Advanced Quantum Mechanics
2020-21

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Part I: Complex Analysis

1.1 Differentiation in the complex plane: the Cauchy-Riemann equations

1.1.1 Introduction

In this part we digress from the main theme of this course on Advanced Quantum Mechanics to introduce a set of important mathematical tools: how to differentiate and integrate functions of a complex variable. We study this topic not really for its own sake, but for the neat method it gives us of performing integrals in real analysis, such as those we have just encoutered in deriving Green's functions in scattering theory. Such integrals would otherwise be difficult (or sometimes, to the best of our knowledge, impossible) to evaluate.

1.1.2 Notation

Our position in the complex plane will be denoted by a complex number z, which we may write as

$$z = x + iy, \tag{1}$$

where x is the real part of z ($x = \operatorname{Re} z$) and y is the imaginary part ($y = \operatorname{Im} z$).[1] A function of this complex variable will in general itself be complex, and hence can also be written as a sum of a real and imaginary part,

$$f(x, y) = u(x, y) + iv(x, y). \tag{2}$$

1.1.3 When is a function of a complex variable differentiable?

When considering a function $g(x)$ of a real variable x, we would define the derivative as

$$\frac{dg}{dx} \equiv \lim_{\delta \to 0} \left(\frac{g(x + \delta) - g(x)}{\delta} \right). \tag{3}$$

Strictly, we should specify the direction in which we approach the point x: we could take the limit $\delta \to 0^+$ (approaching it from the right) or the limit $\delta \to 0^-$ (approaching it from the left). However, for a differentiable function these give the same answer; if they don't give the same answer (see Fig. 1 for an example), we say that the function is not differentiable at the point x.

In the complex plane, this issue of the direction from which to approach the point of interest becomes rather more severe, since in the two-dimensional plane there is an

[1] Note that the imaginary part of z is a real number, i.e. that the i part is not included.

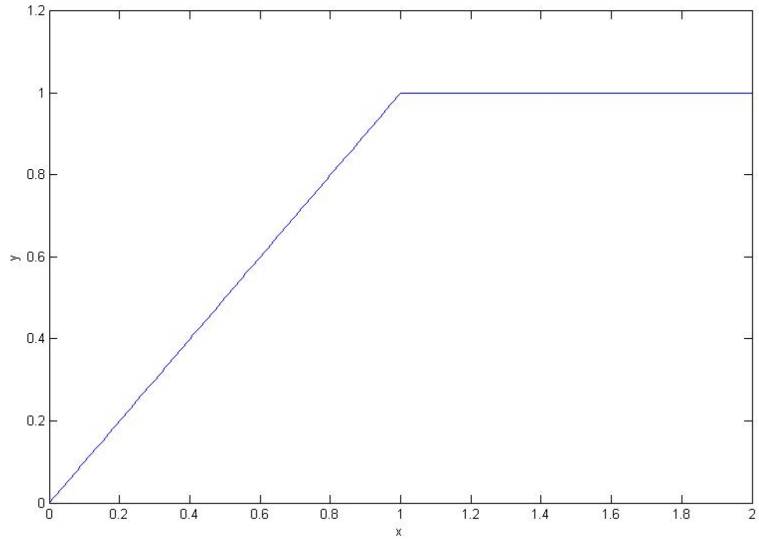

Figure 1: An example of a function which is not differentiable at the point $x = 1$. Both of the limits $\delta \to 0^-$ and $\delta \to 0^+$ exist, but they give different answers.

infinite number of directions we could choose. Copying our approach in the case of real analysis, we say that a function $f(z)$ as differentiable (or **analytic**) at a certain point if it doesn't matter from which direction we approach that point when calculating the derivative. This requirement places some constraints on the functions $u(x, y)$ and $v(x, y)$, which we shall now derive.

1.1.4 The Cauchy-Riemann equations

Let us try to evaluate the derivative of f at a point $z = x + iy$ in the complex plane. It turns out to be sufficient to consider only two orthogonal directions of approach, and the most natural choices are to approach the point along the x-direction and to approach it along the y-direction. Let us calculate the derivative (using the usual definition) for each of these choices.

1. *Approaching along* x. The derivative of the function f with respect to the complex variable z is written, by direct analogy with the real case, as

$$\frac{df}{dz} = \lim_{\delta \to 0} \left(\frac{f(z+\delta) - f(z)}{\delta} \right). \tag{4}$$

In the case when we approach along x, δ is just a real variable, so $z+\delta = (x+\delta)+iy$. Hence

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y); & (5) \\ f(z+\delta) &= u(x+\delta, y) + iv(x+\delta, y). & (6) \end{aligned}$$

But since δ is infinitesimal, we may Taylor-expand $u(x+\delta, y)$ and $v(x+\delta, y)$:

$$u(x+\delta, y) = u(x,y) + \delta \frac{\partial u}{\partial x}(x,y) + O(\delta^2); \tag{7}$$
$$v(x+\delta, y) = v(x,y) + \delta \frac{\partial v}{\partial x}(x,y) + O(\delta^2). \tag{8}$$

Hence

$$f(z+\delta) - f(z) = \delta \frac{\partial u}{\partial x} + i\delta \frac{\partial v}{\partial x} + O(\delta^2), \tag{9}$$

and thus

$$\frac{df}{dz} = \lim_{\delta \to 0} \left(\frac{f(z+\delta) - f(z)}{\delta} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \tag{10}$$

2. *Approaching along y.* In the case where we approach along y, δ is purely imaginary; let us say that $\delta = i\epsilon$. So in this case our small displacement gets added to y rather than x:

$$f(z) = u(x,y) + iv(x,y); \tag{11}$$
$$f(z+\delta) = u(x, y+\epsilon) + iv(x, y+\epsilon). \tag{12}$$

Again, since ϵ is infinitesimal, we may Taylor-expand $u(x, y+\epsilon)$ and $v(x, y+\epsilon)$:

$$u(x, y+\epsilon) = u(x,y) + \epsilon \frac{\partial u}{\partial y}(x,y) + O(\epsilon^2); \tag{13}$$
$$v(x, y+\epsilon) = v(x,y) + \epsilon \frac{\partial v}{\partial y}(x,y) + O(\epsilon^2). \tag{14}$$

Hence

$$f(z+\delta) - f(z) = \epsilon \frac{\partial u}{\partial y} + i\epsilon \frac{\partial v}{\partial y} + O(\epsilon^2). \tag{15}$$

However, this time when we divide by δ we are dividing by $i\epsilon$, and hence

$$\frac{f(z+\delta) - f(z)}{\delta} = \frac{f(z+\delta) - f(z)}{i\epsilon} = -i \frac{f(z+\delta) - f(z)}{\epsilon} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} + O(\epsilon). \tag{16}$$

Thus

$$\frac{df}{dz} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \tag{17}$$

But if the function is analytic at the point (x, y), then these two results must agree. This implies that the real and imaginary parts of Eq. 10 and Eq. 17 must separately be equal to each other, i.e. that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{18}$$

These are the **Cauchy-Riemann equations**. If they are satisfied at a point (x, y), then the function is said to be analytic (= differentiable) at that point.

