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Trémaux trees and planarity☆

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ABSTRACT

We present a characterization of planarity based on Trémaux trees (i.e. DFS trees), from which we deduce a rather simple planarity test algorithm. We finally recall a theorem on "cotree critical non-planar graphs" which very much simplifies the search for a Kuratowski subdivision in a non-planar graph.

1. Introduction

It has been well known since the publications in 1973–74 by Hopcroft and Tarjan [13] that the time complexity of the problem of graph planarity is linear in the number of edges. In the late seventies, we produced the so-called Left–Right algorithm which has been recognized as the fastest among the ones implemented, on the basis of comparative tests performed by graph specialists [1]. In 1982 [8] we gave a characterization of planarity in terms of Trémaux trees which eases the justification of the Left–Right algorithm, although a formal proof of its validity remains rather tricky. Two years ago while working on the correctness of the Left–Right algorithm we were puzzled to realize that we did not use our simpler characterization of planarity published in 1985 [11]. Looking at it closely, we introduced a third characterization of planarity [7] — again based on Trémaux trees — which checks planarity by considering at most Δm configurations instead of a cubic number. That characterization suggests an algorithm which straightforwardly computes the embedding of the graph if it is planar. It is quite a lot simpler than the Left–Right algorithm and even more efficient. That characterization will also simplify our linear time algorithm for extracting a Kuratowski subdivision in a non-planar graph. Actually this new algorithm can be depicted as a fast non-recursive version of the Hopcroft–Tarjan one.

After presenting properties of Trémaux trees related to planarity and the main ideas of our new planarity algorithm, we shall recall some newer results [6] which very much simplify the search for a Kuratowski subdivision in a non-planar graph.

In this paper we only consider finite simple graphs.

☆ This work was initiated with Pierre Rosenstiehl.

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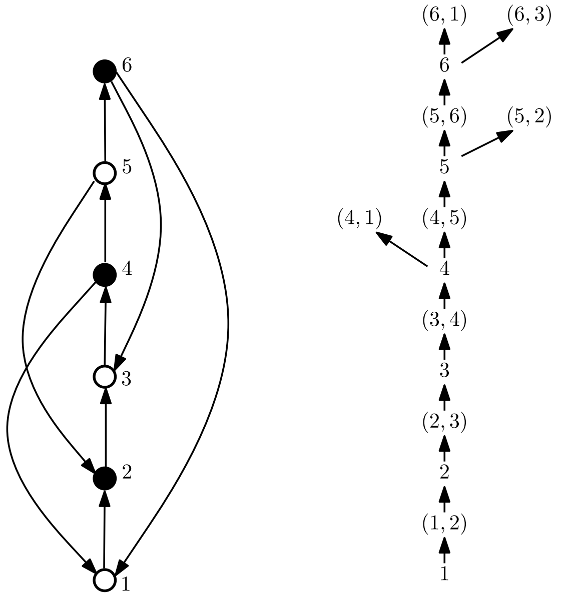


Fig. 1. The partial order $\prec$ defined by a Trémaux tree of $K_{3,3}$ (the cover relation correspond to bottom-up arrows as usual).

2. Trémaux trees

Depth-first search (DFS) is a fundamental graph searching technique known since the 19th century (see for instance Luca's report on Trémaux's work [14] or Tarry's publication of the Trémaux's algorithm [16]) and popularized by Hopcroft and Tarjan [12,15] in the seventies. The structure of DFS enables efficient algorithms for many other graph problems [2]. Performing a DFS on a graph defines a spanning tree with specific properties (also known as a *Trémaux tree*) and an embedding of it as a rooted planar tree, the edges going out of a vertex being circularly ordered according to the discovery order of the DFS.

A rooted spanning tree $\mathcal{T}$ of a graph $G = (V, E)$ defines a partition of the edge set of G into two classes, the set of *tree edges* T and the set of *cotree edges* $E \setminus T$. For every cotree edge e, $T + e$ contains a unique cycle, the *fundamental cycle* of e. A rooted tree defines a partial order $\preceq$ on V: $x \preceq y$ if the tree path linking y to the root of $\mathcal{T}$ includes x. The rooted tree $\mathcal{T}$ is a *Trémaux tree* if every cotree edge is incident to two comparable vertices (with respect to $\preceq$). A Trémaux tree $\mathcal{T}$ defines an orientation of the edges of the graph: an edge $\{x, y\}$ (with $x \prec y$) is oriented from x to y (upwards) if it is a tree edge and from y to x (downwards) if it is a cotree edge. According to this orientation, circuits are exactly the fundamental cycles. We will denote by $\omega^+(v)$ the set of the edges incident to a vertex v and going out of v. When $\mathcal{T}$ is a Trémaux tree, the partial order $\preceq$ is extended to $V \cup E$ (or to G for short) as follows: for any edge $e = (x, y)$ oriented from x to y, put $x \prec e$ and if $x \prec y$ (that is, if e is a tree edge) also put $e \prec y$; see Fig. 1.

Notice that in the partial order $\preceq$, all elements α and β of G have a unique greatest lower bound (meet) $\alpha \wedge \beta$.

Definition 2.1. Although it is usually defined on vertices in the literature, we define here "low" as a mapping from E to V:

$$\mathrm{low}(e) = \begin{cases} \min\ \{v \in V : \exists (u, v) \in E,\ (u, v) \succeq e\} & \text{if } e \text{ is a tree edge,} \\ y & \text{if } e = (x, y) \text{ is a cotree edge.} \end{cases}$$

In other words, low(e) is the minimum vertex that a directed path starting with e and containing at most one cotree edge can reach.

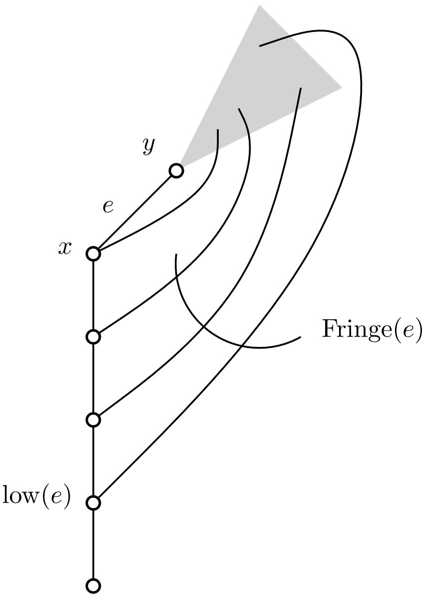


Fig. 2. The fringe of a tree edge *e*.

