Fluids

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1 introduction

1.1 continuum assumption

fluid element: a patch over which we define local variables

1. A fluid element L_{el} is small enough that we can ignore systematic varations across it:
$$L_{el} \ll L_{\text{scale}} \approx \frac{q}{|\boldsymbol{\nabla} q|}$$
q is any quantity and L_{el} is a length scale over which q varies by order unity.
2. A fluid element is large enough that it contains enough particles that you can ignore fluctuations due to a finite number of particles.
$$nL_{el}^3 \gg 1$$
n is the number density (m$^-3$).
3. A fluid element is large enough that consituent particles "know" about local conditions through colliding with each other:
$$L_{el} \gg \lambda$$
λ is the mean free path.

$\therefore$ all quantities are constant throughout a fluid element. (temp, pressure, density)

2 relating Eulerian and Lagrangian descriptions

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{Q(t+\delta t) - Q(t)}{\delta t} \quad \longrightarrow \quad \frac{\mathrm{d}Q(r,t)}{\mathrm{d}t} = \frac{Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t - Q(\mathbf{r},t)}{\delta t}$$

$$\begin{aligned}
Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - Q(\mathbf{r},t) &= \underbrace{Q(\mathbf{r}, t+\delta t) - Q(\mathbf{r},t)}_{\text{variation in } t \text{ at fixed } \mathbf{r}} + \underbrace{Q(\mathbf{r}+\delta\mathbf{r}, t+\delta t) - Q(\mathbf{r}, t+\delta t)}_{\text{variation in } \mathbf{r} \text{ at fixed } t+\delta t} \\
&= \delta t \frac{\partial Q}{\partial t} + \ldots + \delta\mathbf{r} \cdot \boldsymbol{\nabla} Q + \ldots \\
&\approx \left.\frac{\partial Q}{\partial t}\right|_t \delta t + \delta\mathbf{r} \cdot \left.\boldsymbol{\nabla} Q\right|_{t+\delta t} \\
&\approx \left.\frac{\partial Q}{\partial t}\right|_t \delta t + \delta\mathbf{r} \cdot \left[\boldsymbol{\nabla} Q + \delta t \frac{\partial}{\partial t} \boldsymbol{\nabla} Q\right]_t
\end{aligned}$$

$$\therefore \frac{\mathrm{d}Q}{\mathrm{d}t} \approx \frac{\partial Q}{\partial t} + \frac{\delta\mathbf{r}}{\delta t} \cdot \boldsymbol{\nabla} Q$$

streamline: a curve that has $\mathbf{u}$ in the tangential direction: where mass element travels to in a steady flow ($\frac{\partial}{\partial t} = 0$)

$$\begin{aligned}
\frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y} &\Rightarrow u_y \mathrm{d}x - u_x \mathrm{d}y = 0 \\
&\Rightarrow \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial y}\mathrm{d}y = \mathrm{d}\psi = 0
\end{aligned}$$

Hence the stream function ψ is constant on a streamline, but differs for different stream lines.

3 flows through surfaces

3.1 Gauss' theorem

For a box of volume $\Delta x \Delta y \Delta z$, the x-component of $\mathbf{u}$ at the center of the 'front' and 'back' face is:

$$\begin{aligned}
u_b &\approx u_x - \frac{\Delta x}{2}\frac{\partial u_x}{\partial x} \\
u_f &\approx u_x + \frac{\Delta x}{2}\frac{\partial u_x}{\partial x}
\end{aligned}$$

$$\text{volume crossing back face per time} = \underbrace{\left(u_x - \frac{1}{2}\frac{\partial u_x}{\partial x}\Delta x\right)}_{\text{distance moved/sec}} \underbrace{\Delta y \Delta z}_{\text{area of face}}$$

How much fluid is transported through a surface area A is the flux through the surface $A = \Delta y \Delta z$.

