Lecture 2 Forms & connections

Dimension 4 is special in differential geometry:

1) 2-forms on 4 manifolds have special properties
2) 2-forms naturally arise as curvature of connections.

· X any smooth manifold

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \dots \to \Omega^n(X) \to 0 \quad \text{deRham complex.} \quad (d^2=0).$$

→ cohomology is $H^i(X;\mathbb{R})$.

Hodge theory $(X,g) \Rightarrow H^i(X;\mathbb{R})$ comes with natural representatives.

$(V, \langle\ \rangle, \text{orientation})$: $\quad * : \Lambda^k V \to \Lambda^{n-k} V$. (more intrinsically, sends a volume form of a plane to
↳ dim n

$e_1\wedge\dots\wedge e_k \to e_{k+1}\wedge\dots\wedge e_n$
ON vectors, ON vectors s.t. $e_1\wedge\dots e_k \dots e_{k+1}\dots\wedge e_n$ is oriented basis.

example $V=\mathbb{R}^3$. $\quad *e_1 = e_2\wedge e_3$
$*e_2 = e_3\wedge e_1$

Exercise $*^2 = (-1)^{(n-1)k}$

$(X, g, \text{oriented})$: $\quad * : \Omega^k(X) \to \Omega^{n-k}(X)$.

↳ $\langle \alpha, \beta\rangle = \alpha\wedge *\beta$. (in particular: $\|\alpha\|_{L^2(X)} = \int \alpha\wedge *\alpha$.)
inner product

$d : \Omega^{k-1}(X) \to \Omega^k(X)$

$d^* = (-1)^{n(k+1)+1} * d *$.
↳ is the formal L^2-adjoint $\langle \alpha, d\beta\rangle_{L^2(X)} = \langle d^*\alpha, \beta\rangle_{L^2(X)}$.
$\alpha\in\Omega^k$, $\beta\in\Omega^{k-1}$

Exercise Hint $0 = \int d(\beta\wedge *\alpha) = \dots$

<u>Hodge theorem</u> There is an L^2 orthogonal decomposition

$$\underset{\substack{\text{~~exact~~} \\ \text{exact}}}{\Omega^k = d\Omega^{k-1}} \oplus \underset{\substack{\shortparallel \\ \ker(d+d^*)}}{\mathcal{H}_k} \oplus \underset{\substack{\text{coexact} \\ \text{~~coclosed~~}}}{d^*\Omega^{k+1}}$$
→ needs some analytical work

↳ harmonic. $\mathcal{H}_k \cong H^k(X;\mathbb{R})$.

$\boxed{\text{Rmk}}$
$(d+d^*)^2 = dd^* + d^*d = \Delta$ is the <u>Hodge laplacian</u>.

↳ Δ is the square of an operator! (a Dirac)

<u>On a 4-manifold</u>: $*^2 : \Omega^2 \circlearrowleft$ is the identity $\Rightarrow \Omega^2 = \Omega^+ \oplus \Omega^-$

self-dual, anti-self-dual

$(+1)$ eigenspace, (-1)

$$\Omega^+ = \operatorname{Span}\left\langle \begin{matrix} dx_1\wedge dx_2 + dx_3\wedge dx_4 \\ dx_1\wedge dx_3 + dx_4\wedge dx_2 \\ dx_1\wedge dx_4 + dx_2\wedge dx_3 \end{matrix} \right\rangle.$$

↑

$SO(4) \cong SO(3) \oplus SO(3)$.

↗

the only reducible $SO(n)$.
($\Leftarrow A_4$ is the only non-simple A_n)

$\Rightarrow \mathcal{H}_2 = \mathcal{H}^+ \oplus \mathcal{H}^-$.

α, β ↘ $\langle\,,\,\rangle_{L^2}$ $\int \alpha\wedge\beta \Leftrightarrow Q_X \otimes \mathbb{R}$ on $H^2(X;\mathbb{Z})\otimes\mathbb{R}$.

$\alpha = *\alpha \Rightarrow \int \alpha\wedge\alpha = \int \alpha\wedge *\alpha = \|\alpha\|^2_{L^2(X)} \geq 0$

$\alpha = -*\alpha \Rightarrow \int \alpha\wedge\alpha = -\int \alpha\wedge *\alpha = -\|\alpha\|^2_{L^2}(X) < 0$.

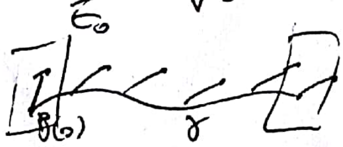
$d^+ = \pi_{\Omega^+}\circ d$.

<u>Exercise</u> $0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega^+(X) \to 0$
has homology $\mathbb{R}$, $H^1(X)$, $H^+(X)$. <u>Hint</u> (what's the adjoint of d^+?).

$\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ vector bundle. A <u>connection</u> is

$\nabla : \Omega^0(E) \to \Omega^1(E)$.
- $\nabla_{fX} s = f\nabla_X s$
- $\nabla_X fs = df \otimes s + f\nabla_X s$.

$\nabla s = 0$

<u>geometrically</u>:  parallel transport.

If ∇ is a connection, any other is $\nabla + a$, $a \in \Omega^1(gl(E))$. (affine space).

Curvature of a connection A is $F_A \in \Omega^2(gl(E))$ <u>local invariant</u>

$$F_A(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[Y,X]} s$$

$\rightarrow$ vector fields — to make things covariant.

measures how parallel transport is commutative

$[X,Y]=0$

<u>Rmk</u> If E is unitary, we can ask ∇ to preserve metric.
$\Rightarrow F_A \in \Omega^2(u(E))$.

<u>Chern-Weil theory</u> The curvature can recover global invariants of E.

E complex $\Rightarrow$ $c_i(E) \in H^{2i}(X;\mathbb{Z})$.

$F_A \in \Omega^2(gl(E))$. If we pick basis of local sections of E (well defined up to conjugacy).
$\Rightarrow F_A$ is an $n \times n$ matrix of 2-forms

$P_k : gl_n(\mathbb{C}) \to \mathbb{C}$. invariant polynomial of degree k. $(P(YXY^{-1}) = P(X))$

$\Rightarrow P_k(F_A)$ is a well defined form (~~of~~ possibly $\in \Omega^{2k}(X;\mathbb{C})$)

<u>Thm</u> $P_k(F_A)$ is a closed form; $[P_k(F_A)]$ is independent of A
$\Rightarrow$ invariant of E.

If $P_k(X) = \left(\frac{i}{2\pi}\right)^k \operatorname{tr}(\Lambda^k X)$ $\Rightarrow$ $c_k(E)$.

(we will only use: $\frac{i}{2\pi} F_A$ is a closed 2-form representing $c_1(L)$).

<u>Exercise</u> show that $c_1(\text{tautological}_{\mathbb{CP}^1}) = -1 \in H^2(\mathbb{CP}^1;\mathbb{Z})$