Project 4: A Bayesian model for hurricane trajectories.

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Introduction

Hurrican Data

hurrican356.csv collected the track data of 356 hurricanes in the North Atlantic area since 1989. For all the storms, their location (longitude & latitude) and maximum wind speed were recorded every 6 hours. The data includes the following variables

1. **ID**: ID of the hurricans
2. **Season**: In which **year** the hurricane occurred
3. **Month**: In which **month** the hurricane occurred
4. **Nature**: Nature of the hurricane

- ET: Extra Tropical
- DS: Disturbance
- NR: Not Rated
- SS: Sub Tropical
- TS: Tropical Storm

5. **time**: dates and time of the record

6. **Latitude** and **Longitude**: The location of a hurricane check point
7. **Wind.kt** Maximum wind speed (in Knot) at each check point

Method

Let t be time (in hours) since a hurricane began, and for each hurrican i, we denote $Y_i(t)$ to be the wind speed at time t. The following Baysian model was suggested.

$$Y_{i,j}(t+6) = \mu_{i,j}(t) + \rho_j Y_{i,j}(t) + \epsilon_{i,j}(t)$$

where $\mu_{i,j}(t)$ is the funtional mean, and the errors $(\epsilon_{i,1}(t), \epsilon_{i,2}(t), \epsilon_{i,3}(t))$ follows a multivariate normal distributions with mean zero and covariance matrix Σ, independent across t. We further assume that the mean functions $\mu_{i,j}(t)$ can be written as

$$\mu_{i,j}(t) = \beta_{0,j} + x_{i,1}(t)\beta_{1,j} + x_{i,2}\beta_{2,j} + x_{i,3}\beta_{3,j} + \sum_{k=1}^{3} \beta_{3+k,j}\Delta_{i,k}(t-6)$$

where $x_{i,1}(t)$, ranging from 0 to 365, is the day of year at time t, $x_{i,2}$ is the calenda year of the hurrican, and $x_{i,3}$ is the type of hurrican, and

$$\Delta_{i,k}(t-6) = Y_{i,k}(t) - Y_{i,k}(t-6), k = 1, 2, 3$$

are the change of latitude, longitude, and wind speed between $t-6$ and t.

Prior distributions

We assume the following prior distributions

For $\boldsymbol{\beta} = (\beta_{k,j})_{k=0,...,6,j=1,2,3}$, we assume $\pi(\boldsymbol{\beta})$ is jointly normal with mean 0 and variance $diag(1, p)$.

We assume that $\pi(\rho_j)$ follows a trucated normal $N_{[0,1]}(0.5, 1/5)$

$\pi(\sigma^2)$ follows a $Wishart(3, diag(0.1, 3))$

Likelihood

The log-likelihood of $Y(t+6)$ is

$$l(Y(t+6)|\mathbf{X}, \beta, \rho, Y(t)) = -n \log\left(\sigma\sqrt{2\pi}\right) - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mathbf{X}^T\beta - \rho Y_i(t)\Big)^2$$

Posterior of β

Since each β_k is mutually-independent distributed, we can look at their posterior distribution individually. Note that $k = 0, 1, 2, ..., 10$ because there are five categories for x_{i3}. \

The prior of β_k has the log-likelihood function:

$$\log \pi(\beta_k) = -\log(\sqrt{2\pi}) - \frac{1}{2}\beta_k^2$$

The posterior of β_k has the log-likelihood function:

$$\begin{aligned}
\log \pi(\beta_k|Y(t+6), \mathbf{X}, \beta_{-k}, \rho, Y(t)) &= \log \pi(\beta_k) + l(Y(t+6)|\mathbf{X}, \beta, \rho, Y(t)) \\
&\propto \text{const} - \frac{1}{2}\beta_k^2 - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big[\beta_k^2 x_{ik}^2 - 2\beta_k x_{ik}\Big(Y_i(t+6) - \mathbf{X}_{-k}^T\beta_{-k} - \rho Y_i(t)\Big)\Big] \\
&= \text{const} - \Big[\beta_k^2\Big\{\sum_{i=1}^{n}\frac{1}{2\sigma^2}(x_{ik}^2 + \frac{\sigma^2}{n})\Big\} - 2\beta_k\frac{1}{2\sigma^2}\Big\{\sum_{i=1}^{n} x_{ik}\big[Y_i(t+6) - \mathbf{X}_{-k}^T\beta_{-k} - \rho Y_i(t)\big]\Big\}\Big] \\
&= \text{const} - \frac{1}{2}\Big\{\sum_{i=1}^{n}\frac{1}{\sigma^2}(x_{ik}^2 + \frac{\sigma^2}{n})\Big\}\Big\{\beta_k - \frac{\sum x_{ik}[Y_i(t+6) - \mathbf{X}_{-k}^T\beta_{-k} - \rho Y_i(t)]}{\sum(x_{ik}^2 + \frac{\sigma^2}{n})}\Big\}^2
\end{aligned}$$

