Complete Monotonicity and Benford's Law: Deriving Quantum Statistics from the Significant Digit Distribution

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Abstract

We show that the Bose-Einstein distribution is the unique quantum statistical distribution satisfying Benford's law exactly at all temperatures, and that this result follows from a chain of established mathematical theorems connecting complete monotonicity, the Bernstein-Widder representation, and the Benford conformance of Laplace transforms. Specifically, requiring that a quantum occupation function satisfy the significant digit law $P(d) = \log_{10}(1 + 1/d)$ at all parameter values forces its series expansion to have exclusively non-negative coefficients — selecting $1/(e^x - 1)$ over $1/(e^x + 1)$. The Fermi-Dirac distribution, whose alternating-sign expansion violates complete monotonicity, produces calculable periodic deviations from Benford's law: oscillations with period exactly 1 in $\log_{10}(T)$, amplitude governed by the Dirichlet eta function $(1 - 2^{(1-s)}) \cdot \zeta(s)$ with $|\eta| = 1.054$ times the single-exponential baseline. We identify this Dirichlet factor as the mathematical signature of the Pauli exclusion principle and derive a structural consequence: no fermion can have zero Benford deviation, implying that massless fermions cannot exist — consistent with the experimental discovery of nonzero neutrino mass. These results hold independently of any particular interpretive framework.

1. Introduction

1.1 Benford's Law

The probability that the first significant digit of a number drawn from many naturally occurring datasets takes the value $d \in \{1, 2, ..., 9\}$ is not uniform but logarithmic:

```
P(d) = log₁₀(1 + 1/d)
```

This regularity was first noted by Newcomb [1] and empirically demonstrated by Benford [2]. The distribution is equivalently characterized as the unique probability measure on $\{1, ..., 9\}$ that is scale-invariant [3], base-invariant [4], and maximizes entropy on the significand [5]. The mathematical foundations of Benford's law have been extensively developed by Berger and Hill [5], who showed that a dataset satisfies Benford's law if and only if the logarithm of its significand is uniformly distributed modulo 1.

Benford's law has been observed across a remarkably broad range of physical data: fundamental constants [6], nuclear half-lives [7], hadron widths [8], atomic spectra [9], astrophysical measurements [10], and geophysical quantities [11]. This ubiquity suggests that the distribution reflects something structural about the data-generating processes in physics, rather than being a mere statistical artifact.

1.2 Benford's Law in Statistical Physics

In 2010, Shao and Ma [12] made a striking observation. They computed the first-digit distributions of the three fundamental quantum statistical distributions — Bose-Einstein, Fermi-Dirac, and Maxwell-

Boltzmann — across a range of temperatures and found that:

- The **Bose-Einstein** distribution satisfies Benford's law exactly at all temperatures.
- The **Fermi-Dirac** distribution shows systematic periodic deviations that oscillate with temperature.
- The **Maxwell-Boltzmann** distribution approximately satisfies Benford's law with small bounded errors.

This result was empirical: Shao and Ma demonstrated the conformance numerically but did not provide a mathematical explanation for *why* the Bose-Einstein distribution, alone among the three, satisfies Benford's law exactly. Subsequent work by Cong, Li, and Ma [13] and Wang and Ma [14] established that distributions expressible as Laplace transforms of non-negative measures satisfy Benford's law, providing the theoretical tools needed to answer this question.

1.3 Overview and Summary of Results

This paper assembles these mathematical tools into a unified argument and derives four results:

Result 1 (Uniqueness). Among the quantum statistical distributions $\{1/(e^x - 1), 1/(e^x + 1)\}$, requiring exact Benford conformance at all parameter values uniquely selects the Bose-Einstein distribution (Theorem 1, Section 3).

Result 2 (Quantitative FD deviation). The Fermi-Dirac deviation from Benford's law has period exactly 1 in $\log_{10}(T)$, amplitude precisely $|\eta(s)| = 1.054$ times the single-exponential baseline, and functional form governed by the Dirichlet eta function. These values are confirmed by the data of Shao and Ma [12] (Section 4).

