UM–SJTU Joint Institute

Bayesian Analysis
(VE414)

Assignment 1

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1 Question 1

a)

Let $A =$ The lower face of the coin is a head.
Let $A_{11} =$ Picks a double-headed coin.
Let $A_{12} =$ A double-headed coin with a head at the bottom.
Let $A_{21} =$ Picks a double-tailed coin.
Let $A_{22} =$ A double-tailed coin with a head at the bottom.
Let $A_{31} =$ Picks a normal coin.
Let $A_{32} =$ A normal coin with a head at the bottom.

$$
\begin{aligned}
P(A) &= P(A_{11}) \cdot P(A_{12}) + P(A_{21}) \cdot P(A_{22}) + P(A_{31}) \cdot P(A_{32}) \\
&= 0.4 \cdot 1 + 0.2 \cdot 0 + 0.4 \cdot 0.5 = 0.6
\end{aligned}
$$

b)

Let $B =$ Opens his eye and finds a coin with a head on the top.

$$
P(A \mid B) = \frac{P(A, B)}{P(B)} = \frac{P(A_{11})}{P(B)} = \frac{0.4}{0.6} = 0.667
$$

c)

Let $B_1 =$ First throw with a head on the top.
Let $B_2 =$ Second throw with a head on the top.

$$
\begin{aligned}
P(A \mid B_1, B_2) &= \frac{P(A, B_1, B_2)}{P(B_1, B_2)} \\
&= \frac{P(A_{11})}{P(A_{11}) \cdot P(A_{12})^2 + P(A_{21}) \cdot P(A_{22})^2 + P(A_{31}) \cdot P(A_{32})^2} \\
&= \frac{0.4}{0.5} = 0.8
\end{aligned}
$$

2 Question 2

Let $X_i = j$ be the random variable that j heads are shown in i throws.

We define the likelihood of the coin throwing as

$$L(p;x) = f_{X|P}(x \mid p) = \frac{k!}{x!(k-x)!}p^x(1-p)^{k-x}$$

Thus we have

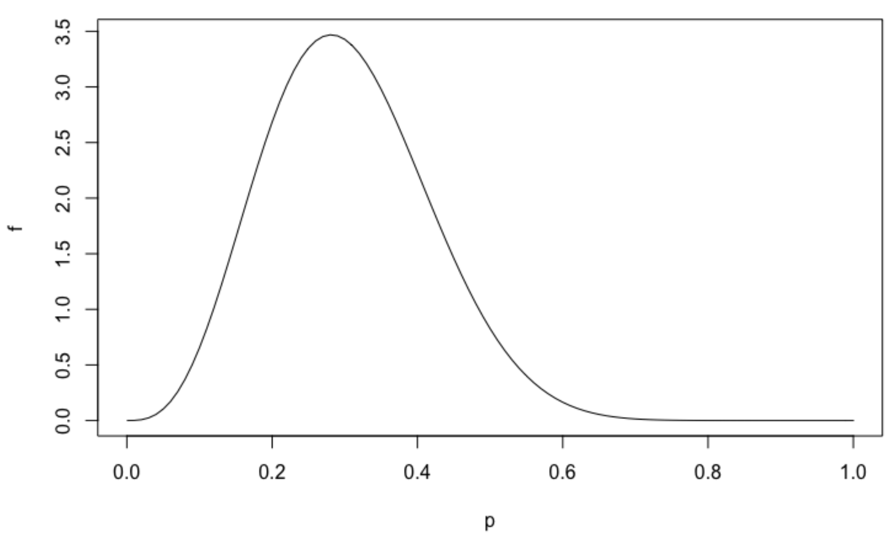
$$L(p;x<3) = f_{X|P}(x<3 \mid p) = \sum_{x=0}^{2} \frac{10!}{x!(10-x)!}p^x(1-p)^{10-x}$$

And we know that

$$\begin{aligned} f_{P|X}(p \mid x) &\propto L(p;x<3)f_P(p) \\ &\propto (45 \cdot p^2(1-p)^8 + 10 \cdot p(1-p)^9 + (1-p)^{10}) \cdot (p^3(1-p)^3) \\ &\propto 45 \cdot p^5(1-p)^{11} + 10 \cdot p^4(1-p)^{12} + p^3(1-p)^{13} \end{aligned}$$

With calculation by mathematica, we get

$$c = 7735/8$$



3 Question 3

a)

We use R to calculate the credible interval.

```
    a.prior = 1
    b.prior = 1

    a.posterior = a.prior + 40
    b.posterior = b.prior + 10
    l = qbeta(0.025, a.posterior, b.posterior)
    u = qbeta(0.975, a.posterior, b.posterior)

    cat(paste("(",l, sep = ""),
        paste(u,")", sep =""),
        "\n", sep = ",")
```

And we get the result that the credible interval for choosing F is $(0.669, 0.887)$. It shows that the choice of the people a biased towards F.

b)

We use R to calculate the credible interval.

```
    a.prior = 1
    b.prior = 1

    a.posterior = a.prior + 15
    b.posterior = b.prior + 35
    l = qbeta(0.025, a.posterior, b.posterior)
    u = qbeta(0.975, a.posterior, b.posterior)

    cat(paste("(",l, sep = ""),
        paste(u,")", sep =""),
        "\n", sep = ",")
```

And we get the result that the credible interval for choosing F is $(0.191, 0.438)$. It shows that the choice of the people a biased towards J.

4 Question 4

a)

We have

$$f_{X|P}(x \mid p) = p(1-p)^{x-1}$$

So the likelihood should be

$$L(p;x) = \Pi_{i=1}^{n} f_{X_i|P}(x_i \mid p) = p^n(1-p)^{(\sum x_i)-n}$$

Then we have the log likelihood

$$l = \log L(p;x) = n \ln p + (-n + \sum x_i)\ln(1-p)$$

To calculate the estimator, we should let the derivative to be 0

$$\frac{\partial l(p;x)}{\partial p} = 0 \longrightarrow \frac{n}{p} - \frac{\sum x_i - n}{1-p} = 0$$

So

$$\hat{p} = \frac{n}{\sum x_i}$$

b)

Since

$$q = p(1-p)$$

So

$$\hat{q} = \hat{p}(1-\hat{p}) = \frac{n}{\sum x_i}(1 - \frac{n}{\sum x_i})$$

c)

Suppose a uniform prior, so the prior can also be the beta distribution

$$p \sim \text{Beta}(1,1)$$

Thus we have

$$f_{P|X}(p \mid x) \propto L(p;x) \cdot f_P(p) \propto p^n(1-p)^{-n+\sum x_i}$$

So the posterior is still a Beta Distribution, with

$$p \mid x \sim \text{Beta}(n+1, -n+1+\sum x_i)$$