# Project 4: A Bayesian model for hurricane trajectories.

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## Introduction

## Hurrican Data

hurrican356.csv collected the track data of 356 hurricanes in the North Atlantic area since 1989. For all the storms, their location (longitude & latitude) and maximum wind speed were recorded every 6 hours. The data includes the following variables

1. **ID**: ID of the hurricans
2. **Season**: In which **year** the hurricane occurred
3. **Month**: In which **month** the hurricane occurred
4. **Nature**: Nature of the hurricane

- ET: Extra Tropical
- DS: Disturbance
- NR: Not Rated
- SS: Sub Tropical
- TS: Tropical Storm

5. **time**: dates and time of the record
6. **Latitude** and **Longitude**: The location of a hurricane check point
7. **Wind.kt** Maximum wind speed (in Knot) at each check point

In this project we are interested in studying 356 hurricanes in the North Atlantic area since 1989, especially about how they are formed and what predictors will be of high prediction power of the hurricane properties. We have several measuremnt on each hurrianc such as the time when it was formed, the nature of the hurricane , the location of a hurricane check point and the maximum wind speed (in Knot) at each check point. To increase the flexibility of our model, we planned to use Bayesian approaching for modeling the outcome of interest, which is the wind speed. By assuming relatively simple structure of priors, we could use Gibbs sampling in our model computation and the result will be pretty robust.

## Method

Let $t$ be time (in hours) since a hurricane began, and for each hurrican $i$, we denote $Y_i(t)$ to be the wind speed at time $t$. The following Baysian model was suggested.

$$Y_{i,j}(t+6) = \mu_{i,j}(t) + \rho_j Y_{i,j}(t) + \epsilon_{i,j}(t)$$

where $\mu_{i,j}(t)$ is the funtional mean, and the errors $(\epsilon_{i,1}(t), \epsilon_{i,2}(t), \epsilon_{i,3}(t))$ follows a multivariate normal distributions with mean zero and covariance matrix $\Sigma$, independent across $t$. We further assume that the mean

functions $\mu_{i,j}(t)$ can be written as

$$\mu_{i,j}(t) = \beta_{0,j} + x_{i,1}(t)\beta_{1,j} + x_{i,2}\beta_{2,j} + x_{i,3}\beta_{3,j} + \sum_{k=1}^{3} \beta_{3+k,j}\Delta_{i,k}(t-6)$$

where $x_{i,1}(t)$, ranging from 0 to 365, is the day of year at time $t$, $x_{i,2}$ is the calenda year of the hurrican, and $x_{i,3}$ is the type of hurrican, and

$$\Delta_{i,k}(t-6) = Y_{i,k}(t) - Y_{i,k}(t-6), k = 1, 2, 3$$

are the change of latitude, longitude, and wind speed between $t-6$ and $t$.

**Prior distributions**

We assume the following prior distributions

For $\beta = (\beta_{k,j})_{k=0,...,6,j=1,2,3}$, we assume $\pi(\beta)$ is jointly normal with mean 0 and variance $diag(1, p)$.

We assume that $\pi(\rho_j)$ follows a trucated normal $N_{[0,1]}(0.5, 1/5)$

$\pi(\sigma^2)$ follows a $Wishart(3, diag(0.1, 3))$

**Likelihood**

The log-likelihood of $Y(t+6)$ is

$$l(Y(t+6)|\mathbf{X},\ ,\rho, Y(t)) = -n\log\left(\sigma\sqrt{2\pi}\right) - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mathbf{X}^T\ - \rho Y_i(t)\Big)^2$$

**Posterior of $\beta$**

Since each $\beta_k$ is mutually-independent distributed, we can look at their posterior distribution individually. Note that $k = 0, 1, 2, ..., 10$ because there are five categories for $x_{i3}$. \