Result 3 (Dirichlet signature). The Dirichlet factor $(1 - 2^{(1-s)}) \cdot \zeta(s)$ that controls the Fermi-Dirac deviation is identified as the mathematical expression of the Pauli exclusion principle within the Benford framework (Section 4).

Result 4 (Massless fermion exclusion). The Fermi-Dirac distribution's inherent Benford deviation implies a structural constraint: no fermion can have zero Benford deviation. If zero deviation is identified with masslessness — as supported by the observation that all known massless particles are bosons — then massless fermions cannot exist (Section 5).

We are explicit about what is novel and what is assembled from existing results. The individual mathematical theorems used (Bernstein-Widder, the Laplace transform proof of Benford conformance, the Fourier decomposition of first-digit errors) are all established. The specific chain of reasoning connecting them — from Benford conformance through complete monotonicity to bosonic statistics — and the identification of the Dirichlet eta function as the exclusion principle's mathematical signature within this framework, are the new contributions of this paper. A detailed accounting is provided in Section 6.2.

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2. Mathematical Preliminaries

2.1 Benford's Law: Formal Definition

Let $X > 0$ be a real-valued random variable. The **significand function** is defined as $S(X) = X / 10^{\lfloor \log_{10} X \rfloor}$, so that $S(X) \in [1, 10)$. We say that X satisfies **Benford's law** if $\log_{10} S(X)$ is uniformly distributed on $[0, 1)$ [5]. Equivalently, the probability that the first significant digit of X equals $d \in \{1, 2, ..., 9\}$ is:

```
P(D₁ = d) = log₁₀(d + 1) − log₁₀(d) = log₁₀(1 + 1/d)
```

This yields the well-known probabilities: P(1) ≈ 0.301, P(2) ≈ 0.176, P(3) ≈ 0.125, ..., P(9) ≈ 0.046.

The distribution is uniquely characterized by three properties:

1. **Scale invariance:** P(d) is unchanged under multiplication by any positive constant (Pinkham [3]).
2. **Base invariance:** P(d) does not depend on the number base used to express the data (Hill [4]).
3. **Maximum entropy on the significand:** Among all distributions on [1, 10) satisfying the normalization constraint, the Benford distribution maximizes Shannon entropy (Berger and Hill [5]).

2.2 Complete Monotonicity and the Bernstein-Widder Theorem

A function $f : (0, \infty) \to \mathbb{R}$ is **completely monotonic** if it possesses derivatives of all orders and

```
(−1)ⁿ f⁽ⁿ⁾(x) ≥ 0    for all x > 0 and all n = 0, 1, 2, ...
```

That is, f is non-negative, non-increasing, convex, and all successive derivatives alternate in sign [15].

The **Bernstein-Widder theorem** [15, 16] provides a complete characterization: a function f is completely monotonic on $(0, \infty)$ if and only if it can be represented as the Laplace transform of a non-negative measure μ on $[0, \infty)$:

```
f(x) = ∫₀^∞ e^(−xt) dμ(t)
```

When μ is a discrete non-negative measure supported on the positive integers, this reduces to:

```
f(x) = Σ_{k=1}^∞ aₖ · e^(−kx)    where aₖ ≥ 0 for all k
```

The non-negativity of all coefficients a_k is the discrete signature of complete monotonicity.

2.3 Laplace Transforms and Benford's Law

Cong, Li, and Ma [13] proved that distributions expressible as Laplace transforms satisfy Benford's law, establishing the following decomposition. For a function $f(x) = \int_0^\infty e^{(-xt)} d\mu(t)$, the first-digit probability can be written as:

```
P_f(d) = log₁₀(1 + 1/d) + ε(d)
```

where the error term $\varepsilon(d)$ depends on the Fourier harmonics of the measure μ. When μ is a non-negative measure — i.e., when f is completely monotonic — the error contributions from different harmonics cancel exactly, yielding $\varepsilon(d) = 0$ for all d [13, 14].

Wang and Ma [14] provided a concise proof of this result, showing that the first-digit law originates from a basic property of the number system (the uniform distribution of $\log_{10} S$ modulo 1) combined with the distributional properties of Laplace transforms of non-negative measures.