$$\text{volume crossing front face per time} = \underbrace{\left(u_x + \frac{1}{2}\frac{\partial u_x}{\partial x}\Delta x\right)}_{\text{distance moved/sec}} \underbrace{\Delta y \Delta z}_{\text{area of face}}$$

$$\therefore \text{ net vol/sec flowing in } x\text{-direction} = \frac{\partial u_x}{\partial x}\Delta x \Delta y \Delta z$$

$$\therefore \text{ total net vol/sec} = (\boldsymbol{\nabla} \cdot \mathbf{u})\Delta x \Delta y \Delta z$$

$$\text{total flux} = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}V$$

$$\text{volume that flows through } \mathrm{d}S \text{ each second} = (\mathbf{u} \cdot \mathbf{n})\mathrm{d}S$$

$$\text{total flux} = \oiint_S \mathbf{u} \cdot \mathrm{d}\mathbf{S} \qquad \mathrm{d}\mathbf{S} = \mathbf{n} \, \mathrm{d}S$$

In the absence of sources or sinks of mass, the total rate at which mass (density ρ) flows through the surface S:

$$\begin{aligned}
-\sum_i \rho\mathbf{u} \cdot \mathrm{d}\mathbf{S}_i &= -\int_S \rho\mathbf{u} \cdot \mathrm{d}\mathbf{S} \\
&= -\int_V \boldsymbol{\nabla} \cdot (\rho\mathbf{u})\mathrm{d}V \\
&= \frac{\partial}{\partial t}\int_V \rho \mathrm{d}V
\end{aligned}$$

$$\therefore \int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u})\right)\mathrm{d}V = 0$$

$$\text{equation of continuity:} \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\mathbf{u}) = 0$$

$$\text{mass conservation in co-moving frame:} \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\boldsymbol{\nabla} \cdot \mathbf{u}$$

4 the momentum equation

For any surface within a fluid there is a momentum flux across it (from each side) *that has nothing to do with any bulk flow* but is a consequence of its thermal properties.

Microscopically, assume a perfect gas. Meaning that the finite temperature imparts molecules with random motions. The pressure is the associated (one-sided) momentum flux.

Thermal motion is isotropic, hence, the local momentum flux is independent of the orientation of the surface and always perpendicular to the surface (the parallel components cancel out)

For a lump of fluid subject to gravity and the inward pressure of the surrounding fluid, pressure force on $\mathrm{d}\mathbf{S}$ is $-p\mathbf{S}$

component of inward pressure force: $\quad -p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S}$

$$-\oiint_S p\hat{\mathbf{n}}\cdot\mathrm{d}\mathbf{S} = -\iiint_V \boldsymbol{\nabla}\cdot p\hat{\mathbf{n}}\,\mathrm{d}V$$

total momentum in volume V: $\quad \int_V \rho\mathbf{u}\,\mathrm{d}V$

rate of change of momentum in V: $\quad \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V$

The equation of motion in direction $\hat{\mathbf{n}}$ is the rate of change of momentum equated to the sum of forces:

$$\therefore \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V\right)\cdot\hat{\mathbf{n}} = -\int_V \boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}})\,\mathrm{d}V + \int_V \rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V$$

$$\boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}}) = \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \cancelto{0}{p\boldsymbol{\nabla}\cdot\hat{\mathbf{n}}}$$

assuming lump is small,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\mathbf{u}\delta V)\cdot\hat{\mathbf{n}} = \cancelto{0}{\mathbf{u}\cdot\hat{\mathbf{n}}\tfrac{\mathrm{d}}{\mathrm{d}t}(\rho\delta V)} + \rho\delta V\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\cdot\hat{\mathbf{n}}$$

$$\therefore \rho\delta V\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\cdot\hat{\mathbf{n}} = \delta V(-\boldsymbol{\nabla}p + \rho\mathbf{g})\cdot\hat{\mathbf{n}}$$

$$\delta V\left(\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \boldsymbol{\nabla}p - \rho\mathbf{g}\right)\cdot\hat{\mathbf{n}} = 0$$

equation of motion,momentum/conservation equation (Lagrangian form):

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g}$$

Eulerian form:

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}p + \rho\mathbf{g}$$

The momentum contained in a fixed grid cell changes in response to external forces (pressure and gravitational forces) plus any imbalance in the momentum flux in and out of the cell.

