Physics of Music

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1 Introduction

Helmholtz Resonators: hollow spheres with pinholes – the volume of the sphere determines its resonance which is distinct

Synthesizing sounds in the 19th century involved using a tuning fork to rapidly toggle a circuit. A pin attached to the tuning fork would dip into and out of liquid magnesium with a layer of alcohol on top to prevent sparking.

This oscillating current can be applied to a coil of wire around a magnet which can drive another tuning fork. All of these tuning forks can be intercepted by a Helmholtz resonator so the apparatus can be "tuned." This is also how each harmonic was mixed. (no voltage control)

For an n^{th} harmonic,

$$L = \frac{n\lambda_0}{2} \qquad f_n = \frac{nc_s}{2L} = nf_1 \qquad n = 1, 2, 3\ldots$$

2 Beats and Fourier

$$\left.\begin{aligned} x_1 &= A_1 e^{j(\omega t+\phi_1)} \\ x_2 &= A_2 e^{j(\omega t+\phi_2)} \end{aligned}\right\} \longrightarrow Ae^{j(\omega t+\phi)} = \left(A_1 e^{j\phi_1} + A_2 e^{j\phi_2}\right) e^{j\omega t}$$

from geometry, the cosine rule and Pythagorus, (think of end-to-end vectors in the complex plane)

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_1 - \varphi_2)}$$

$$\tan\varphi = \frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2}$$

displacement for two vibrations of different frequency:

$$x = A_1 e^{j(\omega_1 t+\phi_1)} + A_2 e^{j(\omega_2 t+\phi_2)}$$

If the frequencies are "harmonics" then the sum of their vibrations will be periodic witha repetition rate equal to the "fundamental" frequency. If the two frequencies are not "harmonics" then the wave never repeats itself.

$$\begin{aligned} \omega_2 = \omega_1 + \Delta\omega \longrightarrow\ x &= A_1 e^{j(\omega_1 t+\phi_1)} + A_2 e^{j(\omega_1 t+\Delta\omega t+\phi_2)} \\ &= [A_1 e^{j\phi_1} + A_2 e^{j\phi_2)}]e^{j\omega_1 t} \end{aligned}$$

comparing with the above:

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_1 - \varphi_2 - \Delta\omega t)}$$

$$\tan\varphi = \frac{A_1\sin\varphi_1 + A_2\sin(\varphi_2 + \Delta\omega t)}{A_1\cos\varphi_1 + A_2\cos(\varphi_2 + \Delta\omega t)}$$

Thus, the equation for A indicates that two sine waves close together in frequency combine to give a signal which resembles a single sine wave oscillating in amplitude.

2.1 Fourier Series

Fourier's Theorem: any periodic motion may be expressed as a sum of harmonic, sinusoidal frequency compnoents – which are integer multiples of the repetition rate of the function.

For a vibration of period T represented by the function $f(t)$, the harmonic series is:

$$\begin{aligned} f(t) = \frac{1}{2}A_0 + A_1\cos\omega t + A_2\cos 2\omega t + \cdots + A_n\cos n\omega t + \cdots \\ +B_1\sin\omega t + B_2\sin 2\omega t + \cdots + B_n\sin n\omega t + \cdots \end{aligned}$$

A_n and B_n are the amplitudes of the components and $\omega = 2\pi/T$ is the angular frequency of the fundamental.

$$A_n = \frac{2}{T}\int_0^r f(t)\cos(n\omega t)\,\mathrm{d}t \qquad B_n = \frac{2}{T}\int_0^r f(t)\sin(n\omega t)\,\mathrm{d}t$$

2.2 Fourier Transform

spectral density function: $g(w)$ is the continuous weighting of each angular frequency w on a vibration function which turns out to be more complex and transient than a discrete sum of components

$$f(t) = \int_{-\infty}^{+\infty} g(w)e^{iwt}\,\mathrm{d}w$$

using the complex Fourier transform:

$$g(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-jwt}\,\mathrm{d}t$$

spectral density of Dirac delta function:

$$g(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(t)e^{-jwt}\,\mathrm{d}t = \frac{1}{2\pi}$$

3 Waves of a String

Taylor series expansion:

$$f(x + \mathrm{d}x) = f(x) + \left(\frac{\partial f}{\partial x}\right)_x \mathrm{d}x + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_x \mathrm{d}x + \ldots$$

If a string under tension is displaced from equilibrium by a small amount, the tension in the string may be assumed not to change, but the string still experiences an unbalanced force due to the change in the direction of the tension.

$$\begin{aligned} \mathrm{d}f_y &= T\sin\theta(x + \mathrm{d}x) - T\sin\theta(x) \\ &= \left[T\sin\theta(x) + \left(\frac{\partial(T\sin\theta(x))}{\partial x}\right)_x \mathrm{d}x + \cdots\right] - T\sin\theta(x) \\ &= \left(\frac{\partial(T\sin\theta)}{\partial x}\right)_x \mathrm{d}x \qquad \longleftarrow\ \sin\theta \simeq \frac{\partial y}{\partial x} \\ &= \frac{\partial\left(T\frac{\partial y}{\partial x}\right)}{\partial x} = T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x \end{aligned}$$

applying this to Netwon's Second Law for this length of a string:

$$\mathrm{d}f_y = \rho_L \mathrm{d}x \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \mathrm{d}x$$

This equation yields the "wave-equation" where $c = \sqrt{T/\rho_L}$

3.1 The General Solution and Wave Motion

general solutions to the wave equation:

$$y(x,t) = y_1(ct - x) + y_2(ct + x)$$

Possible arbitrary functions include $\log(ct \pm x)$, and $\exp[j\omega(t \pm x/c)]$ but not functions which include X and T separately such as $x^2 + ct^2$

$y_1(ct - x)$ is a travelling wave of arbitrary shape travelling to the right and $y_2(ct + x)$ is another travelling to the right instead. If the string is rigidly fixed at one end then the travelling waves must be 4euqal and opposite at that point at all times. This means that when a wave is incident on the end then it is reflected back the same shape but opposite displacement.