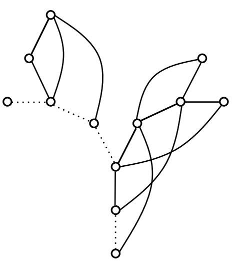
Definition 2.2. The *fringe* Fringe(e) of an edge $e = (x, y)$ is defined by

$$\text{Fringe}(e) = \{f \in E \setminus T : f \succeq e \ \text{ and } \ \text{low}(f) \prec x\}.$$

In other words, the fringe of a tree edge $e = (x, y)$ is the set of all cotree edges linking a vertex in the subtree rooted at y and a vertex strictly smaller than x (see Fig. 2), while the fringe of a cotree edge f is just $\{f\}$.

Definition 2.3. Le e be a tree edge:

- if $\text{Fringe}(e) = \emptyset$, the edge e is a *block edge* (dotted in the picture);
- otherwise, if all the edges in Fringe(e) have the same low, the edge e is a *thin edge* (light in the picture);
- otherwise, the edge e is a *thick edge* (fat in the picture).



Definition 2.4. A "*Low-order*" $\prec^\star$ is a partial order on E such that, for any $v \in V$ and any $e, f \in \omega^+(v)$:

- if $\text{low}(e) \prec \text{low}(f)$ then $e \prec^\star f$,
- if $\text{low}(e) = \text{low}(f)$, and f is a thick tree edge but e is not, then $e \prec^\star f$.

Otherwise, if e and f are not going out of the same vertex, e and f are not comparable with respect to $\prec^\star$. (See Fig. 3.)

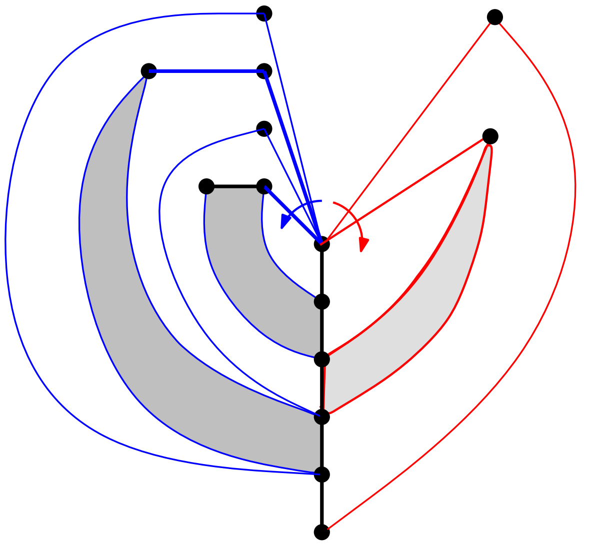


Fig. 3. If G is planar, the Low-order $\prec^\star$ is compatible with the circular order of outgoing edges whose low incidences are on the same side of the tree (left or right).

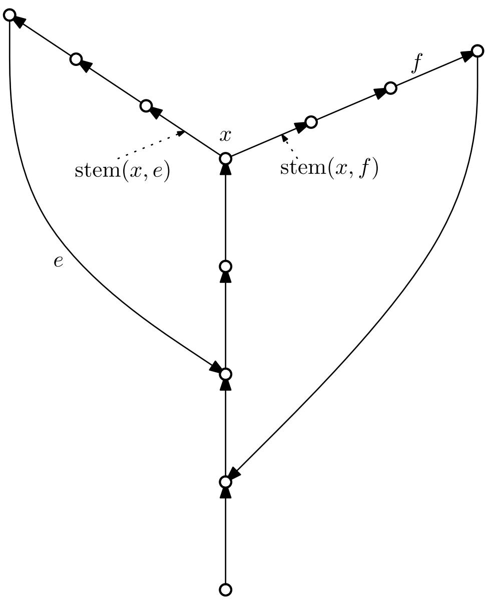


Fig. 4. Definition of "stem".

It will also be helpful to introduce notation for the first edge in the directed chain from a vertex x to an edge e (see Fig. 4):

Definition 2.5. For $x \prec e$, where $x \in V$ and $e \in E$, we define

$$\mathrm{stem}(x, e) = \min\{f \in E : x \prec f \preceq e\}. \tag{1}$$

Notice that in this definition, as in the remainder of the paper, "min" will always be related to the partial order $\prec$ defined by the Trémaux tree considered.

3. Trémaux trees and planarity

Planarity has been related to Trémaux trees by de Fraysseix and Rosenstiehl in a series of articles [11,10,8].

We give here a new characterization of planarity based on a bicoloration of the cotree edges of a Trémaux tree with constraints expressed by a single configuration.

Definition 3.1. Let v be a vertex and let $e_1, e_2 \in \omega^+(v)$.

The *fringe-opposed subset* $\mathrm{Fop}_{e_2}(e_1)$ is defined by

$$\mathrm{Fop}_{e_2}(e_1) = \{f \in \mathrm{Fringe}(e_1) : \mathrm{low}(f) \succ \mathrm{low}(e_2)\}$$

Definition 3.2. Given a graph G and a Trémaux tree $\mathcal{T}$ of G, a coloring $\lambda : E \setminus T \to \{-1, 1\}$ is an *F-coloring* if, for every vertex v and any edges $e_1, e_2 \in \omega^+(v)$, $\mathrm{Fop}_{e_2}(e_1)$ and $\mathrm{Fop}_{e_1}(e_2)$ are monochromatic and colored differently.

If λ is an F-coloring, it is easily checked that $-\lambda$ is also an F-coloring.

In a planar drawing, an F-coloring is defined by the partition of the cotree edges into two sets, the edges f having their low incidence on the left (resp. the right) of the tree edge $\mathrm{stem}(\mathrm{low}(f), f)$. Formally, a cotree f is a *left edge* (resp. a *right edge*) if the fundamental cycle of f is directed counterclockwise (resp. clockwise).