1.1.5 The 'function-of-z' test

There is another simple way of checking whether a function of a complex variable is analytic, which turns out to be equivalent to the Cauchy-Riemann approach but is often much easier to use. This is the so-called 'function-of-z' test. It relies on the following property: If a function of a complex variable can be written using only z (and without using its conjugate, z^*), then the function is analytic.

For example, consider the function

$$f(z) = z^3. \tag{19}$$

This can clearly be written using z only (we just have done!), so it passes the 'function-of-z' test. Let's show that it satisfies the Cauchy-Riemann equations. Expanding f in terms of x and y, we obtain

$$\begin{aligned} f(x,y) &= (x+iy)^3 && (20)\\ &= x^3 + 3ix^2y - 3xy^2 - iy^3. && (21) \end{aligned}$$

Hence the real and imaginary parts of f are given by

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2; && (22)\\ v(x,y) &= 3x^2y - y^3. && (23) \end{aligned}$$

The relevant partial derivatives are

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2; & \frac{\partial u}{\partial y} &= -6xy;\\ \frac{\partial v}{\partial x} &= 6xy; & \frac{\partial v}{\partial y} &= 3x^2 - 3y^2. \end{aligned} \tag{24}$$

Comparing with Eq. 18, we can see that the Cauchy-Riemann equations are indeed satisfied. It is not too hard to extend this to a proof that any function that passes the 'function-of-z' test will also satisfy the Cauchy-Riemann equations.

1.2 Singularities of meromorphic functions: poles and their types

1.2.1 Introduction

Having explored what it means for a function of a complex variable to be analytic at a point in the complex plane, we now move on to define two important classes of functions of a complex variable. These are **holomorphic** and **meromorphic** functions. We will go on to discuss their properties we shall also investigate their properties and introduce the idea of **pole classification** in this section.

1.2.2 Holomorphic functions

If a function of a complex variable is analytic everywhere in the complex plane, it is said to be **holomorphic**. Here are a few examples of holomorphic functions:

- $f(z) = z^4$;
- $g(z) = \sin(z)$;
- $h(z) = \cos(z^2 + 3)$;
- $p(z) = e^{-z^2}$;
- $q(z) = z^3 - 3z + 4i$.

If you would like an exercise in complex analysis, you could check explicitly that each of these functions satisfies the Cauchy-Riemann equations Eq. 18 at every point in the complex plane. (Though of course you don't need to, since they clearly all pass the function-of-z test.)

Holomorphic functions are allowed to have zeros; indeed, all of the examples above except the fourth one have one or more zeros at certain points in the complex plane. (Again, if you'd like an exercise, find those zeros!) However, they are not allowed to diverge (i.e. go to plus or minus infinity) anywhere.

1.2.3 Meromorphic functions

If a function is analytic everywhere in the complex plane *except at a discrete set of points*, it is said to be **meromorphic**. The points where it is not differentiable are called its **singular points**. There are several ways in which a point may be singular, but one of the most common is that the function might diverge there. A divergence at a certain point means that the Cauchy-Riemann equations are not satisfied there, because the derivatives involved in Eq. 18 can't even be defined. It is possible for a function to pass the function-of-z test, but still be meromorphic rather than holomorphic — so always look explicitly to see whether the function has any singular points!

Here are some meromorphic functions, with a brief discussion of where their singularities are and why. I have included, at the end, a couple of examples where the singularity is *not* due to a divergence, just for your general edification; but we shall say no more about these examples.

- $f(z) = \dfrac{1}{z}$. This clearly diverges when $z = 0$, and hence it has a singular point at the origin.

- $g(z) = \dfrac{1}{z^2+1}$. This diverges when the denominator is zero, i.e. when $z^2 = -1$. The solutions to this equation are $z = \pm i$. Therefore this function has two singular points: one at $z = i$, and the other at $z = -i$.

- $h(z) = \tan z$. This may be written $\dfrac{\sin z}{\cos z}$, and hence it diverges whenever $\cos z = 0$. This occurs when $z = \pi\left(n + \frac{1}{2}\right)$, where n is any integer. Thus the singular points of $\tan z$ are infinite in number, consisting of a set of equally-spaced points on the real axis.

- $s(z) = \tanh z$. This may be written $\dfrac{\sinh z}{\cosh z}$, and hence it diverges whenever $\cosh z = 0$. Using the identity[2] $\cosh z = \cos(iz)$, it follows that the singular points are wherever $\cos(iz) = 0$. This occurs when $iz = \pi\left(n + \frac{1}{2}\right)$, where n is any integer. Multiplying through by $-i$, this becomes $z = -i\pi\left(n + \frac{1}{2}\right)$. Thus the singular points of $\tanh z$ are also infinite in number, consisting of a set of equally-spaced points on the imaginary axis.

- $p(z) = \sqrt{z}$. This doesn't diverge anywhere, but nonetheless it has a singular point at the origin. The reason is that the function is not single-valued: if you go round the origin once, you switch from the $+\sqrt{z}$ to the $-\sqrt{z}$ branch. Because of this, the origin is called a **branch point**. I thought I ought to mention this kind of singularity, though we won't discuss branch points any further in this course (except briefly in the next bullet point).

- $q(z) = \ln z$. The logarithm is particularly nasty at the origin. Not only does it diverge there, but it is also a multivalued function with $z = 0$ as its branch point. To see this, consider the following reasoning. If $q = \ln z$, then $z = e^q$. But since $e^{2i\pi} = 1$, I can add $2i\pi$ to q without changing z. The same applies to any multiple of $2i\pi$, which means that if $q = \ln z$, then $q + 2in\pi = \ln z$ too! Thus the logarithm has an infinite number of branches, for which the origin is the branch point.

1.2.4 Poles and their classification

Consider a meromorphic function $f(z)$, and let $z = z_0$ be one of its singular points. Near that point, let's say it diverges in the following way:

$$f(z) \sim \frac{1}{(z-z_0)^\alpha}, \tag{25}$$

[2] If you don't already know this, you can prove it easily using the expressions for $\cosh z$ and $\cos z$ as sums of exponentials.

where α is an integer. α is referred to as the **order** of the pole: a pole with $\alpha = 1$ is a pole of order one (or **simple pole**); a pole with $\alpha = 2$ is a pole of order two (or **double pole**); and so on.

An important ingredient of complex analysis is to be able to find the poles of a meromorphic function, and to classify them according to their order. Let's take an example:

$$f(z) = \frac{1}{z^2 + (1-i)z - i}. \tag{26}$$

This passes the function-of-z test, so it must be either holomorphic or meromorphic. To check whether it has divergences, we can look at whether the denominator ever goes to zero:

$$z^2 + (1-i)z - i = 0. \tag{27}$$

This may be factorised:[3]

$$(z+1)(z-i) = 0, \tag{28}$$

so that the denominator vanishes (and therefore $f(z)$ diverges) when $z = -1$ or $z = i$. Near the pole $z = -1$, the factor $(z - i)$ is approximately constant, and so

$$f(z) \sim \frac{1}{z+1}; \tag{29}$$

hence this is a simple pole. Similar reasoning shows that the pole at $z = i$ is also simple. Hence $f(z)$ has two simple poles (one at $z = i$, one at $z = -1$), and is otherwise analytic everywhere.

