DEFINITION OF THE HOMOLOGY COBORDISM INVARIANTS

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The point of today will be to see that $\mathrm{SWFH}^{\mathrm{Pin}(2)}(Y, \mathbb{F})$ gives rise to some $\beta : \Theta_3^H \to \mathbb{Z}$. Recall Θ_3^H is the homology cobordism group consisting of Y^3 oriented $\mathbb{Z}HS^3$ (integral homology three spheres). The upshot will be that β satisfies:

$$\beta(-Y) = -\beta(Y) \qquad \mu(Y) = \beta(Y) \pmod 2 \tag{1}$$

where μ is the Rokhlin homomorphism, so the SES:

$$0 \to \ker \mu \to \Theta_3^H \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{2}$$

does not split.

1. The module structure on equivariant homology

Let G be a Lie group acting on a space X. The Borel homology is

$$H_*^G(X) = H_*(X \times_G EG) \tag{3}$$

where EG is the universal bundle. This is a module over $H_G^*(\mathrm{pt}) = H^*(BG)$. For us $G = \mathrm{Pin}(2)$, and we want to get our hands on

$$H^*(B\,\mathrm{Pin}(2), \mathbb{F}) \ . \tag{4}$$

Recall $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{H}$. Also recall $\mathrm{SU}(2)$ is isomorphic to the unit elements of $\mathbb{H}$, so it is diffeomorphic to S^3. This means we get a fibration:

$$\begin{tikzcd} \mathrm{Pin}(2) \arrow[r, "i"] & \mathrm{SU}(2) \arrow[d, "\psi"] \\ & \mathbb{RP}^2 \end{tikzcd} \tag{5}$$

where i is the inclusion and ψ is the Hopf fibration follows by the involution on S^2.

In such a setting, we get another fibration[1]

$$\begin{tikzcd} \mathbb{RP}^2 \arrow[r] & B\,\mathrm{Pin}(2) \arrow[d] \\ & B\,\mathrm{SU}(2) = \mathbb{HP}^\infty \end{tikzcd} \tag{6}$$

where $\mathbb{HP}^\infty$ is the infinite-dimensional quaternionic projective space.

Now we calculate $H^*(B\,\mathrm{Pin}(2), \mathbb{F})$ using a Leray-Serre spectral sequence. Recall that this is

$$E_2^{p,q} = H^p\left(B\,\mathrm{SU}(2), H^q\left(\mathbb{RP}^2\right)\right) \Rightarrow H^{p+q}(B\,\mathrm{Pin}(2)) \ . \tag{7}$$

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[1]Explicitly this is given by the inclusion of orbits. See Peter May's response to this post on MathOverflow.

We know that the cohomology of $\mathbb{RP}^2$ in degree 0, 1, and 2 is given by

$$\begin{tikzcd} \mathbb{F} \arrow[r, bend left, "q"] & \mathbb{F} \arrow[r, bend left, "q"] & \mathbb{F} \end{tikzcd} . \tag{8}$$

We also know that the cohomology of $B\,\mathrm{SU}(2) = \mathbb{HP}^\infty$ looks like

$$\begin{tikzcd} \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & 0 & 0 & \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & 0 & 0 & \mathbb{F} \end{tikzcd} . \tag{9}$$

The spectral sequence associated to the fibration has no room for higher differentials, so the cohomology groups of $B\,\mathrm{Pin}(2)$ look like:

$$\begin{tikzcd} \mathbb{F} \arrow[r, bend right, "q"'] \arrow[rrrr, bend left, "v"] & \mathbb{F} \arrow[r, bend right, "q"'] \arrow[rrrr, bend left, "v"] & \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & \mathbb{F} \arrow[r, bend right, "q"'] & \mathbb{F} \arrow[r, bend right, "q"'] & \mathbb{F} & 0 & \cdots \end{tikzcd} . \tag{10}$$

The multiplicative property of a spectral sequence gives a ring isomorphism

$$H^*\left(B\,\mathrm{Pin}(2), \mathbb{F}\right) \cong \mathbb{F}[q,v] / \left(q^3\right) \tag{11}$$

where $\deg(v) = 4$, $\deg(q) = 1$.

