

Chapter 8

The behaviour of bosons: photons and black-body radiation

If we have light bouncing around in a cavity (say a laser, or even just a blast furnace), the light waves clearly have to satisfy a wave equation subject to the boundary conditions of the cavity. If this sounds a bit familiar to the quantum mechanical particle-in-a-box problem, there's a reason for that: it is. Even these classical light waves end up having a discrete set of solutions due to the boundary conditions, and these can be written in the form (see Glazer & Wark, problem 8.1):

$$\begin{aligned}E_x(x, y, z) &= A_x \cos(k_x x) \sin(k_y y) \sin(k_z z) \\E_y(x, y, z) &= A_y \sin(k_x x) \cos(k_y y) \sin(k_z z) \\E_z(x, y, z) &= A_z \sin(k_x x) \sin(k_y y) \cos(k_z z)\end{aligned}\tag{8.1}$$

where E_x is the x component of the electric field in the cavity, $k_x = n_x \pi / L$, $n_x = 1, 2, 3 \dots$ and similarly for the y and z directions. We are not bothered at this point with the subtleties that change one \sin to a \cos term in each field component. Here, the main point is the similarity between the solutions for the field and that for the wavefunction of a quantum particle in a 3D box (equation 6.17). In particular, we end up with exactly the same form of 'grid' of allowed values of k . We can therefore use equation 6.19 as a starting point:

$$g(k) dk = \frac{\text{vol. of shell at } k}{\text{vol. per } k} = \frac{1}{8} 4\pi k^2 \left(\frac{L}{\pi}\right)^3 dk\tag{6.19}$$

and proceed to calculate the density of electromagnetic modes in a cavity. Again we can convert k to another quantity because James Clerk Maxwell had worked out that the wave-vector k of an electromagnetic wave had a well-defined relationship to its frequency:

$$\omega = ck\tag{8.2}$$

where c is the speed of light. So

$$dk = \frac{d\omega}{c}; \quad k^2 = \frac{\omega^2}{c^2}\tag{8.3}$$

Using (8.3), equation 6.19 becomes

$$g(\omega) d\omega = \frac{1}{8} 4\pi \frac{\omega^2}{c^3} \left(\frac{L}{\pi}\right)^3 d\omega \cdot G \quad (8.4)$$

$$= \frac{V}{\pi^2 c^3} \omega^2 d\omega \quad (8.5)$$

Note that in (8.5) we have acquired a polarisation factor $G = 2$. This is to account for the two possible polarisations of light, each of which will give us the same set of modes for the density of modes.

There is a subtlety in the relationship between k and \mathcal{E} that is not properly addressed in Glazer & Wark: we have just shown that light waves in a box have discrete k and that this means they must also have discrete ω . We tend to associate continuous variables becoming discrete with quantisation. *Does this mean that classical waves become quantum mechanical simply because we are considering them when they're in a bounded box?* With any luck, you'll come to the conclusion that the answer is no! In classical physics, there's no *a priori* reason to believe that discrete standing wave oscillation frequencies corresponds to quantisation of energy. We have made *no* assumptions that that we can only put discretised amounts of energy into the system as a whole. If we can make the cavity mode of frequency ω have any energy \mathcal{E} that we like, then we can give the entire system any energy that we like.

Einstein won his Nobel prize, not for relativity, but for the insight to suggest that the light wave itself was 'quantised', namely that a wave of frequency ω can only have energies

$$\mathcal{E}_n(\omega) = \left(n + \frac{1}{2}\right) \hbar\omega \quad (8.6)$$

By doing this, Einstein introduced the first concept of the quantum particle, namely the photon. This also underscored the importance of \hbar .

We are now in a position to explain the full framework for approaching the thermal physics of photons.

1. Cavity boundary conditions and Maxwell's equations for electromagnetic waves give a set of *classical* light modes with discrete but closely spaced allowed values of k and ω .
2. The quantisation condition is to say that each of these modes can have a discrete set of energies

$$\mathcal{E}_n(\omega) = \left(n + \frac{1}{2}\right) \hbar\omega \quad (8.6)$$

3. The interpretation of this energy spectrum is that **each time we increment/decrement n for a given mode of frequency ω , we create/destroy a photon of energy $\hbar\omega$.**
4. Photons are bosons with spin = 1, but bosons *whose number is not conserved*. They are created and destroyed from the vacuum according to the thermal conditions.