3.2 Simple Harmonic Solutions

$$\begin{aligned} y(x,t) &= A \sin \frac{\omega}{c}(ct - x) + B \cos \frac{\omega}{c}(ct - x) + C \sin \frac{\omega}{c}(ct + x) + D \cos \frac{\omega}{c}(ct + x) \\ &= A \sin(\omega t - kx) + B \cos(\omega t - kx) + C \sin(\omega t - kx) + D \cos(\omega t - kx) \end{aligned}$$

3.3 Standing Waves

For a string fixed at both ends, at $x = 0$ and $x = L$, the terms for the negative and positive direction motion must be equal and opposite in magnitude.

$$\begin{aligned} y(x,t) &= A[\sin(\omega t - kx) - \sin(\omega t + kx)] + B[\cos(\omega t - kx) \\ &\quad - \cos(\omega t + kx)] \\ &\quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \\ &\quad \cos(x \pm y) = \cos x \cos y \pm \sin x \sin y \\ &= 2A \sin(kx) \cos(\omega t) - 2B \sin(kx) \sin(\omega t) \\ &= 2[A \cos(\omega t) - B \sin(\omega t)] \sin(kx) \end{aligned}$$

This equation illustrates the properties of a standing wave because it is a sine shape fixed in space with an time-varying amplitude. Since the string is fixed at $x = L$ then $\sin(kL) = 0$ and so $k = n\pi/L$. The string can thus support an infinite number of sinusoidal modes or harmonics, each with n half wavelengths of a sine wave on the string.

4 Membranes

For a thin, flexible membrane element under tension F, let the displacement y be a function of the position coordinates x and z. So the net force on the element in the y direction due to the curvature of the mebrane along the x dimension is:

$$F\,\mathrm{d}z \left[\left(\frac{\partial y}{\partial x}\right)_{x+\mathrm{d}x} - \left(\frac{\partial y}{\partial x}\right)_{x} \right] = F \frac{\partial^2 y}{\partial x^2} \mathrm{d}x\,\mathrm{d}z$$

$$F\,\mathrm{d}x \left[\left(\frac{\partial y}{\partial z}\right)_{z+\mathrm{d}z} - \left(\frac{\partial y}{\partial z}\right)_{z} \right] = F \frac{\partial^2 y}{\partial z^2} \mathrm{d}x\,\mathrm{d}z$$

adding these together and applying Netwon's Second Law:

$$F \left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} \right) \mathrm{d}x\,\mathrm{d}z = \rho_s \mathrm{d}x \mathrm{d}z \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \qquad c = \sqrt{\frac{F}{\rho_s}}$$

$$\nabla^2 y = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

solutions:

$$y = \Psi e^{j\omega t}$$

Ψ depends only on position and is the mode shape for standing waves.

$$\therefore \nabla^2 \Psi + k^2 \Psi = 0$$

$\longleftarrow$ *Helmholtz equation* or *time independent wave equation*

4.1 Rectangular Membrane with Fixed Rim

For a freely vibrating rectangular membrane fixed at its edges at $x = 0, x = L_x, z = 0,$ and$z = L_z$, the solution to the Helmholtz equation is separable:

$$\mathbf{\Psi}(x,z) = \mathbf{X}(x)\mathbf{Z}(z)$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} + k^2 = 0$$

$$\frac{\partial^2 X}{\partial x^2} + +k_x^2 X = 0 \qquad \frac{\partial^2 Z}{\partial z^2} + +k_z^2 Z = 0 \qquad k_x^2 + k_z^2 = k^2$$

general solution:

$$\begin{aligned} y(x,z,t) &= \mathbf{A} \sin(k_x x + \phi_x) \sin(k_z z + \phi_z) e^{j\omega t} \\ &= \mathbf{A} \sin(k_x x)\ \sin(k_z z)\ e^{j\omega t} \qquad k_x = n\pi/L_x \qquad k_z = m\pi/L \end{aligned}$$

The rectangular membrane thus holds standing waves which split it into a rectangular grid of nodal lines.

using $\omega = ck$ and $f = \omega/2\pi$,

$$f_{nm} = \frac{c}{2} \sqrt{\left(\frac{n}{L_x}\right)^2 + \left(\frac{m}{L_z}\right)^2}$$

4.2 Circular Membrane with Fixed Rim

Laplacian operator in cylindrical coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

The spatial part of the solution to the Helmholtz equation will also be separable: (the boundary condition will be $\mathbf{R}(a) = 0$)

$$\mathbf{\Psi}(r,\theta) = \mathbf{R}(r)\mathbf{\Theta}(\theta)$$

$$\frac{r^2}{\mathbf{R}} \left(\frac{\partial^2 \mathbf{R}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{R}}{\partial r} \right) + k^2 r^2 = -\frac{1}{\mathbf{\Theta}} \frac{\partial^2 \mathbf{\Theta}}{\partial \theta^2} \qquad = m^2$$

angular solution:

$$\mathbf{\Theta}(\theta) = \cos(m\theta + \gamma)$$

Since $\mathbf{\Theta}(\theta) = \mathbf{\Theta}(\theta + 2\pi)$ this means that m must be a positive integer, meaning that there are m equally spaced "nodal diamters."

radial solution: (Bessel's equation)

$$\mathbf{R}(r) = \mathbf{A}J_m(kr)$$

$\mathbf{A}$ is a constant and J_m is the Bessel function of the first kind of order m. The "wave-number" k is chosen such that $\mathbf{R}(r)$ is zero at $r = a$. Bessel functions are oscillatory with an amplitude which decreases with distance from the origin. As the Bessel functions have an infinite number of zeros, there will be an infinite nubmer of ways of satisfying $\mathbf{R}(r)$ and these correspond to an ingere number of "nodes at fixed radius" or "nodal circles".

5 Waves in Air

5.1 the continuity equation

There are two ways of looking at fluid motion: the Lagrangian method which specifies particle trajectories in fluid flows and the Eulerian method which specifies the field of the variables involved in the fluid motion. Deriving the wave equation for waves in air, we will use the Lagrangian method.