Lemma 3.1. *Let G be a planar graph with Trémaux tree $\mathcal{T}$. Then G has an F-coloring.*

Proof. Assume that G is embedded on the plane, the root of T belonging to the unbounded face of G. Define $\lambda : E \setminus T \to \{-1, 1\}$ as follows: for $e \in E \setminus T$, if the fundamental cycle $\gamma(e)$ of e (which is a circuit) is oriented clockwise then $\lambda(e) = -1$, otherwise $\lambda(e) = 1$. Notice that a circuit is oriented clockwise if and only if the disk that it bounds lies on the right of the edges of the circuit. Let $v \in V$ and $e_1, e_2 \in \omega^+(v)$. Let g_1 (resp. g_2) be a cotree edge such that $g_1 \succ e_1$ and $\mathrm{low}(g_1) = \mathrm{low}(e_1)$ (resp. $g_2 \succ e_2$ and $\mathrm{low}(g_2) = \mathrm{low}(e_2)$). Let γ be the cycle formed by the edge g_1, the tree chain between $\mathrm{low}(g_1)$ and $\mathrm{low}(g_2)$, the edge g_2, and the tree chain from the upper incidence vertex of g_2 to the upper incidence vertex of g_1 (through v). Let u be the maximum of $\mathrm{low}(e_1)$ and $\mathrm{low}(e_2)$ with respect to $\prec$. Let D_1 (resp. D_2) be the disk bounded by the fundamental cycle γ_1 of g_1 (resp. by the fundamental cycle γ_2 of g_2). Let P be the tree chain between u and v. Two cases may occur:

- P is in the interior of the disk D_γ bounded by γ. Then γ_1 and γ_2 are oriented in opposite ways, for instance $\lambda(g_1) = 1$ and $\lambda(g_2) = -1$. Then the right of P belongs to D_1 and the left of P belongs to D_2. As $f_1 \in \mathrm{Fop}_{e_2}(e_1)$ (resp. $f_2 \in \mathrm{Fop}_{e_1}(e_2)$) cannot cross γ_2 (resp. γ_1), it belongs to D_1 (resp. D_2); hence $\lambda(f_1) = \lambda(g_1)$ and $\lambda(f_2) = \lambda(g_2)$.
- P is in the exterior of the disk D_γ bounded by γ. Then γ_1 and γ_2 are oriented in the same way and one of D_1, D_2 (say D_1) is included in the other. Then the right of P belongs to D_1 and the left of P belongs to the complement of D_2. As $f_1 \in \mathrm{Fop}_{e_2}(e_1)$ (resp. $f_2 \in \mathrm{Fop}_{e_1}(e_2)$) cannot cross γ_2 (resp. γ_1), it belongs to D_1 (resp. the complement of D_2) and hence $\lambda(f_1) = \lambda(g_1)$ and $\lambda(f_2) = -\lambda(g_2)$.

Whichever case applies, we deduce that $\mathrm{Fop}_{e_2}(e_1)$ and $\mathrm{Fop}_{e_1}(e_2)$ are monochromatic and of different color; hence λ is an F-coloring. $\square$

Define

$$\mathrm{L}(v) = \{f \in E \setminus T : f \succ v \text{ and } \forall g \in E \setminus T, g \succ v \Rightarrow \mathrm{low}(g) \succeq \mathrm{low}(f)\}.$$

Definition 3.3. A coloring $\sigma : E \to \{-1, 1\}$ is a *strong F-coloring* if

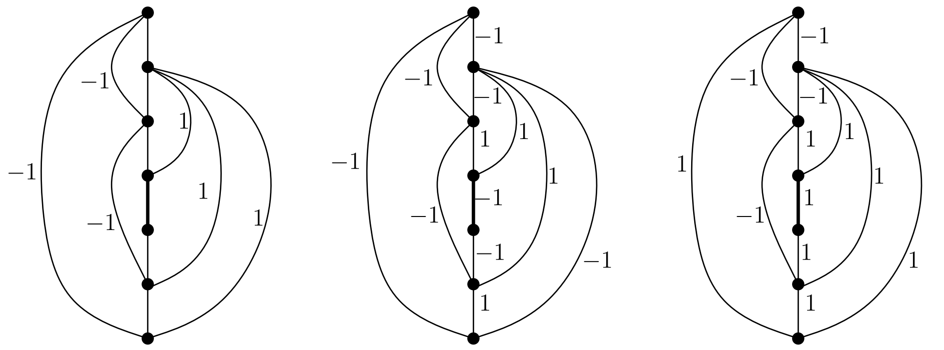


Fig. 5. An F-coloring of G (on the left) may be used to define several strong F-colorings (middle and right).

- the restriction of σ to $E \setminus T$ is an F-coloring,
- for any $v \in V$, $\mathrm{L}(v)$ is monochromatic,
- for any tree edge e such that $\mathrm{Fringe}(e) \neq \emptyset$, there exists an edge $f \in \mathrm{Fringe}(e)$ such that

$$\mathrm{low}(f) = \max\{\mathrm{low}(g) : g \in \mathrm{Fringe}(e)\},$$
$$\sigma(f) = \sigma(e).$$

Again, if σ is a strong F-coloring then so is $-\sigma$.

Lemma 3.2. *If G has an F-coloring then it has a strong F-coloring.*

Proof. We first define σ on the cotree edges. To do that, we start by considering the constraints due to the fact that this restriction has to be an F-coloring. These constraints may be satisfied by assumption, as G has an F-coloring. The addition of the constraints that the sets $\mathrm{L}(v)$ are monochromatic may not lead to a contradiction, as the only constraints involving $f_1 \in \mathrm{L}(v)$ would also involve any $f_2 \in \mathrm{L}(v)$ and would require that f_1 and f_2 actually have the same color. A coloring of the tree edges may then be easily achieved. Notice that the extension of the coloring to the tree edges is usually not uniquely defined (cf Fig. 5). $\square$

Definition 3.4. The "*out-order*" $\prec_{\mathrm{out}}$ is defined as follows: two edges e, f are compared by $\prec_{\mathrm{out}}$ if and only if there exists $v \in V$ such that $e, f \in \omega^+(v)$. In this case, we have $e \prec_{\mathrm{out}} f$ if

- $\sigma(e) = \sigma(f) = 1$ and $f \prec^\star e$,
- $\sigma(e) = 1$ and $\sigma(f) = -1$, or
- $\sigma(e) = \sigma(f) = -1$ and $e \prec^\star f$.

This definition is consistent with the intuition that we have on planar embeddings where the root of the Trémaux tree is on the unbounded face and where (for a cotree edge e) $\sigma(e) = -1$ (resp. $\sigma(e) = 1$) means that the fundamental cycle of e is oriented clockwise (resp. anticlockwise). See Fig. 3. Notice that if one considers the strong F-coloring $-\sigma$ instead of σ, then the out-order is reversed.