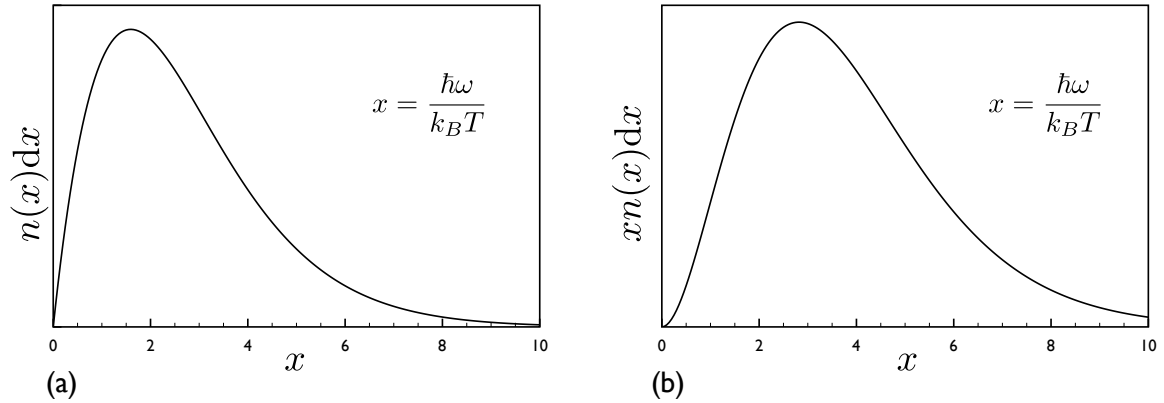


Figure 8.1: (a) Number of the photons per unit frequency interval: $n(x) dx$ as a function of x .
 (b) Energy of the photon gas per unit frequency interval: $x n(x) dx$ as a function of x .

8.1 The Bose-Einstein distribution for non-conserved bosons

Item 4 in the list above allows us to simplify equation 6.15: since we are no longer subject to the constraint of fixed particle numbers, then $\alpha = 0$. Also, from item 3 in the list above, we see that the distribution will be giving us the *number of photons* in the modes with frequency $\omega \rightarrow \omega + d\omega$:

$$n(\omega) d\omega = \frac{g(\omega) d\omega}{e^{\hbar\omega/k_B T} - 1} \quad (8.7)$$

If we combine this with our expression (8.5) for the density of modes, we get

$$n(\omega) d\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\hbar\omega/k_B T} - 1} \quad (8.8)$$

Figure 8.1 (a) shows the form of a plot of $n(x) dx$, where $x = \hbar\omega/k_B T$. From this we can see that the *number of photons* emitted from a cavity in thermal equilibrium at temperature T is strongly peaked at a frequency

$$\omega = 1.59 \frac{k_B T}{\hbar}$$

However, the energy associated with each mode is proportional to ω (as the energy of a photon of frequency ω is $\hbar\omega$) and so to look at the *energy of the photon gas* we require a plot of $x n(x) dx$, shown in figure 8.1 (b), where the peak occurs at (a different) frequency

$$\omega_p = 2.82 \frac{k_B T}{\hbar} \quad (8.9)$$

8.2 Energy of a photon gas

We can also easily calculate the energy of this photon gas between ω and $\omega + d\omega$:

$$u(\omega) d\omega = \hbar\omega \cdot n(\omega) d\omega \quad (8.10)$$

and using 8.8, we obtain *Planck's Radiation Law*:

$$u(\omega) d\omega = \frac{V \hbar}{\pi^2 c^3} \frac{\omega^3 d\omega}{e^{\hbar\omega/k_B T} - 1} \quad (8.11)$$

It gives the energy distribution for radiation emitted from a cavity in thermal equilibrium at temperature T as a function of frequency.

