DEFINITION OF THE HOMOLOGY COBORDISM INVARIANTS

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The point of today will be to see that $\mathrm{SWFH}^{\mathrm{Pin}(2)}(Y, \mathbb{F})$ gives rise to some $\beta : \Theta_3^H \to \mathbb{Z}$. Recall Θ_3^H is the homology cobordism group consisting of Y^3 oriented $\mathbb{Z}HS^3$ (integral homology three spheres). The upshot will be that β satisfies:

$$\beta(-Y) = -\beta(Y) \qquad \mu(Y) = \beta(Y) \pmod 2 \tag{1}$$

where μ is the Rokhlin homomorphism, so the SES:

$$0 \to \ker \mu \to \Theta_3^H \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{2}$$

does not split.

1. The module structure on equivariant homology

Let G be a Lie group acting on a space X. The Borel homology is

$$H_*^G(X) = H_*(X \times_G EG) \tag{3}$$

where EG is the universal bundle. This is a module over $H_G^*(\mathrm{pt}) = H^*(BG)$. For us $G = \mathrm{Pin}(2)$, and we want to get our hands on

$$H^*(B\,\mathrm{Pin}(2), \mathbb{F}) \ . \tag{4}$$

Recall $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{H}$. Also recall $\mathrm{SU}(2)$ is isomorphic to the unit elements of $\mathbb{H}$, so it is diffeomorphic to S^3. This means we get a fibration:

$$\begin{tikzcd} \mathrm{Pin}(2) \arrow[r, "i"] & \mathrm{SU}(2) \arrow[d, "\psi"] \\ & \mathbb{RP}^2 \end{tikzcd} \tag{5}$$

where i is the inclusion and ψ is the Hopf fibration follows by the involution on S^2.

In such a setting, we get another fibration[1]

$$\begin{tikzcd} \mathbb{RP}^2 \arrow[r] & B\,\mathrm{Pin}(2) \arrow[d] \\ & B\,\mathrm{SU}(2) = \mathbb{HP}^\infty \end{tikzcd} \tag{6}$$

where $\mathbb{HP}^\infty$ is the infinite-dimensional quaternionic projective space.

Now we calculate $H^*(B\,\mathrm{Pin}(2), \mathbb{F})$ using a Leray-Serre spectral sequence. Recall that this is

$$E_2^{p,q} = H^p\left(B\,\mathrm{SU}(2), H^q\left(\mathbb{RP}^2\right)\right) \Rightarrow H^{p+q}(B\,\mathrm{Pin}(2)) \ . \tag{7}$$

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[1]Explicitly this is given by the inclusion of orbits. See Peter May's response to this post on MathOverflow.

We know that the cohomology of $\mathbb{RP}^2$ is degree 0, 1, and 2 is given by

$$\begin{tikzcd} \mathbb{F} \arrow[r, bend left, "q"] & \mathbb{F} \arrow[r, bend left, "q"] & \mathbb{F} \end{tikzcd} . \tag{8}$$

We also know that the cohomology of $B\,\mathrm{SU}(2) = \mathbb{HP}^\infty$ looks like

$$\begin{tikzcd} \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & 0 & 0 & \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & 0 & 0 & \mathbb{F} \end{tikzcd} . \tag{9}$$

The spectral sequence associated to the fibration has no room for higher differentials, so the cohomology groups of $B\,\mathrm{Pin}(2)$ look like:

$$\begin{tikzcd} \mathbb{F} \arrow[r, bend right, "q"'] \arrow[rrrr, bend left, "v"] & \mathbb{F} \arrow[r, bend right, "q"'] \arrow[rrrr, bend left, "v"] & \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & \mathbb{F} \arrow[r, bend right, "q"'] & \mathbb{F} \arrow[r, bend right, "q"'] & \mathbb{F} & 0 & \cdots \end{tikzcd} . \tag{10}$$

The multiplicative property of a spectral sequence gives a ring isomorphism

$$H^*(B\,\mathrm{Pin}(2), \mathbb{F}) \cong \mathbb{F}[q, v] / (q^3) \tag{11}$$

where $\deg(v) = 4$, $\deg(q) = 1$.