Definition 3.5. Let G be a graph with Trémaux tree T and strong F-coloring σ. The *σ-traversal order* of T is the linear order $<_{\mathrm{T}}$ defined on E by

$$e <_{\mathrm{T}} f \iff (e \prec f) \quad \text{or} \quad (\mathrm{stem}(e \wedge f, e) \prec_{\mathrm{out}} \mathrm{stem}(e \wedge f, f)).$$

Remark that $<_{\mathrm{T}}$ is the order in which a depth-first approach using $\prec_{\mathrm{out}}$ as a priority indication discovers the edges of the graph.

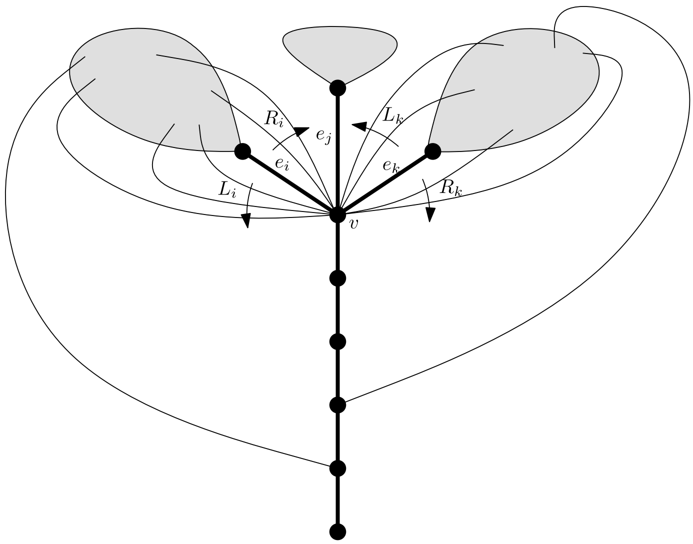


Fig. 6. Circular order of incoming edges. Here $e_i <_T e_j <_T e_k$. The edges of L_i, L_k, corresponding to $\sigma = -1$, are ordered in decreasing $<_T$ order, while the edges in R_i, R_k, corresponding to $\sigma = 1$, are ordered in increasing $<_T$ order.

Definition 3.6. Let G be a graph with Trémaux tree $\mathcal{T}$ and strong F-coloring σ. The *σ-map* of G is defined by the circular order around each vertex v of G (see Fig. 6):

Around v one finds the tree edge entering v (if v is not the root of $\mathcal{T}$) and then $L_1, e_1, R_1, L_2, e_2, R_2, \dots, L_p, e_p, R_p$ where $e_1 <_T e_2 < + \cdots <_T e_p$, L_i (resp. R_i) is the set of incoming cotree edges $f = (x, v)$ such that $\sigma(f) = -1$ (resp. $\sigma(f) = 1$) and $\text{stem}(v, f) = e_i$ ordered in decreasing $<_T$ order (resp. in increasing $<_T$ order).

Notice that if one considers the strong F-coloring $-\sigma$ instead of σ, then the circular orders are reversed.

In any drawing of a σ-map of G in the plane where the tree edges cross no other edges, where two cotree edges may cross at most once, and where the root of $\mathcal{T}$ is on the unbounded face, a cotree edge g with a negative (resp. positive) σ-value is such that the disk bounded by the fundamental cycle of g is on the left of g (resp. on the right of g).

Lemma 3.3. *Let G be a graph with Trémaux tree $\mathcal{T}$ and strong F-coloring σ.*

In a drawing of the σ-map of G in the plane where tree edges cross no other edges and two cotree edges may cross at most once, two cotree edges with the same σ-value cannot cross.

Proof. We prove the lemma by contradiction. Assume that two cotree edges e and f cross, with $e <_T f$ and $\sigma(e) = \sigma(f)$. By considering the strong F-coloring $-\sigma$ instead of σ and a mirror image of the drawing, one may assume that $\sigma(e) = \sigma(f) = -1$. As the fundamental cycle of e and the fundamental cycle of f cannot cross on a single point and as tree edges cross no other edges, we deduce that some of these cycles have some vertex in common. Thus $\text{low}(e)$ and $\text{low}(f)$ are comparable with respect to $\prec$. Let $v = e \wedge f$, $\alpha = \text{stem}(v, e)$, $\beta = \text{stem}(v, f)$. By definition of $<_T$ we have $\alpha <_T \beta$. As $\sigma(e) = -1$, the disk D bounded by the fundamental cycle of e is on the left of e (and hence on the left of α); thus β (being to the right of α) is out of D. As f crosses e, it reaches its lower incidence inside D hence $\text{low}(f) \geq \text{low}(e)$.

- If $\text{low}(e) = \text{low}(f)$ then f is before e in the circular order at $\text{low}(e)$, which contradicts $e <_T f$.
- Otherwise, as $f \in \text{Fop}_\alpha(\beta)$ and $\sigma(e) = \sigma(f)$, we have $e \notin \text{Fop}_\beta(\alpha)$ and hence $\text{low}(e) \leq \text{low}(\beta)$. As $\text{Fop}_\alpha(\beta)$ is monochromatic and non-empty, $\sigma(\beta) = \sigma(f) = -1$ (by the definition of a strong F-coloring). According to the definition of $\prec_{\text{out}}$ we deduce that $\sigma(\alpha) = -1$; hence $\beta \prec^\star \alpha$ and

low(β) $\leq$ low(α) and hence low(α) $=$ low(β) $=$ low(e) as low(α) $\leq$ low(e) $\leq$ low(β). As low(f) $>$ low(β), β is a thick tree edge; hence α is also a thick tree edge (for otherwise $\alpha \prec^* \beta$) and $\mathrm{Fop}_\beta(\alpha) = \mathrm{Fringe}(\alpha) \setminus L(\mathrm{low}(\beta)) = \mathrm{Fringe}(\alpha) \setminus L(\mathrm{low}(\alpha))$ is not empty. Thus it is monochromatic, of color opposed to that of $\mathrm{Fop}_\alpha(\beta)$, and thus $\sigma(\alpha) = 1$ by definition of a strong F-coloring, a contradiction. □

Lemma 3.4. *Let G be a graph with Trémaux tree $\mathcal{T}$.*

In a drawing of G in the plane where tree edges cross no other edges and two cotree edges may cross at most once, two crossing cotree edges e and f have comparable lows (with respect to $\prec$).

