Sheaves on stacks

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Preliminaries, reminders, and notation

- $\mathbf{S}$ is a **site**, i.e. a category equipped with a Grothendieck topology.
- E.g. $S =$ schemes over a fixed scheme B.
- A category $\mathbf{X}$ fibered over $\mathbf{S}$ is a **prestack**.
- For $U \in S$, write $\mathbf{X}(U)$ for the fiber.
- If $\mathbf{X}(U)$ is a **groupoid** for all $U \in \mathbf{S}$ we say this is fibered in groupoids.
- In this case we have a functor:

$$\begin{array}{c} \mathbf{S}^{\mathsf{op}} \longrightarrow \mathbf{Grpd} \\ U \mapsto X(U) \subset \mathbf{X}\,. \end{array} \quad . \tag{1}$$

- In order for a prestack to be a **stack** we require that, for all $U \in S$ with a covering $\{U_i \to U\}_{i \in I}$, the functor

$$\mathbf{X}(U) \to \mathbf{X}\left(\{U_i \to U\}_{i \in I}\right)$$

is an equivalence of categories.

Quasi-coherent sheaves on a prestack

Let $\mathbf{X} \xrightarrow{p} \mathbf{Sch}/B$ be a category fibered in groupoids over schemes over B.

Definition

A **quasi-coherent sheaf** $\mathcal{F}$ on $\mathbf{X}$ is

1. a quasi-coherent sheaf $\mathcal{F}_x$ on $p(x)$ for all $x \in \mathbf{X}$,
2. and for all $f\colon x \to y$ in $\mathbf{X}$, an isomorphism

$$\alpha_f \colon \mathcal{F}_y \xrightarrow{\simeq} (pf)^* \mathcal{F}_x \ .$$

which satisfy the cocycle condition: for any two morphism $f\colon x \to y$ and $g\colon y \to z$, the following diagram commutes:

$$\begin{tikzcd} pf^* \circ pg^* \mathcal{F}_z \arrow[r, equal] \arrow[d, "pf^*\alpha_g"] & (pg \circ pf)^* \mathcal{F}_z \arrow[d, "\alpha_{g\circ f}"] \\ pf^* \mathcal{F}_y \arrow[r, "\alpha_f"] & \mathcal{F}_x \ . \end{tikzcd}$$

Artin stacks

Definition (Artin stack)

An **Artin stack** is a stack in groupoids $\mathbf{X}$ over the fppf site[a] such that

1. the diagonal map of $\mathbf{X}$ is **representable**, and
2. there exists a smooth surjection from (the stack associated to) a scheme to $\mathbf{X}$.

[a] The underlying category is the category $\mathbf{Aff}/S$ for a fixed scheme S.

- Let $\mathbf{A}, \mathbf{B} \to \mathbf{S}$ be categories fibered in groupoids. A morphism $f \colon \mathbf{A} \to \mathbf{B}$ is **representable** if for every scheme U and morphism of stacks $\mathbf{U} \to \mathbf{B}$, the fiber product $\mathbf{A} \times_{\mathbf{B}} \mathbf{U}$ is representable.[1]
- The diagonal morphism $\Delta \colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ is representable if and only if for any pair of morphisms of algebraic spaces $\mathbf{A}, \mathbf{B} \to \mathbf{X}$, the fiber product $\mathbf{A} \times_{\mathbf{X}} \mathbf{B}$ is representable.

[1] I.e. isomorphic to (the stack associated to) some algebraic space.

Hierarchy

$$\begin{array}{c}
\{\text{affine } B\text{-schemes}\} \\
\downarrow\hspace{-0.3em}\text{\tiny$\cap$} \\
\{\text{separated } B\text{-schemes}\} \\
\downarrow \\
\{B\text{-schemes}\} \\
\downarrow \\
\{B\text{-algebraic spaces}\} \\
\downarrow \\
\{\text{algebraic stacks over } S\} \\
\downarrow \\
\{\text{stacks over } S\} \\
\downarrow \\
\{\text{categories fibered in groupoids}\} \ .
\end{array}$$

Quotient stacks

- Let G be an affine smooth group scheme over a scheme B.
- Let X be an S-scheme with an action of G.
- Define $[X/G]$ to be the following category fibered over $\mathbf{Sch}/B$.
  - An object over U is a principal G-bundle $E \to U$ together with an equivariant map $E \to X$.
  - A morphism from $E \to U$ to $E' \to U$ is a bundle map that is compatible with the corresponding equivariant maps.
- This forms an **Artin stack**.
- If the stabilizers of the geometric points are finite and reduced, then this is a Deligne-Mumford stack.

Quotient stack vs. quotient space

Proposition (Stacks Project, Lemma 64.34.1)

*Let X be an algebraic space over a scheme. If the stabilizers are **trivial** (i.e. the action is **free**) the quotient $\overline{X/G}$ exists as an **algebraic space**.*

- There is a (destructive) functor $\Gamma \colon \mathbf{QC}([X/G]) \to \mathbf{QC}\left(\overline{X/G}\right)$.
- The quotient space is generally more **coarse** than the quotient stack.
- We can detect this at the level of $\mathbf{QC}$:

$$\mathbf{QC}([X/G]) = \{M \in \mathcal{O}_X\text{-}\mathbf{Mod} \mid M \text{ is } G \text{ equivariant}\}$$
$$\mathbf{QC}(X/G) = \mathcal{O}_X^G\text{-}\mathbf{Mod} \ .$$

- E.g. if $X = \bullet$ then $\overline{\bullet/G} = \bullet$ and

$$\mathbf{QC}([\bullet/G]) = \mathbf{Rep}(G)$$
$$\mathbf{QC}(\bullet) = \mathbf{Vect} \ .$$