Spherical varieties

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But first. . . ☺

Our building will officially be renamed "PMA"! ☺

See here for the full announcement.

Table of contents

Notation

- k is a field.
- G is an algebraic group over k (an algebraic variety which is also a group).
- X is an algebraic variety over k equipped with an action of G.

We are entering the subsection:

What are they?

We are entering the subsection:

Why define such a thing?

Where do they show up?

We are entering the subsection:

Map of the zoo

$$
\begin{array}{ccccc}
\text{abelian} & \cap & \text{simple} & & = \emptyset \\
\Downarrow & & \Downarrow & & \\
\text{nilpotent} & \cap & \text{semisimple} & & = \emptyset \\
\Downarrow & & \Downarrow & & \\
\text{solvable} & \cap & \text{reductive} & & = \text{abelian}
\end{array}
$$

Definition (solvable)

An algebraic group G is *solvable* if and only if it admits a subnormal series

$$G = G_0 \supset G_1 \supset \cdots \supset G_k = \{1\} \tag{1}$$

such that each G_i / G_{i+1} is abelian. In other words it is built out of abelian groups by extensions.

Simple, semisimple, and reductive

The *radical of G*, written $R(G)$, is the maximal normal subgroup which is connected, and solvable.

(Such a subgroup exists because extensions and quotients of solvable algebraic groups are solvable.)

Definition 1 (simple)

G is *simple* if and only if it does not contain any (proper, nontrivial, and connected) normal subgroups.

Definition 2 (semisimple)

G is *semisimple* if and only if $R(G) = \{1\}$.

Definition 3 (reductive)

G is *reductive* if and only if $R(G) \cong (\mathbb{C}^\times)^n$ for some n.

Tl;dr:

$$\textbf{Simple} = \text{no } \underline{\text{normal}} \text{ subgroups;} \tag{2}$$

$$\textbf{semisimple} = \text{no } \underline{\text{solvable}}\ \underline{\text{normal}} \text{ subgroups;} \tag{3}$$

$$\textbf{reductive} = \text{solvable normal subgroups are } \underline{\text{abelian}}. \tag{4}$$

First observations

- Simple $\implies$ semisimple $\implies$ reductive.
- $G/R(G)$ is always semisimple.
- If G is abelian then G is reductive.
- If G is solvable and nonabelian then G is not reductive.
- If G is unipotent then G is not reductive.

Unipotent characterization

Recall an operator T is called *unipotent* if and only if there is some $N \in \mathbb{Z}_+$ such that

$$(T-1)^N = 0 \ . \tag{5}$$

An algebraic group is called *unipotent* if it acts by unipotent operators in any rational representation.

Fact 1

G is reductive if and only if it does not contain any normal subgroups which are (proper, connected, and) unipotent.

Just as $G/R(G)$ was semisimple, the quotient of any algebraic group by its maximal normal subgroup which is (connected and) unipotent is reductive.

Examples of reductive algebraic groups

Example 1

$(\mathbb{C}^\times)^n$ is reductive.

Example 2

The algebraic groups

- SL_n (for $n \geq 2$),
- Sp_{2n}, and
- SO_n (for $k = \overline{k}$)

are all simple, so reductive.

Example 3

GL_n and O_n are reductive.

Examples which are not reductive

Example 4

The additive group, and product of it, are not reductive. This is because we can view $a \in \mathbb{G}_a$ as

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} . \tag{6}$$

So this is actually an unipotent group, and therefore cannot be reductive.

Example 5

The Borel subgroup B of GL_n is not reductive. This consists of upper triangular matrices, and has nontrivial unipotent normal subgroup consisting of upper-triangular matrices with 1 on the diagonal.

Example of semisimple quotient

The following is from [Mil]. Consider the algebraic group GL_{m+n}. This is given by block matrices

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \tag{7}$$

where A is $m \times m$ and C is $n \times n$.
The radical consists of matrices of the form:

$$\begin{bmatrix} aI_m & B \\ 0 & cI_n \end{bmatrix} \tag{8}$$

and the semisimple quotient is:

$$G/R(G) = \mathrm{PGL}_m \times \mathrm{PGL}_n \ . \tag{9}$$

We are entering the subsection:

Semi-simple representations

A semisimple (or completely reducible) representation is a direct sum of simple (or irreducible) representations.
link
wikipedia

Rep. theoretic characteristic of being reductive

Theorem

Assume char $k = 0$. *G is reductive iff every representation is semisimple.*

The direction($\Longleftarrow$) is easy to show. Normal unipotent subgroups of G act trivially on semisimple representations of G. So if G admits a faithful semisimple representation then G is reductive.

We are entering the subsection:

Usual definition of spherical varieties

Definition (spherical variety)

X is a *spherical variety* if and only if it contains an open dense B orbit.

Now we will identify some equivalent characterizations of spherical varieties. The punchline will be that this is fundamentally some kind of finiteness condition.

Complexity

Definition (complexity)

The complexity of X, written $c(X)$, is the minimal codimension of a B orbit.

Theorem 1

X is spherical if and only if $c(X) = 0$.

Proof.

Open dense orbits are the proper codimension 0 orbits. $\square$

Finitely many B orbits

Theorem 2

X is spherical if and only if it has finitely many B orbits.

Lemma 2 (Theorem 4.5.5 [Per])

If $Y \subset X$ is a closed B-stable subvariety then $c(Y) \leq c(X)$.

Proof.

($\implies$): Let $Y \subseteq X$ be some minimal subvariety containing infinitely many orbits. Lemma 2 implies $c(Y) = 0$. The complement of this orbit is a closed G-stable subvariety which must have infinitely many B-orbits, contradicting minimality of Y.
($\impliedby$): Nonzero complexity implies infinitely many G orbits (and hence B orbits) since any maximal orbit has nonzero codimension, and orbits are disjoint. □

Borel invariant rational functions

Theorem 3

X is spherical if and only if the only B invariant rational functions are constant: $k(X)^B = 0$.

This follows from Rosenlicht's theorem [Ros63] Theorem 2.3.

Theorem (Rosenlicht)

The transcendence degree of $k(X)^B$ over k is $c(X)$.

The idea is that

$$\begin{aligned} c(X) &= \dim(\text{“}X/B\text{”}) && (10) \\ &= \text{transcendence degree}\,(k(\text{“}X/B\text{”})) && (11) \\ &= \text{transcendence degree}\left(k(X)^B\right). && (12) \end{aligned}$$

So $c(X) = 0$ if and only if $k(X)^B = k$.

Tl;dr (equivalent definitions of spherical varieties)

Theorem

The following are equivalent:

- *X is spherical (i.e. X contains an open dense B orbit),*
- *$c(X) = 0$ (i.e. the maximal B orbit is codimension 0),*
- *X has finitely many B orbits,*
- *$k(X)^B = k$.*

We are entering the subsection:

Good features of spherical varieties

References

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Maxwell Rosenlicht, *A remark on quotient spaces*, An. Acad. Brasil. Ci. **35** (1963), 487–489. MR 171782