Thus, the posterior of β_k follows a normal distribution with

$$\begin{aligned}
\mu_k &= \frac{\sum x_{ik}[Y_i(t+6) - \mathbf{X}_{-k}^T\beta_{-k} - \rho Y_i(t)]}{\sum(x_{ik}^2 + \frac{\sigma^2}{n})} \\
\sigma_k^2 &= \Big\{\sum_{i=1}^{n}\frac{1}{\sigma^2}(x_{ik}^2 + \frac{\sigma^2}{n})\Big\}^{-1}
\end{aligned}$$

Posterior of ρ

The prior of ρ has the log-likelihood function:

$$\log \pi(\rho) = -\log(\sqrt{\frac{2\pi}{5}}) - \frac{25}{2}(\rho - \frac{1}{2})^2$$

The posterior of ρ is proportional to

$$
\begin{aligned}
&\text{const} - \frac{25}{2}(\rho - \frac{1}{2})^2 - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mathbf{X}^T\beta - \rho Y_i(t)\Big)^2 \\
&= \text{const} - \frac{n}{2\sigma^2}\Big(\frac{25\sigma^2}{n}\rho^2 - \frac{25\sigma^2}{4n}\rho\Big) - \sum_{i=1}^{n} \frac{1}{2\sigma^2}\Big(\rho^2 Y_i(t)^2 - 2\rho Y_i(t)[Y_i(t+6) - \mathbf{X}^T\beta]\Big) \\
&= \text{const} - (\frac{25}{2} + \frac{1}{2\sigma^2}\sum Y_i(t)^2)\rho^2 + \frac{1}{\sigma^2}\rho \sum \Big\{\Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\beta]\Big) + \frac{25\sigma^2}{8n}\Big\} \\
&= \text{const} - \frac{1}{2}\Big\{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2\Big\}\Big\{\rho - \frac{\frac{1}{\sigma^2}\sum\Big\{\Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\beta]\Big) + \frac{25\sigma^2}{8n}\Big\}}{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2}\Big\}^2
\end{aligned}
$$

Thus, the posterior of ρ follows a normal distribution with

$$
\begin{aligned}
\mu_\rho &= \frac{\frac{1}{\sigma^2}\sum\Big\{\Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\beta]\Big) + \frac{25\sigma^2}{8n}\Big\}}{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2} \\
&= \frac{\sum\Big(Y_i(t)[Y_i(t+6) - \mathbf{X}^T\beta]\Big) + \frac{25\sigma^2}{8}}{25\sigma^2 + \sum Y_i(t)^2} \\
\sigma_\rho^2 &= \Big\{25 + \frac{1}{\sigma^2}\sum Y_i(t)^2\Big\}^{-1}
\end{aligned}
$$

Posterior of σ^2

The prior of σ^2 has the log-likelihood function:

$$
\begin{aligned}
\log \pi(\sigma^2) &= \text{const} - (\alpha+1)\log\frac{1}{\sigma^2} + \frac{-\alpha'}{\sigma^2} \\
&= \text{const} - 2(\alpha+1)\log(\sigma) - \alpha'\sigma^{-2}
\end{aligned}
$$

The posterior of σ^2 is proportional to

$$
\begin{aligned}
&\text{const} - 2(\alpha+1)\log(\sigma) - \alpha'\sigma^{-2} - n\log\Big(\sigma\sqrt{2\pi}\Big) - \sum_{i=1}^{n}\frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mathbf{X}^T\beta - \rho Y_i(t)\Big)^2 \\
&= \text{const} - \Big(n + 2(\alpha+1)\Big)\log(\sigma) - \sigma^{-2}\Big\{\alpha' + \sum_{i=1}^{n}\frac{1}{2}\Big(Y_i(t+6) - \mathbf{X}^T\beta - \rho Y_i(t)\Big)^2\Big\}
\end{aligned}
$$

where $\alpha = \alpha' = 0.001$.