The prior of $\beta_k$ has the log-likelihood function:

$$\log \pi(\beta_k) = -\log(\sqrt{2\pi}) - \frac{1}{2}\beta_k^2$$

The posterior of $\beta_k$ has the log-likelihood function:

$$\begin{aligned}
\log \pi(\beta_k|Y(t+6), \mathbf{X},\ _{-k}, \rho, Y(t)) &= \log \pi(\beta_k) + l(Y(t+6)|\mathbf{X},\ ,\rho, Y(t)) \\
&\propto \text{const} - \frac{1}{2}\beta_k^2 - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big[\beta_k^2 x_{ik}^2 - 2\beta_k x_{ik}\Big(Y_i(t+6) - \mathbf{X}_{-k\ -k}^T - \rho Y_i(t)\Big)\Big] \\
&= \text{const} - \Big[\beta_k^2\Big\{\sum_{i=1}^{n} \frac{1}{2\sigma^2}(x_{ik}^2 + \frac{\sigma^2}{n})\Big\} - 2\beta_k \frac{1}{2\sigma^2}\Big\{\sum_{i=1}^{n} x_{ik}[Y_i(t+6) - \mathbf{X}_{-k\ -k}^T - \rho Y_i(t)]\Big\}\Big] \\
&= \text{const} - \frac{1}{2}\Big\{\sum_{i=1}^{n} \frac{1}{\sigma^2}(x_{ik}^2 + \frac{\sigma^2}{n})\Big\}\Big\{\beta_k - \frac{\sum x_{ik}[Y_i(t+6) - \mathbf{X}_{-k\ -k}^T - \rho Y_i(t)]}{\sum(x_{ik}^2 + \frac{\sigma^2}{n})}\Big\}^2
\end{aligned}$$

Thus, the posterior of $\beta_k$ follows a normal distribution with

$$\begin{aligned}
\mu_k &= \frac{\sum x_{ik}[Y_i(t+6) - \mathbf{X}_{-k\ -k}^T - \rho Y_i(t)]}{\sum(x_{ik}^2 + \frac{\sigma^2}{n})} \\
\sigma_k^2 &= \Big\{\sum_{i=1}^{n} \frac{1}{\sigma^2}(x_{ik}^2 + \frac{\sigma^2}{n})\Big\}^{-1}
\end{aligned}$$

**Posterior of $\rho$**

The prior of $\rho$ has the log-likelihood function:

$$\log \pi(\rho) = -\log(\sqrt{\frac{2\pi}{5}}) - \frac{25}{2}(\rho - \frac{1}{2})^2$$

The posterior of $\rho$ is proportional to

$$\begin{aligned}
&\text{const} - \frac{25}{2}(\rho - \frac{1}{2})^2 - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mathbf{X}^T\ - \rho Y_i(t)\Big)^2 \\
&= \text{const} - \frac{n}{2\sigma^2}\Big(\frac{25\sigma^2}{n}\rho^2 - \frac{25\sigma^2}{4n}\rho\Big) - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big(\rho^2 Y_i(t)^2 - 2\rho Y_i(t)[Y_i(t+6) - \mathbf{X}^T\ ]\Big) \\
&= \text{const} - (\frac{25}{2} + \frac{1}{2\sigma^2}\sum Y_i(t)^2)\rho^2 + \frac{1}{\sigma^2}\rho \sum \Big\{\Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\ ]\Big) + \frac{25\sigma^2}{8n}\Big\} \\
&= \text{const} - \frac{1}{2}\Big\{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2\Big\}\Big\{\rho - \frac{\frac{1}{\sigma^2}\sum \Big\{\Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\ ]\Big) + \frac{25\sigma^2}{8n}\Big\}}{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2}\Big\}^2
\end{aligned}$$

Thus, the posterior of $\rho$ follows a normal distribution with

$$\begin{aligned}
\mu_\rho &= \frac{\frac{1}{\sigma^2}\sum \Big\{\Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\ ]\Big) + \frac{25\sigma^2}{8n}\Big\}}{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2} \\
&= \frac{\sum \Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\ ]\Big) + \frac{25\sigma^2}{8}}{25\sigma^2 + \sum Y_i(t)^2} \\
\sigma_\rho^2 &= \Big\{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2\Big\}^{-1}
\end{aligned}$$

**Posterior of $\sigma^2$**

The prior of $\sigma^2$ has the log-likelihood function:

$$\begin{aligned}
\log \pi(\sigma^2) &= \text{const} - (\alpha + 1)\log\frac{1}{\sigma^2} + \frac{-\alpha'}{\sigma^2} \\
&= \text{const} - 2(\alpha + 1)\log(\sigma) - \alpha'\sigma^{-2}
\end{aligned}$$