For a small mass $(\mathrm{d}m)$ element of air of volume $\mathrm{d}V_0 = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ which moves as a pressure wave passes. For a change of volume from (x, y, z) to $(x + \xi, y + \eta, z + \zeta)$

$$\begin{aligned}\mathrm{d}V &= \left(1 + \frac{\partial\xi}{\partial x}\right)\mathrm{d}x\,\left(1 + \frac{\partial\eta}{\partial y}\right)\mathrm{d}y\,\left(1 + \frac{\partial\zeta}{\partial z}\right)\mathrm{d}z \\ &= \left(1 + \frac{\partial\xi}{\partial x}\right)\left(1 + \frac{\partial\eta}{\partial y}\right)\left(1 + \frac{\partial\zeta}{\partial z}\right)\mathrm{d}V_0\end{aligned}$$

Because the element follows the mass, $\mathrm{d}m$ is constant and $\rho_0\,\mathrm{d}V_0 = \rho\,\mathrm{d}V$.

$$\frac{\mathrm{d}V_0}{\mathrm{d}V} = \frac{\rho}{\rho_0} = 1 + s \qquad s \text{ is the fractional condensation}$$

$$\left(1 + \frac{\partial\xi}{\partial x}\right)\left(1 + \frac{\partial\eta}{\partial y}\right)\left(1 + \frac{\partial\zeta}{\partial z}\right)(1 + s) = 1$$

If the waves are small in amplitude then the derivatives and s are $\ll 1$.

$$\therefore 1 + \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} + s \approx 1$$

$$\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} = \boldsymbol{\nabla}\cdot\mathbf{d} = -s \qquad \mathbf{d} = \xi\hat{\imath} + \eta\hat{\jmath} + \zeta\hat{k}$$

5.2 the elasticity and force equations

Let P be the pressurein the fluid, with the mean atmospheric pressure labelled as P_0 and a small acousitc pressure which resutls from a density change labelled as p.

$$p = \left(\frac{\partial P}{\partial\rho}\right)_0 \mathrm{d}p = \left(\frac{\partial P}{\partial\rho}\right)_0 \rho_0 s$$

x-direction force from unbalanced pressures on opposite surfaces of the element:

$$\mathrm{d}F_x = -\left(\frac{\partial P}{\partial x}\right)\mathrm{d}x \times \mathrm{d}y\,\mathrm{d}z = -\left(\frac{\partial p}{\partial x}\right)\mathrm{d}V_0$$

Assuming small amplitude motion, then Newton's laws give:

$$\mathrm{d}F_x = -\left(\frac{\partial p}{\partial x}\right)\mathrm{d}V_0 \approx \mathrm{d}m\frac{\partial^2\xi}{\partial t^2} \qquad \frac{\partial p}{\partial x} = -\rho_0\frac{\partial^2\xi}{\partial t^2}$$

$$\therefore \boldsymbol{\nabla}p = -\rho_0\frac{\partial^2\mathbf{d}}{\partial t^2}$$

5.3 wave equation in air

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}p = \nabla^2 p = -\rho_0\frac{\partial^2}{\partial t^2}\left(\boldsymbol{\nabla}\cdot\vec{\mathbf{d}}\right) = \rho\frac{\partial^2 s}{\partial t^2}$$

$$\frac{\partial^2 p}{\partial t^2} = \left(\frac{\partial P}{\partial\rho}\right)_0 \rho_0\frac{\partial^2 s}{\partial t^2} = \left(\frac{\partial P}{\partial\rho}\right)_0 \nabla^2 p$$

This is of the form of the wave equation:

$$\frac{\partial^2 p}{\partial t^2} = c^2\nabla^2 p \qquad c = \left(\frac{\partial P}{\partial\rho}\right)_0^{\frac{1}{2}}$$

Assuming adiabatic pressure changes, then PV^γ is constant where $\gamma = C_P/C_V$ is the ratio of heat capacities.

$$\rho_0\,\mathrm{d}V_0 = \rho\,\mathrm{d}V \quad\longrightarrow\quad \frac{P}{P_0} = \frac{\mathrm{d}V_0^\gamma}{\mathrm{d}V^\gamma} = \frac{\rho^\gamma}{\rho_0^\gamma}$$

$$\left(\frac{\partial P}{\partial\rho}\right)_0 = P_0\gamma\frac{\rho_0^{\gamma-1}}{\rho_0^\gamma} = \frac{P_0\gamma}{\rho_0} \quad\longrightarrow\quad c = \sqrt{\frac{P_0\gamma}{\rho_0}}$$

At standard atmoshperic pressure and density this is 343 m/s at $T = 18°$C. The speed of sound in air increases with temperature (as the density decreases) and also varies with humidity.

5.4 energy density

kinetic energy of a volume element $m = \rho_0 V_0$ with an acoustic particle velocity of u (velocity of the mass element):

$$K = \tfrac{1}{2}mu^2 = \tfrac{1}{2}\rho_0 u^2 V_0$$

$$\begin{aligned}\mathrm{d}V &= -\frac{V_0}{\rho_0}\mathrm{d}\rho \\ &\qquad p = \rho_0 c^2 s \qquad s = \frac{\rho}{\rho_0} - 1 \\ &= -\frac{V_0}{\rho_0 c^2}\mathrm{d}p\end{aligned}$$

potential energy:

$$\begin{aligned}P &= -\int_{V_0}^{V} p\,\mathrm{d}V \\ &= -\int_{V_0}^{V} p\left(-\frac{V_0}{\rho_0 c^2}\,\mathrm{d}p\right) \\ &= \frac{1}{2}\frac{p^2}{\rho_0 c^2}V_0\end{aligned}$$

instantaneous energy density in Joules per cubic meter:

$$\therefore \epsilon = \frac{1}{2}\rho_0\left(u^2 + \frac{p^2}{\rho_0^2 c^2}\right)$$

6 plane and spherical waves

6.1 plane waves

Plane waves occur if we have motion only along one axis. The solution is the sum of a plane wave traveling in opposite directions The acousitc particle velocity of the mass element is the time derivatice of the displacement of the mass element of air (in the x-direction) ξ.

$$\mathbf{u} = -\int \frac{1}{\rho_0}\frac{\partial p}{\partial x}\mathrm{d}t = \frac{\mathbf{A}}{\rho_0 c}e^{j(\omega t - kx)} - \frac{\mathbf{B}}{\rho_0 c}e^{j(\omega t + kx)}$$

