with $b_2^+ \geq 2$

Lecture 6 Gluing and Floer homology

$SW(X,s) \in \mathbb{Z}$.

$\rightsquigarrow m(X,h)$ function on $H_2(X;\mathbb{R})$.

$$m(X,h) = \sum_{s \in Spin^c(X)} SW(X,s)\, e^{\langle c_1(s), h\rangle} \in \mathbb{R}.$$

(this determines $SW(X,s)$ if $H^2(X)$ is torsion-free).

Example $m(K3,h) \equiv 1$. (only the canonical class has non-trivial SW).

$E(2)$.

$\boxed{n \geq 2}$

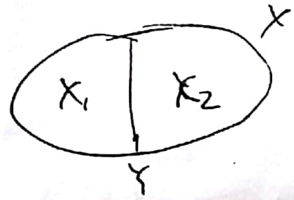
Thm $$m(E(n)_{p,q}, h) = 2^{n-2} \frac{\sinh(F\cdot h)^{n}}{\sinh\left(\frac{F\cdot h}{p}\right)\cdot \sinh\left(\frac{F\cdot h}{q}\right)}$$

F is the class of the fiber.

(~~many of these are~~ not

these are obtained by cutting / gluing along tori. T^3

In general



can we assign a vector space with $\langle\ \rangle$ to Y, $\rightsquigarrow (HM(Y), \langle\ \rangle)$

$X_1 |_Y \rightsquigarrow \Psi_{X_1} \in HM$

$X_2 |_Y \rightsquigarrow \Psi_{X_2} \in HM$

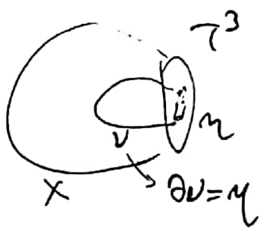
such that $m(X_1 \cup X_2, h) = \langle \Psi_{X_1}, \Psi_{X_2}\rangle$?

Yes $HM(Y)$ is called reduced monopole Floer homology.

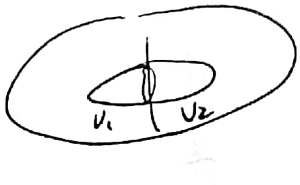
(do not talk about local coefficients..)

In our case: $HM(T^3, \Gamma_\eta) \cong \mathbb{R}$ (Floer homology with twisted local coefficients).

$\hookrightarrow$ of $[\eta] \in H_1(T^3)$.



$\Rightarrow \Psi_{X,\nu} \in HM$ with the properties above



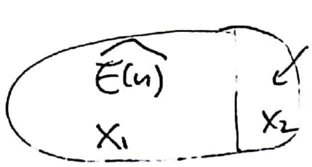
$m(X_1 \cup X_2, \nu_1 \cup \nu_2) =$

$= \langle \Psi_{X_1,\nu_1}, \Psi_{X_2,\nu_2} \rangle$.

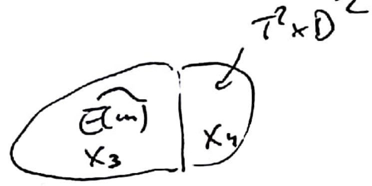
product

For example, to compute $E(n+m)$ from $E(n)$ and $E(m)$.

$E(n)$

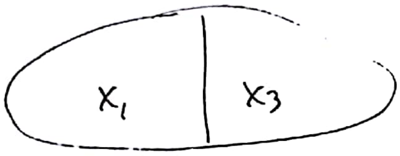


$E(m)$

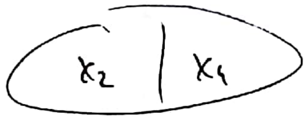


$m(E(n), \nu_1\cup\nu_2) = \psi_{X_1,\nu_1}\cdot\psi_{X_2,\nu_2}$

$m(E(m), \nu_3, \nu_4) = \psi_{X_3,\nu_3}\cdot\psi_{X_4,\nu_4}$.



$E(n+m)$



$T^2\times S^2$

attention!

$\Downarrow$ $b_2^+=1$.

$m(S^2\times T^2, \nu_2\cup\nu_4) = \psi_{X_2,\nu_2}\cdot\psi_{X_4,\nu_4}$.

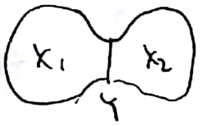
$m(E(n+m), \nu_1\cup\nu_3) = \psi_{X_1,\nu_1}\cdot\psi_{X_3,\nu_3}$

$$\Rightarrow m(E(n+m),\nu_1\cup\nu_3)\cdot \cancel{m(S^2\times T^2} = \frac{m(E(n),\nu_1\cup\nu_2)\cdot m(E(m),\nu_3,\nu_4)}{m(S^2\times T^2, \nu_2\cup\nu_4)}.$$

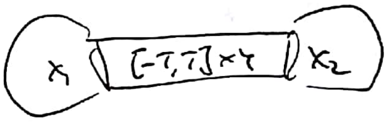
Similarly: $HM(S^3)=0$ $\Rightarrow$ If $X = X_1\# X_2$ with $b_2^+(X_i)\geq 1 \Rightarrow SW_X\equiv 0$.

consequence of PSC.