The posterior of $\sigma^2$ is proportional to

$$\begin{aligned}
&\text{const} - 2(\alpha + 1)\log(\sigma) - \alpha'\sigma^{-2} - n\log\Big(\sigma\sqrt{2\pi}\Big) - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mathbf{X}^T\ - \rho Y_i(t)\Big)^2 \\
&= \text{const} - \Big(n + 2(\alpha + 1)\Big)\log(\sigma) - \sigma^{-2}\Big\{\alpha' + \sum_{i=1}^{n} \frac{1}{2}\Big(Y_i(t+6) - \mathbf{X}^T\ - \rho Y_i(t)\Big)^2\Big\}
\end{aligned}$$

where $\alpha = \alpha' = 0.001$.

Thus, the posterior of $\rho$ follows an inverse-gamma distribution with

$$\begin{aligned}
\alpha_{post} &= n + 2\alpha + 1 \\
\alpha'_{post} &= \alpha' + \sum_{i=1}^{n} \frac{1}{2}\Big(Y_i(t+6) - \mathbf{X}^T\ - \rho Y_i(t)\Big)^2
\end{aligned}$$

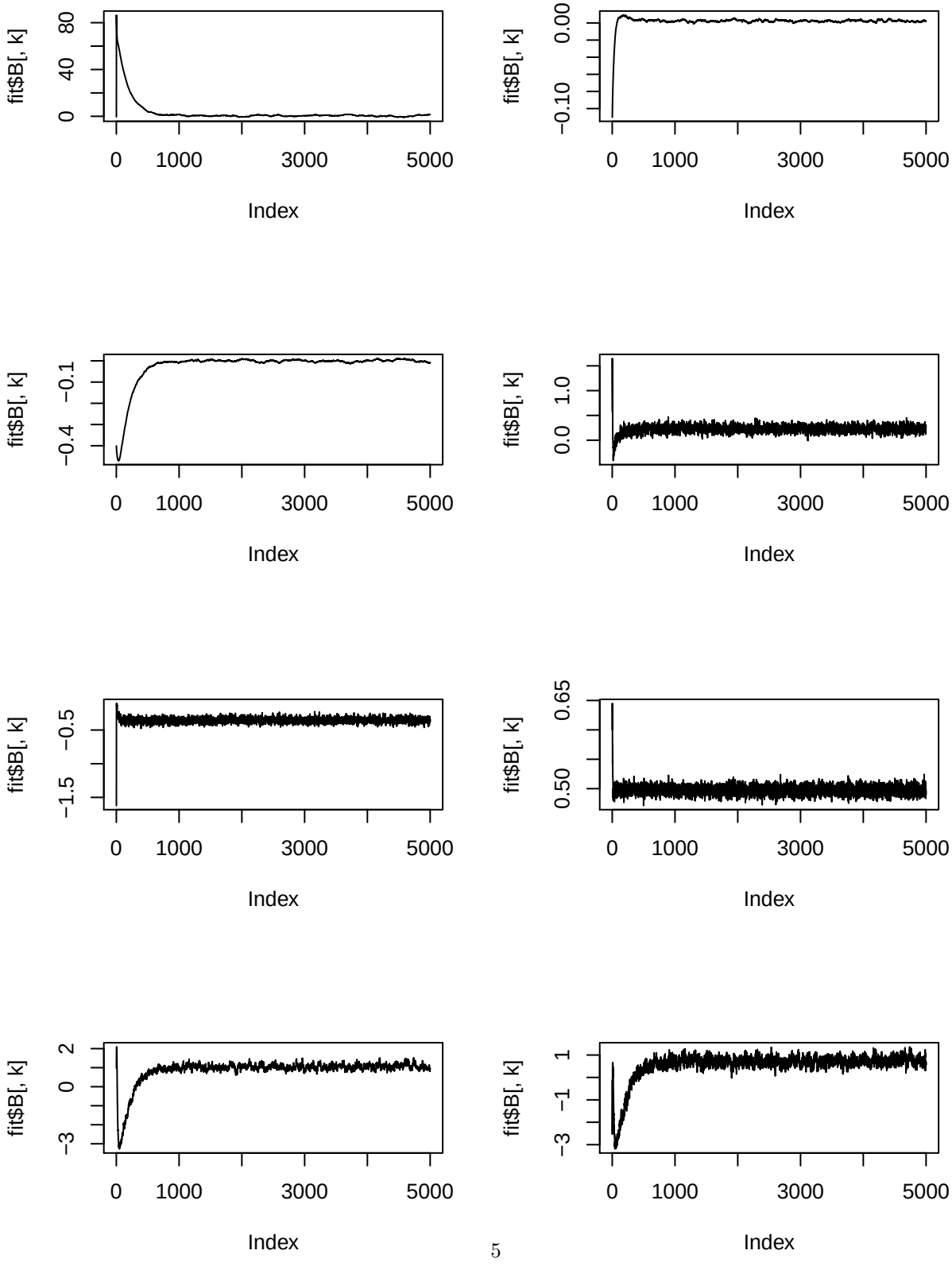
**Gibbs sampling algorithm**

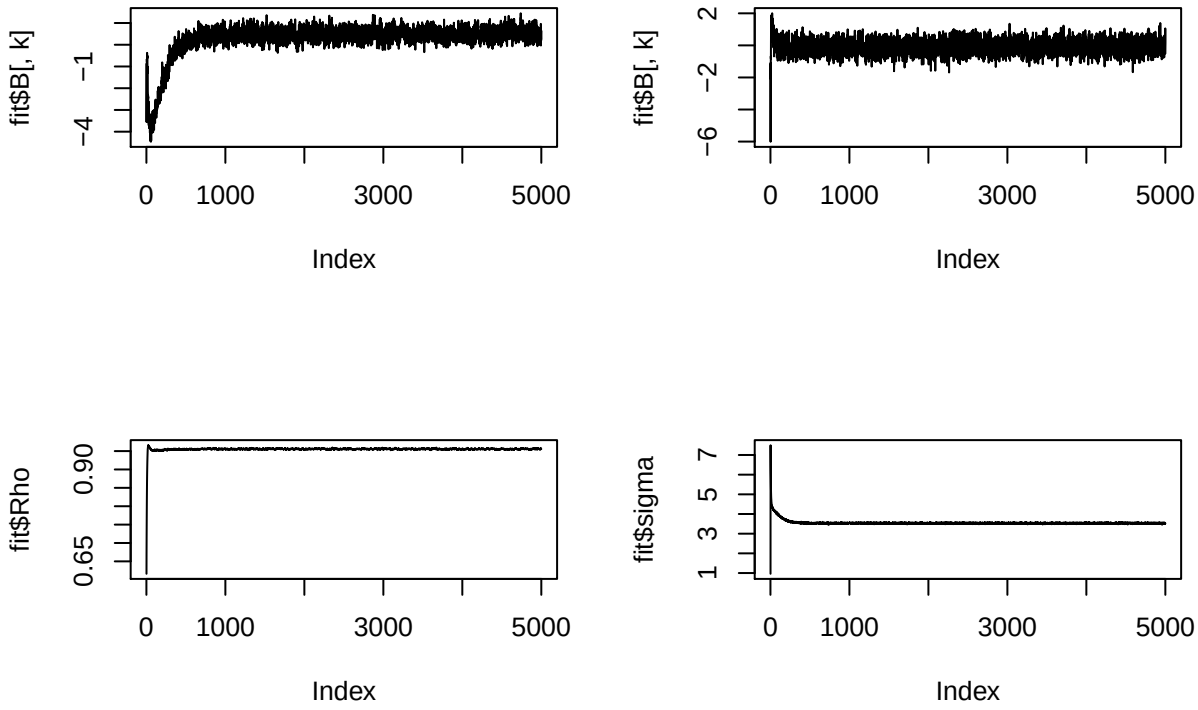
Denote $\theta = (\beta_0, \beta_1, ..., \beta_9, \rho, \sigma^2)$. We proceed as follows:
1. Begin with some initial values of $\theta^0$.
2. Sample each component of the vector, $\theta$, from the distribution of that component conditioned on all other components sampled so far. For example, for $k \geq 1$, Generate $\beta_0^{(k)}$ from $\pi(\beta_0|\beta_1^{(k-1)}, ..., \beta_9^{(k-1)}, \rho^{(k-1)}, \sigma^{2(k-1)}, Y, \mathbf{X})$. Then generate $\beta_1^{(k)}$ from $\pi(\beta_1|\beta_0^{(k)}, \beta_2^{(k-1)}, ..., \beta_9^{(k-1)}, \rho^{(k-1)}, \sigma^{2(k-1)}, Y, \mathbf{X})$.
3. Repeat the above step $k$ times.

