UM–SJTU Joint Institute

Bayesian Analysis
(VE414)

Assignment 3

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1 Question 1

a)

We know that

$$f_{\{Y,\alpha,\beta\}|X} \propto f_{X|\{Y,\alpha,\beta\}} \cdot f_{Y|\{\alpha,\beta\}} \cdot f_{\alpha,\beta}$$

And we have

$$f_{X|\{Y,\alpha,\beta\}} = \prod_{i=1}^{n} \text{Possion}(y_i) = \prod_{i=1}^{n} \frac{y_i^{x_i} e^{-y_i}}{x_i!}$$

$$f_{\alpha,\beta} = f_\alpha \cdot f_\beta = \text{Exp}(a)\text{Gamma}(b,c) = a\exp(-a\alpha) \cdot \frac{c^\beta}{\Gamma(b)}\beta^{b-1}\exp(-c\beta)$$

$$f_{Y|\{\alpha,\beta\}} = \prod_{i=1}^{n} \text{Gamma}(\alpha,\beta) = \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} y_i^{\alpha-1}\exp(-\beta y_i)$$

So we have

$$f_{\{Y,\alpha,\beta\}|X} \propto \beta^{n\alpha+b-1} y^{\Sigma x+n\alpha-1}\exp(-(n+\beta)y_i - c\beta - a\alpha) \cdot \frac{c^\beta}{\Gamma(\alpha)^n}$$

b)

We've made several assumptions of independence here:

1. X is independent of α and β.
2. α is independent of β.
3. x_i is independent of x_j if $i \neq j$.
4. y_i is independent of y_j if $i \neq j$.

c)

We have

$$f_{\{Y,\beta\}|\{X,\alpha\}} = \frac{f_{\{Y,\alpha,\beta\}|X}}{f_{\alpha|x}}$$

Since we know that α is independent from x, so $f_{\alpha|x} = f_\alpha$, so

$$f_{\{Y,\beta\}|\{X,\alpha\}} \propto f_{\{Y,\alpha,\beta\}|X}$$

d)

We have

$$f_{\{\alpha,\beta\}|X} = \frac{f_{\{Y,\alpha,\beta\}|X}}{f_{Y|\{X,\alpha,\beta\}}} \sim \text{Gamma}(-2\boldsymbol{x} + \alpha, \beta + 2)$$

2 Question 2

a)

Let the case that the new data point is considered as a tiger to be $y = 0$, and considered as a greasy to be $y = 1$.

So we have

$$\delta(x) = \arg\min \left\{ z\lambda f_{Y|x}(0 \mid x) + (z-1)(1-\lambda) f_{Y|x}(1 \mid x) \right\}$$

Since we have

$$f_{Y|x}(y = 0 \mid x) = \frac{L(0; x) f_Y(y = 0)}{f_X(x)} \qquad f_{Y|x}(y = 1 \mid x) = \frac{L(1; x) f_Y(y = 1)}{f_X(x)}$$

And if we times a positive constant to the arg min function, the result will not be changed, so

$$\delta(x) = \arg\min \{ z\lambda L(0; x) f_Y(y = 0) + (z-1)(1-\lambda) L(1; x) f_Y(y = 1) \}$$

Since we have

$$\boldsymbol{\mu}_0 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \boldsymbol{\Sigma}_0 = \begin{bmatrix} 12 & 1 \\ 1 & 2 \end{bmatrix} \qquad \boldsymbol{\mu}_1 = \begin{bmatrix} 13 \\ 10 \end{bmatrix} \quad \boldsymbol{\Sigma}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Therefore, the optimal decision rule should be