Proof. As the fundamental cycle of e and the fundamental cycle of f cannot cross at a single point, and as tree edges cross no other edges and e and f cross exactly once, we deduce that these cycles have a vertex v in common. As the path from the root of $\mathcal{T}$ to v includes both low(e) and low(f), they are comparable with respect to $\prec$. □

Lemma 3.5. *Let G be a graph with Trémaux tree $\mathcal{T}$ and strong F-coloring σ.*

In a drawing of the σ-map of G in the plane where tree edges cross no other edges and two cotree edges may cross at most once, two cotree edges with the different σ-value cannot cross.

Proof. We prove the lemma by contradiction. Assume that two cotree edges e and f cross, with $e <_{\mathrm{T}} f$ and $\sigma(e) \neq \sigma(f)$. By considering the strong F-coloring $-\sigma$ instead of σ, and a mirror image of the drawing, and by exchanging the labels e and f, one may also assume that low(f) $\preceq$ low(e) (as these lows are comparable, according to Lemma 3.4).

Let $v = e \wedge f$, $\alpha = \mathrm{stem}(v, e)$, $\beta = \mathrm{stem}(v, f)$. By definition of $<_{\mathrm{T}}$ we have $\alpha <_{\mathrm{T}} \beta$. If $\sigma(e) = -1$ and $\sigma(f) = 1$ then the disk bounded by the fundamental cycles of e is on the left of e (and hence on the left of α); thus β (being to the right of α) is out of D. As f crosses e, it reaches its lower incidence inside D and hence $\sigma(f) = -1$, a contradiction. It follows that $\sigma(e) = 1$ and $\sigma(f) = -1$.

We first prove that we may assume that $f \in L(v)$. Otherwise, let $g \in L(v)$ and let π be the intersection point of e and f in the drawing. As f and g do not cross the tree, the union of the disks bounded by the fundamental cycles of e and f is a disk which contains v in its interior. Thus g crosses at least one of these fundamental cycles and hence crosses at least one of e and f. According to Lemma 3.3, g cannot cross a cotree edge with the same σ-value; hence it crosses the cotree edge with opposite σ-value. By considering g and the cotree edge that it crosses, we reduce to the case where $f \in L(v)$.

The edge e cannot belong to $L(v)$ as this set is monochromatic, by definition of a strong F-coloring. Hence low(e) $>$ low(f) and $e \in \mathrm{Fop}_\beta(\alpha)$. It follows that $\sigma(\alpha) = \sigma(e) = 1$; thus $\sigma(\beta) = 1$ and $\alpha \prec^\star \beta$. By the definition of $\prec^\star$ and as $f \in L(v)$, we deduce low(α) $\leq$ low(β) $\leq$ low(f) $\leq$ low(α). As low(e) $>$ low(f), it follows that α is a thick tree edge; hence β is a thick tree edge (for otherwise $\beta \prec^\star \alpha$) and $\mathrm{Fop}_\alpha(\beta)$ is not empty, and is monochromatic, with σ-value equal to that of β and opposed to the σ-value of $\mathrm{Fop}_\beta(\alpha)$, i.e. to $\sigma(e)$. Hence $\sigma(e) = -\sigma(\beta) = -1$, a contradiction. □

Our main theorem will now follow from the existence of a drawing of any σ-map of G in the plane where tree edges cross no other edges and two cotree edges may cross at most once, and two cotree edges with the different σ-value cannot cross. How to obtain such a drawing is displayed in Fig. 7 and is the subject of the next section.

4. How to draw a map on the plane

Theorem 4.1. *Let G be a connected graph and let $\mathcal{T}$ be a rooted spanning tree of G. Assume that a circular order of the edges around the vertices (i.e. a map of G) is given. Then there exists a drawing of G in the plane with prescribed circular orders such that tree edges cross no other edges, two cotree edges may cross at most once and the root of $\mathcal{T}$ belongs to the unbounded face.*

Moreover, such a drawing may be computed in $O(m)$-time, where m is the number of edges of G.

Proof. Consider the following classical construction of the double-occurrence sequence S:

