Problem 1

$$
\begin{aligned}
Var(Y_{ij}) &= var(\mu + b_i + e_{ij}) \\
&= var(b_i) + Var(e_{ij}) \quad (b_i \perp e_{ij}) \\
&= \sigma_b^2 + \sigma_e^2
\end{aligned}
$$

$$
\begin{aligned}
Cov(Y_{ij}, Y_{ik}) &= E(Y_{ij} - EY_{ij})(Y_{ik} - EY_{ik}) \\
&= E(Y_{ij} - \mu)(Y_{ik} - \mu) \\
&= E(b_i + e_{ij})(b_i + e_{ik}) \\
&= E(b_i^2 + b_i e_{ik} + b_i e_{ij} + e_{ij} e_{ik}) \\
&= Eb_i^2 + Eb_i Ee_{ik} + Eb_i Ee_{ij} + Ee_{ij} Ee_{ik} \quad (e_{ij} \perp e_{ik},\ b_i \perp e_{i\cdot}) \\
&= Eb_i^2 \\
&= Var(b_i) + (Eb_i)^2 \\
&= \sigma_b^2
\end{aligned}
$$

$$
\begin{aligned}
corr(Y_{ij}, Y_{ik}) &= \frac{cov(Y_{ij}, Y_{ik})}{\sqrt{var(Y_{ij}) \cdot var(Y_{ik})}} \\
&= \frac{\sigma_b^2}{\sigma_b^2 + \sigma_e^2}
\end{aligned}
$$

Both the variance and correlation is constant across visits, so this corresponds to compound symmetry pattern.

Problem 2 Question 2
Note: Answers to Questions 1 & 3 of Problem 2 are attached after this.

$$Y_{ij} = \beta_0 + a_i + b_0 * I_{(sex_i=0)} + b_1 * I_{(sex_i=1)} + \beta_1 * age_{ij} + e_{ij}$$

$$E(Y_{ij}) = E(\beta_0 + a_i + b_0 I(sex_i=0) + b_1 I(sex_i=1) + \beta_1 \cdot age_{ij} + e_{ij})$$
$$= \beta_0 + \beta_1 \cdot age_{ij}$$

$$var(Y_{ij}) = var(a_i + b_k + e_{ij}) = var(a_i) + var(b_k) + var(e_{ij}) = \sigma_a^2 + \sigma_b^2 + \sigma_e^2$$

Variance-covariance matrices:

① Within-subject covariance:

$$cov(Y_{ij}, Y_{ik}) = E[(Y_{ij} - EY_{ij})(Y_{ik} - EY_{ik})]$$
$$= E[(a_i + b_k + e_{ij})(a_i + b_k + e_{ik})]$$
$$= E[a_i^2 + b_k^2 + e_{ij}e_{ik}]$$
$$= \sigma_a^2 + \sigma_b^2 \qquad (\text{since } e_{ij} \perp e_{ik})$$

② Between-subject covariance:

(i) if the subjects are of the same gender:

$$cov(Y_{mj}, Y_{nk}) = E[(Y_{mj} - E_{mj})(Y_{nk} - E_{nk})]$$
$$= E[(a_m + b_l + e_{mj})(a_n + b_l + e_{nk})] \quad (l=0,1)$$
$$= E\, b_l^2$$
$$= \sigma_b^2$$

(ii) if the subjects are of different genders:

$$cov(Y_{mj}, Y_{nk}) = E[(Y_{mj} - E_{mj})(Y_{nk} - E_{nk})]$$
$$= E[(a_m + b_1 + e_{mj})(a_n + b_0 + e_{nk})]$$
$$= 0 \qquad (\text{since } b_1 \perp b_0)$$

Based on the abve calculation, We construct matrices A and B:

$$A = \begin{bmatrix} \sigma_a^2+\sigma_b^2+\sigma_e^2 & \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2 \\ \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2+\sigma_e^2 & \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2 \\ \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2+\sigma_e^2 & \sigma_a^2+\sigma_b^2 \\ \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2 & \sigma_a^2+\sigma_b^2+\sigma_e^2 \end{bmatrix}_{4\times4}, \quad B = \begin{bmatrix} \sigma_b^2 & \sigma_b^2 & \sigma_b^2 & \sigma_b^2 \\ \sigma_b^2 & \sigma_b^2 & \sigma_b^2 & \sigma_b^2 \\ \sigma_b^2 & \sigma_b^2 & \sigma_b^2 & \sigma_b^2 \\ \sigma_b^2 & \sigma_b^2 & \sigma_b^2 & \sigma_b^2 \end{bmatrix}_{4\times4},$$

where $A_{4\times4}$ is the covariance within a subject, and B is the covariance between measurements from 2 subjects of the same gender.

Therefore, we get

$$C_{boy} = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & B & A \end{bmatrix}_{16\times16}, \quad D_{girl} = \begin{bmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & B & A \end{bmatrix}_{11\times11},$$

and finally

$$var(\underset{\sim}{Y}) = \begin{bmatrix} C_{boy} & 0 \\ 0 & D_{girl} \end{bmatrix}_{108\times108} \quad [108=(16+11)\times4]$$