Thus, the posterior of ρ follows an inverse-gamma distribution with

$$
\begin{aligned}
\alpha_{post} &= n + 2\alpha + 1 \\
\alpha'_{post} &= \alpha' + \sum_{i=1}^{n}\frac{1}{2}\Big(Y_i(t+6) - \mathbf{X}^T\beta - \rho Y_i(t)\Big)^2
\end{aligned}
$$

Gibbs sampling algorithm

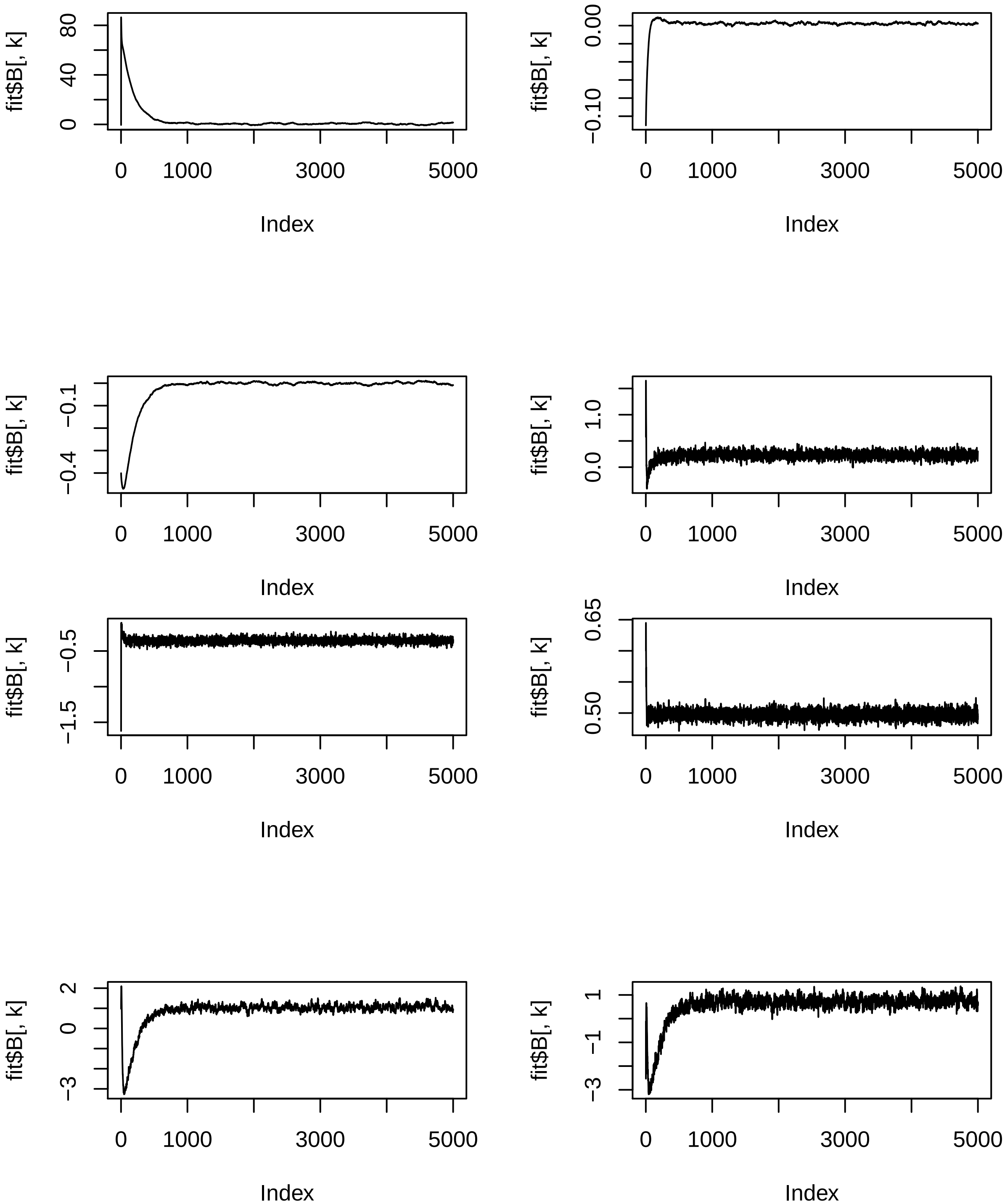
Denote $\theta = (\beta_0, \beta_1, ..., \beta_9, \rho, \sigma^2)$. We proceed as follows:
1. Begin with some initial values of θ^0.
2. Sample each component of the vector, θ, from the distribution of that component conditioned on all other components sampled so far. For example, for $k \geq 1$, Generate $\beta_0^{(k)}$ from