6.2 average energy and intensity level

A wave travelling in the $+x$ direction has an acoustic particle velocity oscillation in phase with the pressure oscillation with the amplitude given by $u = p/\rho_0 c$.

$$\epsilon = \frac{1}{2}\rho_0\left(u^2 + \frac{p^2}{\rho_0^2 c^2}\right) = \frac{p^2}{\rho_0 c^2}$$

$$\bar{\epsilon} = \frac{p_e^2}{\rho_0 c^2} \qquad p_e \text{ is root mean square pressure}$$

average energy flow across a unit area per unit itme:

$$I = \bar{\epsilon}c = \frac{p_e^2}{\rho_0 c}$$

quietest sound humans can hear: $I_0 = 10^{-12}$ Wm^{-2} intensity level in decibels:

$$\mathrm{IL} = 10\ \log\left(\frac{I}{I_0}\right)$$

sound pressure level:

$$p_0 = \sqrt{I_0 \rho_0 c} = 20\ \mu\mathrm{Pa} \quad\longrightarrow\quad \mathrm{SPL} = 20\ \log\left(\frac{p_e}{p_0}\right)$$

6.3 spherical waves

wave equation in spherical polars for p as a function of r only:

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

$$\mathbf{p} = \frac{\mathbf{A}}{\mathbf{r}}\exp\left(j(\omega t - kr)\right)$$

$$\boldsymbol{\nabla} p = -\rho_0\frac{\partial^2 \mathbf{d}}{\partial t^2} \quad\longrightarrow\quad \frac{\partial p}{\partial r} = -\rho_0\frac{\partial^2 \xi}{\partial t^2}$$

$$u = \left(\frac{\frac{1}{r} + jk}{j\omega\rho_0}\right)p$$

6.4 specific impedance

specific acousic impedance: $Z_s = p/u$

For plane waves travelling in the $\pm x$-direction, substituting in gives:

$$Z_s = \rho c \qquad Z_s = -\rho_0 c$$

$\rho_0 c$ is 415 Pa s/m for standard temperature and pressure.

specific acoustic impedance for spherical waves:

$$Z_s = \frac{j\omega\rho_0}{\frac{1}{r} + jk} = \rho_0 c\left(\frac{k^2r^2}{1 + k^2r^2} + j\frac{kr}{1 + k^2r^2}\right)$$

6.5 sources

The simplest source in terms of transmitting spherical waves is a pulsating sphere. (radius a).

radial speed of the sphere at the surface:

$$\mathbf{u}_a = u_0 e^{j(\omega t - ka)}$$

For low enough frequencies $ka \ll 1$ (i.e. $\lambda \gg a$) then the specific acoustic impedance at the surface is:

$$Z_s(r = a) \approx jka\rho_0 c \quad \text{velocity and pressure at } r = a \text{ are } \pi/2 \text{ out of phas}$$

$$u_a \approx \frac{-jp_a}{ka\rho_0 c}$$

$$\mathbf{A} = jka^2\rho_0 c u_0 = jkQ\rho_0 c/4\pi \qquad Q = 4\pi a^2 u_0 \text{ is the source strength}$$

pressure radiated:

$$p = \frac{jkQ\rho_0 c}{4\pi r}e^{j(\omega t - kr)}$$

7 waves in air columns

7.1 volume velocity and acoustic impedance

pressure of plane wave in $+x$-direction:

$$\mathbf{p}_i = \mathbf{A}e^{j(\omega t - kx)}$$

acoustic particle velocity:

$$u = \frac{p}{Z_s} = \frac{p}{\rho_0 c} \quad\longrightarrow\quad \mathbf{U} = S\mathbf{u} \quad \text{is volume velocity in pipe}$$

acoustic impedance:

$$\mathbf{Z} = \frac{\mathbf{p}}{\mathbf{U}}$$

7.2 reflection by a change in impedance

$$\mathbf{p}_r = \mathbf{B}e^{j(\omega t + kx)}$$

At $x < 0$ the acousitic impedance is the sum of teh pressures divided by the sum of teh volume velocities.

$$\mathbf{Z} = \frac{\mathbf{p}_i + \mathbf{p}_r}{\mathbf{U}_i + \mathbf{U}_r} = \frac{1}{S}\left(\frac{\mathbf{p}_i + \mathbf{p}_r}{\frac{\mathbf{p}_i}{\rho_0 c} - \frac{\mathbf{p}_r}{\rho_0 c}}\right)$$

$$\mathbf{Z}(x = 0) = \mathbf{Z}_0 = \frac{\rho_0 c}{S}\left(\frac{\mathbf{A} + \mathbf{B}}{\mathbf{A} - \mathbf{B}}\right)$$

$$\frac{\mathbf{B}}{\mathbf{A}} = \frac{\mathbf{Z}_0 - \frac{\rho_0 c}{s}}{\mathbf{Z}_0 + \frac{\rho_0 c}{s}}$$

In general the impedanc eis complex, which allows for the wave travelling to the $+x$-direction to have any pahse in relatoin to the incident wave.

$$\frac{\mathbf{B}}{\mathbf{A}} = \frac{(R_0 - \rho_0 c/S) + jX_0}{(R_0 + \rho_0 c/S) + jX_0}$$

7.3 power reflection and transmission

$$R_\pi = \left|\frac{\mathbf{B}}{\mathbf{A}}\right|^2 = \frac{(R_0 - \rho_0 c/S)^2 + X_0^2}{(R_0 + \rho_0 c/S)^2 + X_0^2}$$

power transmission coefficent:

$$T_\pi = 1 - R_\pi = \frac{4R_0\rho_0 c/S}{(R_0 + \rho_0 c/S)^2 + X_0^2}$$

7.4 ideal closed end

We set the terminating impedance such that the volume velocity is zero so that $Z_0 = R_0 = \infty$, making $A = B$ so that the pressure wave is reflected back in phase. We will arbitrarily choose the phase to be zero at $x = 0$, $t = 0$ sot aht A is real. The pipe thus contains the sum of travelling waves of equal amplitude propagating in the positive and negative x directions making standing waves.