We will randomly select 80% hurricanes and applied the proposed Gibbs sampling algorithm to estiamte the posterior distributions of the model parameters. Then we will apply the model to track the remaining 20% hurricans, and evaluate model performance in terms of how well could predict and track these hurricanes.

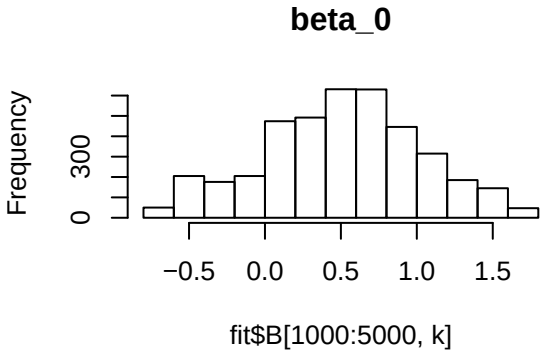
# Results

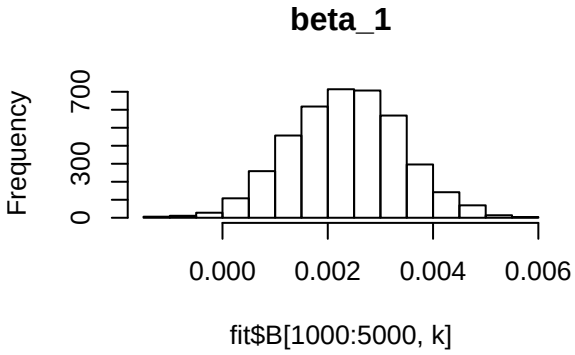
### Parameter estimation of posterior distributions

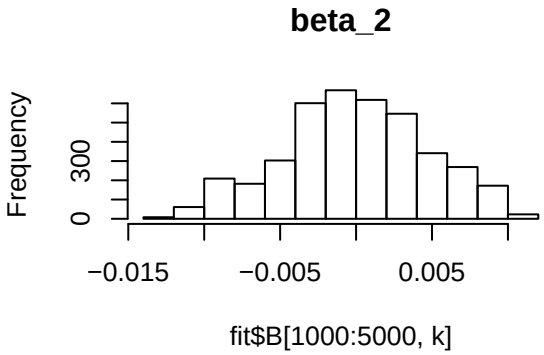


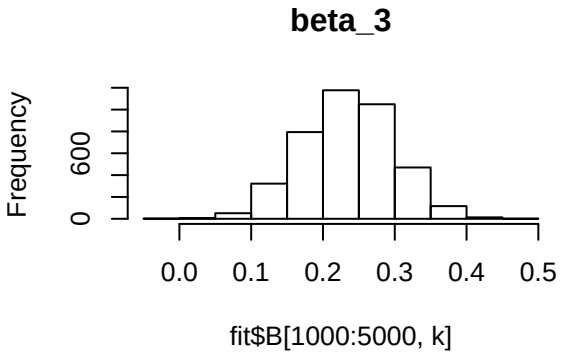


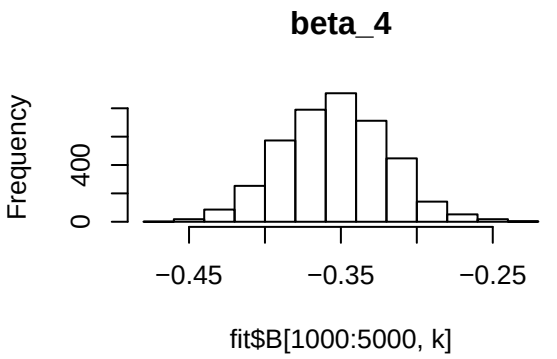

The plots above shows the length of burn-in period and stationary stage of each Markov chains. On average, burn-in period is about 500 runs.