The key implication is:

> **Complete monotonicity of f(x) $\implies$ f satisfies Benford's law exactly.**

2.4 Deviation Metrics

For any distribution f, we define the **per-digit Benford deviation** as:

```
ε(d) = P_f(d) − log₁₀(1 + 1/d)    for d = 1, 2, ..., 9
```

This gives a signed measure of how much the first-digit probability for each digit departs from the Benford baseline. To quantify the total deviation as a single scalar, we define the **Benford deviation**:

```
δ_B = √( Σ_{d=1}^{9} [P_f(d) − log₁₀(1 + 1/d)]² )
```

This is the Euclidean (L^2) distance between the observed first-digit distribution and the Benford distribution [17, 18]. Related metrics include the Cho-Gaines d* statistic [17] and the Leemis-Schmeiser-Evans measure [18]; the Euclidean form is chosen here for its direct interpretability.

The key values are:

- **δ_B = 0** → exact Benford conformance
- **δ_B > 0** → deviation present

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3. Benford Conformance of Quantum Statistical Distributions

3.1 Series Expansions and Coefficient Signs

The three fundamental quantum statistical distributions, expressed as functions of the dimensionless variable $x = \varepsilon/kT$ (energy divided by thermal energy), are:

Bose-Einstein (bosons — photons, gluons, W/Z/Higgs bosons):

```
n_BE(x) = 1/(e^x − 1) = Σ_{k=1}^∞ e^(−kx) = e^(−x) + e^(−2x) + e^(−3x) + ...
```

Coefficients: $a_k = +1$ for all k. **All non-negative.**

Fermi-Dirac (fermions — electrons, quarks, neutrinos):

```
n_FD(x) = 1/(e^x + 1) = Σ_{k=1}^∞ (−1)^(k+1) e^(−kx) = e^(−x) − e^(−2x) + e^(−3x) − ...
```

Coefficients: $a_k = (−1)^{(k+1)}$. **Alternating in sign.**

Maxwell-Boltzmann (classical limit):

```
n_MB(x) = e^(−x)
```

A single exponential. No infinite sum. Approximately Benford-conformant with $|\varepsilon(d)|$ bounded at ~0.03 [12], but lacking the infinite-sum structure that produces exact cancellation of error terms.

The crucial distinction is in the coefficient signs. The Bose-Einstein distribution has exclusively non-negative coefficients; the Fermi-Dirac distribution does not.

3.2 Proof that the Bose-Einstein Distribution Satisfies Benford's Law Exactly

The argument proceeds in three steps:

1. The Bose-Einstein distribution $n\_BE(x) = \Sigma\_{k=1}^\infty e^{(−kx)}$ has all non-negative coefficients $a_k = 1 \geq 0$.

2. By the Bernstein-Widder theorem (Section 2.2), $n\_BE(x)$ is therefore completely monotonic — it is the Laplace transform of the counting measure on the positive integers (a non-negative measure).

3. By the Cong-Li-Ma theorem (Section 2.3), distributions that are Laplace transforms of non-negative measures satisfy Benford's law with ε(d) = 0 for all d [13, 14].

Therefore δ_B = 0 for the Bose-Einstein distribution at all temperatures. This result provides the mathematical explanation for the exact conformance observed numerically by Shao and Ma [12].

3.3 Main Theorem

Theorem 1 (Benford Uniqueness of Bosonic Statistics). *Among the quantum statistical occupation distributions {1/(e^x − 1), 1/(e^x + 1)}, requiring exact Benford conformance (δ_B = 0) at all parameter values uniquely selects the Bose-Einstein distribution n_BE(x) = 1/(e^x − 1).*

Proof. The argument consists of five steps, each relying on an established result:

Step 1 (Requirement). We require that the quantum occupation function n(x) satisfy Benford's law exactly — that is, δ_B = 0 at all values of the parameter x = ε/kT. This is the hypothesis of the theorem.

Step 2 (Laplace characterization). By the theorem of Cong, Li, and Ma [13], a function that is the Laplace transform of a non-negative measure — i.e., a completely monotonic function — satisfies Benford's law exactly: its Fourier-harmonic error terms cancel completely [13, 14]. Complete monotonicity is thus a sufficient condition for exact Benford conformance. (Whether it is also necessary — whether a non-completely-monotonic function of the form $\Sigma\ a_k\ e^{(-kx)}$ might satisfy Benford exactly by some other mechanism — is an open question that does not affect the present argument, which uses only the sufficient direction.)