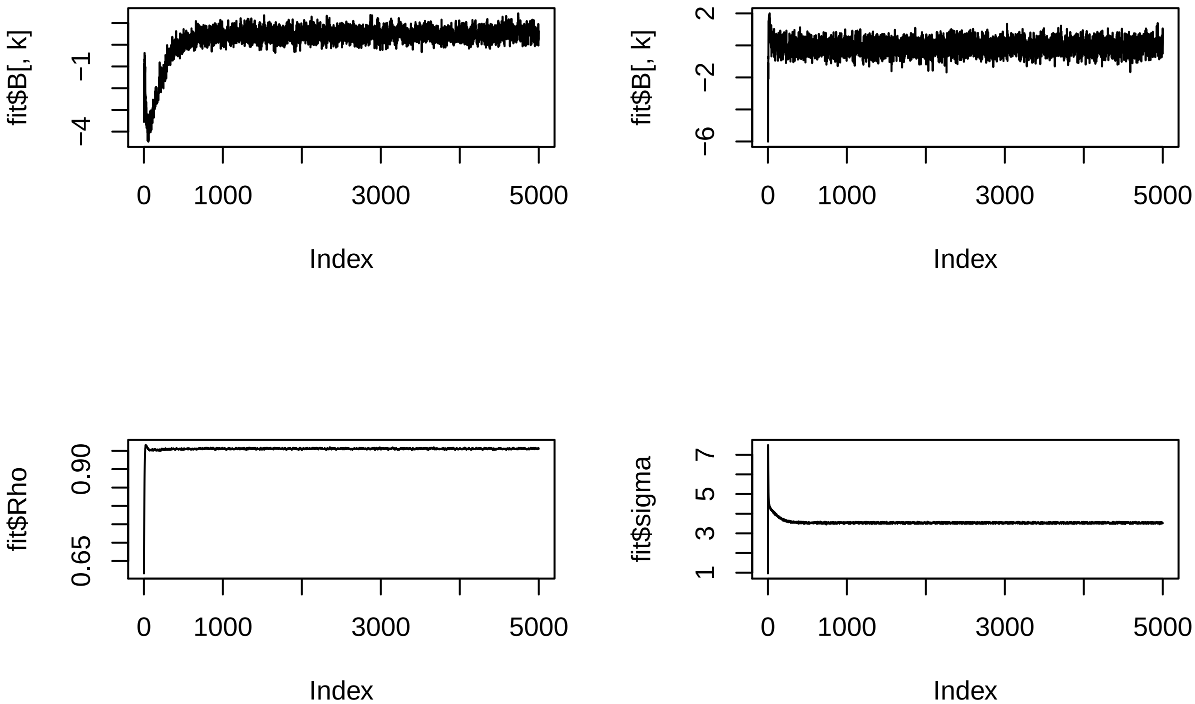
$\pi(\beta_0|\beta_1^{(k-1)}, ..., \beta_9^{(k-1)}, \rho^{(k-1)}, \sigma^{2(k-1)}, Y, \mathbf{X})$. Then generate $\beta_1^{(k)}$ from $\pi(\beta_1|\beta_0^{(k)}, \beta_2^{(k-1)}, ..., \beta_9^{(k-1)}, \rho^{(k-1)}, \sigma^{2(k-1)}, Y, \mathbf{X})$.
3. Repeat the above step k times.

We will randomly select 80% hurricanes and applied the proposed Gibbs sampling algorithm to estiamte the posterior distributions of the model parameters. Then we will apply the model to track the remaining 20% hurricans, and evaluate model performance in terms of how well could predict and track these hurricanes.