Therefore for X with a $\mathrm{Pin}(2)$-action, its Borel homology has an action by the ring above, with q and v decreasing the grading by 1 and 4 respectively.

2. Three infinite towers

Let $\left(I_\lambda^\mu\right)^{S^1}$ denote the fixed points set of I_λ^μ under the action of the subgroup $S^1 \subset \mathrm{Pin}(2)$. These fixed points pick up the part of the flow that live in the reducible locus, i.e. $\{(a,\varphi)\, \varphi = 0\}$.[2] As it turns out:

$$\left(I_\lambda^\mu\right)^{S^1} = S^{\dim V_\lambda^0} . \tag{12}$$

Since we defined

$$\mathrm{SWF}(Y) = \Sigma^{\mathbb{H}n(Y,g)/4} \Sigma^{-V_\lambda^0} I_\lambda^\mu \tag{13}$$

we get that

$$\left(\mathrm{SWF}(Y)\right)^{S^1} = S^{n(Y,g)} . \tag{14}$$

We should think of $\mathrm{SWF}(Y)$ as being made up of a reducible part $S^{n(Y,g)}$ and some free cells as the irreducible part. I.e. If we mod out by the sphere we get a free $\mathrm{Pin}(2)$ action.

The $\mathrm{Pin}(2)$-equivariant Seiberg-Witten Floer homology

$$\mathrm{SWFH}_*^{\mathrm{Pin}(2)}\left(\mathrm{SWF}(Y); \mathbb{F}\right) = \tilde{H}_*^{\mathrm{Pin}(2)}\left(\mathrm{SWF}(Y); \mathbb{F}\right) \tag{15}$$

is a module over $\mathbb{F}[q,v]/(q^3)$ where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Now we have a *localization theorem* in equivariant cohomology which tells us that

$$V^{-1}\tilde{H}^*_{\mathrm{Pin}(2)}\left(\mathrm{SWF}(Y); \mathbb{F}\right) = V^{-1}\tilde{H}^*_{\mathrm{Pin}(2)}\left(S^{n(Y,g)}; \mathbb{F}\right) . \tag{16}$$

Note that

$$\tilde{H}^*_{\mathrm{Pin}(2)}\left(S^{n(Y,g)}; \mathbb{F}\right) = H^{*-n(Y,g)}\left(B\,\mathrm{Pin}(2); \mathbb{F}\right) . \tag{17}$$

Now we interpret this localization theorem as concerning Borel homology rather than cohomology. Since we work over $\mathbb{F}$, the Borel homology is just the dual space to the cohomology

[2] Recall $a \in \Omega^1(Y, i\mathbb{R})$ and $\varphi \in \Gamma(\mathbb{S})$ is a section of our spin bundle.

in each grading. Furthermore, we can recover the module action on Borel homology from the one on Borel cohomology. The upshot is that $\mathrm{SWFH}_*^{\mathrm{Pin}(2)}$ is the homology of a complex (the equivariant cellular complex of the Conley index) of the form:

$$\begin{tikzcd}
\vdots & & & & \\
\mathbb{F} \arrow[dddd, bend right, "v"'] & & & & \\
0 & & \text{(finite part)} & & \\
\mathbb{F} \arrow[d, "q"] \arrow[dddd, bend right, "v"'] & \mathbb{F} \arrow[dr, bend left, "q"] & & & \\
\mathbb{F} \arrow[d, "q"] \arrow[dddd, bend right, "v"'] & & \mathbb{F} \arrow[dr, bend left, "q"] & & . \\
\mathbb{F} \arrow[dddd, bend right, "v"'] & & & \mathbb{F} & \\
0 & \mathbb{F} \arrow[dr, bend left, "q"] \arrow[ddl, "\partial"] & & & \\
\mathbb{F} \arrow[d, "q"] & & \mathbb{F} \arrow[ddll, "\partial"] & & \\
\mathbb{F} \arrow[d, "q"] & & & & \\
\mathbb{F} & & & &
\end{tikzcd} \tag{18}$$

The finite part can be any finite dimensional vector space, with some v and q actions. There can be various other differentials both inside it and relating to the infinite towers. The important part is that there are always three infinite v towers in the complex, and they produce three infinite towers in homology. These towers correspond to the S^1-fixed point set of $\mathrm{SWF}(Y)$, and the finite part comes from the cells with a free $\mathrm{Pin}(2)$ action.