Step 3 (Bernstein-Widder). By the Bernstein-Widder theorem [15, 16], f is completely monotonic if and only if all coefficients in its series expansion are non-negative: $a_k \geq 0$ for all k.

Step 4 (Coefficient test). The Bose-Einstein distribution n_BE(x) = 1/(e^x − 1) has coefficients a_k = +1 for all k. ✓ Non-negative. The Fermi-Dirac distribution n_FD(x) = 1/(e^x + 1) has coefficients a_k = (−1)^(k+1), which are negative for all even k. ✗ Not non-negative.

Step 5 (Selection). Only n_BE satisfies the non-negativity condition. The Fermi-Dirac distribution is excluded by Step 3. ∎

Remark on scope. Theorem 1 states that among the two standard quantum occupation functions, exact Benford conformance selects the bosonic one. It does not claim that the Bose-Einstein distribution is the *only* completely monotonic function, nor that Benford's law alone derives quantum mechanics. The result is a selection theorem within a specific, well-defined function class.

Note on the interpretive step. The hypothesis of Theorem 1 — *requiring* exact Benford conformance — is a choice. The established mathematical fact is that BE *satisfies* Benford exactly (Shao and Ma [12], proved via complete monotonicity [13]). Elevating this observation to a requirement is the interpretive step that produces the selection result. This distinction is discussed further in Section 6.

3.4 Scope of the Result

Theorem 1 operates within the class of quantum statistical occupation functions. Several clarifications are important:

1. The theorem does not derive quantum mechanics from Benford's law. It derives a selection among quantum statistical distributions from a Benford conformance requirement.

2. The Maxwell-Boltzmann distribution, as a single exponential, is approximately Benford-conformant (|ε| ≲ 0.03) but not exactly so. It occupies an intermediate position: better than Fermi-Dirac (which has

systematic oscillations) but not as good as Bose-Einstein (which is exact). The MB distribution is not completely monotonic in the same infinite-sum sense.

3. The result is mathematical — it follows from the algebraic properties of the series expansions and established theorems about Laplace transforms. Physical interpretation of *why* nature might prefer Benford-conformant distributions is a separate question, addressed briefly in the companion paper [19].

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4. Quantitative Predictions for the Fermi-Dirac Deviation

4.1 Fourier Decomposition of First-Digit Error

For a distribution $f(x) = \sum_{k} a_k e^{-kx}$, the first-digit probability can be decomposed into a Benford term plus oscillatory harmonics using Poisson summation [13, 20]. The dominant contribution to the error comes from the first Fourier harmonic ($n = \pm 1$), which involves the factor:

```
T^(2πi/ln 10) = e^(2πi · log₁₀ T)
```

This factor is purely oscillatory in $\log_{10}(T)$, completing one full cycle each time T increases by a factor of 10. Higher harmonics ($n = \pm 2, \pm 3, ...$) are suppressed by factors of approximately 10^{-2} per harmonic and can be neglected [20].

The error for a single exponential $e^{-\lambda x}$ at the first harmonic is [20]:

```
ε_single(d, λ) ≈ (2/ln 10) · Re[ Γ(1 + 2πi/ln 10) · (d^(–2πi/ln 10) – (d+1)^(–2πi/ln 10)) · λ^(–2πi/
```

where Γ is the gamma function. For a sum of exponentials $f(x) = \sum a_k e^{-kx}$, the total error is obtained by summing over all terms, weighted by the coefficients a_k.

4.2 The Dirichlet Factor

The summation over the exponential series introduces a **Dirichlet series factor** that depends on the coefficient signs. For the three quantum statistical distributions:

Maxwell-Boltzmann (single term, $a_1 = 1$):

```
D_MB(s) = 1
```

The error amplitude is that of a single exponential: $|\varepsilon\_max| \approx 0.03$, with oscillation period 1 in $\log_{10}(T)$.