If we integrate equation 8.11 over all frequencies ω , we obtain the total energy for the black-body radiation:

$$U = \frac{V \hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3 d\omega}{e^{\hbar\omega/k_B T} - 1} \quad (8.12)$$

To solve this, we make a transformation of variables to turn the definite integral into a dimensionless number:

$$x = \frac{\hbar\omega}{k_B T} \quad \omega = \frac{k_B T x}{\hbar} \quad d\omega = \frac{k_B T}{\hbar} dx$$

This gives us

$$U = \frac{V \hbar}{\pi^2 c^3} \left(\frac{k_B T}{\hbar} \right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \quad (8.13)$$

The integral can be looked up in a table of integrals: in fact it is $\pi^4/15$. So, we finally arrive at

$$U = \left(\frac{V \pi^2 k_B^4}{15 \hbar^3 c^3} \right) T^4 \quad (8.14)$$

which is called the *Stefan-Boltzmann law*.

We can borrow a result from kinetic theory: the energy flux through a hole of unit area is

$$\eta = \frac{1}{4} \bar{c} \frac{U}{V} \quad (8.15)$$

and in this case averaging the speed is really easy since all the photons have the speed of light. This allows us to define the famous *Stefan's law*, the experimental determination of which had preceded the theory. The energy emitted per unit time per unit area by a black body in thermal equilibrium at temperature T is given by

$$\eta = \left(\frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \right) T^4 = \sigma T^4 \quad (8.16)$$

where σ is *Stefan's constant*.

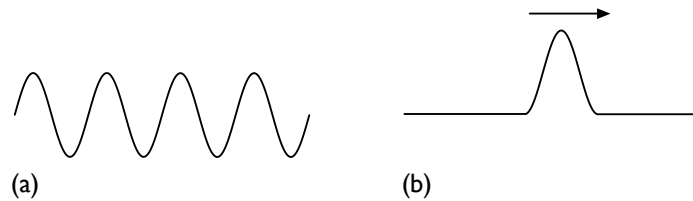


Figure 8.2: A string under tension. (a) One of the modes of vibration when plucked. (b) A wavepacket travelling along it.

8.3 The quantum-classical crossover

Early in our derivation of the quantum distributions we noted (equation 6.13) that they both limited to the classical Boltzmann distribution at high temperatures. That discussion was correct, but not particularly intuitive. We now want to re-examine the issue, and in order to make our conclusions intuitive (which is always good), we also need to re-examine our understanding of classical and quantum mechanics.

So what do we mean by a ‘classical particle’? It is an entity whose translational motion can be described using position and momentum coordinates, which can be known simultaneously with infinite precision. A *quantum* particle, on the other hand, can never satisfy this condition due to the uncertainty principle.

So, what *is* a quantum particle? We are always talking about eigenstates which are spread out across the whole system, and don’t really seem like particles at all... To answer this, we can think of a string under tension: we can pluck it to produce its modes of vibration, which are a series of ‘spread out’ sine or cosine functions, depending on the boundary conditions, as schematically illustrated in figure 8.2 (a). Alternatively, we can ‘flick’ one end of it to produce a pulse, or wave-packet, that travels along the string (figure 8.2 b)). The pulse looks much more like a particle, and can be formed from a weighted sum of normal modes or ‘eigenstates’, centred on the one whose wavelength is closest to the pulse width.

This shows that the basis of eigenstates which is the one that emerges most naturally from the solution of Schrödinger’s equation, but is often *not* the most sensible one to use when trying to visualise what’s going on.

For most purposes when considering assemblies of quantum particles, it’s best to use the eigenstates to construct an equivalent set of wave-packets. Each wave-packet has a central wave-vector \mathbf{k}' . The wave-packet’s width in k (the number of k states that we mix in to form it) is up to us to choose, but there is a penalty to pay in that the sharper the wave-packet is in k , the broader it is in x , consistent with the uncertainty principle. The average wavelength of each wave-packet will be the central one, namely $\lambda = 2\pi/k'$, and its width in space will be of this order of magnitude, i.e. of the order of magnitude of the de Broglie wavelength λ_{dB} . We now have the real-space picture of a gas in quantum particles shown in figure 8.3: they are slightly fuzzy objects whose momenta and position are not perfectly known, but specified to some compromise accuracy.