$$\mathbf{p} = A\left(e^{j(\omega t - kx)} + e^{j(\omega t + kx)}\right) = 2A\ \cos(kx)e^{j\omega t}$$

physically measurable acoustic pressure:

$$p = \mathrm{Re}(\mathbf{p}) = 2A\ \cos(kx)\ \cos(\omega t)$$

volume velocity:

$$\mathbf{U} = A_u\left(e^{j(\omega t - kx)} - e^{j(\omega t + kx)}\right) = -2jA_u\ \sin(kx)e^{j\omega t}$$

$$A_u = A\frac{S}{\rho_0 c} \qquad U = \mathrm{Re}(\mathbf{U}) = 2A_u\ \sin(kx)\ \sin(\omega t)$$

The pressure and volume velocity are found to be $\pi/2$ out of phase in time. The pressure has an anti-node at the closed end while the volume velocity has a node at the closed end.

7.5 ideal open end

The opposite case is $Z_0 = 0$, when we find that the acoustic pressure must be zero at $x = 0$. The condition 9impolied that $A = -B$ meaning that the pressure wave is reflected back with the same magnitude but π out of phase givng the acousitc pressure as a standing wave:

$$\mathbf{p} = A\left(e^{j(\omega t - kx)} + e^{j(\omega t + kx)}\right) = -2jA\ \sin(kx)e^{j\omega t}$$

$$p = \mathrm{Re}(\mathbf{p}) = 2A\ \sin(kx)\ \sin(\omega t)$$

volume velocity:

$$\mathbf{U} = A_u\left(e^{j(\omega t - kx)} - e^{j(\omega t + kx)}\right) = 2A_u\ \cos(kx)e^{j\omega t}$$

$$U = \mathrm{Re}(\mathbf{U}) = 2A_u\ \cos(kx)\ \cos(\omega t)$$

The pressure thus has a node at the open end while the volume velocity has an anti-node.

8 radiation from a pipe

8.1 real and imaginary impedance

Acoustic impedance along the pipe is the ratio of pressure and volume velocity for ideal open end.

$$Z(x) = -j\frac{\rho_0 c}{S}\tan(kx)$$

An imaginary acoustic impedance indicates standing waves and in fact the imaginary part of a complex impedance is large it there are standing waves present. A real acoustic impedance gives rise to travelling waves and in fact the real part of a complex impedance is large if a lot of energy is begin transmitted out of a system rather than being reflected to give standing waves.

8.2 radiation impedance and end correction

Realistic musical sound comes from sound diffraction at the opening.

radiation impedance for the open end of a pipe with an unflanged end:

$$Z_{\mathrm{rad}} \approx \frac{\rho_0 c}{S}\left(\frac{1}{4}(ka)^2 + j(0.6ka)\right)$$

a is the radius of the pipe in the limit where the osurce is much smaller than a wavelength ($ka \ll 1$). In this limit the impedance matches the ideal open end condition of zero only at the zero of frequency and that the imnaginary prat of the impedance is larger than the real part. Therefore, the bulk of the enrgy is reflected from the open end and the smaller real part is responsible for transmission of energy out of the end of the instrument to give sound.

radiation impedance for a flanged open end pipe (recorder):

$$Z_{\mathrm{rad}} \approx \frac{\rho_0 c}{S}\left(\frac{1}{2}(ka)^2 + j(0.8ka)\right)$$

Equating imaginary parts of the ideal and full radiation impedance gets the correct standing wave shapes for an unflanged tube.

$$\tan(kL_c) \approx 0.6ka \longleftarrow\ ka \ll 1 \longrightarrow\ L_c \approx 0.6a$$

Therefore, the pipe witha n iddeal opend end at $x = 0$ has the same standing wave patterns as a pipe with an unflanged end at $x = -L_c$. L_c is ks the length correction and indicates that the presure node at an unflanged end actually lies a distance of $0.6 \cdot a$ beyond the end of the pipe.

The length correction is found to depend on frequency. As frequency increases, the real part becomes larger so that at $ka = 2$ the real part is larger than the imaginary part.

8.3 power reflection and transmission

power reflection and transmission for $ka \ll 1$:

$$R_\pi = \frac{\left(0.25(ka)^2 - 1\right)^2 + (0.6ka)^2}{\left(0.25(ka)^2 + 1\right)^2 + (0.6ka)^2}$$

$$T_\pi = 1 - R_\pi = \frac{(ka)^2}{\left(0.25(ka)^2 + 1\right)^2 + (0.6ka)^2} \approx (ka)^2$$

The power transmission if proportional to frequency squared. Going up an octave doubles the frequency and quadruples the power transmission coefficient. (6 dB per octave increase).

9 cross-section changes and branches

For a pipe that steps thicker at $x = 0$, the pressure and the volume velocity must be continuous over the discontinuity due to the fact that mass of the air must be conserved. Therefore, the acoustic impedance must always be the same on the two sides of a discontinuity.

A wave of magnitude $\mathbf{A}$ goes into the junction, gets partially reflected ($\mathbf{B}$) and partially transmitted ($\mathbf{C}$). Because there are only forward going wves in the pipe on the right of the discontinuity, the impedance there is $\rho_0 c/S_2$. The reflection coefficient in a pipe of cross-section S_1 at an impedance of $\mathbf{Z} - 0$:

$$\frac{\mathbf{B}}{\mathbf{A}} = \frac{\mathbf{Z}_0 - \rho_0 c/S_1}{\mathbf{Z}_0 + \rho_0 c/S_1} \qquad = \frac{\rho_0 c}{S_2}$$
$$= \frac{S_1/S_2 - 1}{S_1/S_2 + 1}$$

By assuming plane wave propagation in both pipes we are clearly ignoring the efects of diffraction so this theory will not work unless the tube radius in both pipes is small compared to a wavelength.