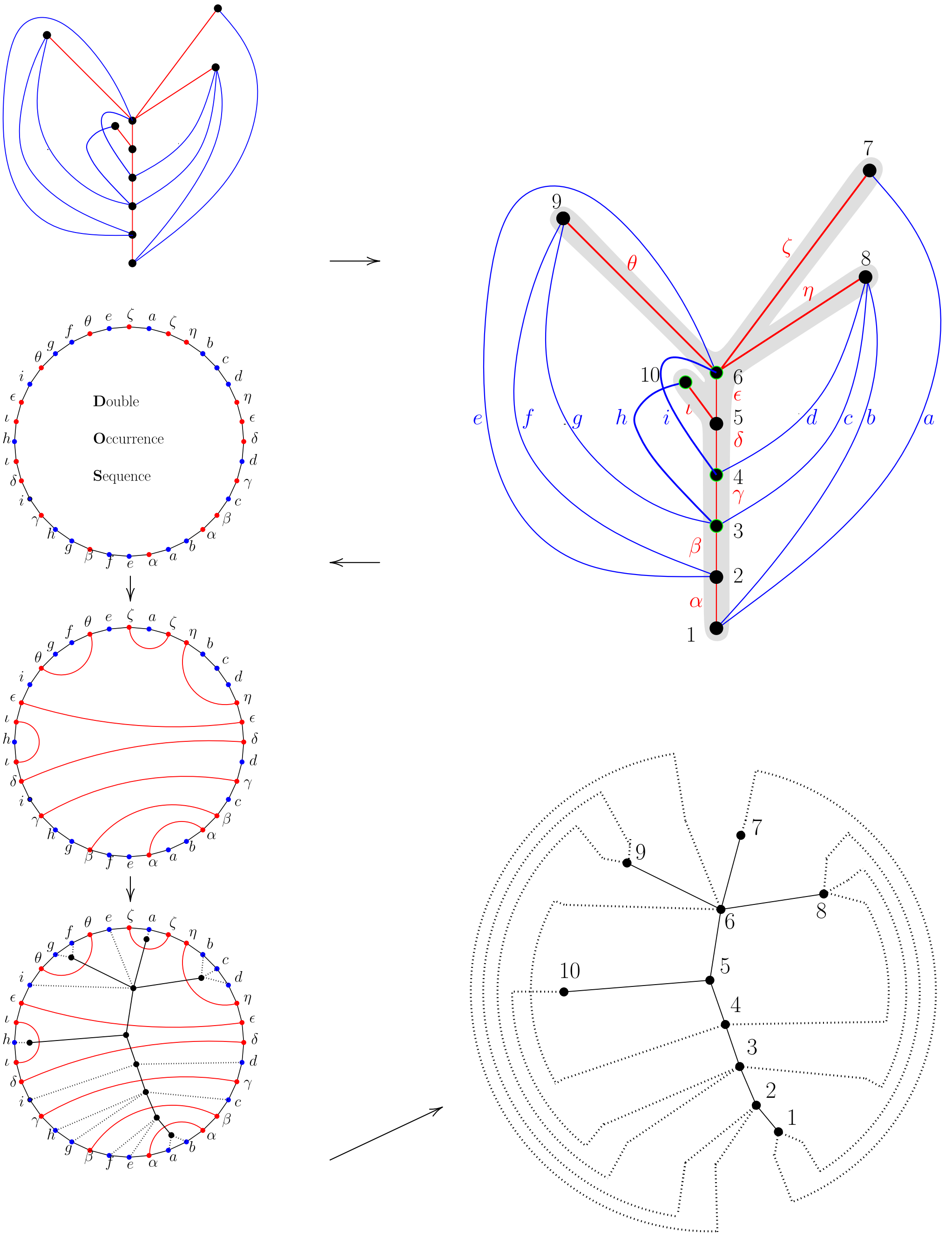


Fig. 7. Drawing a map on the plane.

- Start from an empty sequence S. Let $i = j = 1$, let v_1 be the root of $\mathcal{T}$ and let e_1 be an edge incident to the root. Add e_1 to the sequence S.
- While $j < 2m$ repeat the following:

– If e_j is a tree edge, let v_{i+1} be the other incidence of e_j and let $i \leftarrow i + 1$.
– Let e_{j+1} be the edge following e_j clockwise around v_i and let $j \leftarrow j + 1$.
– Add e_j to the sequence S.

If one only considers the occurrences of tree edges, this is the classical traversal sequence of a so-called planar tree. In such a case, the occurrences of tree edges are not interlaced; that is, if e and f are tree edges, the circular sequence does not contain *efef* as a subsequence.

We place the $2m$ occurrences of the sequence S regularly spaced on a circle. For each tree edge e, we connect the two occurrences of e by a straight line segment[1]. As occurrences of tree edges are not interlaced, the corresponding chords will not cross. Then each region inside the circle corresponds to a vertex of the tree. To represent a vertex, we choose a point in the corresponding region (for instance as a barycenter). Then we connect these points to obtain a drawing of $\mathcal{T}$, and each point corresponding to a vertex v of $\mathcal{T}$ is connected to the occurrences of the cotree edges incident to v on the cycle (which obviously induces no crossings and respects the circular order around v). Routing the connections between the two occurrences of cotree edges in such a way that no two cotree edge will cross more than once and preserving the property that v_1 will belong to the unbounded face is easy. Notice that the left and right cotree edges are as expected. □

From these lemmas, a theorem follows directly:

Theorem 4.2. *Let G be a graph, and let $\mathcal{T}$ be a Trémaux tree of G. The following conditions are equivalent:*

(i) *G is planar,*
(ii) *G admits an F-coloring,*
(iii) *G admits a strong F-coloring.*

Moreover, if G is planar, any strong F-coloring σ defines a planar embedding of G with the root of $\mathcal{T}$ on the unbounded face such that each cotree edge e has its fundamental cycle oriented clockwise if $\sigma(e) = 1$ and anticlockwise if $\sigma(e) = -1$.

5. The planarity algorithm

Let G be a graph of size m (that is a graph with m edges). The planarity algorithm involves three steps that are performed in $O(m)$-time.

5.1. Preprocessing

The first phase of the planarity algorithm is the following preprocessing phase: do a DFS search, compute the low function, compute the status of edges (thick, thin or block), sort all edges according to the Low-order $\prec^\star$ (this is efficiently performed using a bucket sort), and at each vertex define a reference edge as the smallest outgoing edge according to $\prec^\star$ (cf Fig. 8).

5.2. The planarity test

We now examine the step of the algorithm which tests the planarity of the graph.

We shall consider some data structure DS responsible for maintaining a set of bicoloring constraints on a set of cotree edges. We assign to each edge of the graph e such a data structure DS(e). These structures are initialized as follows: DS(e) is empty if e is a tree edge and includes e (with no bicoloring constraints) if e is a cotree edge. We say that all the cotree edges have been *processed* and that the tree edges are still *unprocessed*.

- While there exists a vertex v, different from the root, such that all the edges in $\omega^+(v)$ have been processed and such that the tree edge $e = (u, v)$ entering v has not been processed, we do the following. Let $e_1 \prec^\star e_2 \prec^\star \cdots \prec^\star e_k$ be the edges in $\omega^+(v)$ ($k \geq 1$).

[1] In Fig. 7, we used curves instead of straight line segments to improve the readability of the drawing.