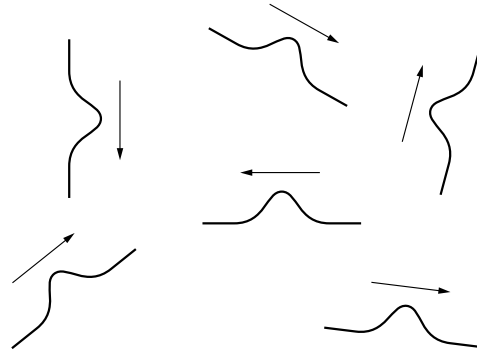


Figure 8.3: A cartoon of how we might think of a real-space picture of a gas of quantum particles.

How well do we *need* to know the particle coordinates in order for the classical particle-like description to be valid? Clearly, if their average spacing is much greater than the uncertainty in their position (their ‘fuzziness’, if you like), then they can still be sensibly treated as distinct classical particles. From this point of view, the condition for the classical limit is

$$V \gg N \lambda_{\text{dB}}^3 \quad (8.17)$$

But

$$\lambda_{\text{dB}} = \frac{2\pi}{k_{\text{dB}}} = \frac{2\pi\hbar}{\sqrt{2m \mathcal{E}_{\text{dB}}}} \quad (8.18)$$

so we can say that

$$\frac{N}{V} \ll \frac{(2m \mathcal{E}_{\text{dB}})^{3/2}}{(2\pi\hbar)^3} \quad (8.19)$$

and therefore

$$\mathcal{E}_{\text{dB}} \gg \frac{(2\pi\hbar)^2}{2m} \left(\frac{N}{V} \right)^{2/3} \quad (8.20)$$

The right hand side of the above expression should look familiar: comparing it to the expression for the Fermi energy for a gas of fermions at zero temperature, you can see that (8.20) is equivalent (within a factor of 4 or so, which, given our approximate treatment, is unimportant) to

$$\mathcal{E}_{\text{dB}} \gg \mathcal{E}_{\text{F}} \quad (8.21)$$

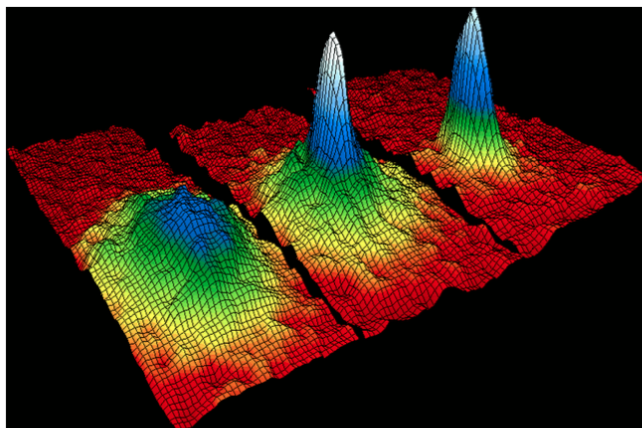
What *physics* actually imposes the above condition? Well, \mathcal{E}_{dB} is the kinetic energy of the particle, which is the sum of the ‘quantum’ kinetic energy due to the Pauli principle and the thermal energy $k_B T$. Since \mathcal{E}_{F} is just the maximum ‘quantum’ contribution to the kinetic energy of any quantum particle, the only way to satisfy $\mathcal{E}_{\text{dB}} \gg \mathcal{E}_{\text{F}}$ is for $k_B T \gg \mathcal{E}_{\text{F}}$, or, equivalently, $T \gg T_{\text{F}}$.

This last discussion has been for fermions, but develops a logic that holds for both fermions and bosons: *the classical limit of a gas of quantum particles is reached when the gas is so dilute that the particle-like wave-packets have negligible overlap. The condition for this is that the temperature be much greater than the chemical potential.* For fermions, we know the consequences of not being in the classical limit (i.e. the degenerate Fermi gas with its huge zero-point energy); we will study them in the next chapter for bosons (Bose-Einstein condensation). After that we’ll take a look at quantum gases in the classical limit, i.e. ‘ideal gases’.