The *thermal pressure* is associated with random motions in the fluid which are isotropic, and is scalar (acts the same way in any direction).

The *ram pressure* is associated with bulk motion of the fluid which is oriented. Only a surface whose normal has some component along the direction of flow feels the ram pressure.

2^{nd} *law of thermodynamics:*:

$$T\mathrm{d}S = \mathrm{d}Q\mathrm{d}U + p\mathrm{d}V$$

$$\left.\begin{aligned}\mathrm{d}Q &= \delta m\,\mathrm{d}q\\ \mathrm{d}U &= \delta m\,\mathrm{d}e\\ \mathrm{d}V &= \delta m\,\mathrm{d}\left(\tfrac{1}{p}\right)\end{aligned}\right\} \longrightarrow T\mathrm{d}s = \mathrm{d}q = \mathrm{d}e + p\,\mathrm{d}\left(\frac{1}{p}\right)$$

$$\mathrm{d}s = \frac{\mathrm{d}S}{\delta m} \qquad e = \frac{p}{(\gamma-1)\rho} \qquad \gamma = \frac{c_p}{c_v}$$

L is the sum of sources and sinks of energy, we can differentiate by $\mathrm{d}t$ (and multiply by density) to get:

$$\rho T\frac{\mathrm{d}s}{\mathrm{d}t} = \rho\left[\frac{\mathrm{d}e}{\mathrm{d}t} + p\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\right)\right] = -L$$

$$\begin{aligned}\rho T\frac{\mathrm{d}s}{\mathrm{d}t} &= \rho\left[\frac{\mathrm{d}e}{\mathrm{d}t} + p\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\right)\right] = -L\\ &= \rho\frac{\mathrm{d}e}{\mathrm{d}t} + p\rho\left(\frac{-1}{p^2}\right)\frac{\mathrm{d}\rho}{\mathrm{d}t}\\ &= \rho\frac{\mathrm{d}e}{\mathrm{d}t} - \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \qquad \text{(apply mass conservation)}\\ &= \rho\frac{\mathrm{d}e}{\mathrm{d}t} + p\boldsymbol{\nabla}\cdot\mathbf{u} = -L\end{aligned}$$

$$\mathbf{u}\cdot\left(\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g}\right)$$

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}u^2\right) = -u\cdot\boldsymbol{\nabla}p + \mathbf{u}\cdot\rho\mathbf{g}$$

$$\begin{aligned}\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}u^2 + e\right) &= -L - p\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\cdot\boldsymbol{\nabla}p + \mathbf{u}\cdot\rho\mathbf{g}\\ &= -L - \boldsymbol{\nabla}\cdot p\mathbf{u} + \mathbf{u}\cdot\rho\mathbf{g}\end{aligned}$$

