

Sheaves on stacks

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Preliminaries, reminders, and notation

- \mathbf{S} is a **site**, i.e. a category equipped with a Grothendieck topology.
- E.g. $S =$ schemes over a fixed scheme B .
- A category \mathbf{X} fibered over \mathbf{S} is a **prestack**.
- For $U \in S$, write $\mathbf{X}(U)$ for the fiber.
- If $\mathbf{X}(U)$ is a **groupoid** for all $U \in \mathbf{S}$ we say this is fibered in groupoids.
- In this case we have a functor:

$$\begin{aligned} \mathbf{S}^{\text{op}} &\longrightarrow \mathbf{Grpd} \\ U &\mapsto \mathbf{X}(U) \subset \mathbf{X} . \end{aligned} \tag{1}$$

- In order for a prestack to be a **stack** we require that, for all $U \in S$ with a covering $\{U_i \rightarrow U\}_{i \in I}$, the functor

$$\mathbf{X}(U) \rightarrow \mathbf{X}(\{U_i \rightarrow U\}_{i \in I})$$

is an equivalence of categories.

Quasi-coherent sheaves on a prestack

Let $\mathbf{X} \xrightarrow{p} \mathbf{Sch}/B$ be a category fibered in groupoids over schemes over B .

Definition

A **quasi-coherent sheaf** \mathcal{F} on \mathbf{X} is

- ① a quasi-coherent sheaf \mathcal{F}_x on $p(x)$ for all $x \in \mathbf{X}$,
- ② and for all $f: x \rightarrow y$ in \mathbf{X} , an isomorphism

$$\alpha_f: \mathcal{F}_y \xrightarrow{\cong} (pf)^* \mathcal{F}_x .$$

which satisfy the cocycle condition: for any two morphism $f: x \rightarrow y$ and $g: y \rightarrow z$, the following diagram commutes:

$$\begin{array}{ccc} pf^* \circ pg^* \mathcal{F}_z & \xlongequal{\quad} & (pg \circ pf)^* \mathcal{F}_z \\ \downarrow pf^* \alpha_g & & \downarrow \alpha_{g \circ f} \\ pf^* \mathcal{F}_y & \xrightarrow{\alpha_f} & \mathcal{F}_x . \end{array}$$

Definition (Artin stack)

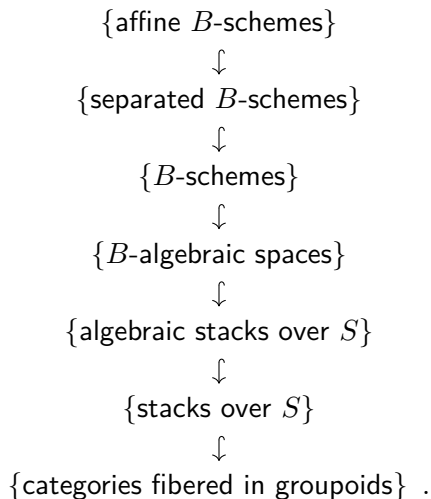
An **Artin stack** is a stack in groupoids \mathbf{X} over the fppf site^a such that

- 1 the diagonal map of \mathbf{X} is **representable**, and
- 2 there exists a smooth surjection from (the stack associated to) a scheme to \mathbf{X} .

^aThe underlying category is the category \mathbf{Aff}/S for a fixed scheme S .

- Let $\mathbf{A}, \mathbf{B} \rightarrow \mathbf{S}$ be categories fibered in groupoids. A morphism $f: \mathbf{A} \rightarrow \mathbf{B}$ is **representable** if for every scheme U and morphism of stacks $U \rightarrow \mathbf{B}$, the fiber product $\mathbf{A} \times_{\mathbf{B}} U$ is representable.¹
- The diagonal morphism $\Delta: \mathbf{X} \rightarrow \mathbf{X} \times \mathbf{X}$ is representable if and only if for any pair of morphisms of algebraic spaces $\mathbf{A}, \mathbf{B} \rightarrow \mathbf{X}$, the fiber product $\mathbf{A} \times_{\mathbf{X}} \mathbf{B}$ is representable.

¹I.e. isomorphic to (the stack associated to) some algebraic space.



- Let G be an affine smooth group scheme over a scheme B .
- Let X be an S -scheme with an action of G .
- Define $[X/G]$ to be the following category fibered over \mathbf{Sch}/B .
 - An object over U is a principal G -bundle $E \rightarrow U$ together with an equivariant map $E \rightarrow X$.
 - A morphism from $E \rightarrow U$ to $E' \rightarrow U$ is a bundle map that is compatible with the corresponding equivariant maps.
- This forms an **Artin stack**.
- If the stabilizers of the geometric points are finite and reduced, then this is a Deligne-Mumford stack.

Quotient stack vs. quotient space

Proposition (Stacks Project, Lemma 64.34.1)

Let X be an algebraic space over a scheme. If the stabilizers are **trivial** (i.e. the action is **free**) the quotient $\overline{X/G}$ exists as an **algebraic space**.

- There is a (destructive) functor $\Gamma: \mathbf{QC}([X/G]) \rightarrow \mathbf{QC}(\overline{X/G})$.
- The quotient space is generally more **coarse** than the quotient stack.
- We can detect this at the level of \mathbf{QC} :

$$\begin{aligned}\mathbf{QC}([X/G]) &= \{M \in \mathcal{O}_X\text{-Mod} \mid M \text{ is } G \text{ equivariant}\} \\ \mathbf{QC}(X/G) &= \mathcal{O}_X^G\text{-Mod} .\end{aligned}$$

- E.g. if $X = \bullet$ then $\bullet/G = \bullet$ and

$$\begin{aligned}\mathbf{QC}([\bullet/G]) &= \mathbf{Rep}(G) \\ \mathbf{QC}(\bullet) &= \mathbf{Vect} .\end{aligned}$$