Chapter 9

Conserved bosons and Bose-Einstein condensation

In 1924-25 Albert Einstein and Satyendra Nath Bose published a theory that predicted what we now call Bose-Einstein condensation (BEC). They found in their calculations that when a given number of particles approach each other sufficiently closely and move sufficiently slowly, they will convert to the lowest energy state. Ever since the publication of this pioneering work, physicists wanted to be able to observe this new fundamental state of matter. But seventy years were to pass before BEC was finally achieved in 1995, using very advanced methods. Eric Cornell, Wolfgang Ketterle and Carl Wieman were jointly awarded the Nobel Prize in 2001 “for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms” (^{87}Rb).



Observation of Bose-Einstein condensation by absorption imaging. (JILA 1995)

Bose-Einstein condensation is the low temperature behaviour of an ideal gas of bosons of non-zero mass, whose numbers are fixed. This BE gas has a unique feature: it undergoes a thermodynamic phase transition, although the bosons here are considered to be non-interacting particles. This phase change is driven only by the particle statistics and it can be treated in an exact mathematical way.

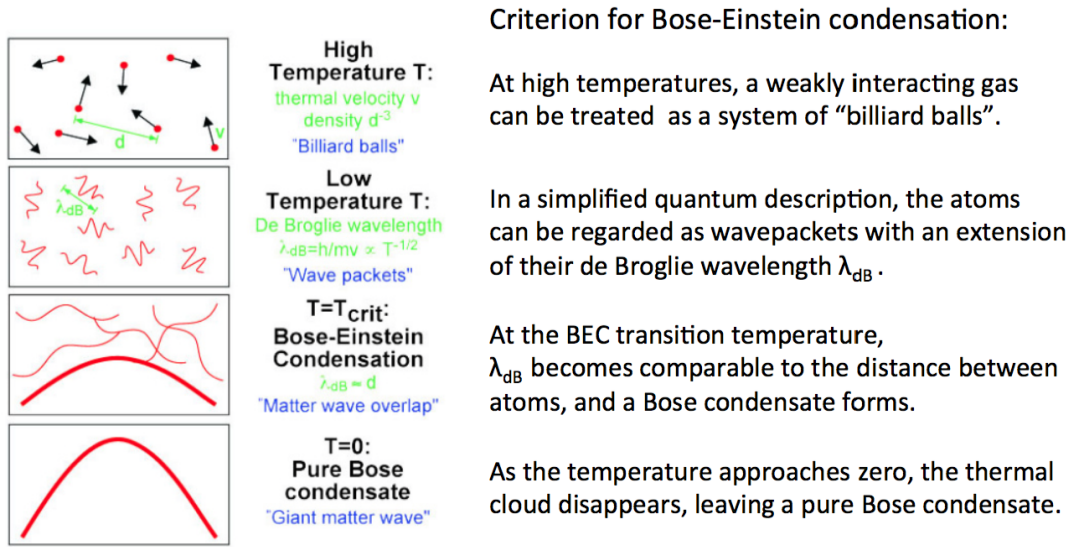


Figure 9.1: Wolfgang Ketterle in his Nobel Lecture on 8 December 2001.

At the phase transition all the thermodynamic observables abruptly change in character. This defines the critical temperature T_{BE} . The term "condensation" is used (by analogy with the normal liquid-gas phase transition) to describe how below the critical temperature T_{BE} in BEC 'normal gas' particles coexist in equilibrium with 'condensed' particles. But, unlike a liquid droplet in a gas, here the 'condensed' particles are not separated in space from the normal particles. Instead, they are separated in momentum space (k -space), where the condensed particles all occupy a single quantum state of zero momentum, while the normal particles all have finite momentum.

The properties of a Bose-Einstein gas follow from the Bose-Einstein distribution for conserved particles:

$$n(\mathcal{E}) d\mathcal{E} = \frac{g(\mathcal{E}) d\mathcal{E}}{e^{(\mathcal{E}-\mu)/k_B T} - 1} . \quad (9.1)$$

The total number of particles in the system can be obtained by integrating (9.1) over all energies

$$N = \int_0^\infty n(\mathcal{E}) d\mathcal{E} = \int_0^\infty \frac{g(\mathcal{E}) d\mathcal{E}}{e^{(\mathcal{E}-\mu)/k_B T} - 1} . \quad (9.2)$$

We substitute (6.23) for the energy density of states $g(\mathcal{E}) d\mathcal{E}$, where we used $\mathcal{E} = \frac{\hbar^2 k^2}{2m}$ (m being the mass of a particle of the gas confined to a box of volume V).