Let's take a slightly trickier example:

$$g(z) = \frac{\coth z}{z}. \tag{30}$$

This passes the function-of-z test, but clearly has points where it diverges; therefore it is a meromorphic function. Where are those divergences?

One obvious candidate is $z = 0$.[4] This looks from Eq. 30 as if it is a simple pole, but don't be fooled! — let's check explicitly. The function $g(z)$ may be written more explicitly as

$$g(z) = \frac{\cosh z}{z \sinh z}. \tag{31}$$

Now we see that actually *both* factors in the denominator vanish at the origin. Furthermore, near the origin $\sinh z \approx z$, and $\cosh z \approx 1$, so that

$$g(z) \sim \frac{1}{z^2}, \tag{32}$$

i.e. this function actually has a *double* pole at $z = 0$.

[3] If you couldn't do that by inspection, you could always use the quadratic equation to find the roots.

[4] However, you need some analysis to prove that! For example, the function $(\tanh z)/z$ also apparently has a divergence at $z = 0$, but it doesn't really, because $\tanh z$ vanishes there.

Are there other divergences? Yes, because there are other places where the $\sinh z$ part vanishes. Specifically:

$$\begin{aligned}
& \sinh z = 0 && (33)\\
\longrightarrow\ & e^z - e^{-z} = 0 && (34)\\
\longrightarrow\ & e^z = e^{-z} && (35)\\
\longrightarrow\ & e^{2z} = 1 && (36)\\
\longrightarrow\ & 2z = \ln 1 && (37)\\
\longrightarrow\ & 2z = 2in\pi. && (38)
\end{aligned}$$

So $g(z)$ also has an infinite string of equally spaced poles up the imaginary axis!: $z = in\pi$. Let us classify these poles. To do so, let $z = z_0 + \epsilon$, where z_0 is the location of one of the poles, and ϵ is small. We can then Taylor-expand $\sinh z$ around the pole in question:

$$\sinh(z_0 + \epsilon) = \sinh(z_0) + \epsilon \cosh(z_0) + O(\epsilon^2). \tag{39}$$

But since $z_0 = in\pi$, $\sinh(z_0) = 0$ (which is what made z_0 a pole in the first place!), so

$$\sinh(z_0 + \epsilon) \approx \epsilon \cosh(z_0). \tag{40}$$

Now,

$$\begin{aligned}
\cosh(z_0) &= \frac{1}{2}\left(e_0^z + e^{-z_0}\right) && (41)\\
&= \frac{1}{2}\left(e^{in\pi} + e^{-in\pi}\right) && (42)\\
&= \frac{1}{2}\left((-1)^n + (-1)^n\right) && (43)\\
&= (-1)^n, && (44)
\end{aligned}$$

so that

$$\sinh(z_0 + \epsilon) \approx (-1)^n \epsilon. \tag{45}$$

Substituting this into Eq. 31, we find that near the pole z_0

$$g(z) \approx \frac{(-1)^n}{in\pi(-1)^n \epsilon} = \frac{1}{in\pi}\frac{1}{\epsilon}. \tag{46}$$

The exponent of ϵ in this expression is 1, and hence this is a simple pole.

Thus the function $g(z)$ has a double pole at the origin, and simple poles at $z = in\pi$, where n is any non-zero integer. Anywhere else in the complex plane, it is analytic.

1.3 Integration in the complex plane: the residue theorem

1.3.1 Introduction

In the last two sections of this part, we have derived the conditions for a function to be differentiable in the complex plane, introduced the idea of meromorphic functions, and learned how to classify their singular points. In this section we shall show that the integral of a holomorphic function round a closed contour is always zero, and that the integral of a meromorphic function around a closed contour depends on whether the contour encloses any of its poles. This will lead us to the famous *residue theorem*.

1.3.2 Integration in the complex plane

Integration in the complex plane is a bit like line integration with vectors, but without the dot product. In particular, to integrate a function in the complex plane one must specify a **contour** along which the integration will be taken. Then the integral can just be completed directly, using $dz = dx + i\,dy$.

For example, consider the integral

$$\int_C f(z)dz, \tag{47}$$

where $f(z) = z^2 + 4$ and C is the straight-line contour from the origin to $z = i$. Along this contour, $z = iy$, and hence $dz = i\,dy$. Substituting these into Eq. 47, we find

$$\int_C f(z)dz = \int_0^1 \left((iy)^2 + 4\right) i\,dy = i \int_0^1 \left(-y^2 + 4\right) dy = i\left[-\frac{1}{3}y^3 + 4y\right]_0^1 = \frac{11i}{3}. \tag{48}$$

1.3.3 Integration of a holomorphic function round a closed loop

Consider now the integral of a holomorphic function $f(z)$ round a closed contour. To begin with, let's take that closed contour to be an infinitesimal square. The four parts of this contour are: C_1, a straight line from $x_0 + iy_0$ to $(x_0 + dx) + iy_0$; C_2, a straight line from $(x_0 + dx) + iy_0$ to $(x_0 + dx) + i(y_0 + dy)$; C_3, a straight line from $(x_0 + dx) + i(y_0 + dy)$ to $x_0 + i(y_0 + dy)$; and C_4, a straight line from $x_0 + i(y_0 + dy)$ to $x_0 + iy_0$. Evaluating the first of these contributions explicitly, we obtain:

$$\int_{C_1} f(z)dz = \int_{x_0}^{x_0+dx} f(z)dx \tag{49}$$

$$= \int_{x_0}^{x_0+dx} u(x, y_0)dx + i \int_{x_0}^{x_0+dx} v(x, y_0)dx \tag{50}$$

$$= u(x_0, y_0)dx + iv(x_0, y_0)dx, \tag{51}$$

where to get to the last line we have used the fact that there is no integral to do, since the interval of integration is already infinitesimal. Similarly, we obtain for the other contributions:

$$\begin{aligned}
\int_{C_2} f(z)dz &= iu(x_0+dx,y_0)dy - v(x_0+dx,y_0)dy; & (52)\\
\int_{C_3} f(z)dz &= -u(x_0,y_0+dy)dx - iv(x_0,y_0+dy)dx; & (53)\\
\int_{C_4} f(z)dz &= -iu(x_0,y_0)dy + v(x_0,y_0)dy. & (54)
\end{aligned}$$

Collecting these terms together, we find

$$\begin{aligned}
\oint_C f(z)dz &= \left[u(x_0,y_0) + iv(x_0,y_0) - u(x_0,y_0+dy) - iv(x_0,y_0+dy)\right]dx \\
&\quad + \left[iu(x_0+dx,y_0) - v(x_0+dx,y_0) - iu(x_0,y_0) + v(x_0,y_0)\right]dy.
\end{aligned} \tag{55}$$

Using Taylor's theorem to write[5]

$$u(x_0,y_0+dy) \approx u(x_0,y_0) + \frac{\partial u}{\partial y}dy, \tag{56}$$

and similarly in the other terms, we find that this becomes

$$\oint_C f(z)dz = \left[\left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right)\right]dx\,dy. \tag{57}$$

But if $f(z)$ satisfies the Cauchy-Riemann equations, then both the real and imaginary parts of this expression are zero. And since any contour can be made up out of infinitesimal squares, the integral of a holomorphic function around any closed contour must also vanish!