The upshot is that since $(\mathrm{SWF}(Y))^{S^1} = S^{n(Y,g)}$ and $n(Y,g) \equiv 2\mu \pmod 4$, we have:

- All elements in the first tower are in degree $2\mu \pmod 4$,
- all elements in the second tower are in degree $2\mu + 1 \pmod 4$, and
- all elements in the third tower are in degree $2\mu + 2 \pmod 4$.

3. Definition of the invariants

Write $A, B, C \in \mathbb{Z}$ for the lowest degrees of each infinite v-tower in homology. Now define the following:

$$\alpha := \frac{A}{2} \qquad \beta := \frac{B-1}{2} \qquad \gamma := \frac{C-2}{2} \; . \tag{19}$$

The point is that:

$$A\,, \qquad B-1\,, \qquad C-2\,, \qquad \equiv 2\mu \pmod 4 \tag{20}$$

so therefore

$$\alpha, \beta, \gamma \equiv \mu \pmod 2\;. \tag{21}$$

Note that the module structure requires

$$\alpha \geq \beta \geq \gamma \ . \tag{22}$$

4. Descent to homology cobordism

All we need to do now is check that α, β, and γ descend to maps $\Theta_3^H \to \mathbb{Z}$. This will use the construction of cobordism maps on Seiber-Witten Floer spectra.

Let W^4 be a smooth oriented $\mathrm{Spin}(4)$ cobordism with $b_1(W) = 0$, and with $\partial W = (-Y_0) \cup Y_1$. (We really just care about when W is a homology cobordism between homology three spheres.) We can consider the Seiberg-Witten equations on W, and do a finite-dimensional approximation to the solution space in the same way that we did in three dimensions. This gets more complicated...

In the end we get a stable equivariant map between two suspension spectra

$$\Psi_W : \Sigma^{m\mathbb{H}} \mathrm{SWF}(Y_0) \to \Sigma^{n\tilde{\mathbb{R}}} \mathrm{SWF}(Y_1) \ . \tag{23}$$

(Recall $\tilde{\mathbb{R}}$ is the rep where S^1 acts trivially, and j acts by multiplication by -1.) In particular:

$$m = \frac{-\sigma(W)}{8} = \mathrm{ind}\, \not{D} \qquad\qquad n = b_2^+(W) = \mathrm{ind}\left(d^+\right) \ . \tag{24}$$

In the case that W is a smooth oriented homology cobordism between homology spheres, there is a unique $\mathrm{Spin}(4)$ structure, $b_1(W) = 0$, and $n = m = 0$. Let F_W be the homomorphism induced on $\mathrm{Pin}(2)$-equivariant homology by the map Ψ_W.

It follows from equivariant localization that in degree $k \gg 0$, the map F_W is an isomorphism. Further F_W is a module map, so we have a commutative diagram

$$\begin{tikzcd} \mathbb{F} \arrow[r, "F_W"] \arrow[d, "v"] & \mathbb{F} \arrow[d, "v"] \\ \mathbb{F} \arrow[r, "F_W"] & \mathbb{F} \end{tikzcd} \ . \tag{25}$$

Because of the module structure, we cannot have $\alpha(Y_1) < \alpha(Y_0)$, and likewise for β and γ. In conclusion,

$$\alpha(Y_1) \geq \alpha(Y_0) \qquad \beta(Y_1) \geq \beta(Y_0) \qquad \gamma(Y_1) \geq \gamma(Y_0) \ . \tag{26}$$

Now we can simply repeat the entire argument with reversed orientation and reversed direction of W. Therefore we get

$$\alpha(Y_1) \leq \alpha(Y_0) \qquad \beta(Y_1) \leq \beta(Y_0) \qquad \gamma(Y_1) \leq \gamma(Y_0) \ , \tag{27}$$

and therefore we have equalities. Therefore α, β, and γ all descend to maps

$$\Theta_3^H \to \mathbb{Z} \ . \tag{28}$$