Equation (9.2) becomes

$$\frac{N}{V} = \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty \frac{\sqrt{\mathcal{E}} d\mathcal{E}}{e^{(\mathcal{E}-\mu)/k_B T} - 1} , \quad (9.3)$$

which gives the particle density N/V as a function of temperature T and the chemical potential μ .

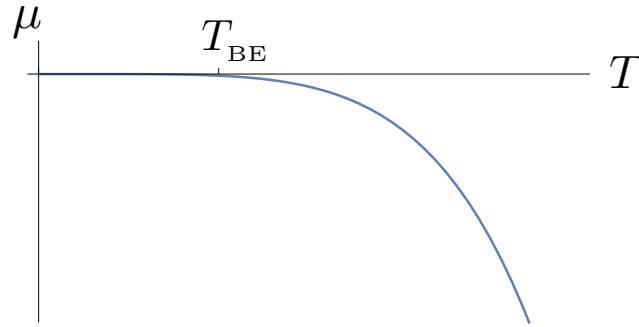


Figure 9.2: The chemical potential μ of a Bose gas as a function of temperature.

Let us vary the temperature of the gas, keeping N and V (and hence N/V) constant.

The RHS of equation (9.3) should not only be constant but also be positive, as we are expecting non-negative occupation numbers. For the BE case this means that $e^{(\mathcal{E}-\mu)/k_B T} > 1$, or $\mathcal{E} - \mu > 0$,

which is ensured for $\mu < \mathcal{E}$ for all energies \mathcal{E} . Having used $\mathcal{E} = \frac{\hbar^2 k^2}{2m}$, the energy of the ground state is zero

$$\mathcal{E}_G = 0, \quad (9.4)$$

from which follows that $\mu < \mathcal{E}_G = 0$, i.e. the chemical potential of a BE gas must always be less than or equal to the lowest, or ‘ground state’ energy \mathcal{E}_G , and thus it must always be *negative*,

$$\mu < 0, \quad (9.5)$$

because otherwise we would have a negative number of particles for some energy level.

If we start cooling the BE gas, the temperature T in (9.3) is lowered. But for the particle density N/V to remain constant, the chemical potential μ must increase (i.e. $|\mu|$ must decrease). Thus the chemical potential μ tends to zero: $\mu \rightarrow \mathcal{E}_G = 0$.

Let’s have a closer look at the temperature dependence of the chemical potential. It can be shown¹ that for high temperatures

$$\mu(T) \approx -\frac{3}{2} k_B T \ln \left(\frac{mk_B T}{2\pi\hbar^2} \left(\frac{V}{N} \right)^{2/3} \right), \quad (9.6)$$

which gives a negative chemical potential, as shown in fig. (9.2).

However, we assume that at a certain *critical temperature* T_{BE} the chemical potential effectively vanishes: $\mu \approx 0$ (it is very close to zero, but still negative) and the occupation number for the ground state is still very small in comparison to N : $N_G \approx 0$.

¹see James F. Annett: Superconductivity, Superfluids and Condensates, Oxford University Press, 2004

We can find an expression for T_{BE} , if we set $\mu = 0$ in equ. (9.3) and evaluate the integral

$$\begin{aligned} \frac{N}{V} &\cong \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty \frac{\sqrt{\mathcal{E}} d\mathcal{E}}{e^{\mathcal{E}/k_B T_{\text{BE}}} - 1} \\ &= \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} (k_B T_{\text{BE}})^{3/2} \int_0^\infty \frac{\sqrt{x} dx}{e^x - 1} \end{aligned} \quad (9.7)$$

where in the last line we introduced a new dimensionless variable $x \equiv \frac{\mathcal{E}}{k_B T_{\text{BE}}}$ (note the T_{BE}). The integral in equ. (9.7) can be looked up to give