This result is called **Cauchy's integral theorem**:

$$\oint_C f(z)dz = 0 \tag{58}$$

for any holomorphic $f(z)$.

This motivates us to think about meromorphic functions; might we get a non-zero answer if the contour of integration encloses a non-analytic point?

[5] we can use Taylor's theorem without worrying here since u is a real function of two real variables. Later on we will see how to derive Taylor-like series for functions of a complex variable.

1.3.4 Cauchy's integral formula

To begin our discussion of integration around a non-analytic point, let us consider the following:

$$\oint_C g(z)\, dz = \oint_C \frac{f(z)}{z - z_0} dz \tag{59}$$

where $f(z)$ is analytic on the contour C, and everywhere within the interior of C. However, the denominator means that the entire integrand is not analytic at the point $z = z_0$, which we assume lies inside C. Thus the function $g(z)$ is meromorphic and we cannot apply Cauchy's integral theorem.

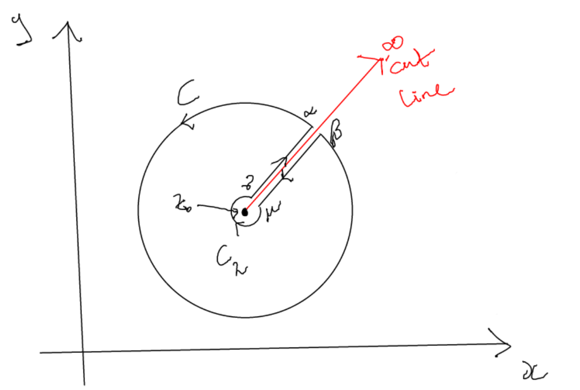

Figure 2: Contour used to exclude a singular point at z_0.

However, we can be sneaky and avoid the pole in our integration, by deforming a contour to go around it - see Fig. 2. C is the original contour, but we make a *cut line* the runs from z_0 to outside C. Now we run our deformed contour either side of the cut line, and encircle the pole with a (cut) circle C_2. Let us call the deformed contour C'. Now we can apply Eq. 58 to C', since C' does not enclose any poles:

$$\oint_{C'} \frac{f(z)}{z - z_0} dz = \int_\alpha^\beta \frac{f(z)}{z - z_0} dz + \int_\beta^\mu \frac{f(z)}{z - z_0} dz + \int_\mu^\nu \frac{f(z)}{z - z_0} dz + \int_\nu^\alpha \frac{f(z)}{z - z_0} dz \tag{60}$$

$$= 0. \tag{61}$$

However, the second and fourth terms on the RHS of the first equation corresponds to two sections that run along the cut line (in opposite directions). They can be made arbitrarily close to one another, and since the integrand is analytic at all points along the line (remembering the pole is excluded), then these integrals must cancel each other. The other two integrals are around the cut circles, which are infinitesimally close to full circles. Therefore we can write for the smaller circular contour (the 3rd term on the RHS above):

$$\int_\mu^\nu \frac{f(z)}{z - z_0} dz = -\oint_{C_2} \frac{f(z)}{z - z_0} dz. \tag{62}$$

Where we have used a negative since since we will define the symbol $\oint$ to mean integration in an anti-clockwise direction, and the inner circle needs integrating in the clockwise direction. We now evaluate this integral by switching to polar coordinates relative to the position of the pole: $z = z_0 + re^{i\theta}$:

$$\oint_{C_2} \frac{f(z)}{z - z_0} dz = \oint_{C_2} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta. \tag{63}$$

Note that we only need to vary θ as we move around the circle: r is fixed. Taking the limit as $r \to 0$ leads to:

$$\oint_{C_2} \frac{f(z)}{z - z_0} dz = if(z_0) \int_0^{2\pi} d\theta = 2\pi i f(z_0). \tag{64}$$

Therefore, from Eq. 67, we have

$$\oint_{C'} \frac{f(z)}{z - z_0} dz = \int_\alpha^\beta \frac{f(z)}{z - z_0} dz + \int_\nu^\alpha \frac{f(z)}{z - z_0} dz \tag{65}$$

$$= \oint_C \frac{f(z)}{z - z_0} dz - \oint_{C_2} \frac{f(z)}{z - z_0} dz \tag{66}$$

$$= \oint_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = 0 \tag{67}$$

and thus

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0). \tag{68}$$

This is **Cauchy's integral formula** (as opposed to Cauchy's integral theorem which was derived in the previous section!) and it really is a remarkable result. It tells us that once the values of an analytic function on a surrounding contour are known, then the value of a function at a point inside the contour is determined.

Cauchy's integral formula may be extended to find derivatives of $f(z)$ at the point z_0 too. To see this, consider the first derivative $f'(z_0)$, which is the limit as δz_0 goes to zero of:

$$\frac{f(z_0 + \delta z_0) - f(z_0)}{\delta z_0} = \frac{1}{2\pi i \delta z_0} \left[\oint \frac{f(z)}{z - z_0 - \delta z_0} dz - \oint \frac{f(z)}{z - z_0} dz \right] \tag{69}$$

and thus, when we take the limit $\delta z_0 \to 0$ we get

$$f'(z_0) = \lim_{\delta z_0 \to 0} \frac{1}{2\pi i \delta z_0} \oint \frac{\delta z_0 f(z)}{(z - z_0 - \delta z_0)(z - z_0)} dz, \tag{70}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz. \tag{71}$$

One can repeat this process to find the second derivative:

$$f''(z_0) = f^{(2)}(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^3} dz, \tag{72}$$

and then to the nth order derivative:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz. \tag{73}$$

Thus we can conclude that since we have defined $f(z)$ to be analytic in the region containing z_0 then this guarantees that all derivatives of $f(z)$ are analytic in that region too.

1.3.5 Laurent series

We know that real functions can be expanded in a Taylor series around a desired point, and over a region of convergence. In this section we will derive the complex variable analogy of this: the Laurent series.

Let us first try find a Taylor series for a function of a complex variable $f(z)$, around an expansion point z_0 and in some region where we know $f(z)$ is analytic. This is the first step to the Laurent series, which will allow us to find a series expansion when z_0 is the position of a pole.