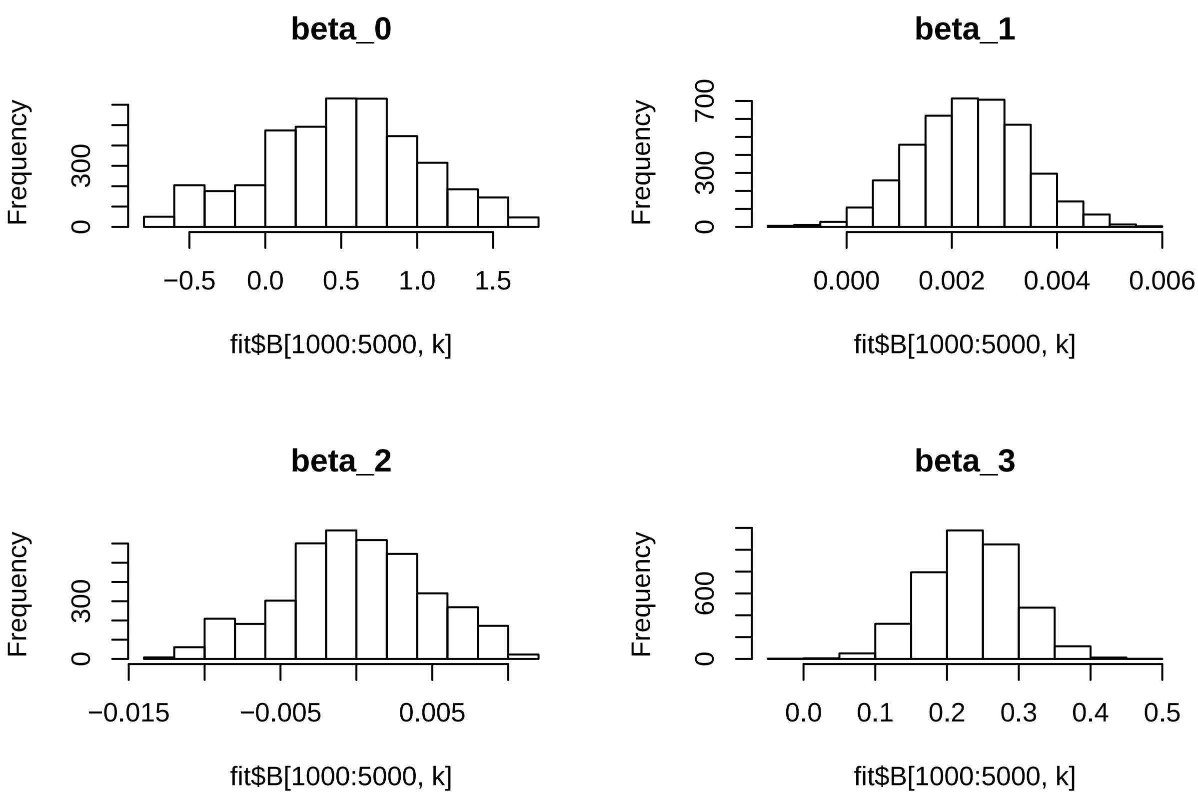
Results

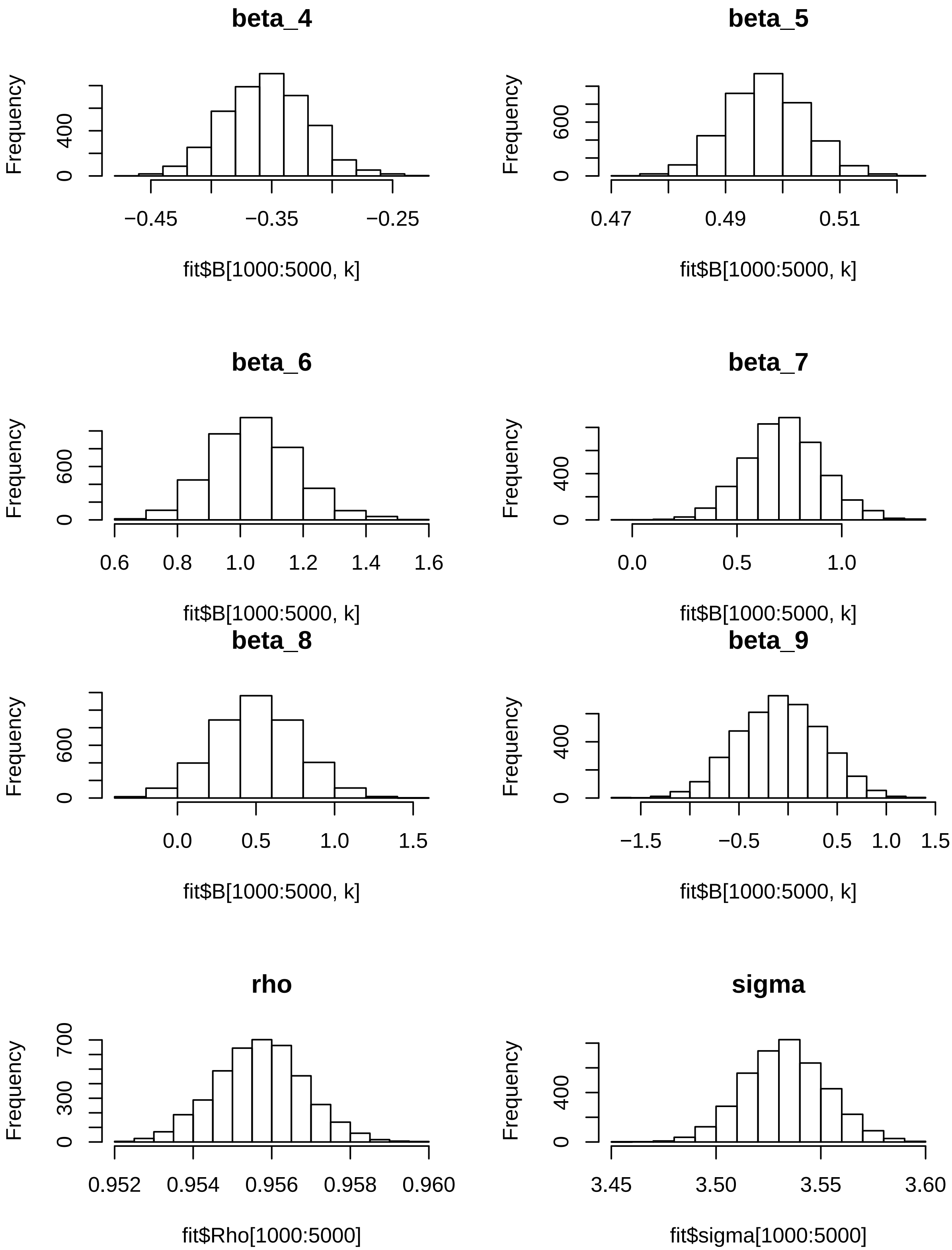
Parameter estimation of posterior distributions





The plots above shows the length of burn-in period and stationary stage of each Markov chains. On average, burn-in period is about 500 runs.





The histograms above show the distribution of parameter values after each chain enters stationary stage.

parameter	posterior.mean	CrI.low	CrI.high
(Intercept)	0.6485750	-0.5045427	36.3870108

parameter	posterior.mean	CrI.low	CrI.high
Yday	0.0023540	0.0000803	0.0063911
Year	-0.0013267	-0.3295719	0.0085877
DeltaLatitude	0.2346255	0.0747606	0.3566071
DeltaLongitude	-0.3539148	-0.4224030	-0.2818217
DeltaSpeed	0.4972567	0.4839647	0.5110733
NatureTS	1.0301557	-1.7909435	1.3176497
NatureET	0.7123291	-1.8519958	1.0847332
NatureSS	0.4867485	-2.5668894	1.0083927
NatureNR	-0.0839264	-0.9149674	0.7809101
Y(t)	0.9551738	0.9524984	0.9579976
sigma	3.5562687	3.4960875	3.8657191

The table above shows the estimated posterior mean of each parameter in the Bayesian model, with the associated 95% credibility intervals (CrI). According to this model, it seems that `DeltaSpeed`(change in wind speed) and `Y(t)` (the wind speed at current time point) are highly predictive of the wind speed at the next time point. More specifically, they are both positively associated with `Y(t+6)`, the wind speed after 6 hours. `Yday` (the day of a year at current time point) and `DeltaLatitude` (the change in latitude) also show significant association with the wind speed after 6 hours.

Discussion