Idea



$\rightsquigarrow$ study the equations on



Equations on $\mathbb{R}\times Y$

$$\begin{cases} \rho(\frac{\partial}{\partial t}): S^+\cong S^- \Rightarrow \begin{matrix} S \\ \downarrow \\ Y\end{matrix} \text{ spinor bundle.} \\ \nabla_A = \frac{d}{dt} + \nabla_{B(t)} \quad B(t) \text{ a family of spin}^c \text{ connections on } \begin{matrix} S \\ \downarrow \\ Y\end{matrix} \\ \Phi = \Phi(t) \text{ section of } \begin{matrix} S \\ \downarrow \\ Y\end{matrix}. \end{cases}$$

$$\boxed{\begin{matrix}\rho: TY \to su(S) \\ e_i \mapsto \sigma_i.\end{matrix}}$$ 3-dim spin^c structure.

Fact 1 $D_A\Phi = \frac{d}{dt}\Phi + D_B\Phi$.

$F_{A^t} = dt\wedge\left(\frac{d}{dt}B^t\right) + F_{B^t} \Rightarrow *_4 F_{A^t} = *\left(\frac{d}{dt}B^t\right) + dt\wedge * F_{B^t}$

$$\Rightarrow F_{A^t}^+ = \frac{1}{2}(F_{A^t} + *_4 F_{A^t}) = \frac{1}{2}\left(*\left(\frac{d}{dt}B^t\right) + F_{B^t} + dt\wedge\left(\frac{d}{dt}B^t + *F_{B^t}\right)\right).$$

$$\rho_{\mathbb{R}\times Y}(F_{A^t}^+) = -\rho_Y\left(\frac{d}{dt}B^t + *F_{B^t}\right).$$

For $D_A = \frac{d}{dt} + D_B$, $\Phi = \Psi(t)$

$$\begin{cases} \frac{1}{2}\rho(F_{A^t}^+) = (\Phi\Phi^*)_0 \\ D_A^+ \Phi = 0 \end{cases} \iff \begin{cases} \frac{d}{dt} B^t = -*F_{B^t} - 2\rho^{-1}(\Psi\Psi^*)_0 \\ \frac{d}{dt}\Psi = -D_B\Psi. \end{cases} \qquad (*)$$

Key fact $(*)$ are $\frac{d}{dt}(B,\Psi) = -\operatorname{grad}\mathcal{L}(B,\Psi)$ where (for fixed B_0)

$$\mathcal{L}(B,\Psi) = -\frac{1}{8}\int (B^t - B_0^t)\wedge(F_{B^t} + F_{B_0^t}) + \frac{1}{2}\int \langle D_B\Psi, \Psi\rangle \, dvol.$$

Chern-Simons   dirac.

Rmk D_B in dimension 3 behaves like dirac in dim 1

$-i\frac{d}{d\theta} : C^\infty(S^1, \mathbb{C})$ ⟲.

$\Rightarrow$ complete orthonormal basis of eigenfunction. $\{e^{in\theta}\}_{n\in\mathbb{Z}}$

· eigenvalues discrete, infinite in both directions

~~Proof~~ ~~Exercise~~ ~~write D~~

$$\cancel{-\frac{1}{8}\left(-2b\wedge(F_{B^t} + F_{B_0^t}) + (B^t\wedge B_0^t)\wedge 2db\right) + \frac{1}{2}\int \langle c\rho(b)\Psi, \Psi\rangle + \int Re\langle\Psi, D_B\Psi\rangle}$$

$$\mathcal{C}(Y,s) = \left\{(B,\Psi) \;\middle|\; \begin{array}{l} B \text{ spin}^c \text{ connection} \\ \Psi \text{ spinor} \end{array}\right\} \xrightarrow{\ \mathcal{L}\ } \mathbb{R}.$$

⟲ $\mathcal{G}(Y,s) = \{u : Y \to S^1\}$ $u(B,\Psi) = (B - u^{-1}du, u\cdot\Psi)$.

$\left(\mathcal{L}(u(B,\Psi)) - \mathcal{L}(B,\Psi) = 2\pi^2([u]\cup c_1(s))[Y]\right)$

Again, we have reducible/irreducible dychotomy.

$\mathcal{G}_0 = \{u : Y \to S^1 \text{ s.t. } u(y_0) = 1\}$. $\mathcal{C}(Y,s)/\mathcal{G}(Y,s) \xrightarrow{\ \mathcal{L}\ } \mathbb{R}/(2\pi^2\mathbb{Z})$.

"smooth manifold"   ⟲ S^1-action

$\Rightarrow$ use Morse theory!