Let's imagine that the nearest non-analytic point to z_0 is z_1. Let us construct a circle that is centred on z_0 and intersects z_1 (see Fig. 3). Now let's draw another circle, labelled C, that is also centred on z_0 but with a smaller radius. A point on this circle will be called z' and so by definition $|z' - z_0| < |z_1 - z_0|$.

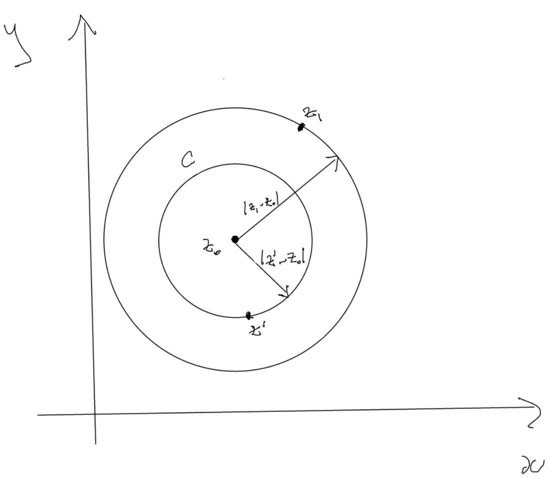

Figure 3: We develop a Taylor expansion about z_0; the nearest non analytic point to z_0 is z_1.

Now, by Cauchy's integral formula, Eq. 68, we know that for a point z within C we can write

$$f(z) = \frac{1}{2\pi i} \oint\limits_C \frac{f(z')}{z'-z} dz' \tag{74}$$

$$= \frac{1}{2\pi i} \oint\limits_C \frac{f(z')}{(z'-z_0)-(z-z_0)} dz' \tag{75}$$

$$= \frac{1}{2\pi i} \oint\limits_C \frac{f(z')}{(z'-z_0)\left[1-\dfrac{z-z_0}{z'-z_0}\right]} dz'. \tag{76}$$

The term in the denominator in square brackets can be expanded since it is a geometric series of the form

$$\frac{1}{1-x} = 1 + x + x^2 + ... = \sum_{n=0}^{\infty} x^n. \tag{77}$$

This series is convergent for $|x| < 1$, and we know $|z-z_0| < |z'-z_0|$ and so we are justified in using the expansion. Then Eq. 76 becomes

$$f(z) = \frac{1}{2\pi i} \oint\limits_C \sum_{n=0}^{\infty} \frac{(z-z_0)^n f(z')}{(z'-z_0)^{n+1}} dz' \tag{78}$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint\limits_C \frac{f(z')}{(z'-z_0)^{n+1}} dz'. \tag{79}$$

Now by using Eq. 73 this becomes:

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}. \tag{80}$$

This is the **Taylor expansion** we were looking for.

We are now in a position to extend our analysis to the case where $f(z)$ is not analytic about z_0. In fact, in general we can consider it to only be analytic in an annular shaped region, with inner radius r and outer radius R both centred on z_0 (see Fig. 4). In this case, we can construct a closed contour C within the analytic region by again using our 'cut line' trick (see again Fig. 4)). This contour has an inner circle (call it C_2, radius r_2) joined by parallel cut lines to an outer circle (C_1 radius r_1) that are both centred on z_0.

Since we know the function we are expanding is analytic on the cut line then the parts of the contour integral along the cut cancel. Therefore Cauchy's integral formula leads to

$$f(z) = \frac{1}{2\pi i} \oint\limits_C \frac{f(z')}{z'-z} dz' \tag{81}$$

$$= \frac{1}{2\pi i} \oint\limits_{C_1} \frac{f(z')}{z'-z} dz' - \frac{1}{2\pi i} \oint\limits_{C_2} \frac{f(z')}{z'-z} dz' \tag{82}$$

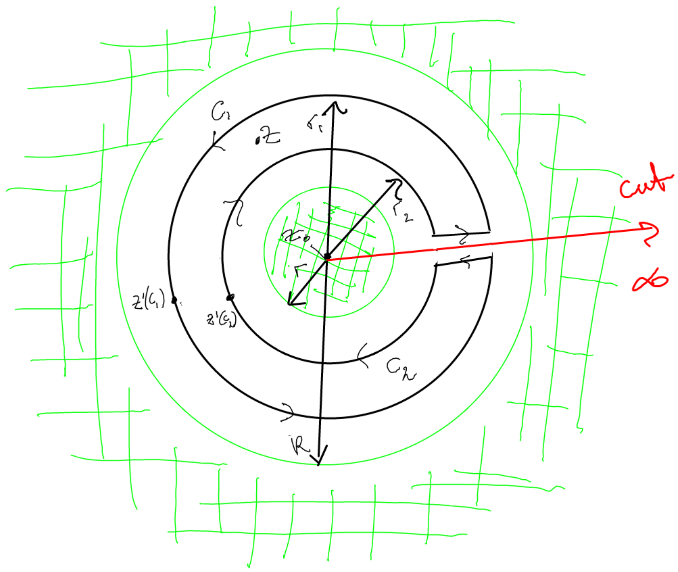

Figure 4: We develop a Laurent expansion in the annular region between the green hatched areas. We perform a contour integration around the black contour which includes circles C_1 and C_2.

where the minus sign again results from the opposite (clockwise) rotation sense of the integration round the inner circle. Now we can treat both terms on the RHS here in the same way we treated similar terms in the Taylor series. In fact, the C_1 term can be treated in exactly the same way, since points z' along it satisfy $|z' - z_0| > |z - z_0|$ and so

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z} dz' = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')}{(z' - z_0)^{n+1}} dz'. \tag{83}$$

For the C_2 term, we instead express the denominator $z' - z = (z' - z_0) - (z - z_0)$ as before but now take out a factor $(z - z_0)$ instead of $(z' - z_0)$: i.e. write $z' - z = -(z - z_0)[1 - (z' - z_0)/(z - z_0)]$. The resulting series is then convergent for $|z - z_0|/|z' - z_0| < 1$, which is true along C_2. Thus

$$\frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z' - z} dz' = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^{-(n+1)} \oint_{C_2} f(z')(z' - z_0)^n dz' \tag{84}$$

$$= \frac{1}{2\pi i} \sum_{m=-\infty}^{-1} (z - z_0)^m \oint_{C_2} f(z')(z' - z_0)^{-(m+1)} dz' \tag{85}$$

where the final equation results from the substitution $m = -(n+1)$: By writing the result for C_2 in this form, and by allowing C_1 and C_2 to become infinitesimally close to each other around point z, we can substitute Eq. 83 and Eq. 85 into Eq. 82 and make a single series for $f(z)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{86}$$

with

$$a_n = \frac{1}{2\pi i}\oint\limits_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'. \tag{87}$$

Eq. 86 is the **Laurent series** we ave been working towards. C is the limit of the now identical C_1 and C_2 contours and can lie anywhere within the annular region of convergence between the circles of radius r and R. Eq. 86 allows us to find expansions of functions of a complex variable around non-analytic points. To make things clearer, let us try an example.