Therefore for X with a $\mathrm{Pin}(2)$-action, its Borel homology has an action by the ring above, with q and v decreasing the grading by 1 and 4 respectively.

2. Three infinite towers

Let $(I^\mu_\lambda)^{S^1}$ denote the fixed points set of I^μ_λ under the action of the subgroup $S^1 \subset \mathrm{Pin}(2)$. These fixed points pick up the part of the flow that live in the reducible locus, i.e. $\{(a, \varphi)\, \varphi = 0\}$.[2] As it turns out:

$$(I^\mu_\lambda)^{S^1} = S^{\dim V^0_\lambda} . \tag{12}$$

Since we defined

$$\mathrm{SWF}(Y) = \Sigma^{\mathbb{H}n(Y,g)/4} \Sigma^{-V^0_\lambda} I^\mu_\lambda \tag{13}$$

we get that

$$(\mathrm{SWF}(Y))^{S^1} = S^{n(Y,g)} . \tag{14}$$

We should think of $\mathrm{SWF}(Y)$ as being made up of a reducible part $S^{n(Y,g)}$ and some free cells as the irreducible part. I.e. If we mod out by the sphere we get a free $\mathrm{Pin}(2)$ action.

The $\mathrm{Pin}(2)$-equivariant Seiberg-Witten Floer homology

$$\mathrm{SWFH}^{\mathrm{Pin}(2)}_*(\mathrm{SWF}(Y); \mathbb{F}) = \tilde{H}^{\mathrm{Pin}(2)}_*(\mathrm{SWF}(Y); \mathbb{F}) \tag{15}$$

is a module over $\mathbb{F}[q, v] / (q^3)$ where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Now we have a *localization theorem* in equivariant cohomology which tells us that

$$V^{-1} \tilde{H}^*_{\mathrm{Pin}(2)}(\mathrm{SWF}(Y); \mathbb{F}) = V^{-1} \tilde{H}^*_{\mathrm{Pin}(2)}\left(S^{n(Y,g)}; \mathbb{F}\right) . \tag{16}$$

Note that

$$\tilde{H}^*_{\mathrm{Pin}(2)}\left(S^{n(Y,g)}; \mathbb{F}\right) = H^{*-n(Y,g)}(B\,\mathrm{Pin}(2); \mathbb{F}) . \tag{17}$$

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[2] Recall $a \in \Omega^1(Y, i\mathbb{R})$ and $\varphi \in \Gamma(\mathbb{S})$ is a section of our spin bundle.

Now we interpret this localization theorem as concerning Borel homology rather than cohomology. Since we work over $\mathbb{F}$, the Borel homology is just the dual space to the cohomology in each grading. Furthermore, we can recover the module action on Borel homology from the one on Borel cohomology. The upshot is that $\mathrm{SWFH}_*^{\mathrm{Pin}(2)}$ is the homology of a complex (the equivariant cellular complex of the Conley index) of the form:

(18)
$$
\begin{tikzcd}
\vdots & & & \\
\mathbb{F} \arrow[dddd, bend right, "v"'] & & & \\
0 & & \text{(finite part)} & \\
\mathbb{F} \arrow[d, bend left, "q"] \arrow[dddd, bend right, "v"'] & \mathbb{F} \arrow[dr, bend left, "q"] & & \\
\mathbb{F} \arrow[d, bend left, "q"] \arrow[dddd, bend right, "v"'] & & \mathbb{F} \arrow[dr, bend left, "q"] & \\
\mathbb{F} \arrow[dddd, bend right, "v"'] & & & \mathbb{F} \\
0 & \mathbb{F} \arrow[ddl, "\partial"] \arrow[dr, bend left, "q"] & & \\
\mathbb{F} \arrow[d, bend left, "q"] & \arrow[ddl, "\partial"] & \mathbb{F} & \\
\mathbb{F} \arrow[d, bend left, "q"] & & & \\
\mathbb{F} & & &
\end{tikzcd}
.
$$

The finite part can be any finite dimensional vector space, with some v and q actions. There can be various other differentials both inside it and relating to the infinite towers. The important part is that there are always three infinite v towers in the complex, and they produce three infinite towers in homology. These towers correspond to the S^1-fixed point set of $\mathrm{SWF}(Y)$, and the finite part comes from the cells with a free $\mathrm{Pin}(2)$ action.