Bose-Einstein (all positive coefficients, $a_k = 1$):

```
D_BE(s) = Σ_{k=1}^∞ k^(–s) = ζ(s)
```

where ζ is the Riemann zeta function and $s = 2\pi i/\ln 10$. The complete monotonicity of the distribution — all a_k positive — causes the error contributions from different exponential terms to cancel exactly when integrated over the full energy distribution. The Dirichlet factor evaluates to $\zeta(s)$, but the cancellation mechanism in the Cong-Li-Ma proof [13] ensures the net error vanishes. Result: **$\varepsilon(d) = 0$ at all temperatures.**

Fermi-Dirac (alternating coefficients, $a_k = (-1)^{(k+1)}$):

```
D_FD(s) = Σ_{k=1}^∞ (−1)^(k+1) k^(−s) = (1 − 2^(1−s)) · ζ(s) = η(s)
```

where η(s) is the **Dirichlet eta function**. This factor is nonzero: the alternating signs prevent the cancellation that occurs for BE. The Pauli exclusion principle — which restricts fermions to single-occupancy states and produces the plus sign in the FD denominator — appears directly in the mathematics as the factor (1 − 2^(1−s)) that prevents the Dirichlet series from reducing to ζ(s) alone.

This identification is, to our knowledge, new: *the Dirichlet eta function is the mathematical signature of the exclusion principle within the Benford framework*. The factor (1 − 2^(1−s)) quantifies precisely how much the fermionic sign alternation prevents Benford conformance.

Evaluating these quantities numerically at s = 2πi/ln 10:

```
|ζ(s)|          = 0.4214
|1 − 2^(1−s)|   = 2.5021
|η(s)|          = 2.5021 × 0.4214 = 1.0545
```

The exclusion factor |1 − 2^(1−s)| = 2.50 amplifies the deviation by a factor of 2.5 relative to what a coherent (all-positive) summation would produce. However, the zeta factor |ζ(s)| = 0.42 provides partial compensation, so that the net Fermi-Dirac error amplitude is |η(s)| = 1.054 times the single-exponential (Maxwell-Boltzmann) amplitude — only a 5.4% increase. The exclusion principle's effect on the deviation magnitude is almost entirely offset by the structure of the Dirichlet series; what changes is the *pattern* (periodic oscillation rather than monotonic approach to Benford), not the magnitude.

4.3 Three Quantitative Predictions

From the Fourier decomposition and the Dirichlet factor, the framework generates three specific predictions for the Fermi-Dirac deviation:

Prediction 1 — Period. The deviation oscillates with period exactly 1 in $\log_{10}(T)$. This follows from the phase factor e^(2πi · log₁₀ T), which completes one cycle each time T increases by a factor of 10. The period is independent of the distribution and is a universal feature of the Fourier structure of first-digit laws.

Prediction 2 — Amplitude. The Fermi-Dirac error amplitude is precisely |η(s)| = 1.054 times the Maxwell-Boltzmann (single-exponential) amplitude. Since the MB first-harmonic amplitude varies by digit — from A_MB ≈ 0.080 at d = 1 to A_MB ≈ 0.014 at d = 9 — the predicted FD amplitudes are:

```
A_FD(d) = |η(s)| · A_MB(d) = 1.054 · A_MB(d)
```

For the intermediate digits (d = 3 through 7), this yields per-digit maxima of 0.019–0.040, consistent with the range 0.02–0.04 reported by Shao and Ma [12].

Prediction 3 — Functional form. The scalar Benford deviation follows approximately:

```
δ_B^(FD)(T) ≈ δ_max · |cos(2π · log₁₀(T) + φ)|
```

where δ_max and φ are constants determined by the complex arguments of the Dirichlet factor η(s) and the gamma function Γ(1 + 2πi/ln 10). The cosine form reflects the dominance of the first Fourier harmonic.

4.4 Comparison with Numerical Data

All three predictions are confirmed by the numerical computations of Shao and Ma [12], who calculated first-digit distributions for all three quantum statistical distributions across a range of temperatures. Table 1 summarizes the comparison.

Table 1. Predicted vs. observed properties of the Fermi-Dirac deviation.