Example: Find the Laurent series expansion of the function $f(z) = [z(z-1)]^{-1}$, about the point $z_0 = 0$, for the annular region defined by circles centred on z_0 and with radii $r = 0$ and $R = 1$.

First, let us remark that if we choose the inner circle radius to be $r = 0$ then the outer can only extend to $R = 1$ since $f(z)$ has a second pole at $z = 1$, a point which lies on this circle. Thus if we moved it any further out, then the region we were expanding in would contain a singular point, which is not allowed.

Now, to the expansion itself. Eq. 87 becomes, for $f(z) = [z(z-1)]^{-1}$:

$$a_n = \frac{1}{2\pi i}\oint\limits_C \frac{1}{(z'-1)(z')^{n+2}}dz'. \tag{88}$$

Now, the geometric series expansion gives:

$$\frac{1}{z'-1} = -\sum_{m=0}^{\infty} z^m \tag{89}$$

and thus

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint\limits_C \frac{1}{(z')^{n+2-m}}dz'. \tag{90}$$

Using polar coordinates $z' = \rho e^{i\theta}$, the integral over the circular contour just becomes an integral over θ:

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint\limits_C \frac{\rho i e^{i\theta}}{\rho^{n+2-m}e^{i(n+2-m)\theta}}d\theta, \tag{91}$$

$$= -\frac{1}{2\pi}\sum_{m=0}^{\infty}\frac{1}{\rho^{n+1-m}}\int_0^{2\pi} e^{-i(n+1-m)\theta}d\theta. \tag{92}$$

Now the integral is zero for all non zero $n+1-m$, but becomes $\int_0^{2\pi} d\theta = 2\pi$ when $n+1 = m$. Thus

$$a_n = \begin{cases} -1 & \text{if } n \geq -1 \\ 0 & \text{if } n < -1. \end{cases} \tag{93}$$

Written out in long hand, the series expansion is:

$$\frac{1}{z(z-1)} = -\frac{1}{z} - 1 - z - z^2 - \ldots . \tag{94}$$

For this easy example, a simple binomial expansion would have been easier and yielded the same result.

1.3.6 The Residue Theorem

We are now in a position to derive the most useful result in complex analysis, which enables the solution of a whole host of physics problems: The Residue Theorem.

We start with a Laurent series expansion of a function $f(z)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n . \tag{95}$$

Now let us integrate around a closed contour that encloses the point z_0 (which may be singular, depending on the coefficients in the expansion). We wish to evaluate the integral for each term in the series, i.e. the quantities I_n

$$I_n = a_n \oint (z - z_0)^n dz. \tag{96}$$

Now, if $n \geq 0$ then the integrand is analytic everywhere inside a finite contour. We know that $\oint g(z)dz = 0$ for any holomorphic function $g(z)$ and so in these cases $I_n = 0$.

Now recall that (Eq. 73):

$$g^{(m)}(z_0) = \frac{m!}{2\pi i} \oint \frac{g(z)}{(z - z_0)^{m+1}} dz \tag{73}$$

where we are now using an analytic function $g(z)$ to avoid confusion with the $f(z)$ defined above in Eq. 95. We can evaluate I_n for negative n by setting $m + 1 = -n$, and using the constant function $g(z_0) = 1$ over the whole plane. Then

$$I_n \propto g^{(-n-1)}(z_0) \tag{97}$$

which for any $n \leq -2$ must zero (any derivative of a constant is zero). The only case that we have not yet dealt with (and not shown to be zero) is $n = -1$. In this case however, we can use Cauchy's integral theorem, Eq.68 directly, again by defining $g(z) = 1$:

$$I_{-1} = a_{-1} \oint \frac{1}{z - z_0} dz = a_{-1} \oint \frac{g(z)}{z - z_0} dz = a_{-1} 2\pi i g(z_0) = a_{-1} 2\pi i \tag{98}$$

Bringing all our results together gives

$$\frac{1}{2\pi i} \oint f(z)dz = a_{-1} = R_{z_0} \tag{99}$$

The coefficient a_{-1} of the Laurent series is called the **residue** R_{z_0} of $f(z)$ at $z = z_0$.

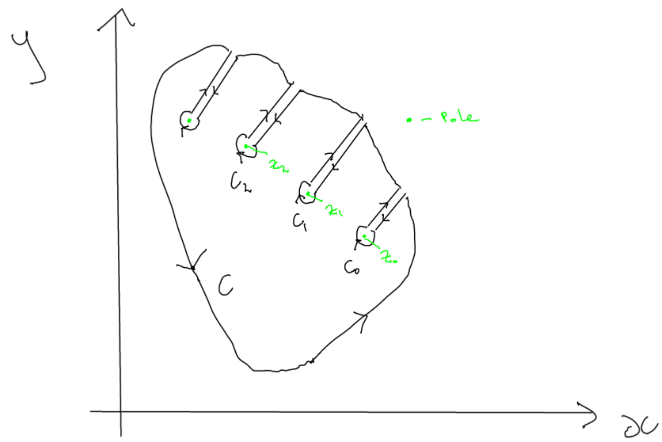

Figure 5: Excluding multiple poles.

Now, let us consider what happens if our contour C contains several poles of a meramorphic function; see Fig. 5. We can do our usual trick of deforming our contour around the poles by using cut lines. Using Cauchy's theorem, we know that if we exclude these singular points then our total contour integral will be zero, and so write:

$$\oint_C f(z)dz - \oint_{C_0} f(z)dz - \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz + \cdots = 0. \tag{100}$$

Where the minus signs come since we are using the convention that the closed contour integral sign means that is it traced in an anticlockwise direction - and all the polresmeres here are encircled in a clockwise sense.

Now, we know that each integral around a pole is just $2\pi i$ multiplied by the residue at that pole. Thus

$$\oint_C f(z)dz = 2\pi i(R_{z_0} + R_{z_1} + R_{z_2} + \dots) \tag{101}$$

$$= 2\pi i \times (\text{sum of enclosed residues}). \tag{102}$$

This is the **residue theorem**. It shows that in order to calculate a contour integral in the complex plane, all we need do is compute all of the residues enclosed by the contour. Eq. 102 is without doubt the most important result in complex analysis. It enables us just to write down the answers to even very complicated-looking integrals, without needing to do anything more than calculate the residues at the poles enclosed by the integration contour.

We will show examples of how we can use this result shortly, but let us first describe some practical ways of finding residues.

1.3.7 Finding residues

Obviously, it is possible to find the residue of a particular point by deriving the Laurent series around that point, and then finding the coefficient a_{-1}. However, this can be a lengthy process and is simply not necessary for many functions. In this section we will discuss how to find residues more easily.

First, let us imagine we have a function $f(z)$ that has a simple pole at position z_0. We could then write the function as

$$f(z) = \frac{g(z)}{z - z_0} \tag{103}$$

where the function $g(z)$ is analytic at z_0 (if it wasn't then $f(z)$ would not have a simple pole at z_0!). $g(z)$ may have other poles away from z_0, but we can always draw a circular contour C centred on z_0 of sufficiently small radius that we will avoid any other poles.