The upshot is that since $(\mathrm{SWF}(Y))^{S^1} = S^{n(Y,g)}$ and $n(Y,g) \equiv 2\mu \pmod 4$, we have:

- All elements in the first tower are in degree $2\mu \pmod 4$,
- all elements in the second tower are in degree $2\mu + 1 \pmod 4$, and
- all elements in the third tower are in degree $2\mu + 2 \pmod 4$.

3. Definition of the invariants

Write $A, B, C \in \mathbb{Z}$ for the lowest degrees of each infinite v-tower in homology. Now define the following:

$$\alpha := \frac{A}{2} \qquad \beta := \frac{B-1}{2} \qquad \gamma := \frac{C-2}{2} \ . \tag{19}$$

The point is that:

$$A\ , \qquad B-1\ , \qquad C-2\ , \qquad \equiv 2\mu \pmod 4 \tag{20}$$

so therefore

$$\alpha, \beta, \gamma \equiv \mu \pmod 2\ . \tag{21}$$

Note that the module structure requires

$$\alpha \geq \beta \geq \gamma\ . \tag{22}$$

4. Descent to homology cobordism

All we need to do now is check that α, β, and γ descend to maps $\Theta_3^H \to \mathbb{Z}$. This will use the construction of cobordism maps on Seiber-Witten Floer spectra.

Let W^4 be a smooth oriented $\mathrm{Spin}(4)$ cobordism with $b_1(W) = 0$, and with $\partial W = (-Y_0) \cup Y_1$. (We really just care about when W is a homology cobordism between homology three spheres.) We can consider the Seiberg-Witten equations on W, and do a finite-dimensional approximation to the solution space in the same way that we did in three dimensions. This gets more complicated. . .

In the end we get a stable equivariant map between two suspension spectra

$$\Psi_W : \Sigma^{m\mathbb{H}}\,\mathrm{SWF}(Y_0) \to \Sigma^{n\tilde{\mathbb{R}}}\,\mathrm{SWF}(Y_1)\ . \tag{23}$$

(Recall $\tilde{\mathbb{R}}$ is the rep where S^1 acts trivially, and j acts by multiplication by -1.) In particular:

$$m = \frac{-\sigma(W)}{8} = \mathrm{ind}\,\not{D} \qquad n = b_2^+(W) = \mathrm{ind}\left(d^+\right)\ . \tag{24}$$

In the case that W is a smooth oriented homology cobordism between homology spheres, there is a unique $\mathrm{Spin}(4)$ structure, $b_1(W) = 0$, and $n = m = 0$. Let F_W be the homomorphism induced on $\mathrm{Pin}(2)$-equivariant homology by the map Ψ_W.

It follows from equivariant localization that in degree $k \gg 0$, the map F_W is an isomorphism. Further F_W is a module map, so we have a commutative diagram

$$\begin{tikzcd} \mathbb{F} \arrow[r, "F_W"] \arrow[d, "v"] & \mathbb{F} \arrow[d, "v"] \\ \mathbb{F} \arrow[r, "F_W"] & \mathbb{F} \end{tikzcd}\ . \tag{25}$$

Because of the module structure, we cannot have $\alpha(Y_1) < \alpha(Y_0)$, and likewise for β and γ. In conclusion,

$$\alpha(Y_1) \geq \alpha(Y_0)\ , \tag{26}$$

$$\beta(Y_1) \geq \beta(Y_0)\ , \tag{27}$$

$$\gamma(Y_1) \geq \gamma(Y_0)\ . \tag{28}$$

Now we can simply repeat the entire argument with reversed orientation and reversed direction of W. Therefore we get

$$\alpha\left(Y_1\right) \leq \alpha\left(Y_0\right) \ , \tag{29}$$

$$\beta\left(Y_1\right) \leq \beta\left(Y_0\right) \ , \tag{30}$$

$$\gamma\left(Y_1\right) \leq \gamma\left(Y_0\right) \ , \tag{31}$$

and therefore we have equalities. Therefore α, β, and γ all descend to maps

$$\Theta_3^H \to \mathbb{Z} \ . \tag{32}$$