$$\int_0^\infty \frac{\sqrt{x} dx}{e^x - 1} = 1.306 \sqrt{\pi} \quad (9.8)$$

and so (9.7) becomes

$$\frac{N}{V} = 2.612 \left(\frac{mk_B T_{\text{BE}}}{2\pi\hbar^2} \right)^{3/2}, \quad (9.9)$$

from where we can extract the critical temperature. Hence the temperature at which the Bose-Einstein condensation starts is

$$T_{\text{BE}} = \frac{2\pi\hbar^2}{mk_B} \left(\frac{N}{2.612 V} \right)^{2/3}. \quad (9.10)$$

Above this temperature, effectively all the particles will be in excited levels above the ground state: $N \approx N_{\text{exc}}$, $N_{\text{G}} \approx 0$. The system is still in the ‘classical’ limit: $N_{\text{exc}} \gg N_{\text{G}}$. Upon reaching $T = T_{\text{BE}}$ the system starts to populate the ground level (lowest state) and BEC will have begun. As the system is cooled further, the temperature decreases ($T < T_{\text{BE}}$) and more and more bosons will be in the ground state and the system begins a macroscopic condensation into a single, identical quantum state; it reached the quantum limit. For $T/T_{\text{BE}} \rightarrow 0$ all bosons are in the ground state and BEC is complete: $N \approx N_{\text{G}}$.

This is illustrated in fig. (9.3), which shows the proportion of particles occupying the ground state as a function of temperature relative to T_{BE} .

When we look back at equ. (9.3) we can’t help but notice that it is valid only for $T > T_{\text{BE}}$. As the temperature of the gas is lowered below T/T_{BE} , huge numbers of particles will occupy the ground state with zero energy: $\mathcal{E}_{\text{G}} = 0$. But just this state is *completely* neglected in (9.3) as the factor $\sqrt{\mathcal{E}}$ in the nominator gives the integrand zero weight. At higher temperatures this does not introduce an error, but at low temperatures we must not simply omit this state, since it will contain a significant number of particles. When writing down an expression for the total number of particles, we must explicitly put in a term for the ground state, which is a unique state ($g_0 = 1$, i.e. non-degenerate) with zero energy. From the Bose-Einstein distribution (9.1) follows immediately that

$$N_{\text{G}} = \frac{1}{e^{-\mu/k_B T} - 1} \quad (9.11)$$

is the number of particles in the ground state.

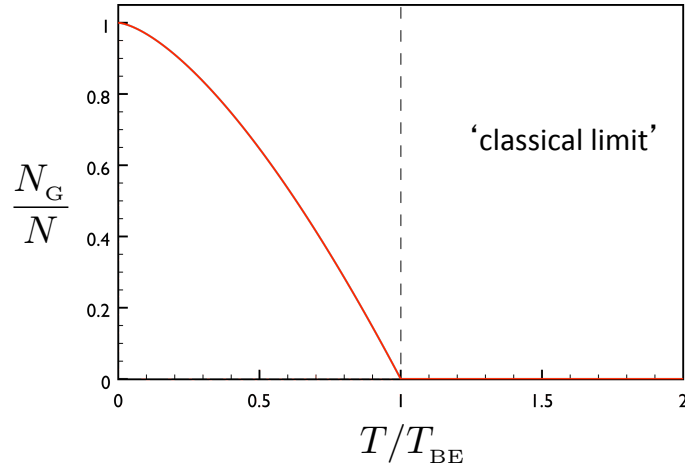


Figure 9.3: The proportion N_G/N of bosons in the ground state as a function of temperature.

Instead of (9.3) we should write

$$N = N_G + N_{\text{exc}} = \frac{1}{e^{-\mu/k_B T} - 1} + V \frac{m^{3/2}}{\sqrt{2} \pi^2 \hbar^3} \int_0^\infty \frac{\sqrt{\mathcal{E}} d\mathcal{E}}{e^{(\mathcal{E}-\mu)/k_B T} - 1} \quad (9.12)$$

where the second term gives the number of particles in the excited states with $\mathcal{E} > 0$.