Now Eq. 99 gives us a direct way of evaluating R_{z_0}:

$$R_{z_0} = \frac{1}{2\pi i} \oint_C f(z) dz, \tag{104}$$

$$= \frac{1}{2\pi i} \oint_C \frac{g(z)}{z - z_0} dz. \tag{105}$$

But now we may use Cauchy's integral theorem, Eq. 68 to find

$$R_{z_0} = g(z_0), \tag{106}$$

or in terms of the original function $f(z)$:

$$R_{z_0} = \lim_{z \to z_0} \left[(z - z_0) f(z) \right]. \tag{107}$$

Example: Find the residue of $f(z) = z^2/(z-2)$ at $z = 2$.
$f(z)$ has a simple pole at $z = 2$ and so we can use the result just derived

$$R_2 = \lim_{z \to 2} \left[(z-2) \frac{z^2}{z-2} \right] = 4. \tag{108}$$

For higher order poles we can use the extension of Cauchy's integral formula to derivatives, Eq. 73. If $f(z)$ has an nth order pole at z_0 then it can be written

$$f(z) = \frac{h(z)}{(z - z_0)^n} \tag{109}$$

with $h(z)$ analytic at z_0. Applying Eq. 73 gives

$$R_{z_0} = \frac{1}{2\pi i} \oint_C f(z) dz \tag{110}$$

$$= \frac{1}{2\pi i} \oint_C \frac{h(z)}{(z-z_0)^n} dz \tag{111}$$

$$= \frac{1}{(n-1)!} h^{(n-1)}(z_0). \tag{112}$$

Thus **residue of a function** $f(z)$ **at a pole of order** n **at** $z = z_0$ is given by:

$$R_{z_0} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z) \right) \right]. \tag{113}$$

Example: Find the residue of $f(z) = 1/z^2$ at $z = 0$.
This time we have a double pole at $z = 0$. Hence

$$R_0 = \lim_{z \to 0} \left[\frac{d}{dz} \left(z^2 \frac{1}{z^2} \right) \right] = \frac{1}{2} \lim_{z \to 0} \left[\frac{d}{dz} 1 \right] = 0. \tag{114}$$

1.3.8 Poles on the contour: the Cauchy principal value

There is one case that we haven't considered yet. A pole lying outside the contour C gives no contribution to the integral; a pole lying inside contributes $2i\pi R$, where R is its residue. But what if the pole lies directly on the contour? (For example, if we were to try and integrate

$$f(z) = \frac{1}{z-1} \tag{115}$$

around the unit circle this problem would occur.)

Strictly speaking, such integrals are not properly defined. However, it would seem natural to allocate them a value that is midway between the other two cases, i.e. to say that a pole on the contour contributes $i\pi R$, where R is its residue. If we do this, we are said to be taking the **Cauchy principal value** of the integral, which is usually denoted by $\mathcal{P}$. Thus the integral

$$\oint_C \frac{dz}{z-1}, \tag{116}$$

where C is the unit circle, is undefined; but its principal value is given by

$$\mathcal{P} \oint_C \frac{dz}{z-1} = i\pi. \tag{117}$$

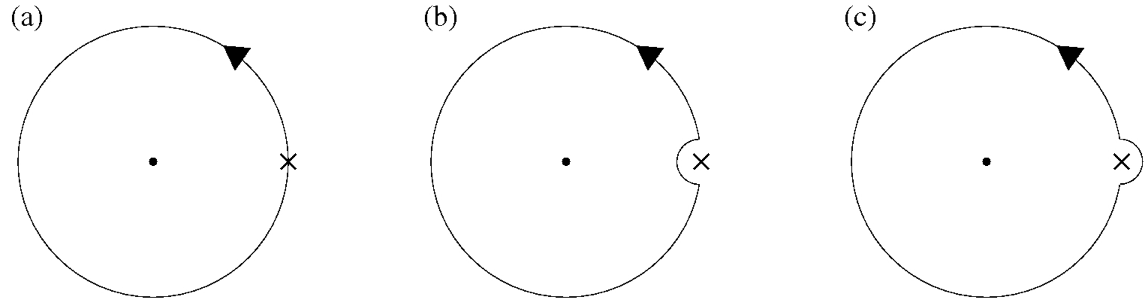

Figure 6: The contours used in the formal definition of the Cauchy principal value. (a) C, the original contour. (b) C_1', a contour deformed to exclude the pole. (c) C_2', a contour deformed to include the pole.

We may formally define the principal value as the average of two other integrals, one using a deformed contour C_1' that passes just inside the pole and the other using a deformed contour C_2' that passes just outside,

$$\mathcal{P}\oint_C f(z)dz \equiv \frac{1}{2}\left\{\oint_{C_1'} f(z)dz + \oint_{C_2'} f(z)dz\right\}. \tag{118}$$

This is illustrated for the case of the integral Eq. divergint in Fig. 6.

1.4 Real integrals by residue methods

1.4.1 Introduction

In the section we shall explore how to use the residue theorem to perform a wide range of otherwise difficult (indeed, sometimes impossible) integrals in real analysis: and indeed find a simple way of evaluating the Green's function integral we encountered in scattering theory before we embarked on our diversion into complex analysis!

1.4.2 Using the residue theorem to evaluate real integrals: closing the contour

Most integrals that come up in physics are of real integrands over a real variable. For example, consider the area of the Lorentzian,

$$I \equiv \int_{-\infty}^{\infty} \frac{dx}{x^2+1}. \tag{119}$$

(We can find the value of this integral by real analysis, using the substitution $x = \tan u$. This gives the answer π, but it's always a good idea to try a new method on a case in which we already know the answer.) If x were a complex variable, the integrand would be a meromorphic function, which suggests that we should use the residue theorem here if we can. There is no problem in promoting this integral to one in a complex variable; we just write

$$I = \int_C \frac{dz}{z^2+1}, \tag{120}$$

where the contour C is the real line. On that line, $z = x$, and so Eq. 119 and Eq. 120 are equal to each other. The integrand is a meromorphic function, with poles where $z^2+1=0$, i.e. at $z = \pm i$. However, we can't use the residue theorem on Eq. 120, because the contour C is not a closed one.

This leads us to the important procedure of **closing the contour**, which will turn Eq. 120 into an integral that can be performed using the residue method. The idea is to add a part to the contour C that makes it closed, but that we can prove has no effect on the actual value of the integral. The two usual choices are either a semicircle at infinity in the upper half-plane (UHP), or a semicircle at infinity in the lower half-plane (LHP). These are shown in Fig. 7. It is necessary to check that adding this curved part to the original real-line contour will not change the value of the integral, and this check has to be done separately for every integral you consider. There are some general rules, however. The basic condition for this trick to work is that the integrand should vanish sufficiently quickly as $|z| \to \infty$.