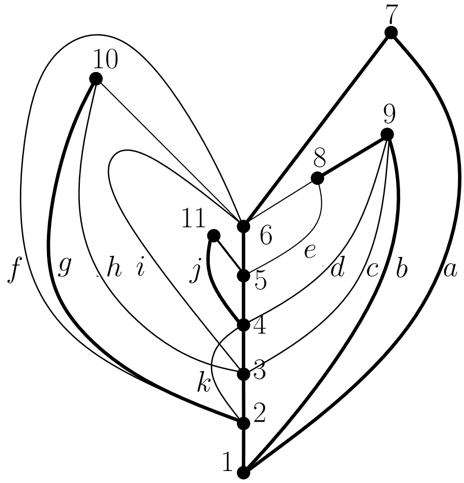
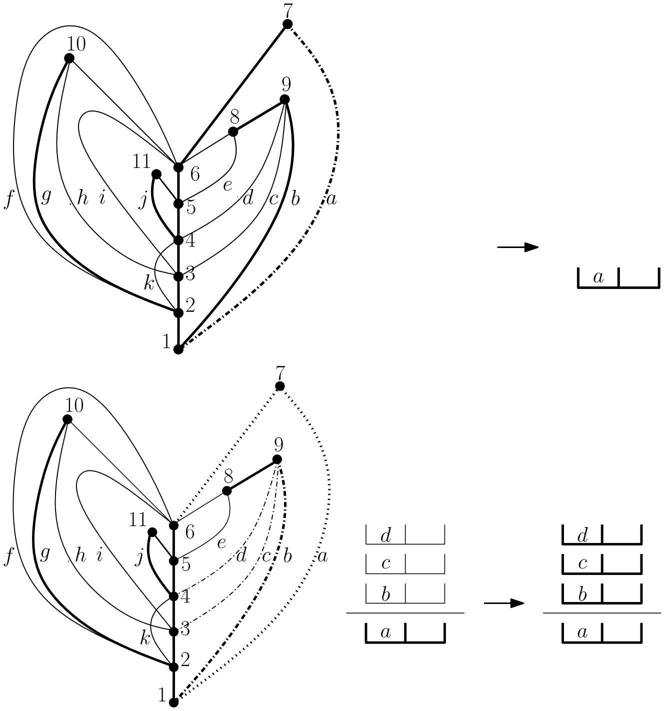
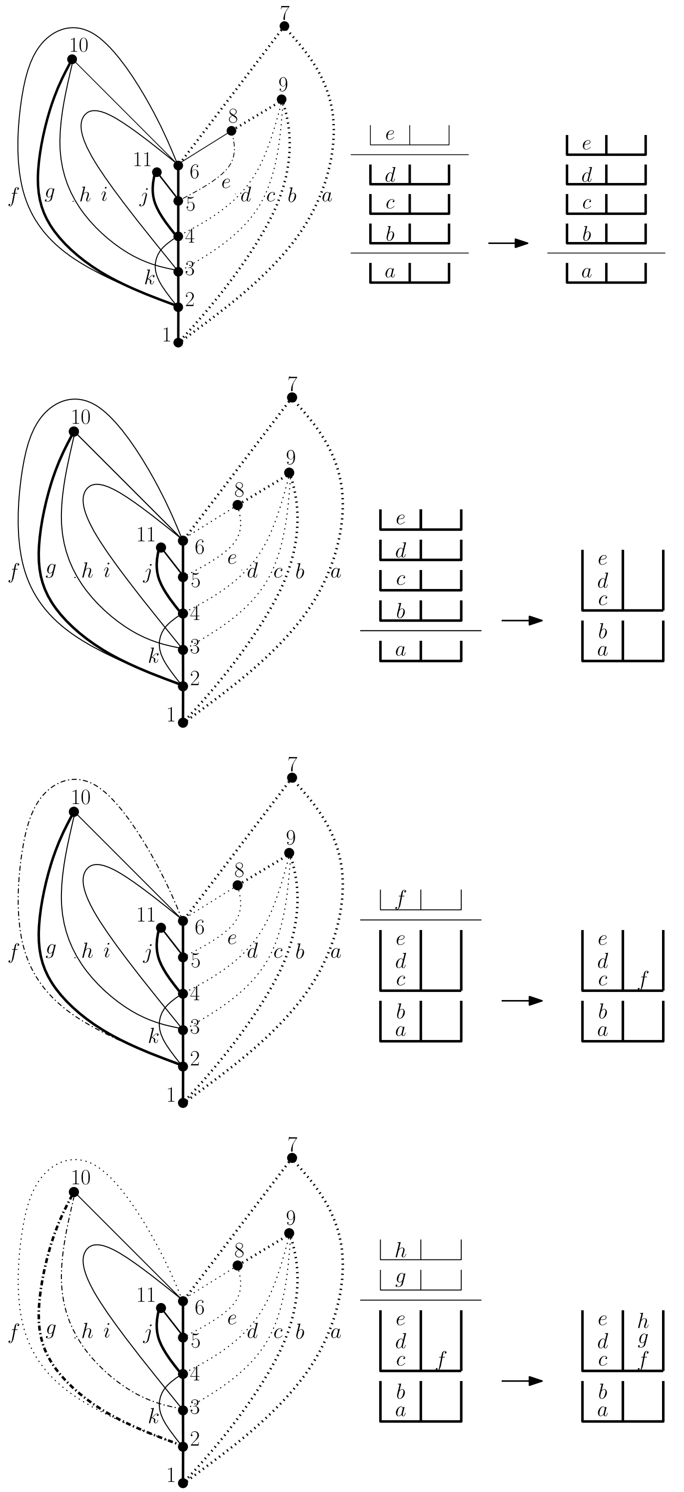


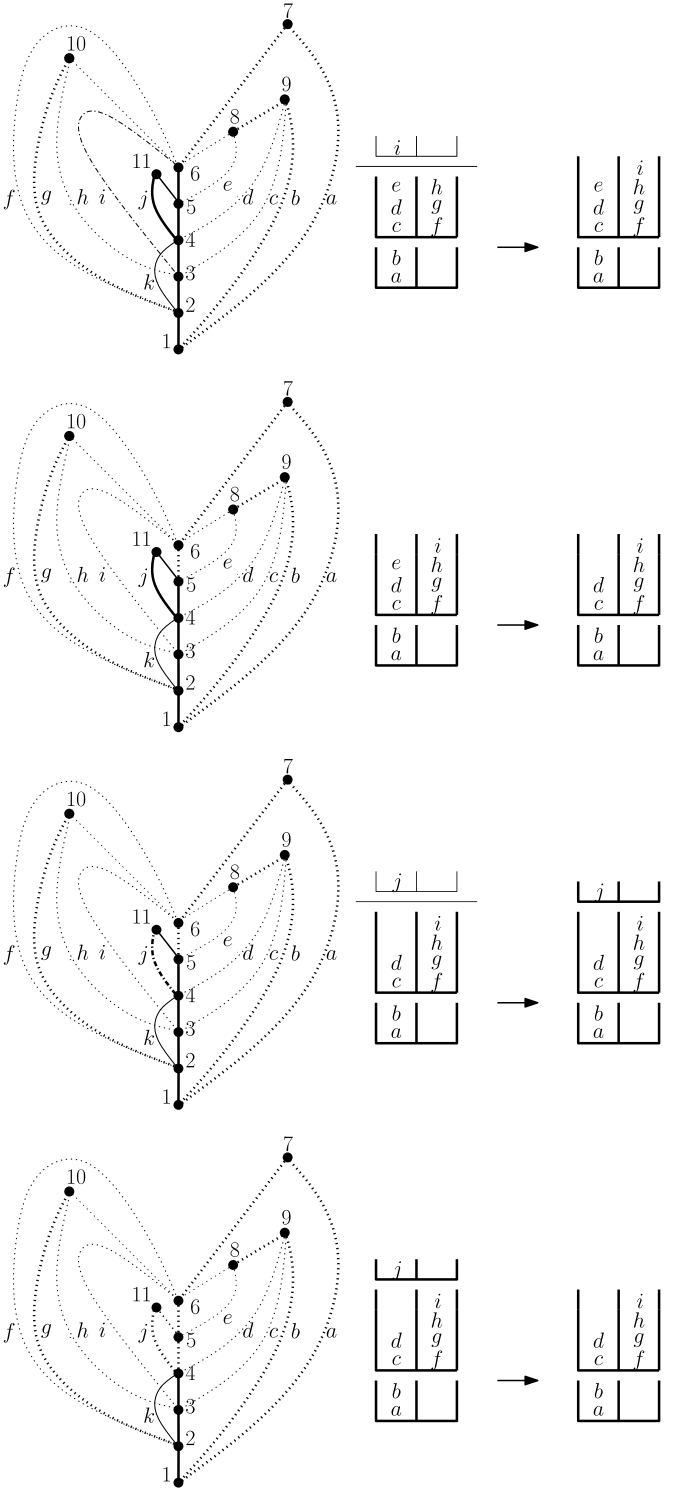
Fig. 8. The reference edge at a vertex v is the smallest edge of $\omega^+(v)$ with respect to $\prec^\star$. Here, reference edges are displayed as fat edges.