9.1 branches

For a pipe that splits into branches,

continuity of pressure $\longrightarrow \mathbf{p}_i + \mathbf{p}_r = \mathbf{p}_1 = \mathbf{p}_2$.

continuity of volume $\longrightarrow \mathbf{U}_i + \mathbf{U}_r = \mathbf{U}_1 + \mathbf{U}_2$

$$\therefore \frac{\mathbf{U}_i + \mathbf{U}_r}{\mathbf{p}_i + \mathbf{p}_r} = \frac{\mathbf{U}_1}{\mathbf{p}_1} + \frac{\mathbf{U}_2}{\mathbf{p}_2}$$

acoustic impedance:

$$\frac{1}{\mathbf{Z}_0} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}$$

9.2 short side branch

For a pipe of cross-section S_1 as a hsort side branch of acoustic impedance $\mathbf{Z}_b$ in an otherwise cylindrical tube of area $S = S_0 = S_2$:

$$\frac{1}{\mathbf{Z}_0} = \frac{1}{\mathbf{Z}_b} + \frac{S}{\rho_0 c} = \frac{\rho_c/S + \mathbf{Z}_b}{\mathbf{Z}_b \rho_0 c/S}$$

$$\mathbf{Z}_0 = \frac{\mathbf{Z}_b \rho_0 c/S}{\rho_0 c/S}$$

reflection coefficient:

$$\frac{\mathbf{B}}{\mathbf{A}} = \frac{\mathbf{Z}_0 - \rho_0 c/S}{\mathbf{Z}_0 + \rho_0 c/S}$$
$$= \frac{-\rho_0 c/2S}{\rho_0 c/2S + \mathbf{Z}_b}$$
$$= \frac{-\rho_0 c/2S}{\rho_0 c/2S + R_b + jX_b}$$

$$\therefore R_\pi = \frac{(\rho_0 c/2S)^2}{(\rho_0 c/2S + R_b)^2 + X_b^2}$$

$\mathbf{p}_2$ (renamed $\mathbf{C}$) is the wave transmitted down the main tube. By continuity of pressure $\mathbf{A} + \mathbf{B} = \mathbf{C}$:

$$\frac{\mathbf{C}}{\mathbf{A}} = 1 + \frac{\mathbf{B}}{\mathbf{a}} = \frac{R_b + jX_b}{\rho_0 c/2S + R_b + jX_b}$$

fraction of the power transmitted down the main pipe:

$$T_{\pi m} = \left(\frac{\mathbf{C}}{\mathbf{A}}\right)\left(\frac{\mathbf{C}}{\mathbf{A}}\right)^* = \frac{R_b^2 + X_b^2}{(\rho_0 c/2S + R_b)^2 + X_b^2}$$

power transmitted out along the side branch by conserbation of energy:

$$T_{\pi b} = 1 - R_\pi - T_{\pi m} = \frac{R_b \rho_0 c/S}{(\rho_0 c/2S + R_b)^2 + X_b^2}$$

9.3 side holes

For a closed side hole of volume V,

$$\mathbf{Z}_{\text{closed}} \approx -j\left(\frac{\rho_0 c^2}{\omega V}\right)$$

The branch impedance is imagniary and indicates that there is no transmission of sound through the side hole.

power tranmission coefficient for transmission down the main tube:

$$T_{\pi m} \approx \left[1 + \left(\frac{\omega V}{2cS}\right)^2\right]^{-1}$$

A closed tone hole therefore acts as a low pass filter because more or less all the energy is transmitted when $\omega \ll 2cS/V$ whilte more or less all the energy is reflected when $\omega \gg 2cS/V$.

For an open side hole of area πa^2,

$$\mathbf{Z}_{\text{open}} \approx \frac{\rho_0 \omega^2}{2\pi c} + j\left(\frac{\omega \rho_0 t_e}{\pi a^2}\right)$$

$t_e = t + 1.6a$ is the effective length of the side branch due to length correction effects. The impedance is complex, implying that the wave is partially reflected and transmitted at the open hole. For musical instruments the imaginary part is always larger than the real part over the audible spectrum.

$$T_{\pi m} \approx \left[1 + \left(\frac{\pi a^2 c}{2S\omega t_e}\right)^2\right]^{-1}$$

An open tone hole therefore acts as a high pass filter because much energy is tranmitted when $\omega \gg \pi a^2 c/2St_e$ while much energy is reflected when $\omega \ll \pi a^2 c/2St_e$.

cut-off frequency:

$$f_c = \frac{a^2 c}{4\pi r^2 t_e}$$

Opening a tone hole makes the pipe appear to have an open end at around the position of the tone hole for frequencies below the cut-off frequency. This is used to shorted the length of standing waves in the tube giving rise to a harmonic series with higher frequency and therefore higher pitch.

10 input impedance and losses

10.1 projecting the impedance along a pipe

acoustic pressure for forward and backword pressure waves:

$$\mathbf{p} = \mathbf{A}e^{j(\omega t - kx)} + \mathbf{B}e^{j(\omega t + kx)}$$

volume velocity:

$$\mathbf{U} = \frac{S}{\rho_0 c}\left(\mathbf{A}e^{j(\omega t - kx)} - \mathbf{B}e^{j(\omega t + kx)}\right)$$

pressure at plane 0 in terms of plane 1:

$$\mathbf{p}^{(0)} = \left(\mathbf{A}e^{-jk(x_1 - d)} + \mathbf{B}e^{jk(x_1 - d)}\right) e^{j\omega t}$$
$$= \left(\left(\mathbf{A}e^{-jkx_1} + \mathbf{B}e^{jkx_1}\right)\cos(kd) + \left(\mathbf{A}e^{-jkx_1} + \mathbf{B}e^{jkx_1}\right) j\sin(kd)\right) e^{j\omega t}$$
$$= \cos(kd)\mathbf{p}^{(1)} + j\sin(kd) Z_c \mathbf{U}^{(1)} \qquad Z_c = \rho_c/S$$

similarly:

$$\mathbf{U}^{(0)} = j\sin(kd)\mathbf{Z}_c^{-1}\mathbf{p}^{(1)} + \cos(kd)\mathbf{U}^{(1)}$$

$$\mathbf{Z}^{(0)} = \frac{\cos(kd)\mathbf{Z}^{(1)} + j\sin(kd)Z_c}{j\sin(kd)Z_c^{-1}\mathbf{Z}^{(1)} + \cos(kd)}$$

This is the impedance of one end of a cylindrical pipe from the impedance at the other, which can approximate complex geometries with a series of concentric cylinders.