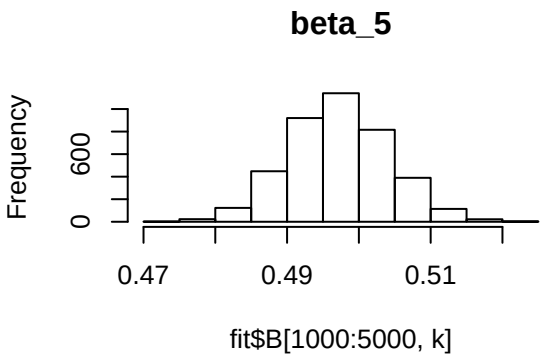

beta_0
Frequency
0
300
−0.5
0.0
0.5
1.0
1.5
fit$B[1000:5000, k]

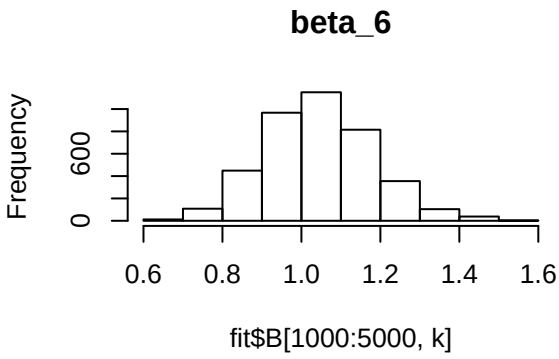

beta_1
Frequency
0
300
700
0.000
0.002
0.004
0.006
fit$B[1000:5000, k]

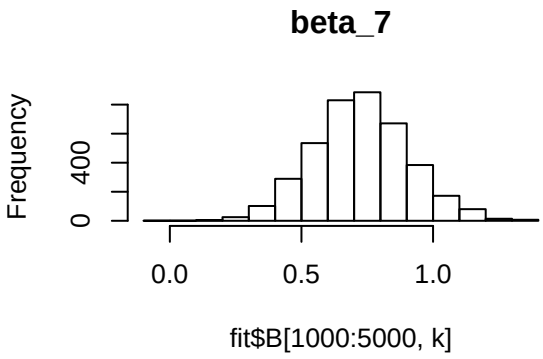

beta_2
Frequency
0
300
−0.015
−0.005
0.005
fit$B[1000:5000, k]