– Initialize DS(e) with DS(e_1).
– For $i : 2 \to k$, *merge* DS(e_i) into DS(e), that is: add to DS(e) the edges in DS(e_i) and add the F-coloring constraints corresponding to the pairs of edges e_j, e_i with $j < i$ (notice that all the cotree edges concerned belong to DS(e)). If some constraint is not satisfied, then the graph is declared non-planar.
– Remove from all of the DS(e) every cotree edge with lower incidence u.
– We declare that edge e has been processed.

- As all the edges have been processed, we declare that the graph is planar.







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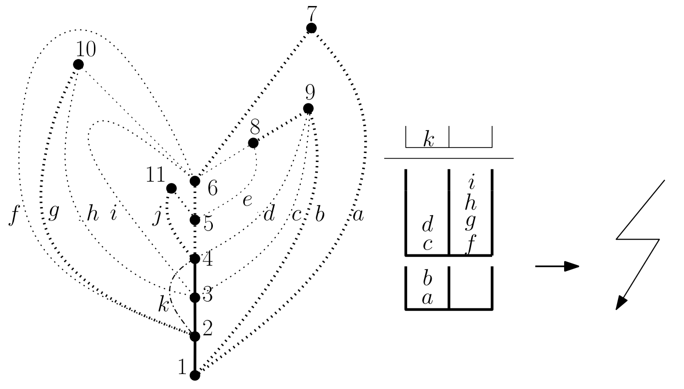

5.3. Embedding computation

We do not describe this step in detail here. Roughly speaking, the embedding is derived from the history data of the planarity test.

6. Trémaux trees and Kuratowski subdivisions

In this last section we present a characterization of a Trémaux-cotree critical graph which is at the basis of our program in Pigale [5], which exhibits Kuratowski subdivisions in non-planar graphs. As the proofs are rather long and technical, we will not even sketch them here, and we refer the reader to [9,3,4,6].

A Trémaux-cotree critical graph is a simple graph of minimum degree 3 having a Trémaux tree, such that any cotree edge is *critical* in the sense that its deletion would lead to a planar graph. A first study of Trémaux-cotree criticality appeared in [9], in which it was proved that a Trémaux-cotree critical graph either is isomorphic to K_5 or includes a subdivision of $K_{3,3}$ and no subdivision of K_5. We shall now recall stronger results that we published later in [6].

A Möbius pseudo-ladder is a natural extension of Möbius ladders allowing triangles. This may be formalized by the following definition.

Definition 6.1. A *Möbius pseudo-ladder* is a non-planar simple graph, which is the union of a polygon $(v_1, \ldots, v_n)$ and chords (called *bars*) such that any two non-adjacent bars are interlaced (recall that two non-adjacent edges $\{v_i, v_j\}$ and $\{v_k, v_l\}$ are *interlaced* if, in circular order, one finds exactly one of $\{v_k, v_l\}$ between v_i and v_j).

This definition means that a Möbius pseudo-ladder may be drawn in the plane as a polygon and internal chords such that any two non-adjacent chords cross (see Fig. 9). Notice that $K_{3,3}$ and K_5 are both Möbius pseudo-ladders.

Theorem 6.1. *Any Trémaux-cotree critical graph is a Möbius pseudo-ladder.*

The following refined theorem gives right away a trivial algorithm for exhibiting a Kuratowski subdivision in a Trémaux-cotree critical graph.

Theorem 6.2. *A simple graph is Trémaux-cotree critical if and only if it is a Möbius pseudo-ladder whose non-critical edges belong to some Hamiltonian path.*

Moreover, if G is Trémaux-cotree critical according to a Trémaux tree $\mathcal{T}$ and G has at least nine vertices, then $\mathcal{T}$ is a chain and G is the union of a cycle of critical edges and pairwise interlaced non-critical chords (see Fig. 10).

The algorithm first computes the set of critical edges of G. For that, we use the property that a tree edge is critical if and only if it belongs to a fundamental cycle of length 4 of some cotree edge to which it is not adjacent. Then, three pairwise non-adjacent non-critical edges are found to complete a Kuratowski subdivision of G isomorphic to $K_{3,3}$.

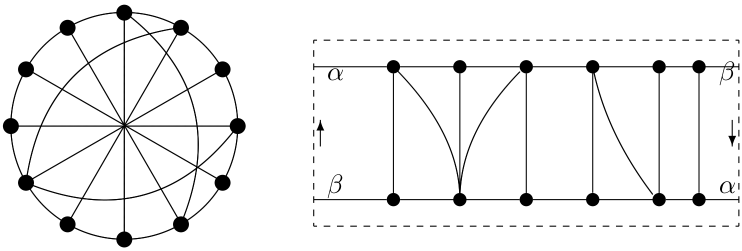

Fig. 9. A Möbius pseudo-ladder on the plane.

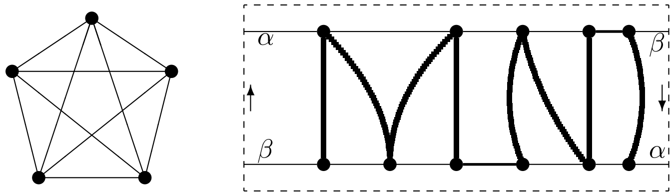

Fig. 10. The Trémaux-cotree critical graphs are either K_5 or Möbius pseudo-ladders having all non-critical edges (thickest) included in a single path.

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