Simulation and reconstruction of in-line holograms acquired with spherical waves, in paraxial approximation

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For more details, please read the full article [1].

1 Introduction

The principle of hologram formation with a spherical wave is illustrated in Fig. 1.

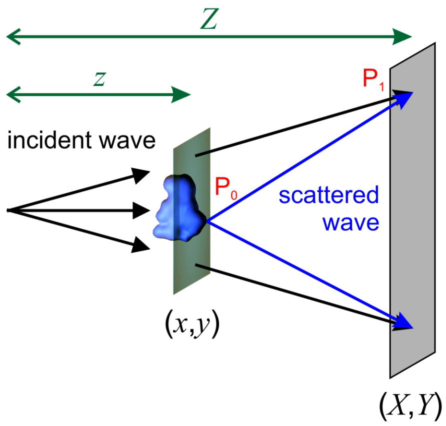

Figure 1: In-line holography schemes realized with a spherical wave.

1.1 Simulation of hologram

The incident wave in the object plane is given by:

$$U_{\mathrm{incident}}(x,y)=\frac{\exp\left(ikr\right)}{r},\tag{1}$$

where $\vec{r}=(x,y,z)$ and z is the distance between source and object plane, as indicated in Fig. 1. The exit wave beyond the object is given by the product of the incident wave and the transmission function of the object:

$$U_{\mathrm{exit\ wave}}(x,y)=U_{\mathrm{incident}}(x,y)\cdot t(x,y)=\frac{\exp\left(ikr\right)}{r}\cdot t(x,y).\tag{2}$$

The propagation of the wave towards the detector is described by the Fresnel-Kirchhoff diffraction formula:

$$U_{\mathrm{detector}}(X,Y)=-\frac{i}{\lambda}\int\int\frac{\exp\left(ikr\right)}{r}\cdot t(x,y)\frac{\exp\left(ik\left|\vec{r}-\vec{R}\right|\right)}{\left|\vec{r}-\vec{R}\right|}\,\mathrm{d}x\mathrm{d}y,\tag{3}$$

where $\vec{r}=(x,y,z)$ is a vector pointing from the source to a point in the object, $\vec{R}=(X,Y,Z)$ is a vector pointing from the source to a point on the detector, and $\left|\vec{r}-\vec{R}\right|$ is the distance between a point in the object plane and a point in the detector plane (see Fig. 1). The hologram is formed as squared amplitude of the wavefront in the detector plane.

1.2 Reconstruction of hologram

The reconstruction of a digital hologram recorded with plane waves begins with illumination with the reference wave $R(X,Y)=\exp(ikR)/R$ followed by propagation backward to the object plane described by the Fresnel-Kirchhoff diffraction formula:

$$U(x,y)\approx\frac{i}{\lambda}\int\int\frac{\exp\left(ikR\right)}{R}H(X,Y)\frac{\exp\left(-ik\left|\vec{r}-\vec{R}\right|\right)}{\left|\vec{r}-\vec{R}\right|}\,\mathrm{d}X\mathrm{d}Y.\tag{4}$$

2 Paraxial Approximation

In the paraxial approximation, the following approximations are valid:

$$r \approx z + \frac{x^2 + y^2}{2z} \tag{5}$$

and

$$\left|\vec{r} - \vec{R}\right| \approx Z + \frac{(x-X)^2 + (y-Y)^2}{2Z}. \tag{6}$$

2.1 Simulation of hologram

Equations 5 and 6 allow the following expansion of Eq. 3:

$$\begin{aligned} U_{\text{detector}}(X,Y) = -\frac{i}{\lambda Z z} \exp\left(\frac{2\pi i}{\lambda}(Z+z)\right) \int\!\!\int \exp\left(\frac{i\pi}{\lambda z}(x^2+y^2)\right) t(x,y)\times \\ \times \exp\left(\frac{i\pi}{\lambda Z}\left((x-X)^2+(y-Y)^2\right)\right) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{7}$$