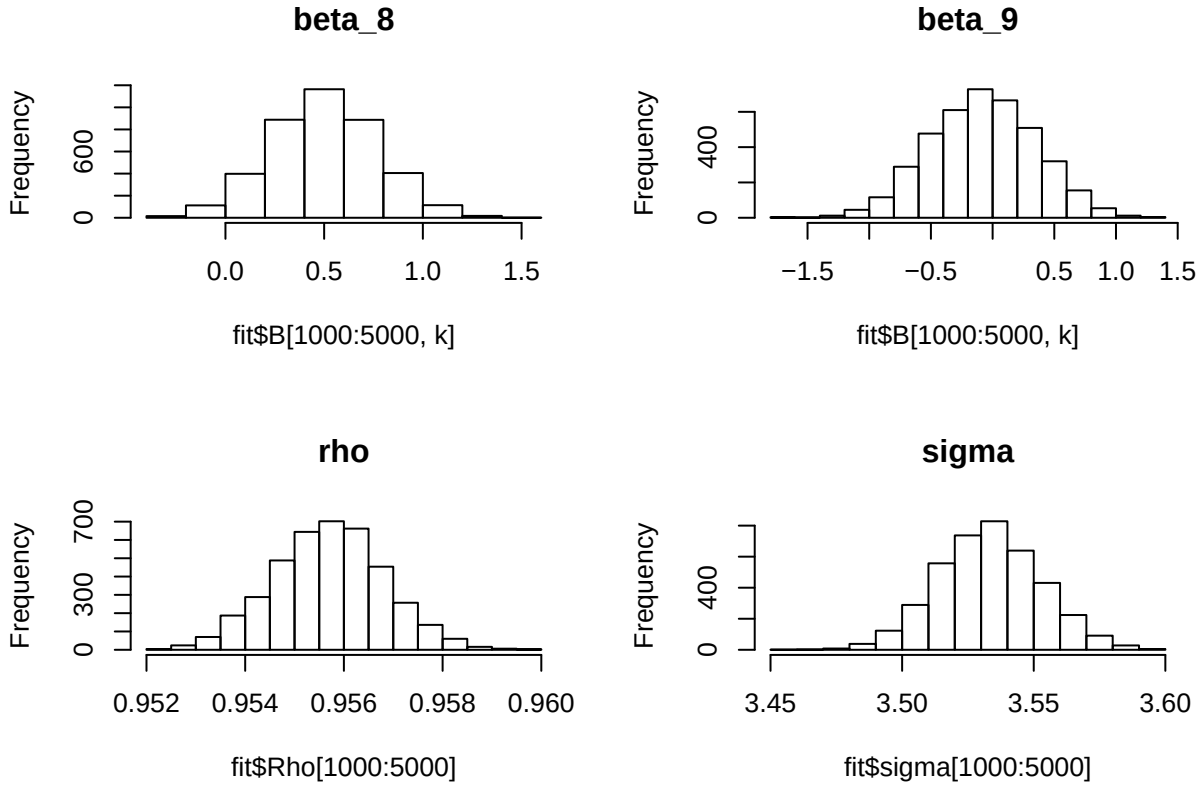

beta_3
Frequency
0
600
0.0
0.1
0.2
0.3
0.4
0.5
fit$B[1000:5000, k]


beta_4
Frequency
0
400
−0.45
−0.35
−0.25
fit$B[1000:5000, k]


beta_5
Frequency
0
600
0.47
0.49
0.51
fit$B[1000:5000, k]


beta_6
Frequency
0
600
0.6
0.8
1.0
1.2
1.4
1.6
fit$B[1000:5000, k]


beta_7
Frequency
0
400
0.0
0.5
1.0
fit$B[1000:5000, k]




The histograms above show the distribution of parameter values after each chain enters stationary stage.

| parameter | posterior.mean | CrI.low | CrI.high |
|---|---|---|---|
| (Intercept) | 0.6485750 | -0.5045427 | 36.3870108 |
| Yday | 0.0023540 | 0.0000803 | 0.0063911 |
| Year | -0.0013267 | -0.3295719 | 0.0085877 |
| DeltaLatitude | 0.2346255 | 0.0747606 | 0.3566071 |
| DeltaLongitude | -0.3539148 | -0.4224030 | -0.2818217 |
| DeltaSpeed | 0.4972567 | 0.4839647 | 0.5110733 |
| NatureTS | 1.0301557 | -1.7909435 | 1.3176497 |
| NatureET | 0.7123291 | -1.8519958 | 1.0847332 |
| NatureSS | 0.4867485 | -2.5668894 | 1.0083927 |
| NatureNR | -0.0839264 | -0.9149674 | 0.7809101 |
| Y(t) | 0.9551738 | 0.9524984 | 0.9579976 |
| sigma | 3.5562687 | 3.4960875 | 3.8657191 |

The table above shows the estimated posterior mean of each parameter in the Bayesian model, with the associated 95% credibility intervals (CrI). According to this model, it seems that `DeltaSpeed`(change in wind speed) and `Y(t)` (the wind speed at current time point) are highly predictive of the wind speed at the next time point. More specifically, they are both positively associated with `Y(t+6)`, the wind speed after 6 hours. `Yday` (the day of a year at current time point) and `DeltaLatitude` (the change in latitude) also show significant association with the wind speed after 6 hours.

### Prediction

We used the remaining 20% hurricans data to conduct predictions using our proposed model. The mean square error (MSE) is 26.8380143.

## Discussion

Our Bayesian regression model and Gibbs sampling seem work well on our hurricane dataset and the predictors we got are having pretty high prediction powers. However, we could still further improve our modeling precedure by considering the following extensions of this project: Firstly we might consider using other MCMC methods like Hamiltonian Monte Carlo and compare its performance with Gibbs sampling. Secondly we might use some more informative priors and see how the final outputs will change. Thirdly we might use more complicated modeling techniques such as hiearchical modeling to account to heterscadesity among different subregions and always remember that is trade-off between model complexicity and computation efficacy/over-fitting.