_1 \wedge \dots \wedge v_k).$$

contraction.

<u>Rmk</u> $Cl(V,\rho)$ is only $\mathbb{Z}/2\mathbb{Z}$-graded! And multiplication changes even/odd by v.

~~Also~~ <u>Def</u> A Clifford module is $Cl(V,\rho) \curvearrowright S$

<u>Rmk</u> just need to check that the action $v \cdot (v \cdot s) = -\rho(v) \cdot s$ for every $v \in V$. (up the wider.)

<u>Let's globalize this</u>. (M,g) Riemannian manifold

$\rightsquigarrow$ $\begin{matrix} Cl(TM,g) \\ \downarrow \\ M \end{matrix}$ bundle of Clifford algebras.

<u>Def</u> A <u>Clifford bundle</u> is $\begin{matrix} S \\ \downarrow \\ M \end{matrix}$ hermitian $(\ )$ bundle of Clifford modules $Cl(T_pM,g) \curvearrowright S_p$. + connection. ∇.

such that

1) $(v \cdot s_1, s_2) + (s, v \cdot s_2) = 0$ (the action of v is skew-adjoint).

2) $\nabla^S_X (Y \cdot s) = (\nabla_X Y) \cdot s + Y \nabla^S_X s$ for all X, Y, s.
Levi-Civita

<u>Example</u> $Cl(TM,g) \cong \Lambda^* TM$ is a Clifford bundle.

<u>Def</u> The <u>Dirac operator</u> associated to $(S, (\ ), \nabla)$ is

$$D : \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{\#} \Gamma(TM \otimes S) \xrightarrow{cl} \Gamma(S).$$

<u>Example</u> The Dirac operator associated to $Cl(TM,g) \cong \Lambda^* TM$.

is $d + d^* : \Omega^*(X) \longrightarrow \Omega^*(X)$. (notice it splits in $\Omega^{odd} \to \Omega^{even}$, $\Omega^{even} \to \Omega^{odd}$).

<u>Exercise</u> check that D^2 looks like a laplacian.

But we can pick different Clifford bundles!

$\boxed{n=2}$ $Cl(\mathbb{R}^2) = \mathbb{H} \curvearrowright \mathbb{H} = S$.
$\qquad \qquad \mathbb{C} \oplus j\mathbb{C}$.

$$\left.\begin{array}{l} \underset{e_1e_2}{e_3} \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \sigma_1 \\ e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma_2 \\ \underset{e_1e_2}{e_3} \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \sigma_3 \end{array}\right\} \text{Pauli matrices.}$$

$\rightsquigarrow$ Dirac is $e_1 \cdot \nabla_{e_1} + e_2 \nabla_{e_2} = \begin{bmatrix} 0 & -\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \end{bmatrix} = 2\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ +\frac{\partial}{\partial \bar z} & 0 \end{bmatrix}$

$\boxed{n=4}$ Our case of interest.

$$\left.\begin{array}{l} \operatorname{rk}_{\mathbb{C}} S = 4. \text{ Locally } e_0 \mapsto \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \qquad e_i \mapsto \begin{bmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{bmatrix} \\ \quad \overset{\shortparallel}{S^+ \oplus S^-} \qquad D : S \to S \text{ splits as } \begin{array}{l} D^+ : S^+ \to S^- \\ D^- : S^- \to S^+ . \end{array} \end{array}\right\} \text{Spin}^c \text{ structure } S$$

Thm S Clifford bundle. D is a first order, elliptic, self adjoint
↳ Nice from the point of view of analysis.
e.g. $\cdot\, Ds = 0 \Rightarrow s \in C^\infty$.
$\cdot$ dim ker, dim coker is finite.

D^+ $\Gamma(S^+) \underset{D^-}{\overset{D^+}{\rightleftarrows}} \Gamma(S^-)$ are adjoint to each other.

$\operatorname{ind}(D^+) = \dim\ker(D^+) - \dim\operatorname{coker}(D^+)$.

Exercise $\cdot \operatorname{ind}(d + d^*) = \chi(X)$.
$\cdot$ is there an natural operator for which $\operatorname{ind} = \sigma(X)$?

Thm $\operatorname{ind} D_A^+ = \frac{1}{8}(c_1^2(S^+) - \sigma(X))$ (Atiyah-Singer index thm).

Fact If (S,ρ) is spin^c structure $\Rightarrow$ $(S \otimes L, \rho \otimes \mathbb{1}_L)$ is also spin^c structure.

$\Rightarrow \operatorname{Spin}^c(X)$ is affine space over $H^2(X;\mathbb{Z}) = \{\text{line bundles}\}$.