SEIBERG-WITTEN EQUATIONS, FINITE DIMENSIONAL APPROXIMATION

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Contents

1. Proof of main theorem

Recall last time we discusses the SES

$$0 \to \ker \mu \to \Theta_3^{H,\mu} \xrightarrow{\mu} \mathbb{Z}/2\mathbb{Z} \to 0 \ .$$

Then we have that μ not splitting is equivalent to the statement that for all Y $\mathbb{Z}HS$ (with $\mu(Y) = 1$) we have $[2Y] \neq 0$ (or equivalently $Y \not\sim -Y$).

To prove this we will invent an invariant β which takes some integral homology sphere and gives us a number. It has the following properties:

(1) $Y_1 \sim Y_2$ implies $\beta(Y_1) = \beta(Y_2)$.
(2) β is the integral lift of the Rocklin invariant: $\mu(Y) = \beta(Y) \pmod 2$.
(3) $\beta(-Y) = -\beta(Y)$.

Warning 1. β is not a group homomorphism.

Proof. If $Y \sim -Y$ then $\beta(Y) = \beta(-Y) = -\beta(-Y)$ so $\beta(Y) = 0$ which implies $\mu(Y) = 0$. □

Some lectures down the road we will get some gadget $\mathrm{SWF}^{\mathrm{pin}(2),*}(Y)$ which will lead to β.

2. Seiberg-Witten theory

2.1. **Setup.** Let Y be an $\mathbb{Z}HS$ and g some Riemannian metric. This will have a (unique) spin^c structure. I.e. we have S a spinor bundle, rank 2 Hermitian and

$$\rho : T^*Y \simeq TY \to \mathfrak{su}(S) \ .$$

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$TY = \langle e_1, e_2, e_3 \rangle$ is trivializable so S is the trivial rank 2 bundle and ρ explicitly sends:

$$\rho(e_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\rho(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\rho(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

This is some kind of gauge theory so we need to specify our fields. Define $\mathcal{A}$ to be the space of $\mathrm{U}(1)$ connections. Since we are doing abelian gauge theory we should think of $\mathcal{A}$ as 1-forms on Y with imaginary coefficients: $\mathcal{A} = \Omega^1(Y; i\mathbb{R})$. A *spinor* is a section $\varphi \in \Gamma(S)$. Then define

$$\partial\!\!\!/ \varphi := \sum \rho(e_i) \frac{\partial \varphi}{\partial x_i} \ .$$

We should think of $\varphi \otimes \varphi^* \in \mathrm{End}(S)$. Then in order to land in $\mathfrak{sl}(S)$ we take the traceless part:

$$(\varphi \otimes \varphi^*)_0 \in \mathfrak{sl}(S) \ .$$

So we can take

$$\rho\left((\rho \otimes \varphi^*)_0\right) \in \Omega^1(Y; i\mathbb{R}) \ .$$

Now we can state the actual SW equations. Define

$$\mathcal{C} := \Omega^1(Y; i\mathbb{R}) \times \Gamma(S) \ni (a, \varphi) \ .$$

Now define:

$$\widetilde{\mathrm{SW}} : \mathcal{C} \to \mathcal{C}$$

to map:

$$(a, \varphi) \mapsto \left(\star\, da\, - 2\rho^{-1}\left((\varphi \otimes \varphi^*)_0\right), \partial\!\!\!/ \varphi + a\varphi\right) \ .$$

Remark 1. This is a Gauge theory, so we should think of $\mathrm{S\tilde{W}} = \nabla\, \mathrm{CSD}$ where CSD is the Chern-Simons-Dirac functional given by:

$$\mathrm{CSD}(a, \varphi) = \frac{1}{2}\left(\int_Y \langle \varphi, (\partial\!\!\!/ + \rho(a))\, \varphi \rangle\, d\,\mathrm{Vol} - \int_Y a \wedge\, da\right) \ .$$

This is a functional on an infinite-dimensional space so we will be completing some sort of infinite-dimensional Morse theory in this setting, i.e. Floer homology.

We now define the Gauge group to be:

$$\mathcal{G} = \mathcal{C}^\infty(Y, \mathrm{U}(1)) \ .$$

For $U \in \mathcal{G}$ we have the action

$$U \cdot (a, \varphi) = \left(a - U^{-1}\, dU\,, U \cdot \varphi\right) \ .$$

Because Y has trivial fundamental group we can write $U = e^\xi$ where $\xi : Y \to i\mathbb{R}$. Therefore we can rewrite the action to be:

$$U_\xi(a, \varphi) = \left(a -\, d\xi\,, e^\xi \cdot \varphi\right) \ .$$

As it turns out we have

$$U_* \widetilde{\mathrm{SW}} = \widetilde{\mathrm{SW}} \ .$$

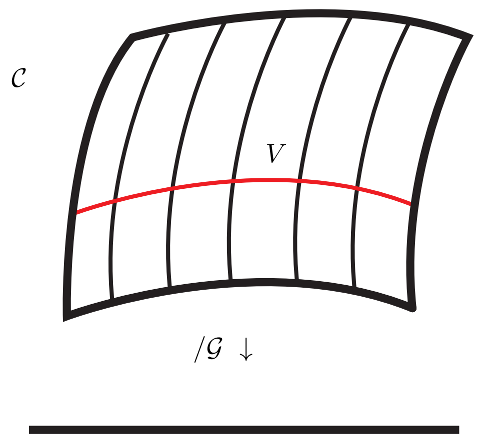

FIGURE 1. The space $\mathcal{C}$ with the orbit of some point under the action of $\mathcal{G}$ pictured as the vertical lines. The red horizontal line is a choice of section V, i.e. a choice of representatives.