$$\begin{aligned}\rho\frac{\mathrm{d}A}{\mathrm{d}T} &= \frac{\mathrm{d}(\rho A)}{\mathrm{d}t} - A\frac{\mathrm{d}\rho}{\mathrm{d}t} \qquad \text{(apply mass conservation)}\\ &= \frac{\mathrm{d}(\rho A)}{\mathrm{dd}t} + \rho A\boldsymbol{\nabla}\cdot\mathbf{u} \qquad \text{(convert into observers frame)}\\ &= \frac{\partial(\rho a)}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}(\rho A) + \rho A\boldsymbol{\nabla}\cdot\mathbf{u}\\ &= \frac{\partial(\rho A)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho A)\mathbf{u}\end{aligned}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho u^2 + \rho e\right)\mathbf{u} = -L - \boldsymbol{\nabla}\cdot p\mathbf{u} - \mathbf{u}\cdot\rho\boldsymbol{\nabla}\psi$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\rho\psi)\mathbf{u} &= (\rho\psi)\boldsymbol{\nabla}\cdot\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}(\rho\psi)\\ &= (\rho\psi)\boldsymbol{\nabla}\cdot\mathbf{u} + \psi\mathbf{u}\cdot\boldsymbol{\nabla}\rho + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\psi\\ &= \psi(\rho\boldsymbol{\nabla}\cdot\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho) + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\psi\\ \text{(use mass conservation)} &= \psi\left(-\frac{\partial\rho}{\partial t}\right) + \rho\mathbf{u}\cdot\boldsymbol{\nabla}\psi\end{aligned}$$

5 equations of state

$$\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\rho u^2 + pe + p + \rho\psi\right)\mathbf{u} = -L \qquad \textit{enthalpy: } \rho e + p = \frac{\gamma}{\gamma-1}p$$

In steady state, the net effect L of the sources and sinks of energy is equal to the flux of energy through the surface of the volume.

In general, $p = p(\rho, T)$ and for an ideal gas:

$$p = nk_BT \quad \text{or} \quad p = \frac{k_B}{m}\rho T$$

barotropic equation of state: p is a function of ρ only.

To approximate a fluid as being isothermal (T = constant $\longrightarrow p\propto\rho$) we require that

- temperature for thermal equilibrium isn't very sensitive to the heating/cooling rate
- in time-dependent problems, therei s time for the system to reach this constant T thermal equilibrium

adiabatic equation of state:

$$p = K\rho^{\gamma} \qquad \gamma = \frac{c_p}{c_b}$$

This is derived from the ideal gas laws assuming no heat excange with surroudings (only $p\mathrm{d}B$ work).

A fluid element behaves adiabatically if K is constant as the element's properties change.

An isentropic fluid is one in which all the elements have the same value of K.

$$\begin{aligned}
-L &= \rho\frac{\mathrm{d}e}{\mathrm{d}t} - \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \qquad e = \frac{p}{(\gamma-1)\rho} \\
&= \frac{\rho}{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho}\right) - \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \\
&= \frac{1}{\gamma-1}\left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{(\gamma-1)p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) \\
&= \frac{1}{\gamma-1}\left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\gamma p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right) \\
&= \frac{\rho^{\gamma}}{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^{\gamma}}\right)
\end{aligned}$$

vorticity: tendency for parcel of fluid to rotate about an axis through its centre of mass

$$\omega = \boldsymbol{\nabla} \times \mathbf{u}$$

$$\mathrm{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

rigid body rotation: reach parcel of fluid changes its orienation as it moves (as opposed to circulation without rotation)
viscosity: the internal stress (force/unit area) from a fluid dragging other fluid

stress: (for a Newtonian fluid)

$$\tau = \mu\frac{\partial u}{\partial y}$$

τ is the coefficient of shear viscosity, and τ is the stress tensor. For a given flow $\mathbf{u}$, thte higher the viscosity, the greater the stress.

viscous force/element of volume (stress):

$$\left[\mu\left.\frac{\partial u}{\partial y}\right|_{y+\delta y} - \mu\left.\frac{\partial u}{\partial y}\right|_{y}\right]\delta x\,\delta z = \left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\delta y\right]\delta x\,\delta z = \mu\frac{\partial^2 u}{\partial y^2}\delta y\,\delta x\,\delta z$$

viscous stress force:

$$F_\nu = \mu\frac{\partial^2 u}{\partial y^2}$$

generalizing:

$$F_\nu = \rho\nu\boldsymbol{\nabla}^2 u$$

equation of motion:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \mathbf{g} + \nu\boldsymbol{\nabla}^2\mathbf{u}$$