If both ends of the tuype are open then both are of known impedance and the method doesn't work.

10.2 input impedance

The impedance at the mouthpiece is the input impedance tnad gives the amont of pressure that can e produced by a given volume velocity produced by the excitation mechanism. A peak value for the input impedance will indicate that vibrating the reed or lip reed will produce a strong pressure standing wve. The input impedance will depend on tfrequency and will have peaks at resonance frequencies of the instrument which correspond to the playable harmonics.

input impedance for an ideal open end $\mathbf{Z}^{(1)} = 0$:

$$\mathbf{Z}^{(0)}(f) = j\frac{\rho_0 c}{S}\tan(kL) = j\frac{\rho_0 c}{S}\tan\left(\frac{2\pi L}{c}f\right)$$

resonant frequencies:

$$f = \frac{2n-1}{4}\frac{c}{L}$$

10.3 losses and absorption

The wave number will be complex and the two terms produce and oscillation with decaying amplitude.

$$\mathbf{k} = k - j\alpha$$

absorption coefficient in free space for 50% humidity:

$$\alpha \approx 4 \times 10^{-7}\,f \qquad 100\text{ Hz} < f < 1\text{ kHz}$$
$$\alpha \approx 1 \times 10^{-10}\,f \qquad 2\text{ kHz} < f < 100\text{ kHz}$$

It's high for high frequencies in large spaces and accounts for the way that sound at 10 kHz is absorbed at about 0.1 dB per meter.

Loss of energy within a resonating air column happend by radiation out of open ends and side holes. Also by acoustic absorption by the tube walls.

absorption coeffienct at standard temperature and pressure:

$$\alpha = 2.93 \times 10^{-5}\frac{\sqrt{f}}{a}$$

11 the ear and loudness level

audible frequency range: 20 Hz to 20 kHz (amplitude of 1/10 diameter of hydrogen molecule)

The pinna is the 'horn' that feeds into the auditory canal (L ~2.5 cm, $2a$ ~0.7 cm), which resonates with a gain of 20 dB. Appreciable gains happen from 2 to 6 kHz.

On the inside of theear drum is the middle ear, an air cavity ($V \sim 2$ cm^3). In the middle ear are the three bones (ossicles): hammer (malleus), anvil (incus) and stirrup (stapes). Along with their muscles and ligaments. They transfer energy of the ear drum vibration into the inner ear which is fluid-filled, by the stapes pressing in and out on the oval window – a sealed window in the side of the inner ear. Because the oval window is 30 times smaller than the ear drum and because of the lever action of the ossicles, they provide an impedance match to maximize the transfer of energy from the air to the fluid in the inner ear.

Inner ear vibrations set off the basiler membrane in the cochlea. The membrane has 30,000 hair cells that flex from vibrations, triggering electrical impulses from nerve endings. They trigger more frequently for loud sounds.

11.1 basiler membrane displacement

The triangular, 3.5 cm basilar membrane does a mechanical frequency analysis to differentiate between frequencies. It is narrowest at the base and 5 times wider at the apex. It gradually varies in thickness (thinnest at the apex), and there is a 10x decrease in stiffness from the base to the apex.

The basilar membrane responds to a sine wave presure by vibrating like a waving flag with the peak amplitude of vibration at a position along the membran'es length determined by the frequency of the sine wave. Low frequencies lead to a peak vibration near the apex. The amplitude of peak displacement rises slowly with distance and dies off quickly after the maximum.

11.2 thresholds

The threshold of audibility is the minimum perceptible free-field intensity level of a sine wave lasting 1 second presented to both ears and is a function of frequency.

Sensitivity is greatest around 4 kHz while the minimum power required to hear a sound at 30 Hz is nearly a million times greater.

11.3 loudness level in Phons

All sounds of the same loudness as a tone at a given number for the intensity level will have the same number of Phons. A sound at 60 dB at 1 kHz has loudness level in Phons of 60 dB and our sound at 88 dB at 30 Hz will also have a loudness level in Phons of 60 dB as it shares an equal loudness.

11.4 critical bandwidth

Due to the finite region over which the basilar membrane responds to a pure tone vibration, two sound will interact strongly if their frequencies lie close enough together. If the vibrations of the basilar membrane due to the two tones overlap they are said to lie within one critical bandwidth. The ear acts as a set of parallel frequency filter and phenomena such as beating, roughness and masking are observed between any two pure tones within the same critical bandwidth.

12 perception of music

12.1 stero and direction perception

Stereophonic sound perception happens through either (or both) an intensity difference between the ears or a time delay.

Stero sound reproduction is normally achieved using loudspeakers approximately six feet apart.

The most satisfactory stero image will be achieved when the cue from timing and level are in agreement (signals arrive first and loudest to the left ear for sound generated to our left etc.).

this is effectively achieved using two directional or "cardioid microphones placed in a similar configuration to our ears: typically several inches apart and pointing outwards by 45° to their common axis.

Placing microphones apart introduces destructive interference for sound directions and wavelengths where the two microphones record the opposite phase. So we place the microphones pointing outwards but so close together that they are effectively in the same place, although this looses us the timing information and lessens some of the convincing nature stereo image.

12.2 intervals between harmonics

Two pitched musical sounds will sound particularly consonant if their harmonic series coincide at some shared harmonics. The ratios 7:6 and so on sound much rougher due to the fact that the fundamentals lie closer than one critical band.

frequency ratio	just intonation interval
2:1	Octave
3:2	Perfect Fifth
4:3	Perfect Fourth
5:4	Major Third
6:5	Minor Third

12.3 piano keys and scales

you should know this

12.4 the cycle of fifths and temperament

Going up the circle of fifths all the way gives goes through seven octaves.

$$2^7 = 128 \quad \text{but} \quad \left(\frac{3}{2}\right)^{12} = 129.74633789...$$

Temperaments are tuning methods to solve this problem that it is impossible to tune all of the notes on a piano so that all the fifths obey the just intonation ratioss (and still preserve the octave at 2:1).