In this case, we have that the integrand goes like $1/z^2$ for large z; so on a semicircle of radius a it will be

$$\sim \frac{1}{a^2 e^{2i\theta}}. \tag{121}$$

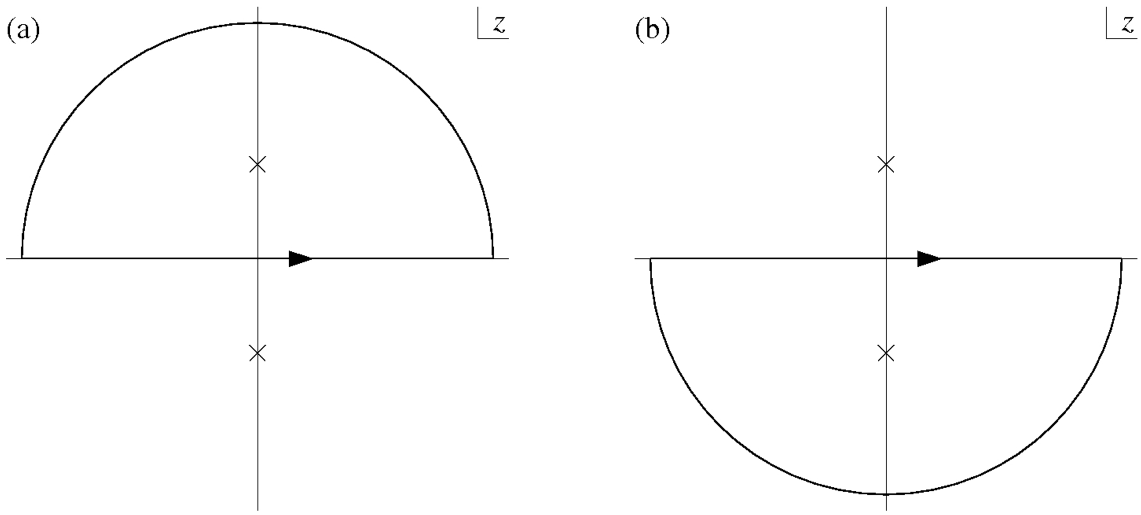

Figure 7: (a) C_1, obtained by closing the real-line contour in the upper half-plane. (b) C_2, obtained by closing the real-line contour in the lower half-plane.

The length of the semicircle is πa, and hence the value of the integral must go roughly like

$$\sim \frac{a}{a^2} = \frac{1}{a} \to 0 \qquad \text{as } a \to \infty. \tag{122}$$

This argument is clearly good for both of the possible closures, so it's up to us whether we close the contour in the upper or the lower half-plane.

Let's choose the upper half-plane. In that case,

$$I = \oint\limits_{C_1} \frac{dz}{z^2+1} = \oint\limits_{C_1} \frac{dz}{(z+i)(z-i)}. \tag{123}$$

The contour C_1 encloses only one of these poles: the one at $z = i$. The residue of the integrand at $z = i$ is

$$R_i = \lim_{z \to i} \left\{ (z-i) \frac{1}{(z+i)(z-i)} \right\} = \frac{1}{2i}. \tag{124}$$

Hence, by the residue theorem,

$$I = 2i\pi R_i = \frac{2i\pi}{2i} = \pi, \tag{125}$$

which is the answer we knew we should get from real analysis. You can easily check that, had we closed the contour in the lower half-plane, we would have got the same answer.[6]

[6] If you try this, remember that for a contour closed in the LHP we're enclosing the poles in the 'wrong' (clockwise) sense, and hence we get $-2i\pi$ (rather than $2i\pi$) times the sum of the residues.

1.4.3 Selecting the appropriate half-plane

The above example is nice, in the sense that we can close the contour in either the UHP or the LHP, but this is not always the case. Consider, for example, the following integral:

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x^2+9)} dx. \tag{126}$$

Again, the integrand looks meromorphic, so we write this as a complex contour integral along the real line:

$$I_2 = \int_C \frac{e^{iz}}{(z^2+4)(z^2+9)} dz. \tag{127}$$

We can't use the residue theorem on this yet, because the contour isn't closed. But which half-plane (if either) are we allowed to close it in? To answer this question, let's examine the behaviour of the integrand in each half-plane in turn:

- *Upper half-plane.* In the upper half-plane, $z = x + iy$, where $y > 0$. Hence

  $$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y}. \tag{128}$$

  This vanishes exponentially as $y \to +\infty$, which means that the curved part of the contour C_1 will give zero.

- *Lower half-plane.* In the lower half-plane, $z = x - iy$, where $y > 0$. Hence

  $$e^{iz} = e^{i(x-iy)} = e^{ix}e^{y}. \tag{129}$$

  This diverges exponentially as $y \to +\infty$. So in this case, not only does the curved part of the contour C_2 give a non-zero contribution — its contribution is actually infinite!

It's clear from this analysis that we have to close the contour in the UHP: only by doing this do we avoid changing the value of I_2 by introducing the extra part of the contour. Hence

$$I_2 = \oint_{C_1} \frac{e^{iz}}{(z^2+4)(z^2+9)} dz. \tag{130}$$

Now the integral has a closed contour, so we can use the residue theorem. The poles of the integrand occur wherever the denominator vanishes, i.e. at $z = \pm 2i$ and at $z = \pm 3i$. But only two of these poles are 'caught' by the contour C_1: those at $z = 2i$ and $z = 3i$.

What are the residues there? To find the residue at $z = 2i$, we should isolate the divergent part of the integrand and remove it. The integrand may be rewritten

$$\frac{e^{iz}}{(z-2i)(z+2i)(z^2+9)}; \tag{131}$$

removing the divergent part $(z-2i)$ and evaluating what remains at the pole, we obtain

$$R_{2i} = \left.\frac{e^{iz}}{(z+2i)(z^2+9)}\right|_{z=2i} = \frac{e^{-2}}{4i(-4+9)} = -i\frac{e^{-2}}{20}. \tag{132}$$

Similarly, at $z = 3i$ we have to isolate the part that diverges there, by rewriting the integrand as

$$\frac{e^{iz}}{(z^2+4)(z+3i)(z-3i)}. \tag{133}$$

Removing the divergent part $(z-3i)$ and evaluating what remains at the pole, we obtain

$$R_{3i} = \left.\frac{e^{iz}}{(z^2+4)(z+3i)}\right|_{z=3i} = \frac{e^{-3}}{6i(-9+4)} = i\frac{e^{-3}}{30}. \tag{134}$$

Putting these results into the residue theorem, we finally obtain

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x^2+9)}dx = \oint_{C_1} \frac{e^{iz}}{(z^2+4)(z^2+9)}dz \tag{135}$$

$$= 2i\pi\left(-i\frac{e^{-2}}{20} + i\frac{e^{-3}}{30}\right) \tag{136}$$

$$= \frac{\pi}{30}\left(3e^{-2} - 2e^{-3}\right). \tag{137}$$

Try and get *that* result by real analysis!