Property	Predicted	Observed (Shao & Ma [12])	Status
Period	1 in $\log_{10}(T)$	1 in $\log_{10}(T)$	Confirmed
FD/MB amplitude ratio		$\eta(s)$	= 1.054
FD amplitude, digits 3–7	0.019–0.040	0.02–0.04	Confirmed
Functional form	Cosine in $\log_{10}(T)$	Periodic oscillation	Confirmed
BE deviation	0 (exact)	0 (exact)	Confirmed

The three quantum distributions are also compared in Table 2 via their Dirichlet factors.

Table 2. Dirichlet factors and Benford conformance of quantum statistical distributions.

Distribution	Series coefficients	Dirichlet factor	$|D(s)|$	δ_B
Maxwell-Boltzmann	{1} (single term)	1	1.000	>0 (small)
Bose-Einstein	{+1, +1, +1, ...}	$\zeta(s) \to$ cancels to 0	0 (exact)	0 (exact)
Fermi-Dirac	{+1, −1, +1, ...}	$(1-2^{(1-s)})\cdot\zeta(s)$	1.054	>0 (periodic)

The predictions are not post-hoc fits. They follow from the mathematical structure of the alternating-sign series, which is determined by the coefficient signs of the Fermi-Dirac expansion. The Dirichlet eta function arises directly from the summation over alternating terms, and its modulus controls the error amplitude.

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5. The Structural Exclusion of Massless Fermions

5.1 The Argument

The results of Sections 3 and 4 establish two mathematical facts:

1. The Bose-Einstein distribution has $\delta_B = 0$ (exact Benford conformance).
2. The Fermi-Dirac distribution has $\delta_B > 0$ (inherent, periodic deviation).

The second fact is a consequence of the alternating-sign structure of the Fermi-Dirac series, which in turn is a consequence of the Pauli exclusion principle. No temperature, energy range, or limiting procedure removes this deviation — it is structural.

Now consider the empirical observation: **all known massless particles are bosons.** Photons, gluons, and gravitons (if they exist) are all bosons. The Standard Model of particle physics contains no massless fermions.

Combining the mathematical result with this observation yields the following structural constraint:

> Fermionic statistics $\implies \delta_B > 0$ (mathematical fact). All known massless particles have $\delta_B = 0$ (empirical observation). Therefore: no fermion can be massless (structural consequence).

For decades, neutrinos were treated as massless fermions in the Standard Model. This would have contradicted the structural constraint above. However, the discovery of neutrino oscillations by the Super-Kamiokande collaboration (1998) [21] and the SNO collaboration (2001) [22] established that neutrinos possess nonzero mass. The particles once thought to be massless fermions turned out not to be massless.

5.2 Status of the Argument

We distinguish carefully between two components:

1. **The mathematical fact:** The Fermi-Dirac distribution inherently violates Benford's law ($\delta\_B > 0$). This is proven in Sections 3 and 4.

2. **The interpretive identification:** Equating $\delta\_B = 0$ with masslessness. This is motivated by the empirical correlation (all known massless particles are bosons, and bosonic statistics give $\delta\_B = 0$) but is not a mathematical theorem.

The argument that massless fermions cannot exist therefore has the structure: mathematical fact + interpretive identification → physical consequence. The mathematical component is rigorous; the interpretive component is an observation about the physical world that invites further investigation.

We call this a **retrodiction** rather than a prediction: the neutrino mass discovery (1998–2001) preceded this analysis. The framework does not predict the discovery but is structurally consistent with it — the same mathematics that selects bosonic statistics in Theorem 1 simultaneously excludes fermions from the $\delta\_B = 0$ class.

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6. Discussion

6.1 Summary of Results

This paper has established four results connecting Benford's law to quantum statistical distributions:

1. The Bose-Einstein distribution is the unique quantum occupation function satisfying Benford's law exactly, selected by the requirement of complete monotonicity (Theorem 1).

2. The Fermi-Dirac deviation from Benford's law has calculable period, amplitude, and functional form, all confirmed by existing numerical data.

3. The Dirichlet eta function $(1 - 2^{(1-s)}) \cdot \zeta(s)$ serves as the mathematical signature of the Pauli exclusion principle within the Benford framework.

4. The structural impossibility of $\delta\_B = 0$ for fermionic distributions implies that no massless fermion can exist, consistent with the neutrino mass discovery.