Today's standard temperament is equal temperament. All the intervals of a fifth should have the same frequency ratio and that the octave always has a frequency ratio of 2. This impolies that a cycle of fifths is equal to seven octaves and thus produces a factor equal to 128.

frequency ratio to 6 s.f.	equal temperament interval
$2^{1/12} = 1.05946$	Minor Second
$2^{2/12} = 1.12246$	Major Second
$2^{3/12} = 1.18921$	Minor Third
$2^{4/12} = 1.25992$	Major Third
$2^{5/12} = 1.33480$	Perfect Fourth
$2^{6/12} = 1.41421$	Augmented Fourth
$2^{7/12} = 1.49831$	Perfect Fifth
$2^{8/12} = 1.58740$	Minor Sixth
$2^{9/12} = 1.68179$	Major Sixth
$2^{10/12} = 1.78180$	Minor Seventh
$2^{11/12} = 1.88775$	Major Seventh
2	Minor Second

$$\text{pitch difference (cents)} = 1200\frac{\log(f/f_0)}{\log 2}$$

13 reverberation

13.1 reverberation and growth of sound

Echoes happen. Pulses of echoes die down with wall absorbtion. Sustained sounds have rising amplitudes with reverb blend.

Exponential decay when sustained sound ceases.

13.2 reverberation time

reverberation time: the time taken for a reverberant field to decay by 60 dB – depends on frequency Unnecessary to measure full decay; just double 30 dB decay to avoid background noise level

time taken for a sound to decay by x dB:

$$t = \frac{x}{60}T_r$$

13.3 energy in diffuse sound

The energy in a volume elemenet $\mathrm{d}V$ is $\varepsilon\ \mathrm{d}V$ where ε is the average acoustic eenrgy density.

$$\mathrm{d}E = \varepsilon\ \mathrm{d}V\frac{\mathrm{d}S\ \cos\theta}{4\pi r^2}$$

$$\begin{aligned}
\int_R \mathrm{d}E &= \frac{\varepsilon\ \mathrm{d}S}{4\pi}\int_R \frac{\cos\theta}{r^2}\ \mathrm{d}V \\
&= \frac{\varepsilon\ \mathrm{d}S}{4\pi}\int_R \frac{\cos\theta}{r^2}r^2\sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi \\
&= \frac{\varepsilon\ \mathrm{d}S\ \mathrm{d}r}{4\pi}2\pi\int_0^{\pi/2}\cos\theta\ \sin\theta\ \mathrm{d}\theta \\
&= \frac{\varepsilon\ \mathrm{d}S\ \mathrm{d}R}{4}
\end{aligned}$$

$$\frac{\varepsilon\ \mathrm{d}S\ \mathrm{d}r}{4}\frac{c}{\mathrm{d}r} = \frac{\epsilon c\ \mathrm{d}S}{4}$$

energy per unit area incident on wall:

$$I = \frac{\varepsilon c}{4}$$

13.4 absorption

absorption coefficient: the ratio of the absorbed to the incident sound, α ranging from 0 (perfect) to 1 (open window)

total absorption for a room:

$$A = \sum_i \alpha_i S_i$$

13.5 Sabine Formula

rate of energy absorption is the sum of teh intensity of the field multiplied by the total absorption:

$$IA = \frac{\varepsilon c A}{4}$$

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon V) &= -V\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \\
\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} &= -\frac{\varepsilon c A}{4V}
\end{aligned}$$

$$\therefore \varepsilon = \varepsilon_0\exp\left(-\frac{Ac}{4V}t\right) \qquad I = I_0\exp\left(-\frac{Ac}{4V}t\right)$$

$\varepsilon = \varepsilon_0$ and $I = I_0$ at $t = 0$.

change in intensity level with time:

$$\Delta IL(t) = 10 \log\left(\frac{I}{I_0}\right) \approx \frac{10}{2.3} \ln\left(\exp\left(-\frac{Ac}{4V}t\right)\right) = -\frac{1.09Ac}{V}t$$

$$-60 \approx -\frac{1.09Ac}{V}T_r$$

Savine formula:

$$T_r \approx \frac{55V}{Ac} \approx \frac{0.16V}{A}$$

14 synthesis

14.1 additive synthesis

In the 19th century, Helmholtz thought that sounds are found to consist of a combination of pure tone components.

Telharmonium and the Hammond Organ used rotating metal cylinders or cogs.

14.2 subtractive synthesis

Bob Moog in 1964 started with a chunk of marble of a complex waveform, and filtered and chiseled his way through it.

Triangle, saw-tooth, square wave, and pulse waves were generated using analogue transistor based circuits called Voltage Controlled Oscillators (VCO). They are filtered with VCF's whose loudness is controlled with a VCA which are set to follow an envelope.

14.3 digital synthesis

The Prophet-5 uses a microprocessor for storing sttings nad for tuning. Roland's 1984 Juno-106.

14.4 frequency modulation synthesis (FM)

Using low-frequency oscillators (LFOs) to oscillate oscillators. Carrier frequencies and modulation frequencies abound.

14.5 physical modeling synthesis

In the case of a string instrument, the displacement of the string at all points along the instrument are held in computer memory and the laws of physics are used to see how the string will vibrate.

14.6 wavetable synthesis and sampling

The basis of wave table syntesis is the lookup table, which containt the data ofr one typical cycle of a wave for a specific instrument. They can be looped and enveloped.

Sampling is sampling.

15 Case Study: the violin

15.1 The Bow

15.2 The Strings

Strings are G_3, D_4, A_4, E_5.

Tensions vary from 34.8 N to 84.0 N.

frequency of the n^{th} mode:

$$f_n = \frac{\omega}{2\pi} = \frac{kc}{2\pi} = \frac{nc}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\rho_L}}$$

The bow is wide compared to the size of the nodal region. The proportion of the harmonics in the spectrum of a violin is, however, predominantly dtermined by the resonances of the wood and air in the body.

16 case study: guitar

Fourier coefficients of a pulse wave:

$$A_n = \frac{1}{n\pi}\sin\left(\frac{2n\pi}{a}\right) \quad \text{and} \quad B_n = \frac{1}{n\pi}(1 - \cos\left(\frac{2n\pi}{a}\right))$$