By taking into account that $z \ll Z$, we rewrite:

$$\begin{aligned} U_{\text{detector}}(X,Y) = -\frac{i}{\lambda Z z} \exp\left(\frac{2\pi i}{\lambda}(Z+z)\right) \exp\left(\frac{i\pi}{\lambda Z}\left(X^2+Y^2\right)\right) \int\!\!\int \exp\left(\frac{i\pi}{\lambda z}(x^2+y^2)\right)\times \\ \times t(x,y) \exp\left(-\frac{2\pi i}{\lambda Z}\left(xX+yY\right)\right) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{8}$$

Next, we rewrite Eq. 8 in the form of a convolution [2]

$$\begin{aligned} U_{\text{detector}}(X,Y) \approx -\frac{i}{\lambda Z z} \exp\left(\frac{2\pi i}{\lambda}(Z+z)\right) \exp\left(\frac{i\pi}{\lambda Z}\left(X^2+Y^2\right)\right) \int\!\!\int t(x,y)\times \\ \times \exp\left(\frac{i\pi}{\lambda z}\left(\left(x - X\frac{z}{Z}\right)^2 + \left(y - Y\frac{z}{Z}\right)^2\right)\right) \mathrm{d}x\mathrm{d}y \end{aligned} \tag{9}$$

of the transmission function with the Fresnel function $s(x,y)$, whereby the latter is given by

$$s(x,y) = -\frac{i}{\lambda z} \exp\left(\frac{i\pi}{\lambda z}\left(x^2+y^2\right)\right), \tag{10}$$

and its Fourier transform is given by

$$S(u,v) = -\frac{i}{\lambda z} \int\!\!\int \exp\left(\frac{i\pi}{\lambda z}\left(x^2+y^2\right)\right) \exp\left(-2\pi i\left(xu+yv\right)\right) \mathrm{d}x\mathrm{d}y = \exp\left(-i\pi\lambda z\left(u^2+v^2\right)\right), \tag{11}$$

The hologram is then calculated as:

$$H(X,Y) = \left|U_{\text{detector}}(X,Y)\right|^2 = \left|t(X,Y) \otimes s(X,Y)\right|^2. \tag{12}$$

The coordinates in the Fourier domain (u,v) are sampled with the pixel size:

$$\Delta_F = \frac{1}{N\Delta_{\text{Object}}} = \frac{1}{S_{\text{Object}}}, \tag{13}$$

where $\Delta_{\text{Object}} = S_{\text{Object}}/N$ is the pixel size in the object plane and $S_{\text{Object}} \times S_{\text{Object}}$ is the object area size.

Thus, a hologram is simulated by:

(a) Calculating the Fourier transform of $t(x,y)$.
(b) Simulating $S(u,v) = \exp\left(-i\pi\lambda z(u^2+v^2)\right)$.
(c) Multiplying the results of (a) and (b).
(d) Calculating the inverse Fourier transform of (c).
(e) Taking the square of the absolute value of the result (d).

The size of the simulated hologram is equal to the size of the object area multiplied by the magnification factor:

$$M = \frac{Z}{z}. \tag{14}$$

2.2 Reconstruction of hologram

The hologram is reconstructed in the reciprocal order by:

(a) Calculating the inverse Fourier transform of $H(X, Y)$.

(b) Simulating $S^*(u, v) = \exp\big(i\pi\lambda z(u^2 + v^2)\big)$.

(c) Multiplying the results of (a) and (b).

(d) Calculating the Fourier transform of (c).

The size of the reconstructed object area is equal to the size of the hologram divided by the magnification factor M.

References

[1] T. Latychevskaia and H.-W. Fink, "Practical algorithms for simulation and reconstruction of digital in-line holograms," Applied Optics **54,** 2424 – 2434 (2015).

[2] T. Latychevskaia, J.-N. Longchamp, and H.-W. Fink, "When holography meets coherent diffraction imaging," Opt. Express **20,** 28871 – 28892 (2012).