Let us define the state of the system as

$$\left\{\text{orientation } \mathbf{R} \in SO(3), \text{ velocity } \mathbf{v} \in \mathbb{R}^3, \text{ position } \mathbf{p} \in \mathbb{R}^3\right\}$$

and without bias for convenience. We call it an *extended pose*. Extended poses may be described by 5×5 matrices in the following Lie group

$$SE_2(3) := \left\{ \mathbf{T} = \left[\begin{array}{c|cc} \mathbf{R} & \mathbf{v} & \mathbf{p} \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{I}_2} \end{array} \right] \in \mathbb{R}^{5\times 5} \middle| \begin{array}{c} \mathbf{R} \in SO(3) \\ \mathbf{v}, \mathbf{p} \in \mathbb{R}^3 \end{array} \right\}.$$

Our Kalman filter is an *error-state* filter. We define the state error as

$$\mathbf{T} := \exp(\boldsymbol{\xi})\hat{\mathbf{T}}, \tag{1}$$

where $\hat{\mathbf{T}}$ is a noise-free "mean", $\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}_9, \boldsymbol{\Sigma})$ is a zero-mean multivariate Gaussian in $\mathbb{R}^9$, and $\exp(\cdot)$ is the $SE_2(3)$ exponential map.

Assume the filter obtains a measurement $\mathbf{y} = h(\boldsymbol{\chi})$, how computing the Jacobian

$$\mathbf{H} = \frac{\partial h}{\partial \boldsymbol{\xi}}|_{\hat{\boldsymbol{\chi}}} \tag{2}$$

to use this measurement in the filter?

First, developing the error, we have

$$\exp(\boldsymbol{\xi})\hat{\mathbf{T}} \Leftrightarrow \begin{cases} \exp(\boldsymbol{\xi}^{\mathbf{R}}) = \mathbf{R}\hat{\mathbf{R}}^T \\ \boldsymbol{\xi}^{\mathbf{v}} = \mathbf{v} - \mathbf{R}\hat{\mathbf{R}}^T\hat{\mathbf{v}} \\ \boldsymbol{\xi}^{\mathbf{p}} = \mathbf{p} - \mathbf{R}\hat{\mathbf{R}}^T\hat{\mathbf{p}} \end{cases}. \tag{3}$$

Linearizing the above results, i.e. applying

$$\exp(\boldsymbol{\xi}) \simeq \mathbf{I}_9 + \left[\begin{array}{c|cc} \boldsymbol{\xi}^{\mathbf{R}}_{\times} & \boldsymbol{\xi}^{\mathbf{v}} & \boldsymbol{\xi}^{\mathbf{p}} \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{0}_{2\times 2}} \end{array} \right], \tag{4}$$

we obtain

$$\left(\mathbf{I}_9 + \left[\begin{array}{c|cc} \boldsymbol{\xi}^{\mathbf{R}}_{\times} & \boldsymbol{\xi}^{\mathbf{v}} & \boldsymbol{\xi}^{\mathbf{p}} \\ \hline \mathbf{0}_{2\times 3} & \multicolumn{2}{c}{\mathbf{0}_{2\times 2}} \end{array} \right] \right) \hat{\mathbf{T}} \Leftrightarrow \begin{cases} \boldsymbol{\xi}^{\mathbf{R}}_{\times} = \mathbf{R}\hat{\mathbf{R}}^T - \mathbf{I}_3 \\ \boldsymbol{\xi}^{\mathbf{v}} = \mathbf{v} - \hat{\mathbf{v}} - \boldsymbol{\xi}^{\mathbf{R}}_{\times}\hat{\mathbf{v}} \\ \boldsymbol{\xi}^{\mathbf{p}} = \mathbf{p} - \hat{\mathbf{p}} - \boldsymbol{\xi}^{\mathbf{R}}_{\times}\hat{\mathbf{p}} \end{cases} \tag{5}$$

Looking at the residual and Linearizing it as

$$\begin{aligned} \mathbf{r} &= \mathbf{y} - \hat{\mathbf{y}} \\ &= h(\boldsymbol{\chi}) - h(\hat{\boldsymbol{\chi}}) \\ &= h(\exp(\boldsymbol{\xi})\hat{\boldsymbol{\chi}}) - h(\hat{\boldsymbol{\chi}}) \\ &\simeq h((\mathbf{I}_9 + \boldsymbol{\xi}^{\wedge})\hat{\boldsymbol{\chi}}) - h(\hat{\boldsymbol{\chi}}) \end{aligned}$$

we can then identify the Jacobian term by term.

Example Assume we measure velocity in the vehicle frame, we have

$$\mathbf{y} = h(\boldsymbol{\chi}) = \mathbf{R}^T \mathbf{v}, \tag{6}$$

such that the residual is computed as

$$\begin{aligned}
\mathbf{r} &= h(\boldsymbol{\chi}) - h(\hat{\boldsymbol{\chi}}) \\
&= \mathbf{R}^T \mathbf{v} - \hat{\mathbf{R}}^T \hat{\mathbf{v}} \\
&\simeq (\hat{\mathbf{R}} + \hat{\mathbf{R}} \boldsymbol{\xi}^{\mathbf{R}}_{\times})(\hat{\mathbf{v}} - \boldsymbol{\xi}^{\mathbf{R}}_{\times} \hat{\mathbf{v}} + \boldsymbol{\xi}^{\mathbf{v}}) - \hat{\mathbf{R}}^T \hat{\mathbf{v}} \\
&\simeq \hat{\mathbf{R}} \hat{\mathbf{v}} - \hat{\mathbf{R}} \hat{\mathbf{v}} - \hat{\mathbf{R}} \boldsymbol{\xi}^{\mathbf{R}}_{\times} \hat{\mathbf{v}} + \hat{\mathbf{R}} \boldsymbol{\xi}^{\mathbf{R}}_{\times} \hat{\mathbf{v}} + \hat{\mathbf{R}} \boldsymbol{\xi}^{\mathbf{v}} \\
&\simeq \hat{\mathbf{R}} \boldsymbol{\xi}^{\mathbf{v}}
\end{aligned}$$

and the Jacobian is given as

$$\mathbf{H} = \begin{bmatrix} \mathbf{0}_{3\times3} & \hat{\mathbf{R}} & \mathbf{0}_{3\times3} \end{bmatrix}.$$