Lecture 5 Explicit examples.

1) Positive scalar curvature.

Key formula $D_A^- D_A^+ \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{2}\rho_X(F_{A^t}^+)\Phi + \frac{1}{4} s \Phi$ (+)

scalar curvature.

In general, $\begin{matrix} S \\ \downarrow \\ X \end{matrix}$ Clifford bundle. Then $D^2 s = \nabla^* \nabla s + (K) s$
$Cl(TX) \curvearrowright S \Rightarrow \nabla, D$.

curvature term
$\Rightarrow$ 0th order operator.

Proof Fix frame $\nabla_{e_i} e_k = 0$.

$Ds = \sum_i e_i \nabla_{e_i} s$.

at p ∇ compatible

$\Rightarrow D^2 s = \sum_{i,j} e_j \nabla_{e_j}(e_i \nabla_{e_i} s) = \sum_{i,j} e_j e_i \nabla_j \nabla_i s =$

$e_i^2 = -1$
$e_i e_j = -e_j e_i$

$= -\sum_i \nabla_i^2 s + \underbrace{\sum_{i<j} e_j e_i (\nabla_j \nabla_i - \nabla_i \nabla_j) s}_{K}$.

$\nabla_A^* \nabla_A$ Exercise Hint: show $\nabla_A^* = -*\nabla_A *$.

Thm If X $b_2^+ \geq 2$ admits a metric of positive scalar curvature $\Rightarrow SW_X \equiv 0$.

Proof Let's look at the unperturbed equations. Integrate $\int \langle (*), \Phi \rangle$.

$D_A^+ \Phi = 0 \Rightarrow 0 = \int \langle \nabla_A^* \nabla_A \Phi, \Phi \rangle_{L^2} + \langle \frac{1}{2} \underbrace{\rho_X(F_{A^t})}_{(\Phi\Phi^*)_0} \Phi, \Phi \rangle_{L^2} + \frac{1}{4} \langle s\Phi, \Phi \rangle_{L^2}$.

$= \|\nabla_A \Phi\|_{L^2}^2 + \underbrace{\frac{1}{4}\|\Phi\|_{L^4}^4}_{\text{exercise}} + \frac{1}{4}\int s|\Phi|^2 \geq 0$. If $s > 0$.

$\Rightarrow \Phi \equiv 0$.

$\Rightarrow$ no irreducible solutions.

Check The same estimate still holds if we perturb with ω small (compared to S).

2) Kähler manifolds

"Kobayashi–Hitchin correspondence": on a Kähler manifold →

$\{$gauge theoretic$\}$ $\longleftrightarrow$ $\{$complex geometry$\}$

(X, J) complex manifold + symplectic form ω s.t. $\omega(\cdot, J\cdot)$ is Riemannian metric

Thm If X is Kähler, $b_2^+ \geq 2$, $SW_X(K_X) = 1$. (canonical)

If K is minimal general type $\Rightarrow SW(\mathfrak{s}) = 0$ for $\mathfrak{s} \neq \pm K_X$.

E.g. any projective surface $\subset \mathbb{CP}^N$, i.e. K3.

In general X, $J : TX \circlearrowleft$ $\begin{cases} J^2 = -\mathrm{Id} \\ J \text{ orthogonal} \end{cases}$ $\Rightarrow$ $spin^c$ structure on X.

Rmk From the principal bundle viewpoint, this is because there exists natural embedding $U(n) \to Spin^c(2n)$.

key ingredients ① spinor bundle.

We can write this very explicitly,

Recall $\Omega^0 \otimes \mathbb{C}$, $\Omega^1 \otimes \mathbb{C} = \Omega^{1,0} \oplus \Omega^{0,1}$

locally $dz_1 = dx$, dz_2 ; locally $d\bar{z}_1$, $d\bar{z}_2$.

$\Omega^2 \otimes \mathbb{C} = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$ ($\dim_{\mathbb{C}} = 1$)

$dz_1 \wedge dz_2$, $dz_1 \wedge d\bar{z}_1$, …, $d\bar{z}_1 \wedge d\bar{z}_2$

$S^+ = \Omega^0 \oplus \Omega^{0,2}$

$\rho \circlearrowleft S^- = \Omega^{0,1}$

$\rho(v) \cdot (\alpha_1 \wedge \dots \wedge \alpha_t) = \sqrt{2}\,(v^{0,1} \wedge \alpha_1 \dots \wedge \alpha_t + v^{0,1} \lrcorner\, \alpha_1 \dots \alpha_t)$.

$T \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$

Dirac operator is just $D^+ : S^+ \to S^-$

$\sqrt{2}(\bar{\partial} + \bar{\partial}^*) : \Omega^0 \oplus \Omega^{0,2} \to \Omega^{0,1}$.

③ Self-duality also interacts well with the Kähler structure.

$$\Omega^2 \otimes \mathbb{C} = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}.$$

$$\Omega^2 \otimes \mathbb{C} = \Omega^+ \oplus \Omega^-.$$

$\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$.

<u>emma</u> $\begin{cases} \Omega^+ = \Omega^{2,0} \oplus \Omega^0 \omega \oplus \Omega^{0,2}. \\ \Omega^- = \Omega^{1,1}_0 \end{cases}$ is pointwise orthogonal to ω. <u>Exercise!</u>

$\Omega^+(i\mathbb{R})$ are of the form $if\omega + \mu - \bar{\mu}$ $\quad f \in C^\infty(X;\mathbb{R})$, $\mu \in \Omega^{0,2}$.

<u>Fact</u> $F_{A^t}^{0,2} = 0 \Rightarrow A^t$ determines a holomorphic structure on $\det(S^+)$.
integrability theorem.

3) Symplectic manifolds

<u>Thm</u> X symplectic $b_2^+ \geq 2 \Rightarrow SW_X(K_X) = \pm 1$.
(+ constraints of s with $SW_X(s) \neq 0$).

<u>Idea</u> ω is locally $dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \Rightarrow$ can find metric g such that ω is self-dual.

$$\begin{cases} \frac{1}{2}\rho(F_{A^t}^+ - 4\omega_t^+) = \Phi\Phi^* \\ D_A^+ \Phi. \end{cases}$$

use Large perturbation $4\omega_t^+ = F_{A_0^t} + it\omega$
some carefully chosen A_0^t.

For t large enough, ~~can~~ find there exists only one solution.