For $T > T_{\text{BE}}$, N_G is negligible small and the first term can be omitted. For $T < T_{\text{BE}}$, the chemical potential is extremely close to zero (but still $\mu < 0$), so when calculating N_{exc} we can set $\mu = 0$ as approximation:

$$N_{\text{exc}} = V \frac{m^{3/2}}{\sqrt{2} \pi^2 \hbar^3} \int_0^\infty \frac{\sqrt{\mathcal{E}} d\mathcal{E}}{e^{\mathcal{E}/k_B T} - 1}. \quad (9.13)$$

We can introduce a new variable $z \equiv \frac{\mathcal{E}}{k_B T}$ in the integral above and use (9.8) to find its value, so that

$$\frac{N_{\text{exc}}}{V} = 2.612 \left(\frac{mk_B T}{2\pi \hbar^2} \right)^{3/2}. \quad (9.14)$$

From eqs. (9.9) and (9.14) we get

$$\frac{N_{\text{exc}}}{N} = \left(\frac{T}{T_{\text{BE}}} \right)^{3/2} \quad (9.15)$$

as the fraction of particles in states with energy $\mathcal{E} > 0$, and with $N = N_G + N_{\text{exc}}$ we obtain the expression for the population of the ground state for $T < T_{\text{BE}}$,

$$N_G = N \left[1 - \left(\frac{T}{T_{\text{BE}}} \right)^{3/2} \right], \quad (9.16)$$

the shape of which is shown in fig. (9.3).

It is instructive to compare expression (9.10) for T_{BE} to equ. (8.20), in which we calculated the classical limit criterion, and (7.3) rewritten as Fermi temperature ($T_{\text{F}} = \mathcal{E}_{\text{F}}/k_{\text{B}}$):

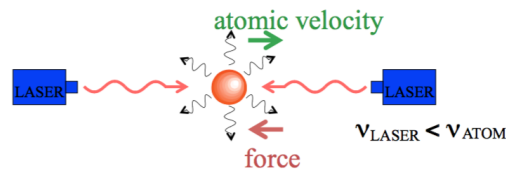
$$\mathcal{E}_{\text{dB}} \gg \frac{(2\pi\hbar)^2}{2m} \left(\frac{N}{V}\right)^{2/3} \quad \text{and} \quad T_{\text{F}} = \frac{\hbar^2}{2mk_{\text{B}}} \left(\frac{3\pi^2 N}{V}\right)^{2/3}.$$

Note that all three scale as $(N/V)^{2/3}$.

When the quantum mechanical statistics of fermions ‘take over’ a system below T_{F} , the Pauli exclusion principle gives them a huge zero-point energy \mathcal{E}_{F} . If the system contains bosons, they begin a macroscopic condensation into a single, identical quantum state at the critical temperature T_{BE} .

Although predicted in 1924-25, Bose-Einstein condensation was experimentally realised only in 1995 for dilute gases of alkali atoms. Using advanced techniques for trapping and cooling atoms in magnetic and laser traps, alkali atoms such as ^7Li , ^{23}Na , ^{87}Rb with atom densities in the traps of around 10^{20} m^{-3} are used to observe BEC at temperatures of the order of $10 \text{ nK} - 1 \mu\text{K}$.

LASER COOLING



EVAPORATIVE COOLING

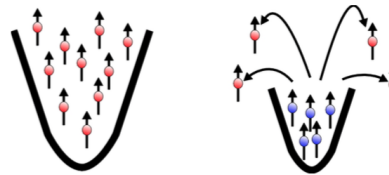
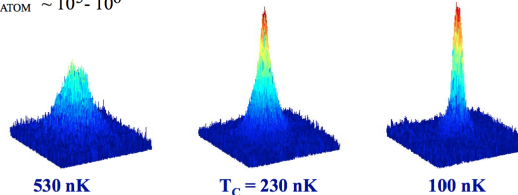


Figure 9.4: The two cooling techniques used for achieving BEC.

Onset of BEC

Rb-87 BEC
 $N_{\text{ATOM}} \sim 10^5 - 10^6$



Donatella Cassettari (Oxford, 2001)
<http://coldatoms.wp.st-andrews.ac.uk/>

Figure 9.5: Donatella Cassettari's results.