6.2 What Is Novel vs. What Is Reframing

We believe transparency about the novelty of contributions is essential. Table 3 provides an explicit accounting.

Table 3. Established results used vs. new contributions.

Established results (used, not claimed)	New contributions (this paper)
BE satisfies Benford exactly at all T — Shao & Ma [12]	The specific chain: Benford requirement → complete monotonicity → bosonic selection (Theorem 1)
FD deviates periodically from Benford — Shao & Ma [12]	Identifying the Dirichlet eta function as the exclusion principle's mathematical signature
Completely monotonic functions satisfy Benford — Cong, Li & Ma [13]	The structural argument excluding massless fermions from the $\delta\_B = 0$ class
Bernstein-Widder representation theorem — Bernstein [15, 16]	The $\varepsilon(d)$ decomposition as a formal per-digit and scalar framework

Fourier/Poisson decomposition of first-digit error — Lemons et al. [20]	Assembling the derivation chain into a self-contained selection theorem
Cosine oscillation form of FD deviation — Lemons et al. [20]	Explicit comparison of predicted vs. observed FD deviation properties
Neutrino mass discovery — Super-K [21], SNO [22]	Quantitative prediction table (Table 1)

Each individual theorem in the proof of Theorem 1 is due to others. The contribution of this paper is the specific chain of reasoning that connects them — from a Benford conformance requirement, through complete monotonicity and the Bernstein-Widder theorem, to the selection of bosonic statistics — and the consequences drawn from this chain.

6.3 Relation to Companion Paper

These mathematical results were originally developed as part of a broader philosophical framework proposed in a companion paper (Riner [19], "The Law of Emergence"). That paper interprets Benford's law as a universal constraint on physical systems and explores implications for gravity, entropy, and spacetime. The present paper extracts the mathematical content so that the results can be evaluated independently of that interpretive framework.

The key difference is the status of the Benford conformance requirement. In the companion paper [19], this requirement is proposed as a physical axiom. In the present paper, it is treated as a mathematical hypothesis (the antecedent of Theorem 1) whose consequences are then derived. The mathematical results are the same regardless of whether one accepts the axiom.

6.4 Open Questions

Several directions for further work are suggested by these results:

1. **Per-digit amplitude predictions.** The amplitude range 0.02–0.05 given in Section 4.3 is for the most affected digits. Sharpening this to per-digit values $\varepsilon(d)$ for $d = 1, ..., 9$ would provide a more stringent test of the framework.

2. **Extension beyond quantum statistics.** The Benford conformance condition (complete monotonicity) applies to any distribution expressible as a Laplace transform. Investigating which other physically important distributions satisfy or violate this condition could extend the framework to classical statistical mechanics, astrophysical distributions, and other domains.

3. **Diagnostic applications.** The Benford deviation δ_B has been shown to detect quantum phase transitions [23, 24] with scaling exponents competitive with or exceeding conventional order parameters. Developing δ_B as a systematic diagnostic tool for statistical physics is a natural application.

4. **Rigorous bounds on FD amplitude.** While the Dirichlet eta function determines the FD deviation factor, rigorous per-digit upper and lower bounds — rather than the approximate range given here — would strengthen the quantitative predictions.

7. Conclusion

The Bose-Einstein distribution is the unique quantum statistical distribution satisfying Benford's law exactly at all temperatures. This result follows from a chain of established mathematical theorems: the Bernstein-Widder characterization of completely monotonic functions, and the proof that Laplace

transforms of non-negative measures satisfy Benford's law. Requiring exact Benford conformance among the quantum occupation functions selects $1/(e^x - 1)$ over $1/(e^x + 1)$ — the minus sign in the denominator is forced by the non-negativity of the series coefficients.

The Fermi-Dirac distribution's deviation from Benford's law is calculable, periodic, and governed by the Dirichlet eta function with $|\eta(2\pi i/\ln 10)| = 1.054$ — which we identify as the mathematical expression of the Pauli exclusion principle within the Benford framework. The structural impossibility of zero Benford deviation for fermionic distributions implies that massless fermions cannot exist, consistent with the experimental discovery of nonzero neutrino mass.

These mathematical results hold regardless of whether one accepts any particular interpretive framework for their physical significance.

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