3. Gauge fixing

We now describe the process of gauge fixing. For a point in our $\mathcal{C}$ we get an orbit of $\mathcal{G}$, and then Gauge fixing is a choice of a section V as in fig. 1. We will choose the Coulomb gauge, i.e. just some particular V.

Recall for $U \in G$, $a \in \mathcal{A}$ we have $U \cdot a = a - U \cdot dU = a - d\xi$. As it turns out we can write $\mathcal{G} = \mathcal{G}_0 \times S^1$ where

$$\mathcal{G}_0 := \left\{ U = e^{\xi} \,\middle|\, \int_Y \xi = 0 \right\} .$$

In general this integral won't be 0, so we write:

$$\xi' = \xi - \frac{\int \xi}{\operatorname{Vol} Y} .$$

Then we get that U_ξ corresponds to

$$\left(e^{\xi'}, e^{\int \xi / \operatorname{Vol} Y} \right) \in \mathcal{G}_0 \times S^1 .$$

For $a \in \mathcal{A}$ we write $[a] \in \mathcal{A}/\mathcal{G}_0$. To understand this we do a bit of Hodge theory.

$$\mathcal{A} = \Omega^1 (Y; i\mathbb{R}) = d\Omega^0 \oplus d^* \left(\Omega^2\right) = \ker d \oplus \ker d^* .$$

From this point of view we write

$$a = da_0 + d^* a_2 .$$

Formally we have that for any $a \in \mathcal{A}$ there exists a unique b such that $[b] = [a]$ and $b \in \ker d^*$.

Then our choice of section is given by:

$$\mathcal{C} \supset V := \{ (a, \varphi) \in \mathcal{C} \mid d^* a = 0 \} .$$

Then what we have shown is that

$$\mathcal{C}/\mathcal{G}_0 = V \ .$$

Explicitly this map is:

$$\begin{aligned} C &\xrightarrow{\quad\pi\quad} V \\ (da_0 + d^* a_2, \varphi) &\longmapsto (d^* a_2, e^{a_0}\varphi) \end{aligned}$$

Remark 2. The point of this is that we went from studying $\mathcal{C}/\mathcal{G}$ to studying V/S^1.

Now we might be worried about $\widetilde{\mathrm{SW}}\Big|_V : V \to \mathcal{C}$ not landing in V. So we will do the obvious thing and post-compose with the projection. Really we should take some kind of tangential projection. For $(a,\varphi) \in V$ we define

$$\pi_{(a,\varphi)} : T_{(a,\varphi)}\mathcal{C} \to T_{(a,\varphi)}V \ .$$

So now we define our real $\mathrm{SW} : V \to V$ to be given by:

$$\mathrm{SW}\,(a,\varphi) = \pi_{(a,\varphi)} \circ \widetilde{\mathrm{SW}} \ .$$

So we have an honest vector field on an (infinite-dimensional) space and we will study the flow-lines.

Lemma 1. *We can split* $\mathrm{SW} = l + c$ *such that* l *is a linear first order elliptic operator and* c *is quadratic.*

Explicitly these will be given by:

$$\begin{aligned} V &\xrightarrow{\qquad l \qquad} V \\ (a,\varphi) &\longmapsto \left(\star\, da\,, \partial\!\!\!/\varphi\right) \end{aligned}$$

$$\begin{aligned} V &\xrightarrow{\qquad c \qquad} V \\ (a,\varphi) &\longmapsto \pi_{(a,\varphi)} \circ \left(-2\rho^{-1}\left((\varphi\otimes\varphi^*)_0\right), a\cdot\varphi\right) \ . \end{aligned}$$

So via gauge fixing we sent $\tilde{\mathrm{SW}}$ to SW on V modulo S^1. So we have an S^1 action, but in fact we actually have a $\mathrm{pin}\,(2) = \mathrm{U}\,(1) \cup j \cdot \mathrm{U}\,(1) \subset \mathbb{H}$ action where $j^2 = -1$ and $ij = -j$. So to get a $\mathrm{pin}\,(2)$ action we just need to specify the action of j:

$$j\,(a,\varphi) = (-a, \varphi j) \ .$$

for $\varphi \in S = \mathbb{C}^2 = \mathbb{H}$.

Looking ahead, we need a finite dimensional approximation to do Morse theory which we will talk about next week.