$$\begin{aligned}
\delta(x) = \arg\min\{ & z \cdot \frac{9}{10} \cdot \frac{2}{3} \cdot (2\pi)^{-k/2} (\det \boldsymbol{\Sigma_0})^{-1/2} \exp(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_0})^T \boldsymbol{\Sigma_0}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_0})) \\
& + (z-1) \cdot \frac{1}{10} \cdot \frac{1}{3} \cdot (2\pi)^{-k/2} (\det \boldsymbol{\Sigma_1})^{-1/2} \exp(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_1})^T \boldsymbol{\Sigma_1}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_1}))\} \\
= \arg\min & \left\{ z \cdot \frac{18}{\sqrt{23}} \exp(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_0})^T \boldsymbol{\Sigma_0}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_0})) + (z-1) \cdot \frac{1}{\sqrt{3}} \exp(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_1})^T \boldsymbol{\Sigma_1}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_1})) \right\}
\end{aligned}$$

So the decision rule should be

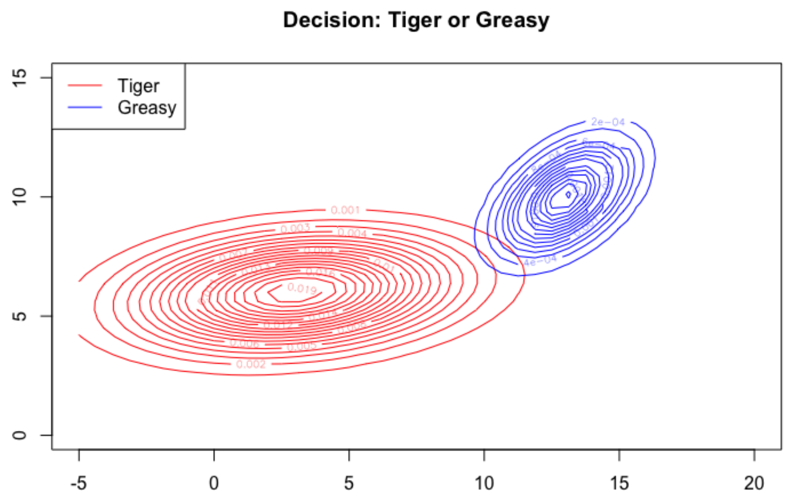
If

$$\frac{18}{\sqrt{23}} \exp(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_0})^T \boldsymbol{\Sigma_0}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_0})) < \frac{1}{\sqrt{3}} \exp(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_1})^T \boldsymbol{\Sigma_1}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_1}))$$

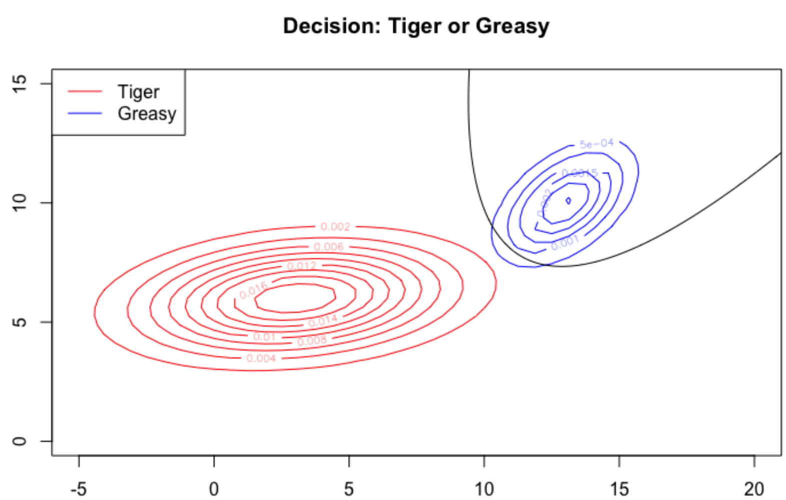
Choose as $y = 1$, which mark the species as a greasy. Otherwise tiger.

b)

We plot the contour of the two values for decision, if the red value is larger than the blue value, we consider this as a tiger, and vice versa.



We draw the contour of the two distribution, and the decision bondary.



c)

The left part is the region where a fish should be classified as a tiger, and the right part is the reigion where the fish should be classified as a greasy.